4.5: Summary of Curve Sketching

You've learned about domain, range, and symmetry of functions previously; limits, continuity, and asymptotes in Chapter 2; derivatives and tangents in Chapters 2 and 3; and extreme values, intervals of increase and decrease, concavity, points of inflection, and l'Hôspitals Rule in this chapter. Now, we put all of this information together to sketch graphs that reveal the important features of functions.

Guidelines for Sketching a Curve:

A. **Domain** Determine the domain of f, that is, the values of x for which $f(x)$ is defined.

B. **Intercepts** $f(0)$ is the y-intercept of the graph, the value where the graph intersects the y-axis. The values of x that satisfy $f(x) = 0$ are the x-intercepts of the graph, the values where the graph intersects the x-axis.

C. **Symmetry**

   (i) If $f(-x) = f(x)$ for all x in the domain, then f is an **even** function and the graph of f is symmetric about the y-axis. If we can determine the graph for $x \geq 0$, we can reflect about the y-axis to obtain the full graph.

   (ii) If $f(-x) = -f(x)$ for all x in the domain, then f is an **odd** function and the graph of f is symmetric about the origin. If we can determine the graph for $x \geq 0$, we can reflect about the y- and x-axes to obtain the full graph.

   (iii) If $f(x + p) = f(x)$ for all x in the domain, where p is a positive constant, then f is a **periodic** function and the smallest such number p is the period. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph.

D. **Asymptotes**

   (i) *Horizontal Asymptotes.* If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of the graph of f.

   (ii) *Vertical Asymptotes.* If $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then $x = a$ is a vertical asymptote of the graph of f.

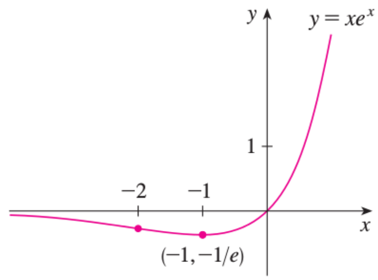
   (iii) *Slant Asymptotes.* See below.

E. **Intervals of Increase or Decrease** The graph of f is increasing on the intervals such that $f'(x) > 0$, and the graph of f is decreasing on the intervals such that $f'(x) < 0$.

F. **Local Minimum and Maximum Values** The graph of f may have local minima or maxima at the crititcal numbers of f, the values of c such that $f'(c) = 0$ or $f'(c)$ does not exist. If f' switches from negative to positive at a critical number c then $f(c)$ is local minimum of f, and if f' switches from positive to negative at a critical number c then $f(c)$ is a local maximum of f (The First Derivative Test). If $f'(c) = 0$ and $f''(c) > 0$ then $f(c)$ is a local minimum of f, otherwise if $f'(c) = 0$ and $f''(c) < 0$ then $f(c)$ is a local maximum of f (The Second Derivative Test).

G. **Concavity and Inflection Points** The graph of f is concave up on the intervals such that $f''(x) > 0$, and the graph of f is concave down on the intervals such that $f''(x) < 0$. If f is continous at c and f'' switches sign at c, then $(c, f(c))$ is an inflection point.

Example 1. *Use the above guidelines to sketch the graph of $f(x) = xe^x$.*

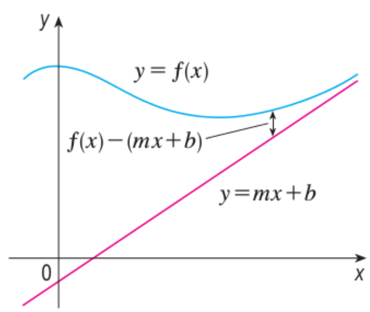


Slant Asymptotes

Some graphs have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and $y = mx + b$ approaches 0 as x approaches ∞. For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case, the equation of the asymptote can be found by long division.



Example 2. *Sketch the graph of $f(x) = \dfrac{x^3}{x^2+1}$.*