Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Monday August 15, 2005 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Root finding:

1. Assume that $f(x)$ is a real-valued function of a real variable, x, that has a root $x = a$.

   a. Describe Newton's method for finding this root.

   b. Assuming whatever smoothness you need, show that Newton's method converges quadratically in a neighborhood of a if this root is simple.

   c. Give an example to show how Newton's method can be slow when the root is multiple.

   d. Describe a remedy that restores quadratic convergence when the degree of multiplicity is known.

Numerical quadrature:

2. a. Use the standard orthogonal polynomial approach to determine the nodes x_1, x_2, x_3 in the Gaussian quadrature formula

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \tag{1}$$

   (The weights can then be found easily by solving a linear system - you do not need to carry that out).

   b. Gaussian quadrature formulas can also be calculated without any reference to orthogonal polynomials. In the case of (1), we want to enforce that the formula becomes exact for the increasing order monomials $f = 1, f = x, \ldots, f = x^5$, resulting in the system of equations

$$\begin{array}{ll} 2 = w_1 \quad + w_2 \quad + w_3 & (2) \\ 0 = w_1 x_1 + w_2 x_2 + w_3 x_3 & (3) \\ \ldots. & . \\ \ldots. & . \\ \ldots. & . \\ 0 = w_1 x_1^5 + w_2 x_2^5 + w_3 x_3^5 & (7) \end{array}$$

   Although nonlinear, systems of this form can be solved by a simple technique. Carry this out, and show that you again get the same result (for the nodes) as in part (a).

   <u>Hint:</u> Consider the polynomial

$$p(x) = (x - x_1)(x - x_2)(x - x_3) = x^3 + c_2 x^2 + c_1 x + c_0$$

   (where c_0 , c_1 , c_2 become our new unknowns, in place of x_1, x_2, x_3). Multiply the equations (2)-(5) by c_0 , c_1 , c_2 , 1 , respectively, and add. Then do the same with equations (3)-(6) and with equations (4)-(7).

Interpolation / Approximation:

3. Let $\{t_k\}_{k=0}^{N}$ and $\{s_k\}_{k=0}^{N}$ be sequences of complex numbers.

   a. Show that there exists a unique interpolating polynomial $p(z) = \Sigma_{k=0}^{N} a_k z^k$ such that $p(t_k) = s_k$, $k = 0, 1, \ldots, N$, where t_k are assumed to be distinct nodes.

   Let $\{t_k\}_{k=0}^{2N} \subseteq [0, 2\pi)$ and consider the function $f_N(t) = a_0 + \sum_{m=1}^{N} a_m \cos(m\,t) + b_m \sin(m\,t)$ for some set of coefficients a_m and b_m such that $|a_N| + |b_N| \neq 0$.

   b. Rewrite the function $f_N(t)$ as $Q(z) = \sum_{m=-N}^{N} c_m z^m$ where $z = e^{it}$.

   c. Show that for a given sequence $\{s_k\}_{k=0}^{2N}$ there exists a unique set of coefficients $\{a_k\}_{k=0}^{N}$ and $\{b_k\}_{k=1}^{N}$ such that $f_N(t_k) = s_k$.

Linear algebra:

4. Consider the $(n+m)\times(n+m)$ real matrix defined in block form by

$$A = \begin{bmatrix} I & X \\ X^T & O \end{bmatrix},$$

   where I is the $n \times n$ identity matrix, X is a full-rank $n \times m$ matrix, O is the $m \times m$ zero matrix, and $m \leq n$.

   a. Show that A is nonsingular.

   b. Find the eigenvalues of A, some of which are in terms of the singular values of X.

   c. Under what conditions on X would the iteration

$$x_{n+1} = x_n - (Ax_n - b)$$

   converge to the solution of $Ax = b$ for any $(n+m)\times(n+m)$ real vector b?

   Hint: For (a) and (b), consider multiplying the first block row of A by X^T.

Numerical ODE:

5. Consider the implicit midpoint scheme

$$u_{n+1} = u_n + hf(t_n + \tfrac{h}{2}, \tfrac{1}{2}(u_n + u_{n+1})) \tag{8}$$

for solving the ODE $u' = f(t,u)$.

a. Show that the scheme is convergent.

b. Find the stability region of the scheme.

Consider numerically solving the pendulum equation

$$u'' + 2bu' + \omega^2 u = 0 \tag{9}$$

where $b, \omega \geq 0$, along with initial conditions, over $0 \leq t \leq T$ where T is large.

c. If $b = 0$ (no damping), is the scheme (8) suitable for solving (9)? If $b > \omega > 0$ (strong damping), is the scheme in (8) suitable for solving (9)?

d. The stability domain for the forward Euler method is inside a circle of radius 1 centered at $x = -1$ in the complex plane. If $b = 0$ (no damping), is the forward Euler scheme suitable for solving (9)? If $b > \omega > 0$ (strong damping), is it then suitable for solving (9)?

Numerical PDE:

6. a. Write down the second order accurate (in time and space) ADI scheme for approximating

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

(You may assume the steps to be the same in the two spatial directions).

b. Perform von Neumann analysis to show that the scheme is unconditionally stable (in the absence of boundaries - as is always assumed in this type of analysis).