Set 4: Polynomial Interpolation – Part 4

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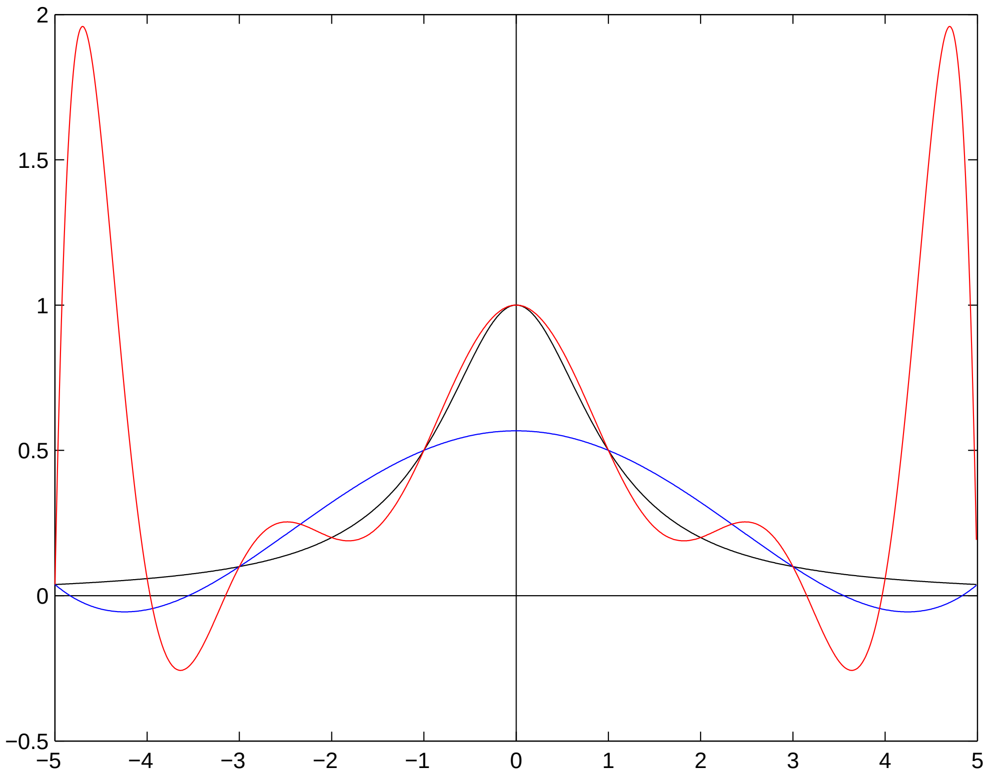
Hermite Interpolation and Osculatory Polynomials



Note. function values OK at points, derivatives are not, sometimes even wrong sign

Approach

Solution:

- specify function values $f(x_i) = y_i$
- specify derivative values $f'(x_i) = y'_i$

Repeat approaches:

- power basis: $p_n(x) = \sum_{i=0}^{n} \alpha_i x^i$
- Lagrange form: $p_n(x) = \sum_{i=0}^{n} \Big[y_i \psi_i(x) + y'_i \Psi_i(x) \Big]$
- Newton form: $p_n(x) = \sum_{i=0}^{n} \alpha_i \Omega_i(x)$

Constrain Derivatives

For example, given 4 constraints construct $p_3(x) = \sum_{i=0}^{3} \alpha_i x^i$:

$$y(0) = p_3(0)$$
$$y'(0) = p_3'(0)$$
$$y(1) = p_3(1)$$
$$y'(1) = p_3'(1)$$
$$\Downarrow$$
$$\alpha_0 = y(0)$$
$$\alpha_1 = y'(0)$$
$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = y(1)$$
$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = y'(1)$$

Example

$$\alpha_0 = y(0)$$

$$\alpha_1 = y'(0)$$

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = y(1)$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = y'(1)$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \\ y(1) \\ y'(1) \end{pmatrix}$$

Example

$$y(0) = 3, \;\; y'(0) = 2$$

$$y(1) = 6, \;\; y'(1) = 1$$

$$\Downarrow$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ -3 \end{pmatrix}$$

$$p_3(x) = -3x^3 + 4x^2 + 2x + 3, \;\; p_3'(x) = -9x^2 + 8x + 2$$

$$p_3(0) = 3, \;\; p_3'(0) = 2, \;\; p_3(1) = 6, \;\; p_3'(1) = 1$$

Monomial Form – Hermite Interpolation

$$p_d(x_i) = y_i \text{ and } p'_d(x_i) = y'_i \quad 0 \le i \le n$$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^d \\ 0 & 1 & 2x_0 & \dots & dx_0^{d-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^d \\ 0 & 1 & 2x_1 & \dots & dx_1^{d-1} \\ \vdots & \vdots & & \vdots & \\ 1 & x_n & x_n^2 & \dots & x_n^d \\ 0 & 1 & 2x_n & \dots & dx_n^{d-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-1} \\ \alpha_d \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \\ y_1 \\ y'_1 \\ \vdots \\ y_n \\ y'_n \end{pmatrix}$$

$$V^T a = y \text{ and } d = 2n+1$$

Monomial Form – Hermite Interpolation

- V is a confluent Vandermonde matrix.
- The confluent columns corrrespond to derivative value constraints.
- V is nonsingular if the values defining the “nonconfluent” columns are distinct.
- Existence and uniqueness of the Hermite interpolating polynomial of degree $2n + 1$ follows.

Lagrange Form – Hermite Interpolation

Constraints

$$p(x_0) = y_0, \;\; p'(x_0) = y'_0$$
$$p(x_1) = y_1, \;\; p'(x_1) = y'_1$$
$$p(x_2) = y_2, \;\; p'(x_2) = y'_2$$
$$\vdots$$
$$p(x_n) = y_n, \;\; p'(x_n) = y'_n$$

$2n + 2$ conditions $\rightarrow$ degree of $p(x)$ is $2n + 1$

Lagrange Form – Hermite Interpolation

Constraints on basis functions

$$p(x) = \sum_{i=0}^{n} \Big[y_i \psi_i(x) + y_i' \Psi_i(x) \Big] \text{ and } p'(x) = \sum_{i=0}^{n} \Big[y_i \psi_i'(x) + y_i' \Psi_i'(x) \Big]$$

$$\delta_{ii} = 1 \quad \delta_{ij} = 0 \text{ for } i \neq j, \;\; 0 \leq i, j \leq n$$

$$\psi_i(x_j) = \delta_{ij}, \qquad \Psi_i(x_j) = 0 \rightarrow p(x_i) = y_i$$

$$\psi_i'(x_j) = 0, \qquad \Psi_i'(x_j) = \delta_{ij} \rightarrow p'(x_i) = y_i'$$

Lagrange Form – Hermite Interpolation

$$\psi_i(x) = \ell_i^2(x)\Big[1 - 2\ell_i'(x_i)(x - x_i)\Big]$$

$$\psi_i(x_j) = \delta_{ij} \text{ as desired}$$

$$\psi_i'(x) = 2\ell_i'(x)\ell_i(x)\Big[1 - 2\ell_i'(x_i)(x - x_i)\Big] - 2\ell_i'(x_i)\ell_i^2(x)$$

$$\psi_i'(x_j) = 0, \quad \text{as desired}$$

$$\psi_i'(x_i) = 2\ell_i'(x_i) \times 1\Big[1 - 0\Big] - 2\ell_i'(x_i) \times 1 = 0 \quad \text{as desired}$$

Derivation of $\psi_i(x)$

$\psi_i(x)$ has degree $2n+1$ and has double roots at x_j, $i \neq j$

$\ell_i^2(x)$ has degree $2n$ with

$$\ell_i^2(x_j) = \delta_{ij} \quad n+1 \text{ conditions}$$

$$\left[\ell_i^2(x_j)\right]' = 0 \quad i \neq j \quad \text{but also} \quad \left[\ell_i^2(x_i)\right]' \neq 0 \quad \text{generally}$$

We have a free degree so consider a linear function $g(x)$ and take

$$\psi_i(x) = \ell_i^2(x)g(x)$$

Check conditions and determine $g(x)$.

Derivation of $\psi_i(x)$

We have

$$\psi_i(x_j) = \ell_i^2(x_j)g(x_j) = 0 \quad i \neq j$$

$$g(x_i) = 1 \rightarrow \psi_i(x_i) = \ell_i^2(x_i)g(x_i) = 1$$

$$\therefore \quad \text{take the form } g(x) = 1 + \beta(x - x_1)$$

$$\psi_i(x) = \ell_i^2(x_j)(1 + \beta(x - x_i))$$

Derivation of $\psi_i(x)$

$$\psi_i(x) = \ell_i^2(x_j)\left[1 + \beta(x - x_i)\right]$$

$$\psi_i'(x) = \beta\ell_i^2(x) + 2\left[1 + \beta(x - x_i)\right]\ell_i(x)\ell_i'(x)$$

$$\psi_i'(x_j) = 0 \quad i \neq j$$

So β must be chosen to satisfy $\psi_i'(x_i) = 0$.

Derivation of $\psi_i(x)$

$$\psi_i(x) = \ell_i^2(x_j)\left[1 + \beta(x - x_i)\right]$$

$$\psi_i'(x) = \beta\ell_i^2(x) + 2\left[1 + \beta(x - x_i)\right]\ell_i(x)\ell_i'(x)$$

$$\psi_i'(x_i) = \beta + 2\ell_i'(x_i)$$

$$\therefore \beta = -2\ell_i'(x_i) \rightarrow \psi_i'(x_i) = 0$$

Lagrange Form – Hermite Interpolation

$$\Psi_i(x) = \ell_i^2(x)(x - x_i)$$

$$\Psi_i(x_j) = 0, 0 \leq i, j \leq n \quad \text{as desired}$$

$$\Psi_i'(x) = \ell_i^2(x) + 2\ell_i'(x)\ell_i(x)(x - x_i)$$

$$\Psi_i'(x_j) = \delta_{ij}, \quad i \neq j \quad \text{as desired}$$

Lagrange Form – Hermite Interpolation

Theorem 4.1. *Given the constraints,* $0 \leq i \leq n$*,*

$$H_d(x_i) = y_i, \;\; H_d'(x_i) = y_i', \;\; x_i \in [a,b], \;\; x_i \neq x_j$$

The unique Hermite interpolation polynomial of degree $d = 2n+1$ *is*

$$H_d(x) = \sum_{i=0}^{n} \Big[y_i \psi_i(x) + y_i' \Psi_i(x) \Big]$$

$$\psi_i(x) = \ell_i^2(x) \Big[1 - 2\ell_i'(x_i)(x - x_i) \Big]$$

$$\Psi_i(x) = \ell_i^2(x)(x - x_i)$$

Further, if $y(x) \in \mathcal{C}^{(d+1)}$ *defines the* y_i *and* y_i' *then* $\exists \xi \in [a,b]$ *such that*

$$y(x) - H_d(x) = \frac{y^{(d+1)}(\xi)}{(d+1)!} \prod_{i=0}^{n} (x - x_i)^2$$

Lagrange Form – Hermite Interpolation

- Construction of the Hermite interpolant requires computing the $m_i(x_i)$ values as before for the Lagrange form.
- Construction of the Hermite interpolant requires computing the $\ell_i'(x_i)$ which requires $m_i'(x_i)$ values.
- $O(n^2)$ incremental construction via recurrences like the forms of Lagrange.
- Complexity of evaluation of the Hermite interpolant is left as an exercise.

Example

$$\psi_i(x) = \ell_i^2(x)\Big[1 - 2\ell_i'(x_i)(x - x_i)\Big], \qquad \Psi_i(x) = \ell_i^2(x)(x - x_i)$$

$$x_0 = 1, \;\; y_0 = 3, \;\; y_0' = 2,$$

$$x_1 = 2, \;\; y_1 = 6, \;\; y_1' = 1$$

$$\psi_0(x) = (x-2)^2(2x-1), \quad \psi_1(x) = (x-1)^2(5-2x)$$

$$\Psi_0(x) = (x-2)^2(x-1), \quad \Psi_1(x) = (x-1)^2(x-2)$$

$$H_3(x) = 3(x-2)^2(2x-1) + 2(x-1)(x-2)^2$$

$$+6(x-1)^2(5-2x) + (x-1)^2(x-2)$$

$$H_3(x) = 8 - 15x + 13x^2 - 3x^3$$

$$H_3'(x) = -15 + 26x - 9x^2$$

Example

$$x_0 = 1, \;\; y_0 = 3, \;\; y_0' = 2,$$

$$x_1 = 2, \;\; y_1 = 6, \;\; y_1' = 1$$

$$H_3(x) = 8 - 15x + 13x^2 - 3x^3$$

$$H_3(1) = 8 - 15 + 13 - 3 = 3$$

$$H_3(2) = 8 - 30 + 52 - 24 = 6$$

$$H_3'(x) = -15 + 26x - 9x^2$$

$$H_3'(1) = -15 + 26 - 9 = 2$$

$$H_3'(2) = -15 + 52 - 36 = 1$$

Newton Form – Hermite Interpolation

The question is what do we do with divided differences of the form $y[x_i, x_i]$?

$$y[x_i, x_i] = \lim_{x_j \to x_i} y[x_i, x_j]$$

$$= \lim_{x_j \to x_i} \frac{y(x_j) - y(x_i)}{x_j - x_i}$$

$$= y'(x_i)$$

This can be used to define the necessary form of the divided difference table.

Newton Form – Hermite Interpolation

Given, $n = 2$, $(x_0, y_0), (x_0, y_0'), (x_1, y_1), (x_1, y_1')$, we create the table for

$$\hat{n} = 3, \quad (\hat{x}_0, y_0), (\hat{x}_1, y_0), (\hat{x}_1, y_1), (\hat{x}_2, y_1')$$

and use derivative values for divided differences with repeated values of $\hat{x}_i$, e.g., $y[\hat{x}_0, \hat{x}_1] = y[x_0, x_0]$.

i	0	1	2	3
$\hat{x}_i$	x_0	x_0	x_1	x_1
f_i	y_0	y_0	y_1	y_1
$y[*,*]$	–	$y[x_0, x_0] = y_0'$	$y[x_0, x_1]$	$y[x_1, x_1] = y_1'$
$y[*,*,*]$	–	–	$y[x_0, x_0, x_1]$	$y[x_0, x_1, x_1]$
$y[*,*,*,*]$	–	–	–	$y[x_0, x_0, x_1, x_1]$

Newton Form – Hermite Interpolation

Using the Newton form in terms of $\hat{x}_i$ first

$$
\begin{aligned}
H_3(x) = y_0 &+ (x - \hat{x}_0) y[\hat{x}_0, \hat{x}_1] \\
&+ (x - \hat{x}_0)(x - \hat{x}_1) y[\hat{x}_0, \hat{x}_1, \hat{x}_2] \\
&+ (x - \hat{x}_0)(x - \hat{x}_1)(x - \hat{x}_2) y[\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3]
\end{aligned}
$$

Now substitute differences and derivatives knowing

$$
\hat{x}_0 = \hat{x}_1 = x_0 \quad \text{and} \quad \hat{x}_2 = \hat{x}_3 = x_1
$$

Newton Form – Hermite Interpolation

$$
\begin{aligned}
H_3(x) &= y_0 + (x - x_0) y[x_0, x_0] \\
&\quad + (x - x_0)^2 y[x_0, x_0, x_1] \\
&\quad + (x - x_0)^2 (x - x_1) y[x_0, x_0, x_1, x_1] \\
&= y_0 + (x - x_0) y_0' + (x - x_0)^2 y[x_0, x_0, x_1] \\
&\quad + (x - x_0)^2 (x - x_1) y[x_0, x_0, x_1, x_1]
\end{aligned}
$$

where the remaining divided differences are defined as in the table.

Note as before, other paths through the table can be used.

Example

$$x_0 = 1, \;\; y_0 = 3, \;\; y_0' = 2,$$

$$x_1 = 2, \;\; y_1 = 6, \;\; y_1' = 1$$

$$y[x_0, x_0] = 2, \;\; y[x_0, x_1] = 3, \;\; y[x_1, x_1] = 1$$

$$y[x_0, x_0, x_1] = 1, \;\; y[x_0, x_1, x_1] = -2$$

$$y[x_0, x_0, x_1, x_1] = -3$$

$$H_3(x) = 3 + 2(x-1) + 1(x-1)^2 - 3(x-1)^2(x-2)$$

$$H_3(x) = 8 - 15x + 13x^2 - 3x^3$$

Osculating Polynomial

Definition 4.1. Let $x_i \in [a, b],\ 0 \le i \le n$ be distinct points, $m_i \in \mathbb{Z}^+,\ 0 \le i \le n$, and $f(x) \in \mathcal{C}^{(m)}[a.b]$ with $m = \max_i m_i$. The unique osculating polynmial, $p_d(x)$, interpolating $f(x)$ satisfies

$$\frac{d^k p}{dx^k}(x_i) = \frac{d^k f}{dx^k}(x_i)$$

$$0 \le i \le n \text{ and } 0 \le k \le m_i$$

$$d + 1 = \sum_{i=0}^{n}(m_i + 1)$$

Osculating Polynomial

Cases of Osculating Polynomials:

- $n = 0$: Taylor polynomial of degree m_0 at x_0.
- $\forall i \;\; m_i = 0$: Lagrange/Newton interpolating polynomial
- $\forall i \;\; m_i = 1$: Hermite interpolating polynomial
- General case: Hermite-Birkoff interpolating polynomial (see text p. 349 for basis functions)