Foundations of Computational Math I Exam 1
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Tuesday 19 October, 2010

Question	Points Possible	Points Awarded
1. Basics	15	
2. Bases and Orthogonality	20	
3. Factorization Complexity	30	
4. Backward Stability	30	
5. Conditioning and Backward Error	25	
Total Points	120	

Name:__________________________________

Alias: __________________________________

to be used when posting anonymous grade list.

Problem 1

(15 points)

Each question below has a brief answer and justification.

1.a. **(5 points)** Explain the idea of a "hidden bit" in a floating point system with base $\beta = 2$ and the benefit achieved by using it.

1.b. **(5 points)** Can the idea of a "hidden bit" be usefully generalized to a floating point system with base $\beta \neq 2$?

1.c. **(5 points)** Suppose you have a problem whose condition number is $\kappa \approx 10^5$. Given that you want at least 2 digits of accuracy in the solution how many decimal digits would you recommend be used in the floating point system used to solve the problem?

Problem 2

(20 points)

Consider the vector space $\mathbb{R}^3$ and the subspace $\mathcal{S}$ of dimension 1 given by

$$\mathcal{S} = span[v_1], \qquad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2.a. Determine a basis $\{v_2, \;\; v_3\}$ of the subspace $\mathcal{S}^\perp$ where the vectors v_2 and v_3 are **not** orthogonal.

2.b. Derive from the basis $\{v_2, \;\; v_3\}$ a second basis $\{q_2, \;\; q_3\}$ of the subspace $\mathcal{S}^\perp$ where the vectors q_2 and q_3 are orthonormal vectors.

Problem 3

(30 points)

3.a

(15 points)

Consider $A \in \mathbb{R}^{n\times n}$ whose nonzero elements are restricted to the main diagonal, the strict upper triangular part, and the first subdiagonal. For $n = 8$ the locations that must be zero are indicated and the positions that may be nonzero are indicated by α_{ij}:

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} & \alpha_{18} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} & \alpha_{27} & \alpha_{28} \\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} & \alpha_{37} & \alpha_{38} \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & \alpha_{47} & \alpha_{48} \\ 0 & 0 & 0 & \alpha_{54} & \alpha_{55} & \alpha_{56} & \alpha_{57} & \alpha_{58} \\ 0 & 0 & 0 & 0 & \alpha_{65} & \alpha_{66} & \alpha_{67} & \alpha_{68} \\ 0 & 0 & 0 & 0 & 0 & \alpha_{76} & \alpha_{77} & \alpha_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{87} & \alpha_{88} \end{pmatrix}$$

(i) **(5 points)** Suppose the subdiagonal elements $\alpha_{i+1,i} \neq 0$ (this is called an unreduced Hessenberg matrix). Determine a necessary and sufficient condition for A to be nonsingular.

(ii) **(10 points)** Describe an efficient algorithm to solve $Ax = b$ via factorization and determine the order computational complexity, i.e., give k in $O(n^k)$. Your solution should include a description of how you exploit the structure of the matrix and how it influences the structure of your factors.

3.b

(15 points)

Consider $S \in \mathbb{R}^{n\times n}$ whose nonzero elements have the following pattern for $n = 8$:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) **(5 points)** Determine a necessary and sufficient condition for S to be nonsingular.

(ii) **(10 points)** Describe an efficient algorithm to solve $Sx = b$ via factorization and determine the order computational complexity, i.e., give k in $O(n^k)$. Your solution should include a description of how you exploit the structure of the matrix and how it influences the structure of your factors.

Problem 4

(25 points)

4.a

(10 points)

Let $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^n$ be two vectors with

$$x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \quad y = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}.$$

$$|\xi_i| \geq 1 \qquad |\eta_i| \geq 1$$

Consider the evaluation of the two inner products

$$\mu = x^T x$$

$$\gamma = x^T y$$

Which of the two inner products would you expect to be less sensitive to the perturbations caused by the finite precision of IEEE floating point arithmetic?

4.b

(15 points)

Use the notation from the first part of the problem and assume the following lemma is true.

Lemma 4.1. *The computed inner product satisfies the following error bounds:*

$$fl(x^Ty) = x^T(y + \Delta y) = (x + \Delta x)^T y, \quad |\Delta x| \leq \omega_n |x|, \quad |\Delta y| \leq \omega_n |y|$$

$$|x^T y - fl(x^T y)| \leq \omega_n \sum_{i=1}^{n} |\xi_i \eta_i| = \omega_n |x|^T |y|$$

$$\omega_n = \frac{nu}{1 - nu}$$

where u is unit roundoff.

Prove the following backward error lemma:

Lemma 4.2. *If $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$ then the matrix vector product $\hat{y} \in \mathbb{R}^n$ computed in finite precision satisfies*

$$\hat{y} = (A + \Delta A)x, \quad |\Delta A| \leq \omega_n |A|, \quad \omega_n = \frac{nu}{1 - nu}$$

Problem 5

(30 points)

5.a

(15 points)

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Determine or bound the condition number for inversion, i.e., solving a system of linear equations, for the matrix A.

5.b

(15 points)

Suppose that

$$Ax \neq b \quad \text{and} \quad A = A^T$$

i.e., the matrix A is symmetric. Let $r = b - Ax$.

Show that if $r^T x \neq 0$ then there exists a backward error $E \in \mathbb{R}^{n \times n}$ such that

$$(A + E)x = b$$

where E is a symmetric rank-1 matrix, i.e.,

$$E = \sigma v v^T, \quad v \in \mathbb{R}^n, \quad \sigma \in \mathbb{R}.$$