1. Root Finding:

See parts a and b of Theorem 2.9 of Atkinson and the uniqueness proof for part a.

2. Numerical Quadrature:

a. Consider the Legendre polynomials, $P_n(x)$, on $[-1,1]$. Take the nodes, x_m, to be the zeros of $P_n(x_m) = 0$, $m = 1, \ldots, n$, and the weights to be solutions of the linear system

$$\int_{-1}^{1} P_k(x)dx = \sum_{j=1}^{n} w_j P_k(x_j), \quad k = 0, \ldots, n-1.$$

The zeros of P_n are all inside the interval $[-1,1]$ and the weights are positive. The product of two polynomials from $\mathcal{P}_{n-1}$ is a polynomial of degree less or equal to $2n-2$. For any polynomial G of degree less or equal to $2n-1$, using Euclidean division we have

$$G(x) = q(x)P_n(x) + r(x),$$

where the degree of polynomials q and r is less or equal to $n-1$. For the inner product we have

$$\int_{-1}^{1} G(x)dx = \int_{-1}^{1} q(x)P_n(x)dx + \int_{-1}^{1} r(x)dx = \int_{-1}^{1} r(x)dx,$$

where the first integral vanishes since q may be represented via Legendre polynomials of degree less or equal to $n-1$. For the discrete inner product we have

$$\sum_{j=1}^{n} w_j G(j) = \sum_{j=1}^{n} w_j q(x_j) P_n(x_j) + \sum_{j=1}^{n} w_j r(x_j) = \sum_{j=1}^{n} w_j r(x_j),$$

where the sum vanishes due to the choice of the nodes. Finally, due to the choice of the weights and since r may be viewed as a linear combination of Legendre polynomials of degree less or equal to $n-1$, the sum and the integral are the same.

b. Consider the collection of Lagrange interpolating polynomials, $L_m(x)$, of degree $n-1$ for the nodes $x_1, \ldots, x_n$. To prove orthogonality, use equivalence of inner products for polynomials of degree less or equal to $2n-1$:

$$\int_{-1}^{1} L_m(x)L_k(x)dx = \sum_{j=1}^{n} w_j L_m(x_j)L_k(x_j).$$

If $m \neq k$, then the sum is zero since $L_m(x_j) = \delta_{mj}$. If $m = k$, then we have

$$\int_{-1}^{1} L_m^2(x)dx = \sum_{j=1}^{n} w_j L_m^2(x_j) = w_m.$$

This also shows that the weights are positive. Finally, a dimensionality argument shows that the L_m form a basis of $\mathcal{P}_{n-1}$. (Alternatively, one may compute Legendre polynomials as linear combinations of the L_m.)

3. Interpolation:

Start with the most obvious ones:

(E) The error should be zero at all the interpolation nodes, which include the two end points. Hence (E) matches (iv).

(D) The leading error should be the very close to the next order Chebyshev polynomial, which oscillates across the interval with equal amplitudes. Thus it matches Figure (ii).

(F) This is a standard truncated orthogonal expansion based on the functions $e^{i\pi kx}$, $k = 0, \pm 1, \pm 2, \ldots, \pm n$. The error must be orthogonal to each of these. In particular (choosing $k = 0$), the integral of the error over [-1,1] is zero. With (ii) and (iv) already eliminated, only (i) can satisfy this.

(A,B) Both sums are Taylor expansions around the origin, and hence the error will be extremely small in a quite wide range surrounding the origin, but then grow rapidly, i.e. this fits with Figures (v) and (vi). The difference $e^x - \sum_{k=0}^{n} \frac{x^k}{k!}$ can never be negative (since it would decrease to zero if we continued the sum to infinitely large n. Hence (A) corresponds to (vi) and (B) to (v). Alternatively, leading to the same answer: The error for (A) should be about equally big at both ends, but for (B) it should be $e^2 \approx 7.4$ times larger at $x = 1$ than at $x = -1$.

(C) The only figure that is left to match is (iii). We can also support this by a direct approximation for when $n >> |x|$ as follows (keeping only the leading order term in the estimates):

$$e^x - (1 + \tfrac{x}{n})^n = e^x - e^{n \log(1+\frac{x}{n})} = e^x - e^{n(\frac{x}{n} - \frac{x^2}{2n^2} + \ldots)} =$$
$$= e^x (1 - e^{-\frac{x^2}{2n} + \cdots}) \approx \frac{x^2 e^x}{2n}.$$

This looks indeed like a very good match with (iii).

4. Linear Algebra:

a. One of various ways to define Gauss-Seidel is as follows:

$$x_i \leftarrow \frac{1}{a_{i,i}}\left(b_i - \sum_{j\neq i} a_{i,j}x_j\right).$$

b. The formula in (a) can be rewritten as

$$\mathbf{x} \leftarrow \mathbf{x} + \frac{1}{a_{i,i}}\left(b_i - \sum_{j} a_{i,j}x_j\right)\epsilon_i = \mathbf{x} + \frac{<\epsilon_i, \mathbf{b} - A\mathbf{x}>}{<\epsilon_i, A\epsilon_i>}\epsilon_i = \mathbf{x} + \frac{<\epsilon_i, A\mathbf{e}>}{<\epsilon_i, A\epsilon_i>}\epsilon_i.$$

Subtracting $\mathbf{x}^*$ from both sides and changing signs yields the following error propagation equation:

$$\mathbf{e} \leftarrow \mathbf{e} - s^*\epsilon_i, \quad s^* = \frac{<\epsilon_i, A\mathbf{e}>}{<\epsilon_i, A\epsilon_i>}.$$

To see that s^* minimizes the new error $\|\mathbf{e} - s\epsilon_i\|_A$ over s, note that

$$\|\mathbf{e} - s\epsilon_i\|_A^2 = \|\mathbf{e}\|_A^2 - 2s <\epsilon_i, A\mathbf{e}> + s^2 <\epsilon_i, A\epsilon_i>.$$

The first and second derivative tests confirm the assertion.

c. To conclude now that Gauss-Seidel converges, note that the A-norm of the error forms a nonincreasing sequence, so the error sequence is bounded in the A-norm (by the initial A-norm of the error). Then there must be a convergent subsequence. Its limit must be A-orthogonal to every ϵ_i or else it would be decreased by a Gauss-Seidel step. But that means that this vector is $\mathbf{0}$, which shows that the A-norm of the error converges to 0, proving the assertion.

5. Numerical ODE:

a. Consider the difference

$$\mathbf{d}(t, \mathbf{y}) = a_2\mathbf{y}(t+2h) + a_1\mathbf{y}(t+h) + a_0\mathbf{y}(t) - h\left(b_2\mathbf{y}'(t+2h) + b_1\mathbf{y}'(t+h) + b_0\mathbf{y}'(t)\right).$$

This difference should vanish for monomials $\mathbf{y}(t)$ 1, t, t^2,..., leading to conditions

$$\begin{aligned}
\sum_{m=0}^{2} a_m &= 0, \\
\sum_{m=0}^{2} m a_m &= \sum_{m=0}^{2} b_m, \\
\sum_{m=0}^{2} m^2 a_m &= 2\sum_{m=0}^{2} m b_m, \\
\cdots & \cdots \cdots \\
\sum_{m=0}^{2} m^p a_m &= p\sum_{m=0}^{2} m^{p-1} b_m,
\end{aligned}$$

where $p \geq 1$ is the order (given that the next equation of this type does not hold).

b. Choose $a_2 = 0$, $a_1 = 1$, $a_0 = -1$, $b_2 = 0$, $b_1 = b_0 = 1/2$ to obtain the trapezoidal rule,

$$\mathbf{y}_{n+1} - \mathbf{y}_n = \frac{1}{2}h\left(\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \mathbf{f}(t_n, \mathbf{y}_n)\right).$$

The order is $p = 2$. To construct the region of absolute stability, consider the test problem

$$y' = \lambda y$$

for which the trapezoidal rule gives

$$y_{n+1} - y_n = \frac{1}{2}\lambda h(y_{n+1} + y_n)$$

or

$$y_{n+1} = \frac{1 + \frac{1}{2}\lambda h}{1 - \frac{1}{2}\lambda h} y_n = \left(\frac{1 + \frac{1}{2}\lambda h}{1 - \frac{1}{2}\lambda h}\right)^{n+1} y_0.$$

Let $z = \lambda h$. If $\mathcal{R}e(z) < 0$, then

$$\left|\frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}\right| < 1$$

(the last inequality may be stated without proof) and, thus, the scheme is A-stable.

c. Implicit Euler (of order $p = 1$) is such a scheme:

$$\mathbf{y}_{n+1} - \mathbf{y}_n = h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}).$$

For the test problem, we have

$$y_{n+1} = \frac{1}{1 - \lambda h} y_n$$

and its region of absolute stability is the complex plane outside the disk of radius 1 centered at 1, that is, $|1 - z| > 1$.

6. Numerical PDE:

a. Taylor expansion gives

$$\frac{u(x,t+k)-u(x,t)}{k} = u_t + \tfrac{k}{2}u_{tt} + O(k^2),$$

$$\frac{\tfrac{3}{2}u(x,t)-2u(x-h,t)+\tfrac{1}{2}u(x-2h,t)}{h} = u_x - \tfrac{h^2}{3}u_{xxx} + O(h^3).$$

Therefore, the stencil is consistent with $u_t + u_x = 0$.

b. It follows immediately from the expansions above that the scheme is first order accurate in time and second order accurate in space.

c. The characteristic to $u_t + u_x = 0$ corresponds to a velocity of one in the positive x-direction. Given the stencil shape, information can travel with that speed as long as $k \le 2h$, i.e. the CFL condition is $\lambda = \tfrac{k}{h} \le 2$.

d. The standard von Neumann analysis starts by substituting $u(x,t) = \xi^{t/k} e^{i\omega x}$ into the difference scheme, and then simplifying. This gives in the present case

$$\xi = 1 - \lambda\{\tfrac{3}{2} - 2e^{-i\omega h} + \tfrac{1}{2}e^{-2i\omega h}\}$$

where $\lambda = k/h$. Writing $\omega h = s$, we have $\xi = 1 - \lambda f(s)$, with $f(s)$ as defined and illustrated in the hint to the problem. The only possible way that $\xi = 1 - \lambda f(s)$ will not obey the stability requirement $|\xi| \le 1$ for all real s and λ small enough would clearly be some trouble around $s = 0$. Hence, we Taylor expand:

$$f(s) = i\,s + O(s^3)$$

from which follows

$$\xi = 1 - \lambda\, i\, s + O(s^3)$$

and

$$|\xi|^2 = 1 + (\lambda s)^2 + O(s^3).$$

The last equation tells that, no matter how small λ is ($\lambda > 0$), the quantity $|\xi|^2$ will exceed one for some small value of s. Therefore, the scheme is unconditionally unstable.