6.5: Average Value of a Function

It's easy to calculate the average of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{ave} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

But, this formula is not valid for calculating the average value of a function $y = f(x)$ over an interval $[a, b]$ (where $a < b$) since f takes infinitely many values on the interval. To approximate the average value of f over $[a, b]$ we first divide $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$ each of width $\Delta x = (b-a)/n$. Then, we choose sample points x_k^* in $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$ and compute the average value of f at the sample points:

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}.$$

Since $n = (b-a)/\Delta x$, we can rewrite this approximation as

$$\begin{aligned} f_{ave} &\approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\frac{b-a}{\Delta x}} \\ &= \frac{1}{b-a}[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]\Delta x \\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x \end{aligned}$$

This approximation becomes exact as $n \to \infty$, that is

$$f_{ave} = \lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x.$$

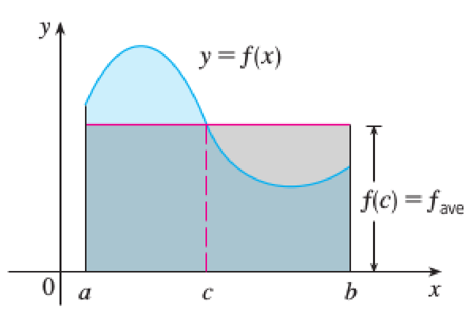
Thus, the **average value of** f on the interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

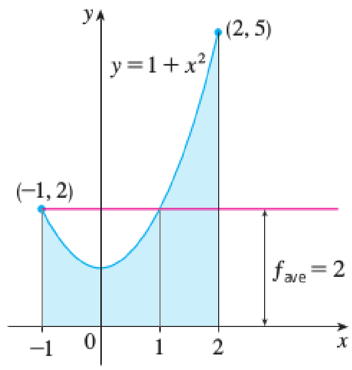
Example 1. *Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.*

Theorem 1. *(Mean Value Theorem for Integrals) If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that*

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$



Example 2. *Find all c that satisfy the conclusion of the Mean Value Theorem for Integrals for $f(x) = 1 + x^2$ on the interval $[-1, 2]$.*



Example 3. *Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.*