Preliminary Exam

August 20, 2002

Do **FOUR** of the following six problems **ONLY**! Show all relevant work!

1. Consider the boundary value problem

$$\begin{aligned} u''(x) + a(x)u'(x) + b(x)u(x) &= f(x) \quad , \quad 0 < x < 1 \\ u(0) &= \alpha \\ u(1) &= \beta \end{aligned}$$

   (a) Use a centered finite difference approximation for the derivatives to write down a system of N finite difference equations corresponding to the problem. Explicitly write the matrix and vectors.

   (b) In a special case, we are led to the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

      i. What does the fact that A is symmetric tell you about the eigenvalues of A?

      ii. Locate the interval in which the eigenvalues of A lie using Gerschgorin's theorem.

      iii. Determine whether A is singular or not.

2. (a) Suppose R^N is equipped with a norm $\| \cdot \|$ and let A be a $N \times N$ non-singular matrix. Define the condition number of A for solving a linear system of equations and the one for determining eigenvalues.

   (b) Show that if u is the solution of $Au = b$ and $u + \delta u$ solves $A(u + \delta u) = b + \delta b$, then

$$\frac{\| \delta u \|}{\| u \|} \leq \text{cond}(A) \frac{\| \delta b \|}{\| b \|}$$

   Also, show that if we perturb the coefficient matrix A, instead of b , then

$$\frac{\| \delta u \|}{\| u + \delta u \|} \leq \text{cond}(A) \frac{\| \delta A \|}{\| A \|}$$

(c) Suppose $N = 2$ and $\|\cdot\|$ is the Euclidean (l_2) norm. Use this information to find the corresponding condition number for the matrix

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}.$$

3. (a) Write down the formula for Newton's iteration in the case of finding a root to the scalar equation $f(x) = 0$ and, also, in the case of a *system* of nonlinear equations.

(b) Write down the formula for the secant method for a scalar equation.

(c) Show that the secant error, to leading order, decays like

$$\varepsilon_{n+1} = \varepsilon_n \cdot \varepsilon_{n-1} \frac{f''(\alpha)}{2\, f'(\alpha)},$$

where α is the root, and $\varepsilon_n = x_n - \alpha$.

(d) The formula above can be shown to imply that the error converges approximately like

$$\varepsilon_{n+1} = c \cdot \varepsilon_n^d.$$

Determine c and d.

(No detailed rigor is required for parts (c) and (d); plausible arguments suffice, as long as they convincingly arrive at the required forms).

4. A *cubic* B-spline, with node points at the integers, takes the values $\{0, \frac{1}{6}, \frac{2}{3}, \frac{1}{6}, 0\}$ at five adjacent nodes, i.e. its support extends over four subintervals.

(a) Define what is meant by a B-spline (of arbitrary order).

(b) Determine the node values and number of subintervals for a *quadratic* spline (recalling that the standard normalization is that $\int_{-\infty}^{\infty} B(x)\, dx = 1$).

(c) To be uniquely determined, a *cubic* spline needs two extra conditions beyond the function values at the nodes. Determine how many (if any) extra conditions a *quadratic* spline requires.

(d) With cardinal data (one at one node point, say at the origin, and zero at the others), a *cubic* spline on the infinite interval will be oscillatory and decay as we move away from the center. Show that the rate of decay is approximately $c \cdot (2 - \sqrt{3})^k \approx c \cdot 0.27^k$ where k is the distance (number of nodes) away from the origin.

Hint: Given that the B-spline node values are $\{0, \frac{1}{6}, \frac{2}{3}, \frac{1}{6}, 0\}$, the data values y_k and B-spline expansion coefficients b_k become related by $\frac{1}{6}b_{k+1} + \frac{2}{3}b_k + \frac{1}{6}b_{k-1} = y_k$.

5. Consider the backward differentiation formula,

$$y_{n+2} - \frac{4}{3} y_{n+1} + \frac{1}{3} y_n = \frac{2}{3}\, h\, f(t_{n+2}, y_{n+2})\,.$$

(a) Determine the order of this method.

(b) Define what is meant by a region of absolute stability, and provide an equation which describes this region in the case of the method above.

(c) Show that the whole negative real axis is in the region of absolute stability. Extra credit is given for a proof that the method is A-stable.

6. (a) Determine the order of Störmer's method,

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f(t_{n+1}, y_{n+1})\,, \quad n \geq 0\,,$$

for solving the second order system of ODE's

$$y'' = f(t, y) \quad , t \geq 0\,,$$

with the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$.

(b) Using the second order cenral differences in space and Störmer's method in time, construct a scheme to solve the wave equation,

$$u_{tt} = u_{xx}\,.$$

(c) Determine the condition for its stability.