Foundations of Computational Math II Exam 1
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Friday, 1 March, 2013

Question	Points Possible	Points Awarded
1. Interpolation	30	
2. Piecewise Linear Interpolation	25	
3. Minimax	25	
4. Splines	35	
5. Optimization via Approximation	25	
Total Points	140	

Name: **Alias:**

Problem 1 (30 points)

1.a

Consider, $r_4(x)$, the unique polynomial of degree 4 that interpolates the data:

$$(x_0, 0),\ (x_1, 0),\ (x_2, 0),\ (x_3, z_3),\ (x_4, z_4)$$

where the x_i are distinct. Let $\omega_{i:i+k}(x) = (x - x_i) \dots (x - x_{i+k})$.

(i) Mark the positions in the divided difference table with 0 if the entry is guaranteed to be 0 and $*$ if it may be nonzero. **Solution:**

i	0		1		2		3		4
x_i	x_0		x_1		x_2		x_3		x_4
z_i	0		0		0		z_3		z_4
$z[-,-]$		0		0		$*$		$*$	
$z[-,-,-]$			0		$*$		$*$		
$z[-,-,-,-]$				$*$		$*$			
$z[-,-,-,-,-]$					$*$				

(ii) Give the Newton form of $r_4(x)$ using the divided differences $z[x_0, \dots, x_i]$.

(iii) Use the appropriate identities to rewrite the divided differences into a form that makes it clear that $r_4(x)$ satisfies the required interpolation conditions.

Solution: The Newton form is:

$$r_4(x) = 0 + 0 + 0 + \omega_{0:2}(x) z[x_0, x_1, x_2, x_3] + \omega_{0:3}(x) z[x_0, x_1, x_2, x_3, x_4]$$

Applying the identities

$$z[x_0, \dots, x_i] = \sum_{j=0}^{i} \frac{z_j}{\omega'_{0:i}(x_j)} \quad \text{and} \quad \omega'_{0:k+1}(x_{k+1}) = \omega_{0:k}(x_{k+1})$$

yields

$$z[x_0, x_1, x_2, x_3] = \frac{z_3}{\omega'_{0:3}(x_3)} = \frac{z_3}{\omega_{0:2}(x_3)}$$
$$z[x_0, x_1, x_2, x_3, x_4] = \frac{z_3}{\omega'_{0:4}(x_3)} + \frac{z_4}{\omega'_{0:4}(x_4)} = \frac{z_3}{\omega'_{0:4}(x_3)} + \frac{z_4}{\omega_{0:3}(x_4)}$$

$$r_4(x) = \omega_{0:2}(x) \frac{z_3}{\omega_{0:2}(x_3)} + \omega_{0:3}(x) \left(\frac{z_3}{\omega'_{0:4}(x_3)} + \frac{z_4}{\omega_{0:3}(x_4)} \right)$$

Checking the interpolation conditions yields

$$0 \leq i \leq 2, \quad \omega_{0:2}(x_i) = \omega_{0:3}(x_i) = 0 \rightarrow r_4(x_i) = 0$$

$$\omega_{0:3}(x_3) = 0 \rightarrow r_4(x_3) = z_3$$

$$\begin{aligned} r_4(x_3) &= \omega_{0:2}(x_4)\frac{z_3}{\omega_{0:2}(x_3)} + \omega_{0:3}(x_4)\left(\frac{z_3}{\omega'_{0:4}(x_3)} + \frac{z_4}{\omega_{0:3}(x_4)}\right) \\ &= \frac{\omega_{0:2}(x_4)}{\omega_{0:2}(x_3)}\left(1 + \frac{(x_4 - x_3)}{(x_3 - x_4)}\right) z_3 + \frac{\omega_{0:3}(x_4)}{\omega_{0:3}(x_4)} z_4 = z_4 \end{aligned}$$

as desired.

1.b

Consider the data:

$$(x_0, f_0),\ (x_1, f_1),\ (x_2, f_2),\ (x_3, f_3),\ (x_4, f_4)$$

where the x_i are distinct. Let $p_4(x)$ be the unique interpolating polynomial of degree 4 that interpolates these 5 data points. Let $p_2(x)$ be the unique interpolating polynomial of degree 2 that interpolates the first 3 data points

$$(x_0, f_0),\ (x_1, f_1),\ (x_2, f_2).$$

Let $\omega_{i:i+k}(x) = (x - x_i)\ldots(x - x_{i+k})$.

(i) Let $a_4(x)$ be the polynomial of degree 4 such that

$$p_4(x) = p_2(x) + a_4(x).$$

We know $a_4(x)$ can be expressed as

$$a_4(x) = f[x_0, x_1, x_2, x_3]\omega_{0:2}(x) + f[x_0, x_1, x_2, x_3, x_4]\omega_{0:3}(x)$$

Find values of z_3 and z_4 that show that $a_4(x)$ can also be expressed as a interpolating polynomial with interpolating conditions like those imposed on $r_4(x)$ in the first part of the problem.

(ii) Show the relationships between the coefficients of $a_4(x)$ expressed in terms of the z_i, $0 \le i \le 4$, and the divided differences of the f_i, $0 \le i \le 4$. What derivation of the divided differences $f[x_0, \ldots, x_i]$ does this exercise generalize?

Solution:

The interpolation conditions on $p_4(x)$ yield the conditions on $a_4(x)$:

$$0 \le i \le 2, \quad f_i = p_4(x_i) = p_2(x_i) + a_4(x_i) = f_i + a_4(x_i) \to a_4(x_i) = 0$$

$$3 \le i \le 4, \quad f_i = p_4(x_i) = p_2(x_i) + a_4(x_i) \to a_4(x_i) = f_i - p_2(x_i)$$

Therefore applying the form from the first part of the question and equating with the form above yields

$$a_4(x) = \omega_{0:2}(x)\frac{z_3}{\omega_{0:2}(x_3)} + \omega_{0:3}(x)\left(\frac{z_3}{\omega'_{0:4}(x_3)} + \frac{z_4}{\omega_{0:3}(x_4)}\right)$$

$$= f[x_0, x_1, x_2, x_3]\omega_{0:2}(x) + f[x_0, x_1, x_2, x_3, x_4]\omega_{0:3}(x)$$

$$f[x_0, x_1, x_2, x_3] = \frac{f_3 - p_2(x_3)}{\omega_{0:2}(x_3)}$$

$$f[x_0, x_1, x_2, x_3, x_4] = \left(\frac{f_3 - p_2(x_3)}{\omega'_{0:4}(x_3)} + \frac{f_4 - p_2(x_4)}{\omega_{0:3}(x_4)}\right) = \left(\frac{f_3 - p_2(x_3)}{\omega_{0:2}(x_3)(x_3 - x_4)} + \frac{f_4 - p_2(x_4)}{\omega_{0:2}(x_4)(x_4 - x_3)}\right)$$

$$= \left(\frac{f[x_0, x_1, x_2, x_3]}{(x_3 - x_4)} + \frac{f_4 - p_2(x_4)}{\omega_{0:2}(x_4)(x_4 - x_3)}\right)$$

$$= \left(\frac{-\omega_{0:2}(x_4)f[x_0, x_1, x_2, x_3] + f_4 - p_2(x_4)}{\omega_{0:2}(x_4)(x_4 - x_3)}\right) = \left(\frac{f_4 - p_2(x_4) - \omega_{0:2}(x_4)f[x_0, x_1, x_2, x_3]}{\omega_{0:3}(x_4)}\right)$$

$$= \left(\frac{f_4 - p_3(x_4)}{\omega_{0:3}(x_4)}\right)$$

where $p_3(x_4)$ is the unique cubic that interpolates (x_i, f_i), $0 \le i \le 3$. These are clearly consistent with the definition that results from the incremental correction construction of the divided differences in Set 1 (page 21).

Problem 2 (25 points)

Let $f(x) = \sin x$ and consider using a piecewise linear interpolating polynomial$g_1(x)$ to approximate $f(x)$ on $-\pi \leq x \leq \pi$. In Set 5 of the class notes a sufficent bound on a uniform separation $h = 2\pi/n$ between the x_i was derived to guarantee that

$$\|f(x) - g_1(x)\|_\infty \leq 10^{-d}.$$

If that bound is applied with $d = 6$ the resulting bound is $h \leq 0.0028$ and the number of points required is over 2200.

Show that by careful consideration of the structure of the problem and removing the restriction of uniform spacing the number of points required for a piecewise linear interpolating polynomial can be reduced substantially while still achieving

$$\|f(x) - g_1(x)\|_\infty \leq 10^{-6}.$$

Solution:

The result derived in the class notes for uniform intervals was

$$\|f(x) - g_1(x)\|_\infty \leq \frac{h^2}{8}\|f^{(2)}(x)\|_\infty$$

and for $f(x) = sinx$ using $\|f^{(2)}(x)\|_\infty \leq 1$

$$h \leq \sqrt{8} \times 10^{-d/2} \rightarrow \|f(x) - g_1(x)\|_\infty \leq 10^{-d}$$

Given $d = 6$ and $h = (2\pi)/n$ this implies $h \approx 2220$.

However, this is very pessimistic in practice. The following observations can be used to reduce the number of points.

The sign has two significant symmetries on the interval $-\pi \leq x \leq \pi$

$$\forall 0 \leq x \leq \pi, \quad \sin -x = -\sin x$$

$$\forall 0 \leq x \leq \pi, \quad \sin \pi - x = -\sin x$$

Therefore, any $\sin x$ on $-\pi \leq x \leq \pi$ can be recovered from the sin on $0 \leq x \leq \pi/2$ and we can restrict $g_1(x)$ to discretizing that interval. This reduces the number of points by a factor of 4 if uniform spacing is used.

Of course, uniform spacing is the result of using $\|f^{(2)}(x)\|_\infty \leq 1$. However, $\|f^{(2)}(x)\|_\infty = \|\sin x\|_\infty$ has significant variation on the interval $0 \leq x \leq \pi/2$ so nonuniform spacing can yield further saving by tightening the bound on each subinterval and then use a local uniform spacing. For example, if $\|\sin x\|_\infty \leq B$ on an interval of width L then

$$h \leq \sqrt{8} \times B^{-1/2} \times 10^{-d/2} \rightarrow n \approx \frac{L \times B^{1/2} \times 10^{d/2}}{2.83}$$

So for example, if we take several subintervals of $0leqx \leq \pi/2$ we have for $d = 6$

$$
\begin{gathered}
0 \leq x \leq 0.1, \quad B = 0.1 = L_1 \rightarrow n_1 = 11 \\
0.1 \leq x \leq 0.2, \quad B = 0.2, L_2 = 0.1 \rightarrow n_2 = 16 \\
0.2 \leq x \leq 0.3, \quad B = 0.3, L_3 = 0.1 \rightarrow n_3 = 19 \\
0.3 \leq x \leq 0.4, \quad B = 0.4, L_4 = 0.1 \rightarrow n_4 = 22 \\
0.4 \leq x \leq 0.5, \quad B = 0.5, L_5 = 0.1 \rightarrow n_5 = 25 \\
0.5 \leq x \leq 0.6, \quad B = 0.56, L_6 = 0.1 \rightarrow n_6 = 26 \\
0.6 \leq x \leq 0.7, \quad B = 0.65, L_7 = 0.1 \rightarrow n_7 = 28 \\
0.7 \leq x \leq 0.8, \quad B = 0.71, L_8 = 0.1 \rightarrow n_8 = 29 \\
0.8 \leq x \leq 0.9, \quad B = 0.78, L_9 = 0.1 \rightarrow n_9 = 31 \\
0.9 \leq x \leq 1.0, \quad B = 0.84, L_{10} = 0.1 \rightarrow n_{10} = 32 \\
1.0 \leq x \leq 1.1, \quad B = 0.89, L_{11} = 0.1 \rightarrow n_{11} = 33 \\
1.1 \leq x \leq 1.57, \quad B = 1.0, L_{12} = 0.47 \rightarrow n_{12} = 166
\end{gathered}
$$

$$\therefore n = 438$$

If we used one interval $0 \leq x \leq \pi/2$ we would have $n = 555$. This is compared to the original of

$$-\pi \leq x \leq \pi, \quad B = 1, L = 2\pi \rightarrow n = 2220$$

In the discussion above, we have used piecewise linear based on the linear interpolant for the endpoints of the interval, i.e., x_{i-1}, x_i. The same intervals can be used but on each a near-minimax linear interpolant could be used. This could still exploit the variability of the $\|f^{(2)}(x)\|_\infty \leq 1$ but also use the lowest bound on $\|\omega_2(x)\|_\infty$.

Problem 3 (25 points)

For this problem let $f(x) = \sin x$.

3.a. Derive the linear minimax approximation, $p_1(x) = \alpha x + \beta$, to $f(x)$ on $0 \leq x \leq \pi$.

3.b. Derive the linear near-minimax polynomial approximation, $c_1(x)$, to $f(x)$ on $0 \leq x \leq \pi$.

3.c. Compare $c_1(x)$ and $p_1(x)$.

Solution:

This is an example of the linear minimax of a concave function seen in the homework.

First determine the critical points (we need 3)

$$
\begin{gathered}
e(x) = \sin x - \alpha x - \beta \\
e'(x) = \cos x - \alpha \\
e'(c) = \cos c - \alpha = 0 \rightarrow c = \arccos \alpha \\
0 \leq c \leq \pi \rightarrow \alpha \geq 0
\end{gathered}
$$

Therefore the endpoints and $0 \leq c \leq \pi$ are the extrema of interest

$$
e(0) = e(\pi) = -e(c)
$$

$$
\begin{gathered}
e(0) = e(\pi) \rightarrow -\beta = -\alpha\pi - \beta \\
\therefore \alpha = 0, \quad c = \frac{\pi}{2}
\end{gathered}
$$

$$
\begin{gathered}
e(0) = -e(c) \rightarrow -\beta = \sin\frac{\pi}{2} - \beta \\
\therefore \beta = \frac{1}{2}
\end{gathered}
$$

$$
p_1(x) = \frac{1}{2}
$$

The linear minimax is a constant as expected for a concave function.

For the near-minimax or Chebyshev interpolant we need the roots of $T_2(z)$ for $-1 \leq z \leq 1$ and covert to $0 \leq x \leq \pi$ via

$$
x_i = \frac{1}{2}\left(\pi z_i + \pi\right).
$$

This yields

$$z_1 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \to x_1 = \frac{\pi}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$z_2 = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} \to x_2 = \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Due to the symmetry of $\sin x$ around $\pi/2$ we have

$$\sin x_1 = \sin x_2 \approx 0.44 < 0.5$$

$$c_1(x) = \sin x_1 < p_1(x).$$

So the near-minimax is also a constant and is below the minimax constant.

Problem 4 (35 points)

Consider a interpolatory quadratic spline, $s(x)$, that satisfies the following interpolation conditions and single boundary condition:

$$s(x_i) = f(x_i) = f_i, \quad 0 \le i \le n$$

$$s'(x_0) = f'(x_0) = f'_0$$

where the x_i are distinct.

4.a. Derive a linear system of equations that yields the values

$$s'(x_i) = s'_i \quad 0 \le i \le n$$

that are used as parameters to define the quadratic spline $s(x)$.

4.b. Identify important structure in the linear system and show that it defines a uniqiue quadratic spline.

4.c. Use the structure of the system to show that if $f(x)$ is a quadratic polynomial then $s(x) = f(x)$.

Solution:

The space of quadratic splines on $n+1$ points has dimension $n+2$. We have n quadratic polynomials, $p_i(x)$ on $[x_{i-1}, x_i]$ for $1 \le i \le n$ and therefore $3n$ parameters available. The continuity of $s(x)$ and $s'(x)$ yields $2n-2$ constraints. The interpolation $s(x_i) = f(x_i)$, $0 \le i \le n$ yields $n+1$ constraints. This leaves 1 boundary condition available to define a unique quadratic interpolating spline.

Using the continuity of $s'(x)$ and the fact that $s(x)$ is piecewise quadratic yields, letting $h_i = x_i - x_{i-1}$,

$$p'_i(x) = \frac{(x - x_{i-1})}{h_i} s'_i - \frac{(x - x_i)}{h_i} s'_{i-1}, \quad 1 \le i \le n$$

$$p_i(x) = \frac{(x - x_{i-1})^2}{2h_i} s'_i - \frac{(x - x_i)^2}{2h_i} s'_{i-1} + \gamma_i, \quad 1 \le i \le n$$

Using $p_i(x_i) = f_i, \ 1 \le i \le n$ we have

$$f_i = p_i(x_i) = s'_i \frac{h_i}{2} + \gamma_i$$

$$\gamma_i = f_i - s'_i \frac{h_i}{2}, 1 \le i \le n$$

$$p_i(x) = \left(\frac{(x - x_{i-1})^2}{2h_i} - \frac{h_i}{2} \right) s'_i - \frac{(x - x_i)^2}{2h_i} s'_{i-1} + f_i, \quad 1 \le i \le n$$

The constraints of continuity of $s(x)$ and $p_1(x_0) = f_0$ yield the n equations

$$p_i(x_{i-1}) = f_{i-1}, \quad 1 \le i \le n$$

$$-s'_i \frac{h_i}{2} - s'_{i-1} \frac{h_i}{2} + f_i = f_{i-1}$$

$$\frac{1}{2}\left(s'_i + s'_{i-1}\right) = f[x_{i-1}, x_i]$$

Note the scale is consistent on both sides of the equation.

Finally, the addition of the boundary conditionn $s'_0 = f'_0$ yields $n+1$ equations and $n+1$ unknowns with the matrix form illustrated here for $n = 5$,

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} s'_0 \\ s'_1 \\ s'_2 \\ s'_3 \\ s'_4 \\ s'_5 \end{pmatrix} = \begin{pmatrix} f'_0 \\ f[x_0, x_1] \\ f[x_1, x_2] \\ f[x_2, x_3] \\ f[x_3, x_4] \\ f[x_4, x_5] \end{pmatrix}$$

The matrix is lower triangular and banded. It is clearly nonsingular. The resulting quadratic spline is therefore unique.

Suppose $f(x) = \alpha x^2 + \beta x + \gamma$ and consider the first order divided difference $f[x_{i-1}, x_i]$. It follows that

$$f[x_{i-1}, x_i] = \alpha(x_i + x_{i-1}) + \beta = f'((x_i + x_{i-1})/2).$$

Now consider the recurrence defined by the linear system.

$$s'_0 = f'_0$$
$$\frac{1}{2}\left(s'_i + s'_{i-1}\right) = f[x_{i-1}, x_i] = \alpha(x_i + x_{i-1}) + \beta$$

$$i = 1, \quad \frac{1}{2}(s'_1 + f'_0) = f[x_0, x_1] = \alpha(x_1 + x_0) + \beta$$
$$(s'_1 + f'_0) = 2\alpha(x_1 + x_0) + 2\beta$$
$$s'_1 = 2\alpha(x_1 + x_0) + 2\beta - f'_0 = 2\alpha(x_1 + x_0) + 2\beta - (2\alpha x_0 + \beta) = f'(x_1)$$

By induction, it follows that $s'_i = f'_i$ as desired. The interpolation at any one point implies that $s(x) = f(x)$ when $f(x)$ is quadratic.

Problem 5 (25 points)

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function with at least 4 continuous derivatives and with a unique minimizer, x^*. Assume that you do not have $f(x)$ or any of its derivatives analytically but you do have a routine that allows you to get values of f for any value of x. You may assume that the computational cost of the evaluation of $f(x)$ is small.

Consider solving the problem

$$\min_{x \in \mathbb{R}} f(x)$$

numerically using Newton's method.

5.a. Clearly, since by assumption, $f(x)$ and its derivatives are not available some method that approximates Newton must be used. Describe a method that uses techniques discussed in class to approximates Newton's method to solve the unconstrained optimization problem.

5.b. Show that the method is parameterized so that the method must approach the performance of Newton's method as the parameter is moved toward a limit. Your argument need not be a formal proof but it must contain all the essential facts. You may assume that you have an initial guess $x^{(0)}$ that is sufficiently close to x^*.

Solution:

The unique minimizer must be a critical point of f^pr. Newton's method requires a suffiently good initial guess since it is a locally convergent method. By assumption, we assume we can always find a close enough initial condition.

To find a root of f' via Newton we need f''. We have neither. To use data and interpolation we must have a sufficiently smooth interpolant. Given that f has at least 4 continuous derivatives, this is pointing to using a cubic interpolatory spline as a surrogate for f.

We also know that under these circumstances $s_h' \to f'$ and $s_h'' \to f''$ as the number of points used to generate the spline increases (at the rate of $O(h^3)$ and $O(h^2)$ respectively).

Therefore, we can get arbitrarily close to the two functions required for Newton's method. The iteration function is

$$\phi_h(x) = x - \frac{s_h'(x)}{s_h''(x)}$$

which given the convergence of the functions above must be such that $|\phi_h'(x^*)| < 1$ at some sufficiently small h.

An alternative approach is to esimate $f'(x_k)$ and $f''(x_k)$ where x_k is the iterate from the pseudo-Newton's method via a difference-based differentiation formula. The first approach chooses a stepsize h and uses first and second order divided differences around x_k, i.e.,

$$f'(x_k) \approx \frac{f(x_k + h) - f(x_k)}{h}$$

$$f''(x_k) \approx \frac{f(x_k + 2h) - 2f(x_k + h) + f(x_k)}{h2}$$

One can get higher order approximations using other difference operators such as centered ones

$$f'(x_k) \approx \frac{f(x_k + h) - f(x_k - h)}{2h}$$
$$f''(x_k) \approx \frac{f(x_k + h) - 2f(x_k) + f(x_k - h)}{h2}$$

It is even possible to use higher order polynomials to approximate both first and second order derivatives. Choose some set of k points to the left and right of x_k so you have $21k+1$ points. Interpolate with a degree $2k$ polynomial $p_{2k}(x)$. The approximations are then

$$f'(x_k) \approx p'_{2k}(x_k)$$
$$f''(x_k) \approx p''_{2k}(x_k)$$

This is a more reliable method than interpolating over a large grid globally without controlling smoothness of $p(x)$ as you would do with a spline.

The total amount of work is not necessarily smaller than the spline form however. This is of course difficult to predict in general.