Set 9: Minimax (Best) and Near-minimax Polynomial Approximation

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Approximation Outline

- Best (Minimax) Polynomial Approximation – 10.8
- Chebyshev (Near Minimax) Approximation – 10.8
- Generalized Fourier Series – 10.1
- Orthogonal Polynomials – 10.1
- Least Squares approximation – 10.1,10.7 and notes
- Chebyshev Economization – 10.8 and notes
- Discrete Least Squares approximation – 10.7 and notes

Best Polynomial Approximation

Let $\mathbb{P}_n$ be the space of polynomials of degree at most n.

Problem 9.1.

$$\min_{p_n \in \mathbb{P}_n} \|f - p_n\|_\infty$$

$$p_n^* = \text{argmin}\|f - p_n\|_\infty \text{ denotes the minimizer}$$

$$E_n^*(f) = \|f - p_n^*\|_\infty \text{ denotes the minimal error}$$

Note. Bounds in this norm guarantee that the pointwise error does not exceed a particular amount.

Lower Bound Characterization

Theorem 9.1. *(De La Vallée-Poussin) Let $p_n(x)$ be a polynomial of degree n with deviations from $f(x)$ on $[a, b]$*

$$f(x_j) - p_n(x_j) = (-1)^j \epsilon_j, \quad 0 \le j \le n+1$$

$$a \le x_0 < x_1 < \cdots < x_n < x_{n+1} \le b$$

$$\text{either } \forall j \;\; \epsilon_j > 0 \text{ or } \forall j \;\; \epsilon_j < 0.$$

The error $E_n^*(f)$ is bounded below by

$$\min_j |\epsilon_j| \le E_n^*(f)$$

Lower Bound Characterization

Proof. (Isaacson and Keller) Suppose $\tilde{p}_n(x)$ has degree n and is such that

$$\|f - \tilde{p}_n\|_\infty < \min_j |\epsilon_j| = \mu$$

Consider the polynomial of degree n, $\tilde{p}_n - p_n$ at x_j, $0 \le j \le n+1$

$$\begin{aligned} \tilde{p}(x_j) - p_n(x_j) &= (f(x_j) - p_n(x_j)) - (f(x_j) - \tilde{p}_n(x_j)) \\ &= (-1)^j \epsilon_j - (f(x_j) - \tilde{p}_n(x_j)) \\ \text{by assumption} \quad & |f(x_j) - \tilde{p}_n(x_j)| < \mu \le |\epsilon_j| \\ \therefore sign(\tilde{p}(x_j) - p_n(x_j)) &= sign(f(x_j) - p_n(x_j)) \end{aligned}$$

$n+2$ points of alternating sign for $\tilde{p}(x) - p_n(x)$ implies $n+1$ roots and $\tilde{p}_n(x) \equiv p_n(x)$ which is a contradiction. $\square$

Optimal Characterization

Theorem 9.2. *(Chebyshev)*

A polynomial of degree at most n, $p_n^*(x)$ is an optimal approximation of $f(x)$ on $[a, b]$ with respect to $\|f - p_n\|_\infty$ if and only if $f(x) - p_n^*(x) = \pm E_n^*(f)$, with alternating sign changes, at least $n + 2$ times in $[a, b]$. The polynomial $p_n^*(x)$ is unique.

Proof. See Isaacson and Keller. $\square$

Optimal Characterization

Note. Theorem 9.1 says $E_n^*(f)$ can be bounded from below using a polynomial of degree n that oscillates around $f(x)$ at least $n+1$ times. This is most easily done with an interpolating polynomial of degree n where we choose the points. A good choice of points yields a tight bound.

Corollary. *The optimal approximation $p_n^*(x)$ interpolates $f(x)$ at $n+1$ points, $\tilde{x}_k$ where $x_k < \tilde{x}_k < x_{k+1}$ for $0 \le k \le n$ and x_k for $0 \le k \le n+1$ are the $n+2$ points of maximum deviation.*

Proof. Follows immediately from the conditions of Theorem 9.2. $\square$

Error Bound

The $n+1$ interpolating points, $\tilde{x}_k$ where $x_k < \tilde{x}_k < x_{k+1}$ are not known but since the interpolating polynomial is unique we have the following error bound when $f \in \mathcal{C}^{n+1}[a,b]$.

Theorem 9.3. *Let* $f \in \mathcal{C}^{n+1}[a,b]$ *and let* $p_n^*(x)$ *be the optimal approximation of* f *of degree at most* n*. There exists* $n+1$ *points* $\tilde{x}_k \in [a,b]$ *such that* $\forall x \in [a,b]$

$$f(x) - p_n^*(x) = \frac{(x-\tilde{x}_0)(x-\tilde{x}_1)\cdots(x-\tilde{x}_n)}{(n+1)!} f^{(n+1)}(\xi(x))$$

with $\xi(x) \in [a,b]$.

Example

Problem 9.2. Find $p_0^*(x) = e^\xi$ as the best approximation to e^x on $[0, 1]$.

$E(x) = e^x - e^\xi$ is monotonic and we have the maximum deviation at $x = 0$ and $x = 1$.

$$E(1) = -E(0)$$

$$e - e^\xi = -1 + e^\xi$$

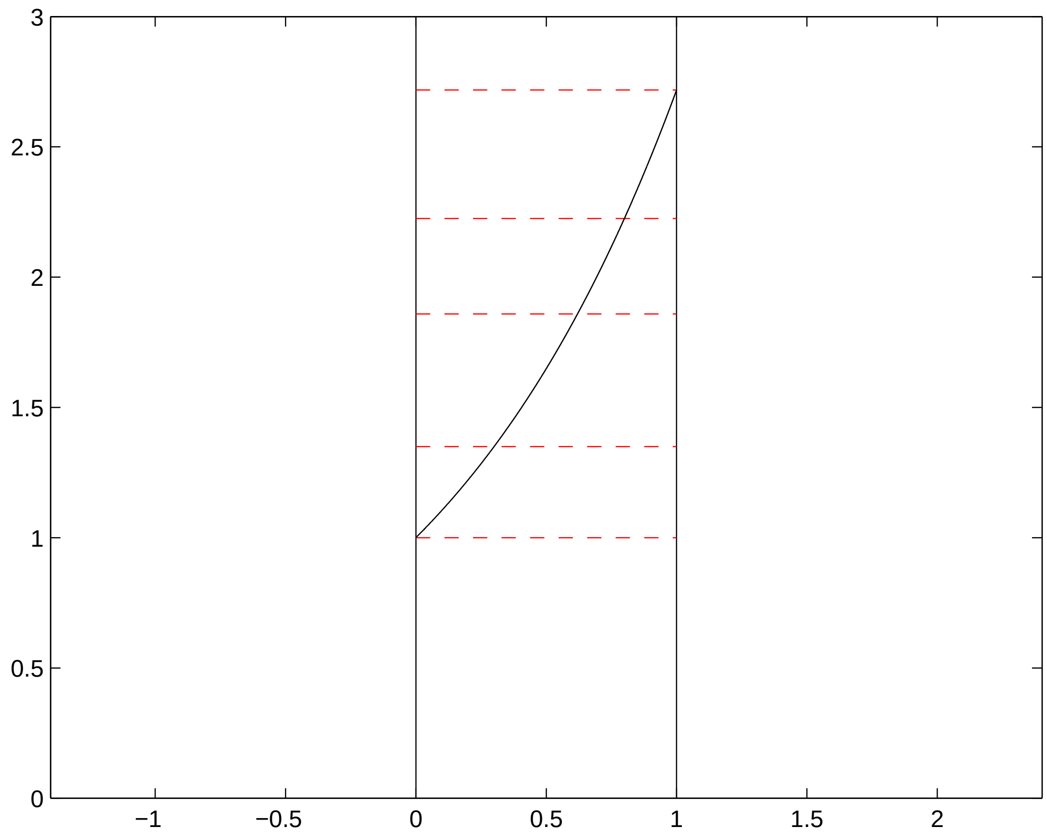
$$-2e^\xi = -(e + 1)$$

$$e^\xi = \frac{(e+1)}{2}$$

$$\xi = \ln\{\frac{(e+1)}{2}\}$$

$$\xi \approx 0.620114507$$

Example



Error trends for constant approximating polynomial

Example

Problem 9.3. Find $p_1^*(x) = \alpha + \beta x$ as the best approximation to e^x on $[0, 1]$.

We need 3 extrema. Let $E(x) = e^x - \alpha - \beta x$.

$$\frac{\partial E}{\partial x} = e^x - \beta, \quad \beta \leq 0 \rightarrow \quad \text{no real critical point}$$

$$\beta > 0 \quad e^x - \beta = 0 \rightarrow x_1 = \ln(\beta) > 0$$

$$\frac{\partial^2 E}{\partial x^2} = e^x > 0$$

Single local minimum in interval $\rightarrow x_0 = 0, \quad x_2 = 1$ are extrema.

Example

We require that $E(0) = -E(\ln\beta) = E(1)$ Therefore,

$$E(0) = E(1) \to 1 - \alpha = e - \alpha - \beta \to \beta = e - 1$$

$$E(0) = -E(\ln\beta) \to 1 - \alpha = -(\beta - \alpha - \beta\ln\beta)$$

$$= -(e - 1 - \alpha - (e-1)\ln(e-1))$$

$$\therefore \alpha = \frac{1}{2}(e - (e-1)\ln(e-1))$$

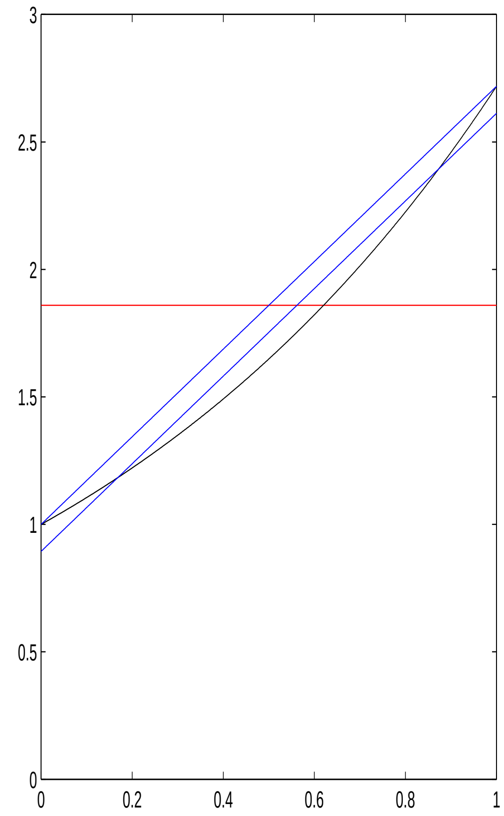
$$\alpha \approx 0.89406658 \quad \text{and} \quad \beta \approx 1.718281828$$

$$p_1^*(x) \approx 0.89406658 + 1.718281828x$$

$$\text{maximum error } = 0.106$$

$$\text{maximum error for endpoint interpolant, } p_1(x) \ = 0.212$$

Example



$$f(x) = e^x,\, p_0^*(x),\, p_1^*(x),\, p_1(x)$$

Best Polynomial

- Best polynomial also called minimax polynomial.
- No general characterization-based algorithm
- Optimal known for certain $f(x)$
- Used to create an iterative algorithm that converges to the best polynomial – Remes Algorithm
- Best polynomials, known solutions and algorithms linked through Chebyshev polynomials and points.

Interpolation Error

For the best polynomial approximation we have for some $\xi(x) \in [a, b]$

$$f(x) - p_n^*(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!} f^{(n+1)}(\xi(x))$$
$$= \frac{\omega_{n+1}(x)}{(n+1)!} f^{(n+1)}(\xi(x))$$

- Interpolating points x_k and $\xi(x)$ are not known.
- Simpler problem: ignore $f^{(n+1)}(x)$, i.e., assume boundable.
- Find points x_k so that we get minimum $\|\omega_{n+1}(x)\|_\infty$.
- Interpolating $f(x)$ at the x_k yields a **near-best** or **near-minimax** polynomial.
- Given a bound on $f^{(n+1)}(x)$ you have a bound on maximum error of near-minimax polynomial.

Best Monic Polynomial

Problem 9.4. Among all polynomials of degree $n+1$ with leading coefficient 1, find the polynomial that deviates least from 0 on $[a, b]$.

In other words, find the best approximation to $g(x) \equiv 0$ among monic polynomials of degree $\leq n+1$:

$$\min \|g(x) - (x^{n+1} - p_n(x))\|_\infty$$

A constrained minimax approximation problem?

Best Monic Polynomial

Reconsider the problem:

$$\min \|x^{n+1} - p_n(x)\|_\infty$$

- The monic polynomial can be viewed as the error function of a minimax approximation.
- Find the best approximation to $f(x) = x^{n+1}$ over the polynomials of degree $\leq n$.
- We must find an error function that is a monic polynomial of degree $n + 1$ that has $n + 2$ extrema to satisfy Chebyshev's theorem as the error function that determines $p_n(x)$.
- Equal magnitude extrema $\Rightarrow$ sinusoidal oscillation?

Best Monic Polynomial

change of variables $x = \cos\theta$

$$-1 \leq x \leq 1 \leftrightarrow 0 \leq \theta \leq \pi$$

$$\cos 0 = 1, \quad \cos\pi = -1$$

sometimes $\quad x = -\cos\theta \quad$ is used

Define $\quad t_{n+1}(\theta) = \alpha_{n+1}\cos\left[(n+1)\theta\right]$

Best Monic Polynomial

$t_{n+1}(\theta)$ takes its maximum magnitude, α_{n+1}, at $n+2$ successive points with alternating signs at the Chebyshev extrema

$$\theta_j = j\left(\frac{\pi}{n+1}\right) \quad \to t_{n+1}(\theta_j) = (-1)^j \alpha_{n+1}$$

$$\therefore t_{n+1}(x) = \alpha_{n+1} \cos\left[(n+1) \arccos x\right]$$

has the extrema pattern required for the error.

Is $t_{n+1}(x)$ a polynomial of degree $n+1$?

Best Monic Polynomial

$$T_n(x) = \cos[n\theta] = \cos[n \arccos x], \quad n = 0, 1, \ldots$$

$$T_0(x) = 1, \quad T_1(x) = x$$

$$\cos[A] + \cos[B] = 2\cos[(A - B)/2]\cos[(A + B)/2]$$

$$\cos[(n+1)\theta] + cos[(n-1)\theta] = 2\cos[\theta]\cos[n\theta], \quad n = 0, 1, \ldots$$

$$T_{n+1}(x) = 2T_1(x)T_n(x) - T_{n-1}(x), \quad n = 1, 2, \ldots$$

$$T_{n+1}(x) = 2^n x^{n+1} + q_n(x), \quad degree(q_n(x)) \leq n \quad \text{for} \quad n = 0, 1, \ldots$$

$$\therefore T_{n+1}(x) \text{ is polynomial of degree } n+1$$

Chebyshev Polynomials (First Kind)

$$\|T_n(x)\|_\infty = 1, \quad -1 \le x \le 1, \quad T_{n+1} = 2xT_n - T_{n-1}$$

$$T_0 = 1, \quad T_1 = x$$

$$T_2 = 2x^2 - 1$$

$$T_3 = 4x^3 - 3x$$

$$T_4 = 8x^4 - 8x^2 + 1$$

$$T_5 = 16x^5 - 20x^3 + 5x$$

$$T_6 = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_7 = 64x^7 - 112x^5 + 56x^3 - 7x$$

$$T_8 = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

Best Monic Polynomial

Choose $\alpha_{n+1} = 2^{-n}$

We have

$$t_{n+1}(x) = 2^{-n} T_{n+1}(x) = x^{n+1} + 2^{-n} q_n(x), \quad n = 0, 1, \ldots$$

Let $\xi_k = \cos \frac{k\pi}{n+1}, \quad k = 0, 1, \ldots, n+1$ in $[-1, 1]$

$$t_{n+1}(\xi_k) = 2^{-n} \cos k\pi = 2^{-n} (-1)^k$$

Best Monic Polynomial

- By Chebyshev's Theorem, $t_{n+1}(x)$ is error function for minimax problem $min\|x^{n+1} - p_n(x)\|_\infty$.
- We only need the error function. We do not need $p_n(x)$ explicitly.
- $\therefore$ $t_{n+1}(x)$ deviates least from $g(x) \equiv 0$ on $[-1, 1]$
- $\|t_{n+1}(x)\|_\infty = 2^{-n}$.

Corollary. *Let $p_n(x)$ be any monic polynomial of degree n. On $[-1, 1]$*

$$\|p_n\|_\infty \geq \frac{1}{2^{(n-1)}}$$

Near-minimax (Chebyshev) Approximation

On $[-1, 1]$ the factor in the error

$$\frac{\omega_{n+1}(x)}{(n+1)!} = \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!}$$

is defined by the roots of $t_{n+1}(x)$.

These are

$$t_{n+1}(x_i) = 0 = \cos\left[(n+1)\arccos x_i\right]$$

$$x_i = \cos\left[\frac{(2i+1)\pi}{(2n+2)}\right], \quad 0 \le i \le n$$

Near-minimax (Chebyshev) Approximation

For $a \leq y \leq b$ change variables to $-1 \leq x \leq 1$

$$a \leq y \leq b \leftrightarrow -1 \leq x \leq 1$$

$$x = \frac{a - 2y + b}{a - b}$$

$$y = \frac{1}{2}\left[(b-a)x + (a+b)\right]$$

$$y_i = \frac{1}{2}\left[(b-a)x_i + (a+b)\right]$$

$$\max_{a \leq y \leq b} \prod_{i=0}^{n} |y - y_i| = \frac{1}{2^n} \left|\frac{b-a}{2}\right|^{n+1}$$

Relationship with Optimal

Lemma. *(Powell, Rivlin)*

If $f(x) \in \mathcal{C}^0[-1,1]$ *and* $p_n(x)$ *is the Chebyshev interpolating polynomial of degree* n *then*

$$\|f - p_n\|_\infty \leq 4E_n^*(f), \textit{ for } n \leq 20$$

$$\|f - p_n\|_\infty \leq 5E_n^*(f), \textit{ for } n \leq 100$$

$$\textit{asymptotically } \|f - p_n\|_\infty \leq (\frac{2}{\pi}\ln n + 2)E_n^*(f)$$

Note. Recall, we have uniform convergence with $n \to \infty$ when $f(x) \in \mathcal{C}^2[-1,1]$ or Lipschitz continuous.