NUMERICAL ANALYSIS WRITTEN QUALIFYING EXAM
August 31, 2002

Instructions:

- Put your Social Security Number on each page of your exam. **Do not** put your name on the exam.
- Solve a total of 6 problems as completely as possible. These 6 problems should be chosen using the following criteria:
  - · choose 2 problems from Part I (linear algebra)
  - · choose 1 problem from Part II (nonlinear equations)
  - · choose 2 problems from Part III (approximation theory, interpolation and numerical integration)
  - · choose 1 problem from Part IV (numerical ODEs)
- Only 6 problems will be graded. If you attempt more than 6 problems, indicate which 6 are to be graded. If you fail to do this, your first 6 problems will be graded.

PART I - Linear Algebra
Complete 2 problems

1. Let $A,\ B \in \mathbf{R}^{n\times n}$ be two lower triangular matrices.

(a) Show that $C = AB$ is also a lower triangular matrix.

(b) Give an algorithm (in a pseudo code) for computing $C = AB$ with the least possible floating point operations by exploiting the sparsity structures of A, B, and C.

(c) Compute the total number of floating point operations of your algorithm. Justify your answer and compare it with general full matrix multiplication.

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2. Matrices in $\mathbf{R}^{n\times n}$ of the form $N(\mathbf{y}, k) = I - \mathbf{y}\mathbf{e}_k^T$ are called Gauss-Jordan transformations, where $\mathbf{y} \in \mathbf{R}^n$ and $\mathbf{e}_k$ is the unit coordinate vector.

(a) Give a formula for $N(\mathbf{y}, k)^{-1}$ assuming it exists.

(b) Let $\mathbf{x} \in \mathbf{R}^n$. Give conditions that ensure the existence of a $\mathbf{y}$ satisfying $N(\mathbf{y}, k)\mathbf{x} = \mathbf{e}_k$?

(c) Give an algorithm that computes A^{-1} for $A \in \mathbf{R}^{n\times n}$ using Gauss-Jordan transformations.

(d) What conditions on A ensure the success of your algorithm?

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3. Consider the following conjugate gradient algorithm for solving the linear system of equations $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbf{R}^{n\times n}$ is a symmetric and positive definite matrix and $\mathbf{b} \in \mathbf{R}^n$.

Choose $\mathbf{x}_0 \in \mathbf{R}^n$.
Set $\mathbf{p}_0 = \mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$.
For $k = 0, 1, \ldots,$ repeat
 if $\mathbf{p}_k = 0$, stop. Otherwise,

$$a_k = \frac{\mathbf{r}_k^T\mathbf{r}_k}{\mathbf{p}_k^T A\mathbf{p}_k},$$
$$\mathbf{x}_{k+1} = \mathbf{x}_k + a_k\mathbf{p}_k,$$
$$\mathbf{r}_{k+1} = \mathbf{r}_k - a_k A\mathbf{p}_k,$$
$$\mathbf{b}_k = \frac{\mathbf{r}_{k+1}^T\mathbf{r}_{k+1}}{\mathbf{r}_k^T\mathbf{r}_k},$$
$$\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + b_k\mathbf{p}_k.$$

Let l be the smallest nonnegative integer such that $p_l = 0$. Show that the vectors $\mathbf{x}_k,\ \mathbf{p}_k,\ \mathbf{r}_k,\ k \le l$, generated by the algorithm, have the following properties:

(a) $\mathbf{r}_i^T\mathbf{p}_j = 0$ for $0 \le j < i \le l$, $\mathbf{r}_i^T\mathbf{p}_i = \mathbf{r}_i^T\mathbf{r}_i$ for $i \le l$,

(b) $\mathbf{p}_i^T A\mathbf{p}_j = 0$ for $0 \le i < j \le l$, $\mathbf{p}_i^T A\mathbf{p}_i > 0$ for $i \le l$,

(c) $\mathbf{r}_i^T\mathbf{r}_j = 0$ for $0 \le i < j \le l$, $\mathbf{r}_i^T\mathbf{r}_i > 0$ for $i < l$,

(d) $\mathbf{r}_i = \mathbf{b} - A\mathbf{x}_i$ for $i \le l$,

(e) $A\mathbf{x}_l = \mathbf{b}$, $l \le n$ ($\mathbf{x}_l$ solves the equation $A\mathbf{x} = \mathbf{b}$).

PART II - Nonlinear Equations
Complete 1 problem

4. Let f be a twice continuously differentiable function on $[a,b]$ and assume that f has a zero $x_* \in (a,b)$. Consider the Newton iterations

$$\begin{cases} x_0 \in (a,b) \\ x_{n+1} = x_n - f(x_n)/f'(x_n) \qquad n = 0,1,2,\cdots \end{cases}$$

(a) Prove that if $x_0 \in (x_* - \delta, x_* + \delta)$ for some sufficiently small δ, then

$$|x_{n+1} - x_*| \le |x_n - x_*|/2 \qquad n = 0,1,2,\cdots$$

(b) Based on (a) prove that Newton's iterations converge to x_* provided x_0 is sufficiently close to x_*. (Do not simply quote the contraction mapping theorem. Prove directly.)

(c) Prove that Newton's iterations converge quadratically, i.e.,

$$|x_{n+1} - x_*| \le C|x_n - x_*|^2 \qquad n = 0,1,2,\cdots$$

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5. Let $\mathbf{F} : \mathbf{R}^m \to \mathbf{R}^m$ be a continuous function and assume that there exists an $\mathbf{x}_*$ such that $\mathbf{F}(\mathbf{x}_*) = \mathbf{0}$. Assume further that

$$\frac{\partial F_i}{\partial x_i} \ge \sum_{j \ne i} \left| \frac{\partial F_i}{\partial x_j} \right| \qquad i = 1,\cdots,m,\ \forall\, \mathbf{x} \in \mathbf{R}^m$$

and

$$\left| \frac{\partial F_i}{\partial x_j} \right| \le C \qquad I,j = 1,\cdots,m,\ \forall\, \mathbf{x} \in \mathbf{R}^m.$$

(a) Let $\mathbf{x}_0 \in \mathbf{R}^m$ be given. Define a fixed point iteration

$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n) \qquad n = 0,1,2\cdots$$

such that $\mathbf{x}_* = \mathbf{G}(\mathbf{x}_*)$ and

$$\left\| \left\{ \frac{\partial G_i}{\partial x_j} \right\} \right\|_\infty < 1$$

where $\left\{ \frac{\partial G_i}{\partial x_j} \right\}$ stands for the Jacobian matrix for $\mathbf{G}$.

(b) Prove that

$$||\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})||_\infty \le \alpha ||\mathbf{x} - \mathbf{y}||_\infty \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathbf{R}^m$$

for some $\alpha \in (0,1)$ (so that the Contraction Mapping Theorem implies $\mathbf{x}_n \to \mathbf{x}_*$.)

(c) Give a concrete nonlinear example of $\mathbf{F}$ on $\mathbf{R}^3$ satisfying all assumptions given above.

PART III - Approximation Theory, Interpolation & Numerical Integration
Complete 2 problems

6. (a) Let $[\cdot,\cdot]$ denote the inner product

$$[f,g] = \int_{-1}^{1} (x+1) f(x)\, g(x)\, dx$$

and P_n denote the set of polynomials of degree $\leq n$. Let $Q(x) \in P_n$ be a nontrivial polynomial which is orthogonal to P_{n-1} with respect to the inner product $[\cdot,\cdot]$. Consider the quadrature

$$I_1(g) \equiv \sum_{i=1}^{n} c_i g(x_i) \approx \int_{-1}^{1} (x+1) g(x)\, dx$$

where $\{x_i\}$ is the set of zeros of Q and c_i are chosen so that $I_1(g)$ is exact for P_{n-1}. Prove that $I_1(g)$ is, in fact, exact for P_{2n-1}.

(b) Show that there exists coefficients $\{a_i\}_{i=0}^{n}$ which make hte following formula exact for P_{2n}:

$$I_2(f) \equiv a_0 f(-1) + \sum_{i=1}^{n} a_i f(x_i) \approx \int_{-1}^{1} (x+1) g(x)\, dx$$

where the nodes $\{x_i\}$ are the same as in (a).

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7. Let

$$\|f\|_\omega^2 = \int_{-1}^{1} \frac{|f(x)|^2}{\sqrt{1-x^2}}\, dx$$

denote a norm for P_n on the interval $[-1,1]$. Define $\Pi_n f$ to be the best approximation to f in P_n in the norm $\|\cdot\|_\omega$, i.e.,

$$\|f - \Pi_n f\|_\omega = \min_{\phi \in P_n} \|f - \phi\|_\omega$$

(a) Give a formula for $\Pi_n f$ in terms of the Chebychev polynomials $\{T_k(x)\}_{k=0}^{n}$

(b) Let $f \in P_{n+1}$. Show that

$$\|f - \Pi_n f\|_\infty = \inf_{\phi \in P_n} \|f - \phi\|_\infty$$

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8. Denote by $p_n(x)$ the Lagrange interpolant of f on the set of points $\{x_0, x_1, \cdots, x_n\}$.

(a) Let $\{L_i\}_{i=0}^{n}$ be the Lagrange basis functions. Prove that $\sum_{i=0}^{n} L_i(x) \equiv 1$.

(b) Assume f is $(n+1)$ times continuously differentiable. Prove that the Lagrange interpolation error is expressed by

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

(hint: introduce the auxiliary function $g = f(t) - p_n(t) - [f(x) - p_n(x)] \prod_{i=0}^{n} \frac{(t-x_i)}{(x-x_i)}$.)

PART IV - Numerical ODEs
Complete 1 problem

9. Consider the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

on the interval $[t_0, T]$. Let N be a positive integer and set $h = (T - t_0)/N$ and $t_i = t_0 + ih$ for $i = 0, 1, \cdots, N$.

(a) For any integer i between 0 and $N - 2$, write out the Simpson's rule for approximating the integral $\int_{t_i}^{t_{i+2}} g(t)\,dt$ for an arbitrary continuous function g. Also, if the error between $\int_{t_i}^{t_{i+2}} g(t)\,dt$ and its Simpson's rule approximation is $O(h^s)$, then $s =$?

(b) Derive a 3-level scheme (involving points t_i, t_{i+1}, t_{i+2}) by integrating $y' = f(t, y)$ from t_i to t_{i+2} and then approximating the right-hand side by Simpson's rule.

(c) Check the consistency, stability (0-stability), and absolute stability of the scheme.

(d) Define a set of initial conditions for your scheme.

(e) Is the scheme given in b) together with the initial conditions d) convergent? If yes, what is the order of accuracy of the scheme?

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10. To find an approximate solution to the initial value problem

$$y' = t \quad \text{on } [1, 2], \quad y(1) = y_0,$$

consider the scheme

$$\begin{aligned} &Y_{i+1} - Y_{i-1} = 2ht_i \quad i = 1, \cdots, N - 1 \\ &Y_0 = y_0, \qquad Y_1 = y_0 + h \end{aligned} \tag{$*$}$$

where N is a positive integer and $h = 1/N$.

(a) Verify that $\widetilde{Y}_i = ih + i^2h^2/2$ satisfies $Y_{i+1} - Y_{i-1} = 2ht_i$ for $i = 1, \cdots, N - 1$.

(b) Find the general solution of $Y_{i+1} - Y_{i-1} = 0$ in the form of $Y_i = C_1 r_1^i + C_2 r_2^i$.

(c) Find an explicit formula for the solution of $(*)$.