## 5.5: The Substitution Rule

The antiderivative of $f'(g(x)) \cdot g'(x)$ is $\int f'(g(x)) \cdot g'(x)\ dx = f(g(x))$ since, by the chain rule, $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$. We can use this fact to evaluate integrals such as

$$\int 2x\sqrt{1+x^2}\ dx$$

by performing a *substitution*. That is, if we let $u = 1 + x^2$, the the differential $du$ is $du = 2xdx$, so the integral can be rewritten in terms of $u$ and evaluated

$$\begin{aligned} \int 2x\sqrt{1+x^2}\ dx = \int \sqrt{1+x^2}\ 2xdx &= \int \sqrt{u}\ du \\ &= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{2/3} + C. \end{aligned}$$

**The Substitution Rule**: If $u = g(x)$ is differentiable and $f$ is continuous on the range of $u$, then

$$\int f(g(x)) \cdot g'(x)\ dx = \int f(u)\ du.$$

**Example 1.** *Find* $\int x^3 \cos(x^4 + 2)\ dx$.

**Example 2.** *Evaluate* $\int \sqrt{2x+1}\ dx$.

**Example 3.** *Find* $\int \frac{x}{1-4x^2}\ dx$

**Example 4.** *Calculate* $\int e^{5x}\ dx$

**Example 5.** *Find* $\int \sqrt{1+x^2}\ x^5\ dx$.

**Example 6.** *Calculate* $\int \tan x\ dx$.

### Definite Integrals

**The Substitution Rule for Definite Integrals**: If $u = g(x)$ is differentiable on $[a, b]$ and $f$ is continuous on the range of $u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

**Example 7.** *Evaluate* $\int_0^4 \sqrt{2x+1}\ dx$.

**Example 8.** *Evaluate* $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2)}$.

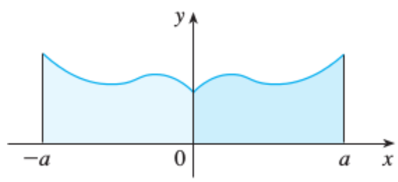
**Example 9.** *Calculate* $\displaystyle\int_1^3 \frac{\ln x}{x}\, dx$.

## Symmetry

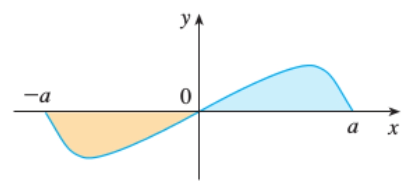
**Integrals of Symmetric Functions**: If $f$ is continuous on $[-a, a]$, then

(a) If $f$ is even, i.e. $f(-x) = f(x)$, then $\int_{-a}^{a} f(x) = 2\int_0^a f(x)\, dx$.

(b) If $f$ is odd, i.e. $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x) = 0$.




(a) $f$ even, $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$




(b) $f$ odd, $\int_{-a}^{a} f(x)\, dx = 0$

**Example 10.** *Evaluate* $\displaystyle\int_{-2}^{2} (x^6 + 1)\, dx$ *using the fact the* $f(x) = x^6 + 1$ *is even.*

**Example 11.** *Evaluate* $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\, dx$ *using the fact the* $f(x) = \frac{\tan x}{1+x^2+x^4}$ *is odd.*