2.8: The Derivative as a Function

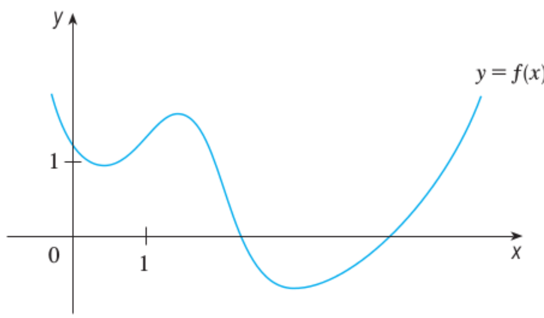
Derivatives

We can regard the **derivative** of f as a function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of f' is the set of x for which the above limits exists and is finite, $\{x \mid f'(x) \text{ exists and } |f'(x)| < \infty\}$. For each x in its domain, $f'(x)$ is the slope of the line tangent to f at the point $P(x, f(x))$.

Example 1. *Use the graph of f below to sketch the graph of f'.*



Example 2. *If $f(x) = x^3 - x$, find a formula for $f'(x)$ using the definition of derivative. Compare the graphs of f and f'.*

Example 3. *If $f(x) = \sqrt{x}$, find a formula for $f'(x)$ using the definition of derivative. Compare the graphs of f and f', and state the domain of each.*

The following are all equivalent notations for f', the derivative function of $y = f(x)$ with respect to x,

$$f' = y' = \frac{df}{dx} = \frac{dy}{dx} = Df$$

A function f is **differentiable** at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 4. *Show that the function $f(x) = |x|$ is differentiable for all real x except $x = 0$.*

Theorem 1. *If f is differentiable at a, then f is continuous at a.*

Notice the converse of this theorem is false, that is, there are continuous functions that are not differentiable. For example, $f(x) = |x|$ is continuous at 0 since $\lim_{x \to 0^-} |x| = \lim_{x \to 0^+} |x| = 0 = f(0)$, yet $f'(0)$ does not exist.

Nondifferentiable Functions

A function f is not differentiable at a if one of the following is true. The graph of f at a

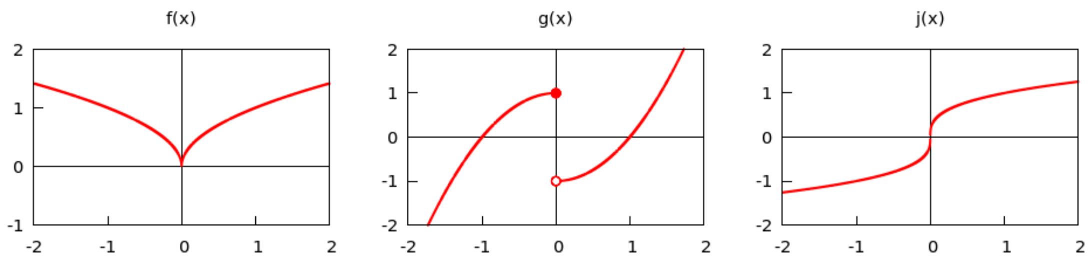
- has a "corner" or "kink" $\left(\lim\limits_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \neq \lim\limits_{h\to 0^+} \frac{f(a+h)-f(a)}{h}\right)$
- is dicontinuous $\left(\lim\limits_{h\to 0^-} \left|\frac{f(a+h)-f(a)}{h}\right| = \infty \text{ or } \lim\limits_{h\to 0^+} \left|\frac{f(a+h)-f(a)}{h}\right| = \infty\right)$
- has a vertical tangent line $\left(\lim\limits_{h\to 0^-} \left|\frac{f(a+h)-f(a)}{h}\right| = \infty \text{ or } \lim\limits_{h\to 0^+} \left|\frac{f(a+h)-f(a)}{h}\right| = \infty\right)$

Example 5. *Show that each function is not differentiable at $x = 0$.*

(a) $f(x) = \sqrt{|x|}$

(b) $g(x) = \begin{cases} -x^2+1 & \text{if } x \leq 0 \\ x^2 - 1 & \text{if } x > 0 \end{cases}$

(c) $j(x) = \sqrt[3]{x}$



Higher Derivatives

Since f' is a function, we can take its derivative to obtain the **second derivative** of f, denoted $(f')' = f''$. The following are all equivalent notations for f'',

$$(y')' = y'' \quad \text{or} \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2} \quad \text{or} \quad D(Df) = D^2 f.$$

The **third derivative** of f obtained by differentiating f'' is denoted $(f'')' = f'''$. The following are all equivalent notations for f''',

$$(y'')' = y''' \quad \text{or} \quad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \quad \text{or} \quad \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) = \frac{d^3f}{dx^3} \quad \text{or} \quad D(D^2 f) = D^3 f.$$

In general the n^{th} **derivative** of f obtained by differentiating f n times is denoted $f^{(n)}$. The following are all equivalent notations for $f^{(n)}$,

$$y^{(n)} = \frac{d^n y}{dx^n} = \frac{d^n f}{dx^n} = D^n f.$$

Example 6. *If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$. Compare the graphs of f, f', f'', and f'''.*