DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO

SAMPLE

NUMERICAL ANALYSIS

QUALIFYING EXAMS

Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Bo Li

10am to 1pm
Tuesday September 8, 2009
2402 AP&M

NAME ______________

#1.1	30	
#1.2	30	
#1.3	30	
#2.1	30	
#2.2	30	
#2.3	30	
#3.1	30	
#3.2	30	
Total	240	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.
- Write your answers to the questions in Section 3 on separate sheets so that they may be graded separately.

1. Norms, Condition Numbers and Linear Equations

Question 1.1.

(a) Consider the subtraction $x = a - b$ of two real numbers a and b such that $a \neq b$. Suppose that $\widetilde{a}$ and $\widetilde{b}$ are the result of making a *relative* perturbation Δa and Δb to a and b. Find the relative error of $\widetilde{x} = \widetilde{a} - \widetilde{b}$ as an approximation to x and hence find a condition number for the operation of subtraction. *Assume that all calculations are done in exact arithmetic.*

(b) State the *standard rounding-error model* for floating-point arithmetic. Given three representable numbers a, b and c, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the expression $s = ab+c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.2. Let A denote a symmetric positive-definite $n \times n$ matrix.

(a) Prove the following:

$$\begin{aligned} a_{ii} &> 0, \quad \text{for all } i \\ |a_{ij}| &\leq \sqrt{a_{ii}a_{jj}}, \quad \text{for all } i \text{ and } j \\ \max_{i,j} |a_{ij}| &= \max_i a_{ii}. \end{aligned}$$

(b) Show that if Gaussian elimination without interchanges is applied to A, then the remaining matrix is symmetric positive definite at every step. Hence show that there exists a unit lower-triangular L and upper triangular U such that $A = LU$.

(c) If A is factorized using Gaussian elimination without interchanges, show that the growth factor ρ_n satisfies $\rho_n \leq 1$.

Question 1.3. Assume that A is an $m \times n$ matrix with rank k ($k < \min(m,n)$).

(a) Define what is meant by a *full-rank factorization* $A = BC$.

(b) Derive a full-rank factorization of A in terms of the singular value decomposition. (You may assume that the decomposition is computed in exact arithmetic.)

(c) Using the singular-value decomposition of part (b), define bases for the subspaces range(A) and null(A). Prove that the proposed bases satisfy the properties of a subspace basis.

(d) Derive the pseudoinverse of A in terms of the full-rank factorization of part (b).

(e) Using the singular-value decomposition of part (b), define orthogonal projections onto range(A) and null(A). Prove that the proposed projections satisfy the properties of an orthogonal projection.

2. Nonlinear Equations and Optimization

Question 2.1.

(a) Derive Newton's method for finding the reciprocal of a given nonzero scalar a.

(b) Determine the exact order of convergence and asymptotic error constant for the method derived in part (a). (Do not attempt to derive the general rate-of-convergence result for Newton's method.)

Question 2.2. Consider the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that

$$f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.$$

(a) Compute the spectral decomposition of the Hessian matrix of second derivatives at $\bar{x} = (0, 1, 0)^T$.

(b) Compute the Newton direction p^N and modified Newton direction p^M at $\bar{x}$. Determine if p^N and p^M are descent directions.

(c) Find a direction of negative curvature that is a direction of decrease for f at $\bar{x}$.

Question 2.3. Let $f : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function on an open convex set $\mathcal{D}$. Let $\nabla f(x)$ denote the gradient of f at any $x \in \mathcal{D}$. If x_k is any point in $\mathcal{D}$, Consider the quadratic model

$$q_k(x) = f(x_k) + \nabla f(x_k)^T (x - x_k) + \frac{1}{2}(x - x_k)^T B (x - x_k),$$

where B is a given *fixed* symmetric positive-definite matrix.

(a) Find the vector p_k such that $x = x_k + p_k$ minimizes $q_k(x)$, and show that p_k is a descent direction for $f(x)$ at x_k.

(b) Show that p_k is a solution of the problem

$$\underset{\substack{p \in \mathbb{R}^n \\ p \neq 0}}{\text{minimize}} \quad \frac{\nabla f(x)^T p}{\|p\|_B}$$

where $\|p\|_B = (p^T B p)^{1/2}$.

(c) Given the direction p_k of part (a), formulate a back-tracking line search that will guarantee a reduction in f that is no worse than η_s times the reduction predicted by the quadratic model q_k, where η_s is a pre-assigned constant such that $0 < \eta_s < 1$. Show that the quadratic model predicts a decrease in f for all α_k such that $0 < \alpha_k < 2$.

3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We denote by $\mathcal{P}$ the set of all real polynomials. For any integer $n \geq 0$, we denote by $\mathcal{P}_n$ the set of all real polynomials of degree $\leq n$.

Question 3.1.

(a) Let $f \in C[a, b] \setminus \mathcal{P}$. For any integer $n \geq 0$, denote

$$E_n(f) = \min_{p_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - p_n(x)|.$$

Prove that the sequence $\{E_n(f)\}_{n=0}^{\infty}$ is *strictly* decreasing (i.e., $E_n(f) > E_{n+1}(f)$ for all $n \geq 0$) and converges to 0.

(b) Find the real numbers A, B, and C so that the numerical quadrature

$$\int_{-2}^{2} f(x)\, dx \approx Af(-1) + Bf(0) + Cf(1)$$

has the highest possible degree of precision. What is this highest possible degree of precision?

Question 3.2. Let $x_1^{(n)}, \ldots, x_n^{(n)}$ be the n distinct roots of orthogonal polynomials Q_n in $L^2(a, b)$ $(n = 1, 2, \ldots)$.

(a) For each $n \geq 2$, let $l_1^{(n)}, \ldots, l_n^{(n)}$ be the Lagrange basis polynomials associated with $x_1^{(n)}, \ldots, x_n^{(n)}$. Prove the following:

$$\int_a^b l_j^{(n)}(x) l_k^{(n)}(x)\, dx = 0 \qquad \text{if } 1 \leq j, k \leq n, \text{ and } j \neq k;$$

$$\sum_{k=1}^{n} \int_a^b \left[l_k^{(n)}(x)\right]^2 dx = b - a.$$

(b) Let $L_{n-1} : C[a, b] \to \mathcal{P}_{n-1}$ be the Lagrange interpolation operator associated with $x_1^{(n)}, \ldots, x_n^{(n)}$. Prove that

$$\lim_{n \to \infty} \int_a^b \left[f(x) - (L_{n-1} f)(x)\right]^2 dx = 0 \qquad \forall f \in C[a, b].$$

Ph.D./Masters Qualifying Examination
in Numerical Analysis

Examiners: Philip E. Gill and Bo Li

1–4pm
Friday May 22, 2009
6402 AP&M

NAME ____________________

#1.1	30	
#1.2	30	
#1.3	30	
#2.1	30	
#2.2	30	
#2.3	30	
#3.1	30	
#3.2	30	
Total	240	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition Numbers and Linear Equations

Question 1.1.

(a) Let $\mathbf{u}$ denote the unit roundoff, and assume that $n\mathbf{u} < 1$ for the positive integer n. If $\{\delta_i\}$ are n scalars such that $|\delta_i| \le \mathbf{u}$, prove that

$$\prod_{i=1}^{n}(1+\delta_i) = 1 + \theta_n, \quad \text{where} \quad |\theta_n| \le \gamma_n,$$

with $\gamma_n = n\mathbf{u}/(1 - n\mathbf{u})$.

(b) State the *standard rounding-error model* for floating-point arithmetic. Given representable numbers a and b, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the expression $s = \sqrt{a^2 + b^2}$. (You may assume that the square root function conforms to the standard rounding-error model for floating-point arithmetic.)

Question 1.2.

(a) Define the spectral condition number $\mathrm{cond}_2(A)$ for any $A \in \mathbb{R}^{m\times n}$. State (but do not prove) an expression for $\mathrm{cond}_2(A)$ in terms of the singular values of A.

(b) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Find a solution of the problem

$$\min_{E\in\mathbb{R}^{n\times n}} \{\|E\|_2 : A + E \ \text{ singular}\}.$$

(c) Prove that $1/\mathrm{cond}_2(A)$ is the relative two-norm distance of A to the nearest singular matrix.

Question 1.3. Assume that A is an $m \times n$ matrix with rank k ($k < \min(m, n)$).

(a) Define what is meant by a *full-rank factorization* $A = BC$.

(b) State the full-rank factorization of A in terms of the QR decomposition with column interchanges. (You may assume that the decomposition is computed in exact arithmetic.)

(c) Using the QR decomposition of part (b), define bases for the subspaces range(A) and null(A). Prove that the proposed bases satisfy the properties of a subspace basis.

(d) Using the QR decomposition of part (b), define orthogonal projections onto range(A) and null(A). Prove that the proposed projections satisfy the properties of an orthogonal projection.

(e) Derive the pseudoinverse of A in terms of the full-rank factorization of part (b).

2. Nonlinear Equations and Optimization

Question 2.1. Let A denote an $n \times n$ matrix, and let s and y be arbitrary n-vectors.

(a) Find all the eigenvalues of the matrix $I + \gamma uv^T$, where γ is a scalar and u and v are n vectors.

(b) Consider the Broyden update formula

$$A_+ = A + \frac{1}{s^T s}(y - As)s^T.$$

If $\|\cdot\|_F$ denotes the Frobenius norm, show that A_+ minimizes $\min \|B - A\|_F$ over all B such that $Bs = y$.

(c) If A is nonsingular, find a condition on A, s and y that will ensure that A_+ is nonsingular.

Question 2.2. Let $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on an open convex set $\mathcal{D}_0 \subseteq \mathcal{D}$.

(a) State the first- and second-order *necessary* conditions for $x^* \in \mathbb{R}^n$ to be an unconstrained minimizer of f.

(b) State first and second-order *sufficient* conditions for $x^* \in \mathbb{R}^n$ to be an unconstrained minimizer of f.

(c) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = (x_3 - 1)^2 \sin x_1 + x_1^2 + x_2^2 - \pi x_1.$$

(i) Write down the quadratic model $q(x)$ that interpolates f, ∇f and $\nabla^2 f$ at the point $x_0 = (-\pi/2,\, 0,\, \pi + 1)^T$.

(ii) Find the step p_N from x_0 to a stationary point of the quadratic model.

(iii) Determine if p_N is a descent direction for f at x_0.

(iv) Find a descent direction of negative curvature at x_0 (if one exists).

Question 2.3. Given an $n \times n$ symmetric matrix B and vectors y and s, consider the symmetric rank-one formula

$$B_+ = B + \frac{1}{(y - Bs)^T s}(y - Bs)(y - Bs)^T.$$

(a) Let $f(x)$ be a quadratic function with positive-definite Hessian H. Let $s = x_+ - x$ and $y = \nabla f(x_+) - \nabla f(x)$. If vectors $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{y} = \nabla f(\bar{x}_+) - \nabla f(\bar{x})$ satisfy $B\bar{s} = \bar{y}$, show that $B_+\bar{s} = \bar{y}$.

(b) Show that if B is symmetric and positive definite, then B_+ will be positive definite if and only if

$$\frac{y^T B^{-1} y - y^T s}{y^T s - s^T B s} > 0.$$

3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ for any integer $n \geq 0$.

Question 3.1.

(a) Let $n \geq 0$ be an integer and T_n the nth Chebyshev polynomial of first kind. Let $P \in \mathcal{P}_n$ satisfy that $|P(x)| \leq 1$ for all $x \in [-1, 1]$. Show that

$$|P(y)| \leq |T_n(y)| \qquad \forall y \notin [-1, 1].$$

(b) Let $\mathbb{F}$ denote the class of functions $a_0 + a_1 \cos x + a_2 \cos 2x$ with $a_0,\ a_1,\ a_2 \in \mathbb{R}$. Find $T \in \mathbb{F}$ such that

$$\int_0^\pi |T(x) - x|^2 dx \leq \int_0^\pi |S(x) - x|^2 dx \qquad \forall S \in \mathbb{F}.$$

Question 3.2.

(a) Use the error formula for the Lagrange interpolation of $f \in C^2[a, b]$ at the two points a and b to derive the error for the trapezoidal numerical integration rule

$$\int_a^b f(x)\, dx \approx \frac{1}{2}(b - a)\, [f(a) + f(b)]\, .$$

(b) Assume that $f \in C^2[a, b]$. Derive an error formula for the composite trapezoidal numerical integration rule

$$\int_a^b f(x)\, dx \approx \frac{h}{2}\, [f(a) + f(b)] + h \sum_{j=1}^{N-1} f(x_j).$$

Here $N \geq 1$ is an integer, $h = (b - a)/N$, and $x_j = a + jh$ $(j = 0, \ldots, N)$.

(c) Apply the composite trapezoidal numerical integration rule to

$$\int_0^{10} \sin x\, dx.$$

How large N is needed so that the error of the numerical integration is less than 10^{-6}? (Ignore the round-off error.) Justify your answer.

Numerical Analysis Qualifying Examination
9:00–12:00, Tuesday, September 9, 2008, AP&M 5829

Part C: Approximation, Interpolation, and Numerical Quadrature. We assume that $a, b \in \mathbb{R}$ with $a < b$. For any integer $n \geq 0$, we denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ and by $\overline{\mathcal{P}}_n$ the set of all polynomials in $\mathcal{P}_n$ with leading coefficient 1.

Question 3.1 [20 points]

Let $n \geq 1$ be an integer. Let $Q_k \in \overline{\mathcal{P}}_k$ $(k = 0, \ldots, n)$ be such that

$$\int_a^b Q_j(x) Q_k(x)\, dx = 0$$

for any indices j and k with $0 \leq j < k \leq n$. Let $P_n \in \overline{\mathcal{P}}_n$. Prove the following:

(a) The identity

$$P_n(x) = c_0 Q_0(x) + \cdots + c_{n-1} Q_{n-1}(x) + Q_n(x)$$

holds true for a unique set of real numbers $c_0, c_1, \ldots, c_{n-1}$. Moreover,

$$\int_a^b |P_n(x)|^2 dx = c_0^2 \int_a^b |Q_0(x)|^2 dx + \cdots + c_{n-1}^2 \int_a^b |Q_{n-1}(x)|^2 dx + \int_a^b |Q_n(x)|^2 dx;$$

(b) The inequality

$$\int_a^b |Q_n(x)|^2 dx \leq \int_a^b |P_n(x)|^2 dx$$

holds true. Moreover, this inequality becomes equality if and only if $P_n = Q_n$.

Question 3.2 [20 points]

(1) Calculate the Lagrange interpolation polynomial that interpolates the function $f(x) = x^{10}$ at points $x = 0, 1, \ldots, 20$. Justify your answer.

(2) Let $N \geq 1$ be an integer, $h = (b - a)/N$, and $x_j = a + jh$ $(j = 0, \ldots, N)$. The composite mid-point quadrature is given by

$$\int_a^b f(x)\, dx \approx h \sum_{j=1}^{N} f\left(\frac{x_{j-1} + x_j}{2}\right).$$

Suppose $f \in C^2[a, b]$. Prove that there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, dx - h \sum_{j=1}^{N} f\left(\frac{x_{j-1} + x_j}{2}\right) = \frac{1}{24}(b - a) h^2 f''(\xi).$$

NA Qual: Parts A and B

Sept. 9, 2008

Name ______________________

#1	30	
#2	35	
#3	50	
#4	35	
A-B	160	
Part C	40	
Total	200	

1. (a) (10) Prove that if $T \in \mathbb{C}^{n\times n}$,

$$T = \begin{array}{cc} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & \quad q \end{array} & \end{array},$$

then $\lambda(T) = \lambda(T_{11}) \cup \lambda(T_{22})$.

(b) (20) Prove that if $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{p\times p}$, and $X \in \mathbb{C}^{n\times p}$ satisfy $AX = XB$, $\text{rank}(X) = p$, then there exists a unitary $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^H AQ = T = \begin{array}{cc} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} & \begin{array}{c} p \\ n-p \end{array} \\ \begin{array}{cc} p & \quad n-p \end{array} & \end{array},$$

where $\lambda(T_{11}) = \lambda(A) \cup \lambda(B)$. (Hint: consider QR decomposition of X, and use (a).)

2. (a) (25) State and prove the Schur Decomposition Theorem for $A \in \mathbb{C}^{n\times n}$. (Hint: use induction and 1(b).)

(b) (10) Use 2(a) to prove that $A \in \mathbb{C}^{n\times n}$ has n orthonormal eigenvectors iff $A^H A = AA^H$.

3. (a) (25) State and prove the SVD Existence Theorem for $A \in \mathbb{R}^{m\times n}$.

(b) (15) Let $A \in \mathbb{R}^{m\times n}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{11} is $k \times k$. Use $S = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$, $m \times n$, to show that $\sigma_{k+1}(A) \le \|A_{22}\|_2$.

(c) (10) Let $A \in \mathbb{R}^{m\times n}$, $m \ge n$. Show that there exist an orthogonal Q and a symmetric positive semi-definite P such that $A = QP$.

4. (a) (20) Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$ iff $\hat{x}$ satisfies the normal equations, where A is $m \times n$, $m \ge n$.

(b) (15) Let A by $n \times n$, symmetric positive definite. Let $u_1, \ldots, u_n$ be an orthonormal basis of eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$. Let $w = \sum_{j=1}^{n} \alpha_j u_j$, $S(\mu) \equiv (A + \mu I)^{-1} w$, $\mu > 0$. Show that

$$\frac{d}{d\mu} \|S(\mu)\|_2 = -\frac{S(\mu)^T (A + \mu I)^{-1} S(\mu)}{\|S(\mu)\|_2}.$$

NA Qual: Parts A and B

May 27, 2008

Name ______________________

#1	25	
#2	20	
#3	30	
#4	30	
#5	55	
Subtotal	160	
Part C	40	
Total	200	

(25) 1. State and prove the SVD Existence Theorem (for real $m \times n$ matrices).

(20) 2. Let A be the $m \times n$, $\operatorname{rank}(A) = r$. Use the SVD of A, $U\Sigma V^T$, to show:

(a) Nullspace $(A) = \operatorname{span}\{v_{r+1}, \ldots, v_n\}$

(b) Range $(A) = \operatorname{span}\{u_1, \ldots . u_r\}$

(30) 3. (a) Let D be an $m \times n$ diagonal matrix. Prove $\|D\|_p = \max_i |d_{ii}|$ for $1 \leq p \leq \infty$.

(b) Prove that if A is $m \times n$, $\operatorname{rank}(A) = n$ and $\|E\|_p \|A^+\|_p < 1$ for some p, $1 \leq p \leq \infty$, then $\operatorname{rank}(A + E) = n$.

(c) Let A be $n \times n$, nonsingular, and $A = QR$, where Q is orthogonal and R is upper triangular with positive diagonal. Prove that Q and R are unique.

(30) 4. Suppose the computed $a_{kk}^{(k)} \neq 0$ for $1 \leq k \leq n-1$, where A is $n \times n$; then the computed L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on E:

$$|E_{ij}| \leq \left\{ \begin{array}{c} (3+u)(i-1)gu \text{ for } i \leq j \\ [(3+u)(j-1)+1]gu \text{ for } i > j \end{array} \right\}, \text{ where } g \equiv \max_k \max_{i,j} |a_{ij}^{(k)}|$$

a typo

and u = unit roundoff.

(55) 5. (20) (a) Show that if the single shift QR method converges, then the convergence is: (a) quadratic for general matrices, (b) cubic for symmetric matrices.

(25) (b) Let $A_0 = A$, where A is symmetric positive definite,

for $k = 1, 2, \ldots$

$A_{k-1} = G_k G_k^T$ (Cholesky)

$A_k \equiv G_k^T G_k$

Prove that if $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with $a \geq c$ then $A_k \to \operatorname{diag}(\lambda_1, \lambda_2)$, where $\lambda_1 \geq \lambda_2 > 0$.

(10) (c) Let $S = \begin{bmatrix} 0 & -B^T \\ B & 0 \end{bmatrix}$, where B is $n \times n$. Relate the eigenvalues and eigenvectors of S to the SVD of B, $B = U\Sigma V^T$.

NA Qual. Part C: Approximation, Interpolation, and Numerical Quadrature.

Question 3.1. [20 points]

(1) Let $f \in C[-1,1]$ be an even function. Let $p_n \in \mathcal{P}_n$ be the best uniform approximation of f in $\mathcal{P}_n$. Prove that p_n is also an even function.

(2) Let $n \geq 1$ be an integer. Let $l_0(x), \ldots, l_n(x)$ be the Lagrange basic interpolation polynomials associated with $n+1$ distinct points $x_0, \ldots, x_n$, i.e.,

$$l_k(x) = \prod_{j=1, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}, \qquad k = 0, \ldots, n.$$

Prove that

$$x^m = \sum_{j=0}^{n} x_j^m l_k(x), \qquad m = 1, \ldots, n.$$

j

Question 3.2. [20 points]

Let $n \geq 1$ be an integer and $-\infty < a < b < \infty$. Consider the numerical quadrature

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} A_k f(x_k),$$

where $x_1, \ldots, x_n \in [a,b]$ are distinct points and $A_1, \ldots, A_n \in \mathbb{R}$. Let m denote the degree of precision of this numerical quadrature. Prove the following:

(i) $m \leq 2n-1$;

(ii) If this is an interpolatory quadrature, then $m \geq n-1$;

(iii) That $m = 2n-1$ if and only if this is a Gaussian quadrature.

Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Bo Li

9am–Noon
Wednesday September 5, 2007
5402 AP&M

NAME ____________________

#1.1	30	
#1.2	30	
#1.3	30	
#2.1	30	
#2.2	30	
#2.3	30	
#3.1	30	
#3.2	30	
Total	240	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition Numbers and Linear Equations

Question 1.1.

(a) Assume that $A \in \mathbb{C}^{m\times n}$. Define the one-norm $\|A\|_1$, two-norm $\|A\|_2$, and infinity norm $\|A\|_\infty$ of A. Show that

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|.$$

(b) Assume that $D \in \mathbb{C}^{n\times n}$ with $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$. Prove that the matrix p-norm is such that $\|D\|_p = \max_{1\le i\le n} |d_i|$ for all $1 \le p \le \infty$.

Question 1.2.

(a) Consider the subtraction $x = a - b$ of two real numbers a and b such that $a \ne b$. Suppose that $\widetilde{a}$ and $\widetilde{b}$ are the result of making a *relative* perturbation Δa and Δb to a and b. Find the relative error of $\widetilde{x} = \widetilde{a} - \widetilde{b}$ as an approximation to x and hence find a condition number for the operation of subtraction. *Assume that all calculations are done in exact arithmetic.*

(b) State the *standard rounding-error model* for floating-point arithmetic. Given three representable numbers a, b and c, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the expression $s = ab+c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.3. Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite.

(a) Prove the following:
  (i) $a_{ii} > 0$, for all i;
  (ii) $|a_{ij}| \le \sqrt{a_{ii}a_{jj}}$, for all i and j; and
  (iii) $\max_{i,j} |a_{ij}| = \max_i a_{ii}$.

(b) Prove that A may be factorized as $A = LDL^T$, where L is unit lower-triangular and D is diagonal with positive diagonal elements.

2. Nonlinear Equations and Optimization

Question 2.1.

(a) Define the Frobenius norm of $A \in \mathbb{C}^{m\times n}$.

(b) Prove that for any $s \in \mathbb{R}^n$, it holds that $\|ss^T\|_F = \|s\|_2^2$. Hence show that $U^* = (y - As)s^T/s^Ts$ solves the optimization problem

$$\min \{ \|U\|_F \mid U \in \mathbb{R}^{n\times n},\ (A+U)s = y \},$$

where $A \in \mathbb{R}^{n\times n}$, and s and y are given fixed vectors in $\mathbb{R}^n$.

(c) Give a brief discussion of the significance of part (b) in reference to Broyden's method for multivariate zero finding.

Question 2.2. Let $f : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a twice continuously differentiable function with gradient vector $\nabla f(x)$.

(a) Consider the quadratic model $q_k(x) = b_k + a_k^T(x - x_k) + \frac{1}{2}(x - x_k)^T B(x - x_k)$, where b_k is a scalar, a_k is an n-vector and B is a given *fixed* symmetric positive-definite matrix. Find the values of a_k and b_k that define a model with function value and gradient equal to $f(x_k)$ and $\nabla f(x_k)$. Find the solution p_k of the quadratic subproblem

$$\underset{p\in\mathbb{R}^n}{\text{minimize}}\ \ q_k(x_k + p).$$

Prove that p_k is a descent direction for $f(x)$ at x_k.

(b) Prove that p_k is a solution of the problem

$$\underset{\substack{p\in\mathbb{R}^n\\ p\neq 0}}{\text{minimize}}\ \ \frac{p^T\nabla f(x_k)}{\|p\|_B},$$

where $\|p\|_B = (p^TBp)^{1/2}$. Briefly discuss the significance of this result.

Question 2.3. Let $F : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ be a continuously differentiable function on an open convex set $\mathcal{D}$. We seek a zero of F by minimizing the scalar-valued function $f(x) = \|F(x)\|_2$. Let x_k and p_k denote vectors in $\mathbb{R}^n$ such that $x_k \in \mathcal{D}$ and $p_k \neq 0$.

(a) If $F(x_k) \neq 0$ and φ is the univariate function $\varphi(\alpha) = \|F(x_k + \alpha p_k)\|_2$, find an expression for the directional derivative $\varphi'(\alpha)$ in terms of $F(x_k + \alpha p_k)$ and $F'(x_k + \alpha p_k)$.

(b) If p_k is the least-length solution of $\min_p \|F(x_k) + F'(x_k)p\|_2$, derive the conditions under which p_k is a descent direction for $\|F\|_2$ at x_k.

(c) Derive the termination criterion for a backtracking line search to be used in conjunction with the direction p_k defined in part (b). Derive the backtracking termination criterion for the special case where $F : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\text{rank}(F') = n$.

3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ for an integer $n \geq 0$.

Question 3.1.

(a) Let $f \in C^1[a,b]$ and $\varepsilon > 0$. Prove that there exists a polynomial p such that

$$\max_{a\leq x\leq b} |f(x) - p(x)| < \varepsilon \qquad \text{and} \qquad \max_{a\leq x\leq b} |f'(x) - p'(x)| < \varepsilon.$$

(b) Find the least-squares approximation in $\mathcal{P}_1$ of the function $f(x) = x^4$ in $L^2[-1,1]$.

Question 3.2.

(a) Let $k \geq 1$ be an integer. Suppose $p_k, q_k \in \mathcal{P}_k$ are the Lagrange interpolation polynomials that interpolate $f_0, \ldots, f_k$ at $x_0, \ldots, x_k$ and $f_1, \ldots, f_{k+1}$ at $x_1, \ldots, x_{k+1}$, respectively. Define

$$r_{k+1}(x) = \frac{(x - x_0)q_k(x) - (x - x_{k+1})p_k(x)}{x_{k+1} - x_0}.$$

Prove that $r_{k+1}(x)$ is the Lagrange interpolation polynomial that interpolates $f_0, \ldots, f_k$, and f_{k+1} at $x_0, \ldots, x_k$, and x_{k+1}.

(b) The trapezoidal numerical integration rule is given by

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\,[f(a) + f(b)].$$

Let $f \in C^2[a,b]$.
Prove that there exists $\xi \in (a,b)$ such that

$$\int_a^b f(x)\,dx = \frac{1}{2}(b-a)\,[f(a)+f(b)] - \frac{1}{12}(b-a)^3 f''(\xi).$$

Let $N \geq 1$ be an integer, $h = (b-a)/N$, and $x_k = a + kh$, $k = 0, \ldots, N$. Prove that there exists $\eta \in (a,b)$ such that

$$\int_a^b f(x)\,dx = \left\{\frac{h}{2}\,[f(a)+f(b)] + h\sum_{k=1}^{N-1} f(x_k)\right\} - \frac{(b-a)f''(\eta)}{12}h^2.$$

Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Bo Li

10am–1pm
Wednesday May 30, 2007
5402 AP&M

NAME ____________________

#1.1	30	
#1.2	30	
#1.3	30	
#2.1	30	
#2.2	30	
#2.3	30	
#3.1	30	
#3.2	30	
Total	240	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition Numbers, Linear Equations and Linear Least-Squares

In Parts 1 and 2, $\|\cdot\|_p$ refers to the vector p-norm or its subordinate matrix norm.

Question 1.1.

(a) Given any $x \in \mathbb{C}^m$, find positive constants c_1 and c_2, independent of x such that

$$c_1\|x\|_2 \le \|x\|_\infty \le c_2\|x\|_2.$$

(b) If $A \in \mathbb{C}^{m\times n}$, prove that $\|A\|_2 = \sigma_1$, where σ_1 is the largest singular value of A.

(c) Assume that $A \in \mathbb{C}^{m\times n}$ has rank r. Find a scalar σ ($\sigma > 0$), independent of p, such that

$$\|Ap\|_\infty \ge \sigma\|p\|_2 \quad \text{for all } p \in \text{range}(A^T).$$

Question 1.2.

(a) State the *standard rounding-error model* for floating-point arithmetic.

(b) Let $\mathbf{u}$ denote the unit round-off. Let n be a positive integer such that $n\mathbf{u} < 1$. If $\{\delta_i\}$ is a set of n numbers such that $|\delta_i| \le \mathbf{u}$, and $\{s_i\}$ are integers such that $s_i = \pm 1$, prove that

$$\prod_{i=1}^{n}(1+\delta_i)^{s_i} = 1 + \theta_n,$$

where $|\theta_n| \le \gamma_n$, with $\gamma_n = n\mathbf{u}/(1-n\mathbf{u})$.

(c) Given two n-vectors x and y, let $\widehat{Z}$ denote the *computed* version of the rank-one matrix $Z = xy^T$. Apply the standard rounding error model to derive a bound for the component-wise forward error in $\widehat{Z}$ as an approximation to Z. Is the calculation of $\widehat{Z}$ backward stable? Justify your answer.

Question 1.3. Assume that $A \in \mathbb{R}^{n\times n}$.

(a) Suppose that r ($r < n$) steps of Householder reduction with column interchanges gives the decomposition

$$AP = Q\begin{pmatrix} R_{11} & R_{12} \\ & 0 \end{pmatrix},$$

where Q is orthogonal, P is a permutation and R_{11} is an $r \times r$ nonsingular upper triangle. Define bases for null(A) and range(A^T) in terms of the QR factors above. Verify that the proposed bases satisfy the properties of a basis.

(b) Now assume that r steps of Householder reduction give:

$$AP = Q\begin{pmatrix} R_{11} & R_{12} \\ & E \end{pmatrix},$$

where Q is orthogonal, P is a permutation and R_{11} is an $r\times r$ nonsingular upper triangle. Show that σ_n, the smallest singular value of A, satisfies $\sigma_n \le \|E\|_2$. Give a *brief* discussion of the implication of this result.

2. Nonlinear Equations, Nonlinear Least-Squares and Optimization

Question 2.1.

(a) Let $F : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ be continuously differentiable on the open convex set $\mathcal{D}$. Compute the Fréchet derivative for the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that $f(x) = \|x\|_2$.

(b) Given a real $n \times n$ symmetric matrix A, find the Fréchet derivative of the function $G : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$ such that

$$G(x, \lambda) = \begin{pmatrix} Ax - \lambda x \\ \|x\|_2 - 1 \end{pmatrix}.$$

Hence define an iteration of Newton's method for finding an eigenvalue of A and its associated eigenvector.

(c) An eigenvalue of a matrix is *simple* if it has algebraic multiplicity 1. If λ^* is a simple eigenvalue of A and x^* is its corresponding normalized eigenvector, prove that $G'(x^*, \lambda^*)$ is nonsingular. Give a *brief* discussion of the implication of this result when finding x^* and λ^* using Newton's method.

Question 2.2. Consider the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that

$$f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.$$

(a) Compute the spectral decomposition of the Hessian matrix of second derivatives at $\bar{x} = (0, 1, 0)^T$.

(b) Compute the Newton direction p^N and modified Newton direction p^M at $\bar{x}$. Determine if p^N and p^M are descent directions.

(c) Find a direction of negative curvature that is a direction of decrease for f at $\bar{x}$.

Question 2.3.

(a) Find all the eigenvalues of the matrix $I + \gamma uv^T$, where γ is a scalar and u and v are n vectors.

(b) Given an $n \times n$ symmetric positive-definite matrix B, and n-vectors y and s, consider the symmetric rank-one quasi-Newton update

$$B_+ = B + \frac{1}{(y - Bs)^T s}(y - Bs)(y - Bs)^T. \tag{2.1}$$

(i) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a quadratic function with a symmetric positive-definite Hessian matrix. Let $s = x_+ - x$ and $y = \nabla f(x_+) - \nabla f(x)$, where $\nabla f(x)$ is the gradient of f evaluated at x. If vectors $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{y} = \nabla f(\bar{x}_+) - \nabla f(\bar{x})$ satisfy $B\bar{s} = \bar{y}$, prove that $B_+\bar{s} = \bar{y}$.

(ii) Find a condition on the vectors y and s that will guarantee the positive definiteness of B_+.

3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ for any integer $n \geq 0$.

Question 3.1.

(a) Prove for any $f \in C[a,b]$ that

$$\lim_{n\to\infty} \inf_{q_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - q_n(x)| = 0,$$

$$\lim_{n\to\infty} \inf_{q_n \in \mathcal{P}_n} \int_a^b [f(x) - q_n(x)]^2 \, dx = 0.$$

(b) Let $p_2 \in \mathcal{P}_2$ be the best uniform approximation in $\mathcal{P}_2$ of the function $g(x) = x^3 - 2x^2 + 1$ with respect to the $C[-1,1]$-norm. What is the value of $p_2(1)$? Justify your answer.

(c) Let $Q_0, \ldots, Q_n, \ldots$ be orthogonal polynomials in $L^2[a,b]$. Fix $n \geq 1$. Prove that Q_n has n simple roots in $[a,b]$.

Question 3.2.

(a) Find the degree of precision of the numerical quadrature

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)] \qquad \forall f \in C^1[a,b].$$

(b) Consider a sequence of interpolatory numerical integration formulas

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}), \qquad n = 1, \ldots .$$

Suppose all the coefficients $A_k^{(n)}$ $(k = 1, \cdots, n;\ n = 1, \cdots)$ are positive. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) = \int_a^b f(x)\,dx \qquad \forall f \in C[a,b].$$

Numerical Analysis Qualifying Examination

September 8, 2006

NAME ________________

SIGNATURE ________________

#1	20	
#2	20	
#3	20	
Total	60	

Question 1. In this problem we will analyze the case of Lagrange interpolation on a set of distinct knots $x_0 < x_1 < \ldots < x_n$ with corresponding function values $f(x_i)$, $0 \le i \le n$.

a. Show how to construct the Lagrange interpolant $p_n(x)$ satisfying $p_n(x_i) = f(x_i)$, $0 \le i \le n$ using divided differences.

b. Prove

$$f(x) - p_n(x) = f[x, x_0, x_1, \ldots, x_n] \prod_{i=0}^{n} (x - x_i).$$

Question 2. Define the terms:

a. Consistency

b. Stability

c. Convergence

as they relate to a multistep formula. Apply these concepts to analyze the two step formula

$$y_{k+1} = y_{k-1} + 2hf(y_k)$$

Question 3. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x) \approx w_1 f(x_1) + w_2 f(x_2) = \mathcal{Q}(f)$$

a. Compute the knots and weights such that $\mathcal{Q}(f)$ is the two point Gaussian quadrature formula.

b. Determine the order of the quadrature formula computed in part a.

c. Write down an expression for the error $|\mathcal{I}(f) - \mathcal{Q}(f)|$.

Numerical Analysis Qualifying Exam

Parts B and C

September 8, 2006

Name ______________________________

#1	20	
#2	20	
#3	20	
#4	20	
#5	20	
B-C	100	
A	60	
Total	160	

(20) 1. Let the *computed* L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on E : $|E_{ij}| \leq (3+u)u \max(i-1, j)g$, $g = \max\limits_{k} \max\limits_{i,j} |a_{ij}^{(k)}|$.

(20) 2. Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$, where A is $m \times n$ and $m \geq n$, iff $\hat{x}$ satisfies the normal equations.

(20) 3. (a) Prove that if A is positive semi-definite, then its eigenvalues are non-negative.

(b) Prove that if A is real symmetric, then A is positive definite iff its eigenvalues are positive.

(c) Let $B = \begin{bmatrix} A \\ a^T \end{bmatrix}$, where A is $m \times n$, $m \geqslant n$. Prove that $\sigma_n(B) \geqslant \sigma_n(A)$ and $\sigma_1(A) \leqslant \sigma_1(B) \leqslant \sqrt{\sigma_1(A)^2 + \|a\|_2{}^2}$.

(20) 4. Let A be $m \times n$.

(a) Prove that if $A^+ = \begin{cases} (A^TA)^{-1}A^T & \text{if } \operatorname{rank}(A) = n \\ A^T(AA^T)^{-1} & \text{if } \operatorname{rank}(A) = m. \end{cases}$

(b) Prove that $\|B(\lambda - A^+)\|_2 = \dfrac{\lambda}{\sigma_r({\sigma_r}^2 + \lambda)}$, where $B(\lambda) = (A^TA + \lambda I)^{-1}A^T$, $\lambda > 0$, $m \geqslant n$, $\operatorname{rank}(A) = r$.

(20) 5. Let A be symmetric positive definite.

(a) Prove that $a_{ii} > 0$ for all i and $|a_{ij}| < (a_{ii} + a_{jj})/2$ for $i \neq j$.

(b) Prove that $A = LDL^T$ exists, where L is unit lower triangular and D is diagonal with positive diagonal elements.

(c) Prove that $\max\limits_{k} \max\limits_{i,j} |a_{ij}^{(k)}| = \max\limits_{i,j} |a_{ij}|$

Numerical Analysis Qualifying Examination

June 2, 2006

NAME ________________

SIGNATURE ________________

#1	20	
#2	20	
#3	20	
Total	60	

Question 1. In this problem we will analyze the case of continuous piecewise *quadratic* interpolation on a mesh of $n+1$ knots $x_0 < x_1 < \ldots < x_n$. We will also need the interval midpoints $x_{i+1/2} = (x_i + x_{i+1})/2$. The dimension of the space is $N = 2n + 1$.

- **a.** Define the *nodal* basis functions. Note there are two types: *hat functions* and *bump functions.*
- **b.** Let f^* be the continuous piecewise quadratic interpolant for f. Using the Peano Kernel Theorem, prove

$$\| f - f^* \|_\infty \leq Ch^3 \| f''' \|_\infty$$

Question 2. Let Let $y' = f(y)$, $y(0) = y_0$. Euler's method for solving this ordinary differential equation is given by

$$y_{k+1} = y_k + h\, f(y_k)$$

for $k = 0, 1, \ldots$ and $t_k = kh$. Let $T_f = nh$ denote the final time.

- **a.** Compute the *local truncation error* for Euler's method.
- **b.** Compute the *region of absolute stability* for Euler's method.
- **c.** Using (discrete) Gronwall's Lemma, prove

$$\max_{0\leq k\leq n} |y(t_k) - y_k| \leq Ch \max_{0\leq t\leq T_f} |y''|$$

Question 3. The Euler-Maclaurin summation formula is

$$\int_a^b f(x)\,dx = T(h) + \sum_{k=1}^{r} C_k h^{2k} \{f^{(2k-1)}(b) - f^{(2k-1)}(a)\} + O(h^{2r+2})$$

where $h = (b-a)/n$, $x_k = a + kh$, C_k is a constant independent of f and h, $f \in C^{2r-1}[a, b]$, and

$$T(h) = \frac{h}{2} \sum_{k=1}^{n} f(x_{k-1}) + f(x_k)$$

is the composite trapezoid rule. Using this information derive a Richardson Extrapolation scheme for computing a high order approximation of $\int_a^b f(x)\,dx$. Be sure to define all terms carefully and explicitly state the order of each intermediate approximation.

Numerical Analysis Qualifying Exam
Parts B and C
June 2, 2006
Name ______________________________

#1	20	
#2	20	
#3	20	
#4	20	
#5	20	
B-C	100	
A	60	
Total	160	

(20) 1. State and prove the SVD Existence Theorem (for real $m \times n$ matrices).

(20) 2. Let the *computed* L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on E : $|E_{ij}| \leq (3+u)u \max(i-1, j)g$, $g = \max_k \max_{i,j} |a_{ij}^{(k)}|$.

(20) 3. Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$, where A is $m \times n$ and $m \geq n$, iff $\hat{x}$ satisfies the normal equations.

(20) 4. (a) Prove that if A is positive definite then its eigenvalues are positive.
(b) Prove that if A is normal and its eigenvalues are positive then A is positive definite.
(c) Prove that A is similar to a diagonal matrix iff A has n linearly independent eigenvectors, where A is $n \times n$.
(d) Prove that if A is real, then λ is a real eigenvalue of A iff it has a real corresponding eigenvector.

(20) 5. (a) State the Schur Decomposition Theorem.
(b) Use it to prove: if A is $n \times n$ then A has n orthonormal eigenvectors iff $A^H A = AA^H$.
(c) Show that if the single shift QR method converges, then the convergence is quadratic for general matrices.

Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Michael Holst

9am–12 Noon
Wednesday May 25, 2005
5829 AP&M

NAME ____________________

#1.1	20	
#1.2	20	
#1.3	20	
#2.1	20	
#2.2	20	
#2.3	20	
#3.1	20	
#3.2	20	
Total	160	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition numbers and Linear Equations

Question 1.1.

(a) Let $\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. Prove that for all $1 \le p \le \infty$,

$$\|\Delta\|_p = \max_{1 \le i \le n} |\delta_i|.$$

(b) Let A and B be any pair of matrices such that the product AB is defined. Prove that $\|AB\|_F \le \|A\|_2 \|B\|_F$.

(c) Let $\|\cdot\|$ and $\|\cdot\|_D$ denote any vector norm and its corresponding dual norm. If $A \in \mathbb{C}^{n\times n}$, let $\|A\|_D$ denote the matrix norm subordinate to $\|\cdot\|_D$. Prove that if $x, y \in \mathbb{C}^n$ then

$$\|xy^H\| = \|x\|\,\|y\|_D.$$

Question 1.2.

(a) Consider the subtraction $x = a - b$ of two real numbers a and b such that $a \ne b$. Suppose that $\widetilde{a}$ and $\widetilde{b}$ are the result of making a *relative* perturbation Δa and Δb to a and b. Find the relative error in $\widetilde{x} = \widetilde{a} - \widetilde{b}$ as an approximation to x and hence find a condition number for the operation of subtraction. *Assume that all calculations are done in exact arithmetic.*

(b) State the *standard rounding-error model* for floating-point arithmetic. Given three representable numbers a, b and c, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the calculation $s = ab + c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.3.

(a) Prove that every nonsingular symmetric matrix A can be written in the form $PAP^T = LBL^T$, where P is a permutation, L is unit lower triangular and B is a block-diagonal matrix with diagonal blocks of order at most one or two.

(b) *Briefly* describe the diagonal *complete* pivoting method for finding the factorization $PAP^T = LBL^T$. Show that $\|L\|$ is bounded independently of A.

2. Least-Squares and Eigenvalues

Question 2.1. Let A be an $m \times n$ with rank r. Assume that $b \in \text{range}(A)$.

(a) Derive necessary and sufficient conditions for x to be the least-length solution of $Ax = b$ and prove that the least-length solution is unique.

(b) Define an algorithm for computing the general solution of $Ax = b$ using the QR factorization of A^T with column interchanges.

(c) Use part (b) to define the least-length solution. Verify that your algorithm gives the solution of least length.

Question 2.2. Consider a non-defective matrix $A \in \mathbb{C}^{2\times 2}$ such that

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$

(a) Find the left and right eigenvectors of A.

(b) Find the condition number of each of the eigenvalues of A.

(c) Briefly discuss the situation where A is close to a defective matrix.

Question 2.3. Let $A \in \mathbb{C}^{n\times n}$. Given an approximate eigenpair (λ, u), describe how you would use one step of inverse iteration to find an improved eigenvector v of A. Hence show that (λ, v) is an exact eigenpair of $A + E$ where E may be chosen to satisfy

$$\|E\|_F = \frac{\|u\|_2}{\|v\|_2}.$$

3. Interpolation, Approximation and ODEs

Question 3.1. Consider the function $f(x) = 2x^3 - x^2 + 1$ on $[0, 2]$.

(a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates $f(x)$ at $x = 0, 1, 2$.

(b) Derive a bound on the error $|f(x) - p_2(x)|$ which is valid over the interval $[0, 2]$.

(c) Use Simpson's rule based on $p_2(x)$ to compute an approximation to

$$\mathcal{I}(f) = \int_0^2 f(x)dx,$$

and give an expression for the error in the approximation.

(d) Derive a bound on the error in the finite difference approximation:

$$f'(x) = \left[\frac{f(x+h) - f(x-h)}{2h}\right].$$

Question 3.2. Consider the problem of best L^p-approximation of a (continuous) function $u(x)$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$: Find $u^* \in V$ such that

$$\|u - u^*\|_{L^p} = \inf_{v \in V} \|u - v\|_{L^p},$$

where

$$\|u\|_{L^p} = \left(\int_0^1 |u|^p \, dx\right)^{1/p}, \ 1 \le p < \infty, \qquad \|u\|_{L^\infty} = \sup_{x \in [0,1]} |u(x)|.$$

We wish to find the best L^p-approximation of the specific function $u(x) = x^4$.

(a) Determine the best L^2-approximation in the subspace of quadratic functions; i.e., $V = \text{span}\{1, x, x^2\}$, and justify the technique you use.

(b) Why (specifically) does this problem become tremendously more difficult if we consider the case $p \neq 2$?

(c) Prove that the decomposition of an element of a Hilbert space using the Projection Theorem is unique.

Numerical Analysis Qualifying Examination

Instructor: Randolph E. Bank

September 9, 2004

NAME ________________

SIGNATURE ________________

#1	25	
#2	25	
#3	25	
#4	25	
#5	25	
#6	25	
Total	150	

Question 1. Let A be a $n \times n$ nonsingular matrix. Prove by induction that $PA = LU$, where P is a permutation matrix, L is unit lower triangular, and U is upper triangular.

Question 2. Let A and B be symmetric, positive definite $N \times N$ matrices. Assume there exist positive constants α and β such that

$$\alpha \leq \frac{x^t A x}{x^t B x} \leq \beta$$

for all $x \neq 0$. Consider the solution of $Ax = b$ by the iterative method:

$$B(x_{k+1} - x_k) = \omega(b - Ax_k)$$

where x_0 is given and $\omega = 2/(\alpha + \beta)$.

a. Derive the error propagator G for this iteration.

b. Prove:

$$\|e_k\|_A \leq \left(\frac{\beta - \alpha}{\beta + \alpha}\right)^k \|e_0\|_A$$

where $e_k = x - x_k$.

Question 3. Let $\phi(\vec{x})$ be a scalar function of the vector variable $\vec{x}$. Suppose $\phi(\vec{x})$ is continuous with continuous first and second partial derivatives, and suppose that the Hessian is symmetric and uniformly positive definite.

1. Formally define Newton's method with line search for solving the optimization problem $\min_{\vec{x}} \phi(\vec{x})$.
2. Let $\vec{p}_k$ be the Newton search direction. Show that $\partial\phi(\vec{x}_k + \alpha\vec{p}_k)/\partial\alpha < 0$ at $\alpha = 0$. Why is this fact significant for the line search?

Question 4. Consider the fundamental quadrature formula (Simpson's Rule)

$$\mathcal{I}(f) = \int_0^1 f(x)\,dx, \qquad \mathcal{Q}(f) = \frac{f(0) + 4f(1/2) + f(1)}{6}.$$

a. Using the Peano Kernel Theorem, prove

$$|\mathcal{E}(f)| = |\mathcal{I}(f) - \mathcal{Q}(f)| \le \frac{\|f^{iv}\|_{\infty[0,1]}}{2880}.$$

You may prove this result for generic C rather than $C = 1/2880$.

b. Derive the *Composite* Simpson's Rule $\mathcal{Q}_c(f)$ for approximating

$$\int_a^b f(x)\,dx$$

with a uniform mesh of size h.

c. Using part **a**, prove

$$\left| \int_a^b f(x)\,dx - \mathcal{Q}_c(f) \right| \le \frac{h^4\,|b-a|\,\|f^{iv}\|_{\infty[a,b]}}{2880}.$$

Question 5. Consider the initial value problem:

$$y' = f(y)$$
$$y(x_0) = y_0$$

and the consider the second backward difference formula:

$$y_{k+1} = \alpha_1 y_k + \alpha_2 y_{k-1} + h\beta_0 f(y_{k+1})$$

a. Find α_1, α_2 and β_0 to maximize the order.

b. Find the local truncation error.

c. Find the region of absolute stability for the method. Is the method A-stable? L-stable?

Question 6. Consider the 2-point boundary value problem $-u'' + u = f$, $u(0) = u(1) = 0$, and its Ritz formulation: Find $u \in H_0^1$ such that

$$a(u, u) - 2(f, u) = \min_{v \in H_0^1} a(v, v) - 2(f, v)$$

$$a(u, v) = \int_0^1 u'v' + uv\, dx \qquad (f, v) = \int_0^1 fv\, dx$$

$$\|u\|^2 = a(u, u) \qquad \|u\|^2 = (u, u)$$

Let $S_0 \subset H_0^1$ be the space of continuous piecewise linear polynomials and let $u_h \in S_0$ be the finite element approximation satisfying

$$a(u_h, u_h) - 2(f, u_h) = \min_{v \in S_0} a(v, v) - 2(f, v).$$

a. Prove the Ritz formulation is equivalent to the Galerkin formulation: Find $u \in H_0^1$ such that

$$a(u, v) = (f, v)$$

for all $v \in H_0^1$. A similar result holds for u_h; you may assume that result and need not prove it.

b. Using part **a**, prove the best approximation result

$$\|u - u_h\| = \min_{v \in S_0} \|u - v\|.$$

Hint: the important step is to show u_h is an orthogonal projection, $a(u - u_h, v) = 0$ for all $v \in S_0$.

Numerical Analysis Qualifying Examination

Instructor: Randolph E. Bank

May 21, 2004

NAME ____________________

SIGNATURE ____________________

#1	25	
#2	25	
#3	25	
#4	25	
#5	25	
#6	25	
Total	150	

Question 1. Let A be an $n \times n$ symmetric positive definite matrix. Consider the partitioning

$$A = \begin{pmatrix} \alpha & c^t \\ c & B \end{pmatrix}$$

where α is a scalar and B is $n-1 \times n-1$.

a. Prove the (Schur complement) matrix $B - cc^t/\alpha$ is positive definite.

b. Using part a, prove by induction that $A = LDL^t$, where L is unit lower triangular, and D is diagonal with positive diagonal elements.

Question 2. Let $x \in \mathbb{R}^n$, and $x \neq 0$.

a. Find a Householder transformation Q such that $Qx = \sigma e_1$ where σ is a scalar and e_1 the unit vector $e_1^T = (1\,0\,0\ldots0)$.

b. Show how to obtain an orthogonal matrix Q such that $Qx = \sigma e_1$ using a sequence of Givens rotations.

Question 3. Let A and B be symmetric, positive definite $N \times N$ matrices. Assume there exist positive constants α and β such that

$$\alpha \leq \frac{x^t A x}{x^t B x} \leq \beta$$

for all $x \neq 0$. Consider the solution of $Ax = b$ by the iterative method:

$$B(x_{k+1} - x_k) = \omega(b - Ax_k)$$

where x_0 is given and $\omega = 2/(\alpha + \beta)$.

a. Derive the error propagator G for this iteration.

b. Prove:

$$\|e_k\|_A \leq \left(\frac{\beta - \alpha}{\beta + \alpha}\right)^k \|e_0\|_A$$

where $e_k = x - x_k$.

Question 4. Let $F(x)$ be a vector function of a vector variable x. Here we study the solution of $F(x^*) = 0$ by Newton's method. Assume that $F(x)$ is continuously differentiable with Jacobian matrix $J(x) \equiv \partial F/\partial x$. Assume that $J(x)$ has a uniformly bounded inverse ($\|J^{-1}\| \leq \gamma$), and is Lipschitz continuous ($\|J(x) - J(y)\| \leq L\|x - y\|$).

a. Define Newton's method for solving $F(x^*) = 0$.

b. Let $e_k = x^* - x^k$ denote the error. Prove

$$\|e_{k+1}\| \leq \frac{\gamma L}{2}\|e_k\|^2.$$

Hint: you may assume the identity

$$F(y) = F(x) + \int_0^1 J(x + \theta(y - x))(y - x)\, d\theta$$

Question 5. Let $f \in \mathcal{C}^2(I)$, $I = [a, b]$, and let $x_i = a + ih$, $0 \le i \le n$, $h = (b-a)/n$ be a uniform mesh on I. Let $\mathcal{S}$ be the space of continuous piecewise linear polynomials with respect to this uniform mesh and let $\tilde{f}$ denote the continuous piecewise linear polynomial interpolant of f.

a. Compute the dimension of $\mathcal{S}$ and define the standard *nodal basis* functions $\{\phi_i\}$ for $\mathcal{S}$.

b. Using the Peano Kernel Theorem, prove:

$$\|f - \tilde{f}\|_{\mathcal{L}^2(I)} \le Ch^2 \|f''\|_{\mathcal{L}^2(I)}$$

(You do NOT need to explicitly evaluate the constant C.)

Question 6. Let $y' = f(y)$, $y(0) = y_0$. Assume $|f(w)-f(z)| \leq \mathcal{K}|w-z|$ for all $w, z \in \mathcal{R}$, and the stepsize h is constant. Let $x_k = kh$, and $y_k \approx y(x_k)$, $k = 0, 1, \ldots$ be the approximate solution generated by the Predictor-Corrector scheme based on Euler's method and the Backward Difference method

$$y_k^* = y_{k-1} + hf(y_{k-1})$$
$$y_k = y_{k-1} + hf(y_k)$$

a. Show how y_k and y_k^* can be combined to estimate the local truncation error (L.T.E.) ℓ_k for the Backward Difference method.

b. Find the region of absolute stability for each method. Note whether each method is A-stable and/or L-stable.

c. Using a (discrete) Gronwall lemma, prove that, for h sufficiently small,

$$\max_k |y(x_k) - y_k| \leq Ch\|y''\|_\infty$$