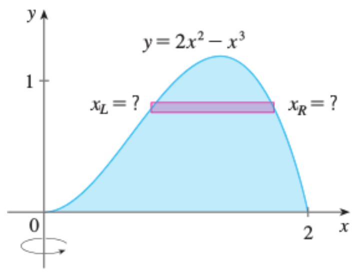
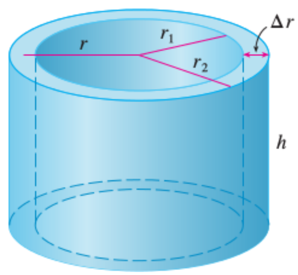
6.3: Volumes by Cylindrical Shells

Some volume problems are difficult to solve using the method of Section 6.2. For example, consider calculating the volume of the solid obtained by rotating the area bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. If we take cross-sections perpendicular to the y-axis, we get annuli, but to calculate the inner and outer radii we'd have to solve $y = 2x^2 - x^3$ for x, which is messy.



Instead, it is easier to use the **method of cylindrical shells**, which consists of approximating the volume with cylindrical shells. Depicted below is a cylindrical shell with inner radius r_1, outer radius r_2, and height h.



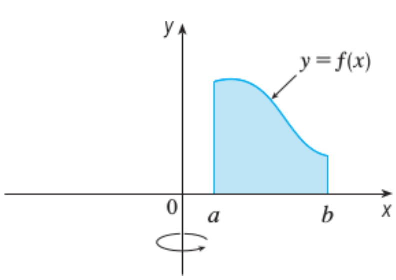
The volume of the shell is obtained by subtracting the volume of the inner cylinder from the outer cylinder:

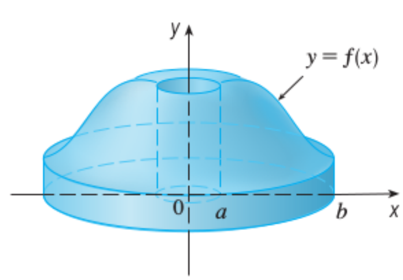
$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h. \end{aligned}$$

If we let $\Delta r = r_2 - r_1$ be the thickness of the shell and let $r = (r_2 + r_1)/2$ be the average radius of the shell, then the volume is

$$\begin{aligned} V &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \\ &= 2\pi r h \Delta r \end{aligned}$$

and it can be remembered as $V = \text{(circumference)} \times \text{(height)} \times \text{(thickness)}$. Now, let S be the solid obtained by rotating the region bounded by $y = f(x)$ (where $f(x) \geq 0$), $y = 0$, $x = a$, and $x = b$ (where $0 \leq a < b$) about the y-axis.





To approximate the volume V of S, we divide the interval $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$, each of width $\Delta x = (b - a)/n$ and let $\bar{x}_k$ be the midpoint of each interval. Rotating the rectangle with base $[x_{k-1}, x_k]$ and height $f(\bar{x}_k)$ about the y-axis forms a cylindrical shell with average radius $\bar{x}_k$, height $f(\bar{x}_k)$, and thickness Δx, so its volume is

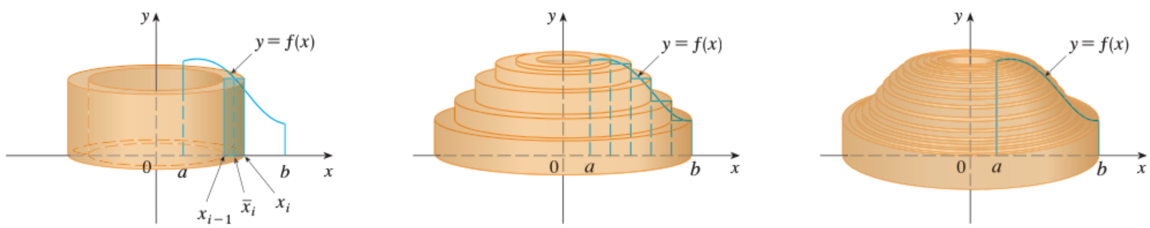
$$V_k = (2\pi\bar{x}_k)[f(\bar{x}_k)]\Delta x.$$

Therefore, adding up the volumes of the cylindrical shells formed from all the subintervals gives us an approximation to the volume V of S:

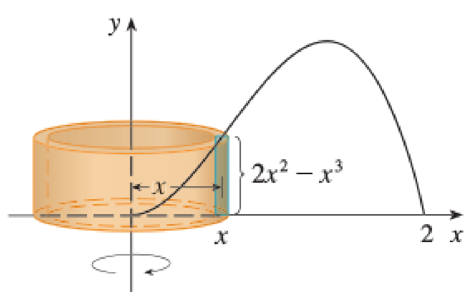
$$V \approx \sum_{k=1}^{n} 2\pi\bar{x}_k f(x_k)\Delta x.$$

The approximation becomes exact as $n \to \infty$. That is,

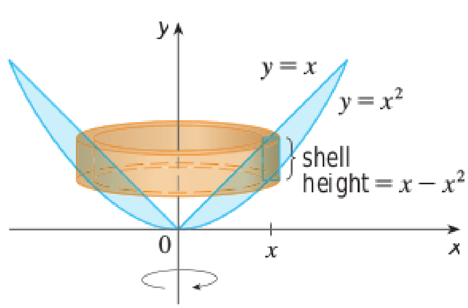
$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi\bar{x}_k f(\bar{x}_k)\Delta x \\ &= \int_a^b 2\pi x f(x)\ dx. \end{aligned}$$



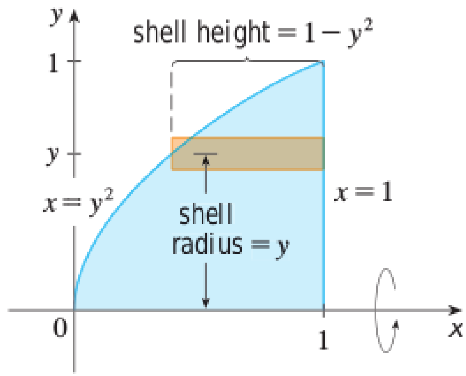
Example 1. *Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.*



Example 2. *Find the volume of the solid obtained by rotating the region between $y = x$ and $y = x^2$ about the y-axis.*



Example 3. *Use cylindrical shells to find the volume of the solid obtained by rotating the region under the curve $y = \sqrt{x}$ from 0 to 1 about the x-axis.*



Example 4. *Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.*