Solutions for Homework 8 Foundations of Computational Math 2 Spring 2012

Problem 8.1

This is not a programming assignment and you need not turn in any code. This problem considers the used of discrete least squares for approximation by a polynomial. Recall, the distinct points $x_0 < x_1 < \cdots < x_m$ are given and the metric

$$\sum_{i=0}^{m} \omega_i (f(x_i) - p_n(x_i))^2$$

with $\omega_i > 0$ is used to determine the polynomial, $p_n^*(x)$, of degree n that achieves the minimal value. We will assume $\omega_i = 1$ for this exercise. Typically, $m \gg n$. If $m = n$ then the unique interpolating polynomial is the solution.

If we let

$$p_n(x) = \sum_{j=0}^{n} \phi_j(x)\gamma_j$$

then the conditions are

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{pmatrix} - \begin{pmatrix} \phi_0(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \dots & \phi_n(x_1) \\ \vdots & & \vdots \\ \phi_0(x_m) & \dots & \phi_n(x_m) \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$r = (b - Ag)$$

Use the Chebyshev polynomials to form an orthonormal basis, i.e.,

$$\phi_i(x) = \alpha_i T_i(x)$$

and the roots of $T_{m+1}(x)$ as the x_i.

1. Consider the i-th row of A. Show that row i can be determined by solving an $n+1 \times n+1$ system of linear equations. Also show that the matrix that determines this system has structure such that the system can be solved in $O(n)$ comptuations.

2. Use your solution to implement a code that assembles the least squares problem and make sure to exploit the algebraic properties of the matrix A to have an efficient solution.

3. Apply your code to several $f(x)$ choices and use multiple n and m values to explore the accuracy of the approximation. Approximate $\|f - p_n^*\|_\infty$ by sampling the difference between f and the polynomial at a large number of points in the interval and taking the maximum magnitude.

Solution:

The matrix

$$\tilde{A} = \begin{pmatrix} T_0(x_0) & \dots & T_n(x_0) \\ T_0(x_1) & \dots & T_n(x_1) \\ \vdots & & \vdots \\ T_0(x_m) & \dots & T_n(x_m) \end{pmatrix}$$

has all 1's in its first column since $T_0(x) = 1$. Since $T_1(x) = x$ it has as its second column $(x_0, \dots, x_m)^T$ where $T_{m+1}(x_i) = 0$, $0 \le i \le m$. Elements $3 \le k \le n+1$ in row i can then be evaluated via the recurrence

$$T_{k+1}(x_i) = 2x_i T_k(x_i) - T_{k-1}(x_i).$$

Evaluating the linear recurrence is equivalent to solving a linear system defined by a lower triangular matrix with a single nonzero subdiagonal and a nonzero main diagonal. The complexity is clearly $O(n)$ per row.

The columns of the matrix $\tilde{A}$ do not have norm 1. They can be normalized by dividing the first column by $\sqrt{m+1}$ and the remaining columns by $\sqrt{m+1}/2$. These discrete vector norms are known analytically and therefore need not be evaluated.

Once b is computed by evaluating $f(x_i)$ at the roots x_i the least squares problem is solved trivially via

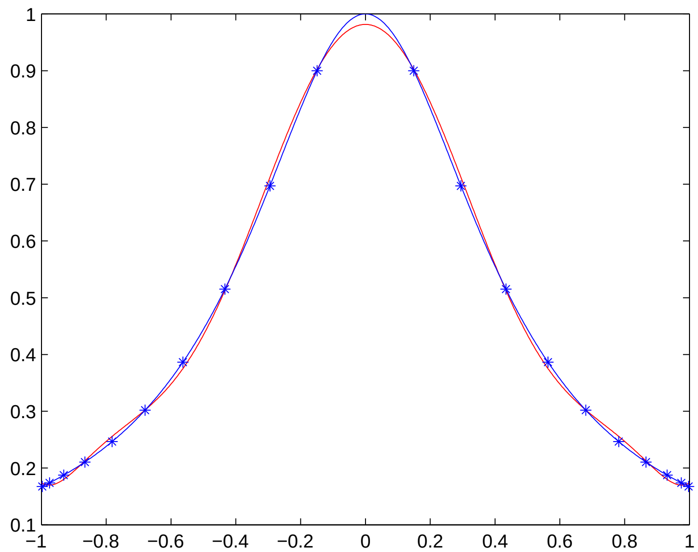
$$g = A^T b$$

As an example consider

$$f(x) = \frac{1}{1+5x^2}$$

with $m = 20$ and $n = 9$. The least squares polynomial $p_9^*(x)$, $f(x)$ and the Chebyshev points are plotted in the figure. The error is

$$\|f - p_9^*\|_\infty \approx 0.018$$



Problem 8.2

Consider the two quadrature formulas

$$I_2(f) = \frac{2}{3}\left[2f(-1/2) - f(0) + 2f(1/2)\right]$$
$$I_4(f) = \frac{1}{4}\left[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)\right]$$

- What is the degree of exactness when $I_2(f)$ is used to approximate $I(f)$ on $[-1, 1]$?
- What is the degree of exactness when $I_2(f)$ is used to approximate $I(f)$ on $[-1/2, 1/2]$?
- What is the degree of exactness when $I_4(f)$ is used to approximate $I(f)$ on $[-1, 1]$?

Solution:

We have

$$I = \int_{-1}^{1} x^k dx = \begin{cases} 0 \text{ if } k \text{ is odd} \\ \frac{2}{k+1} \text{ if } k \text{ is even} \end{cases}$$
$$I_2(f) = \begin{cases} \frac{4}{3}\left[(-1/2)^k - 1 + (1/2)^k\right] = 2 \text{ if } k = 0 \\ \frac{4}{3}\left[(-1/2)^k + (1/2)^k\right] \text{ if } k \geq 1 \end{cases}$$

So clearly $I = I_2(f) = 0$ if $k \geq 1$ is odd. For $k \geq 2$ and even we must have

$$3 \times 2^k = 4(k+1)$$

which is equal for $k=2$ and not equal for $k=4$. Therefore the degree of exactness for $i_2(f)$ on $[-1,1]$ is $p=3$.

We have

$$I = \int_{-1/2}^{1/2} x^k dx = \begin{cases} 0 \text{ if } k \text{ is odd} \\ \frac{1}{(k+1)2^k} \text{ if } k \text{ is even} \end{cases}$$

On the other hand, for $k=0$ we have $I_2=2$ and $I=1$ therefore, $I_2(f)$ is useful as an open formula on $[-1,1]$ with degree of exactness 3.

For $I_4(f)$ on $[-1,1]$ we have

k	I_4	I
0	2	2
1	0	0
2	$\frac{2}{3}$	$\frac{2}{3}$
3	0	0
4	$\frac{4}{27}$	$\frac{2}{5}$

$I_2(f)$ is a closed formula on $[-1,1]$ with degree of exactness 3.

Problem 8.3

Consider the quadrature formula

$$I_0(f) = (b-a)f(a) \approx \int_a^b f(x)dx$$

- What is the degree of exactness?
- What is the order of infinitesimal?

Solution:

Expanding $f(x)$ and integrating we have

$$\begin{aligned} \int_a^b (f(a) + (x-a)f'(\xi(x)))dx &= I_0 + \int_a^b (x-a)f'(\xi(x))dx \\ &= I_0 + f'(\eta)\int_a^b (x-a)dx \quad \text{since } (x-a) \geq 0 \\ \therefore |I - I_0| &= f'(\eta)\frac{(b-a)^2}{2} = O(h^2) \end{aligned}$$

So the degree of exactness is 0 and the order of infinitesimal is 2.

Problem 8.4

Consider

$$\int_a^b \omega(x) f(x) dx \approx \alpha f(x_0)$$

where $\omega(x) = \sqrt{x}$.

Determine α and x_0 such that the degree of exactness is maximized.

Solution:

We asusme $0 \leq a < b$ to keep the weight real. First set $f(x) = x^0 \equiv 1$ and require the quadrature to be exact. Second, set $f(x) = x \equiv 1$ and require the quadrature to be exact. This yields the following

$$\begin{aligned}
f(x) = 1 \longrightarrow \int_a^b x^{1/2} dx &= \frac{2}{3}\left[x^{3/2}\right]_a^b \\
&= \frac{2}{3}\left[b^{3/2} - a^{3/2}\right] = \alpha f(x_0) = \alpha \\
f(x) = x \longrightarrow \int_a^b x^{3/2} dx &= \frac{2}{5}\left[x^{5/2}\right]_a^b \\
&= \frac{2}{5}\left[b^{5/2} - a^{5/2}\right] = \alpha f(x_0) = \alpha x_0 \\
\therefore x_0 &= \frac{3}{5}\left[\frac{(b^{5/2} - a^{5/2})}{(b^{3/2} - a^{3/2})}\right]
\end{aligned}$$

We have no other parameters so the maximum degree of exactness is 1.

Note that if $a = 0$ then $\forall b, \;\; x_0 = (3/5)b$. Also note that

$$0 \leq a \leq \frac{3}{5}\left[\frac{(b^{5/2} - a^{5/2})}{(b^{3/2} - a^{3/2})}\right] \leq b$$

i.e., $a \leq x_0 \leq b$, is easily shown.