3.5: Implicit Differentiation

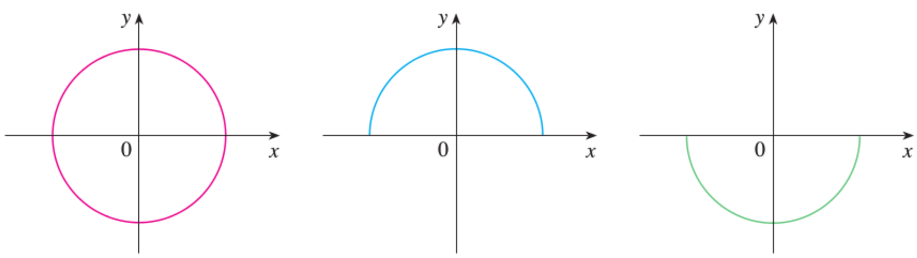
So far, we have focused on differentiating *explicit* functions of x of the form $y = f(x)$. Using **implicit differentiation**, we can solve for y' without knowing the form of $y = f(x)$. That is, we can solve for y' from *implicit* functions of x and y such as

$$x^2 + y^2 = 25, \quad x^3 + y^3 = 6xy, \quad \text{and} \quad \sin(x + y) = y^2 \cos x.$$

Example 1.

(a) If $x^2 + y^2 = 25$, *find* $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ *at the point* $(3, 4)$.



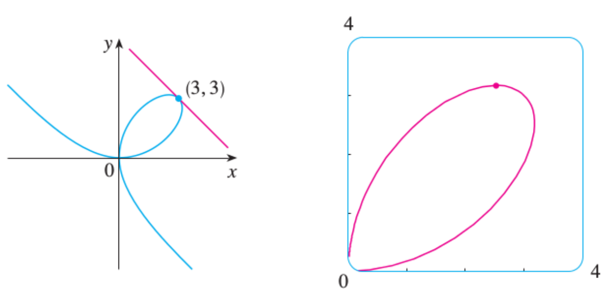
(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Example 2.

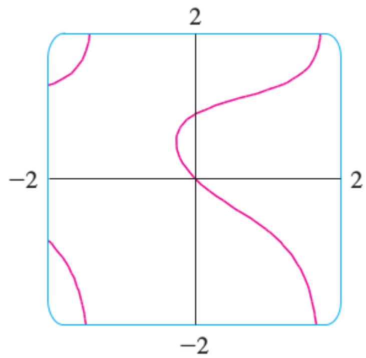
(a) Find y' *if* $x^3 + y^3 = 6xy$.

(b) Find the tangent to the curve $x^3 + y^3 = 6xy$ *at the point* $(3, 3)$.

(c) At what point in the first quadrant is the tangent line horizontal?



Example 3. *Find* y' *if* $\sin(x + y) = y^2 \cos x$.



Derivatives of Inverse Trigonometric Functions

Implicit deifferentiation is especially useful for differentiating the inverse trigonometric functions.

Example 4. *Use implicit differentiation to find the derivatives of*

(a) $\sin^{-1} x$.

(b) $\tan^{-1} x$.

The derivatives of the inverse trigonometric functions are

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$
5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
6. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Example 5. *Differentiate (a)* $y = \frac{1}{\sin^{-1} x}$ *and (b)* $f(x) = x \arctan x$.