Solutions for Homework 6 Foundations of Computational Math 2 Spring 2012

Problem 6.1

Consider a minimax approximation to a function $f(x)$ on $[a, b]$. Assume that $f(x)$ is continuous with continuous first and second order derivatives. Also, assume that $f''(x) < 0$ on for $a \leq x \leq b$, i.e., f is concave on the interval.

6.1.a. Derive the equations you would solve to determine the linear minimax approximation, $p_1(x) = \alpha x + \beta$, to $f(x)$ on $[a, b]$ and describe their use to solve the problem.

6.1.b. Apply your approach to determine $p_1(x) = \alpha x + \beta$ for $f(x) = -x^2$ on $[-1, 1]$.

6.1.c. How does $p_1(x)$ relate to the quadratic monic Chebyshev polynomial $t_2(x)$?

6.1.d. Apply your approach to determine $\tilde{p}_1(x) = \tilde{\alpha} x + \tilde{\beta}$ for $f(x) = -x^2$ on $[0, 1]$.

6.1.e. How could the quadratic monic Chebyshev polynomial $t_2(y)$ on $-1 \leq y \leq 1$ be used to provide and alternative derivation of $\tilde{p}_1(x)$ on $0 \leq x \leq 1$?

6.1.f. Suppose you adapt your approach to derive a constant approximation, $p_0(x)$. What points will you use as the extrema of the error?

Solution:

We have $f(x) \in \mathcal{C}^2[a, b]$ with $f''(x) < 0$ on $[a, b]$. Define the error function

$$\begin{aligned} e(x) &= f(x) - \alpha x - \beta \\ e'(x) &= f'(x) - \alpha \\ e''(x) &= f''(x) < 0 \end{aligned}$$

So $e(x)$ is also concave on $[a, b]$. It therefore has **at most** one point, $x = c$, in the interval where $e'(c) = 0$.

Determining $p_1(x)$ requires three points where $e(x)$ is maximal and of alternating sign. Since $e(x)$ is concave the potential extrema are a, b, and c (if it exists). Assume that c exists and that $a < c < b$. We then have the equations

$$\begin{aligned} e(a) &= -e(c) \\ e(b) &= -e(c) \text{ or } e(a) = e(b) \\ f'(c) &= \alpha \end{aligned}$$

to determine the unknowns α, β, and c. The discussion of $f(x) = e^x$ on $[0, 1]$ in the notes is an example of this case.

Consider $f(x) = -x^2$ on $[a, b]$.

$$f'(c) = \alpha \rightarrow c = -\frac{\alpha}{2}$$
$$e(a) = e(b) \rightarrow -a^2 - \alpha a = -b^2 - \alpha b$$
$$\rightarrow \alpha = \frac{(a^2 - b^2)}{(b - a)} \rightarrow c = \frac{1}{2}(a + b)$$
$$-e(a) = e(c) \rightarrow a^2 + \alpha a + \beta = -c^2 - \alpha c - \beta$$
$$\rightarrow \beta = \frac{\alpha^2}{8} - a\frac{\alpha}{2} - \frac{a^2}{2}$$

Taking $a = -1$ and $b = 1$ yields:

$$\alpha = 0$$
$$c = 0$$
$$\beta = -\frac{1}{2}$$
$$p_1(x) = p_0(x) = -\frac{1}{2}$$

It is easy to verify that $e(-1) = e(1) = -1/2$ and $e(c) = e(0) = 1/2$ satisfying the necessary and sufficient minimax conditions.

It is interesting to note that $p_1(x)$ is in fact a constant. However, this is consistent with the alternative approach to deriving a minimax approximation to $-x^2$ on $[-1, 1]$. This was done when we derived the monic polynomial with minimum maximum magnitude on $[-1, 1]$. We saw that this was given by the Chebyshev polynomial of degree 2 scaled so as to have leading coefficient 1, $t_2(x) = x^2 - 1/2$. Here, we have

$$e(x) = -t_2(x) = -x^2 + \frac{1}{2} = -x^2 - p_1(x)$$

so we know that we have derived the minimax $p_1(x)$ and it happens to be a constant.

Taking $a = 0$ and $b = 1$ in the equations above yields:

$$\alpha = -1$$
$$c = \frac{1}{2} \rightarrow a < c < b$$
$$\beta = \frac{1}{8}$$
$$\tilde{p}_1(x) = -x + \frac{1}{8}$$
$$e(x) = -x^2 + x - \frac{1}{8}$$

It is easily verified that the minimax conditions are satisfied by $\tilde{p}_1(x)$:

$$e(0) = -e(1/2) = e(1) = -1/8$$

To use $t_2(y) = y^2 - 1/2$ on $-1 \le y \le 1$ to derive $\tilde{p}_1(x)$ on $0 \le x \le 1$ we use a change of variables

$$x = \frac{y+1}{2} \quad \text{and} \quad y = 2x - 1$$

We are looking for

$$\tilde{p}_1(x) = \alpha x + \beta = \underset{q_1 \in \mathbb{P}_1}{\operatorname{argmin}} \|-x^2 - q_1(x)\|_\infty \quad 0 \le x \le 1$$

So we have the minimum value

$$E_n^*(-x^2) = \|-x^2 - \tilde{p}_1(x)\|_\infty = \|-x^2 - \alpha x - \beta\|_\infty = \|-\left(\frac{y+1}{2}\right)^2 - \alpha\,\frac{y+1}{2} - \beta\|_\infty$$

for $0 \le x \le 1$ and $-1 \le y \le 1$. We can rewrite the optimal error E_n^* as a function of y yielding

$$\|-\left(\frac{y+1}{2}\right)^2 - \alpha\,\frac{y+1}{2} - \beta\|_\infty = \frac{1}{4}\|-y^2 - (2\alpha+2)y - 4\beta - 2\alpha - 1\|_\infty$$

The term inside the norm is a monic polymomial that is also the minimax error function in y for the linear minimax approximation $\hat{p}_1(y)$ to $-y^2$ on $-1 \le y \le 1$. We know from our derivation of the monic Chebyshev polynomials that we have

$$\frac{1}{4}\|-y^2 - (2\alpha+2)y - 4\beta - 2\alpha - 1\|_\infty = \frac{1}{4}\|-t_2(y)\|_\infty = \frac{1}{4}\|-y^2 + 1/2\|_\infty$$

Therefore we can determine α and β via the equations

$$\begin{gathered} 2 + 2\alpha = 0 \rightarrow \alpha = -1 \\ -4\beta - 2\alpha - 1 = \frac{1}{2} \rightarrow \beta = \frac{1}{8} \end{gathered}$$

Therefore,

$$\tilde{p}_1(x) = \alpha x + \beta = -x + \frac{1}{8}$$

which is as derived above by other means.

To determine $p_0(x) = \beta$ for $f(x)$ on $[a, b]$ where $f''(x) < 0$ we set $\alpha = 0$ in the equations:

$$\begin{gathered} e(a) = -e(c) \\ e(b) = -e(c) \text{ or } e(a) = e(b) \\ f'(c) = e'(c) = 0 \end{gathered}$$

to determine the unknowns β, and c. Which equations we use depends on more details of $f(x)$. The error $e(x)$ is still concave but now $f'(x) = e'(x)$. This may not have a solution c.

There are 4 possiblilities:

- c does not exist, i.e., $f(x)$ is monotonically increasing or decreasing on $[a, b]$. In this case we have the extrema of $e(x)$ at the endpoints and we determine β from

$$-e(a) = e(b)$$

- $c = a$ In this case we have the extrema of $e(x)$ at the endpoints and we determine β from

$$-e(a) = e(b)$$

- $c = b$ In this case we have the extrema of $e(x)$ at the endpoints and we determine β from

$$-e(a) = e(b)$$

- $a < c < b$ and we can determine β from either

$$-e(a) = e(c) \text{ or } e(c) = -e(b)$$

The last case is the situation we have for $f(x) = -x^2$ on $[-1, 1]$.
If $f'(x) > 0$, i.e., monotonically increasing, then we have

$$\begin{aligned} -e(a) &= e(b) \\ -f(a) + \beta &= f(b) - \beta \\ \beta &= \frac{f(b) - f(a)}{2} \end{aligned}$$

The same holds for a constant approximation of a monotonically decreasing $f(x)$.

Problem 6.2

Show that the Chebyshev polynomial of degree n can be written

$$T_n(x) = \frac{1}{2}\left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\right]$$

Solution: Clearly,

$$T_n(x) = \frac{1}{2}\left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\right]$$

is such that $T_0(x) = 1$ and $T_1(x) = x$. We need only verify that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Let $a = \sqrt{x^2 - 1}$ then $a^2 - x^2 + 1 = 0$. We have

$$
\begin{aligned}
2xT_n(x) - T_{n-1}(x) &= \big[x(x+a)^n + x(x-a)^n\big] - \frac{1}{2}\big[(x+a)^{n-1} + (x-a)^{n-1}\big] \\
&= \frac{1}{2}\big[2x(x+a)^n + 2x(x-a)^n\big] - \frac{1}{2}\big[(x+a)^{n-1} + (x-a)^{n-1}\big] \\
&= \frac{1}{2}\big[2x(x+a)^n + 2x(x-a)^n - (x+a)^{n-1} - (x-a)^{n-1}\big] \\
&= \frac{1}{2}\big[2x(x+a)(x+a)^{n-1} + 2x(x-a)(x-a)^{n-1} - (x+a)^{n-1} - (x-a)^{n-1}\big] \\
&= \frac{1}{2}\big[(x+a)^{n-1}(2x(x+a) - 1) + (x-a)^{n-1}(2x(x-a) - 1)\big] \\
&= \frac{1}{2}\big[(x+a)^{n-1}(2x^2 + 2ax - 1) + (x-a)^{n-1}(2x^2 - 2ax - 1)\big] \\
&= \frac{1}{2}\big[(x+a)^{n-1}(2x^2 + 2ax - 1 + a^2 - x^2 + 1) + (x-a)^{n-1}(2x^2 - 2ax - 1 + a^2 - x^2 + 1)\big] \\
&= \frac{1}{2}\big[(x+a)^{n-1}(x^2 + 2ax + a^2) + (x-a)^{n-1}(x^2 - 2ax + a^2)\big] \\
&= \frac{1}{2}\big[(x+a)^{n-1}(x+a)^2 + (x-a)^{n-1}(x-a)^2\big] \\
&= \frac{1}{2}\big[(x+a)^{n+1} + (x-a)^{n+1}\big] \\
&= T_{n+1}(x) \quad \square
\end{aligned}
$$

Problem 6.3

6.3.a. Suppose you are given an arbitrary polynomial of degree 3 or less with the form

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3.$$

Show that there are unique coefficients, γ_i, $0 \le i \le 3$, for $p(x)$ in the representation of the form

$$p(x) = \gamma_0 T_0(x) + \gamma_1 T_1(x) + \gamma_2 T_2(x) + \gamma_3 T_3(x)$$

where $T_i(x)$, $0 \le i \le 3$, are the Chebyshev polynomials.

6.3.b. Is this true for any degree n? Justify your answer.

6.3.c. Consider $T_{32}(x)$, the Chebyshev polynomial of degree 32 and $T_{51}(x)$, the Chebyshev polynomial of degree 51. What is the coefficient of x^{13} in $T_{32}(x)$? What is the coefficient of x^{20} in $T_{51}(x)$?

Solution:

To go from

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3.$$

to

$$p(x) = \gamma_0 T_0(x) + \gamma_1 T_1(x) + \gamma_2 T_2(x) + \gamma_3 T_3(x)$$

uniquely, we show that the coefficients are related via a nonsingular matrix. To do this we need to write the monomials in terms of the T_i. We have

$$\begin{aligned}
T_0 &= 1 \\
T_1 &= x \\
T_2 &= 2x^2 - 1 \\
T_3 &= 4x^3 - 3x \\
x^0 &= T_0 \\
x^1 &= T_1 \\
x^2 &= \frac{1}{2}T_2 + \frac{1}{2} \\
x^3 &= \frac{1}{4}T_3 + \frac{3}{4}T_1
\end{aligned}$$

Therefore,

$$\begin{aligned}
p(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \\
&= \alpha_0 T_0 + \alpha_1 T_1 + \alpha_2(\frac{1}{2}T_2 + \frac{1}{2}T_0) + \alpha_3(\frac{1}{4}T_3 + \frac{3}{4}T_1) \\
&= (\alpha_0 + \frac{1}{2}\alpha_2)T_0 + (\alpha_1 + \frac{3}{4}\alpha_3)T_1 + (\frac{1}{2}\alpha_2)T_2 + (\frac{1}{4}\alpha_3)T_3 \\
\gamma_0 &= (\alpha_0 + \frac{1}{2}\alpha_2) \\
\gamma_1 &= (\alpha_1 + \frac{3}{4}\alpha_3) \\
\gamma_2 &= (\frac{1}{2}\alpha_2) \\
\gamma_3 &= (\frac{1}{4}\alpha_3)
\end{aligned}$$

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

The matrix has nonzeros on the diagonal and is upper triangular therefore nonsingular and there is a unique correspondence between the α_i and the γ_i.

This clearly generalizes to all n since we must get a triangular matrix and the diagonal element must be nonzero since it comes from the relationship between x^n and $T_n(x)$.

What is the coefficient of x^{13} in $T_{32}(x)$? What is the coefficient of x^{20} in $T_{51}(x)$? The answer to both of these is 0. The pattern can be seen from the first few T_i:

$$
\begin{gathered}
T_0 = 1 \\
T_1 = x \\
T_2 = 2x^2 - 1 \\
T_3 = 4x^3 - 3x \\
T_4 = 8x^4 - 8x^2 + 1 \\
T_5 = 16x^5 - 20x^3 + 5x \\
T_6 = 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7 = 64x^7 - 112x^5 + 56x^3 - 7x
\end{gathered}
$$

We have some simple properties that can be proven in general:

- The coefficient of x^n in T_n is 2^{n-1}.
- The signs of the terms in T_n alternate.
- If n is even the terms in T_n are only the even powers of x less than n.
- If n is odd the terms in T_n are only the odd powers of x less than n.

The last is applicable here. x^{13} in $T_{32}(x)$ is an odd power for an even n and x^{20} in $T_{51}(x)$ is an even power for an odd n and therefore both coefficients are 0. To prove the statement note that it true for all of the T_i listed. The induction hypothesis and the recurrence $T_{n+1} = 2xT_n - T_{n-1}$ say that if n is even $2xT_n$ has only odd powers and by assumption T_{n-1} has only odd powers therefore T_{n+1} has only odd powers. Therefore all odd degree T_i have only odd powers and all even degree T_i have only even powers.