Qualifying Exam

Computational Mathematics

August 2008

Do all five problems. Each problem is worth 20 points.

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1. (20 points) Consider the iteration method

$$x^{(k+1)} = Mx^{(k)} + b\,,$$

where $x^{(k)}$ and b are vectors in R^n, $M \in R^{n\times n}$, and $x^{(0)}$ is a given initial guess. Assume that $\|M\| < 1$, where $\|M\|$ is a matrix norm subordinate to (induced by) the vector norm $\|x\|$. Show that

(a) (5 points) The process is convergent to the unique solution of the linear system $x = Mx + b$.

(b) (7 points) Prove that

$$\|x^{(k)} - x\| \le \|(I-M)^{-1}\| \cdot \|x^{(k+1)} - x^{(k)}\|\,.$$

(c) (8 points) Show also that

$$\|x^{(k)} - x\| \le \|M\|^k \|x^{(0)}\| + \frac{\|M\|^k \|b\|}{1-\|M\|}\,.$$

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2. (20 points) The differential equation $y' = f(x,y)$ can be approximated by the finite difference scheme

$$y_{n+1} = y_n + \frac{h}{2}[y'_n + y'_{n+1}] + \frac{h^2}{12}[y"_n - y"_{n+1}]\,,$$

where $y'_n = f(x_n, y_n)$ and $y"_n = f_x(x_n,y_n) + f(x_n,y_n)f_y(x_n,y_n)\,.$

(a) (10 points) Show that this scheme is fourth-order accurate

(b) (10 points) State what it means for $h\lambda$ to belong to the region of absolute stability for this scheme, and show that the region of absolute stability contains the entire negative real axis.

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3. (20 points) Consider the partial differential eqaution

$$v_t = v_{xx} + bv\,, \quad t > 0, \quad x \in R$$

where $b > 0$. Analyze the convergence of the following difference scheme for the above partial differential equation

$$u_k^{n+1} = ru_{k-1}^n + (1 - 2r + b\Delta t)u_k^n + ru_{k+1}^n\,, \quad k = \pm 1, \cdots$$

where $r = \Delta t/(\Delta x)^2$.

4. (20 points) For nonlinear conservation law

$$u_t + f(u)_x = 0$$

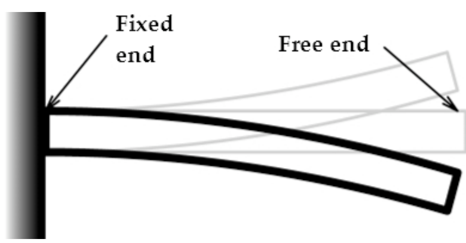
(a) (5 points) Write down the entropy condition for the differential equation and the discrete entropy condition for a conservative numerical scheme

(b) (5 points) Write down the Godunov scheme for this equation

(c) (10 points) Prove that the Godunov scheme satisfies the discrete entropy condition

5. (20 points) Consider the Beam equation from mechanics with boundary conditions that model a *cantilever* beam:

$$\begin{aligned} \textbf{PDE}: &\quad u^{(iv)} = f(x) \quad x \in (0,1)\,, \\ \textbf{BC}: &\quad u(0) = 0, \quad u'(0) = 0, \quad u''(1) = 0, \quad u'''(1) = 0. \end{aligned}$$



(a) (4 points) Recast this problem as a variational problem. Clearly state the test and trial function spaces.

(b) (6 points) Develop a finite element method for this problem.

(c) (2 points) In words, explain what an *a priori* error estimate is.

(d) (2 points) In words, explain what an *a posteriori* error estimate is.

(e) (6 points) Prove an *a posteriori* error estimate for the method developed in Part (a) in the following *energy norm*:

$$\|e\|_E^2 = \int_0^1 (e'')^2 \, dx.$$