Normal forms of vector fields

Python notebook at `github.com/joepatmckenna/gss`

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Graduate Student Seminar

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Outline

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Introduction

Overview

- Goal: Transform the o.d.e. $\dot{x} = f(x)$ near an equilibrium x_0 to the simplest orbitally equivalent equation.
- Method: Perform a sequence of substitutions $x = \psi(\xi)$ that zero out terms in the Taylor series of $f(x)$.
- Result: A module describing the 'essential' Taylor series terms. A diffeomorphism that maps the invariant manifolds of x_0 to the eigenspaces of the linearized equation $\dot{x} = f'(x_0)x$.

Substitution

- How does the substitution $x = \psi(\xi)$ transform the o.d.e. $\dot{x} = f(x)$?

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\psi'(\xi)\,\dot{\xi} &= \dot{x} && \text{differentiate } x = \psi(\xi) \\
\dot{\xi} &= \psi'(\xi)^{-1}\,\dot{x} && \text{left-multiply by } \psi'(\xi)^{-1} \\
&= \psi'(\xi)^{-1}\; f(\psi(\xi)) && \text{substitute } \dot{x} = f(x) = f(\psi(\xi)) \\
\dot{x} &= (\psi')^{-1}\; f \circ \psi(x) && \text{relabel } \xi \leftarrow x \\
&\stackrel{\text{def}}{=} \mathrm{S}_\psi f(x)
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- By a 'similarity' transformation.

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- If g generates the flow ψ, then the substitution $x = \psi(\xi)$ transforms $\dot{x} = f(x)$ to

$$\dot{x} = \mathsf{S}_\psi f(x) \sim e^{\mathsf{L}_g} f(x)$$

$$= \left(I + \mathsf{L}_g + \frac{1}{2}\mathsf{L}_g^2 + \frac{1}{3!}\mathsf{L}_g^3 + \cdots \right) (f_1 + f_2 + f_3 + \cdots)$$

where $\mathsf{L}_g f = f'g - g'f$
(Murdock, 2003)

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Substitution: one degree at a time

- If g_j has degree j, then the substitution $x = \psi_j(\xi)$ leaves the r.h.s. $f(x)$ unchanged up to degree $j-1$:

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\dot{x} \sim &(I + \mathsf{L}_{g_j} + \cdots)(f_1 + f_2 + \cdots + f_{j-1} + f_j + \cdots) \\
&= \underbrace{f_1 + \cdots + f_{j-1}}_{\text{unchanged}} + \underbrace{f_j + \mathsf{L}_{g_j} f_1}_{h_j = \text{ new deg } j \text{ term}} + \cdots
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- Which g_j should we choose?

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The algorithm

- At the j^{th} step of normalization, $\mathsf{L}_{f_1} g_j = f_j - h_j$

  1. find the degree j generator g_j:

     set h_j to projection of f_j onto $\overline{\mathrm{im}\ \mathsf{L}_{f_1}}$ and solve $\mathsf{L}_{f_1} g_j = f_j - h_j$

  2. perform the substitution $x = \psi_j(\xi)$ where g_j generates ψ_j:

     $\dot{x} \sim \left(I + \mathsf{L}_{g_j} + \frac{1}{2}\mathsf{L}_{g_j}^2 + \cdots \right) (f_1 + f_2 + f_3 + \cdots)$

- The composed substitution $x = \psi_k \circ \cdots \circ \psi_3 \circ \psi_2(\xi)$ modifies $\dot{x} = f(x)$ to

$$\dot{x} = f_1(x) + h_2(x) + h_3(x) + \cdots + h_k(x) + \cdots$$

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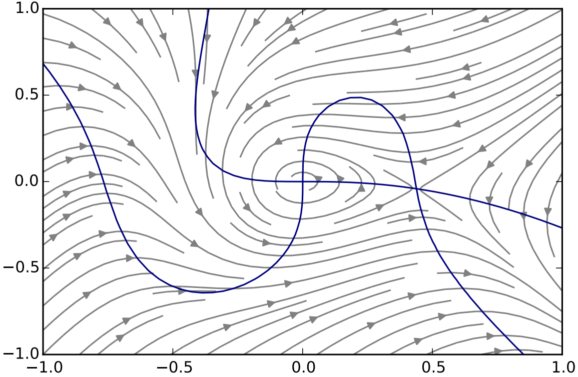
An example

$$\dot{x}_1 = -x_2 - 2x_1x_2 - x_1^3 - x_1^2x_2 - x_1x_2^2$$
$$\dot{x}_2 = x_1 - x_1^2 - 3x_1^3 - x_1^2x_2 - x_2^3$$



An example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-x_2\\ x_1\end{bmatrix}}_{f_1} + \underbrace{\begin{bmatrix}0 & -2 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1^2\\ x_1x_2\\ x_2^2\end{bmatrix}}_{f_2} + \underbrace{\begin{bmatrix}-1 & -1 & -1 & 0\\ -3 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2x_2\\ x_1x_2^2\\ x_2^3\end{bmatrix}}_{f_3}$$

$$f_1' = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

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L is (bi)linear, so L_{f_1} must have matrix representation

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Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 2x_1 & 0 \\ 0 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix}}_{g} = \begin{bmatrix} -2x_1x_2 \\ -x_1^2 \end{bmatrix}$$

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\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
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0 & 1 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} x_1 x_2 & 0 \end{bmatrix}^T \\
0 & -1 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1 x_2 \end{bmatrix}^T \\
0 & 0 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} x_2 & x_1 \\ 0 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix}}_{g} = \begin{bmatrix} x_1^2 - x_2^2 \\ -x_1x_2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|c}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
0 & 1 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & ? & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} x_2 & x_1 \\ 0 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix}}_{g} = \begin{bmatrix} x_1^2 - x_2^2 \\ -x_1x_2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|l}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
0 & 1 & ? & ? & ? & ? & \leftarrow \quad \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & ? & ? & ? & ? & \leftarrow \quad \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & ? & ? & ? & ? & \leftarrow \quad \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & ? & ? & ? & ? & \leftarrow \quad \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & ? & ? & ? & ? & \leftarrow \quad \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & ? & ? & ? & ? & \leftarrow \quad \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 2x_2 \\ 0 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}}_{g} = \begin{bmatrix} 2x_1x_2 \\ -x_2^2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|l}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
0 & 1 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & 2 & ? & ? & ? & \leftarrow \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & -1 & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 2x_2 \\ 0 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}}_{g} = \begin{bmatrix} 2x_1x_2 \\ -x_2^2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|l}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
0 & 1 & 0 & ? & ? & ? & \leftarrow \quad \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & 2 & ? & ? & ? & \leftarrow \quad \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & 0 & ? & ? & ? & \leftarrow \quad \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & 0 & ? & ? & ? & \leftarrow \quad \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & 0 & ? & ? & ? & \leftarrow \quad \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & -1 & ? & ? & ? & \leftarrow \quad \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 2x_2 \\ 0 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}}_{g} = \begin{bmatrix} 2x_1x_2 \\ -x_2^2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|l}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
0 & 1 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & 2 & ? & ? & ? & \leftarrow \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & 0 & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & -1 & ? & ? & ? & \leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 0 \\ 2x_1 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}}_{g} = \begin{bmatrix} x_1^2 \\ -2x_1x_2 \end{bmatrix}$$

$\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix}$ $\downarrow$	$\begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix}$ $\downarrow$	$\begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}$ $\downarrow$	$\begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}$ $\downarrow$	$\begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix}$ $\downarrow$	$\begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$ $\downarrow$	
0	1	0	1	?	?	$\leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T$
-2	0	2	0	?	?	$\leftarrow \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T$
0	-1	0	0	?	?	$\leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T$
-1	0	0	0	?	?	$\leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T$
0	-1	0	-2	?	?	$\leftarrow \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T$
0	0	-1	0	?	?	$\leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 0 \\ 2x_1 & 0 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}}_{g} = \begin{bmatrix} x_1^2 \\ -2x_1x_2 \end{bmatrix}$$

$\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} \downarrow$	$\begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} \downarrow$	$\begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} \downarrow$	$\begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} \downarrow$	$\begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} \downarrow$	$\begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} \downarrow$	
0	1	0	1	?	?	$\leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T$
-2	0	2	0	?	?	$\leftarrow \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T$
0	-1	0	0	?	?	$\leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T$
-1	0	0	0	?	?	$\leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T$
0	-1	0	-2	?	?	$\leftarrow \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T$
0	0	-1	0	?	?	$\leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix}0\\ x_1^2\end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix}0 & 0\\ 2x_1 & 0\end{bmatrix}}_{g'}\underbrace{\begin{bmatrix}-x_2\\ x_1\end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{f_1'}\underbrace{\begin{bmatrix}0\\ x_1^2\end{bmatrix}}_{g} = \begin{bmatrix}x_1^2\\ -2x_1x_2\end{bmatrix}$$

$$\begin{array}{ccc|ccc|c}
\begin{bmatrix}x_1^2\\ 0\end{bmatrix} & \begin{bmatrix}x_1x_2\\ 0\end{bmatrix} & \begin{bmatrix}x_2^2\\ 0\end{bmatrix} & \begin{bmatrix}0\\ x_1^2\end{bmatrix} & \begin{bmatrix}0\\ x_1x_2\end{bmatrix} & \begin{bmatrix}0\\ x_2^2\end{bmatrix} & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
0 & 1 & 0 & 1 & ? & ? & \leftarrow \quad \begin{bmatrix}x_1^2 & 0\end{bmatrix}^T \\
-2 & 0 & 2 & 0 & ? & ? & \leftarrow \quad \begin{bmatrix}x_1x_2 & 0\end{bmatrix}^T \\
0 & -1 & 0 & 0 & ? & ? & \leftarrow \quad \begin{bmatrix}x_2^2 & 0\end{bmatrix}^T \\
\hline
-1 & 0 & 0 & 0 & ? & ? & \leftarrow \quad \begin{bmatrix}0 & x_1^2\end{bmatrix}^T \\
0 & -1 & 0 & -2 & ? & ? & \leftarrow \quad \begin{bmatrix}0 & x_1x_2\end{bmatrix}^T \\
0 & 0 & -1 & 0 & ? & ? & \leftarrow \quad \begin{bmatrix}0 & x_2^2\end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix}0\\ x_1x_2\end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix}0 & 0\\ x_2 & x_1\end{bmatrix}}_{g'}\underbrace{\begin{bmatrix}-x_2\\ x_1\end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{f_1'}\underbrace{\begin{bmatrix}0\\ x_1x_2\end{bmatrix}}_{g} = \begin{bmatrix}x_1x_2\\ x_1^2 - x_2^2\end{bmatrix}$$

$$\begin{array}{ccc|ccc|cc}
\begin{bmatrix}x_1^2\\ 0\end{bmatrix} & \begin{bmatrix}x_1x_2\\ 0\end{bmatrix} & \begin{bmatrix}x_2^2\\ 0\end{bmatrix} & \begin{bmatrix}0\\ x_1^2\end{bmatrix} & \begin{bmatrix}0\\ x_1x_2\end{bmatrix} & \begin{bmatrix}0\\ x_2^2\end{bmatrix} & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\hline
0 & 1 & 0 & 1 & 0 & ? & \leftarrow & \begin{bmatrix}x_1^2 & 0\end{bmatrix}^T \\
-2 & 0 & 2 & 0 & 1 & ? & \leftarrow & \begin{bmatrix}x_1x_2 & 0\end{bmatrix}^T \\
0 & -1 & 0 & 0 & 0 & ? & \leftarrow & \begin{bmatrix}x_2^2 & 0\end{bmatrix}^T \\
\hline
-1 & 0 & 0 & 0 & 1 & ? & \leftarrow & \begin{bmatrix}0 & x_1^2\end{bmatrix}^T \\
0 & -1 & 0 & -2 & 0 & ? & \leftarrow & \begin{bmatrix}0 & x_1x_2\end{bmatrix}^T \\
0 & 0 & -1 & 0 & -1 & ? & \leftarrow & \begin{bmatrix}0 & x_2^2\end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 0 \\ x_2 & x_1 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix}}_{g} = \begin{bmatrix} x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|cc}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\hline
0 & 1 & 0 & 1 & 0 & ? & \leftarrow & \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & 2 & 0 & 1 & ? & \leftarrow & \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & 0 & 0 & 0 & ? & \leftarrow & \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & 0 & 0 & 1 & ? & \leftarrow & \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & 0 & -2 & 0 & ? & \leftarrow & \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & -1 & 0 & -1 & ? & \leftarrow & \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 0 \\ x_2 & x_1 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix}}_{g} = \begin{bmatrix} x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

$$\begin{array}{cccc|ccc|l}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} && \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & \\
\downarrow & \downarrow & \downarrow && \downarrow & \downarrow & \downarrow & \\
\hline
0 & 1 & 0 && 1 & 0 & ? & \leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & 2 && 0 & 1 & ? & \leftarrow \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & 0 && 0 & 0 & ? & \leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & 0 && 0 & 1 & ? & \leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & 0 && -2 & 0 & ? & \leftarrow \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & -1 && 0 & -1 & ? & \leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2x_2 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}}_{g} = \begin{bmatrix} x_2^2 \\ 2x_1x_2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|cc}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\hline
0 & 1 & 0 & 1 & 0 & 0 & \leftarrow & \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & 2 & 0 & 1 & 0 & \leftarrow & \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & 0 & 0 & 0 & 1 & \leftarrow & \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & 0 & 0 & 1 & 0 & \leftarrow & \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & 0 & -2 & 0 & 2 & \leftarrow & \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & -1 & 0 & -1 & 0 & \leftarrow & \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2x_2 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}}_{g} = \begin{bmatrix} x_2^2 \\ 2x_1x_2 \end{bmatrix}$$

$$\begin{array}{ccc|ccc|cc}
\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} & \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} & \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\hline
0 & 1 & 0 & 1 & 0 & 0 & \leftarrow & \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T \\
-2 & 0 & 2 & 0 & 1 & 0 & \leftarrow & \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T \\
0 & -1 & 0 & 0 & 0 & 1 & \leftarrow & \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T \\
\hline
-1 & 0 & 0 & 0 & 1 & 0 & \leftarrow & \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T \\
0 & -1 & 0 & -2 & 0 & 2 & \leftarrow & \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T \\
0 & 0 & -1 & 0 & -1 & 0 & \leftarrow & \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_2^2$

$$g = \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} \Rightarrow \mathsf{L}_{f_1} g = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2x_2 \end{bmatrix}}_{g'} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{f_1'} \underbrace{\begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}}_{g} = \begin{bmatrix} x_2^2 \\ 2x_1x_2 \end{bmatrix}$$

$\begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} \downarrow$	$\begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix} \downarrow$	$\begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} \downarrow$	$\begin{bmatrix} 0 \\ x_1^2 \end{bmatrix} \downarrow$	$\begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix} \downarrow$	$\begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} \downarrow$	
0	1	0	1	0	0	$\leftarrow \begin{bmatrix} x_1^2 & 0 \end{bmatrix}^T$
-2	0	2	0	1	0	$\leftarrow \begin{bmatrix} x_1x_2 & 0 \end{bmatrix}^T$
0	-1	0	0	0	1	$\leftarrow \begin{bmatrix} x_2^2 & 0 \end{bmatrix}^T$
-1	0	0	0	1	0	$\leftarrow \begin{bmatrix} 0 & x_1^2 \end{bmatrix}^T$
0	-1	0	-2	0	2	$\leftarrow \begin{bmatrix} 0 & x_1x_2 \end{bmatrix}^T$
0	0	-1	0	-1	0	$\leftarrow \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T$

An example

- $(j = 2)$ Step 1: Find degree 2 generator g_2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_2\\ x_1\end{bmatrix} + \begin{bmatrix}0 & -2 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1^2\\ x_1x_2\\ x_2^2\end{bmatrix} + \begin{bmatrix}-1 & -1 & -1 & 0\\ -3 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2x_2\\ x_1x_2^2\\ x_2^3\end{bmatrix}$$

$$\underbrace{\left[\begin{array}{ccc|ccc} 0 & 1 & & 1 & & \\ -2 & 0 & 2 & & 1 & \\ & -1 & 0 & & & 1 \\ \hline -1 & & & 0 & 1 & \\ & -1 & & -2 & 0 & 2 \\ & & -1 & & -1 & 0 \end{array}\right]}_{\mathsf{L}_{f_1}} g_2 = \underbrace{\left[\begin{array}{c} 0\\ -2\\ 0\\ \hline -1\\ 0\\ 0\end{array}\right]}_{f_2} - h_2 \quad \Rightarrow \begin{cases} h_2 = 0\\ g_2 = \begin{bmatrix}x_1^2\\ 0\end{bmatrix}\end{cases}$$

An example

- $(j=2)$ Step 1: Find degree 2 generator g_2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_2\\ x_1\end{bmatrix} + \begin{bmatrix}0 & -2 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1^2\\ x_1x_2\\ x_2^2\end{bmatrix} + \begin{bmatrix}-1 & -1 & -1 & 0\\ -3 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2x_2\\ x_1x_2^2\\ x_2^3\end{bmatrix}$$

$$\underbrace{\left[\begin{array}{ccc|ccc} 0 & 1 & & 1 & & \\ -2 & 0 & 2 & & 1 & \\ & -1 & 0 & & & 1 \\ \hline -1 & & & 0 & 1 & \\ & -1 & & -2 & 0 & 2 \\ & & -1 & & -1 & 0 \end{array}\right]}_{\mathsf{L}_{f_1}} g_2 = \underbrace{\left[\begin{array}{c} 0\\ -2\\ 0\\ \hline -1\\ 0\\ 0\end{array}\right]}_{f_2} - h_2 \qquad \Rightarrow \begin{cases} h_2 = 0 \\ g_2 = \begin{bmatrix}x_1^2\\ 0\end{bmatrix}\end{cases}$$

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$$\underbrace{\left[\begin{array}{ccc|ccc} 0 & 1 & & 1 & & \\ -2 & 0 & 2 & & 1 & \\ & -1 & 0 & & & 1 \\ \hline -1 & & & 0 & 1 & \\ & -1 & & -2 & 0 & 2 \\ & & -1 & & -1 & 0 \end{array}\right]}_{\mathsf{L}_{f_1}} g_2 = \underbrace{\left[\begin{array}{c} 0\\ -2\\ 0\\ \hline -1\\ 0\\ 0\end{array}\right]}_{f_2} - h_2 \qquad \Rightarrow \begin{cases} h_2 = 0 \\ g_2 = \begin{bmatrix}x_1^2\\ 0\end{bmatrix}\end{cases}$$

An example

- $(j = 2)$ Step 2: perform substitution $x = \psi_2(\xi)$ where g_2 generates ψ_2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} \sim \left(I + \mathsf{L}_{g_2} + \frac{1}{2}\mathsf{L}_{g_2}^2 + \cdots\right)(f_1 + f_2 + f_3 + \cdots)$$

$$\begin{array}{cccccccc}
 & \text{deg } 1 & & \text{deg } 2 & & \text{deg } 3 & & \text{deg } \geq 4 \\
= & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & + & \mathsf{L}_{g_2} f_1 & + & \mathsf{L}_{g_2} f_2 & + & \cdots \\
 & & & & + & \frac{1}{2}\mathsf{L}_{g_2}^2 f_1 & + & \cdots \\
= & \begin{bmatrix}-x_2\\ x_1\end{bmatrix} & & & + & \begin{bmatrix}-1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2x_2\\ x_1x_2^2\\ x_2^3\end{bmatrix} & + & \cdots
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= & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & + & \mathsf{L}_{g_2} f_1 & + & \mathsf{L}_{g_2} f_2 & + & \cdots \\
 & & & & + & \frac{1}{2}\mathsf{L}_{g_2}^2 f_1 & + & \cdots \\
= & \begin{bmatrix}-x_2\\ x_1\end{bmatrix} & & & + & \begin{bmatrix}-1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2x_2\\ x_1x_2^2\\ x_2^3\end{bmatrix} & + & \cdots
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= & \begin{bmatrix}-x_2\\ x_1\end{bmatrix} & & & + & \begin{bmatrix}-1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2 x_2\\ x_1 x_2^2\\ x_2^3\end{bmatrix} & + & \cdots
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 & & & & + & \frac{1}{2}\mathsf{L}_{g_2}^2 f_1 & + & \cdots
\end{array}$$

$$= \begin{bmatrix}-x_2\\ x_1\end{bmatrix} + \begin{bmatrix}-1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2x_2\\ x_1x_2^2\\ x_2^3\end{bmatrix} + \cdots$$

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$$\begin{array}{cccccccc}
 & \text{deg } 1 & & \text{deg } 2 & & \text{deg } 3 & & \text{deg } \geq 4 \\
= & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & + & \mathsf{L}_{g_2} f_1 & + & \mathsf{L}_{g_2} f_2 & + & \cdots \\
 & & & & + & \frac{1}{2}\mathsf{L}_{g_2}^2 f_1 & + & \cdots \\
= & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & & & + & \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix} & + & \cdots
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= & \begin{bmatrix}-x_2\\ x_1\end{bmatrix} & & & + & \begin{bmatrix}-1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2 x_2\\ x_1 x_2^2\\ x_2^3\end{bmatrix} & + & \cdots
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= & \begin{bmatrix}-x_2\\ x_1\end{bmatrix} & & & + & \begin{bmatrix}-1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2 x_2\\ x_1 x_2^2\\ x_2^3\end{bmatrix} & + & \cdots
\end{array}$$

Matrix representation of L_{f_1} on $\mathcal{V}_3^2$

$$\mathsf{L}_{f_1}|_{\mathcal{V}_3^2} = \left[\begin{array}{cccc|cccc} 0 & 1 & & & 1 & & & \\ -3 & 0 & 2 & & & 1 & & \\ & -2 & 0 & 3 & & & 1 & \\ & & -1 & 0 & & & & 1 \\ \hline -1 & & & & 0 & 1 & & \\ & -1 & & & -3 & 0 & 2 & \\ & & -1 & & & -2 & 0 & 3 \\ & & & -1 & & & -1 & 0 \end{array}\right]$$

- $\dim(\ker \mathsf{L}_{f_1}|_{\mathcal{V}_3^2}) = 2$
- basis for $\overline{\mathsf{im}\ \mathsf{L}_{f_1}|_{\mathcal{V}_3^2}}$:

$$\begin{matrix} \begin{bmatrix}1 & 0 & 1 & 0 & 0 & 1 & 0 & 1\end{bmatrix}^T \\ \begin{bmatrix}0 & -1 & 0 & -1 & 1 & 0 & 1 & 0\end{bmatrix}^T \end{matrix} \Rightarrow h_3 = \alpha_3(x_1^2+x_2^2)\begin{bmatrix}x_1\\x_2\end{bmatrix} + \beta_3(x_1^2+x_2^2)\begin{bmatrix}-x_2\\x_1\end{bmatrix}$$

Matrix representation of L_{f_1} on $\mathcal{V}_3^2$

$$\mathsf{L}_{f_1}|_{\mathcal{V}_3^2} = \left[\begin{array}{cccc|cccc}
0 & 1 & & & 1 & & & \\
-3 & 0 & 2 & & & 1 & & \\
 & -2 & 0 & 3 & & & 1 & \\
 & & -1 & 0 & & & & 1 \\
\hline
-1 & & & & 0 & 1 & & \\
 & -1 & & & -3 & 0 & 2 & \\
 & & -1 & & & -2 & 0 & 3 \\
 & & & -1 & & & -1 & 0
\end{array}\right]$$

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Matrix representation of L_{f_1} on $\mathcal{V}_3^2$

$$\mathsf{L}_{f_1}\big|_{\mathcal{V}_3^2} = \left[\begin{array}{cccc|cccc} 0 & 1 & & & 1 & & & \\ -3 & 0 & 2 & & & 1 & & \\ & -2 & 0 & 3 & & & 1 & \\ & & -1 & 0 & & & & 1 \\ \hline -1 & & & & 0 & 1 & & \\ & -1 & & & -3 & 0 & 2 & \\ & & -1 & & & -2 & 0 & 3 \\ & & & -1 & & & -1 & 0 \end{array}\right]$$

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0 & 1 & & & 1 & & & \\
-3 & 0 & 2 & & & 1 & & \\
 & -2 & 0 & 3 & & & 1 & \\
 & & -1 & 0 & & & & 1 \\
\hline
-1 & & & & 0 & 1 & & \\
 & -1 & & & -3 & 0 & 2 & \\
 & & -1 & & & -2 & 0 & 3 \\
 & & & -1 & & & -1 & 0
\end{array}\right]$$

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An example

- $(j=3)$ Step 1: Find degree 3 generator g_3

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_2\\ x_1\end{bmatrix} + \begin{bmatrix}-1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1^3\\ x_1^2x_2\\ x_1x_2^2\\ x_2^3\end{bmatrix} + \cdots$$

$$\underbrace{\left[\begin{array}{cccc|cccc} 0 & 1 & & & 1 & & & \\ -3 & 0 & 2 & & & 1 & & \\ & -2 & 0 & 3 & & & 1 & \\ & & -1 & 0 & & & & 1 \\ \hline -1 & & & & 0 & 1 & & \\ & -1 & & & -3 & 0 & 2 & \\ & & -1 & & & -2 & 0 & 3 \\ & & & -1 & & & -1 & 0 \end{array}\right]}_{\mathsf{L}_{f_1}} g_3 = \underbrace{\begin{bmatrix}-1\\ 0\\ -1\\ 0\\ 0\\ -1\\ 0\\ -1\end{bmatrix}}_{f_3} - h_3 \Rightarrow \begin{cases} h_3 = f_3\\ g_3 = 0\end{cases}$$

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix} + \cdots$$

$$\underbrace{\left[\begin{array}{cccc|cccc} 0 & 1 & & & 1 & & & \\ -3 & 0 & 2 & & & 1 & & \\ & -2 & 0 & 3 & & & 1 & \\ & & -1 & 0 & & & & 1 \\ \hline -1 & & & & 0 & 1 & & \\ & -1 & & & -3 & 0 & 2 & \\ & & -1 & & & -2 & 0 & 3 \\ & & & -1 & & & -1 & 0 \end{array}\right]}_{\mathsf{L}_{f_1}} g_3 = \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}}_{f_3} - h_3 \Rightarrow \left\{ \begin{array}{l} h_3 = f_3 \\ g_3 = 0 \end{array} \right.$$

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An example

- $(j = 3)$ Step 2: perform substitution $x = \psi_3(\xi)$ where g_3 generates ψ_3

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \sim (I + \mathsf{L}_{g_3} + \cdots)\,(f_1 + h_2 + f_3 + \cdots)$$

$$\begin{array}{cccccccc}
 & \text{deg } 1 & & \text{deg } 2 & & \text{deg } 3 & & \text{deg } \geq 4 \\
= & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & & & + & \mathsf{L}_{g_3} f_1 & + & \cdots \\
= & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & & & - & (x_1^2 + x_2^2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & + & \cdots
\end{array}$$

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= & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & & & + & \mathsf{L}_{g_3} f_1 & + & \cdots \\
= & \begin{bmatrix}-x_2\\ x_1\end{bmatrix} & & & - & (x_1^2+x_2^2)\begin{bmatrix}x_1\\ x_2\end{bmatrix} & + & \cdots
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= & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & & & + & \mathsf{L}_{g_3} f_1 & + & \cdots \\
= & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & & & - & (x_1^2 + x_2^2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & + & \cdots
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$$\begin{array}{ccccccccc}
 & \text{deg } 1 & & \text{deg } 2 & & \text{deg } 3 & & \text{deg } \geq 4 \\
 = & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & & & + & \mathsf{L}_{g_3} f_1 & + & \cdots \\
 = & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & & & - & (x_1^2 + x_2^2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & + & \cdots
\end{array}$$

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 & \text{deg } 1 & & \text{deg } 2 & & \text{deg } 3 & & \text{deg } \geq 4 \\
 = & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & & & + & \mathsf{L}_{g_3} f_1 & + & \cdots \\
 = & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & & & - & (x_1^2 + x_2^2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & + & \cdots
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$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} \sim \left(I + \mathsf{L}_{g_3} + \cdots\right)\left(f_1 + h_2 + f_3 + \cdots\right)$$

$$\begin{array}{cccccccc}
 & \text{deg } 1 & & \text{deg } 2 & & \text{deg } 3 & & \text{deg } \geq 4 \\[2ex]
= & f_1 & + & f_2 & + & f_3 & + & \cdots \\
 & & & & + & \mathsf{L}_{g_3} f_1 & + & \cdots \\[2ex]
= & \begin{bmatrix}-x_2\\ x_1\end{bmatrix} & & & - & (x_1^2+x_2^2)\begin{bmatrix}x_1\\ x_2\end{bmatrix} & + & \cdots
\end{array}$$

An example

- ($j = 3$) Step 2: perform substitution $x = \psi_3(\xi)$ where g_3 generates ψ_3

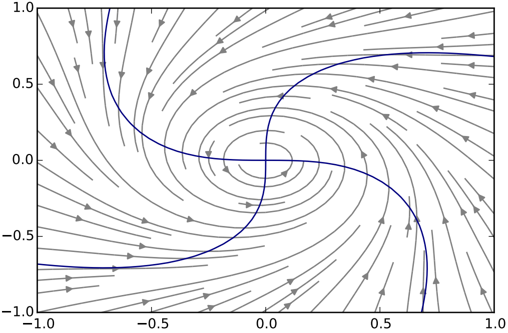
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \sim (I + \mathsf{L}_{g_3} + \cdots)(f_1 + h_2 + f_3 + \cdots)$$

$$\begin{array}{ccccccccc} & \text{deg } 1 & & \text{deg } 2 & & \text{deg } 3 & & \text{deg } \geq 4 \\ = & f_1 & + & f_2 & + & f_3 & + & \cdots \\ & & & & + & \mathsf{L}_{g_3} f_1 & + & \cdots \\ = & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & & & - & (x_1^2 + x_2^2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & + & \cdots \end{array}$$

An example

- Normal form truncated at degree 3:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_2\\x_1\end{bmatrix} - \left(x_1^2 + x_2^2\right)\begin{bmatrix}x_1\\x_2\end{bmatrix}$$



Normal form module

- $\dim(\ker \mathsf{L}_{f_1}|_{\mathcal{V}_n^2}) = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$

- basis for $\overline{\operatorname{im} \mathsf{L}_{f_1}|_{\mathcal{V}_n^2}}$ when n odd and $m = (n-1)/2$:

$$\begin{bmatrix} \binom{m}{0} & 0 & \binom{m}{1} & 0 & \cdots & \binom{m}{m} & 0 & 0 & \binom{m}{0} & 0 & \binom{m}{1} & 0 & \cdots & \binom{m}{m} \end{bmatrix}^T$$

$$\begin{bmatrix} 0 & -\binom{m}{0} & 0 & -\binom{m}{1} & 0 & \cdots & -\binom{m}{m} & \binom{m}{0} & 0 & \binom{m}{1} & 0 & \cdots & \binom{m}{m} & 0 \end{bmatrix}$$

$$\Rightarrow h_n = \alpha_n (x_1^2 + x_2^2)^m \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta_n (x_1^2 + x_2^2)^m \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

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$$\ker \mathsf{L}_{f_1} = \mathbb{R}[[r^2]] v_1 \oplus \mathbb{R}[[r^2]] v_2$$

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- $x_1^2 + x_2^2$ is a polynomial *invariant* of the flow of $f_1 = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$

$$\left(\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \cdot \nabla \right) (x_1^2 + x_2^2) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \cdot \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 0$$

- $v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ are *equivariants* of the flow of $f_1 = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$

$$\mathsf{L}_{f_1} v_i = 0 \quad \implies \quad \text{the flows of } f_1 \text{ and } v_i \text{ commute}$$

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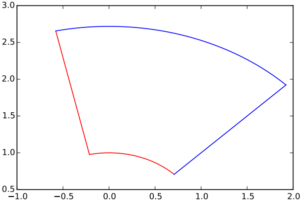
Normal form geometry

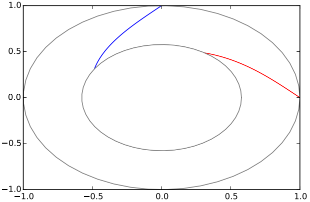
- The flow φ^t of the normal form

$$\dot{x} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + \sum_{n \geq 1} (x_1^2 + x_2^2)^n \left(\alpha_n \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta_n \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \right)$$

preserves the foliation $\mathcal{F}$ induced by the linearization flow $\psi = e^{f_1' t}$.

Assume z_1, z_2 are on same leaf (circle) of $\mathcal{F}$, so $z_2 = \psi(z_1)$.
After some time t, $\varphi^t(z_2) = \varphi^t(\psi(z_1)) = \psi(\varphi^t(z_1))$.
Therefore, $\varphi^t(z_1), \varphi^t(z_2)$ are on the same leaf of $\mathcal{F}$.





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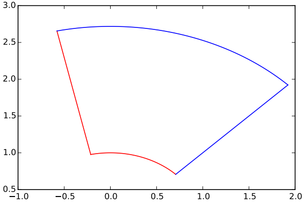

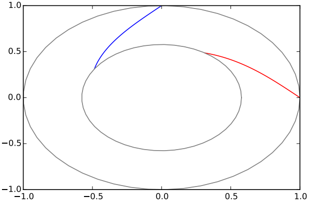

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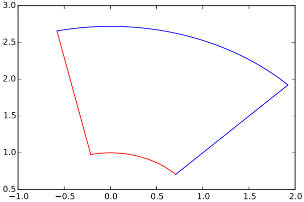

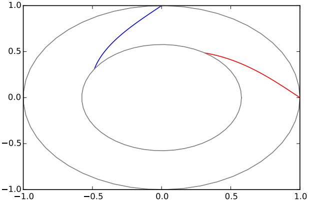

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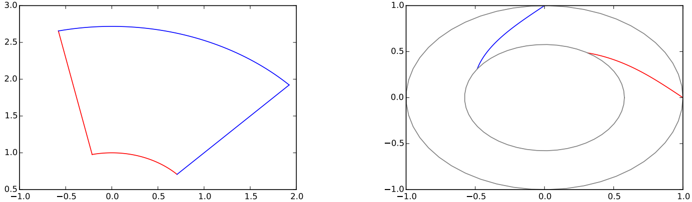

The End

Thanks for coming! Questions?

Appendix

Theorem

$L_g f = f'g - g'f$

$$
\begin{aligned}
\frac{d}{ds}\mathrm{S}_\psi f &= \frac{d}{ds}(\psi')^{-1}\ f\circ\psi && \text{def. of } \mathrm{S}_\psi \\
&= (\psi')^{-1}\ f'\circ\psi\ \frac{d}{ds}\psi + \left[\frac{d}{ds}(\psi')^{-1}\right]\ f\circ\psi && \text{prod. \& chain rules} \\
&= (\psi')^{-1}\ f'\circ\psi\ g\circ\psi - \underbrace{(\psi')^{-1}\ g'\circ\psi}_{i.o.u.}\ f\circ\psi && g \text{ generates } \psi
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\frac{d}{ds}\ \mathrm{S}_\psi f\big|_{s=0} &= f'g - g'f && \text{substitute } s=0 \\
&\stackrel{\text{def}}{=} [f,g] && \text{`Lie bracket'}
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$$\frac{d}{ds}\mathsf{S}_\psi f = \frac{d}{ds}(\psi')^{-1} f \circ \psi = (\psi')^{-1}\,\mathsf{L}_g f \circ \psi \qquad \text{last slide}$$

$$\frac{d^j}{ds^j}\mathsf{S}_\psi f = (\psi')^{-1}\,\mathsf{L}_g^j f \circ \psi \qquad \text{iteration}$$

$$\frac{d^j}{ds^j}\,\mathsf{S}_\psi f\big|_{s=0} = \mathsf{L}_g^j f \qquad \text{substitute } s = 0$$

$$\mathsf{S}_\psi f = \left(I + \mathsf{L}_g + \frac{1}{2!}\mathsf{L}_g^2 + \cdots\right) f \qquad \text{Taylor series}$$

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$$\frac{d^j}{ds^j}\mathsf{S}_\psi f = (\psi')^{-1}\, \mathsf{L}_g^j f \circ \psi \qquad \text{iteration}$$

$$\frac{d^j}{ds^j}\, \mathsf{S}_\psi f\big|_{s=0} = \mathsf{L}_g^j f \qquad \text{substitute } s = 0$$

$$\mathsf{S}_\psi f = \left(I + \mathsf{L}_g + \frac{1}{2!}\mathsf{L}_g^2 + \cdots\right) f \qquad \text{Taylor series}$$

Appendix

$$
\begin{tikzcd}
\psi \arrow[d, "\mathsf{D}_s"'] & \varphi \arrow[r, "\mathrm{C}_\psi"] \arrow[d, "\mathsf{D}_t"'] & \psi^{-1} \circ \varphi \circ \psi \arrow[d, "\mathrm{D}_t"] \\
g & f \arrow[r, "\mathsf{S}_\psi"] \arrow[dr, "\mathsf{L}_g"'] & (\psi')^{-1}\, f \circ \psi \arrow[d, "\mathsf{D}_s"] \\
 & & [f, g]
\end{tikzcd}
$$