# Foundations of Computational Math I Exam 2
## Take-home Exam
## Open Notes, Textbook, Homework Solutions Only
## Calculators Allowed
## No collaborations with anyone
## Due beginning of Class Wednesday, 3 December, 2012

| Question | Points Possible | Points Awarded |
|---|---|---|
| 1. Linear Iteration | 30 | |
| 2. Factorization and Preconditioning | 30 | |
| 3. Nonlinear Iteration | 30 | |
| 4. Nonlinear Iteration | 20 | |
| Total Points | 110 | |

**Name:**__________________________________

**Alias:** __________________________________

**to be used when posting anonymous grade list.**

# Problem 1 (30 points)

## 1.a(10 points)

Given a nonsingular $A \in \mathbb{R}^n$, and a nonsingular $M \in \mathbb{R}^n$, show that for all $b \in \mathbb{R}^n$, $x = A^{-1}b$ is a fixed point of

$$x_{i+1} = M^{-1}Nx_i + M^{-1}b$$

if and only if $A = M - N$.

## 1.b(20 points)

Consider the matrix

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 1 & 0 & 1 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

(i) Will Jacobi's method converge when solving $Ax = b$ for all $b$ and $x_0$? Justify your answer.

(ii) Will Gauss-Seidel converge when solving $Ax = b$ for all $b$ and $x_0$? Justify your answer.

(iii) Does Gauss-Seidel converge in a finite number of steps when solving $Ax = b$ for all $b$ and $x_0$? Justify your answer.

(iv) Does a permutation, $P$, exist such that Gauss-Seidel solving $PAP^TPx = Pb$ for all $b$ and $x_0$ converges in one step? Justify your answer.

# Problem 2 (30 points)

## 2.a(10 points)

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Suppose when computing the Cholesky factorization of $A$ using IEEE floating point arithmetic at some step we have an active part of the matrix (Schur complement) that is identically 0. Since every element in the computed active part has the value of 0, even if we started complete pivoting at this point it would not progress. What can we conclude about the original matrix $A$?

## 2.b(20 points)

Assume the nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is an $M$-matrix and partition it

$$A = \begin{pmatrix} \alpha_{11} & a_{12}^T \\ a_{21} & A_{22} \end{pmatrix}, \quad \alpha_{11} \in \mathbb{R} \quad A_{22} \in \mathbb{R}^{n-1\times n-1}$$

The following facts are true (you need not prove them):

- The matrix $S \in \mathbb{R}^{n-1\times n-1}$ defined by

$$S = A_{22} - a_{21}\alpha_{11}^{-1}a_{12}^T$$

  is also an $M$-matrix.

- If $C \in \mathbb{R}^{n\times n}$ is an $M$-matrix, $B \in \mathbb{R}^{n\times n}$ is a matrix whose off-diagonal elements satisfy $\beta_{ij} \leq 0$, $i \neq j$ and

$$C \leq B$$

  (that is $\alpha_{ij} \leq \beta ij$, $1 \leq i,\ j \leq n$) then $B$ is also an $M$-matrix.

A very effective way to generate a precondtioner $P$ when using an iterative method to solve $Ax = b$ when $A$ is sparse is to create $P = LU$ where $L$ and $U$ are sparse lower and upper triangular matrices respectively such that $A \approx LU$. This is called an incomplete factorization preconditioner.

This preconditioner can be computed, for example, by modifying $LU$ factorization so that on each step some elements in the portion of the matrix updated are set to 0. By doing this the factors $L$ and $U$ are kept sparse. However, in general, this procedure cannot be guaranteed to complete since the updated matrix after removing the elements might not have an $LU$ factorization.

Use the facts above to show that given that $A$ is an $M$-matrix the incomplete $LU$ factorization procedure described above must complete and produce $L$ and $U$. (You need not consider the quality of the approximation to $A$.)

## Problem 3 (30 points)

Suppose $g(x)$ be a smooth function, i.e., continuously differentiable to any order, Let $m > 1$ be an integer and consider the modified Newton iteration

$$\phi(x) = x - m\frac{g(x)}{g'(x)}$$

Suppose $\alpha \in \mathbb{R}$ is a root with multiplicity $d \geq 1$, i.e., $g(\alpha) = 0$ and $g^{(j)}(\alpha) = 0$ for $j = 0, \ldots, d-1$.

**3.a** **(10 points)** Recall that $m = 1$ defines Newton's method which is quadratically convergent for simple roots, i.e., $d = 1$. Under what conditions on $m$ and $d$ can local convergence be guaranteed? Justify your answer.

**3.b** **(10 points)** For cases the iteration is convergent, determine the associated order of convergence. Justify your answer.

**3.c** **(5 points)** Under what circumstances, if any, is the iteration divergent? Justify your answer.

**3.d** **(5 points)** Demonstrate your conclusions by considering $m = 3$ and the two functions $g_1(x) = x^3$ and $g_2(x) = x^3 - 69$. **You do not have to turn in any code. Of course, it is recommended you implement the iteration on whatever machine using whatever language/system that is convenient.**

## Problem 4 (20 points)

Consider the iteration

$$\phi(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

for $a > 0$.

**4.a** **(5 points)** Determine the fixed points of $\phi(x)$. Justify your answer.

**4.b** **(15 points)** For each fixed point, determine if $\phi(x)$ converges to it and when convergent determine the associated order of convergence. Justify your answer.