Solutions for Homework 7 Foundations of Computational Math 2 Spring 2012

Problem 7.1

For this problem, consider the space $\mathcal{L}^2[-1,1]$ with inner product and norm

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx \text{ and } \|f\|^2 = (f,f)$$

Let $P_i(x)$, for $i = 0, 1, \dots$ be the Legendre polynomials of degree i and let $n+1-$st have the form

$$P_{n+1}(x) = \rho_n(x - x_0)(x - x_1)\cdots(x - x_n)$$

i.e., x_i for $0 \le i \le n$ are the roots of $P_{n+1}(x)$.

Let the Lagrange interpolation functions that use the x_i be $\ell_i(x)$ for $0 \le i \le n$. So, for example,

$$L_n(x) = \ell_0(x)f(x_0) + \cdots + \ell_n(x)f(x_n)$$

is the Lagrange form of the interpolation polynomial of $f(x)$ defined by the roots.

Let $\mathbb{P}_n$ be the space of polynomials of degree less than or equal to n. We can write the least squares approximation of $f(x)$ in terms of the $P_i(x)$ using the generalized Fourier series as

$$f_n(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \cdots + \alpha_n P_n(x) \text{ where } \alpha_i = \frac{(f, P_i)}{(P_i, P_i)}$$

7.1.a

Clearly, $(\ell_i, \ell_i) \neq 0$. Show that $(\ell_i, \ell_j) = 0$ when $i \neq j$. Therefore, the functions $\ell_0(x), \dots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$.

Solution:

Recall that $P_{n+1}(x) \perp \mathbb{P}_n$ so $(P_{n+1}, p) = 0$ for any $p(x) \in \mathbb{P}_n$. Also, we have for a constant scale factor γ_i

$$\ell_i(x) = \gamma_i \frac{P_{n+1}(x)}{(x - x_i)}$$

We therefore have

$$\begin{aligned}
(\ell_i, \ell_j) &= \gamma_i\gamma_j \int_{-1}^{1} \left(\frac{P_{n+1}(x)}{(x - x_i)}\right)\left(\frac{P_{n+1}(x)}{(x - x_j)}\right) dx \\
&= \gamma_i\gamma_j \int_{-1}^{1} P_{n+1}(x) \left(\frac{P_{n+1}(x)}{(x - x_i)(x - x_j)}\right) dx \\
&= \gamma_i\gamma_j (P_{n+1}, p_{n-1}) = 0
\end{aligned}$$

where $p_{n-1}(x) \in \mathbb{P}_n$ since it is a polynomial of degree $n - 1$.

7.1.b

Suppose we evaluate $f_n(x)$ at the x_i to obtain the data $f_n(x_0), \ldots, f_n(x_n)$. We can then write $f_n(x)$ in its Lagrange form,

$$f_n(x) = L_n(x) = f_n(x_0)\ell_0(x) + \ldots + f_n(x_n)\ell_n(x)$$

Since the $\ell_0(x), \ldots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$, they also can be used to compute, $f_n(x)$, the unique least squares approximation to $f(x)$. As with the Legendre polynomials, using the generalized Fourier series, yields

$$f_n(x) = \sigma_0\ell_0(x) + \sigma_1\ell_1(x) + \cdots + \sigma_n\ell_n(x) \text{ where } \sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)}$$

Show that these last two forms of $f_n(x)$ give the same polynomial by showing that

$$\sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)} = f_n(x_i)$$

Hint: Consider the relationship between $f(x)$ and $f_n(x)$.

Solution:

We have $f(x) = f_n(x) + r(x)$ where the residual satisfies $r(x) \perp \mathbb{P}_n$ for this inner product. Recall, this is a basic property of least squares approximations.

Since $\ell_i(x) \in \mathbb{P}_n$, we have

$$(f, \ell_i) = (f_n + r, \ell_i) = (f_n, \ell_i) + (r, \ell_i) = (f_n, \ell_i)$$

Now use the Lagrange form

$$f_n(x) = f_n(x_0)\ell_0(x) + f_n(x_1)\ell_1(x) + \cdots + f_n(x_n)\ell_n(x)$$

in the inner product (f_n, ℓ_i) and the orthogonality of the $\ell_i(x)$ functions to get

$$\begin{aligned}
(f_n, \ell_i) &= (\sum_j \ell_j(x) f_n(x_j), \ell_i) \\
&= \int_{-1}^{1} \sum_j \ell_i(x)\ell_j(x) f_n(x_j) dx \\
&= \sum_j \int_{-1}^{1} \ell_i(x)\ell_j(x) f_n(x_j) dx \\
&= \int_{-1}^{1} \ell_i(x)\ell_i(x) f_n(x_i) \\
&= (\ell_i, \ell_i) f_n(x_i)
\end{aligned}$$

$$\sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)} = \frac{(\ell_i, \ell_i) f_n(x_i)}{(\ell_i, \ell_i)} = f_n(x_i)$$

Problem 7.2

Consider $f(x) = e^x$ on the interval $-1 \leq x \leq 1$. Suppose we want to approximate $f(x)$ with a polynomial. Generate the following polynomials:

(a) $F_1(x)$ and $F_3(x)$:the first and third order Taylor series approximations of $f(x)$ expanded about $x = 0$.

(b) $N_1(x)$: the linear near-minimax approximation to $f(x)$ on the interval.

(c) $C_1(x)$ and $C_2(x)$ – the linear and quadratic polynomials that result from Chebyshev economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(d) $p_1(x)$ and $p_2(x)$ – the linear and quadratic polynomials that result from Legendre economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(7.2.a) Derive bounds on the ∞ norm of the error where possible.

(7.2.b) Evaluate the error for each polynomial approximation on a very fine grid on the interval $-1 \leq x \leq 1$ and compare to the bounds.

Solution:

Taylor Series: The Taylor series of degree k about $x = 0$ is

$$F_k(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!}$$

We therefore have the truncated Taylor series

$$F_1(x) = 1 + x$$
$$F_2(x) = 1 + x + \frac{x^2}{2}$$
$$F_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Chebyshev Economization: To compute the Chebyshev economizations of $F_3(x)$ we note the monic forms to generate the approximations of x^2 and x^3

$$T_2 = 2x^2 - 1 \rightarrow t_2 = x^2 - \frac{1}{2} \rightarrow x^2 \approx \frac{1}{2}$$
$$T_3 = 4x^3 - 3x \rightarrow t_3 = x^3 - \frac{3x}{4} \rightarrow x^3 \approx \frac{3x}{4}$$

We therefore have

$$
\begin{aligned}
F_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \\
\approx 1 + x + \frac{x^2}{2} + \frac{1}{6} \times \frac{3x}{4} &= 1 + \frac{9}{8}x + \frac{x^2}{2} = C_2(x) \\
C_2(x) \approx 1 + \frac{9}{8}x + \frac{1}{2} \times \frac{1}{2} &= \frac{5}{4} + \frac{9}{8}x = C_1(x)
\end{aligned}
$$

Legendre Economization: For Legendre polynomials we have

$$x^0 = P_0(x), \quad x = P_1(x)$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

The truncated Taylor series can be rewritten in terms of the Legendre basis for polynomials of degree 3 or less and then truncated appropriately:

$$
\begin{aligned}
F_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \\
&= P_0(x) + P_1(x) + \frac{1}{2}\big[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\big] + \frac{1}{6}\big[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\big] \\
&= \frac{7}{6}P_0(x) + \frac{11}{10}P_1(x) + \frac{1}{3}P_2(x) + \frac{1}{15}P_3(x)
\end{aligned}
$$

$$
\begin{aligned}
p_2(x) &= \frac{7}{6}P_0(x) + \frac{11}{10}P_1(x) + \frac{1}{3}P_2(x) \\
&= \frac{7}{6} + \frac{11}{10}x + \frac{1}{3}\big(\frac{3}{2}x^2 - \frac{1}{2}\big) \\
&= 1 + \frac{11}{10}x + \frac{1}{2}x^2
\end{aligned}
$$

$$
\begin{aligned}
p_1(x) &= \frac{7}{6}P_0(x) + \frac{11}{10}P_1(x) \\
&= \frac{7}{6} + \frac{11}{10}x
\end{aligned}
$$

We can also use the alternate derivation based on the Generalized Fourier Series

$$
\begin{aligned}
\gamma_0 &= \frac{(p, P_0)}{(P_0, P_0)} = \frac{1}{2}\big(\alpha_0(1, P_0) + \alpha_1(x, P_0) + \alpha_2(x^2, P_0) + \alpha_3(x^3, P_0)\big) \\
\gamma_1 &= \frac{(p, P_1)}{(P_1, P_1)} = \frac{3}{2}\big(\alpha_0(1, P_1) + \alpha_1(x, P_1) + \alpha_2(x^2, P_1) + \alpha_3(x^3, P_1)\big) \\
\gamma_2 &= \frac{(p, P_2)}{(P_2, P_2)} = \frac{5}{2}\big(\alpha_0(1, P_2) + \alpha_1(x, P_2) + \alpha_2(x^2, P_2) + \alpha_3(x^3, P_2)\big) \\
\gamma_3 &= \frac{(p, P_3)}{(P_3, P_3)} = \frac{7}{2}\big(\alpha_0(1, P_3) + \alpha_1(x, P_3) + \alpha_2(x^2, P_3) + \alpha_3(x^3, P_3)\big)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_0 &= \frac{1}{2}\left(2\alpha_0 + 0\alpha_1 + \frac{2}{3}\alpha_2 + 0\alpha_3\right) = \alpha_0 + \frac{1}{3}\alpha_2 \\
\gamma_1 &= \frac{3}{2}\left(0\alpha_0 + \frac{2}{3}\alpha_1 + 0\alpha_2 + \frac{2}{5}\alpha_3\right) = \alpha_1 + \frac{3}{5}\alpha_3 \\
\gamma_2 &= \frac{5}{2}\left(0\alpha_0 + 0\alpha_1 + \frac{4}{15}\alpha_2 + 0\alpha_3\right) = \frac{2}{3}\alpha_2 \\
\gamma_3 &= \frac{7}{2}\left(0\alpha_0 + 0\alpha_1 + 0\alpha_2 + \frac{4}{35}\alpha_3\right) = \frac{2}{5}\alpha_3
\end{aligned}
$$

So we have

$$
F_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}
$$

$$
\alpha_0 = 1, \quad \alpha_1 = 1, \quad \alpha_2 = 1/2, \quad \alpha_3 = 1/6
$$

$$
\begin{aligned}
\gamma_0 &= \alpha_0 + \frac{1}{3}\alpha_2 = \frac{7}{6} \\
\gamma_1 &= \alpha_1 + \frac{3}{5}\alpha_3 = \frac{11}{10} \\
\gamma_2 &= \frac{2}{3}\alpha_2 = \frac{1}{3} \\
\gamma_3 &= \frac{2}{5}\alpha_3 = \frac{1}{15}
\end{aligned}
$$

$$
\begin{aligned}
F_3(x) &= \gamma_0 P_0(x) + \gamma_1 P_1(x) + \gamma_2 P_2(x) + \gamma_3 P_3(x) \\
&= \frac{7}{6}P_0(x) + \frac{11}{10}P_1(x) + \frac{1}{3}P_2(x) + \frac{1}{15}P_3(x)
\end{aligned}
$$

Truncation proceeds as above.

Near-minimax: The near-minimax approximation interpolates $f(x)$ at the roots of $T_2(x)$ which are $\pm\sqrt{1/2}$. Letting $\alpha = 1/\sqrt{2}$ and after some manipulation we have

$$N_1(x) = \alpha^2(e^{\alpha} - e^{-\alpha}) + \alpha(e^{\alpha} - e^{-\alpha})x$$

Error Bounds: The Taylor series error on $-1 \leq x \leq 1$

$$|e^x - F_k(x)| \leq \frac{xe^{\xi(x)}}{(k+1)!} \leq \frac{e}{(k+1)!}$$

Therefore we have

$$|e^x - F_3(x)| \leq \frac{e}{24} \leq 0.12$$
$$|e^x - F_2(x)| \leq \frac{e}{6} \leq 0.45$$
$$|e^x - F_1(x)| \leq \frac{e}{2} \leq 1.36$$

For Chebyshev economization the error introduced at each step is given by $\alpha_k|t_k(x)|$ so we have

$$|F_3(x) - C_2(x)| \leq \alpha_3|t_3(x)| = \frac{1}{24}$$

$$|F_3(x) - C_1(x)| \leq \alpha_3|t_3(x)| = \frac{1}{24} + \alpha_2|t_2(x)| = \frac{1}{24} + \frac{1}{4} = \frac{7}{24}$$

$$|e^x - C_2| \leq \frac{e+1}{24} \approx 0.155$$
$$|e^x - C_1| \leq \frac{e+7}{24} \approx 0.405$$

For Legendre economization we have

$$e^x = F_3(x) + E_3(x)$$
$$= \frac{7}{6}P_0(x) + \frac{11}{10}P_1(x) + \frac{1}{3}P_2(x) + \frac{1}{15}P_3(x) + E_3(x)$$

Therefore

$$|e^x - p_2(x)| \leq |E_3(x)| + \frac{1}{15}|P_3(x)|$$
$$|e^x - p_1(x)| \leq |E_3(x)| + \frac{1}{15}|P_3(x)| + \frac{1}{3}|P_2(x)|$$

Since $|P_3(x)| \le 1$ and $|P_2(x)| \le 1$ on $-1 \le x \le 1$ we have

$$|e^x - p_2(x)| \le \frac{e}{24} + \frac{1}{15} \le 0.18$$
$$|e^x - p_1(x)| \le \frac{e}{24} + \frac{1}{15} + \frac{1}{3} \le 0.52$$

For the near-minimax we use the standard interpolation error and exploit the fact that the monic polynomial is $t_2(x)$, i.e.,

$$|e^x - N_1(x)| = \frac{1}{2} t_2(x) e^{\xi} \le \frac{e}{4} \approx 0.68$$

A plot of the errors for the Taylor, Chebyshev and Legendre quadratic polynomials is shown in

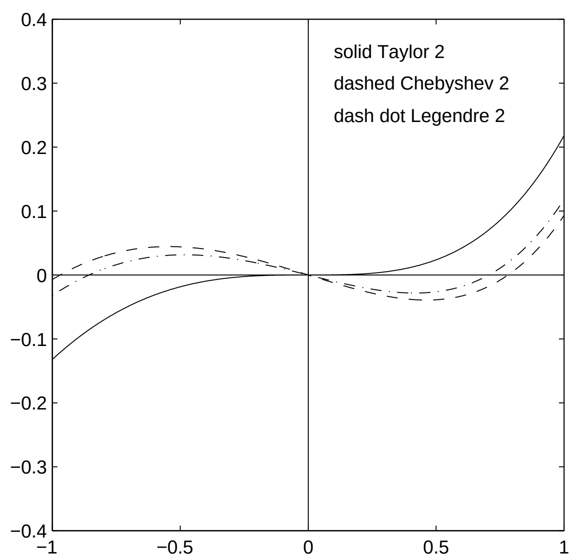

Figure 1: Error for quadratic economizations of e^x on $-1 \le x \le 1$.