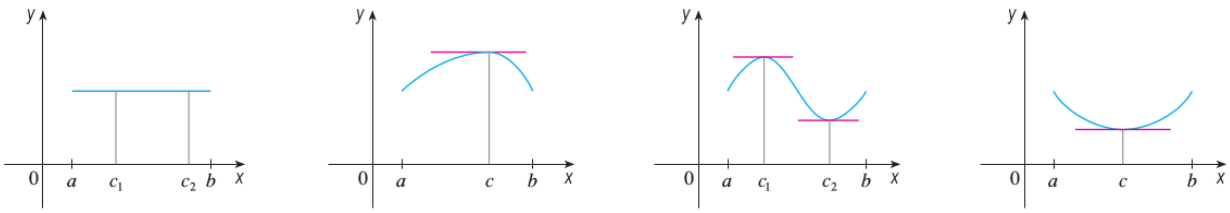
4.2: The Mean Value Theorem

Rolle's Theorem: If f is function that is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there is a number c in (a, b) such that $f'(c) = 0$



Example 1. *Let $s = y(t)$ be the position of an object thrown upwards at time t. If the object is in the same position at two different times, that is, if $f(t_0) = f(t_1)$ for $t_0 \neq t_1$, what can you conclude about the velocity?*

Example 2. *Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.*

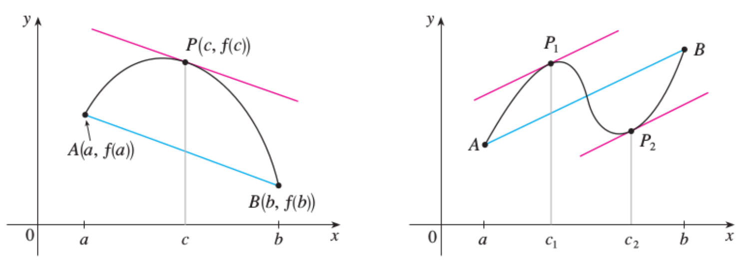
(a) $f(x) = 5 - 12x + 3x^2$, $[1, 3]$

(b) $f(x) = \sqrt{x} - \frac{1}{3}x$, $[0, 9]$

Example 3. *Prove that $x^3 + x - 1 = 0$ has exactly one real root.*

Example 4. *Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?*

The Mean Value Theorem: If f is a function that is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Example 5. *Let $s = f(t)$ be the position in miles of car at time t in hours. If the car traveled 120 mi in the first 2 hours, what can you conclude about the instaneous velocity of the car?*

Example 6. *Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.*

(a) $f(x) = x^3 - x$, $[0, 2]$

(b) $f(x) = \ln x$, $[1, 4]$

(c) $f(x) = \frac{1}{x}$, $[1, 3]$

Example 7. *Let f be a continuous and differentiable function. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?*

Theorem: If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Example 8. *Prove the identity* $\tan^{-1} x + \cot^{-1} x = \pi/2$.