10.5: Matrix Algebra

Supplementary Notes

An $m \times n$ matrix $A = (a_{ij})$ has m *rows* and n *columns*:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

and a_{ij} is the element in the ith row and jth column of A.

Equality: Two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal if and only if their corresponding entries are equal, that is, if $a_{ij} = b_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Addition/Subtraction: For two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, their sum is the $m \times n$ matrix $A + B = (a_{ij} + b_{ij})$, and their difference is the $m \times n$ matrix $A - B = (a_{ij} - b_{ij})$. That is, matrices are added (or substracted) by adding (or substracting) their corresponding entries. As in addition and substraction of real numbers, the following properties of matrix addition and substraction are true.

- $A + B = B + A$ (commutative property)
- $(A + B) + C = A + (B + C)$ (associative property)

Scalar Multiplication: The product of a real number c and an $m \times n$ matrix $A = (a_{ij})$ is the $m \times n$ matrix $cA = (ca_{ij})$. That is, a matrix is multiplied by a real number by multiplying every entry in the matrix by the real number. For real numbers c, c_1 and c_2 and $m \times n$ matrices A and B, the following properties of scalar multiplication are true.

- $c_1(c_2 A) = (c_1 c_2)A$ (associative property)
- $c(A + B) = cA + cB$ (scalar distributive property)
- $(c_1 + c_2)A = c_1 A + c_2 A$ (matrix distributive property)

Matrix Multiplication: The product of a $1 \times n$ matrix $A = (a_j)$ and an $n \times 1$ matrix $B = (b_i)$ is the number calculated by

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

For an $m \times n$ matrix A and $n \times p$ matrix B, the product AB is the $m \times p$ matrix whose entry in row i, column j is the product of the ith row of A and the jth column of B. For $m \times n$ matrix A, $n \times p$ matrices B, B_1, B_2 and $p \times q$ matrix C, the following properties of matrix multiplication are true.

- $(AB)C = A(BC)$ (associative property)
- $A(B_1 + B_2) = AB_1 + AB_2$ (distributive property)

Multiplicative Identity: The $n \times n$ identity matrix I has 1's along is diagonal and 0's elsewhere

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

For any $n \times n$ matrix A, $AI = IA = A$.

Inversion: If the determinant of the $n \times n$ matrix A is nonzero, then A has an $n \times n$ inverse matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. The inverse A^{-1} can be calculated by reducing the augmented matrix

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

to reduced row echelon form

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right].$$

Exercises

1. Find a, b, c and d if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

2. Find a, b, c and d if

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 7 & -7 \end{bmatrix}$$

3. Find $BC - 2B$ if

$$B = \begin{bmatrix} -3 & 1 \\ 2 & -4 \\ 1 & -2 \end{bmatrix} \text{ and } C = \begin{bmatrix} -6 & 3 \\ 9 & -3 \end{bmatrix}$$

4. Find the matrix product ABC if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

5. Find the solution of the system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z & = 1 \\ a_{21}x + a_{22}y + a_{23}z & = -\frac{1}{2} \\ a_{31}x + a_{32}y + a_{33}z & = \frac{1}{2} \end{cases}$$

if the inverse of the coefficient matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is $\begin{bmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ 1 & -\frac{1}{2} & -1 \\ -2 & 2 & 3 \end{bmatrix}$

6. Find the inverse matrix of

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$