Set 20: Homogeneous Linear Recurrence Review

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Homogenous Linear Recurrence

There is no forcing term in a homogeneous recurrence:

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = 0$$

$$y_0 = c_0, \quad y_1 = c_1, \ldots, y_{k-1} = c_{k-1}$$

Example 20.1. Fibonacci sequence

$$y_n = y_{n-1} + y_{n-2}$$

$$y_0 = 1, \quad y_1 = 1$$

$$\alpha_0 = 1, \quad \alpha_1 = \alpha_2 = -1$$

$$1, \quad 1, \quad 2, \quad 3, \quad 5\ldots$$

Homogenous Linear Recurrence

Characteristic polynomial:

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = 0$$

$$p(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \cdots + \alpha_k$$

Example 20.2. Fibonacci sequence

$$y_n = y_{n-1} + y_{n-2}$$

$$p(x) = x^2 - x - 1$$

Homogenous Linear Recurrence

Roots of the characteristic polynomial are homogeneous solutions:

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = 0$$

$$p(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \cdots + \alpha_k$$

$$p(r) = 0 \to y_n = r^n$$

Example 20.3. $k = 2$, $y_n = r^n$

$$\begin{aligned} \alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} &= \alpha_0 r^n + \alpha_1 r^{n-1} + \alpha_2 r^{n-2} \\ &= r^{n-2}\big(\alpha_0 r^2 + \alpha_1 r + \alpha_2\big) \\ &= r^{n-2} p(r) = 0 \end{aligned}$$

Homogenous Linear Recurrence

Roots with multiplicity m give m homogeneous solutions:

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = 0$$

$$p(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \cdots + \alpha_k$$

$$p(r) = p'(r) = \cdots = p^{(m)}(r) = 0$$

$$y_n = r^n, \;\; y_n = nr^n, \ldots, y_n = n^{m-1} r^n$$

Homogenous Linear Recurrence

Example 20.4. $k = 2$, $y_n = r^n$ and r is a double root:

$$p(r) = \alpha_0 r^2 + \alpha_1 r + \alpha_2 = 0, \quad p'(r) = 2\alpha_0 r^2 + \alpha_1 = 0$$

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = \alpha_0 r^n + \alpha_1 r^{n-1} + \alpha_2 r^{n-2} = 0$$

$$\left(\alpha_0 x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2}\right)' = \alpha_0 n x^{n-1} + \alpha_1 (n-1) x^{n-2} + \alpha_2 (n-2) x^{n-3}$$

$$\left(p(x) x^{n-2}\right)' = (n-2) x^{n-3} p(x) + x^{n-2} p'(x)$$

$$\therefore \alpha_0 n r^{n-1} + \alpha_1 (n-1) r^{n-2} + \alpha_2 (n-2) r^{n-3} = 0$$

$$\alpha_0 n r^n + \alpha_1 (n-1) r^{n-1} + \alpha_2 (n-2) r^{n-2} = 0$$

$$y_n = n r^n$$

Homogenous Linear Recurrence

- n fundamental solutions, s_i, given by roots and their multiplicity.
- Any linear combination is a solution

$$s = A_0 s_0 + \cdots + A_{n-1} s_{n-1}$$

- Set A_i based on initial conditions.

Homogenous Linear Recurrence

Example 20.5. Fibonacci sequence

$$y_n = y_{n-1} + y_{n-2}, \quad y_0 = 1, \quad y_1 = 1$$

$$p(x) = x^2 - x - 1, \quad r_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

$$y_n = \gamma_0 \Big(\frac{1-\sqrt{5}}{2}\Big)^n + \gamma_1 \Big(\frac{1+\sqrt{5}}{2}\Big)^n$$

$$y_0 = 1 = \gamma_0 + \gamma_1$$

$$y_1 = 1 = \gamma_0 \Big(\frac{1-\sqrt{5}}{2}\Big) + \gamma_1 \Big(\frac{1+\sqrt{5}}{2}\Big)$$

$$\therefore \gamma_0 = -\frac{1}{\sqrt{5}} \frac{1-\sqrt{5}}{2}, \quad \gamma_0 = \frac{1}{\sqrt{5}} \frac{1+\sqrt{5}}{2}$$

$$y_n = \frac{1}{\sqrt{5}} \Big(\frac{1+\sqrt{5}}{2}\Big)^n - \frac{1}{\sqrt{5}} \Big(\frac{1-\sqrt{5}}{2}\Big)^n$$