Solutions for Homework 2 Foundations of Computational Math 2 Spring 2012

Problem 2.1

Let $p_n(x)$ be the unique polynomial that interpolates the data

$$(x_0, f_0), \dots, (x_n, f_n)$$

Suppose that we assume the form

$$p_n(x) = \alpha_0 + \alpha_1(x - x_0) + \cdots + \alpha_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

and let

$$a = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} \quad y = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

2.1.a. Show that the constraints yield a linear system of equations

$$La = y$$

where L is lower triangular.

2.1.b. Show that the linear system yields a recurrence for the α_i that is equivalent to one of the standard definitions of the divided differences and therefore this is the Newton form of $p_n(x)$.

Solution:

Taking $n = 3$ suffices to show the pattern. We have

$$p_3(x) = \alpha_0 + \alpha_1(x - x_0) + \alpha_2(x - x_0)(x - x_1) + \alpha_3(x - x_0)(x - x_1)(x - x_2)$$

$$\begin{aligned} p_3(x_0) &= \alpha_0 + \alpha_1(x_0 - x_0) + \alpha_2(x_0 - x_0)(x - x_1) + \alpha_3(x_0 - x_0)(x - x_1)(x - x_2) \\ p_3(x_1) &= \alpha_0 + \alpha_1(x_1 - x_0) + \alpha_2(x_1 - x_0)(x_1 - x_1) + \alpha_3(x_1 - x_0)(x_1 - x_1)(x_1 - x_2) \\ p_3(x_2) &= \alpha_0 + \alpha_1(x_2 - x_0) + \alpha_2(x_2 - x_0)(x_2 - x_1) + \alpha_3(x_2 - x_0)(x_2 - x_1)(x_2 - x_2) \\ p_3(x_3) &= \alpha_0 + \alpha_1(x_3 - x_0) + \alpha_2(x_3 - x_0)(x_3 - x_1) + \alpha_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{aligned}$$

$$\begin{gathered} p_3(x_0) = \alpha_0 = y_0 \\ p_3(x_1) = \alpha_0 + \alpha_1(x_1 - x_0) = y_1 \\ p_3(x_2) = \alpha_0 + \alpha_1(x_2 - x_0) + \alpha_2(x_2 - x_0)(x_2 - x_1) = y_2 \\ p_3(x_3) = \alpha_0 + \alpha_1(x_3 - x_0) + \alpha_2(x_3 - x_0)(x_3 - x_1) + \alpha_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = y_3 \end{gathered}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & 0 \\ 1 & (x_3 - x_0) & (x_3 - x_0)(x_3 - x_1) & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

$$La = y$$

Since $x_i \neq x_j$ when $i \neq j$ we have nonzero elements on the diagonal of the lower triangular matrix L implying it is nonsingular. The solution a is therefore unique given the interpolation constraints.

It is easy to see that

$$\begin{gathered} \alpha_0 = y_0 \\ \alpha_1 = \frac{y_1 - y_0}{x_1 - x_0} = y[x_0, x_1] \end{gathered}$$

Therefore, we also have

$$\begin{gathered} \alpha_0 + \alpha_1(x_2 - x_0) + \alpha_2(x_2 - x_0)(x_2 - x_1) = y_2 \\ y_0 + y[x_0, x_1](x_2 - x_0) + \alpha_2(x_2 - x_0)(x_2 - x_1) = y_2 \\ p_2(x_2) + \alpha_2(x_2 - x_0)(x_2 - x_1) = y_2 \end{gathered}$$

$$\alpha_2 = \frac{y_2 - p_1(x_2)}{(x_2 - x_0)(x_2 - x_1)} = y[x_0, x_1, x_2]$$

where $p_1(x)$ is the unique linear polynomial that interpolates (x_i, y_i) for $i = 0, 1$ given in Newton form.

Similarly we have

$$\alpha_0 + \alpha_1(x_3 - x_0) + \alpha_2(x_3 - x_0)(x_3 - x_1) + \alpha_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = y_3$$
$$\Downarrow$$
$$\alpha_3 = \frac{y_3 - p_2(x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = y[x_0, x_1, x_2, x_3]$$

where $p_2(x)$ is the unique quadratic polynomial that interpolates (x_i, y_i) for $i = 0, 1, 2$ given in Newton form.

This is easily turned into a general induction argument that shows the α_i satisfy the standard definition of a divided difference

$$y[x_0, x_1, \ldots, x_n] = \frac{y_n - p_{n-1}(x_n)}{(x_n - x_0) \cdots (x_n - x_{n-1})}$$

where $p_{n-1}(x)$ is the unique polynomial of degree $n - 1$ that interpolates the points (x_i, y_i) for $i = 0, \ldots, n - 1$.

Problem 2.2

Consider a polynomial

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$p_n(\gamma)$ can be evaluated using Horner's rule (written here with the dependence on the formal argument x more explicitly shown)

$c_n(x) = \alpha_n$
for $i = n - 1 : -1 : 0$
$\qquad c_i(x) = x c_{i+1}(x) + \alpha_i$
end

$p_n(x) = c_0(x)$

Note that when evaluating $x = \gamma$ the algorithm produces $n+1$ constants $c_0(\gamma), \ldots, c_n(\gamma)$ one of which is equal to $p_n(\gamma)$.

2.2.a

Suppose that Horner's rule is applied to evaluate $p_n(\gamma)$ and that the constants $c_0(\gamma), \ldots, c_n(\gamma)$ are saved. Show that

$$p_n(x) = (x - \gamma)q(x) + p_n(\gamma)$$
$$q(x) = c_1(\gamma) + c_2(\gamma)x + \cdots + c_n(\gamma)x^{n-1}$$

Solution:

The pattern is clear from considering $p_3(x)$.

We have

$$p_3(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

$$\begin{aligned} c_0(\gamma) &= \gamma c_1(\gamma) + \alpha_0 \\ c_1(\gamma) &= \gamma c_2(\gamma) + \alpha_1 \\ c_2(\gamma) &= \gamma c_3(\gamma) + \alpha_2 \\ c_3(\gamma) &= \alpha_3 \end{aligned}$$

$$\begin{aligned} c_0(\gamma) - \gamma c_1(\gamma) &= \alpha_0 \\ x c_1(\gamma) - x\gamma c_2(\gamma) &= \alpha_1 x \\ x^2 c_2(\gamma) - x^2 \gamma c_3(\gamma) &= \alpha_2 x^2 \\ x^3 c_3(\gamma) &= \alpha_3 x^3 \end{aligned}$$

$$p_3(x) = (c_0(\gamma) - \gamma c_1(\gamma)) + (x c_1(\gamma) - x\gamma c_2(\gamma)) + (x^2 c_2(\gamma) - x^2\gamma c_3(\gamma)) + x^3 c_3(\gamma)$$

$$\begin{aligned} \therefore p_3(x) &= c_0(\gamma) + (x-\gamma)(c_1(\gamma) + x c_2(\gamma) x^2 \gamma c_3(\gamma)) \\ &= p_3(\gamma) + (x-\gamma) q(x) \end{aligned}$$

2.2.b

Suppose that Horner's rule is applied to evaluate $p_n(\gamma)$ and that the constants $c_0^{(1)}(\gamma), \ldots, c_n^{(1)}(\gamma)$ are saved to define $q_{(1)}(x) = c_1^{(1)}(\gamma) + c_2^{(1)}(\gamma)x + \cdots + c_n^{(1)}(\gamma)x^{n-1}$. Suppose further that Horner's rule is applied to evaluate $q_{(1)}(\gamma)$ and that the constants $c_0^{(2)}(\gamma), \ldots, c_{n-1}^{(2)}(\gamma)$ are saved to define $q_{(2)}(x) = c_1^{(2)}(\gamma) + c_2^{(2)}(\gamma)x + \cdots + c_{n-1}^{(2)}(\gamma)x^{n-2}$. This can continue until Horner's rule is applied to evaluate $q_{(n)}(\gamma)$ and $q_{(n+1)}(x) = 0$, i.e., there are no constants other than $c_0^{(n)}(\gamma)$ produced.

Show that

$$\begin{aligned} q_{(1)}(\gamma) &= p_n'(\gamma) \\ q_{(2)}(\gamma) &= p_n''(\gamma)/2 \\ q_{(3)}(\gamma) &= p_n'''(\gamma)/3! \\ &\vdots \\ q_{(n-1)}(\gamma) &= p_n^{(n-1)}(\gamma)/(n-1)! \\ q_{(n)}(\gamma) &= p_n^{(n)}(\gamma)/n! \end{aligned}$$

and therefore form the coefficients of the Taylor form of $p_n(x)$

$$p_n(x) = p_n(\gamma)+(x-\gamma)p_n'(\gamma)+\frac{(x-\gamma)^2}{2}p_n''(\gamma)+\frac{(x-\gamma)^3}{3!}p_n'''(\gamma)\cdots+\frac{(x-\gamma)^{n-1}}{(n-1)!}p_n^{(n-1)}(\gamma)+\frac{(x-\gamma)^n}{n!}p_n^{(n)}(\gamma)$$

Solution:
We have

$$\begin{gathered} p_n(x) = p_n(\gamma) + (x-\gamma)q_{(1)}(x) \\ p_n'(x) = (x-\gamma)q_{(1)}'(x) + q_{(1)}(x) \\ \therefore p_n'(\gamma) = q_{(1)}(\gamma) \end{gathered}$$

The subsequent applications of Horner's rule yield

$$\begin{gathered} q_{(i)}(x) = q_{(i)}(\gamma) + (x-\gamma)q_{(i+1)}(x) \\ q_{(i)}'(x) = (x-\gamma)q_{(i+1)}'(x) + q_{(i+1)}(x) \\ \therefore q_{(i)}'(\gamma) = q_{(i+1)}(\gamma) \end{gathered}$$

To get the Taylor's form of $p_n(x)$ consider $n = 3$ to see the pattern of substitution that can be applied for any n. We have

$$\begin{gathered} p_3(x) = p_3(\gamma) + (x-\gamma)q_{(1)}(x) \\ = p_3(\gamma) + (x-\gamma)(q_{(1)}(\gamma) + (x-\gamma)q_{(2)}(\gamma)) \\ = p_3(\gamma) + (x-\gamma)q_{(1)}(\gamma) + (x-\gamma)^2 q_{(2)}(\gamma) \\ = p_3(\gamma) + (x-\gamma)q_{(1)}(\gamma) + (x-\gamma)^2(q_{(2)}(\gamma) + (x-\gamma)q_{(3)}(\gamma)) \\ = p_3(\gamma) + (x-\gamma)q_{(1)}(\gamma) + (x-\gamma)^2 q_{(2)}(\gamma) + (x-\gamma)^3 q_{(3)}(\gamma) \end{gathered}$$

Since the Taylor' form is unique we have the the constants must be equal, i.e.,

$$q_{(i)}(\gamma) = \frac{p^{(i)}(\gamma)}{i!}$$

One can compute the derivatives of the expression

$$p_3(x) = p_3(\gamma) + (x-\gamma)q_{(1)}(\gamma) + (x-\gamma)^2 q_{(2)}(\gamma) + (x-\gamma)^3 q_{(3)}(\gamma)$$

and evaluate them at γ to get the same result.

Problem 2.3

Text exercise 8.10.8 on page 376
Solution:

We have two polynomials of degree n that we must show are identical. The first polynomial is defined by interpolation conditions:

$$p_n^{(k)}(x_0) = f^{(k)}(x_0), \quad 0 \le k \le n$$

where the superscript denotes the order of the derivative. The second polynomial is the order n Taylor expansion of $f(x)$ around x_0.

$$q_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

There are various ways to show $p_n(x) = q_n(x)$. First we show that the $n+1$ conditions on $p_n(x)$ defines a unique polynomial of degree n. For example, take $n = 4$ take $p_4(x)$ in terms of the monomial basis:

$$p_4(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$$

The conditions

$$p_n^{(k)}(x_0) = f^{(k)}(x_0), \quad 0 \le k \le n$$

can be expressed as a linear system of equations relating the coefficients α_i, $0 \le i \le 4$ to the values $f(x_0)^{(k)}, \quad 0 \le k \le 4$. The system is:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 \\ 0 & 0 & 2 & (3*2)x_0 & (4*3)x_0^2 \\ 0 & 0 & 0 & 3! & (4*3*2)x_0 \\ 0 & 0 & 0 & 0 & 4! \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f'(x_0) \\ f''(x_0) \\ f^{(3)}(x_0) \\ f^{(4)}(x_0) \end{pmatrix}$$

In general, the matrix is an upper triangular matrix with diagonal elements $\mu_{i,i} = (i-1)!$. The matrix is therefore nonsingular and the polynomial is unique.

To show that $q_n(x) = p_n(x)$ we now need only show that $q_n(x)$ satisfies the same $n+1$ interpolation conditions that $p_n(x)$ satisfies. We have

$$q_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

It is easily seen that when evaluating $q_n^{(k)}(x_0)$ all terms except the constant vanish. From scaling the linear system above these constant terms can be shown to be

$$\frac{k!}{k!} f^{(k)}(x_0) = f^{(k)}(x_0)$$

for $0 \le k \le n$. So $q_n(x)$ satisfies the same conditions as $p_n(x)$ and by uniqueness is the same polynomial.