Set 25: Multidimensional Polynomial Interpolation

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Multidimensional Interpolation

Let $f(v) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a given function and domain.

We are given $k+1$ points $v_i \in \mathbb{R}^n$, and values $f(v_i) \in \mathbb{R}$ for $0 \le i \le k$.

We seek a function $p(v)$ in some class of functions such that $p(v_i) = f(v_i)$ for $0 \le i \le k$ and possibly satisfying additional constraints, e.g., smoothness or piecewise polynomials etc.

We consider here the problem of $p(v) \in \mathbb{P}_d^n$, polynomials in n variables of degree at most d.

The Space

$\mathbb{P}_d^n$, the space of polynomials in n variables of degree d, is easily described in terms of the associated monomials of n variables.

Definition 25.1. Let $v \in \mathbb{R}^n$ have $e_i^T v = \nu_i$. A multivariate monomial in $\mathbb{P}_d^n$ has the form

$$v^D = \nu_1^{d_1} \nu_2^{d_1} \cdots \nu_{n-1}^{d_{n-1}} \nu_n^{d_n}$$

where $D = (d_1, \ldots, d_n)$.

The degree of the monomial v^D is $deg(v^D) = \sum_{i=1}^n d_i$.

Any $p(v) \in \mathbb{P}_d^n$ can be expressed uniquely with $\alpha_j \neq 0 \in \mathbb{R}$ as

$$p(v) = \sum_{j=1}^{J} \alpha_j v^{D_j}$$

and $deg(p) = \max_j (deg(D_j)) \leq d$.

Dimension of the Space

Lemma. $\mathbb{P}_d^n$ *is a linear space with dimension*

$$\dim(d, n) = \dim(\mathbb{P}_d^n) = \binom{n+d}{n}$$

$\dim(d, n)$ has the following values:

n	$d = 1$	$d = 5$	$d = 10$	$d = 15$
1	2	6	11	16
5	6	252	3003	15,504
10	11	3003	184,756	3,268,760
15	16	15,504	3,268,760	155,117,520

Dimension Examples

$$(d,n) = (d,1) \rightarrow \dim(d,1) = \binom{d+1}{d} = d+1$$

$$p(\nu) = \alpha_0 + \alpha_1 \nu + \cdots + \alpha_d \nu^d$$

$$(d,n) = (2,1) \rightarrow \dim(2,2) = \binom{4}{2} = 6$$

$$p(\nu_1, \nu_2) = \alpha_0 \nu_1^2 + \alpha_1 \nu_2^2 + \alpha_2 \nu_1 \nu_2 + \alpha_3 \nu_1 + \alpha_4 \nu_2 + \alpha_5$$

Interpolation

Theorem 25.1. *Given $v_1, \ldots, v_k$, there exists and interpolating $p^* \in \mathbb{P}_d^n$ through the points $(v_1, f(v_1)), \ldots, (v_k, f(v_k))$ for any values of $f(v_1), \ldots, f(v_k)$ if and only if the k linear functionals $\mu_i : \mathbb{P}_d^n \to \mathbb{R}$*

$$\mu_i(p) = p(v_i)$$

are linearly independent. The interpolating p^* is unique if and only if $k = \dim(d, n)$.

Note. Since $k \ll \dim(d, n)$ in practice, one might choose a subspace of dimension k and a basis for an approximation. Alternatively, domain $\Omega \subset \mathbb{R}^n$ could be restricted.

Example

(from Ueberhuber)

Suppose $n = 2$ and $d = 1$. We seek $p^*(v) \in \mathbb{P}_1^2$ so we have

$$p(v) = \alpha_0 + \alpha_1\nu_1 + \alpha_2\nu_2, \quad v = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

Given the points

$$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

there is not an interpolating $p^*(v)$ for any set $f(v_1), f(v_2), f(v_3)$.

Example

To see this note:

$$\mu_1(p) = \alpha_0$$

$$\mu_2(p) = \alpha_0 + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$$

$$\mu_3(p) = \alpha_0 + \alpha_1 + \alpha_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} f(v_1) \\ f(v_2) \\ f(v_3) \end{pmatrix}$$

Since the matrix is singular, i.e., the μ_i are dependent, p^* does not exist for all $f(v_i)$ data.

Altered Domain

Supppose $\Omega \subset \mathbb{R}^2$ is taken to be a rectangle

$$\Omega = \mathcal{R} = \left\{(x, y) \mid \; a \leq x \leq b \text{ and } c \leq y \leq d\right\}$$

Further, constrain the interpolation points to be a 2-dimensional mesh within $\mathcal{R}$.

- discretize $a \leq x \leq b \rightarrow x_i, \;\; 0 \leq i \leq n$
- discretize $c \leq y \leq d \rightarrow y_i, \;\; 0 \leq i \leq m$
- $h_x = \max_i(x_{i+1} - x_i)$, $h_y = \max_i(y_{i+1} - y_i)$, $h = \max(h_x, h_y)$

Lagrange in Two Dimensions

We want

$$p(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} f_{ij} \phi_{ij}(x,y)$$

$$\phi_{ij}(x_j, y_k) = \delta_{ij,jk}$$

Easily solved in the subclass of multivariate polymomials $\mathbb{P}_{n,m}$ that are a polynomial of degree n in x and of degree m in y, i.e.,

$$p(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{ij} x^i y^j$$

Note. This is not the same class as $\mathbb{P}_d^n$. Here the maximum degree in each is specified, not the sum of degrees.

Lagrange in Two Dimensions

We easily see that

$$\phi_{ij}(x,y) = \ell_i(x)\ell_j(y)$$

$$\phi_{ij}(x_j,y_k) = \delta_{ij,jk} \;\; 0 \le i \le n, \; 0 \le j \le m$$

$$\ell_i(x) = \frac{\prod_{k \ne i}(x - x_k)}{\prod_{k \ne i}(x_i - x_k)}$$

$$\ell_j(y) = \frac{\prod_{k \ne j}(y - y_k)}{\prod_{k \ne j}(y_j - y_k)}$$

Note. The $\phi_{ij}(x,y)$ are known as 2-d shape functions.

Lagrange in Two Dimensions

Theorem 25.2. *Given a mesh* $x_i, \ \ 0 \leq i \leq n$, $y_i, \ \ 0 \leq i \leq m$ *and function values* $f(x_i, y_j) = f_{ij}$ *at each point in that mesh, the unique interpolating polynomial of degree* n *in* x *and degree* m *in* y *is*

$$p(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} f_{ij} \ell_i(x) \ell_j(y)$$

where $\ell_i(x)$ *and* $\ell_j(y)$ *are the Lagrange characteristic polynomials associated with the* x_i *and* y_j *respectively.*

Basic Results

Lemma. *$\mathbb{P}_{n,m}$ on a rectangular mesh is a linear space of dimension $d = (m+1)(n+1)$ and the shape functions $\phi_{ij}(x, y)$ are a basis.*

Theorem 25.3. *Let $f(x, y) \in \mathcal{C}^{n+1,m+1}$ on the rectangle $\mathcal{R}$ on which the mesh is defined. The interpolating polynomial of degree n in x and degree m in y satisfies*

$$\|f - p\|_\infty \leq \beta_{mn} \max(h_x^{n+1}, h_y^{m+1})$$

β_{mn} is a constant with respect to (x, y) that is a function of $\|D_y^{m+1} f\|_\infty$ and $\|D_x^{n+1} f\|_\infty$, and $D_y^{m+1} f$ and $D_x^{n+1} f$ are the $m+1$-st and $n+1$-t partial derivatives of f in y and x respectively.

Piecewise Lagrange in Two Dimensions

- global 2-d Lagrange interpolating polynomials have similar properties to those of one dimension.
- Runge's phenomenon gets even worse.
- Solution is to define piecewise polynomials on each small rectangle defined by the mesh.
- Each local polynomial is in $\mathbb{P}_{n,m}$.
- Often $m = n$ resulting in
  - $m = n = 1 \rightarrow$ bilinear polynomials
  - $m = n = 2 \rightarrow$ biquadratic polynomials
  - $m = n = 3 \rightarrow$ bicubic polynomials or bicubic splines

Piecewise Bilinear Lagrange Interpolation

Let $x_i \le x \le x_{i+1}$ and $y_{j-1} \le y \le y_j$ define the local rectangle R_{ij}.

The local bilinear interpolating polynomial defined on R_{ij} is

$$p_{i,j}(x,y) = f_{i,j}\ell_{i,j}^{(ij)}(x,y) + f_{i+1,j}\ell_{i+1,j}^{(ij)}(x,y)$$

$$+f_{i,j-1}\ell_{i,j-1}^{(ij)}(x,y) + f_{i+1,j-1}\ell_{i,j-1}^{(ij)}(x,y)$$

$$\ell_{i,j}^{(ij)}(x,y) = \frac{(x - x_{i+1})}{(x_i - x_{i+1})}\frac{(y - y_{j-1})}{(y_j - y_{j-1})}$$

$$\ell_{i,j-1}^{(ij)}(x,y) = \frac{(x - x_{i+1})}{(x_i - x_{i+1})}\frac{(y - y_j)}{(y_{j-1} - y_j)}$$

$$\ell_{i+1,j}^{(ij)}(x,y) = \frac{(x - x_i)}{(x_{i+1} - x_i)}\frac{(y - y_{j-1})}{(y_j - y_{j-1})}$$

$$\ell_{i+1,j-1}^{(ij)}(x,y) = \frac{(x - x_i)}{(x_{i+1} - x_i)}\frac{(y - y_j)}{(y_{j-1} - y_j)}$$

Piecewise Bilinear Lagrange Interpolation

- This can be expressed in terms of basis functions $\phi_{ij}(x, y)$ that have local support.
- an interior point (x_i, y_j) is the meeting point of 4 rectangles
  - upper left – $R_{i-1,j+1}$
  - lower left – $R_{i-1,j}$
  - upper right – $R_{i,j+1}$
  - lower right – $R_{i,j}$
- 9 points are involved
  - upper row – (x_{i-1}, y_{j+1}),(x_i, y_{j+1}),(x_{i+1}, y_{j+1})
  - middle row – (x_{i-1}, y_j),(x_i, y_j),(x_{i+1}, y_j)
  - lower row – (x_{i-1}, y_{j-1}),(x_i, y_{j-1}),(x_{i+1}, y_{j-1})

Piecewise Bilinear Lagrange Interpolation

As before take the coeffient of f_{ij} from the local polynomial of each to define basis function

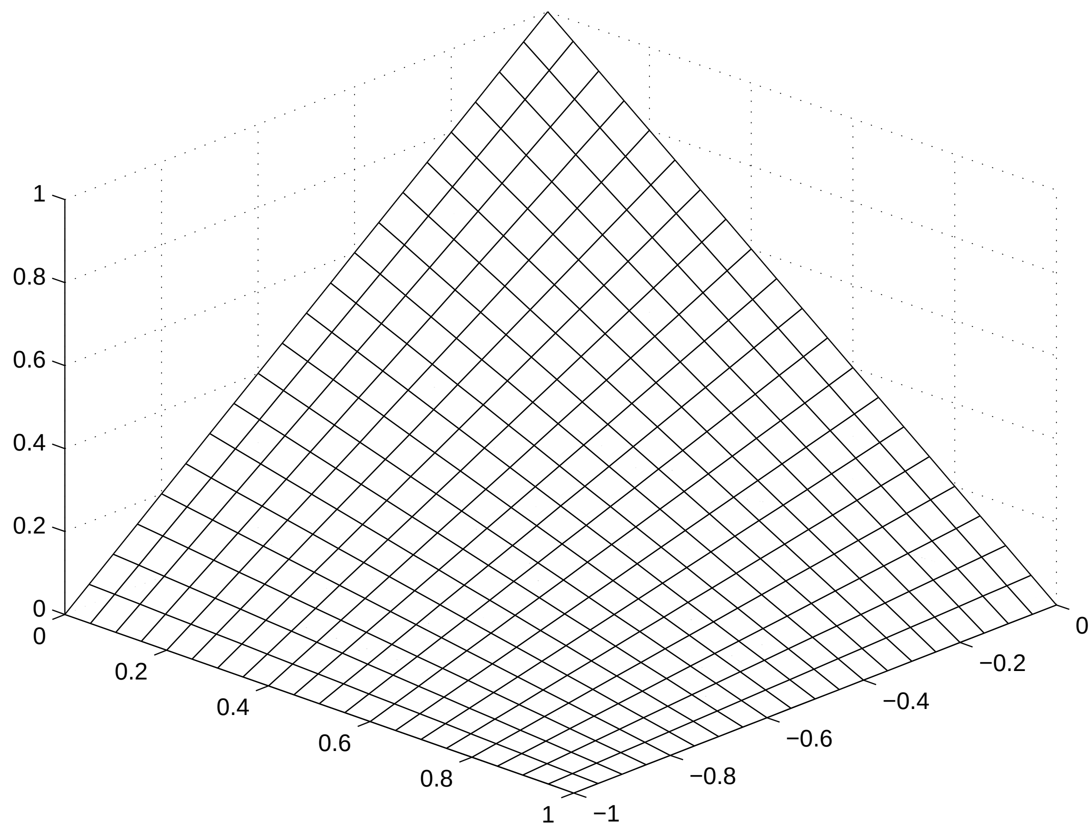
$$\phi_{ij}(x,y) = \frac{(x-x_{i+1})}{(x_i-x_{i+1})}\frac{(y-y_{j-1})}{(y_j-y_{j-1})} \text{ on } R_{i,j}$$

$$\phi_{ij}(x,y) = \frac{(x-x_{i+1})}{(x_i-x_{i+1})}\frac{(y-y_{j+1})}{(y_j-y_{j+1})} \text{ on } R_{i,j+1}$$

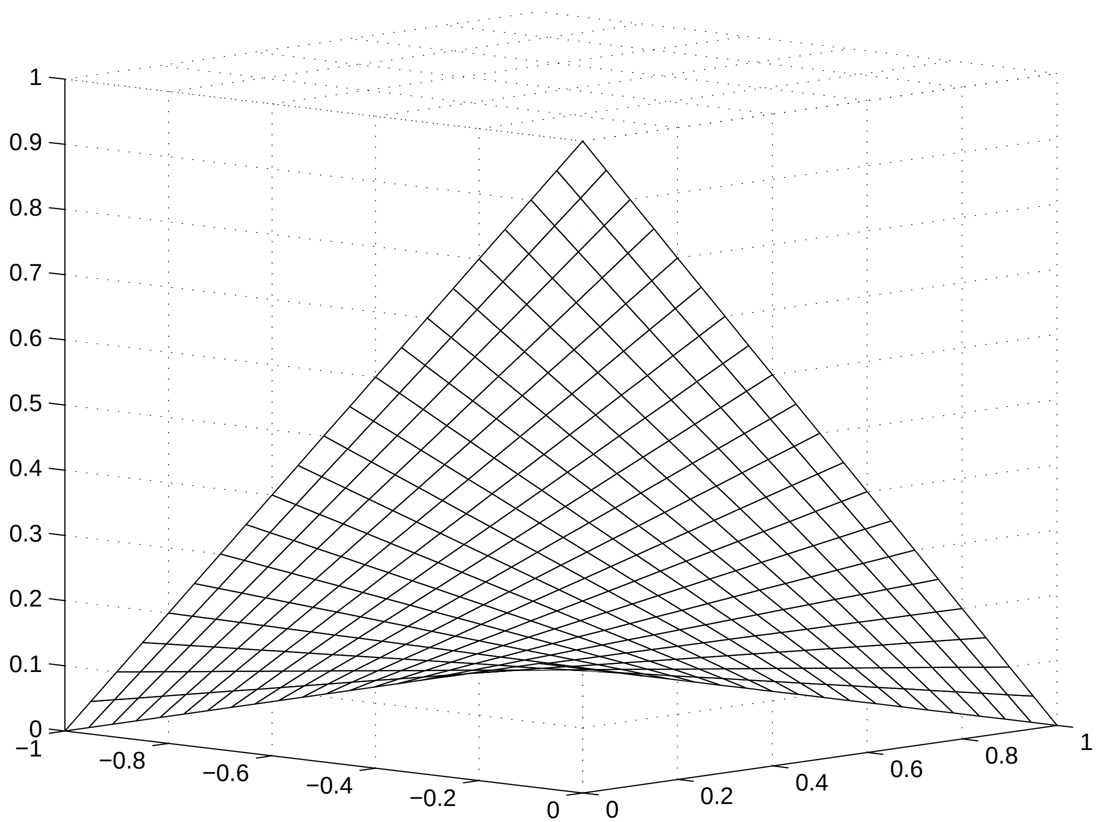
$$\phi_{ij}(x,y) = \frac{(x-x_{i-1})}{(x_i-x_{i-1})}\frac{(y-y_{j+1})}{(y_j-y_{j+1})} \text{ on } R_{i-1,j+1}$$

$$\phi_{ij}(x,y) = \frac{(x-x_{i-1})}{(x_i-x_{i-1})}\frac{(y-y_{j-1})}{(y_j-y_{j-1})} \text{ on } R_{i-1,j}$$

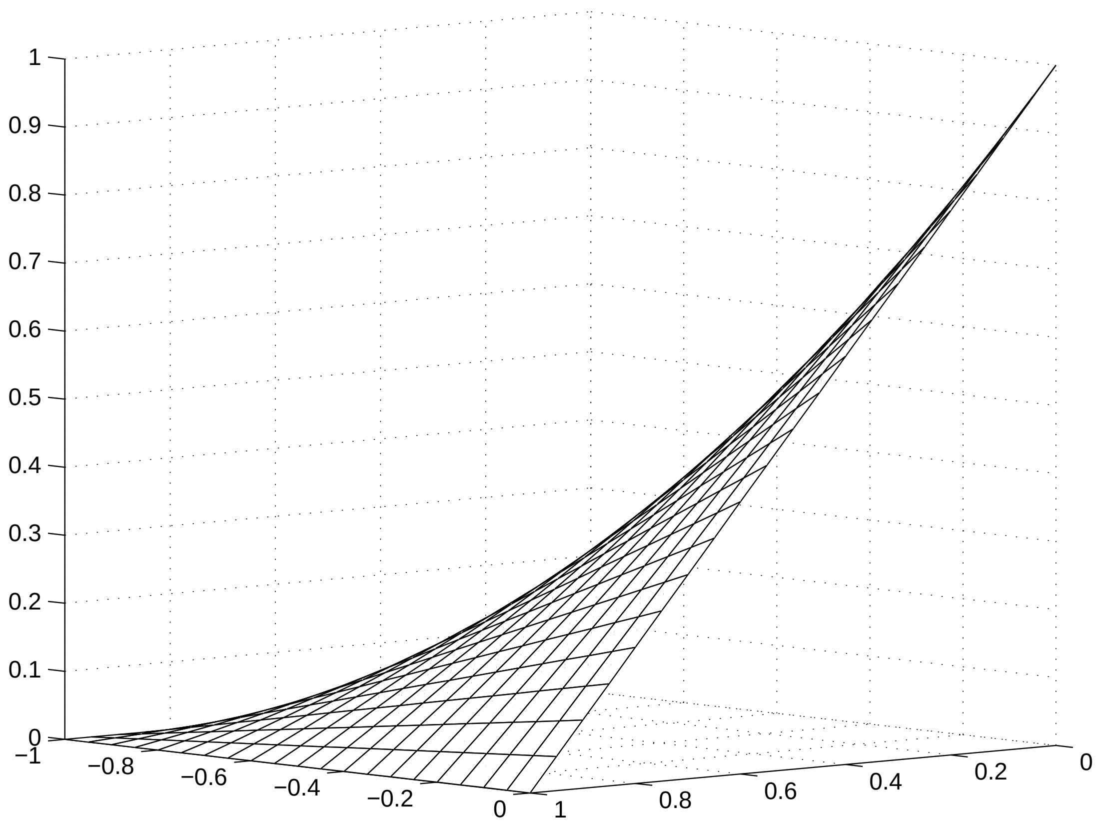
Bilinear Basis



Bilinear Basis



Bilinear Basis



Piecewise Bilinear Lagrange Interpolation

- The error follows trivially from the earlier Lagrange 2-d error

$$\|f - p\|_\infty \le \beta_{11} h^2$$

- Bicubic splines on each R_{ij} possible
- Biquadratic on each set of 4 local rectangles possible.

Triangulation

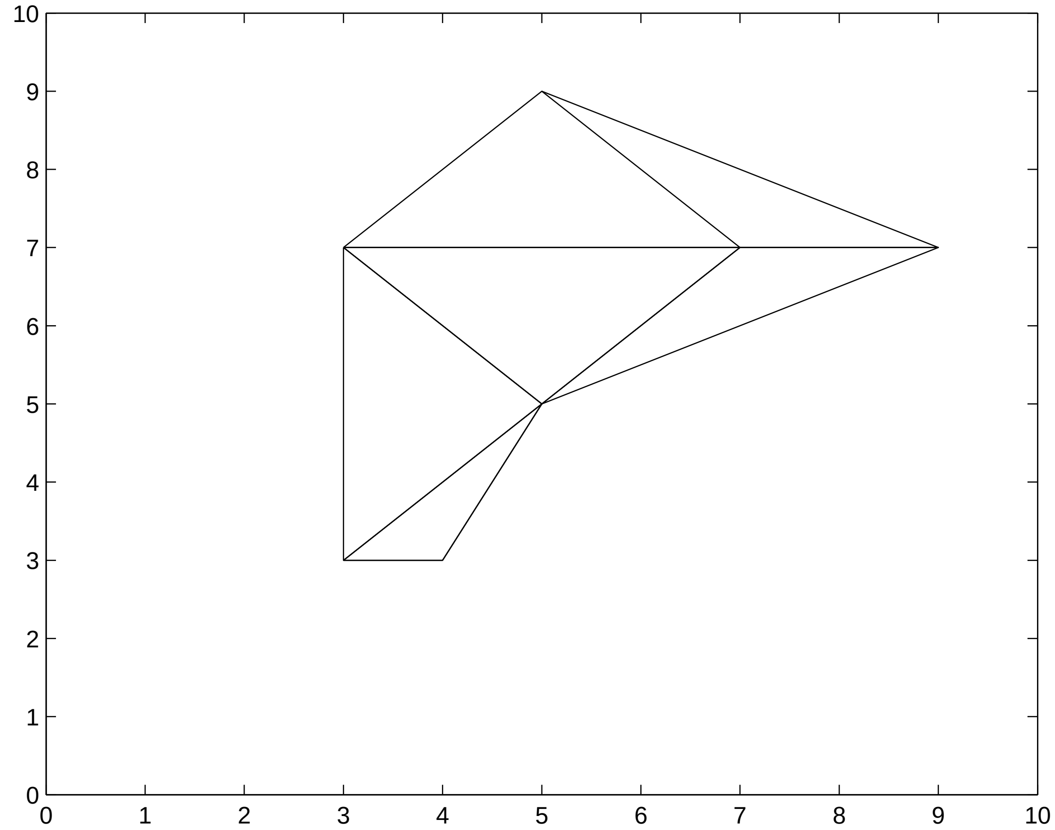
- more complicated domains Ω are disctetized differently
- mesh points define the vertices of disjoint triangles
- the union of the triangles approximates Ω
- triangles vary in area
- easy to adapt and add triangles
- main concern is keeping angles large enough
- a very large literature exists on such mesh generation

Proper Triangulation

A set of triangles must satisfy:

- The set of all vertices is the partition π, i.e., no extra or missing points
- Each pair of triangles (T_i, T_j) either
  - intersect at exactly one vertex
  - intersect at exactly one complete side
  - do not intersect
- the union of the T_i and their interiors is Ω

Triangulations



Method of Plates

- Each triangle, T_i, has three vertices: $p_0^{(i)}$,$p_1^{(i)}$,$p_2^{(i)}$
- Three points and associate function values define a plane surface above $\Omega \subset \mathbb{R}^2$
- The equation has the form

$$g_i(x, y) = \rho_i x + \mu_i y + \gamma_i$$

  where the coefficients depend on $f(p_0^{(i)})$,$f(p_1^{(i)})$,$f(p_2^{(i)})$
- It can also be written as a combination of $f(p_0^{(i)})$,$f(p_1^{(i)})$,$f(p_2^{(i)})$ as before.

Method of Plates

A basis function can be determined for $f(p_j)$ by considering the plane on T with $g(p_0) = 1$,$g(p_1) = g(p_2) = 0$.

$$\tilde{g}(x,y) = 1 - \frac{(y - y_2)}{(y_1 - y_2)} - \frac{(x - x_1)}{(x_2 - x_1)} \text{ and } g(x,y) = \frac{\tilde{g}(x,y)}{\tilde{g}(x_0,y_0)}$$

$$\tilde{g}(x_1,y_1) = 1 - \frac{(y_1 - y_2)}{(y_1 - y_2)} - \frac{(x_1 - x_1)}{(x_2 - x_1)} = 0$$

$$\tilde{g}(x_2,y_2) = 1 - \frac{(y_2 - y_2)}{(y_1 - y_2)} - \frac{(x_2 - x_1)}{(x_2 - x_1)} = 0$$

$$\tilde{g}(x_0,y_0) = 1 - \frac{(y_0 - y_2)}{(y_1 - y_2)} - \frac{(x_0 - x_1)}{(x_2 - x_1)} \neq 0$$

It follows that $\phi_j(x,y)$ is defined by $g_i(x,y)$ on each T_i for which $p_j = p_0^{(i)}$.

Method of Plates

- The interpolating piecewise function $s_1(x, y)$ is

$$s_1(x, y) = \sum_{i}^{n} f(p_i)\phi_i(x, y)$$

- $s_1(x, y)$ is continuous on Ω
- $\frac{\partial s_1}{\partial x}(x, y)$ and $\frac{\partial s_1}{\partial y}(x, y)$ in general are not continuous.
- If h is the length of the longest edge of all T_i and $f \in \mathcal{C}^1[\Omega]$ then

$$\|f - s_1\|_\infty \leq \beta_1 h$$

Other Methods

- piecewise quadratic over each triangle
- often there is a reference triangle and coordinate system
- triangles in Ω are worked with by transforming between coordinate systems
- identifying basis is often important to analyze methods
- local coordinate system might be “spectral” or in terms of a more complicated approximation similar to our discussions later in this class
- finite elements, finite volume, spectral methods, etc.