Problem 1:

a. Newton's method is given by the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2 \ldots$$

b. First note that if a is simple, then $f'(a) \neq 0$ by definition. If f is continuously differentiable in a neighborhood of a, then $f'(x) \neq 0$ near a. Now subtracting a from the above formula and using the fact that a is a root, we have

$$x_{n+1} - a = x_n - a - \frac{f(x_n) - f(a)}{f'(x_n)}, \qquad n = 0, 1, 2 \ldots$$

Assuming now that f is twice continuously differentiable near a, a Taylor series about a yields (for some $\hat{x}_n$ near a)

$$x_{n+1} - a = x_n - a - \frac{f'(x_n)(x_n - a) + f''(\hat{x}_n)(x_n - a)^2}{f'(x_n)}, \qquad n = 0, 1, 2 \ldots$$

from which follows

$$|x_{n+1} - a| \leq \frac{\max |f''|}{\min |f'|}(x_n - a)^2, \qquad n = 0, 1, 2 \ldots,$$

where the max and min are taken in a small neighborhood about a. Our smoothness assumption on f guarantees that this ratio is bounded by a constant, which proves the first result.

c. Consider $f(x) = x^2$. Note then that Newton's method becomes

$$x_{n+1} = x_n - \frac{x_n^2}{2x_n} = \frac{1}{2}x_n, \qquad n = 0, 1, 2 \ldots,$$

which obviously converges to $a = 0$ only linearly.

d. To restore general quadratic convergence for the case of $f(x)$ having a root of multiplicity m at a, you simply apply the usual Newton's method to

$$F(x) = \frac{f(x)}{f'(x)}, \qquad x \neq a.$$

This is a reasonable approach because f must be of the form $f(x) = (x - a)^m g(x)$, where $g(a) \neq 0$, which implies that $F(x)$ is defined at and about a and $F'(a) = \frac{1}{m} \neq 0$.

Problem 2:

a. We can find the successive orthogonal polynomials by Gram-Schmidt:

$p_0(x) = 1$

$p_1(x) = x + a_0 \qquad 0 = \int_{-1}^{1} p_1(x) \cdot p_0(x)dx = 2a_0 \qquad \Rightarrow\ a_0 = 0$

$p_2(x) = x^2 + b_1 x + b_0 \qquad 0 = \int_{-1}^{1} p_2(x) \cdot p_0(x)dx = \frac{2}{3} + 2b_0 \qquad \Rightarrow\ b_0 = -\frac{1}{3}$

$\qquad 0 = \int_{-1}^{1} p_2(x) \cdot p_1(x)dx = \frac{2}{3} b_1 \qquad \Rightarrow\ b_1 = 0$

$p_3(x) = x^3 + c_2 x^2 + c_1 x + c_0 \qquad 0 = \int_{-1}^{1} p_3(x) \cdot p_0(x)dx = 2c_0 + \frac{2}{3} c_2$

$\qquad 0 = \int_{-1}^{1} p_3(x) \cdot p_1(x)dx = \frac{2}{5} + \frac{2}{3} c_1 \qquad \Rightarrow\ c_1 = -\frac{3}{5}$

$\qquad 0 = \int_{-1}^{1} p_3(x) \cdot p_2(x)dx = \frac{8}{45} c_2 \qquad \Rightarrow\ c_0 = c_2 = 0$

$\Rightarrow\ p_3(x) = x^3 - \frac{3}{5} x$; Roots: $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{5}}$.

b. With $p(x) = (x - x_1)(x - x_2)(x - x_3) = x^3 + c_2 x^2 + c_1 x + c_0$ and following the hint, we end up with the new system of equations

$$\begin{cases} 2c_0 + 0\,c_1 + \frac{2}{3} c_2 + 0 \cdot 1 & = [w_1\ \ p(x_1) + w_2\ \ p(x_2) + w_3\ \ p(x_3)] \ = 0 \\ 0c_0 + \frac{2}{3} c_1 + 0c_2 + \frac{2}{5} \cdot 1 & = [w_1 x_1 p(x_1) + w_2 x_2 p(x_2) + w_3 x_3 p(x_3)] \ = 0 \\ \frac{2}{3} c_0 + 0c_1 + \frac{2}{5} c_2 + 0 \cdot 1 & = [w_1 x_1^2 p(x_1) + w_2 x_2^2 p(x_2) + w_3 x_3^2 p(x_3)] \ = 0 \end{cases} ,$$

with the solution $c_0 = c_2 = 0,\ c_1 = -\frac{3}{5}$. The polynomial $p(x)$ is therefore the same as $p_3(x)$ in part (a), and we obtain the same roots as before, i.e. $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{5}}$.

Problem 3: Interpolation/Approximation

a)

By introducing the matrix

$$V = \begin{bmatrix} 1 & t_0 & t_1^2 & \cdots & t_1^N \\ 1 & t_1 & t_2^2 & \cdots & t_2^N \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^N \end{bmatrix},$$

the vector $\mathbf{a} = [a_0, \ldots, a_N]^T$, and the vector $\mathbf{s} = [s_0, \ldots, s_N]^T$ we can write the interpolation problem as the linear system

$$V\mathbf{a} = \mathbf{s}. \tag{1}$$

We want to show that (1) has a unique solution. Thus, we need to show that the system is non-singular, or that $V\mathbf{a} = \mathbf{0}$ only has the trivial solution $\mathbf{a} = \mathbf{0}$.

Assume that $V\mathbf{a} = \mathbf{0}$ has a non-trivial solution $\mathbf{a} \neq \mathbf{0}$. Then

$$a_0 + a_1 t_k + a_2 t_k^2 + \ldots + a_N t_k^N = 0$$

for $k = 0, \ldots, N$. Since the nodes are assumed to be distinct, this means that the N:th degree polynomial $p(z)$ has $N+1$ distinct nodes which is impossible. Hence, $V\mathbf{a} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$ which means that the system (1) has a unique solution.

b)

Using Euler's formula we have that

$$f_N(t) = a_0 + \sum_{m=1}^{N} a_m \cos(mt) + b_m \sin(mt) =$$

$$a_0 + \sum_{m=1}^{N} a_m \frac{e^{imt} + e^{-imt}}{2} + b_m \frac{e^{imt} - e^{-imt}}{2i} =$$

$$a_0 + \sum_{m=1}^{N} \frac{a_m - ib_m}{2} e^{imt} + \frac{a_m + ib_m}{2} e^{-imt} = \sum_{m=-N}^{N} c_m z^m$$

where $z = e^{it}$ and

$$c_m = \begin{cases} a_0, & m = 0 \\ (a_m - ib_m)/2, & m > 0 \\ (a_m + ib_m)/2, & m < 0 \end{cases} .$$

c)

Introduce the $2N$:th degree polynomial $p(z) = z^N Q(z)$. Using the result from b) we can now write the trigonometric interpolation problem as

$$p(t_k) = t_k^N s_k$$

for $k = 0, 1, \ldots, 2N$ which has a unique solution according to a).

Problem 4:

a. Assume that

$$\begin{bmatrix} I & X \\ X^T & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0. \tag{1}$$

Using the hint gives

$$\begin{bmatrix} X^T & X^T X \\ X^T & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Subtracting the second row from the first, we get $X^T X y = 0$, which from the full-rank assumption implies that $y = 0$. We then get $x = 0$ from the first row in equation (1). Thus, A is nonsingular.

b. Assume that

$$\begin{bmatrix} I & X \\ X^T & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

First consider the case that $y = 0$. Then it is easy to see that any $x \neq 0$ in the null space of X^T (which is possible iff we have the strict inequality $m < n$) yields an eigenvector with eigenvalue $\lambda = 1$. Now consider the case $y \neq 0$. Again using the hint we get

$$\begin{bmatrix} X^T & X^T X \\ X^T & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} X^T x \\ y \end{bmatrix}.$$

Then $X^T x = \lambda y$ and it is easy to see that $X^T X y = \lambda(\lambda - 1) y$. So y must be an eigenvector of $X^T X$ belonging to, say, $\mu > 0$ and we must have $\lambda(\lambda - 1) = \mu$ (which has two distinct roots, neither of which are equal to 1, that is, this $\lambda \neq 1$). Note then that $x = (1/(\lambda - 1)) X y$. It's easy to see that these vectors indeed are eigenvectors, so the result is $\lambda_0 = 1$, $\lambda_i = 1 + \sqrt{\mu_i}$, and $\lambda_{-i} = 1 - \sqrt{\mu_i}$ when $m < n$ and just $\lambda_i = 1 + \sqrt{\mu_i}$ and $\lambda_{-i} = 1 - \sqrt{\mu_i}$ when $m = n$.

c. The iteration is generally convergent iff the spectral radius of $I - A$ is strictly less than 1. Clearly, this is true iff all eigenvalues of A are in $(0, 2)$, which is true iff all $|\mu_i| < 1$, that is, all singular values of X are less than 1.

Problem 5: Numerical ODE

a)

To establish convergence, we must show consistency and stability.

To show consistency, it suffices to show that the scheme is exact for solving $u' = f(t,u)$ when $u(t) = 1$ and $u(t) = t$. When $u(t) = 1$ then $f(t,u) = u' = 0$. Plugging this into the scheme gives that

$$u_{n+1} - u_n - hf(t_n + \frac{h}{2}, \frac{u_{n+1} + u_n}{2}) = 1 - 1 - h \times 0 = 0.$$

Hence, the scheme is exact for $u(t) = 1$. When $u(t) = t$ then $f(t,u) = u' = 1$. Plugging this into the scheme gives that

$$u_{n+1} - u_n - hf(t_n + \frac{h}{2}, \frac{u_{n+1} + u_n}{2}) = h - 0 - h \times 1 = 0.$$

Hence, the scheme is exact for $u(t) = t$ and therefore at least first order accurate, which is sufficient to guarantee consistency. (It is in fact second order accurate.)

To show stability, we note that for the root condition, we get the characteristic equation $r = 1$ which only has one root on the unit circle. Hence, the stability condition is also satisfied.

b)

Apply the scheme to the ODE $u' = \lambda u$:

$$u_{n+1} - u_n - h\lambda(\frac{u_n + u_{n+1}}{2}) = 0 \Leftrightarrow$$

$$1 - \frac{\lambda h}{2} u_{n+1} = 1 + \frac{\lambda h}{2} u_n$$

We set $\mu \equiv \lambda h$ which gives the amplification factor

$$r = \frac{1 + \frac{\mu}{2}}{1 - \frac{\mu}{2}}.$$

To find the boundary of the stability domain, we let $r = e^{i\theta}$ and solve for μ:

$$\mu = 2\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i2\tan(\theta/2).$$

Hence, the boundary of the stability domain is the imaginary axis, and it is easily checked that $|r| < 1$ in the left half of the complex plane.

c)

We can write the equation as the first order linear system

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \Leftrightarrow \mathbf{u}' = A\mathbf{v}$$

The eigenvalues of A are $\lambda = -b \pm \sqrt{b^2 - \omega^2}$ which are located in the left half of the complex plain. Hence the implicit midpoint scheme is appropriate for solving the pendelum equation for both no damping and with strong dampling.

d)

Since we're considering solving the equation for long times, we have to make sure that $h\lambda$ lies inside the stability domain even though convergence is guaranteed as $h \to 0$.

For no damping the eigenvalues of the system lie on the imaginary axis which is outside the stability domain of the Euler forward scheme. Therefore Euler forward is not suitable for solving the equation with no damping.

For strong damping, the eigenvalues are real and negative. The Euler forward scheme may be used, but only with small time steps to ensure that the $h\lambda$ lie within the stability region.

Problem 6:

a. If we let $(D_+D_-)_x$ denote the standard second order approximation for $\frac{\partial^2}{\partial x^2}$ and similarly for $\frac{\partial^2}{\partial y^2}$, the ADI method splits the time step k into two stages:

$$u(x,y,t+\tfrac{k}{2})-u(x,y,t) \quad = \tfrac{k}{2}[(D_+D_-)_x\, u(x,y,t+\tfrac{k}{2})+(D_+D_-)_y\, u(x,y,t)]$$

and

$$u(x,y,t+k)-u(x,y,t+\tfrac{k}{2}) = \tfrac{k}{2}[(D_+D_-)_x\, u(x,y,t+\tfrac{k}{2})+(D_+D_-)_y\, u(x,y,t+k)]$$

We note that each of the two stages will only require the solution of tridiagonal linear systems.

b. To use von Neumann analysis in 2-D, we consider solutions of the form $u(x,y,t)=\kappa^{t/k}e^{i\omega_1 x}\, e^{i\omega_2 y}$. Each of the two stages has its own amplification factor κ, which we denote by κ_1 and κ_2 respectively. Substituting the assumed solution form into the two stages gives for the first stage

$$\kappa_1^{1/2}-1=\frac{k}{2}\left[\kappa_1^{1/2}\,\frac{(e^{i\omega_1 h}-2+e^{-i\omega_1 h})}{h^2}+\frac{(e^{i\omega_2 h}-2+e^{-i\omega_2 h})}{h^2}\right],$$

i.e.

$$\kappa_1^{1/2}\left[1+\tfrac{k}{h^2}(1-\cos\omega_1 h)\right]=\left[1-\tfrac{k}{h^2}(1-\cos\omega_2 h)\right],$$

and similarly for the second stage

$$\kappa_2^{1/2}\left[1+\tfrac{k}{h^2}(1-\cos\omega_2 h)\right]=\left[1-\tfrac{k}{h^2}(1-\cos\omega_1 h)\right].$$

The total amplification factor for the two stages can therefore be written

$$\kappa=\kappa_1^{1/2}\kappa_2^{1/2}=\frac{\left[1-\frac{k}{h^2}(1-\cos\omega_2 h)\right]}{\left[1+\frac{k}{h^2}(1-\cos\omega_1 h)\right]}\cdot\frac{\left[1-\frac{k}{h^2}(1-\cos\omega_1 h)\right]}{\left[1+\frac{k}{h^2}(1-\cos\omega_2 h)\right]}\,.$$

Swapping the order of the factors in the numerators, the whole expression becomes a product of two quantities which both have magnitude less than or equal to one. Hence, the scheme is unconditionally stable.