Set 23: Rational Interpolation – Part 1

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The Functions

Definition 23.1. The function

$$r_{nm}(x) = \frac{p_n(x)}{q_m(x)}$$

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$$q_m(x) = \beta_0 + \beta_1 x + \cdots + \beta_m x^m$$

is a rational function. Note that there are $n + m + 1$ degrees of freedom since multiplying p and q by the same constant does not affect r and does not alter the number of coefficients in each.

We consider using rational functions to interpolate and approximate a function $f(x)$.

The Functions

$r_{nm}(x)$ has many representations using pairs of polynomials with different degrees:

$$r_{23}(x) = \frac{2x^2 + x}{2x^3 + x^2 + x + 1} = \frac{x(2x+1)}{(x^2+1)(2x+1)} = \frac{x}{x^2+1} = r_{12}(x)$$

Definition 23.2. A rational function $r_{nm}(x) = p_n(x)/q_m(x)$ is relatively prime if the numerator and denominator polynomials have no common factor other than a constant.

The Functions

So we have an equivalence class on the set of polynomial pairs:

Definition 23.3. $(p_n(x), q_m(x)) \sim (p_s(x), q_t(x))$ if and only if

$$\frac{p_n(x)}{q_m(x)} = \frac{p_s(x)}{q_t(x)}$$

$$p_n(x)q_t(x) - p_s(x)q_m(x) = 0$$

Each equivalence class $\mathcal{E}(r(x))$ has a unique relatively prime member.

Distinct Point Rational Interpolation Problem

The simplest problem revisits one we solved earlier with a unique polynomial

Given $d + 1$ pairs of data (x_i, f_i) find $r_{nm}(x)$ such that

$$r_{nm}(x_i) = f_i \quad 0 \leq i \leq d$$

where $x_i \neq x_j$.

Distinct Point Rational Interpolation Problem

Of course we know that there is a solution with $n = d$ and $m = 0$ given by the Lagrange interpolating polynomial $P_d(x)$.

Other interpolation rational functions also exist

$$r(x) = P_d(x) + \phi(x) \prod_{i=0}^{d} (x - x_i)$$

where $\phi(x)$ is any rational function whose denominator is finite at all x_i.

We need constraints!

A Necessary Condition

Theorem 23.1. *Given n and m, if $d = n + m$ and*

$$r_{nm}(x) = \frac{p_n(x)}{q_m(x)}, \quad p_n(x) = \sum_{i=0}^{n} \alpha_i x^i, \quad q_m(x) = \sum_{i=0}^{m} \beta_i x^i$$

$$r_{nm}(x_i) = f_i \quad 0 \le i \le d$$

then the homogeneous linear system

$$p_n(x_i) - f_i q_m(x_i) = 0 \quad 0 \le i \le d$$

or, in matrix form, $\quad Sv = 0, \quad S \in \mathbb{R}^{n+m+1 \times n+m+2}$

$$v^T = (\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_m)$$

has a nonzero solution, i.e., $\mathcal{N}(S) \neq \emptyset$.

Example

Let $n = 1$ and $m = 2$, use monomial basis (others could be used) and consider the points

$$\left\{(0,0),(1,1/2),(2,2/5),(3,3/10)\right\}$$

$$\begin{pmatrix} 1 & x_0 & -f_0 & -f_0x_0 & -f_0x_0^2 \\ 1 & x_1 & -f_1 & -f_1x_1 & -f_1x_1^2 \\ 1 & x_2 & -f_2 & -f_2x_2 & -f_2x_2^2 \\ 1 & x_3 & -f_3 & -f_3x_3 & -f_3x_3^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \hline \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

Let $n = 1$ and $m = 2$ and consider the points

$$\left\{(0,0),(1,1/2),(2,2/5),(3,3/10)\right\}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1/2 & -1/2 & -1/2 \\ 1 & 2 & -2/5 & -4/5 & -8/5 \\ 1 & 3 & -3/10 & -9/10 & -27/10 \end{pmatrix} \left(\begin{array}{c} \alpha_0 \\ \alpha_1 \\ \hline \beta_0 \\ \beta_1 \\ \beta_2 \end{array}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1/2 & -1/2 & -1/2 \\ 1 & 2 & -2/5 & -4/5 & -8/5 \\ 1 & 3 & -3/10 & -9/10 & -27/10 \end{pmatrix}$$

$$Se_2 + Se_3 + Se_5 = 0 \to \left(\begin{array}{c} 0 \\ \gamma \\ \hline \gamma \\ 0 \\ \gamma \end{array}\right) \in \mathcal{N}(S)$$

Example

$$\therefore r_{12}(x) = \frac{\gamma x}{\gamma + \gamma x^2} = \frac{x}{1+x^2}$$

Check conditions.

$$r_{12}(0) = 0, \quad r_{12}(1) = 1/2, \quad r_{12}(2) = 2/5, \quad r_{12}(3) = 3/10$$

Interpolation problem solved.

Another Example

Let $n = m = 1$ and consider the points $\big\{(0,1),(1,2),(2,2)\big\}$.

$$\begin{pmatrix} 1 & x_0 & -f_0 & -f_0 x_0 \\ 1 & x_1 & -f_1 & -f_1 x_1 \\ 1 & x_2 & -f_2 & -f_2 x_2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -2 & -2 \\ 1 & 2 & -2 & -4 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Simple Constraint Example

Easy to get a 1-dimensional subspace in the null space, for any $\gamma \in \mathbb{R}$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -2 & -2 \\ 1 & 2 & -2 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r_{11}(x) = \frac{\alpha_0 + \alpha_1 x}{\beta_0 + \beta_1 x} = \frac{2\gamma x}{\gamma x} = 2$$

$r_{11}(x)$ solves $Sv = 0$ but **does not** interpolate all points since $r_{11}(0) = 2 \neq 1 = f_0$

Simple Constraint Example

- The condition $Sv = 0$ is necessary but not sufficient.
- For the example, $p_1(x)$ and $q_1(x)$ have a common nonconstant factor, x, i.e., $p_1(x)/q_1(x)$ is not relatively prime.
- This is related to the cause of the problem.
- It is possible to get a form that is not relatively prime and solves the interpolation problem.
- Need some more theory.

The Theory

Theorem 23.2. *Given* n, m *and* $n+m+1$ *data pairs* (x_i, f_i), *we have:*

- *The associated homogeneous system* $Sv = 0$ *with* $S \in \mathbb{R}^{n+m+1 \times n+m+2}$ *always has nontrivial solutions and each solution* $r_{nm}(x) = p_n(x)/q_m(x)$ *defines a rational function, i.e.,* $q_m(x) \not\equiv 0$.

- *If* v_1 *and* v_2 *are nontrivial solutions of* $Sv = 0$ *then they define the same rational function.*

- *If* $v_1 \neq 0$ *and* $v_2 \neq 0$ *define the same rational function and* $Sv_1 = 0$ *it does not follow that* $Sv_2 = 0$.

The Theory

Consider the second condition of the lemma. Let v_1 and v_2 be nontrivial solutions of $Sv = 0$ with $v_1 \leftrightarrow r^{(1)}(x)$ and $v_2 \leftrightarrow r^{(2)}(x)$. We have

$$r^{(1)}(x) = \frac{p^{(1)}(x)}{q^{(1)}(x)}, \quad r^{(2)}(x) = \frac{p^{(2)}(x)}{q^{(2)}(x)}$$

$$0 \le i \le d, \quad p^{(1)}(x_i) = f_i q^{(1)}(x_i), \;\; p^{(2)}(x_i) = f_i q^{(2)}(x_i)$$

$$f_i q^{(1)}(x_i) q^{(2)}(x_i) - f_i q^{(2)}(x_i) q^{(1)}(x_i) = 0$$

$$p^{(1)}(x_i) q^{(2)}(x_i) - p^{(2)}(x_i) q^{(1)}(x_i) = 0$$

$$deg\left(p^{(1)}(x) q^{(2)}(x) - p^{(2)}(x) q^{(1)}(x)\right) = d \quad \text{with} \quad d+1 \quad \text{roots}$$

$$\therefore p^{(1)}(x) q^{(2)}(x) - p^{(2)}(x) q^{(1)}(x) \equiv 0 \to r^{(1)}(x) = r^{(2)}(x)$$

The Theory

The third result of the lemma is proven by the second example. So we have if $v \in \mathcal{N}(S)$ and $v \leftrightarrow r(x)$ then

$$\mathcal{N}(S) \subseteq \mathcal{E}(r(x))$$

but it may be that

$$\mathcal{N}(S) \neq \mathcal{E}(r(x))$$

The Theory

Theorem 23.3. *If* $r_{nm}(x) \sim v \in \mathcal{N}(S)$ *solves the interpolation problem, i.e.,*

$$r(x_i) = f_i, \quad 0 \le i \le d$$

then the relatively prime form

$$\tilde{r}(x) = \tilde{p}(x)/\tilde{q}(x) \leftrightarrow \tilde{v}$$

also solves the interpolation problem and therefore $\tilde{v} \in \mathcal{N}(S)$.

Proof. This follows immediately from $r(x) = \tilde{r}(x)$ and the necessity of the vector associated with an interpolating rational function being in $\mathcal{N}(S)$. $\square$

Sufficient Conditon for Solution

So if we find $v \in \mathcal{N}(S)$ with $v \leftrightarrow r(x) = p(x)/q(x)$ so that

$$r(x_i) = f_i, \quad 0 \leq i \leq d$$

then all $(\hat{p}, \hat{q}) \sim (p, q)$ solve the interpolation problem.

Note that $r(x) = p(x)/q(x)$ may not be relatively prime.

Sufficient Conditon for No Solution

We have the following characterization of an interpolation problem that is not solvable for the given n,m and data.

Theorem 23.4. *If a solution v of $Sv = 0$ defines an $r_{nm}(x)$ that does not intepolate all $n + m + 1$ points then there is no such $r_{st}(x)$ with $s \leq n$ and $t \leq m$.*

Proof. Since by Theorem 23.2 $\forall \hat{v} \in \mathcal{N}(S)$ we have $\hat{v} \leftrightarrow \hat{r}(x) = r_{nm}(x)$, it follows that no vector in $\mathcal{N}(S)$ generates a function that solves the interpolation problem. Since by Theorem 23.1, any solution to the interpolation problem must be in $\mathcal{N}(S)$, no such solution exists. $\square$

Essentially this means that there must be some $\hat{v} \leftrightarrow \hat{r}(x) = r_{nm}(x)$ such that $v \notin \mathcal{N}(S)$.

Sufficient Conditon for No Solution

In particular, this means that the relatively prime form is not in $\mathcal{N}(S)$.

Theorem 23.5. *If a solution v of $Sv = 0$ with $v \leftrightarrow r_{nm}(x)$ is such that $r_{nm}(x) = p_n(x)/q_m(x)$ does not intepolate all $n + m + 1$ points then*

- $p_n(x)/q_m(x)$ *is not relatively prime;*
- *or equivalently, the relatively prime form of $r_{nm}(x) = \tilde{p}(x)/\tilde{q}(x) \leftrightarrow \tilde{v}$ is such that $\tilde{v} \notin \mathcal{N}(S)$.*

Sufficient Conditon for No Solution

Proof. To see that $r_{nm}(x) = p_n(x)/q_m(x)$ is not relatively prime consider each x_i. We assume $r_{nm}(x_i) < \infty$. If $q_m(x_i) \neq 0$ then it follows that $p_n(x_i)/q_m(x_i) = f_i$ so (x_i, f_i) cannot be a point where the interpolation condition is not satisfied.

So consider an (x_i, f_i) is not interpolated. We have, by the reasoning above,

$$p_n(x_i)/q_m(x_i) \neq f_i \rightarrow q_m(x_i) = 0$$

$$\text{also,} \quad v \in \mathcal{N}(S) \rightarrow p_n(x_i) - f_i q_m(x_i) = 0,$$

$$\therefore p_n(x_i) = 0$$

So $p_n(x)$ and $q_m(x)$ share a factor $(x - x_i)^k$ with $k \geq 1$ and $r_{nm}(x) = p_n(x)/q_m(x)$ is not relatively prime. □

A Necessary and Sufficient Condition

Theorem 23.6. *Given* n, m *and* $n+m+1$ *data pairs* (x_i, f_i), *if* $Sv_1 = 0$ *yields a rational function,* $r_{nm}^{(1)}(x)$, *then there exists a rational interpolant for all* $n+m+1$ *points if and only if there is a solution* $Sv_2 = 0$ *that yields a relatively prime rational function,* $r_{nm}^{(2)}(x)$, *that is equivalent to* $r_{nm}^{(1)}(x)$.

Proof. This follows immediately from Theorem 23.3 and Theorem 23.5 .

$\square$

A Necessary and Sufficient Condition

Corollary. *If S has full rank then $r_{nm}(x)$ interpolates all $n+m+1$ points if and only if $v \leftrightarrow r_{nm}(x)$ is relatively prime. Note that in this case the null space has dimension 1 so all solutions are equivalent.*

A Familiar Example

Let $n = m = 2$ and consider the points

$$\{(-2, 1/5), (-1, 1/2), (0, 1), (1, 1/2), (2, 1/5)\}$$

$$\begin{pmatrix} 1 & x_{-2} & x_{-2}^2 & -f_{-2} & -f_{-2}x_{-2} & -f_{-2}x_{-2}^2 \\ 1 & x_{-1} & x_{-1}^2 & -f_{-1} & -f_{-1}x_{-1} & -f_{-1}x_{-1}^2 \\ 1 & x_0 & x_0^2 & -f_0 & -f_0x_0 & -f_0x_0^2 \\ 1 & x_1 & x_1^2 & -f_1 & -f_1x_1 & -f_1x_1^2 \\ 1 & x_2 & x_2^2 & -f_2 & -f_2x_2 & -f_2x_2^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A Familiar Example

Substituting the point values yields

$$\begin{pmatrix} 1 & -2 & 4 & -1/5 & 2/5 & -4/5 \\ 1 & -1 & 1 & -1/2 & 1/2 & -1/2 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1/2 & -1/2 & -1/2 \\ 1 & 2 & 4 & -1/5 & -2/5 & -4/5 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A Familiar Example

Note that $Se_1 + Se_4 + Se_6 = 0$ so

$$Sv = 0$$

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \to r_{22}(x) = \frac{1}{1+x^2}$$

$r_{22}(x)$ is relatively prime. So we can get the exact Runge function as opposed to the divergence with uniform points we get with polynomial interpolation.

Simple Algorithm

The matrix $S \in \mathbb{R}^{n+m+1 \times n+m+2}$ is rectangular – short and fat. Consider S^T.

- Find $Q_k \in \mathbb{R}^{n+m+2 \times k}$ such that $Q_k^T Q_k = I$, $k = rank(S^T)$ and $\mathbb{R}(Q_k) = \mathbb{R}(S^T)$.
- Choose random $v \in \mathbb{R}^{n+m+2}$
- Compute $\hat{v} = (I - Q_k Q_k^T)v$. If $\|\hat{v}\|$ is too small choose new v and repeat until large enough.
- $Q^T \hat{v} = 0 \rightarrow \hat{v} \in \mathcal{N}(S)$ so find $r(x) = p(x)/q(x) \leftrightarrow \hat{v}$.
- Check $r(x_i) = f_i$ for $0 \leq i \leq d$.

In practice, rank revealing factorization can be costly and we are often interested in keeping n and m small or otherwise constraining $r(x)$. So many other approaches are described in the literature.