MAC2311: Calculus 1 - Section 1

Test 3 March 19, 2015

Name: ____

Answer each question in the space provided on the question sheets. If you run out of space for an answer, continue on the back of the page. Credit will only be given if you clearly show all of your work. Calculators may be used for this test.

Question	**Points**	**Score**
1	2	
2	3	
3	5	
4	7	
5	8	
6	5	
7	6	
8	2	
9	9	
10	9	
Total:	56	

1. [2 points] Find the differential dy if $y = \ln(\sin x)$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sin x} \cdot \frac{d}{dx} \sin x \\
\frac{dy}{dx} &= \frac{\cos x}{\sin x} = \cot x \\
dy &= \cot x \; dx
\end{aligned}$$

2. [3 points] Find the critical numbers of the function $f(x) = x^{3/7}(x-5)$.

$$\begin{aligned}
f'(x) &= x^{3/7} + \frac{3}{7}x^{-4/7}(x-5) \\
&= x^{3/7}\frac{7x^{4/7}}{7x^{4/7}} + \frac{3(x-5)}{7x^{4/7}} \\
&= \frac{7x + 3(x-5)}{7x^{4/7}} \\
&= \frac{5(2x-3)}{7x^{4/7}}
\end{aligned}$$

does not exist at $x = 0$ and is 0 when $2x - 3 = 0 \Rightarrow x = 3/2$. Therefore, $x = 0$ and $x = 3/2$ are critical numbers of f.

3. [5 points] Find the absolute minimum and absolute maximum values of $f(x) = x^3 - 3x$ on the interval $[0, 2]$.

$$\begin{aligned} f'(x) &= 3x^2 - 3 = 0 \\ &= x^2 - 1 = 0 \\ &= (x+1)(x-1) = 0 \Rightarrow x = \pm 1 \end{aligned}$$

So, $x = \pm 1$ are critical numbers of f but only $x = 1$ is in the interval $[0, 2]$. Comparing the value of f at the critical numbers and the endpoints of the interval

$$f(0) = 0 \qquad f(1) = -2 \qquad f(2) = 2$$

the absolute minimum is $f(1) = -2$ and the absolute maximum is $f(2) = 2$.

4. (a) [5 points] Find the linearization $L(x)$ of the function $f(x) = \sqrt[9]{x-4}$ at $a = 5$.

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt[9]{a-4} + \frac{1}{9(a-4)^{8/9}}(x-a) \\ &= \sqrt[9]{5-4} + \frac{1}{9(5-4)^{8/9}}(x-5) \\ &= 1 + \frac{1}{9}(x-5) \\ &= \frac{4}{9} + \frac{x}{9} \end{aligned}$$

(b) [2 points] Approximate the number $\sqrt[9]{0.5}$ using the linearization $L(x)$ from part (a). (Write your answer as a reduced fraction or as a decimal number rounded to two places.)

$$\begin{aligned} f(x) &\approx \frac{4}{9} + \frac{x}{9} \qquad \text{for } x \approx 5 \\ \sqrt[9]{.5} = f(4.5) &\approx \frac{4}{9} + \frac{4.5}{9} = \frac{17}{18} = .9\bar{4} \end{aligned}$$

5. [8 points] Let $f(x) = \dfrac{x}{x+1}$.

(a) [1 point] On what interval(s) is f continuous? (Write your answer using interval notation.)

f is continuous on $(-\infty, -1) \cup (-1, \infty)$.

(b) [2 points] On what interval(s) is f differentiable? (Write your answer using interval notation.)

$$\begin{aligned} f'(x) &= \frac{(x+1)\cdot 1 - x \cdot 1}{(x+1)^2} \\ &= \frac{1}{(x+1)^2} \end{aligned}$$

exists for all $x \neq -1$, so f is differentiable on $(-\infty, -1) \cup (-1, \infty)$.

(c) [5 points] Does f satisfy the hypotheses of the Mean Value Theorem on the interval $[-5, -2]$? If so, find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Yes, f is continuous on $[-5, -2]$ and is differentiable on $(-5, -2)$.

$$\begin{aligned} f'(c) &= \frac{f(-2) - f(-5)}{-2 - (-5)} \\ \frac{1}{(c+1)^2} &= \frac{2 - 5/4}{3} = \frac{1}{4} \\ (c+1)^2 &= 4 \\ c + 1 &= \pm 2 \\ c &= \pm 2 - 1 \Rightarrow c = -3, 1 \end{aligned}$$

Since only $c = -3$ is in $(-5, -2)$, $c = -3$ is the only number that satisfies the conclusion of the Mean Value Theorem.

6. [5 points] The area between the graphs of $\sin x$ and $\cos x$ on the interval $[\pi/4, x]$ is

$$A = \sqrt{2} - \cos x - \sin x$$

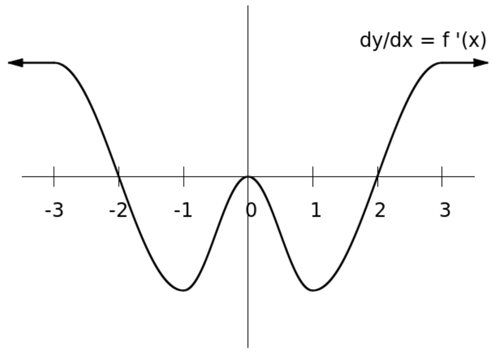
for $\pi/4 \leq x \leq 5\pi/4$. Suppose x is a function of t and changes at the rate $\sqrt{3}+1$. At what rate is A changing with respect to t when $x = \pi/3$?

Given: $\dfrac{dx}{dt} = \sqrt{3} + 1$

Unknown: $\left.\dfrac{dA}{dt}\right|_{x=\pi/3}$.

$$\begin{aligned}
A(t) &= \sqrt{2} - \cos x(t) - \sin x(t) \\
\frac{dA}{dt} &= \sin x \cdot \frac{dx}{dt} - \cos x \cdot \frac{dx}{dt} \\
&= \frac{dx}{dt}(\sin x - \cos x) = (\sqrt{3} + 1)(\sin x - \cos x) \\
\left.\frac{dA}{dt}\right|_{x=\pi/3} &= (\sqrt{3} + 1)\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \\
&= \frac{1}{2}(\sqrt{3} + 1)(\sqrt{3} - 1) = 1
\end{aligned}$$

7. [6 points] Use the graph of $dy/dx = f'(x)$, the derivative of $y = f(x)$, below to answer the following questions.



(a) [2 points] On what interval(s) is f decreasing? (Write your answer using interval notation.)

$f' < 0$ and so f is decreasing on $(-2, 0) \cup (0, 2)$.

(b) [1 point] At what x-value(s) does f have a local minimum?

f' switches from negative to positive at $x = 2$ so f has a local minimum at $x = 2$.

(c) [2 points] On what interval(s) is f concave downward? (Write your answer using interval notation.)

f' is decreasing so $f'' < 0$ and f is concave downward on $(-3, -1) \cup (0, 1)$.

(d) [1 point] Does f have any inflection points? If so, state the x-coordinate(s) of the inflection point(s) of f.

f' switches from decreasing to increasing at $x = -1$ and $x = 1$, and f' switches from increasing to decreasing at $x = 0$, so f has three inflection points with x-coordinates $x = -1, 0$, and 1.

8. [2 points] Suppose f'' is continuous on $(-\infty, \infty)$.

(a) [1 point] If $f'(-2) = 0$ and $f''(-2) = -1$, what can you say about f?

A. At $x = -2$, f has a local maximum.

B. At $x = -2$, f has a local minimum.

C. At $x = -2$, f has neither a maximum nor a minimum.

D. More information is needed to determine if f has a maximum or minimum at $x = -2$.

(b) [1 point] If $f'(0) = 0$ and $f''(0) = 0$, what can you say about f?

A. At $x = 0$, f has a local maximum.

B. At $x = 0$, f has a local minimum.

C. At $x = 0$, f has neither a maximum nor a minimum.

D. More information is needed to determine if f has a maximum or minimum at $x = 0$.

9. [9 points] Let $f(x) = xe^{-x}$.

(a) [2 points] On what interval(s) is f increasing? (Write your answer using interval notation.)

$$\begin{aligned} f'(x) &= x \cdot (-e^{-x}) + 1 \cdot e^{-x} \\ &= e^{-x}(1-x) \end{aligned}$$

Since $e^{-x} > 0$ for all x, $f' > 0$ so f is increasing on $(-\infty, 1)$.

(b) [1 point] At what x-value(s) does f have a local maximum?

$x = 1$ is the only critical number of f and f' switches from positive to negative at $x = 1$ so $f(1) = e^{-1}$ is a local maximum of f.

(c) [2 points] On what interval(s) is f concave upward? (Write your answers using interval notation.)

$$\begin{aligned} f''(x) &= e^{-x} \cdot (-1) + (-e^{-x}) \cdot (1-x) \\ &= e^{-x}(x-2) \end{aligned}$$

Since $e^{-x} > 0$ for all x, $f'' > 0$ so f is concave upward on $(2, \infty)$.

(d) [1 point] Does f have any inflection points? If so, state the inflection point(s) of f.

Since f switches concavity at $x = 2$, f has an inflection point $(2, f(2)) = (2, 2e^{-2})$

(e) [3 points] Evaluate $\lim_{x \to \infty} xe^{-x}$.

$\lim_{x \to \infty} xe^{-x}$ is an indeterminate form of type $\infty \cdot 0$. Rewriting the limit, and applying l'Hôspital's Rule,

$$\begin{aligned} \lim_{x \to \infty} xe^{-x} &= \lim_{x \to \infty} \frac{x}{e^x} \qquad \text{(indeterminate form of type } \frac{\infty}{\infty}\text{)} \\ &= \lim_{x \to \infty} \frac{1}{e^x} \\ &= 0 \end{aligned}$$

10. [9 points] Evaluate the following limits.

(a) [5 points] $\lim_{x\to 1^-}\left(\frac{x}{\ln x}-\frac{1}{x-1}\right)$.

$\lim_{x\to 1^-}\left(\frac{x}{\ln x}-\frac{1}{x-1}\right)$ is an indeterminate form of type $\infty-\infty$. Finding a common denominator and using l'Hôspital's Rule twice,

$$\begin{aligned}
\lim_{x\to 1^-}\left(\frac{x}{\ln x}-\frac{1}{x-1}\right) &= \lim_{x\to 1^-}\left(\frac{x^2-x-\ln x}{x\ln x-\ln x}\right) && \text{(indeterminate form of type } \tfrac{0}{0}\text{)}\\
&= \lim_{x\to 1^-}\left(\frac{2x-1-1/x}{x\cdot 1/x+1\cdot\ln x-1/x}\right)\\
&= \lim_{x\to 1^-}\left(\frac{2x-1-1/x}{1+\ln x-1/x}\right) && \text{(indeterminate form of type } \tfrac{0}{0}\text{)}\\
&= \lim_{x\to 1^-}\left(\frac{2+1/x^2}{1/x+1/x^2}\right)\\
&= \frac{2+1/1^2}{1/1+1/1^2}=\frac{3}{2}
\end{aligned}$$

(b) [4 points] $\lim_{x\to 0^+}(1-3x)^{1/x}$.

$\lim_{x\to 0^+}(1-3x)^{1/x}$ is an indeterminate form of type 1^∞. Rewriting the limit, and applying l'Hôspital's Rule,

$$\begin{aligned}
\lim_{x\to 0^+}(1-3x)^{1/x} &= e^{\lim_{x\to 0^+}\frac{1}{x}\ln(1-3x)} && \text{(limit is indeterminate form of type } \infty\cdot 0\text{)}\\
&= e^{\lim_{x\to 0^+}\frac{\ln(1-3x)}{x}} && \text{(limit is indeterminate form of type } \tfrac{0}{0}\text{)}\\
&= e^{\lim_{x\to 0^+}\frac{1/(1-3x)\cdot(-3)}{1}}\\
&= e^{-3}
\end{aligned}$$