

Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Tuesday January 12, 2010 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Nonlinear equations:

1. Consider the system

$$\begin{aligned}x &= g_1(x, y) = \frac{1}{\sqrt{2}} \sqrt{1 + (x + y)^2} - \frac{2}{3} \\y &= g_2(x, y) = \frac{1}{\sqrt{2}} \sqrt{1 + (x - y)^2} - \frac{2}{3}\end{aligned}$$

Consider a region, D , in the xy -plane for which the fixed point iteration

$$\begin{aligned}x_{n+1} &= \frac{1}{\sqrt{2}} \sqrt{1 + (x_n + y_n)^2} - \frac{2}{3} \\y_{n+1} &= \frac{1}{\sqrt{2}} \sqrt{1 + (x_n - y_n)^2} - \frac{2}{3}\end{aligned}$$

is guaranteed to converge to a unique solution for any $(x_0, y_0) \in D$.

- a. State clearly what properties this region must have.
- b. Find a region with these properties and show that it has these properties.
- c. Restate the fixed point iteration as the solution of a system of nonlinear equations. Symbolically describe Newton's method for this system. Show that for any $(x_0, y_0) \in D$, (x_1, y_1) is well defined. What can be said about the next step of this iteration?

Hint: It is helpful to consider the region in which any solution of this equation must lie.

Numerical quadrature:

2. a. Write down the formulas for the Trapezoidal rule and for Simpson's rule when approximating $\int_a^b f(x)dx$ by means of function values at $x_i = a + hi$, $i = 0, 1, \dots, n$. Here, n is an even number, and $h = (b - a)/n$. Then show that you can obtain Simpson's rule by Richardson extrapolation based on the Trapezoidal rule.

- b. In order to obtain fourth order accuracy, it is not necessary to use weights that oscillate across the whole interval, as shown by the following integration formula

$$\int_a^b f(x)dx = h \left[\left\{ \frac{3}{8}f(x_0) + \frac{7}{6}f(x_1) + \frac{23}{24}f(x_3) \right\} + \{f(x_4) + f(x_5) + f(x_6) + \dots + f(x_{n-3})\} + \left\{ \frac{23}{24}f(x_{n-2}) + \frac{7}{6}f(x_{n-1}) + \frac{3}{8}f(x_n) \right\} \right] + O(h^4)$$

Outline an approach that will lead you to this formula above (and to similar ones of still higher orders of accuracy, using more extensive 'end corrections' but with constant weights across the main part of the interval). It is not essential that you work through the algebra that leads to the exact coefficients shown above, as long as the approach that you outline is one that will lead to the goal.

Interpolation / Approximation:

3. We are given the data
- | | | | |
|-----|----|----|---|
| x | -1 | 0 | 2 |
| y | 1 | -1 | 1 |
- a. Find the interpolating polynomial using Lagrange's method.
 - b. Find the interpolating polynomial using Newton's method.
 - c. Find the best fitting straight line $a + bx$ (minimizing $\sum_{i=1}^3 (a + bx_i - y_i)^2$).
 - d. Determine the *natural cubic spline* (second derivative zero at both ends) that fits the data.

Linear algebra:

4. Let A be an $n \times n$ real matrix and let \underline{b} be an $n \times 1$ vector. Consider the minimization problem

$$\min_{\underline{x} \in \mathbb{R}^n} \|\underline{Ax} - \underline{b}\|^2 + \lambda^2 \|\underline{x}\|^2.$$

Let \underline{x}_λ be a solution.

- a. Characterize the solution by deriving an equivalent linear system.
- b. Show that, for $\lambda > 0$, the solution is always unique.
- c. Characterize the solution of this problem in terms of the singular values and singular vectors of A .
- d. Suppose $\text{rank } A < n$. Describe $\lim_{\lambda \rightarrow 0} \underline{x}_\lambda$.

Hint: When all else fails, use calculus.

Numerical ODE:

5. The *Forward Euler* scheme for $y' = f(t, y)$ can be written as

$$y(t+k) = y(t) + kf(t, y(t)),$$

and is readily obtained from the truncated Taylor expansion $y(t+k) = y(t) + ky'(t) + O(k^2)$. This scheme can be seen as the lowest order example of a *Taylor series scheme* for ODEs. The next order accurate Taylor series scheme becomes

$$y(t+k) = y(t) + kf + \frac{k^2}{2}(f_t + ff_y),$$

with *local* error $O(k^3)$ and *global* error $O(k^2)$.

- a. Derive the Taylor series scheme that is globally of third order.

Although explicit formulas for still higher order Taylor schemes rapidly become excessively complicated, very effective recursive algorithms are available for computing any number of expansion coefficients. One situation however where of the explicit formulations, such as those above, are essential is for deriving the compatibility conditions for Runge-Kutta methods.

- b. The following is a general explicit 2-stage Runge-Kutta method:

$$\begin{aligned}d^{(1)} &= kf(t+0 \cdot k, y) \\d^{(2)} &= kf(t+c \cdot k, y+a \cdot d^{(1)}) \\----- \\y(t+k) &= y(t) + b_1 \cdot d^{(1)} + b_2 \cdot d^{(2)}\end{aligned}$$

Derive the compatibility conditions that the coefficients a, c, b_1, b_2 need to satisfy in order for this scheme to be (globally) second order accurate.

Numerical PDE:

6. Consider the following equations

I. $u_t = bu_x, \quad b > 0 \text{ constant.}$

II. $u_t = u_{xx}$

III. $u_t = u_{xx} + bu_x$

for $x \in (0, 1)$ and $t > 0$.

a. For each equation, develop a finite difference formula using

(i) Forward Euler in time,

(ii) Upwind differences in space for the first-order terms (be careful about the wind direction),

(iii) Centered difference in space for the second-order terms

(Use the notation u_j^ℓ to denote the approximation at time t_ℓ and spatial point x_j .)

b. For each equation, display the stencil using a different symbol for each term. Give reasonable boundary/initial conditions for each equation.

c. For equations I and II, use Von Neumann analysis to obtain a restriction on the time step size that guarantees stability of the finite difference solution. Show your work.

d. For equation III, obtain an equation for the amplification factor. State a simple condition that would guarantee stability and explain in words why this is more complicated than the condition for Equation II (Do not try to solve for the time step restriction).