Set 12: Orthogonality and Approximation- Part 3

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Foundations of Computational Math 2
Spring 2013

Discrete Least Squares

Suppose $x_0 < x_1 < \cdots < x_m$ are given and the metric

$$\sum_{i=0}^{m} \omega_i (f(x_i) - p_n(x_i))^2$$

with $\omega_i > 0$ is used to determine the polynomial, $p_n^*(x)$, of degree n that achieves the minimal value.

Typically, $m \gg n$. If $m = n$ then the unique interpolating polynomial is the solution.

Discrete Least Squares

Suppose we have a basis of polynomials $\big(\phi_0(x), \ldots, \phi_n(x)\big)$ and let

$$p_n(x) = \sum_{j=0}^{n} \phi_j(x)\xi_j$$

then the conditions are

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} = \begin{pmatrix} \omega_0^{1/2} f(x_0) \\ \omega_1^{1/2} f(x_1) \\ \vdots \\ \omega_m^{1/2} f(x_m) \end{pmatrix} - \begin{pmatrix} \omega_0^{1/2}\phi_0(x_0) & \ldots & \omega_0^{1/2}\phi_n(x_0) \\ \omega_1^{1/2}\phi_0(x_1) & \ldots & \omega_1^{1/2}\phi_n(x_1) \\ \vdots & & \vdots \\ \omega_m^{1/2}\phi_0(x_m) & \ldots & \omega_m^{1/2}\phi_n(x_m) \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

$$r = W^{1/2}(b - Ax) = (\tilde{b} - \tilde{A}x)$$

Discrete Least Squares

Two equivalent problems:

$$\min_{x \in \mathbb{R}^n} \|b - Ax\|_W^2$$

$$\text{where } \|v\|_W^2 = v^T W v$$

$$\min_{x \in \mathbb{R}^n} \|\tilde{b} - \tilde{A}x\|_2^2$$

The latter is the standard linear least squares problem. We assume that A has linearly independent columns and therefore we can solve using:

- Householder reflector-based transformation method
- SVD-based method
- Conjugate gradient iterative method

Discrete Least Squares and Polynomials

Back to polynomial approximations.

We assumed a basis of polynomials $\big(\phi_0(x), \dots, \phi_n(x)\big)$ and let

$$p_n(x) = \sum_{j=0}^{n} \phi_j(x)\xi_j$$

Can we select $\phi_j(x)$ so that $\tilde{A} = W^{1/2}A$ has orthogonal columns? If so then the discrete least squares problem is solved by applying $\tilde{A}^T$ to the vector $\tilde{b}$ and scaling.

Discrete Orthogonal Polynomials

Define the inner product

$$(f, g) = (g, f) = \sum_{i=0}^{m} f(x_i)g(x_i)$$

i.e., we have $\omega_i = 1$ for $0 \leq i \leq m$

We want polynomials $P_i(x)$ for $0 \leq i \leq m$ such that

$$(P_r(x), P_s(x)) = \delta_{r,s}, \quad 0 \leq r, s \leq m$$

We restrict the problem further by choosing $-1 \leq x_i \leq 1$ and equally spaced points $x_i = x_0 + ih$, with $x_0 = -1$ and $h = 2/m$.

Gram Polynomials

Theorem 12.1. *Let* $m > 0$ *be given and let* $x_i = x_0 + ih$*, with* $x_0 = -1$ *and* $h = 2/m$ *and define the inner product*

$$(f, g) = (g, f) = \sum_{i=0}^{m} f(x_i) g(x_i).$$

The Gram polynomials, $P_i(x)$*, for* $0 \le n \le m$*, are defined by the recurrence*

$$P_{-1}(x) = 0, \quad P_0(x) = \frac{1}{\sqrt{m+1}}, \quad P_{n+1}(x) = \alpha_n x P_n(x) - \gamma_n P_{n-1}(x)$$

$$\alpha_n = \frac{m}{n+1} \Big(\frac{4(n+1)^2 - 1}{(m+1)^2 - (n+1)^2} \Big)^{1/2} \textit{ and } \gamma_n = \frac{\alpha_n}{\alpha_{n-1}}$$

$$\textit{satisfy } (P_i, P_j) = \delta_{i,j}, \quad 0 \le i, j \le m$$

Proof. See Dahlquist and Bjorck or Isaacson and Keller. $\square$

Gram Polynomials

- Gram Polynomials are the discrete analogs of the Legendre Polynomials
- When $n << \sqrt{m}$ they behave like Legendre polynomials.
- When $n >> \sqrt{m}$ they have large oscillations and large maximum norms.
- When using equidistant data $n < 2\sqrt{m}$ is recommended.

Chebyshev Polynomials

Lemma. *Recall the Chebyshev polynomials*

$$T_n(x) = \cos(n \arccos x), \;\; n \geq 1$$

$$T_0(x) = 1, \;\; T_1(x) = x, \;\; T_{n+1} = 2xT_n(x) - T_{n-1}(x)$$

and consider the roots of $T_{m+1}(x)$ *for some given* $m > 0$*,*

$$x_i = \cos \frac{2i+1}{m+1}\frac{\pi}{2}, \;\; 0 \leq i \leq m$$

The discrete inner product $(T_i, T_j) = \sum_{k=0}^{m} T_i(x_k)T_j(x_k)$ *for* $0 \leq i, j \leq m$ *satisfies (note the weights are all* 1*):*

$$(T_i, T_j) = 0, \;\; i \neq j$$

$$(T_i, T_j) = \frac{m+1}{2}, \;\; i = j \neq 0$$

$$(T_i, T_j) = m+1, \;\; i = j = 0$$

Chebyshev Polynomials

Theorem 12.2. *Let*

$$P_n(x) = \frac{\sqrt{2}}{\sqrt{m+1}} \cos(n \arccos x), \;\; n \geq 1$$

$$P_0(x) = \frac{1}{\sqrt{m+1}}, \;\; P_1(x) = \frac{\sqrt{2}}{\sqrt{m+1}} T_1(x),$$

$$P_2(x) = \frac{\sqrt{2}}{\sqrt{m+1}} T_2(x), \quad P_{n+1} = 2xP_n(x) - P_{n-1}(x), \;\; n \geq 2$$

and, for some given $m > 0$*, let*

$$x_i = \cos \frac{2i+1}{m+1} \frac{\pi}{2}, \;\; 0 \leq i \leq m$$

The discrete inner product $(P_i, P_j) = \sum_{k=0}^{m} P_i(x_k) P_j(x_k)$ *for* $0 \leq i, j \leq m$ *satisfies*

$$(P_i, P_j) = \delta_{i,j}$$