Ph.D. qual. exam. and M.S. comp. exam. on Numerical analysis. Wednesday August 16, 2006.

Answer at least 8 questions with at least 6 having a different number (1-7). Show your calculations and justify your answers.

1a. Consider the linear interpolation polynomial $P_1(x)$ of the function $\ln(x)$ on $[1, e]$ at the points $x_0 = 1, x_1 = e$. Estimate the maximum of the absolute error $|P_1(x) - \ln(x)|$ for $x \in [1, e]$, i.e., give an upper bound for

$$\|P_1 - \ln(.)\|_\infty = \max_{x \in [1,e]} |P_1(x) - \ln(x)| \leq ?$$

1b. Is the following function on the interval $[1, 3]$ a cubic spline? If yes is it a periodic spline?

$$s(x) = \begin{cases} x^3 - 2x^2 - 2x + 5 & \text{for} \quad x \in [1, 2], \\ -x^3 + 10x^2 - 26x + 21 & \text{for} \quad x \in [2, 3]. \end{cases}$$

2a. To obtain an approximation to

$$\int_{-1}^{1} e^{-x^2} dx$$

with the midpoint rule and using an equidistant subdivision of the interval $[-1, 1]$, how many evaluations of the function e^{-x^2} are sufficient to ensure a total error smaller than 10^{-8}?

2b. We consider a quadrature formula

$$\int_\alpha^{\alpha+h} f(x)dx \approx h \sum_{i=1}^{s} b_i f(\alpha + c_i h)$$

given by the coefficients

$$\begin{aligned} (b_1, b_2, b_3, b_4) &= (1/8, 3/8, 3/8, 1/8), \\ (c_1, c_2, c_3, c_4) &= (0, 1/3, 2/3, 1). \end{aligned}$$

(a) What is its order?

(b) Is this quadrature formula symmetric?

1a

$$\| \alpha + \beta x - \ln(x) \| \le E(1) = \frac{\ln(e-1)}{2} + \frac{1}{e-1}\left(\frac{2-e}{2}\right) \approx .0617$$

$$E(x) = \ln(x) - \alpha - \beta x$$

$$E'(x) = \frac{1}{x} - \beta = 0 \Rightarrow x = \frac{1}{\beta}$$

$$E(1) = -E(1/\beta) = E(e)$$

$$\cancel{\alpha} + \beta = \cancel{\alpha} + \beta e - 1$$

$$\beta(1-e) = -1$$

$$\boxed{\beta = \frac{1}{e-1}} \quad \frac{1}{\beta} = e-1$$

$$\alpha + \frac{1}{e-1} = +\left(\ln(e-1) - \alpha - 1\right)$$

$$= +\ln(e-1) - \alpha - 1$$

$$2\alpha = \ln(e-1) - \frac{1}{e-1} - 1$$

$$= \ln(e-1) - \left(\frac{1}{e-1} + \frac{e-1}{e-1}\right)$$

$$= \ln(e-1) - \frac{e}{e-1}$$

$$\boxed{\alpha = \frac{1}{2}\left(\ln(e-1) - \frac{e}{e-1}\right)}$$

$$\frac{1}{2}\left(\ln(e-1) - \frac{e}{e-1}\right) + \frac{x}{e-1} - \ln(x)$$

$x=1$: $$\frac{\ln(e-1)}{2} + \frac{1}{e-1}\left(1 - \frac{e}{2}\right)$$

$x=e$: $$\frac{\ln(e-1)}{2} - \frac{e}{2(e-1)} + \frac{e}{e-1} - \frac{e-1}{e-1}$$

$$\frac{\ln(e-1)}{2} + \frac{1}{e-1}\left(\frac{-e}{2} + \cancel{e} - \cancel{e} + 1\right)$$

(16)

$$s_1(2) = s_2(2)$$

$$(2)^3 - 2(2)^2 - 2(2) + 5 = -(2)^3 + 10(2)^2 - 26(2) + 21$$

$$8 - 8 - 4 + 5 = -8 + 40 - 52 + 21$$

$$1 = 1$$

✓

$$s_1'(2) = s_2'(2)$$

$$3(2)^2 - 4(2) - 2 = -3(2)^2 + 20(2) - 26$$

$$12 - 8 - 2 = -12 + 40 - 26$$

$$2 = 2$$

✓

$$s_1''(2) = s_2''(2)$$

$$6(2) - 4 = -6(2) + 20$$

$$8 = 20 - 12$$

✓

$$3(20+6) = 60 + 18$$
$$78$$

$$12 - 6$$
$$90 \quad - 78$$

$$s_1(1) = 1 - 2 - 2 + 5 = 2 \neq s_2(3) = -27 + 10(9) - 26(3) + 21 = 6$$

No not periodic.

2a

$$\int_{x_i}^{x_{i+1}} e^{-x^2}dx - I_{mid} = \int_{x_i}^{x_{i+1}} \frac{f''(\eta)}{2}(x-x_{i+\frac{1}{2}})^2 dx$$

$h = \cancel{H}$ $\quad \frac{x_{i+1}-x_i}{2}$ $\quad x_{i+\frac{1}{2}} = x_i + h = x_{i+1} - h$

$x = x_i + sh$ $\quad s = \frac{x-x_i}{h}$

$dx = h\,ds$

$$x - x_{i+\frac{1}{2}} = x - x_i - h = sh - h = h(s-1)$$

MVT

$$\frac{f''(\eta)}{2}\int_0^2 (sh-h)^2\, h\,ds$$

$$= \frac{f''(\eta)h^3}{2}\int_0^2 (s-1)^2\,ds = \frac{f''(\eta)h^3}{2}\left[\frac{(s-1)^3}{3}\right|_0^2$$

$$= \frac{f''(\eta)h^3}{6}(1+1)$$

$$= \frac{f''(\eta)h^3}{3}$$

$$= \frac{f''(\eta)(x_{i+1}-x_i)^3}{24}$$

$H = \frac{1+1}{N} = \frac{2}{N}$

$E_{glob} = N \cdot E_{loc}$

$$= N \cdot \frac{f''(\eta)}{24}\left(\frac{2}{N}\right)^3$$

$$= \left|\frac{f''(\eta)}{3}\cdot\frac{1}{N^2}\right| \le \left|\frac{2\cdot 1}{3N^2}\right| \le 10^{-8}$$

$\sqrt{\frac{2}{3}10^8} \le N \approx 8\times 10^3$

function evaluations 1/interval.

(2b)

$\alpha = 0$
$h = 1$

$$\int_0^1 f(x)\,dx \approx \frac{1}{8}f(0) + \frac{3}{8}f\left(\frac{1}{3}\right) + \frac{3}{8}f\left(\frac{2}{3}\right) + \frac{1}{8}f(1)$$

~~f=1~~ $f = 1$

$$\int_0^1 dx = x\Big|_0^1 = 1 = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

$f = x$

$$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} = \frac{1}{8} + \frac{3}{8}\cdot\frac{1}{3} + \frac{3}{8}\cdot\frac{2}{3} + \frac{1}{8}\cdot 1$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2}$$

$f = x^2$

$$\int_0^1 x^2\,dx = \frac{1}{3} = 0 + \frac{3}{8}\cdot\frac{1}{9} + \frac{\cancel{3}}{\cancel{8}_2}\cdot\frac{\cancel{4}^1}{\cancel{9}_3} + \frac{1}{8}$$

$$= \frac{1}{24} + \frac{1}{6} + \frac{1}{8} = \frac{1}{24} + \frac{4}{24} + \frac{3}{24} = \frac{8}{24} = \frac{1}{3}$$

$f = x^3$

$$\int_0^1 x^3\,dx = \frac{1}{4} = 0 + \frac{\cancel{3}}{8}\cdot\frac{1}{\cancel{27}_{3^2}} + \frac{\cancel{3}}{\cancel{8}}\cdot\frac{\cancel{2^3}}{3^{\cancel{3}}2} + \frac{1}{8}$$

$$= \frac{1}{2^3\cdot 3^2} + \frac{1}{3^2} + \frac{1}{2^3} = \frac{1 + \overset{8}{2^3} + \overset{9}{3^2}}{2^3\cdot 3^2} = \frac{18}{81} = \frac{2\cdot 3^2}{2^3\cdot 3^2} = \frac{1}{4}$$

$f = x^4$

2b

$$\int_0^1 x^4 dx = \frac{1}{5} \neq \frac{\cancel{8}}{2^3 \cancel{8}} \cdot \frac{1}{3^{\cancel{4}}3} + \frac{\cancel{8}}{\cancel{2^3}} \cdot \frac{2^{\cancel{4}}}{3^{\cancel{4}}3} + \frac{1}{8}$$

$$\begin{array}{r} 1 \\ 28 \\ +16 \\ \hline 44 \end{array}$$

$$= \frac{1}{2^3 \cdot 3^3} + \frac{2}{3^3} + \frac{1}{2^3}$$

$8(20+7)$

$160 + 56$

d.o.e $= 3$

$$= \frac{1 + \overset{16}{2^4} + \overset{\cancel{8}\,27}{\cancel{2^3}} \cdot 3^3\ 27}{2^3 \cdot 3^3} = \frac{\cancel{237}\ 44}{216} = \frac{\cancel{2}\cdot 11}{2^{\cancel{3}} 3^3}$$

216

$$= \frac{11}{2 \cdot 27}$$

$$= \frac{11}{54}$$

symmetric about the point:

$$\alpha + \frac{h}{2}$$