Homework 6 Foundations of Computational Math 1 Fall 2012

Problem 6.1

Suppose you are attempting to solve $Ax = b$ using a linear stationary iterative method defined by

$$x_k = Gx_{k-1} + f$$

that is consistent with $Ax = b$.

Suppose the eigenvalues of G are real and such that $|\lambda_1| > 1$ and $|\lambda_i| < 1$ for $2 \leq i \leq n$. Also suppose that G has n linearly independent eigenvectors, z_i, $1 \leq i \leq n$.

6.1.a. Show that there exists an initial condition x_0 such that x_k converges to $x = A^{-1}b$.

6.1.b. Does your answer give a characterization of selecting x_0 that could be used in practice to create an algorithm that would ensure convergence?

Problem 6.2

Suppose you are attempting to solve $Ax = b$ using a linear stationary iterative method defined by

$$x_k = M^{-1}Nx_{k-1} + M^{-1}b$$

where $A = M - N$. Suppose further that $M = D + F$ where $D = diag(\alpha_{11}, \ldots, \alpha_{nn})$ and F is made up of any subset of the off-diagonal elements of A. The matrix N is therefore the remaining off-diagonal elements of A after removing those in F.

Show that if A is a strictly diagonally dominant M-matrix then the iteration is convergent to $x = A^{-1}b$.

Problem 6.3

6.3.a. Textbook page 241, Problem 2

6.3.b. Textbook page 241, Problem 4

6.3.c. Textbook page 241, Problem 5

Material in textbook Sections 1.7 and 5.1 is useful for these problems.

Problem 6.4

Consider the $n \times n$ matrix

$$T_\alpha = \begin{pmatrix} \alpha & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & \alpha & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & \alpha & -1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & -1 & \alpha & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & \alpha & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & \alpha \end{pmatrix}$$

(6.4.a) Show that the eigenvalues of T_α are

$$\lambda_j = \alpha - 2\cos j\theta, \qquad \theta = \frac{\pi}{n+1}$$

with an associated eigenvector

$$q_j = \big(\sin(j\theta),\ \ \sin(2j\theta), \dots, \sin(nj\theta)\big)^T$$

(6.4.b) For what values of α is T_α positive definite?

(6.4.c) Show that for $\alpha = 2$ the matrix T_α is an M-matrix.

(6.4.d) What is the rate of convergence for Jacobi's method if $\alpha = 2$?

(6.4.e) What is the rate of convergence for Gauss-Seidel if $\alpha = 2$?

Problem 6.5

The Gershgorin theorems in Section 5.1 of the text and the idea of irreducibility in Section 5.1 are often valuable in analyzing the convergence of iterative methods. Familiarize yourself with both in order to answer this question.

6.5.a. Use the appropriate Gershgorin-related facts to show that if A is symmetric and strictly diagonally dominant then Jacobi converges for all x_0.

6.5.b. Use the appropriate Gershgorin-related facts to show that Jacobi converges for all x_0 for the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Problem 6.6

Consider the block tridiagonal matrix associated with an $n \times n$ grid discretization of the partial differential $u_{\xi,\xi} + u_{\eta,\eta} = g$ on a two-dimensional domain.

The matrix is $n^2 \times n^2$ with $n \times n$ blocks $T_i \in \mathbb{R}^{n \times n}$ $1 \le i \le n$ $E_i = -I_n \in \mathbb{R}^{n \times n}$ $1 \le i \le n$ with block tridiagonal structure given by

$$A = \begin{pmatrix} T_1 & E_1 & 0 & \cdots & \cdots & \cdots & 0 \\ E_2 & T_2 & E_2 & 0 & & & \vdots \\ 0 & E_3 & T_3 & E_3 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & E_{n-1} & T_{n-2} & E_{n-2} & 0 \\ 0 & & \cdots & 0 & E_{n-1} & T_{n-1} & E_{n-1} \\ 0 & & & \cdots & 0 & E_n & T_n \end{pmatrix}$$

where T_i are tridiagonal and E_i are diagonal and dimensions $n \times n$

$$T_i = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & -1 & 4 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 4 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$

$$E_i = -I_n = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 0 & -1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & -1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

6.6.a. Determine the computational complexity of one step of Jacobi for a linear system involving the matrix.

6.6.b. Determine the computational complexity of one step of Gauss-Seidel for a linear system involving the matrix.

Problem 6.7

6.7.a

Consider the two matrices:

$$A_1 = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad and \quad A_2 = \begin{pmatrix} 1 & -\frac{1}{12} \\ -\frac{3}{4} & 1 \end{pmatrix}$$

Suppose you solve systems of linear equations involving A_1 and A_2 using Jacobi's method. For which matrix would you expect faster convergence?

6.7.b

Consider the matrix

$$A = \begin{pmatrix} 4 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix}$$

(i) Will Jacobi's method converge when solving $Ax = b$?

(ii) Will Gauss-Seidel converge when solving $Ax = b$?