Qualifying Exam
Computational Mathematics
August 2010

Do all six problems. Each problem is worth 20 points.

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1. (20 points) Consider a system of ODEs of the form:

$$\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u}),$$

where $\mathcal{L}(\boldsymbol{u})$ is an operator that represents some spatial discretization coming from some PDE. Assume that the spatial discretization represented in $\mathcal{L}(u)$ is chosen so that the forward Euler method in time:

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t \, \mathcal{L}\left(\boldsymbol{u}^n\right),$$

satisfies the *strong stability requirement*:

$$\|\boldsymbol{u}^{n+1}\| \leq \|\boldsymbol{u}^n\|$$

in some norm $\|\cdot\|$, under the CFL condition:

$$\Delta t \leq \Delta t_{\text{FE}}.$$

(a) (10 points) Consider an s-stage Runge-Kutta method of the form:

$$\begin{aligned}
&\boldsymbol{u}^{(0)} = \boldsymbol{u}^n \\
&\text{for} \quad i = 1, \ldots, s \\
&\qquad \boldsymbol{u}^{(i)} = \sum_{k=0}^{i-1} \left\{ \alpha_{ik} \, \boldsymbol{u}^{(k)} + \Delta t \, \beta_{ik} \, \mathcal{L}\left(\boldsymbol{u}^{(k)}\right) \right\} \\
&\text{end} \\
&\boldsymbol{u}^{n+1} = \boldsymbol{u}^{(s)},
\end{aligned}$$

where

$$\alpha_{ik} \geq 0 \quad \forall i, k, \qquad \beta_{ik} \geq 0 \quad \forall i, k, \qquad \sum_{k=0}^{i-1} \alpha_{ik} = 1 \quad \forall i.$$

Prove that under some appropriate time-step restriction that this method also satisfies the *strong stability requirement.*

(b) (10 points) Find a 2-stage Runge-Kutta method of the same form as in part (a) that is second-order accurate and has the largest allowable Δt to still satisfy the *strong stability requirement.*

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2. (20 points) Consider the constant coefficient advection equation in $\mathbb{R}^2$:

$$\begin{aligned} \textbf{PDE:} \quad & q_t + u\, q_x + v\, q_y = 0, \\ \textbf{IC:} \quad & q(x, y, 0) = f(x, y), \end{aligned}$$

where $u > 0$ and $v > 0$. Furthermore, consider a Cartesian grid defined by the grid points

$$x_i = i\Delta x \qquad y_j = j\Delta y,$$

and let

$$Q_{ij}^n \approx q(x_i, y_j, t^n).$$

Construct a single finite difference method that satisfies **ALL** following requirements:

- Second-order accurate in space and time;
- Stable for $\;0 \leq \nu \leq 1$, where

$$\nu = \max\left(\frac{u\Delta t}{\Delta x}, \frac{v\Delta t}{\Delta y}\right);$$

- Makes use of the smallest possible stencil.

You must prove that your method satisfies each of these three requirements.

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3. (20 points) Consider the following nonlinear two-point boundary value problem:

$$\begin{aligned} \textbf{ODE:} \quad & u''(x) = f(x, u(x), u'(x)), \quad x \in [0, 1], \\ \textbf{BCs:} \quad & u(0) = \alpha, \quad u(1) = \beta. \end{aligned}$$

(a) (10 points) Assume that this **BVP** has a unique solution. Explain in detail how you would discretize and solve this problem using a finite difference approach based on second order accurate central finite differences.

(b) (10 points) Consider next an approach that replaces the above **BVP** by an **IVP** of the form:

$$\begin{aligned} \textbf{ODE:} \quad & u''(x) = f(x, u(x), u'(x)), \quad x \in [0, 1], \\ \textbf{ICs:} \quad & u(0) = \alpha, \quad u'(0) = \gamma, \end{aligned}$$

where γ is now also an unknown. Explain in detail how this **IVP** can be used to find a solution to the above **BVP**. Also explain in detail you would discretize and solve this problem.

4. (20 points) Consider the problem of interpolating the following data made up of $n+1$ distinct points:

$$(x_0,\, f_0), \quad (x_1,\, f_1), \quad (x_2,\, f_2), \quad \ldots, \quad (x_n,\, f_n).$$

   (a) (4 points) Prove that there exists a unique **global** polynomial of degree at most n that interpolates the above data.

   (b) (6 points) Consider the Chebyshev polynomials:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2x\, T_n(x) - T_{n-1(x)} \quad n > 0.$$

   Prove all of the following:

      i. (2 points) $T_n(x)$ is a polynomial of degree exactly n with n distinct real roots between $-1 \le x \le 1$;

      ii. (2 points) $-1 \le T_n(x) \le 1$ for all $-1 \le x \le 1$;

      iii. (2 points) The coefficient of x^n in $T_n(x)$ is exactly 2^{n-1} for $n > 0$.

   (c) (8 points) Consider the problem of interpolating the function $f(x)$ at the $n+1$ distinct points $x_0, x_1, \ldots, x_n$, where $-1 \le x_0 < x_1 < x_2 < \cdots < x_n \le 1$, with the global polynomial $p_n(x)$. Prove that the max-norm error:

$$\|f(x) - p_n(x)\|_\infty := \max_{-1 \le x \le 1} \left| f(x) - p_n(x) \right|$$

   is minimized over all possible choices of the points $x_0, x_1, \ldots, x_n$ if these points are the $n+1$ roots of $T_{n+1}(x)$.

   (d) (2 points) How does $\|f(x) - p_n(x)\|_\infty$ decay with increasing n.

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5. (20 points) Consider the following 1D advection-diffusion equation:

$$\begin{aligned}
\textbf{PDE:} &\quad u_t + a u_x = \kappa u_{xx}, \quad \kappa > 0, \quad 0 \le x \le 1,\\
\textbf{BCs:} &\quad u(0,t) = u(1,t) = 0,\\
\textbf{IC:} &\quad u(x,0) = f(x).
\end{aligned}$$

   (a) (4 points) Find a weak formulation for this PDE. Prove that both the PDE and the weak formulation have a unique solution. Prove that the two formulations have the same solution.

   (b) (8 points) Construct a finite element method that uses cG(1) elements in both space and time. Write out in detail the discrete problem that must be solved in order to update the solution.

   (c) (4 points) Prove that the method from part (b) has a unique solution.

   (d) (4 points) What happens to this method as $\kappa \to 0^+$?

6. (20 points) Consider the following 1D heat equation:

$$\begin{aligned} \textbf{PDE:} &\quad u_t = u_{xx}, \quad 0 \le x \le 1, \\ \textbf{BCs:} &\quad u(0,t) = 0, \quad u(1,t) = 0. \\ \textbf{IC:} &\quad u(x,0) = f(x) \end{aligned}$$

(a) (4 points) Construct a Taylor series in time for the solution $u(x, t + \Delta t)$ about $\Delta t = 0$, retaining the $\mathcal{O}(1)$, $\mathcal{O}(\Delta t)$, and $\mathcal{O}(\Delta t^2)$ terms. Replace any time derivatives with spatial derivatives via the PDE.

(b) (8 points) Construct a Galerkin finite element method for the Taylor series computed in part (a) with basis functions that are linear in each element and C^0 across element edges.

(c) (8 points) Construct a Galerkin finite element method for the Taylor series computed in part (a) with basis functions that are cubic in each element and C^1 across element edges.