MAC2312: Calculus 2 - Section 3

Test 2

June 25, 2015

Name: ______________________________

> Answer each question in the space provided on the question sheets. If you run out of space for an answer, continue on the back of the page. Credit will only be given if you clearly show all of your work. Calculators may not be used for this test.

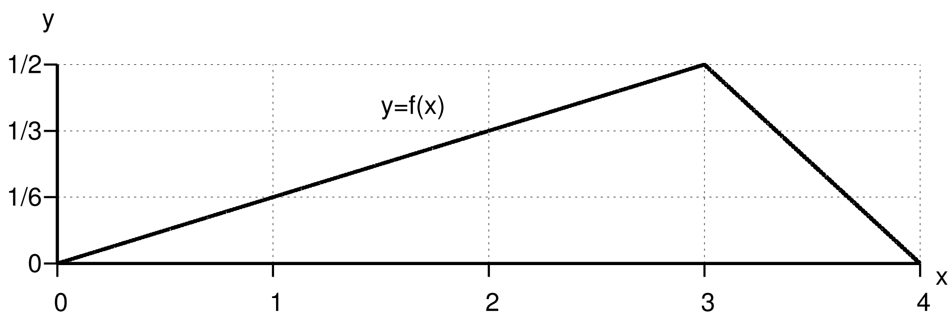
Question	**Points**	**Score**
1	4	
2	4	
3	4	
4	4	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
11	12	
12 (bonus)	–	
Total:	100	

1. [4 points] Circle all of the following integrals that are improper.

$$\int_0^1 \frac{1}{x^2-x-2}\,dx \qquad \int_0^{\pi/2} \sec x\ dx$$

$$\int_0^\infty e^{-x^2}\ dx \qquad \int_0^{10} |2x-10|\ dx$$

2. [4 points] If X is a random variable with the probability density function $f(x)$ defined for $0 \le x \le 4$ depicted below, what is the probability that $X \ge 2$.



$$P(X \ge 2) = \underbrace{\int_2^4 f(x)\ dx}_{\text{area under } f \text{ from } x=2 \text{ to } 4} = \underbrace{\left(1\cdot\frac{1}{3}+\frac{1}{2}\cdot 1\cdot\frac{1}{6}\right)}_{\text{area under } f \text{ from } x=2 \text{ to } 3} + \underbrace{\frac{1}{2}\cdot 1\cdot\frac{1}{2}}_{\text{area under } f \text{ from } x=3 \text{ to } 4} = \tfrac{2}{3}$$

3. A population is modeled by the logistic model

$$\frac{dP}{dt} = 2P\left(1-\frac{P}{3000}\right).$$

(a) [2 points] For what values of P is the population increasing? $\underline{\qquad 0 < P < 3000 \qquad}$

(b) [2 points] What are the equilibrium solutions?

$\frac{dP}{dt} = 0 \Rightarrow P = 0$ and $P = 3000$ are equilibrium solutions.

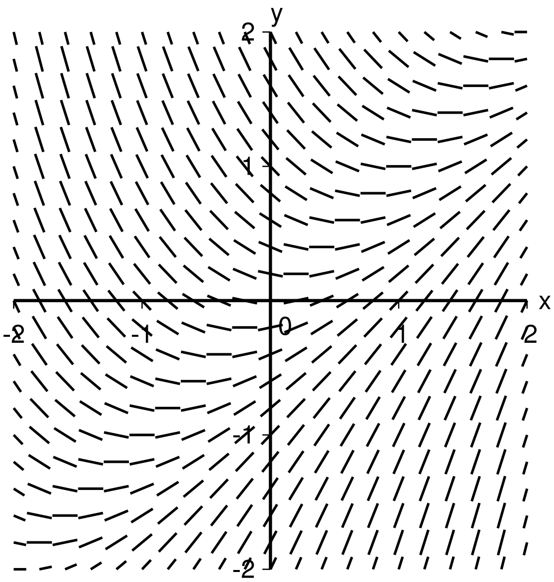
4. [4 points] For what values of c is $y = e^{ct}$ a solution to $y'' - 3y' + 2y = 0$?

Substituting $y = e^{ct}$ into the differential equation,

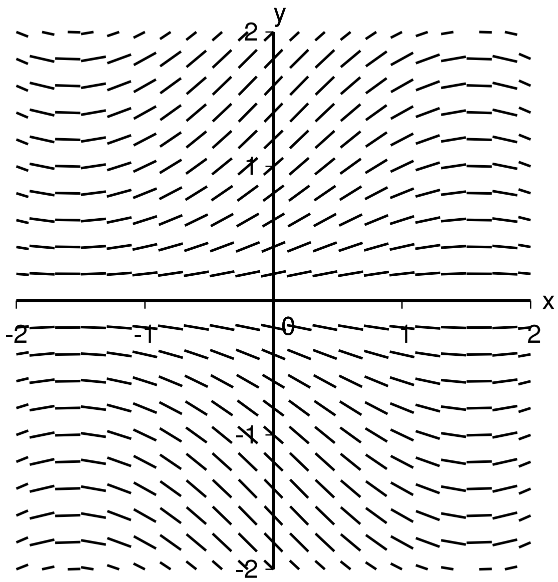
$$\begin{aligned} c^2e^{ct} - 3ce^{ct} + 2e^{ct} &= 0 \\ c^2e^{ct} - 3ce^{ct} + 2e^{ct} &= 0 \\ e^{ct}(c^2-3c+2) &= 0 \\ e^{ct}(c-1)(c-2) &= 0 \Rightarrow c = 1 \text{ or } 2 \end{aligned}$$

5. Use the four direction fields depicted below to answer the following.

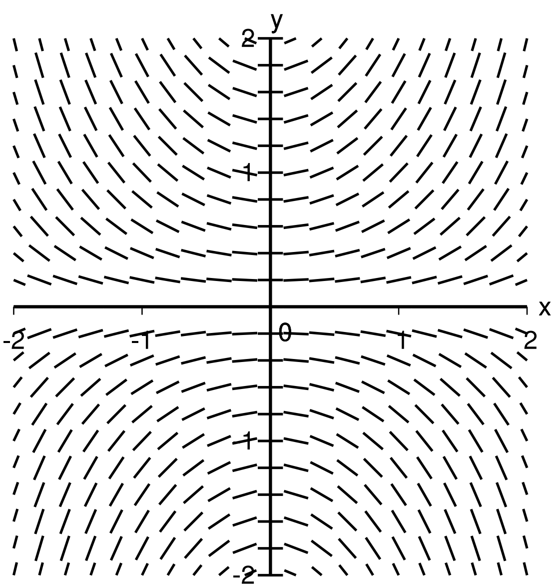
I.



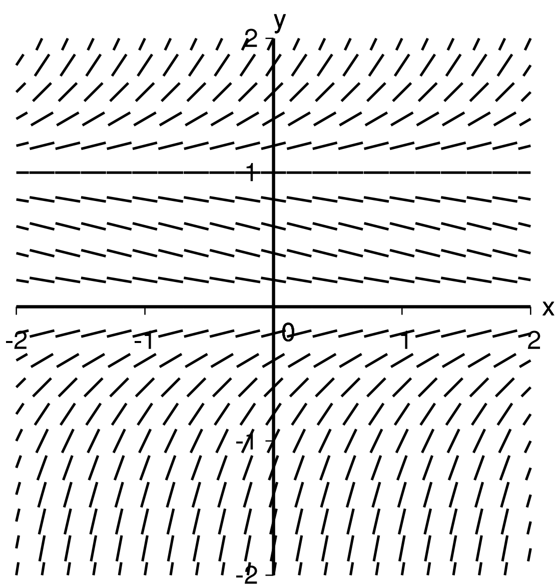
II.



III.



IV.



(a) [4 points] Match the direction field with the differential equation.

A. $y' = xy$: ______III______

B. $y' = \sin y \cos x$: ______II______

C. $y' = y^2 - y$: ______IV______

D. $y' = x - y$: ______I______

(b) [4 points] List all of the differential equations from A, B, C, or D that are separable: ______A, B, and C______

(c) [4 points] List all of the differential equations from A, B, C, or D that are autonomous: ______C______

6. [12 points] Use Euler's Method with step size 0.5 to compute the approximate y-values y_1, y_2, and y_3 of the solution to the initial value problem

$$\begin{cases} y' = xy \\ y(0) = 8. \end{cases}$$

$F(x, y) = xy$, $h = 0.5$, $x_0 = 0$, and $y_0 = 8$, so $x_n = 0.5n$ for $n = 0, 1, 2, \ldots$, and

$$\begin{aligned}
y_1 &= y_0 + hF(x_0, y_0) = 8 + 0.5(0)(8) = 8 \\
y_2 &= y_1 + hF(x_1, y_1) = 8 + 0.5(0.5)(8) = 10 \\
y_3 &= y_2 + hF(x_2, y_2) = 10 + 0.5(1)(10) = 15
\end{aligned}$$

7. [12 points] Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the semicircle $(x-2)^2 + y^2 = 1$ for $y \geq 0$ and $1 \leq x \leq 3$ about the y-axis.

Differentiating with respect to x and substituting $y = \pm\sqrt{1-(x-2)^2}$,

$$\begin{aligned}
2(x-2) + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{2-x}{y} = \pm\frac{2-x}{\sqrt{1-(x-2)^2}} \\
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{(x-2)^2}{1-(x-2)^2} = \frac{1}{1-(x-2)^2}
\end{aligned}$$

and using $ds = \sqrt{1 + (dy/dx)^2}\ dx$, the surface area is

$$S = \int 2\pi x\ ds = \int_1^3 2\pi x\sqrt{\frac{1}{1-(x-2)^2}}\ dx$$

8. [12 points] For what value of c is the function $f(x) = cx^2(1-x)$ defined for $0 \le x \le 1$ a probability density function?

$$\begin{aligned}
\int_0^1 f(x)\ dx &= \int_0^1 cx^2(1-x)\ dx = c\int_0^1 (x^2 - x^3)\ dx \\
&= c\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{x=0}^{1} = c\left[\frac{1}{3} - \frac{1}{4}\right] \\
&= c \cdot \frac{1}{12} = 1 \Rightarrow c = 12
\end{aligned}$$

9. [12 points] Find the orthogonal trajectories of the family of curves $xy = k$.

Differentiating with respect to x, the family of curves satisfies

$$\begin{aligned}
y + x\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{y}{x}
\end{aligned}$$

so the orthogonal trajectories satisfy

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x}{y} \\
\int y\ dy &= \int x\ dx \\
\frac{y^2}{2} &= \frac{x^2}{2} + C \\
y^2 &= x^2 + 2C \\
y^2 - x^2 &= K \qquad\qquad \text{where } K = 2C
\end{aligned}$$

10. [12 points] Find the center of mass (centroid) of the region bounded by the curves $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$.

$$A = \int_0^\pi \sin x \; dx = [-\cos x]_0^\pi = 1 - (-1) = 2$$

$$\begin{aligned} \bar{x} &= \frac{1}{2} \int_0^\pi x \sin x \; dx \\ &= \frac{1}{2} \left[-x \cos x|_{x=0}^{\pi} + \int_0^\pi \cos x \; dx \right] \\ &= \frac{1}{2} \left[\pi + \sin x|_{x=0}^{\pi} \right] = \frac{\pi}{2} \end{aligned} \qquad \left[\begin{array}{ll} u = x & v = -\cos x \\ du = dx & dv = \sin x \end{array} \right]$$

$$\begin{aligned} \bar{y} &= \frac{1}{2} \int_0^\pi \frac{1}{2} \sin^2 x \; dx \\ &= \frac{1}{4} \int_0^\pi [1 - \cos(2x)] \; dx \\ &= \frac{1}{8} \left[x - \frac{1}{2} \sin(2x) \right]_0^\pi = \frac{\pi}{8} \end{aligned}$$

$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$

11. [12 points] For which values of p is $\displaystyle\int_0^1 \frac{1}{x^p} \, dx$ convergent? Justify your answer.

If $p = 1$, $\int_0^1 \frac{1}{x} \, dx = \lim_{t \to 0^+} [\ln x]_{x=t}^1 = \lim_{t \to 0^+} (-\ln t) = \infty$. If $p \neq 1$,

$$\begin{aligned} \int_0^1 \frac{1}{x^p} \, dx &= \lim_{t \to 0^+} \int_t^1 x^{-p} \, dx \\ &= \lim_{t \to 0^+} \frac{1}{1-p} \left[x^{1-p} \right]_{x=t}^1 = \lim_{t \to 0^+} \frac{1}{1-p} \left[1 - t^{1-p} \right] \\ &= \frac{1}{1-p} \left[1 - \lim_{t \to 0^+} t^{1-p} \right] \end{aligned}$$

and

$$\lim_{t \to 0^+} t^{1-p} = \begin{cases} 0 & \text{if } 1 - p > 0 \\ \infty & \text{if } 1 - p < 0 \end{cases} = \begin{cases} 0 & \text{if } p < 1 \\ \infty & \text{if } p > 1 \end{cases}$$

so

$$\int_0^1 \frac{1}{x^p} \, dx \begin{cases} \text{converges} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \ge 1 \end{cases}$$

12. [5 points (bonus)] Are there any solids of revolution that have finite volume but infinite surface area? If so, give an example.

Yes. For example, the solid obtained by rotating the region bounded by $y = \frac{1}{x^p}$ for $1 \leq x < \infty$ about the x-axis has finite volume but infinite surface area if $\frac{1}{2} < p \leq 1$.

The volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x^p}$ for $1 \leq x < \infty$ about the x-axis is

$$\begin{aligned} V &= \int_1^\infty 2\pi \left(\frac{1}{x^p}\right)^2 \, dx \qquad \text{by the method of cross-sectional areas, Sec 6.2 in text} \\ &= 2\pi \int_1^\infty \frac{1}{x^{2p}} \, dx \end{aligned}$$

which converges if $2p > 1 \Rightarrow p > \frac{1}{2}$. The area of the surface obtained by rotating $y = \frac{1}{x^p}$ for $1 \leq x < \infty$ about the x-axis is

$$\begin{aligned} S &= \int_1^\infty 2\pi \frac{1}{x^p} \sqrt{1 + \frac{p^2}{x^{2p+2}}} \, dx = 2\pi \int_1^\infty \sqrt{\frac{1}{x^{2p}} + \frac{p^2}{x^{4p+2}}} \, dx \\ &= 2\pi \int_1^\infty \frac{\sqrt{x^{2p+2} + p^2}}{x^{2p+1}} \, dx \geq 2\pi \int_1^\infty \frac{\sqrt{x^{2p+2}}}{x^{2p+1}} \, dx \\ &= 2\pi \int_1^\infty \frac{x^{p+1}}{x^{2p+1}} \, dx = 2\pi \int_1^\infty \frac{1}{x^p} \, dx \end{aligned}$$

which diverges if $p \leq 1$. So, by the Comparison Theorem, S diverges if $p \leq 1$.

Formula Sheet

Trigonometric Identities

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$
- $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$
- $\sin x_1 \sin x_2 = \frac{1}{2}[\cos(x_1 - x_2) - \cos(x_1 + x_2)]$
- $\cos x_1 \cos x_2 = \frac{1}{2}[\cos(x_1 - x_2) + \cos(x_1 + x_2)]$
- $\sin x_1 \cos x_2 = \frac{1}{2}[\sin(x_1 - x_2) + \sin(x_1 + x_2)]$
- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x$

Trigonometric Integrals

- $\displaystyle\int \tan x \; dx = \ln|\sec x| + C$
- $\displaystyle\int \csc x \; dx = -\ln|\csc x + \cot x| + C$
- $\displaystyle\int \sec x \; dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int \cot x \; dx = \ln|\sin x| + C$
- $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$