Unit I

Supplementary Notes

3.1 Quadratic Functions

$$f(x) = \underbrace{ax^2 + bx + c}_{\text{expanded form}} = \underbrace{a(x-h)^2 + k}_{\text{vertex form}}$$

The graph of f is a *parabola* with the following properties

- opens up or down if $a > 0$ or $a < 0$
- vertex $(h, k) = (-\frac{b}{2a}, f(-\frac{b}{2a})) = (-\frac{b}{2a}, c - \frac{b^2}{4a})$
- 0, 1, or 2 $x-$intercepts if $b^2 - 4ac < 0, = 0,$ or > 0
- $y-$intercept: $f(0) = c$
- vertical axis of symmetry $x = h$
- minimum value of k

3.2 Power Functions

$f(x) = a(x-h)^n + k$ click to see Supplementary Notes

3.3 Polynomial Functions

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 && \} \text{ expanded form} \\ &= a(x - z_1)(x - z_2)\cdots(x - z_{n-1})(x - z_n) && \} \text{ factored form} \end{aligned}$$

- degree n
- $f(z) = 0 \Leftrightarrow z$ is a *zero* of f

The graph of f has the following properties

- $x-$intercepts: real zeros of f
  - crosses or touches $x-$axis if *multiplicity* odd or even
- $y-$intercept: $f(0) = a_0$
- For large $|x|$, behaves like ax^n

3.4 Real Zeros of Polynomial Functions

$$\underbrace{f(x)}_{\text{dividend}} = \underbrace{q(x)}_{\text{quotient}}\underbrace{g(x)}_{\text{divisor}} + \underbrace{r(x)}_{\text{remainder}}$$

- $f(z)$ is the *remainder* of f divided by $(x - z)$
- $(x - z)$ *factor* of $f \Leftrightarrow f(z) = 0$
- f continuous and $f(a)$, $f(b)$ opposite sign $\Rightarrow f$ has zero in (a, b)
- p/q *rational zero* of $f \Rightarrow p$ factor of a_0 and q factor of a_n

3.5 Complex Numbers

$$a + bi \quad \text{where } i^2 = -1$$

- conjugate: $a - bi$
- powers of i:
  - $i^{2n} = (i^2)^n = (-1)^n$
  - $i^{2n+1} = (i^2)^n i = (-1)^n \cdot i$

3.6 Complex Zeros, Fundamental Theorem of Algebra

The zeros of a polynomial f with real coefficients may be real or complex; if $a + bi$ is a zero of f, so is its conjugate $a - bi$

3.7 Rational Functions

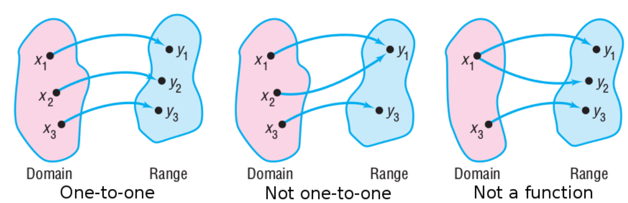
$$f(x) = \frac{g(x)}{h(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

- domain: real x such that $h(x) \neq 0$
- vertical asymptotes: real x such that $h(x) = 0$ and $g(x) \neq 0$
- jump discontinuities: real x such that $h(x) = g(x) = 0$
- If $n \leq m$, horizontal asymptote:
  - $y = 0$ if $n < m$
  - $y = a_n/b_m$ if $n = m$
- If $n = m+1$, oblique asymptote: $y = q(x)$ where $g(x) = q(x)h(x) + r(x)$

3.8 Polynomial and Rational Inequalities

The sign of a function may change at a *zero* or an x-value *not in the domain* of the function

4.1 One-to-one and Inverse Functions



- f is a *function* if any *vertical* line intersects graph of f at most once
- f is *one-to-one* if any *horizontal* line intersects graph of f at most once
- f one-to-one $\Rightarrow f$ has inverse f^{-1} such that
  - $x \xrightarrow{f} y$ if and only if $x \xleftarrow{f^{-1}} y$
  - (domain of f) = (range of f^{-1}); (range of f) = (domain of f^{-1})
  - The graphs of f and f^{-1} are symmetric about the line $y = x$

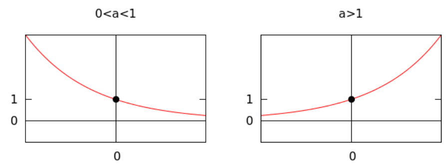
Unit II

Supplementary Notes

4.2 Exponential Functions

$$f(x) = a^x \qquad (a > 0, a \neq 1)$$

- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- $y-$intercept: 1
- horiz. asymp.: $y = 0$ ($x-$axis)
- $\begin{cases} \text{decreasing} & \text{if } 0 < a < 1 \\ \text{increasing} & \text{if } a > 1 \end{cases}$



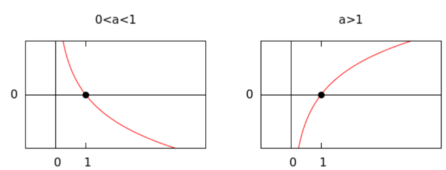
Laws of Exponents	
$a^s \cdot a^t = a^{s+t}$	$(a^s)^t = a^{s \cdot t}$
$(a \cdot b)^s = a^s \cdot b^s$	$a^{-s} = \left(\frac{1}{a}\right)^s = \frac{1}{a^s}$
$1^s = 1$	$a^0 = 1$

Reflection and Translation $(h, k > 0)$, to obtain the graph of		
a^{-x-h}	translate the graph of a^x then reflect the graph of a^{x-h}	rightward h units about the $y-axis$
a^{-x+h}	translate the graph of a^x then reflect the graph of a^{x+h}	leftward h units about the $y-axis$
$-a^x + k$	reflect the graph of a^x then translate the graph of $-a^x$	about the $x-$axis upward k units
$-a^x - k$	reflect the graph of a^x then translate the graph of $-a^x$	about the $x-$axis downward k units

4.3 Logarithmic Functions

$$f(x) = \log_a x \qquad (a > 0, a \neq 1)$$

- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- $x-$intercept: 1
- vert. asymp.: $x = 0$ ($y-$axis)
- $\begin{cases} \text{decreasing} & \text{if } 0 < a < 1 \\ \text{increasing} & \text{if } a > 1 \end{cases}$



Special Logarithms	
$\log x = \log_{10} x$	$\ln x = \log_e x$

4.4 Properties of Logarithms $(a, b, m, n > 0,\ a, b \neq 1$, real number $p)$

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a(mn) = \log_a m + \log_a n$
- $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
- $\log_a n^p = p \log_a n$
- $\log_a n = \frac{\log_b n}{\log_b a}$

4.5 Log. and Exponential Equations $(a > 0,\ a \neq 1$, real numbers $s, t > 0)$

- $x = \log_a y$ $\Leftrightarrow y = a^x$
- $a^s = a^t$ $\Leftrightarrow s = t$
- $\log_a s = \log_a t$ $\Leftrightarrow s = t$

4.6 Compound Interest

- Periodic: $A = P(1 + \frac{r}{n})^{nt}$
- Continuous: $A = Pe^{rt}$
- Effective interest rate: $r_e = \left(1 + \frac{r}{n}\right)^n - 1$

A: future amount ($)
P: initial amount a.k.a. principal ($)
r: annual interest rate (%)
n: periods per year
t: time (years)

4.7 Exponential Growth and Decay

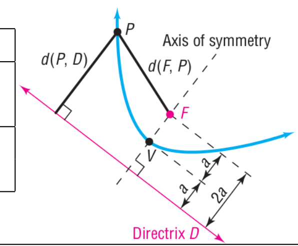
Exponential law a.k.a Law of uninhibited growth $(k > 0)$ or decay $(k < 0)$:

- $A(t) = A_0 e^{kt}$ where $A_0 = A(0)$ $\qquad (k \neq 0)$

9.2 The Parabola

- V: (h, k)
- $d(V, F) = d(V, D) = a$

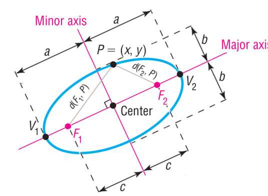
Equation	Opens	F	D
$(x-h)^2 = 4a(y-k)$	up	$(h, k+a)$	$y = k - a$
$(x-h)^2 = -4a(y-k)$	down	$(h, k-a)$	$y = k + a$
$(y-k)^2 = 4a(x-h)$	right	$(h+a, k)$	$x = h - a$
$(y-k)^2 = -4a(x-h)$	left	$(h-a, k)$	$x = h + a$



9.3 The Ellipse

- center: (h, k)
- $c^2 = a^2 - b^2$
- $d(\text{center}, V) = a$
- $d(\text{center}, F) = c$

Equation	M. Ax.	V	F
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	horiz.	$(h \pm a, k)$	$(h \pm c, k)$
$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$	vert.	$(h, k \pm a)$	$(h, k \pm c)$

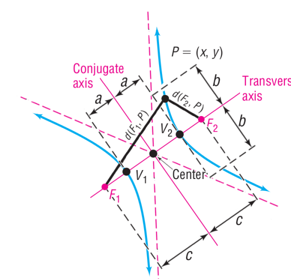


9.4 The Hyperbola

- center: (h, k)
- $c^2 = a^2 + b^2$
- $d(\text{center}, V) = a$
- $d(\text{center}, F) = c$

Equation	Tr. Ax.	V	F
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	horiz.	$(h \pm a, k)$	$(h \pm c, k)$
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$	vert.	$(h, k \pm a)$	$(h, k \pm c)$

- asymptotes: set Equation $= 0$, solve for y



Unit III

Supplementary Notes

10.1 Systems of Linear Equations: Two Equations

$$\begin{cases} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{cases} \xrightarrow{\text{row reduce}} \begin{cases} c_{11}x + c_{12}y &= d_1 \\ c_{22}y &= d_2 \end{cases}$$

- $c_{22} \neq 0 \Rightarrow$ unique solution
- $c_{22} = 0$ and $d_2 \neq 0 \Rightarrow$ no solution
- $c_{22} = d_2 = 0 \Rightarrow$ infinitely many solutions

10.2 Systems of Linear Equations: Three Equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{cases} \xrightarrow{\text{row reduce}} \begin{cases} c_{11}x + c_{12}y + c_{13}z &= d_1 \\ c_{22}y + c_{23}z &= d_2 \\ c_{33}z &= d_3 \end{cases}$$

- $c_{33} \neq 0 \Rightarrow$ unique solution
- $c_{33} = 0$ and $d_3 \neq 0 \Rightarrow$ no solution
- $c_{33} = d_3 = 0 \Rightarrow$ infinitely many solutions

10.3 Matrices

Valid row operations:

- Interchange two rows
- Multiply row by non-zero constant
- Replace row by non-zero multiple of itself plus multiple of another row

Reduced row echelon form:

- Leading non-zero entry in each row is 1, above and below leading 1, in the same column, is all 0's
- Leading 1 in a row is to the right of any leading 1 in above rows
- Any row of all 0's is at the bottom

10.4 Determinants

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} \end{array}$$

10.4 Determinants (ctd.)

<u>Cramer's Rule</u>: For 3×3 matrix A, if $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $|A| \neq 0$, then $x = \frac{|A_x|}{|A|}$, $y = \frac{|A_y|}{|A|}$, and $z = \frac{|A_z|}{|A|}$

For a square matrix A

- $|A|$ switches sign if two rows are interchanged
- $|A|$ is multiplied by a nonzero constant if a row is multiplied by that constant
- $|A|$ does not change if a row is replaced by itself plus a non-zero multiple of another row

10.5 Matrix Algebra

For $m \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ and real numbers c, c_1, and c_2

- $A = B \Leftrightarrow a_{ij} = b_{ij}$
- $A + B = B + A = (a_{ij} + b_{ij})$
- $(A + B) + C = A + (B + C)$
- $cA = (ca_{ij})$
- $c_1(c_2A) = (c_1c_2)A$
- $c(A + B) = cA + cB$
- $(c_1 + c_2)A = c_1A + c_2A$

For $m \times n$ matrix A, $n \times p$ matrices B, B_1, B_2 and $p \times q$ matrix C

- ij^{th} entry of $m \times p$ matrix $AB =$ (row i of A) $\times$ (col. j of B)
- $(AB)C = A(BC)$
- $A(B_1 + B_2) = AB_1 + AB_2$

Inversion: $\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$

11.1 Sequences

recursive:
$a_1 = a$ for real a
$a_n = f(a_{n-1})$ for $n \geq 2$

explicit:
$a_n = f(n)$ for $n \geq 1$

sum of first n terms:
$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

11.2 Arithmetic Sequences

recursive:
$a_1 = a$ for real a
$a_n = a_{n-1} + d$ for $n \geq 2$

explicit:
$a_n = a_1 + (n-1)d$ for $n \geq 1$

$$d = \frac{a_m - a_n}{m - n}$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

11.3 Geometric Sequences and Series

recursive:
$a_1 = a$ for real a
$a_n = a_{n-1}r$ for $n \geq 2$

explicit:
$a_n = a_1 r^{n-1}$ for $n \geq 1$

$$r = \sqrt[m-n]{\frac{a_m}{a_n}}$$

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

geometric series:
$$S = \sum_{k=1}^{\infty} a_k = \frac{a_1}{1 - r}$$

Unit IV

Supplementary Notes

11.4 Mathematical Induction

- To prove by induction that $P(n)$ is true for all positive integers n, we assume $P(k)$ is true for some positive integer k and show that $P(k+1)$ is true.

11.5 The Binomial Theorem

- factorial function:

$$\begin{aligned} 0! &= 1 \\ n! &= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \quad \text{for } n \geq 1 \end{aligned}$$

- binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for } 0 \leq k \leq n$$

- Pascal's triangle:

$$\begin{array}{lccccccccccc} \text{Row 0:} & & & & & & \binom{0}{0} & & & & & \\ \text{Row 1:} & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ \text{Row 2:} & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ \text{Row 3:} & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ \text{Row 4:} & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \text{Row 5:} & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ \vdots & & & & & & \vdots & & & & & \end{array}$$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}$$

- binomial theorem:

$$\begin{aligned} (a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad (n \geq 0) \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{k} a^{n-k} b^k + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n \end{aligned}$$