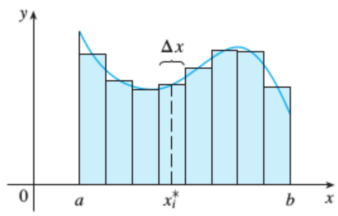
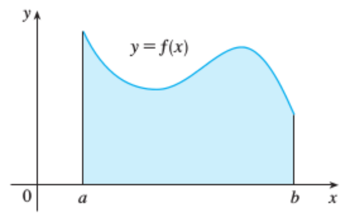
5.2: The Definite Integral

If f is a function defined for all $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of the subintervals and we let $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ be any *sample points* in these subintervals, so x_k^* lies in the k^{th} subinterval $[x_{k-1}, x_k]$. Then the **definite integral of** f **from** a **to** b is

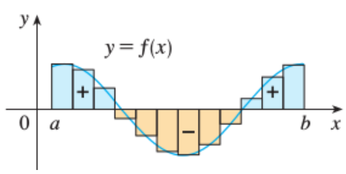
$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

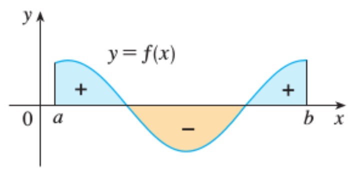
provided this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.





If f takes positive and negative values, then $\int_a^b f(x)\ dx$ is the area of the region below f above the x-axis (shaded blue in the figure below) *minus* the region above f below the x-axis (shaded orange in the figure below).





Theorem 1. *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\ dx$ exists.*

Theorem 2. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \quad \textit{and} \quad x_k = a + k\Delta x$$

for $k = 0, 1, 2, \ldots, n$

Example 1. *Express*

$$\lim_{n\to\infty} \sum_{k=1}^{n} (x_k^3 + x_k \sin x_k)\Delta x$$

on the interval $[0, \pi]$ as a definite integral.

Evaluating Integrals

The following formulas are useful for evaluating integrals using the Riemann sum definition.

- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

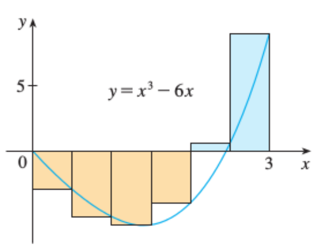
The following formulas are rules for working with sums in sigma notation.

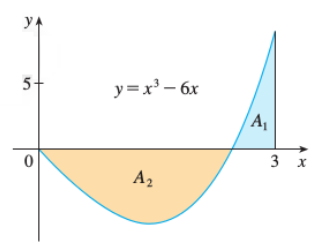
- $\displaystyle\sum_{k=1}^{n} c = nc$
- $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$
- $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Example 2.

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be the right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\ dx$.





Example 3. *Set up an expression for $\int_1^3 e^x\ dx$ as a limit of sums.*

Example 4. *Evaluate the following integrals by interpreting each in terms of areas.*

(a) $\displaystyle\int_0^1 \sqrt{1 - x^2}\ dx$

(b) $\displaystyle\int_0^3 (x - 1)\ dx$

The Midpoint Rule

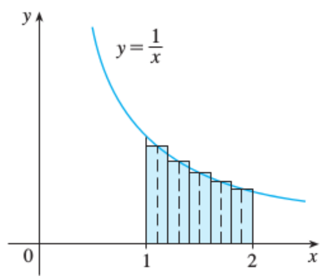
$$\int_a^b f(x)\ dx \approx \sum_{k=1}^{n} f(\bar{x_k})\Delta x = \Delta x[f(\bar{x_1}) + f(\bar{x_2}) + f(\bar{x_3}) + \cdots + f(\bar{x_n})]$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad \bar{x_k} = \frac{1}{2}(x_{k-1} + x_k)$$

for $k = 1, 2, \ldots, n$.

Example 5. *Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\ dx$.*



Properties of the Definite Integral

If we integrate over the interval $[b, a]$ instead of $[a, b]$, then Δx switches from $(b-a)/n$ for the interval $[a, b]$ to $(a-b)/n$ for the interval $[b, a]$. Since these are the same value with opposite signs,

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$

If $a = b$, then $\Delta x = 0$ and

$$\int_a^b f(x)\ dx = 0.$$

The following are properties of the definite integral

- $\int_a^b c\ dx = c(b-a)$ where c is a constant
- $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$ where c is a constant
- $\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
- $\int_a^b [f(x) - g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$

Example 6. *Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\ dx$.*

The following property tells us how to combine definite integrals of the same function over adjacent intervals

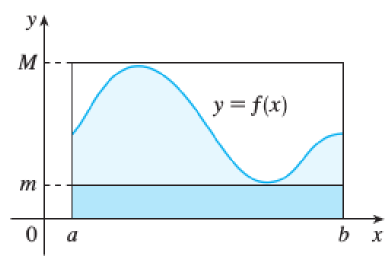
$$\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx.$$

Example 7. *If it is known that* $\int_0^{10} f(x)\ dx = 17$ *and* $\int_0^8 f(x)\ dx = 12$, *find* $\int_8^{10} f(x)\ dx$.

The following comparison properties are useful for placing bounds on definite integrals.

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$



Example 8. *Use the last comparison property above to estimate* $\int_0^1 e^{-x^2}\ dx$.