Homework 7 Foundations of Computational Math 2 Spring 2012

Solutions will be posted 2/29/12

Problem 7.1

For this problem, consider the space $\mathcal{L}^2[-1,1]$ with inner product and norm

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx \text{ and } \|f\|^2 = (f,f)$$

Let $P_i(x)$, for $i = 0, 1, \dots$ be the Legendre polynomials of degree i and let $n+1-$st have the form

$$P_{n+1}(x) = \rho_n (x - x_0)(x - x_1) \cdots (x - x_n)$$

i.e., x_i for $0 \le i \le n$ are the roots of $P_{n+1}(x)$.

Let the Lagrange interpolation functions that use the x_i be $\ell_i(x)$ for $0 \le i \le n$. So, for example,

$$L_n(x) = \ell_0(x)f(x_0) + \cdots + \ell_n(x)f(x_n)$$

is the Lagrange form of the interpolation polynomial of $f(x)$ defined by the roots.

Let $\mathbb{P}_n$ be the space of polynomials of degree less than or equal to n. We can write the least squares approximation of $f(x)$ in terms of the $P_i(x)$ using the generalized Fourier series as

$$f_n(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \cdots + \alpha_n P_n(x) \text{ where } \alpha_i = \frac{(f, P_i)}{(P_i, P_i)}$$

7.1.a

Clearly, $(\ell_i, \ell_i) \ne 0$. Show that $(\ell_i, \ell_j) = 0$ when $i \ne j$. Therefore, the functions $\ell_0(x), \dots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$.

7.1.b

Suppose we evaluate $f_n(x)$ at the x_i to obtain the data $f_n(x_0), \dots, f_n(x_n)$. We can then write $f_n(x)$ in its Lagrange form,

$$f_n(x) = L_n(x) = f_n(x_0)\ell_0(x) + \ldots + f_n(x_n)\ell_n(x)$$

Since the $\ell_0(x), \dots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$, they also can be used to compute, $f_n(x)$, the unique least squares approximation to $f(x)$. As with the Legendre polynomials, using the generalized Fourier series, yields

$$f_n(x) = \sigma_0 \ell_0(x) + \sigma_1 \ell_1(x) + \cdots + \sigma_n \ell_n(x) \text{ where } \sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)}$$

Show that these last two forms of $f_n(x)$ give the same polynomial by showing that

$$\sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)} = f_n(x_i)$$

Hint: Consider the relationship between $f(x)$ and $f_n(x)$.

Problem 7.2

Consider $f(x) = e^x$ on the interval $-1 \leq x \leq 1$. Suppose we want to approximate $f(x)$ with a polynomial. Generate the following polynomials:

(a) $F_1(x)$ and $F_3(x)$:the first and third order Taylor series approximations of $f(x)$ expanded about $x = 0$.

(b) $N_1(x)$: the linear near-minimax approximation to $f(x)$ on the interval.

(c) $C_1(x)$ and $C_2(x)$ – the linear and quadratic polynomials that result from Chebyshev economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(d) $p_1(x)$ and $p_2(x)$ – the linear and quadratic polynomials that result from Legendre economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(7.2.a) Derive bounds on the ∞ norm of the error where possible.

(7.2.b) Evaluate the error for each polynomial approximation on a very fine grid on the interval $-1 \leq x \leq 1$ and compare to the bounds.