3.10: Linear Approximations and Differentials

We can use the tangent line at $(a, f(a))$ to approximate the curve $y = f(x)$ when $x \approx a$. An equation of this tangent line is

$$L(x) = f(a) + f'(a)(x - a)$$

called the **linearization** of f at a and the approximation

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{for } x \approx a$$

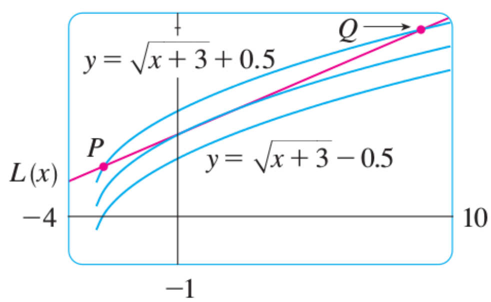
is called the **linear approximation** of f at a.

Example 1. *Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates of underestimates?*

Example 2. *For what values x is the linear approximation*

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5?



Example 3. *Use a linear approximation to estimate the given numbers*

(a) $(1.999)^4$

(b) $e^{-0.015}$

Differentials

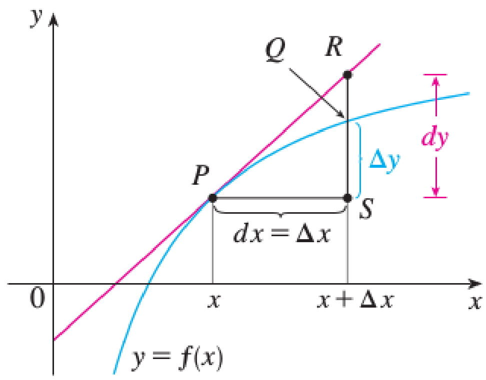
Let $y = f(x)$, the **differential** dx represents a change Δx in x. The corresponding change Δy in y along f is

$$\Delta y = f(x + \Delta x) - f(x).$$

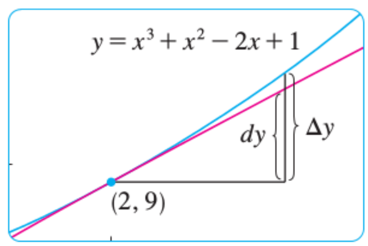
The **differential** dy given by

$$dy = f'(x)dx$$

represents the corresponding change in y along the linearization of f.



Example 4. *Compare the values of* Δy *and* dy *if* $y = x^3 + x^2 - 2x + 1$ *and* x *changes(a) from 2 to 2.05 and (b) from 2 to 2.01.*



Example 5. *Find the differential* dy *of each function.*

(a) $y = x^2 \sin x$

(b) $y = \ln \sqrt{1 + t^2}$

(c) $y = \frac{s}{1+2s}$

(d) $y = e^{-u} \cos u$