APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS

Summer 2011 (June)

(CLOSED BOOK EXAM)

Solve 8 out of 10 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

1	2	3	4	5
6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: June 1st, 2011

Time: 9:00 – 1:00 PM

1. Find the classical solution (smooth solution, no discontinuity of any kind) for the following equation:

$$uu_x + u_t = 0$$

where $u = u(x,t)$, with given initial condition:

$$u(x,0) = h(x)$$

where $h(x)$ is a given smooth solution.

Draw the lines of characteristics for the above equation with the following two different initial conditions:

- $u(x,0) = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$
- $u(x,0) = \begin{cases} 2, & x \leq 0 \\ 1, & x > 0 \end{cases}$

2. Consider the fractional linear transformation

$$T(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}.$$

a) What requirements must be imposed on coefficients to ensure $T(z)$ is a conformal map?

b) How many fixed points can $T(z)$ have? Give geometrical interpretation.

c) How does $T(z)$ map circles and straight lines.

3. Evaluate the integral

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 2i} dx$$

using the residue theory.

4. Let $f(z)$ be analytic on A and let $f'(z_0) \neq 0$, $z_0 \in A$. Show that if γ is a sufficiently small circle centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{dz}{f(z) - f(z_0)}.$$

Hint: Use the Inverse Function Theorem.

5. Consider a vector field $\boldsymbol{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$. The divergence is $\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$.

a) Construct a second-order finite difference approximation to the divergence $\nabla \cdot \boldsymbol{f}$ for the interior points of a rectangular grid over a rectangular domain. Give a justification why the approximation is second order.

b) Derive a second-order accurate finite difference approximations along the boundary of the rectangular grid.

6. Heun's method.

a) Derive Heun's method (i.e., explicit two-stage Runge-Kutta method) for solving scalar initial value problem $y' = f(t, y)$ with $y(0) = y_o$.

b) Show that the method is second order accurate.

c) Consider the special case $f(t, y) = \lambda y$ with $\lambda \in \mathbb{R}$. Find the maximum h such that the method is stable. Note that h may depend on λ.

7. Consider the boundary value problem

$$u'' = f(t), \quad 0 < t < 1,$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 1.$$

Approximate its solution by $u(t) \approx \sum_{i=1}^{n} x_i \phi_i(t)$, where the $\phi_i(t)$ denote the basis functions.

a) Show how to use Galerkin's method to derive its weak form. What conditions does $\phi_j(t)$ need to satisfy for the weak form to be valid?

b) Suppose the hat function is used as the shape function (i.e., piecewise linear shape function), and the domain is decomposed into subintervals of length $h = 0.2$. Compute the element stiffness matrix $\boldsymbol{K}$.

8. For the following system of partial differential equations,

$$
\begin{aligned}
u_t + a v_x &= 0 \\
v_t + b u_x &= 0
\end{aligned}
$$

where a and b are constants.

(a). Under what condition the system is hyperbolic?

(b). If the system is hyperbolic (assume this for all questions below), design an upwind scheme to solve this system of equations.

(c). Find the stability condition for the upwind scheme in (b).

(d). Find the Riemann solution of the system if the initial conditions for u and v are given as

$$
\{u(x,0),\ v(x,0)\} = \begin{cases} \{u_l,\ v_l\} & \text{if } x < 0 \\ \{u_r,\ v_r\} & \text{if } x > 0 \end{cases}
$$

9. A generalized numerical scheme for the linear hyperbolic equation $u_t + au_x = 0$ has the following form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \chi\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0$$

(a). Find the specific value of χ so that the above scheme is the (i) central-explicit, (ii) forward, (iii) backward, (iv) Lax-Friedrich, and (v) Lax-Wendroff scheme.

(b). Analyze the consistency of the generalized scheme. Under what condition the scheme is second order in Δt and Δx.

(c). Find the stability condition for the generalized scheme.

(d). In what range of χ the scheme is unconditionally unstable?

10. The central difference scheme for Poisson's equation $\Delta u = f$ in two dimension can be written in a matrix form $AU = b$.

(a). Describe A, U and b in terms of grid indices i and j.

(b). An iterative method is to let $A = B - C$ and then iterate the equation through $BU^{k+1} = CU^k + b$. Find B and C for Jacobi and Gauss-Seidel method.

(c). Under what condition the iteration in (b) will converge. Show details of your proof.

(d). Show that the Gauss-Seidel method converges twice as fast as the Jacobi method.