Preliminary Exam

August 20, 2002

Do **FOUR** of the following six problems **ONLY**! Show all relevant work!

1. Consider the boundary value problem

$$
\begin{aligned}
u''(x) + a(x)u'(x) + b(x)u(x) &= f(x) \quad , \quad 0 < x < 1 \\
u(0) &= \alpha \\
u(1) &= \beta
\end{aligned}
$$

(a) Use a centered finite difference approximation for the derivatives to write down a system of N finite difference equations corresponding to the problem. Explicitly write the matrix and vectors.

Solution:

$$
\begin{gathered}
u(x_j - h) - 2u(x_j) + u(x_j + h) + a(x_j)\frac{h}{2}\left[u(x_j + h) - u(x_j - h)\right] + h^2 b(x_j)u(x_j) = h^2 f(x_j)\,, \\
x_j = j/(N+1), \quad j = 1, 2, .., N\,, \\
h = 1/(N+1)
\end{gathered}
$$

$$
u(x_j - h)\left[1 - a(x_j)\frac{h}{2}\right] + u(x_j)\left[-2 + h^2 b(x_j)\right] + u(x_j + h)\left[1 + a(x_j)\frac{h}{2}\right] = h^2 f(x_j)\,,
$$

$$
\mathbf{A} = \begin{bmatrix}
-2 + h^2 b(x_1) & 1 + a(x_1)\frac{h}{2} & 0 & 0 & \cdots & 0 \\
1 - a(x_2)\frac{h}{2} & -2 + h^2 b(x_2) & 1 + a(x_2)\frac{h}{2} & 0 & \cdots & 0 \\
0 & 1 - a(x_3)\frac{h}{2} & -2 + h^2 b(x_3) & 1 + a(x_3)\frac{h}{2} & 0 & \vdots \\
\vdots & 0 & 1 - a(x_j)\frac{h}{2} & -2 + h^2 b(x_j) & 1 + a(x_j)\frac{h}{2} & 0 \\
0 & \cdots & 0 & 1 - a(x_{N-1})\frac{h}{2} & -2 + h^2 b(x_{N-1}) & 1 + a(x_{N-1})\frac{h}{2} \\
0 & 0 & \cdots & 0 & 1 - a(x_N)\frac{h}{2} & -2 + h^2 b(x_N)
\end{bmatrix},
$$

$$
\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \quad , \quad \vec{r} = \begin{bmatrix} h^2 f(x_1) - (1 - a(x_1)\frac{h}{2})\alpha \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{N-1}) \\ h^2 f(x_N) - (1 + a(x_N)\frac{h}{2})\beta \end{bmatrix} \quad , \qquad \mathbf{A}\ \vec{u} = \vec{r} \quad .
$$

(b) In a special case, we are led to the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

i. What does the fact that A is symmetric tell you about the eigenvalues of A?
Solution: All eigenvalues to a symmetric matrix are real.

ii. Locate the interval in which the eigenvalues of A lie using Gerschgorin's theorem.
Solution:
The Gerschgorin's circles are all centered at -2. All have radius 2 except two which have radius 1. We know that

$$-4 \leq \lambda_i \leq 0$$

iii. Determine whether A is singular or not.
Solution:
Several solutions are possible. For example,
1. A sharp version of Gerschgorin's theorem tells that if an eigenvalue is at the edge of the Gerschgorin set it must lie on the edge of **all** the circles. In our case, two of the circles do not reach the origin.
2. Consider

$$[-2], \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, etc.$$

The determinants are $D_1 = -2, D_2 = 3, D_3 = -4$. By expansion along row & column

$$\begin{aligned} D_n &= -2D_{n-1} - D_{n-2} \\ D_n &= (-1)^n(n+1) \neq 0 \end{aligned}$$

3. Straightforward Gaussian elimination gives $A = L \cdot U$ with

$$\mathbf{L} = \begin{bmatrix} 1 & & & & \\ -1/2 & 1 & & & \\ & -2/3 & 1 & & \\ & & -3/4 & 1 & \\ & & & \ddots & \ddots \end{bmatrix}, \mathbf{U} = \begin{bmatrix} -2 & 1 & & & \\ & -3/2 & 1 & & \\ & & -4/3 & 1 & \\ & & & -5/4 & 1 \\ & & & \ddots & \ddots \end{bmatrix}$$

Given the sequences in the diagonals there will never occur a zero diagonal element.

4. A is negative definite since

$$x^t A x = -2x_1^2 + 2x_1x_2 - 2x_2^2 + 2x_2x_3 - \cdots + 2x_{n-1}x_n - 2x_n^2$$

or

$$x^t A x = -x_1^2 - (x_1 - x_2)^2 - \cdots - (x_{n-1} - x_n)^2 - x_n^2 \,.$$

5. Suppose $Ax = 0$. If we can show that $x = 0$, then A is non-singular. Say $x_1 > 0$, then

$$x_2 = 2x_1 > x_1$$

$$x_3 = 2x_2 - x_1 > x_2$$

$$\ldots\ldots$$

$$x_n = 2x_{n-1} - x_{n-2} > x_{n-1}$$

and we have a conflict with the last line, telling that $x_{n-1} = 2x_n$. Hence $x_1 = 0$, which implies $x_2 = \cdots = x_n = 0$.

2. (a) Suppose R^N is equipped with a norm $\| \cdot \|$ and let A be a $N \times N$ non-singular matrix. Define the condition number of A for solving a linear system of equations and the one for determining eigenvalues.

Solution:

For linear systems

$$\text{cond } A = \| A^{-1} \| \| A \|$$

For the eigenvalue problem, let P be a matrix such that $P^{-1}AP$ is diagonal (if this is possible then P=eigenvectors(A)). Then,

$$\text{cond } A = \| P^{-1} \| \| P \|$$

(b) Show that if u is the solution of $Au = b$ and $u + \delta u$ solves $A(u + \delta u) = b + \delta b$, then

$$\frac{\| \delta u \|}{\| u \|} \leq \text{cond}(A) \frac{\| \delta b \|}{\| b \|}$$

Also, show that if we perturb the coefficient matrix A, instead of b , then

$$\frac{\| \delta u \|}{\| u + \delta u \|} \leq \text{cond}(A) \frac{\| \delta A \|}{\| A \|}$$

Solution:

$$\begin{aligned} \delta u &= A^{-1} \delta b \\ \| & \delta u \| = \| A^{-1} \delta b \| \leq \| A^{-1} \| \| \delta b \| \end{aligned} \tag{0.1}$$

$$\begin{aligned} Au &= b \\ \| & b \| = \| Au \| \leq \| A \| \| u \| \end{aligned} \tag{0.2}$$

Multiply 0.1 and 0.2

$$\begin{aligned} \| & \delta u \| \| b \| \leq \| A^{-1} \| \| A \| \| \delta b \| \| u \| \\ \frac{\| \delta u \|}{\| u \|} &\leq \text{cond } A \frac{\| \delta b \|}{\| b \|} \end{aligned}$$

$$\begin{aligned} (A + \delta A)(u + \delta u) &= b \\ A(u + \delta u) + \delta A(u + \delta u) &= b \end{aligned}$$

Subtract $Au = b$ and we get

$$\begin{aligned} A \delta u &= -\delta A (u + \delta u) \\ \delta u &= -A^{-1} \delta A (u + \delta u) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\| \delta u \|}{\| u + \delta u \|} &\leq \| A^{-1} \delta A \| \leq \| A^{-1} \| \| \delta A \| = \frac{\| A \| \| A^{-1} \| \| \delta A \|}{\| A \|} \\ \frac{\| \delta u \|}{\| u + \delta u \|} &\leq \text{cond}(A) \frac{\| \delta A \|}{\| A \|} \end{aligned}$$

(c) Suppose $N = 2$ and $\|\cdot\|$ is the Euclidean (l_2) norm. Use this information to find the corresponding condition number for the matrix

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}.$$

Solution:

$\mathrm{cond}_2 A$ is the ratio of the largest singular value of A to the smallest singular value of A. This is the ratio of the square root of the largest eigenvalue of $A^T A$ to the square root of the smallest eigenvalue of $A^T A$.

$$A^T A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}$$

$$\begin{aligned} (5-\lambda)(10-\lambda) - 1 &= 0 \\ \lambda^2 - 15\lambda + 49 &= 0 \end{aligned}$$

so that

$$\lambda_1 = (15 + \sqrt{29})/2\,,$$

$$\lambda_2 = (15 - \sqrt{29})/2\,.$$

Thus, we have

$$\mathrm{cond}_2 A = \sqrt{\frac{15+\sqrt{29}}{15-\sqrt{29}}} = \frac{15+\sqrt{29}}{14}$$

3. (a) Write down the formula for Newton's iteration in the case of finding a root to the scalar equation $f(x) = 0$ and, also, in the case of a *system* of nonlinear equations.

Solution: Scalar: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

System: $\mathbf{x}_{n+1} = \mathbf{x}_n - F^{-1}(\mathbf{x}) f(\mathbf{x}_n)$, where $F(\mathbf{x})$ is the Jacobian matrix.

In more detail, to solve

$$\begin{cases} f(x, y, \ldots, z) &= 0 \\ g(x, y, \ldots, z) &= 0 \\ \vdots & \\ h(x, y, \ldots, z) &= 0 \end{cases}$$

we iterate

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \cdots & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \cdots & \frac{\partial g}{\partial z} \\ \vdots & & & \vdots \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \cdots & \frac{\partial h}{\partial z} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \vdots \\ \Delta z \end{bmatrix} = - \begin{bmatrix} f \\ g \\ \vdots \\ h \end{bmatrix},$$

where the matrix and the RHS are evaluated at the location of the last iterate. We obtain the next iterate through $x_{n+1} - x_n = \Delta x,\ y_{n+1} - y_n = \Delta y,\ \ldots,\ z_{n+1} - z_n = \Delta z$.

(b) Write down the formula for the secant method for a scalar equation.

Solution:

$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

(c) Show that the secant error, to leading order, decays like

$$\varepsilon_{n+1} = \varepsilon_n \cdot \varepsilon_{n-1} \frac{f''(\alpha)}{2\, f'(\alpha)},$$

where α is the root, and $\varepsilon_n = x_n - \alpha$.

Solution:

The idea is to Taylor expand around the root, and then simplify. For brevity of notation, call the last iterates $x_2,\ x_1,\ x_0$. Without loosing generality, we can assume $\alpha = 0$. Then

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \\ &= x_1 - \frac{(f(0) + x_1 f'(0) + \frac{x_1^2}{2} f''(0) + \ldots)(x_1 - x_0)}{(f(0) + x_1 f'(0) + \frac{x_1^2}{2} f''(0) + \ldots - f(0) - x_0 f'(0) - \frac{x_0^2}{2} f''(0) - \ldots)} \end{aligned}$$

With $\alpha = 0$, $\varepsilon_2 = x_2,\ \varepsilon_1 = x_1,\ \varepsilon_0 = x$. Since f and its derivatives are all evaluated at zero, we omit to repeat that. Furthermore, $f(0) = 0$. We have

$$\begin{aligned} \varepsilon_2 &= \varepsilon_1 - \frac{(\varepsilon_1 f' + \frac{\varepsilon_1^2}{2} f'' + \ldots)(\varepsilon_1 - \varepsilon_0)}{(\varepsilon_1 - \varepsilon_0) f' + \frac{1}{2}(\varepsilon_1^2 - \varepsilon_0^2) f'' + \ldots} \\ &= \varepsilon_1 - \varepsilon_1 \frac{f' + \frac{\varepsilon_1}{2} f'' + \ldots}{f' + \frac{1}{2}(\varepsilon_1 + \varepsilon_0) f'' + \ldots} \\ &= \varepsilon_1 - \varepsilon_1 \frac{1 + \frac{\varepsilon_1}{2} f''/f' + \ldots}{1 + \frac{1}{2}(\varepsilon_1 + \varepsilon_0) f''/f' + \ldots} \\ &= \varepsilon_1 - \varepsilon_1 (1 + \frac{\varepsilon_1}{2} f''/f' + \ldots)(1 - \frac{1}{2}(\varepsilon_1 + \varepsilon_0) f''/f' + \ldots) \\ &= \frac{1}{2} \varepsilon_0 \varepsilon_1 \frac{f''}{f'} + \ldots \end{aligned}$$

(d) The formula above can be shown to imply that the error converges approximately like

$$\varepsilon_{n+1} = c \cdot \varepsilon_n^d \,.$$

Determine c and d.

(No detailed rigor is required for parts (c) and (d); plausible arguments suffice, as long as they convincingly arrive at the required forms).

Solution:

With the assumption,

$$\varepsilon_n \approx c \cdot \varepsilon_{n-1}^d \,,$$

$$\varepsilon_{n+1} \approx c \cdot \varepsilon_n^d \approx c^{d+1} \cdot \varepsilon_{n-1}^{d^2} \,,$$

and we get

$$c^{d+1} \cdot \varepsilon_{n-1}^{d^2} \approx c \cdot \varepsilon_{n-1}^d \cdot \varepsilon_{n-1} \cdot \frac{f''(\alpha)}{2\, f'(\alpha)} \,,$$

which simplifies to

$$c^{d+1} \cdot \varepsilon_{n-1}^{d^2} \approx c \cdot \varepsilon_{n-1}^{d+1} \cdot \frac{f''(\alpha)}{2\, f'(\alpha)} \,,$$

i.e., $d^2 = d + 1$ or $d = (1 + \sqrt{5})/2$ (need to choose positive root), and

$$c = \left[\frac{f''(\alpha)}{2\, f'(\alpha)}\right]^{1/((1+\sqrt{5})/2)} = \left[\frac{f''(\alpha)}{2\, f'(\alpha)}\right]^{(\sqrt{5}-1)/2}$$

4. A *cubic* B-spline, with node points at the integers, takes the values $\{0, \frac{1}{6}, \frac{2}{3}, \frac{1}{6}, 0\}$ at five adjacent nodes, i.e. its support extends over four subintervals.

(a) Define what is meant by a B-spline (of arbitrary order).

Solution:

The B-spline is the spline (not everywhere zero) with the narrowest possible support (i.e. non-zero over the shortest interval).

(b) Determine the node values and number of subintervals for a *quadratic* spline (recalling that the standard normalization is that $\int_{-\infty}^{\infty} B(x)\, dx = 1$).

Solution:

Let the nodes be at $x = 0,\ 1,\ 2,\ \ldots$ A quadratic spline has continuous function value and first derivative at the nodes, but can jump in the second derivative. The most general form it can take is therefore

$$\begin{array}{ll} [0,1] & \alpha\, x^2 \\ [1,2] & \alpha\, x^2 + \beta(x-1)^2 \\ [2,3] & \alpha\, x^2 + \beta(x-1)^2 + \gamma(x-2)^2 \\ \ldots & \end{array}$$

The question is how soon we can get back to identically zero. We can readily see that can be achieved for

$$\begin{array}{l} \alpha\, x^2 + \beta(x-1)^2 + \gamma(x-2)^2 + \delta(x-3)^2 = \\ = x^2(\alpha+\beta+\gamma+\delta) + x(-2\beta-4\gamma-6\delta) + 1(\beta+4\gamma+9\delta) \end{array}$$

if we choose for ex. $\alpha = 1,\ \beta = -3,\ \gamma = 3,\ \delta = -1$ (or any multiple of this). To achieve normalization: With the choice above, $\int_0^3 \ldots dx = \int_0^3 x^2 dx - 3\int_1^3 (x-1)^2 dx + 3\int_2^3 (x-2)^2 dx = 2$. So we should use $\alpha = \frac{1}{2},\ \beta = -\frac{3}{2},\ \gamma = \frac{3}{2},\ \delta = -\frac{1}{2}$, giving the B-spline with node values $\{0, \frac{1}{2}, \frac{1}{2}, 0\}$ (i.e. extending over 3 subintervals).

(c) To be uniquely determined, a *cubic* spline needs two extra conditions beyond the function values at the nodes. Determine how many (if any) extra conditions a *quadratic* spline requires.

Solution:

Say we have n nodes, i.e. $n-1$ intervals. Then: Unknowns: 3 in each of $n-1$ intervals, i.e. total $3n-3$. Equations: n node values, and also two connection conditions at each of the $n-2$ interior nodes, total $3n-4$. Therefore, we need one extra condition.

(d) With cardinal data (one at one node point, say at the origin, and zero at the others), a *cubic* spline on the infinite interval will be oscillatory and decay as we move away from the center. Show that the rate of decay is approximately $c \cdot (2-\sqrt{3})^k \approx c \cdot 0.27^k$ where k is the distance (number of nodes) away from the origin.

Hint: Given that the B-spline node values are $\{0, \frac{1}{6}, \frac{2}{3}, \frac{1}{6}, 0\}$, the data values y_k and B-spline expansion coefficients b_k become related by $\frac{1}{6}b_{k+1} + \frac{2}{3}b_k + \frac{1}{6}b_{k-1} = y_k$.

Solution:

Away from the center, $y_k = 0$, so the b_k will satisfy a 3-term recursion relation with characteristic equation $\frac{1}{6}r^2 + \frac{2}{3}r + \frac{1}{6} = 0$, i.e. $r_{1,2} = -2 \pm \sqrt{3}$. Given that the oscillations decay for increasing distance from the center, the growing component must be absent, and the decay (in magnitude) therefore of the form $c \cdot (2-\sqrt{3})^k$.

5. Consider the backward differentiation formula,

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}\,h\,f(t_{n+2}, y_{n+2})\,.$$

(a) Determine the order of this method.

Solution: There are several equivalent ways to demonstrate the order. Here is a way using generating polynomials,

$$\rho(w) = w^2 - \frac{4}{3}w + \frac{1}{3}\,,$$

and

$$\sigma(w) = \frac{2}{3}w^2\,.$$

By verifying that

$$\rho(w) - \sigma(w)\log w = O(|w-1|^3)\,, \quad w \to 1\,,$$

we conclude that the method is of order 2.

(b) Define what is meant by a region of absolute stability, and provide an equation which describes this region in the case of the method above.

Solution:

We apply the scheme to the test problem,

$$y' = \lambda y\,.$$

Setting $\xi = \lambda h$ gives

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}\,\xi\,y_{n+2}\,,$$

with characteristic equation

$$r^2 - \frac{4}{3}r + \frac{1}{3} = \frac{2}{3}\,\xi\,r^2\,.$$

The region of absolute stability is the domain in the complex ξ-plane for points of which the sequence y_n remains bounded as $n \to \infty$. Equivalently, we can require that for such points the roots of the chracteristic equation of the method are less or equal to 1. If a root is equal to 1, then it must be simple.

Here both roots, r_1 and r_2, of the quadratic equation above have to be less or equal to 1.

(c) Show that the whole negative real axis is in the region of absolute stability. Extra credit is given for a proof that the method is A-stable.

Solution:

If ξ is real and $\xi \leq 0$, then by explicitly writing the roots of the characteristic equation one can observe that they are less than 1.

To show A-stability, note that the edge of the stability domain gets traced out if we put $r = \mathrm{e}^{\mathrm{i}\,\theta}$ and solve for ξ. We have

$$\xi = \frac{3}{2}\frac{r^2 - \frac{4}{3}r + \frac{1}{3}}{r^2} = \frac{3}{2} - 2\mathrm{e}^{-\mathrm{i}\,\theta} + \frac{1}{2}\mathrm{e}^{-2\mathrm{i}\,\theta} = [\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta] + \mathrm{i}\,[2\sin\theta - \frac{1}{2}\sin 2\theta]\,,$$

or

$$\xi = (1 - \cos\theta)^2 + \mathrm{i}\,\sin\theta\,(2 - \cos\theta)\,.$$

Since $(1 - \cos\theta)^2 \geq 0$, the whole left half-plane is in the stability domain.

6. (a) Determine the order of Störmer's method,

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f(t_{n+1}, y_{n+1})\,, \quad n \geq 0\,,$$

for solving the second order system of ODE's

$$y'' = f(t, y) \quad , t \geq 0\,,$$

with the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$.

Solution:

Let $Y_n = y(t_n)$, $n \geq 0$, be the exact solution at time t_n. We have (using first several terms of Taylor expansion):

$$Y_{n+2} - 2Y_{n+1} + Y_n - h^2 f(t_{n+1}, Y_{n+1}) = [Y_{n+1} + hY'_{n+1} + h^2 Y''_{n+1}/2 + h^3 Y'''_{n+1}/6 + O(h^4)] -$$

$$2Y_{n+1} + [Y_{n+1} - hY'_{n+1} + h^2 Y''_{n+1}/2 - h^3 Y'''_{n+1}/6 + O(h^4)] - h^2 Y''_{n+1} = O(h^4)\,,$$

which implies the order 2.

(b) Using the second order cenral differences in space and Störmer's method in time, construct a scheme to solve the wave equation,

$$u_{tt} = u_{xx}\,.$$

Solution:

We have

$$u_l^{n+2} - 2u_l^{n+1} + u_l^n = \left(\frac{\Delta t}{\Delta x}\right)^2 (u_{l-1}^{n+1} - 2u_l^{n+1} + u_{l+1}^{n+1})\,, \quad n \geq 0\,,$$

where index l indicates spatial discretization.

(c) Determine the condition for its stability.

Solution:

Set $\mu = \frac{\Delta t}{\Delta x}$ and move to the Fourier domain with respect to the spatial variables, to obtain

$$\hat{u}^{n+2} - 2(1 - 2\mu^2 \sin^2 \tfrac{1}{2}\theta)\hat{u}^{n+1} + \hat{u}^n = 0\,, \quad \theta \in [0, 2\pi]\,.$$

The method is stable if and only if the roots of

$$r^2 - 2(1 - 2\mu^2 \sin^2 \tfrac{1}{2}\theta)r + 1 = 0$$

are inside or on the boundary of the unit disk for all θ. If the roots are on the boundary, then they have to be simple.

Since $r_1 r_2 = 1$, and $|r_1|, |r_2| \leq 1$, for the method to be stable the roots must be on the boundary of the unit disk, $r_1 = \mathrm{e}^{\mathrm{i}\phi}$ and $r_2 = \mathrm{e}^{-\mathrm{i}\phi}$, for some $\phi \neq 0$. If $\phi = 0$, then $r_1 = r_2$ and the root is not simple.

Substituting $r = \mathrm{e}^{\mathrm{i}\phi}$, we obtain equation for ϕ,

$$\cos\phi = 1 - 2\mu^2 \sin^2 \tfrac{1}{2}\theta\,.$$

If $0 < \mu \leq 1$, then it is easy to see that $-1 \leq 1 - 2\mu^2 \sin^2 \frac{1}{2}\theta < 1$ for all θ and, thus, there exists $\phi \neq 0$ so that $r_1 \neq r_2$, $|r_1| = |r_2| = 1$.