Homework 1 Foundations of Computational Math 1 Fall 2012

Problem 1.1

This problem considers three basic vector norms: $\|.\|_1$,$\|.\|_2$,$\|.\|_\infty$.

1.1.a. Prove that $\|.\|_1$ is a vector norm.

1.1.b. Prove that $\|.\|_\infty$ is a vector norm.

1.1.c. Consider $\|.\|_2$.

(i) Show that $\|.\|_2$ is definite.

(ii) Show that $\|.\|_2$ is homogeneous.

(iii) Show that for $\|.\|_2$ the triangle inequality follows from the Cauchy inequality $|x^H y| \leq \|x\|_2 \|y\|_2$.

(iv) Assume you have two vectors x and y such that $\|x\|_2 = \|y\|_2 = 1$ and $x^H y = |x^H y|$, prove the Cauchy inequality holds for x and y.

(v) Assume you have two arbitrary vectors $\tilde{x}$ and $\tilde{y}$. Show that there exists x and y that satisfy the conditions of part (iv) and $\tilde{x} = \alpha x$ and $\tilde{y} = \beta y$ where α and β are scalars.

(vi) Show the Cauchy inequality holds for two arbitrary vectors $\tilde{x}$ and $\tilde{y}$.

Problem 1.2

What is the unit ball in $\mathbb{R}^2$ for each of the vector norms: $\|.\|_1$,$\|.\|_2$,$\|.\|_\infty$?

Problem 1.3

Consider the matrices

$$B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 2 \\ 0 & 2 \\ -1 & 1 \end{pmatrix}$$

1.3.a. Show that they have the same range space.

1.3.b. We have $x = B_1 c_1 = B_2 c_2$ for all x in the range space. Determine the relationship between c_1 and c_2 and express it as a linear transformation.

Problem 1.4

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, i.e.,

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$$

.

1.4.a. Suppose you are given a routine that returns $F(x)$ given any $x \in \mathbb{R}^n$. How would you use this routine to determine a matrix $A \in \mathbb{R}^{m\times n}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^n$?

1.4.b. Show A is unique.

Problem 1.5

Consider the matrix

$$L = \begin{pmatrix} \lambda_{11} & 0 & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix}$$

Suppose that $\lambda_{11} \neq 0$, $\lambda_{33} \neq 0$, $\lambda_{44} \neq 0$ but $\lambda_{22} = 0$.

1.5.a. Show that L is singular.

1.5.b. Determine a basis for the nullspace $\mathcal{N}(L)$.

Problem 1.6

1.6.a. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be nonsingular matrices. Show $(AB)^{-1} = B^{-1}A^{-1}$.

1.6.b. Suppose $A \in \mathbb{R}^{m\times n}$ with $m > n$ and let $M \in \mathbb{R}^{n\times n}$ be a nonsingular square matrix. Show that $\mathcal{R}(A) = \mathcal{R}(AM)$ where $\mathcal{R}()$ denotes the range of a matrix.

Problem 1.7

Let $y \in \mathbb{R}^m$ and $\|y\|$ be any vector norm defined on $\mathbb{R}^m$. Let $x \in \mathbb{R}^n$ and A be an $m \times n$ matrix with $m > n$.

1.7.a. Show that the function $f(x) = \|Ax\|$ is a vector norm on $\mathbb{R}^n$ if and only if A has full column rank, i.e., $rank(A) = n$.

1.7.b. Suppose we choose $f(x)$ from part (1.7.a) to be $f(x) = \|Ax\|_2$. What condition on A guarantees that $f(x) = \|x\|_2$ for any vector $x \in \mathbb{R}^n$?