Homework 9 Foundations of Computational Math 2 Spring 2012

Solutions will be posted Monday, 3/19/12

Problem 9.1

In this problem we consider the numerical approximation of the integral

$$I = \int_{-1}^{1} f(x)dx$$

with $f(x) = e^x$. In particular, we use a priori error estimation to choose a step size h for Newton Cotes or a number of points for a Gaussian integration method.

9.1.a

Consider the use of the composite Trapezoidal rule to approximate the integral I.

- Use the fact that we have an analytical form of $f(x)$ to estimate the error using the composite trapezoidal rule and to determine a stepsize h so that the error will be less than or equal to the tolerance 10^{-2}.
- Approximately how many points does your h require?

9.1.b

Consider the use of the Gauss-Legendre method to approximate the integral I. Use $n = 1$, i.e., two points x_0 and x_1 with weights γ_0 and γ_1.

- Use the fact that we have an analytical form of $f(x)$ to estimate the error that will result from using the two-point Gauss-Legendre method to approximate the integral.
- How does your estimate compare to the tolerance 10^{-2} used in the first part of the question?
- Recall that for $n = 1$ we have the Gauss Legendre nodes $x_0 \approx -0.5774$ and $x_1 \approx 0.5774$. Apply the method to approximate I and compare its error to your prediction. The true value is
$$I = \int_{-1}^{1} e^x dx \approx 2.3504$$

Problem 9.2

Suppose $\phi_i(x)$, $i = 0, 1, \ldots$ are a set of orthonormal polynomials with respect to the inner product

$$(f, g)_\omega = \int_a^b \omega(x) f(x) g(x)\, dx$$

where $\phi_i(x)$ has degree i. Show that

$$\int_a^b \omega(\xi) G_n(x, \xi)\, d\xi = 1$$

where

$$G_n(x, \xi) = \sum_{i=0}^{n} \phi_i(x) \phi_i(\xi)$$

Problem 9.3

Let $U(x)$ and $V(x)$ be polynomials of degree n defined on $x \in [-1, 1]$. Let $x_j, \quad 0 \le j \le n$ and $\gamma_j, \quad 0 \le j \le n$ be the Gauss-Legendre quadrature points and weights. Finally, let $\ell_j(x), \quad 0 \le j \le n$ be the Lagrange characteristic interpolating polynomials defined with nodes at the Gauss-Legendre quadrature points.

Show that the following summation by parts formula holds:

$$\sum_{j=0}^{n} U'(x_j) V(x_j) \gamma_j = (U(1)V(1) - U(-1)V(-1)) - \sum_{j=0}^{n} U(x_j) V'(x_j) \gamma_j$$

Problem 9.4

For the Legendre polynomials, $P_n(x)$, we have the recurrence

$$P_0 = 1, \quad P_1 = x$$
$$P_{n+1} = \frac{2n+1}{n+1} x P_n - \frac{n}{n+1} P_{n-1}$$

This yields a form that is orthogonal but it is not monic and it is not orthonormal. For example, we have

$$P_2 = \frac{3}{2} x^2 - \frac{1}{2}$$

$$P_3 = \frac{5}{2} x^3 - \frac{3}{2} x$$

9.4.a

Let $\tilde{P}_n(x)$ be the Legendre polynomial that is normalized so that the series is orthonormal, i.e.,

$$(\tilde{P}_i,\ \tilde{P}_j) = \delta_{ij}$$

but not necessarily in monic form. Derive a recurrence that relates $\tilde{P}_{n+1}$ to $\tilde{P}_n$ and $\tilde{P}_{n-1}$.

Recall, that the class notes and the reference text by Isaacson and Keller give the following recurrence for the normalized (but not monic) Legendre polynomials

$$\tilde{P}_{n+1} = (A_n x + B_n)\tilde{P}_n - C_n \tilde{P}_{n-1}$$

where

$$\tilde{P}_n = \tilde{a}_n x^n + \tilde{b}_n x^{n-1} + q_{n-2}(x)$$

$$A_n = \frac{\tilde{a}_{n+1}}{\tilde{a}_n}$$

$$B_n = \frac{\tilde{a}_{n+1}}{\tilde{a}_n}\left(\frac{\tilde{b}_{n+1}}{\tilde{a}_{n+1}} - \frac{\tilde{b}_n}{\tilde{a}_n}\right)$$

$$C_n = \frac{\tilde{a}_{n+1}\tilde{a}_{n-1}}{\tilde{a}_n^2}$$

Show that your recurrence is equivalent to this recurrence.

9.4.b

The textbook in equations (10.7) and (10.8) gives the recurrence for the monic, but not necessarily normalized, form of Legendre polynomials. The coefficients of this recurrence gives the coefficients necessary to define the Jacobi matrix whose eigendecomposition give the Gauss Legendre quadrature nodes and weights. Determine the values of α_k and β_k and show that they are consistent with the values used in the MATLAB codes for Gauss Legendre quadrature in Section 10.6 of the textbook.