Set 2: Polynomial Interpolation – Part 2

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Overview

- Complexity measured in terms of number of computations, i.e., sequential computation
- Evaluation of a polynomial in mononmial form
- Interpolation polynomials
- For each form consider:
  - Complexity of constructing the polynomial, i.e., the parameters
  - Complexity of evaluating the polynomial at a point $x \neq x_i$
  - Complexity of updating the polynomial to include a new point

Horner's Rule

Assume the polynomial, $p_n(x)$, is given in terms of monomials, x^i

$$p_n(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$$

For example, let $n = 4$

$$p_4(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$$

$$p_4(x) = \alpha_0 + x(\alpha_1 + x(\alpha_2 + x(\alpha_3 + (x\alpha_4))))$$

Repeated application leads to evaluation of derivatives at x.

Adaptable to other similar forms, e.g., Newton form.

Lagrange Form

- Storing only $x_i, \ y_i$, i.e., no preprocessing, yields an $O(n^2)$ to evaluate $p_n(x)$ via repeated linear interpolation, e.g., Aitken's method.
- Using the basic Lagrange form defined earlier yields
  - $O(n^2)$ to compute the $n + 1$ coefficients that define $p_n(x)$
  - $O(n^2)$ to evaluate $p_n(x)$
- Rewriting yields improvements
  - Barycentric form 1
  - Barycentric form 2

Lagrange Form

Recall, the standard form is sum of n-degree polynomials

$$m_i^{(n)}(x) = \prod_{j=0, j\neq i}^{n} (x - x_j)$$

$$\ell_i^{(n)}(x) = \frac{m_i^{(n)}(x)}{m_i^{(n)}(x_i)}$$

$$p_n(x) = \sum_{i=0}^{n} y_i \, \ell_i^{(n)}(x)$$

Lagrange Form

Lemma. *Given* (x_i, y_i) *for* $0 \le i \le n$ *and defining*

$$m_i^{(n)}(x) = \prod_{j=0, j\neq i}^{n} (x - x_j) \quad \textit{and} \quad \omega_j(x) = \prod_{i=0}^{j-1} (x - x_i)$$

we have

$$m_i^{(n)}(x) = \frac{\omega_{n+1}(x)}{(x - x_i)} \quad \textit{and} \quad m_i^{(n)}(x_i) = \omega'_{n+1}(x_i)$$

and the Lagrange characteristic functions can be written

$$\ell_i(x) = \frac{\omega_{n+1}(x)}{(x - x_i)\omega\prime_{n+1}(x_i)}.$$

Lagrange Form

Proof. Let $i = n$. We have

$$\begin{aligned}
\omega_{n+1}(x) &= \omega_n(x)(x - x_n) \\
&\rightarrow \omega'_{n+1}(x) = \omega'_n(x)(x - x_n) + \omega_n(x) \\
&\rightarrow \omega'_{n+1}(x_n) = \omega_n(x_n)
\end{aligned}$$

$$\therefore \ell_n(x) = \frac{\omega_n(x)}{\omega_n(x_n)} = \frac{\omega_{n+1}(x)}{(x - x_n)\omega'_{n+1}(x_n)}.$$

This adapts trivially to $\ell_i(x)$ for $i \neq n$. $\square$

Barycentric Form 1

Definition 2.1. (Berrut and Trefethen, Siam Review Vol. 46 No. 3)

Given (x_i, y_i) for $0 \leq i \leq n$, the Barycentric interpolation formula form 1 is

$$p_n(x) = \omega_{n+1}(x) \sum_{i=0}^{n} \frac{y_i}{(x - x_i)\omega'_{n+1}(x_i)} = \omega_{n+1}(x) \sum_{i=0}^{n} y_i \frac{\gamma_i}{(x - x_i)}$$

where

$$\gamma_i = 1/\omega'_{n+1}(x_i) \quad \text{and} \quad \omega_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$$

Barycentric Form 1

- Construction of $p_n(x)$ requires the computation of γ_i, for $0 \leq i \leq n$ where

$$\gamma_i^{-1} = \omega'_{n+1}(x_i) = m_i^{(n)}(x_i) = \prod_{j=0, j\neq i}^{n} (x_i - x_j)$$

- Construction of $p_n(x)$ **does not depend on** y_i.
- Construction of $p_n(x)$ can be done in $O(n^2)$ computations.
- Structure can be exploited to keep the constant small.

Lagrange Complexity

Let $n = 4$

$$m_0^{(4)} = \left[(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)\right]_R$$

$$m_1^{(4)} = \left[(x_1 - x_0)\right]_L \left[(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)\right]_R$$

$$m_2^{(4)} = \left[(x_2 - x_0)(x_2 - x_1)\right]_L \left[(x_2 - x_3)(x_2 - x_4)\right]_R$$

$$m_3^{(4)} = \left[(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)\right]_L \left[(x_3 - x_4)\right]_R$$

$$m_4^{(4)} = \left[(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)\right]_L$$

Note. The terms in $[--]_L$ appear earlier with sign changes in $[--]_R$ terms.

Lagrange Complexity

Let $n = 4$ and update to $n = 5$

$$m_0^{(5)} = [(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)]\,(x_0 - x_5)$$

$$m_1^{(5)} = [(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)]\,(x_1 - x_5)$$

$$m_2^{(5)} = [(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)]\,(x_2 - x_5)$$

$$m_3^{(5)} = [(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)]\,(x_3 - x_5)$$

$$m_4^{(5)} = [(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)]\,(x_4 - x_5)$$

$$m_5^{(5)} = [(x_5 - x_0)(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)]$$

- The terms in brackets in the $m_i^{(5)}$, $0 \le i \le 4$ are the $m_i^{(4)}$ (known).
- The terms in $m_5^{(5)}$ are the terms applied to the earlier $m_i^{(4)}$ to get $m_i^{(5)}$ with sign changes.

Lagrange Complexity

We have $O(n)$ computations to update the parameters for $p_n(x)$ to those for $p_{n+1}(x)$ via the algorithm:

$p = 1$
for $i = 0, \ldots, n-1$

$$t = (x_i - x_n)$$
$$m_i^{(n)} = t \times m_i^{(n-1)}$$
$$p = -t \times p$$

end
$m_n^{(n)} = p$

Barycentric Form 1

- Construction of $p_n(x)$ is $O(n^2)$: start with $m_0^{(1)} = x_0 - x_1$ and $m_1^{(1)} = x_1 - x_0$ and apply incremental algorithm.
- Construction of $p_n(x)$ **does not depend on** y_i.
- Update of $p_n(x)$ to $p_{n+1}(x)$ is $O(n)$
- Evaluation of $p_n(x)$ requires
  - $O(n)$ computations to evaluate $\omega_{n+1}(x)$
  - $O(n)$ computations to evaluate $\sum_{i=0}^{n} y_i(\gamma_i/(x - x_i))$

Barycentric Form 2

Lemma. *If*

$$\ell_i(x) = \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)}$$

then

$$\sum_{i=0}^{n} \ell_i(x) = \omega_{n+1}(x) \sum_{i=0}^{n} \frac{\gamma_i}{(x - x_i)} = 1$$

$$\omega_{n+1}(x) = \left[\sum_{i=0}^{n} \frac{\gamma_i}{(x - x_i)} \right]^{-1}$$

Barycentric Form 2

Definition 2.2. (Berrut and Trefethen, Siam Review Vol. 46 No. 3)

Given (x_i, y_i) for $0 \leq i \leq n$ and defining

$$\omega_j(x) = \prod_{i=0}^{j-1}(x - x_i)$$

the Barycentric interpolation formula form 2 or the true form is

$$p_n(x) = \{\sum_{i=0}^{n} y_i \frac{\gamma_i}{(x - x_i)}\} / \{\sum_{i=0}^{n} \frac{\gamma_i}{(x - x_i)}\}$$

where $\gamma_i = 1/\omega'_{n+1}(x_i)$.

Barycentric Form 2

- The term 'barycentric form' in the literature usually refers to Barycentric Form 2.
- Construction of $p_n(x)$ **does not depend on** y_i.
- Update of $p_n(x)$ to $p_{n+1}(x)$ is $O(n)$.
- Evaluation of $p_n(x)$ requires $O(n)$ computations.
- Main advantage: common factors in the γ_i can be cancelled!
- γ_i replaced by β_i which are often simpler and better scaled (although the scale may still be bad)
- Construction of $p_n(x)$ **may be** $O(n)$.

Barycentric Form 2

Consider the interval $[a, b]$ with equally spaced interpolation points $x_i = a + ih$ where $h = (b - a)/n$.

$$\gamma_i^{-1} = \left[\prod_{j=0}^{i-1}(x_i - x_j)\right]\left[\prod_{j=i+1}^{n}(x_i - x_j)\right]$$

$$= \left[\prod_{j=0}^{i-1}(a + ih - a - jh)\right]\left[\prod_{j=i+1}^{n}(a + ih - a - jh)\right]$$

$$\left[h^i\prod_{j=0}^{i-1}(i - j)\right]\left[h^{n-i}\prod_{j=i+1}^{n}(i - j)\right] = h^n\left[\prod_{j=1}^{i}(j)\right]\left[(-1)^{n-i}\prod_{j=1}^{n-i}(j)\right]$$

$$= (-1)^{n-i}h^n(i!)(n - i)!$$

Barycentric Form 2

$$\gamma_i = \frac{(-1)^{n-i}}{h^n (i!)(n-i)!} = \frac{(-1)^{n-i}}{h^n n!} \binom{n}{i}$$

$$= \frac{(-1)^n}{h^n n!} \left[(-1)^i \binom{n}{i} \right] = \frac{(-1)^n}{h^n n!} \beta_i$$

$$\beta_{i+1} = -\beta_i \frac{n-i}{i+1}$$

Barycentric Form

- γ_i can be replaced by β_i in the barycentric formula via cancellation of common factor.
- β_i produced recursively implies construction of $p_n(x)$ requires $O(n)$ computations.
- Note the large range in magnitude of γ_i and β_i for equally spaced nodes (approximately 2^n)
- Reflects ill-conditioning of interpolation on equally spaced nodes
- on $[-1, 1]$ nonuniform points clustered at the endpoints with a density proportional to $1/\sqrt{1-x^2}$ as $n \to \infty$ are required to make the weights comparable in scale, i.e., they do not vary by an exponential in n. The Chebyshev points are a good example.

Barycentric Form and Chebyshev Points

Lemma. *If the nodes for interpolation are Chebyshev points of the first kind given by*

$$x_j = \cos\frac{(2j+1)\pi}{2n+2} \quad 0 \le j \le n$$

then the barycentric coefficients are

$$\beta_j = (-1)^j \sin\frac{(2j+1)\pi}{2n+2} \quad 0 \le j \le n$$

Barycentric Form and Chebyshev Points

Lemma. *If the nodes for interpolation are Chebyshev points of the second kind given by*

$$x_j = \cos \frac{j\pi}{n} \quad 0 \le j \le n$$

then the barycentric coefficients are

$$\beta_j = (-1)^j \delta_j \quad \delta_j = \begin{cases} 1/2 & \textit{if } j = 0 \textit{ or } j = n \\ 1 & \textit{otherwise} \end{cases}$$

Barycentric Form and Chebyshev Points

- Uniform and Chebyshev points of first, second, third, and fourth kind all have simple formulae for barycentric weights.
- Construction of $p_n(x)$ requires $O(n)$ computations in all of these cases.
- Construction of $p_n(x)$ **does not depend on** y_i therefore reusable for other functions.
- Update and evaluation of $p_n(x)$ requires $O(n)$ computations in all of these cases.
- Higham (IMA Journal of Numerical Analysis, Volume 24, 2004) has shown the very satisfactory stability properties of the barycentric form.
- Becoming popular for numerical methods for solving PDEs

Barycentric Form Example

A simple MATLAB code (From Berrut and Trefethen, Siam Review Vol. 46 No. 3) using Chebyshev points of the second kind applied to the problem

$$f(x) = |x| + \frac{1}{2}x - x^2$$

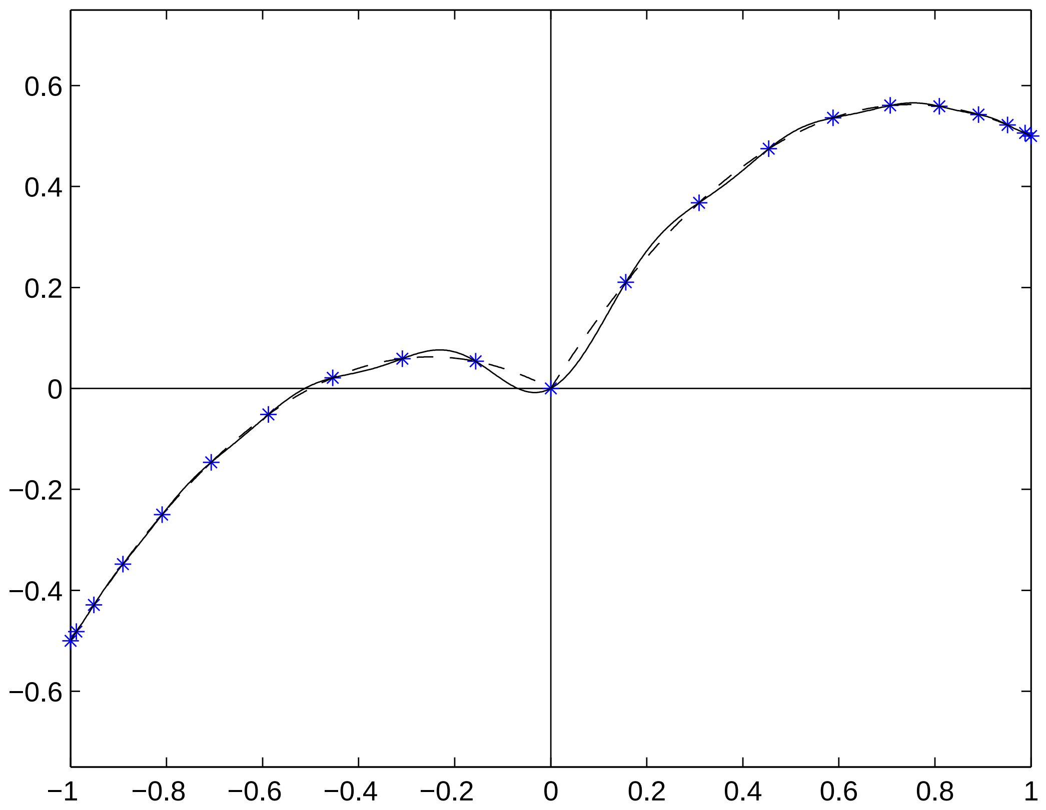
$$-1 \leq x \leq 1$$

$$x_k = \cos(k\pi/n), \quad 0 \leq k \leq n$$

- $n = 20$
- increasing n improves approximation

```
fun = @(z)  abs(z)+0.5*z-z.^2;
n=20;
% Chebyshev of the second kind Barycentric coefficients
c= [0.5; ones(n-1,1); 0.5].*(-1).^((0:n)');
% evaluate  f at the Chebyshev points
x = cos(pi*(0:n)'/n);   xx=linspace(-1,1,1000)';
f = fun(x);    ft=fun(xx);
numer=zeros(size(xx));    denom=zeros(size(xx));
for j=1:n+1
   xdiff=xx-x(j);
   temp=c(j)./xdiff;
   numer = numer+ temp*f(j);
   denom = denom+ temp;
end
ff=numer./denom;
plot(x,f,'*')  % interpolation points
plot(xx,ff,'-k') % interpolating polynomial
plot(xx,ft,'--k') % f(x)
```

Barycentric Example



$f(x)$: dotted line, $p(x)$: solid line, $*$: interpolation points.

Newton Complexity

With $\omega_{i+1}(x) = (x - x_0) \cdots (x - x_i)$, we have

$$p_n(x) = \sum_{i=0}^{n} y[x_0, \ldots, x_i]\omega_i(x)$$

$$D^k y_i = y[x_i, \ldots, x_{i+k}] = \frac{y[x_{i+1}, \ldots, x_{i+k}] - y[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

- Divided difference table requires $\frac{n(n+1)}{2}$ values $D^k y_i \;\; 1 \le k \le n, \;\; 0 \le i \le n - k$
- $O(n^2)$ computations to construct the table.
- adding (x_{n+1}, y_{n+1}) adds a column of $n + 1$ entries and requires $O(n)$ computations

Newton Complexity

- Evaluation of $p_n(x) = \sum_{i=0}^{n} y[x_0, \ldots, x_i]\omega_i(x)$ can be done in a nested fashion in $O(n)$.
- $p_n(x)$ can be specified in terms of multiple "paths" through the divided difference table, i.e., in terms of different sets of divided differences.
- order of points in divided difference does not matter

$$y[x_0, x_1] = \frac{(y_1 - y_0)}{(x_1 - x_0)} = \frac{-(y_1 - y_0)}{-(x_1 - x_0)} = \frac{(y_0 - y_1)}{(x_0 - x_1)} = y[x_1, x_0]$$

$$y[x_0, x_1, x_2] = y[x_1, x_0, x_2] = y[x_2, x_0, x_1]$$

Newton Form

i	0		1		2		3
x_i	0		1		3		4
y_i	-5		1		25^*		55
$y[-,-]$		6		12^*		30	
$y[-,-,-]$			2		6^*		
$y[-,-,-,-]$				1^*			

Newton Form

Create the quadratic polynomial $p_2(x)$ that interpolates (x_0, f_0),(x_1, f_1), and (x_2, f_2) using information from the left side of table.

$$
\begin{aligned}
p_2(x) &= y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2] \\
&= -5 + 6x + 2x(x - 1) \\
&= -5 + 4x + 2x^2
\end{aligned}
$$

Other forms possible – all equivalent by uniqueness.

Newton Form

Create the cubic polynomial $p_3(x)$ that interpolates (x_i, f_i), $0 \leq i \leq 3$.

Use information on left side of table:

$$
\begin{aligned}
p_3(x) = y_0 &+ (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2] \\
&+ (x - x_0)(x - x_1)(x - x_2)y[x_0, x_1, x_2, x_3] \\
&= -5 + 6x + 2x(x - 1) + x(x - 1)(x - 3) \\
&= -5 + 7x - 2x^2 + x^3
\end{aligned}
$$

Newton Form

Create the cubic polynomial $r_3(x)$ that interpolates (x_i, f_i), $0 \leq i \leq 3$.

Use information on right side of table:

$$
\begin{aligned}
r_3(x) &= y_3 + (x - x_3)y[x_3, x_2] + (x - x_3)(x - x_2)y[x_3, x_2, x_1] \\
&\quad + (x - x_3)(x - x_2)(x - x_1)y[x_3, x_2, x_1, x_0] \\
&= y_3 + (x - x_3)y[x_2, x_3] + (x - x_3)(x - x_2)y[x_1, x_2, x_3] \\
&\quad + (x - x_3)(x - x_2)(x - x_1)y[x_0, x_1, x_2, x_3] \\
&= 55 + 30(x - 4) + 6(x - 4)(x - 3) + 1(x - 4)(x - 3)(x - 1) \\
&= -5 + 7x - 2x^2 + x^3
\end{aligned}
$$

Newton Form

Create the cubic polynomial $q_3(x)$ that interpolates (x_i, f_i), $0 \le i \le 3$.

Use information marked by $*$ in table:

$$
\begin{aligned}
q_3(x) &= y_2 + (x - x_2)y[x_2, x_1] + (x - x_2)(x - x_1)y[x_2, x_1, x_3] \\
&+ (x - x_2)(x - x_1)(x - x_3)y[x_2, x_1, x_3, x_0] \\
&= y_2 + (x - x_2)y[x_1, x_2] + (x - x_2)(x - x_1)y[x_1, x_2, x_3] \\
&+ (x - x_2)(x - x_1)(x - x_3)y[x_0, x_1, x_2, x_3] \\
&= 25 + 12(x - 3) + 6(x - 3)(x - 1) + (x - 3)(x - 1)(x - 4) \\
&= -5 + 7x - 2x^2 + x^3
\end{aligned}
$$

Newton and Barycentric Lagrange Complexity

- Construction of $p_n(x)$ is $O(n^2)$ for Newton and Lagrange.
- Construction of $p_n(x)$ is $O(n)$ for some Lagrange.
- Construction of $p_n(x)$ for Newton **depends on** y_i.
- Construction of $p_n(x)$ for Lagrange **does not depend on** y_i.
- Lagrange parameters can be reused for other y_i values.
- Update of $p_n(x)$ to $p_{n+1}(x)$ is $O(n)$ for both.
- Evaluation of $p_n(x)$ is $O(n)$ for both.
- Stability of Lagrange independent of node order.
- Stability of Newton depends on node order.