Program in Applied Mathematics
Preliminary Examination in Numerical Analysis
Friday, August 22, 1997

This test contains problem(s) in four categories - one category on each page. Solve one (and no more than one) in each category/page. Note that there is no choice of problem in Category IV.

The test will last from 10 am to 1 pm.

I. INTERPOLATION/QUADRATURE:

1. Discrete Fourier Transform (DFT): If we start with a set of N complex numbers, apply first the DFT and follow this by an inverse DFT, we get back the very same N complex numbers that we started with (and in the same order). Suppose we again start with N complex numbers but now instead apply the DFT twice in succession (i.e. not invoke its inverse at either stage),

   a. Describe in very simple words how the result relates to the input,
   b. Derive formally your answer to point a. above.

2. Hermite Interpolation: Let $f \in C^6[-1,1]$.

   a. Construct the Hermite interpolating polynomial $p(x)$ on the interval [-1,1] such that

   $$p(x_i) = f(x_i)$$
   $$p'(x_i) = f'(x_i)$$

   for $x_i = -1, 0, 1$.

   b. Give a formula for the interpolation error

   $$E(f) = p(x) - f(x).$$

   c. Show that the quadrature formula

   $$\int_{-1}^{1} f(t)dt \approx \tfrac{7}{15} f(-1) + \tfrac{16}{15} f(0) + \tfrac{7}{15} f(1) + \tfrac{1}{15} f'(-1) - \tfrac{1}{15} f'(1)$$

   is exact for all polynomials of degree $d \leq 5$.

II. FINITE DIFFERENCE / FINITE ELEMENT:

3. Finite differences:

a. Use the basic expression for the relationship between the differential operator D and the forward difference operator Δ_+ and backward difference operator Δ_- to show that $y'(kh) = (y((k+1)h)-y((k-1)h)/(2h)+O(h)$.
[This is an understatement: Taylor series expansion shows readily that the error is $O(h^2)$]

b. Use this same formalism to derive the standard centered second order approximation to $y''(kh)$

c. Prove that the accuracy in the solution of Poisson's equation is $O(h^2)$ when using the approximation in part b.
[Assume a uniform grid on a unit square and Dirichlet boundary conditions. Assume also that the matrix A for the resulting linear system is symmetric and positive definite with a minimun eigenvalue of about $2\pi^2$].

4. Finite elements:

a. Consider the 2-point boundary value problem $L\,y = -y'' + y = f(x)$, $y(0) = y(1) = 0$. Derive the weak form (using integration by parts),

b. Using FEM on a uniform grid with standard chapeau (hat) functions, derive the entries of the associated matrix $A = (a_{k,l})$.

c. Define the bilinear form $L(v,w) := \langle Lv,w\rangle$ and show that L is bounded and coercive with respect to the Sobolev norm $\|v\|_H := \sqrt{\|v\|^2 + \langle Lv, v\rangle}$.

III. LINEAR ALGEBRA:

5. Eigenvalues:

a. The following are techniques for finding eigenvalues and eigenvectors of the $N \times N$ matrix A. Describe each method in detail and characterize the eigenvalues each method is intended to find.

   i. Power method,
   ii. Inverse power method, and
   iii. Shifted inverse power method.

b. Assume A has a complete set of eigenvectors and eigenvalues that satisfy

$$0 < |\lambda_1| < |\lambda_2| < \ldots < |\lambda_n|.$$

Prove the convergence of the inverse power method.

   i. To what will it converge?
   ii. What is the rate of convergence?
   iii. What may happen if all $<$ are replaced by $\leq$ above?

6. Matrix norms:

Consider the matrix

$$A = \begin{bmatrix} -0.4 & 1.0 & -0.8 \\ 1.2 & -2.0 & 1.4 \\ -0.6 & 1.0 & -0.2 \end{bmatrix}$$

with the inverse

$$A^{-1} = \begin{bmatrix} 5.0 & 3.0 & 1.0 \\ 3.0 & 2.0 & 2.0 \\ 0.0 & 1.0 & 2.0 \end{bmatrix}.$$

a. What is $\|A\|_1$?

b. What is the condition number of A in the 1-norm?

c. Suppose $Ax = b$ and $(A+E)\hat{x} = b$, where $\|E\|_1 \leq 0.01$. Give a bound on the relative difference between the two solutions. (This should be a number)

IV. ORDINARY DIFFERENTIAL EQUATIONS

7. Linear multistep methods for ODEs: The following are six suggestions for linear multistep formulas for solving $y' = f(x,y)$:

a. $y^{n+1} = \frac{1}{2}y^n + \frac{1}{2}y^{n-1} + 2hf^n$

b. $y^{n+1} = y^n$

c. $y^{n+1} = y^{n-3} + \frac{4}{3}h(f^n + f^{n-1} + f^{n-2})$

d. $y^{n+1} = y^{n-1} + \frac{1}{3}h(7f^n - 2f^{n-1} + f^{n-2})$

e. $y^{n+1} = \frac{8}{19}(y^n - y^{n-2}) + y^{n-3} + \frac{6}{19}h(f^{n+1} + 4f^n + 4f^{n-2} + f^{n-3})$

f. $y^{n+1} = -y^n + y^{n-1} + y^{n-2} + 2h(f^n + f^{n-1})$

The incomplete table below summarizes their properties. Complete its missing entries (you need not supply any derivations).

case	char. eq.	roots	stability	accuracy	consistency	leading error term	convergence to solution
a	$r^2 - \frac{1}{2}r - \frac{1}{2} = 0$	$1, -\frac{1}{2}$	Yes	0		$-\frac{1}{2}hf'(\xi)$	
b					No		No
c					Yes		
d				3		$\frac{1}{3}h^4 f^{III}(\xi)$	Yes
e	$r^4 - \frac{8}{19}r^3 + \frac{8}{19}r - 1 = 0$	$\pm 1, \frac{4}{19} \pm \frac{\sqrt{345}}{19}i$		6		$-\frac{6}{665}h^6 f^{VII}(\xi)$	
f				2	Yes	$\frac{2}{3}h^3 f^{II}(\xi)$	