Numerical Analysis Qualifier

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INSTRUCTIONS: Do any (and only) 8 of the following 10 problems.

Throughout this exam $\|\cdot\|$ denotes the Euclidean vector norm or the associated induced matrix norm.

1. Condition number:

   (a) Consider the problem $f : X \to Y$, where X and Y are normed vector spaces. Define the (relative) condition number of f at $x \in X$. You may assume that f is differentiable at x.

   (b) Consider the solution of the linear system of equations $Ax = b$, where the right-hand side $b \in \mathbb{R}^n$ is the "data" and the matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular. Determine the (relative) condition number of this problem.

2. Stability: Let X and Y be normed vector spaces. Let $\tilde{f} : X \to Y$ denote an algorithm for the solution of the problem $f : X \to Y$.

   (a) What does it mean that the algorithm $\tilde{f}$ is *stable* for each $x \in X$?

   (b) What does it mean that the algorithm $\tilde{f}$ is *backward stable* for each $x \in X$?

3. Gaussian elimination: Consider the problem of solving $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is nonsingular and $x, b \in \mathbb{R}^n$, by Gaussian elimination with partial pivoting (GEPP).

   (a) Let $PA = LU$, where P is a permutation matrix. Let $\tilde{L}$ and $\tilde{U}$ be the computed factors by GEPP. Then
   $$\tilde{L}\tilde{U} = PA + E,$$
   where the matrix E satisfies
   $$\frac{\|E\|}{\|L\|\|U\|} = \mathcal{O}(\varepsilon_{\text{machine}})$$
   and $\varepsilon_{\text{machine}}$ denotes machine epsilon. Is GEPP backward stable? Justify your answer.

(b) Let $x = A^{-1}b$ and let $\tilde{x}$ be the approximate solution computed with GEPP. Give a bound for $\|x - \tilde{x}\|$. Justify your bound.

4. The SVD: Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$.

   (a) What is the singular value decomposition (SVD) of A? Give properties and sizes of the matrices involved.

   (b) Assume that $1 \leq k < n$ singular values vanish. Describe the rank, range, and null space of A in terms of the SVD.

   (c) Show that the SVD exists.

5. QR factorization: Let $A \in \mathbb{R}^{m\times n}$, $k \geq n$.

   (a) What is the QR factorization of A?

   (b) Describe the computation of the QR factorization with the aid of Householder matrices.

6. Arnoldi and GMRES: Application of ℓ steps of the Arnoldi process to the matrix $A \in \mathbb{R}^{n\times n}$ with initial vector $v_1 = b/\|b\| \in \mathbb{R}^n$ yields the Arnoldi decomposition
$$AV_\ell = V_{\ell+1}\bar{H}_\ell.$$

   (a) Describe the matrices V_ℓ, $V_{\ell+1}$, and $\bar{H}_\ell$. How large are they? What are their properties? What is range(V_ℓ)?

   (b) The GMRES method is a popular iterative method for the solution of linear systems of equations $Ax = b$, with A nonsingular. Describe the method.

7. Polynomial interpolation: Consider the problem of approximating a real-valued function f on the interval $[-1, 1]$ by a polynomial p of degree less than n, which interpolates f at the distinct points $-1 \leq x_1 < x_2 < \ldots < x_n \leq 1$. Assume that f is differentiable sufficiently many times.

   (a) Give an expression for the error $f(x) - p(x)$ for $-1 \leq x \leq 1$.

   (b) Prove the above expression.

8. Chebyshev polynomials:

   (a) Define the family of Chebyshev polynomials $T_0, T_1, T_2, \ldots$ for the interval $[-1, 1]$. What are the zeros of T_n?

(b) Discuss the significance of the zeros of T_n for the polynomial interpolation problem considered above.

9. Orthogonal polynomials and Gauss quadrature:

   (a) Give the inner product with respect to which Chebyshev polynomials are orthogonal.

   (b) Let $x_1, x_2, \ldots, x_n$ be the zeros of the nth orthogonal polynomial with respect to the inner product

   $$(f, g) = \int_a^b f(x)g(x)w(x)dx,$$

   where $w(x)$ is a nonnegative weight function defined on the interval $[a, b]$. Let $w_1, w_2, \ldots, w_n$ be weights of the quadrature rule

   $$\int_a^b f(x)w(x)dx \approx \sum_{k=1}^{n} w_k f(x_k),$$

   constructed to be exact for all polynomials of degree strictly less than n. Prove that this rule is exact for all polynomials of degree strictly less than $2n$.

10. The Newton method: Let $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear differentiable function. The equation $f(x) = 0$ can be solved by Newtons's method.

   (a) Describe Newton's method for the solution of $f(x) = 0$.

   (b) Define quadratic convergence of an iterative method.

   (c) Show that Newton's method yields quadratic convergence.