Homework 2 Foundations of Computational Math 2 Spring 2012

Solutions will be posted Friday, 1/27/12

Problem 2.1

Let $p_n(x)$ be the unique polynomial that interpolates the data

$$(x_0, f_0), \dots, (x_n, f_n)$$

Suppose that we assume the form

$$p_n(x) = \alpha_0 + \alpha_1(x - x_0) + \cdots + \alpha_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

and let

$$a = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} \qquad y = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

2.1.a. Show that the constraints yield a linear system of equations

$$La = y$$

where L is lower triangular.

2.1.b. Show that the linear system yields a recurrence for the α_i that is equivalent to one of the standard definitions of the divided differences and therefore this is the Newton form of $p_n(x)$.

Problem 2.2

Consider a polynomial

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$p_n(\gamma)$ can be evaluated using Horner's rule (written here with the dependence on the formal argument x more explicitly shown)

$c_n(x) = \alpha_n$
for $i = n - 1 : -1 : 0$
    $c_i(x) = x c_{i+1}(x) + \alpha_i$
end

$p_n(x) = c_0(x)$

Note that when evaluating $x = \gamma$ the algorithm produces $n+1$ constants $c_0(\gamma), \dots, c_n(\gamma)$ one of which is equal to $p_n(\gamma)$.

2.2.a

Suppose that Horner's rule is applied to evaluate $p_n(\gamma)$ and that the constants $c_0(\gamma), \ldots, c_n(\gamma)$ are saved. Show that

$$p_n(x) = (x-\gamma)q(x) + p_n(\gamma)$$
$$q(x) = c_1(\gamma) + c_2(\gamma)x + \cdots + c_n(\gamma)x^{n-1}$$

2.2.b

Suppose that Horner's rule is applied to evaluate $p_n(\gamma)$ and that the constants $c_0^{(1)}(\gamma), \ldots, c_n^{(1)}(\gamma)$ are saved to define $q_{(1)}(x) = c_1^{(1)}(\gamma) + c_2^{(1)}(\gamma)x + \cdots + c_n^{(1)}(\gamma)x^{n-1}$. Suppose further that Horner's rule is applied to evaluate $q_{(1)}(\gamma)$ and that the constants $c_0^{(2)}(\gamma), \ldots, c_{n-1}^{(2)}(\gamma)$ are saved to define $q_{(2)}(x) = c_1^{(2)}(\gamma) + c_2^{(2)}(\gamma)x + \cdots + c_{n-1}^{(2)}(\gamma)x^{n-2}$. This can continue until Horner's rule is applied to evaluate $q_{(n)}(\gamma)$ and $q_{(n+1)}(x) = 0$, i.e., there are no constants other than $c_0^{(n)}(\gamma)$ produced.

Show that

$$
\begin{aligned}
q_{(1)}(\gamma) &= p_n'(\gamma) \\
q_{(2)}(\gamma) &= p_n''(\gamma)/2 \\
q_{(3)}(\gamma) &= p_n'''(\gamma)/3! \\
&\vdots \\
q_{(n-1)}(\gamma) &= p_n^{(n-1)}(\gamma)/(n-1)! \\
q_{(n)}(\gamma) &= p_n^{(n)}(\gamma)/n!
\end{aligned}
$$

and therefore form the coefficients of the Taylor form of $p_n(x)$

$$p_n(x) = p_n(\gamma) + (x-\gamma)p_n'(\gamma) + \frac{(x-\gamma)^2}{2}p_n''(\gamma) + \frac{(x-\gamma)^3}{3!}p_n'''(\gamma) \cdots + \frac{(x-\gamma)^{n-1}}{(n-1)!}p_n^{(n-1)}(\gamma) + \frac{(x-\gamma)^n}{n!}p_n^{(n)}(\gamma)$$

Problem 2.3

Text exercise 8.10.8 on page 376