Set 22: Ordinary Differential Equations: Stiffness

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Stiffness: Stability vs. Accuracy

- A-stable methods are neither always effective nor necessary in practice.
- Stability needed for λ with large negative real part, not by the imaginary axis.
- As $Re(\lambda) \to -\infty$ the solution $y(t) = ce^{\lambda t}$ damps at an increasing rate, i.e., $y'(t) \to -\infty$.
- The numerical method should mimic this, i.e., for the test problem

$$\lim_{Re(\lambda) \to -\infty} \left[\frac{y_n}{y_{n-1}} \right] = 0$$

- Such a method has stiff decay.
- Absolute stability is not strong enough.

Stiffness: Stability vs. Accuracy

Stiffness is not a rigorous mathematical idea. It depends on

- the initial value problem
- interval of integration – size and location
- accuracy requirements
- (absolute) stability requirements

Stiffness: Stability vs. Accuracy

Definition 22.1. (informal) An IVP is stiff in an interval of integration if the stepsize required to maintain stability is much smaller than the stepsize required to maintain accuracy.

- The source of the problem is the relative behavior of the solution desired and nearby integral curves.
- If the integral curves decay rapidly relative to the scale of the interval of integration and the smoothness of the desired solution allows stepsizes on a much longer scale then the problem is stiff.
- Stiffness arises from the influence of transient behavior of nearby integral curves that is no longer present in the solution of interest.

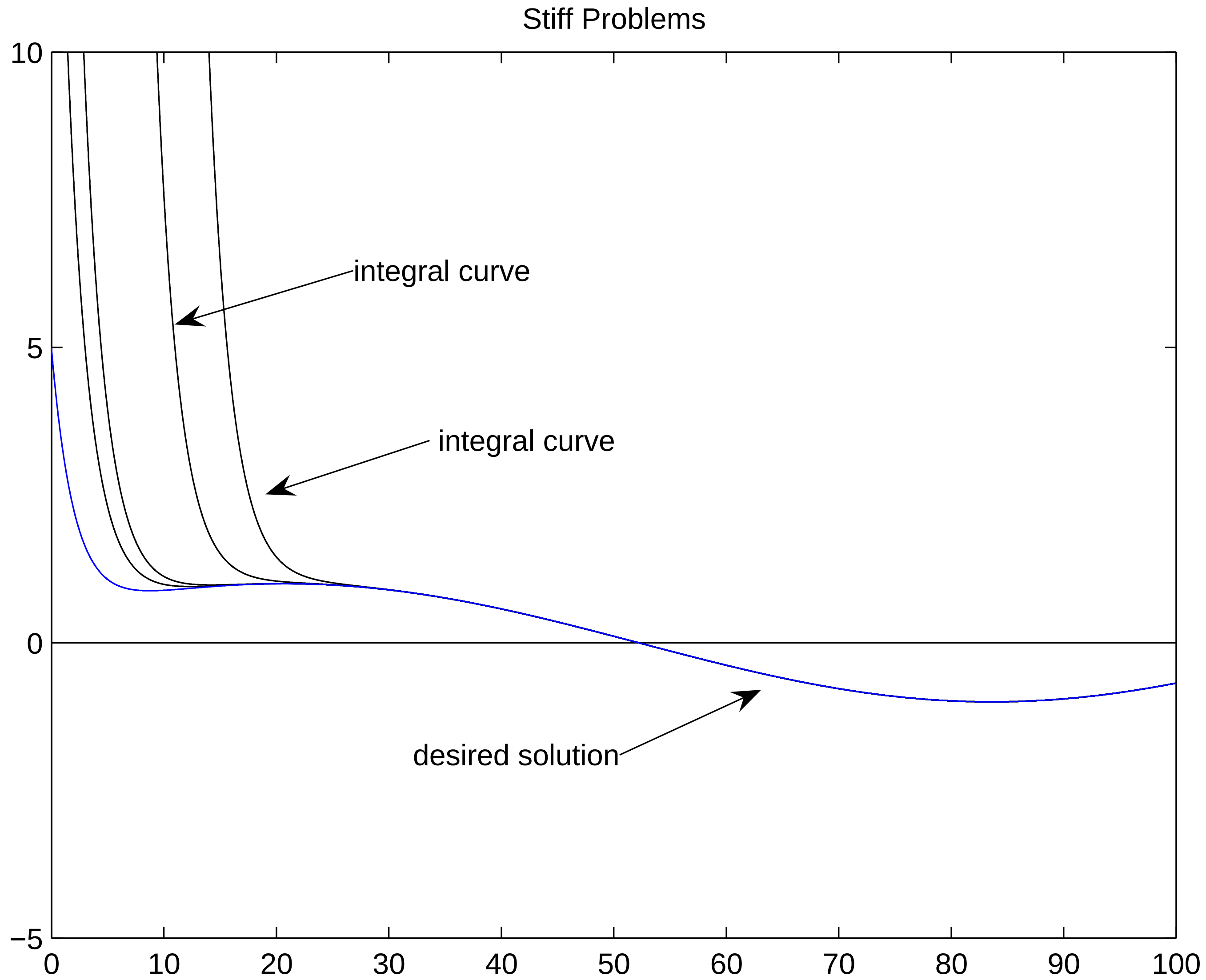
Stiffness: Stability vs. Accuracy

- Typical in multiscale problems and measured for systems by the eigenvalues of the local Jacobian.
- Systems are not required.
- A scalar IVP can be stiff. (Components of the solution have multiple scales.)

Example 22.1.

$$f = \lambda(y - F(t)) + F'(t) \quad y(0) = y_0$$

$$y(t) = (y_0 - F(0))e^{\lambda t} + F(t)$$


Stiff Problems
10
5
0
−5
0
10
20
30
40
50
60
70
80
90
100
integral curve
integral curve
desired solution

Stiffness and Efficiency

Three typical situations:

1. Transient regions alternating with smooth regions where accuracy is required by the user only in the smooth regions.

2. Transient regions alternating with smooth regions where accuracy is required by the user over the entire interval of integration.

3. An initial transient region where accuracy is of required by the user followed by a smooth region where accuracy is required. This is an important special case of (1). Finding a steady state for the system is a typical example.

Stiffness and Efficiency

- When accuracy is required in a transient region, the stepsize due to accuracy is usually smaller than the stepsize due to stability.
- When subsequently moving into smooth region, a “nonstiff” method increasing the stepsize will usually encounter stability limits on h
- When accuracy is not required in a transient region, the key is to exploit methods that damp rapidly solution or integral curve components with rapid decay.

Example

Consider situation (3) for

$$f = -100(y - \sin t) + \cos t \quad y(0) = 1$$

$$y(t) = e^{-100t} + \sin t$$

- Forward Euler: $h = 0.05$ on $0 \leq t \leq 0.5$

  unstable

- Backward Euler: $h = 0.1,\ 0.05$ on $0 \leq t \leq 1.0$

  transient errors damped rapidly and accurate in smooth region

- Trapezoidal Rule: $h = 0.1,\ 0.05$ on $0 \leq t \leq 1.0$

  transient errors damped slowly and then accurate in smooth region; once accurate; it is more accurate than Backward Euler for same h.

Example

Consider situation (2) for

$$f = -100(y - \sin t) + \cos t \quad y(0) = 1$$

$$y(t) = e^{-100t} + \sin t$$

- Backward Euler: $h = 0.01$ on $0 \le t \le 0.5$

  essentially accurate in both regions but clearly adaptive step needed for efficiency

- Trapezoidal Rule: $h = 0.01$ on $0 \le t \le 0.5$

  essentially accurate in both regions but clearly adaptive step needed for efficiency; more accurate than Backward Euler.

- Forward Euler: $h = 0.05$ on $0 \le t \le 0.5$

  unstable

Stiffness

- Trapezoidal Rule seems ideal but it has problems since

$$\text{As } Re(\lambda) \to -\infty, \quad \frac{y_n}{y_{n-1}} = \frac{2 + h\lambda}{2 - h\lambda} \to -1$$

- Backward Euler has stiff decay since

$$\text{As } Re(\lambda) \to -\infty, \quad \frac{y_n}{y_{n-1}} = \frac{1}{1 - h\lambda} \to 0$$

- Backward Euler and similar methods can step over transients in desired solution as well and still get accurate values in stiff intervals.
- Backward Euler is superstable and can supress integral curves that are growing and oscillatory.
- Highly oscillatory problems **are not stiff**. Other methods are required.

Stiffness and Stability

- BDF 1 and 2 A-stable, strongly stable, stiff decay
- BDF's trade absolute stability for stiff decay
- $A(\alpha)-$ stability is stability in a wedge around real axis – still too restrictive

Issues Affecting Method for Stiff Problems

- Implicit methods required (at least on the stiff portion of the system)
- Functional iteration is not efficient when solving for y_n, e.g.,

$$y_n^{(i+1)} = h\beta_0 f(t_n, y_n^{(i)}) - \sum_{j=1}^{k} \alpha_j y_{n-j} + h \sum_{j=1}^{k} \beta_j f_{n-j}$$

$$\therefore \|h\beta_0 \frac{\partial f}{\partial y}\| \leq h\beta_0 L < 1$$

- Newton's method or similar method must be used.
- Jacobian reevaluation and factorization (if direct methods used) required for exact Newton so inexact or quasi-Newton methods used.

Issues Affecting Method for Stiff Problems

- estimate of Lipschitz constant or norm of Jacobian, L, needed
- absolute stability region shape used to set $h_{stab}L \in \mathcal{R}$ for stiff and nonstiff methods
- $h_{fi}\beta_0 L < 1$ set for nonstiff method using functional iteration
- predictor-corrector difference to estimate error and h_{err} set so user tolerance satisfied.
- all of this done for stiff and nonstiff methods of order k, $k-1$, $k+1$
- new step selected and occasionally new order selected and less frequently new method selected
- automatic step, order, and method selection