Homework 1 Foundations of Computational Math 2 Spring 2012

Problem 1.1

Consider the data points

$$(x, y) = \{(0, 2), \quad (0.5, 5), \quad (1, 8)\}$$

Write the interpolating polynomial in both Lagrange and Newton form for the given data.

Solution:

For the Lagrange form we have

$$\begin{aligned} p_2(x) &= 2 \times \frac{(x-0.5)(x-1)}{(-0.5)(-1)} + 5 \times \frac{(x)(x-1)}{(-0.5)(0.5)} + 8 \times \frac{(x)(x-0.5)}{(1)(0.5)} \\ &= 6x + 2 \end{aligned}$$

For Newton's form we have

$$\begin{aligned} f[0, 0.5] &= 6 \\ f[0.5, 1] &= 6 \\ f[0, 0.5, 1] &= 0 \\ p_2(x) = 2 + (x-0)f[0, 0.5] + (x-0)(x-0.5)f[0, 0.5, 1] &= 6x + 2 \end{aligned}$$

Problem 1.2

Use this divided difference table for this problem. Justify all of your answers.

i	0		1		2		3		4		5
x_i	-1		0		2		4		5		6
f_i	13		2		-14		18		67		91
$f[-,-]$		-11		-8		16		49		24	
$f[-,-,-]$			1		6		11		$-25/2$		
$f[-,-,-,-]$				1		1		$-47/8$			
$f[-,-,-,-,-]$					0		$-55/48$				
$f[-,-,-,-,-,-]$						$-55/336$					

1.2.a

Use the divided difference information about the unknown function $f(x)$ and consider the unique polynomial, denoted $p_{1,5}(x)$, that interpolates the data given by pairs (x_1, f_1), (x_2, f_2), (x_3, f_3), (x_4, f_4) , and (x_5, f_5). Use two different sets of divided differences to express $p_{1,5}(x)$ in two distinct forms.

Solution: Two of the standard paths are the left and right edges of the triangle of divided differences defined by the pairs (x_1, f_1), (x_2, f_2), (x_3, f_3), (x_4, f_4) , and (x_5, f_5). The left side uses the points in standard order (x_1, f_1), (x_2, f_2), (x_3, f_3), (x_4, f_4) , and (x_5, f_5):

$$\begin{aligned}
p_{1,5}(x) &= f_1 + (x - x_1)f[x_1, x_2] + (x - x_1)(x - x_2)f[x_1, x_2, x_3] \\
&+ (x - x_1)(x - x_2)(x - x_3)f[x_1, x_2, x_3, x_4] \\
&+ (x - x_1)(x - x_2)(x - x_3)(x - x_4)f[x_1, x_2, x_3, x_4, x_5] \\
&= 2 - 8x + 6x(x - 2) + x(x - 2)(x - 4) - \frac{55}{48}x(x - 2)(x - 4)(x - 5)
\end{aligned}$$

Note that

$$p_{1,5}(6) = 2 - 48 + 144 + 48 - \frac{55}{48}48 = 91$$

and the other points can also be verified.

The right side uses the points in reverse order (x_5, f_5), (x_4, f_4), (x_3, f_3), (x_2, f_2) , and (x_1, f_1):

$$\begin{aligned}
p_{1,5}(x) &= f_5 + (x - x_5)f[x_5, x_4] + (x - x_5)(x - x_4)f[x_5, x_4, x_3] \\
&+ (x - x_5)(x - x_4)(x - x_3)f[x_5, x_4, x_3, x_2] \\
&+ (x - x_5)(x - x_4)(x - x_3)(x - x_2)f[x_5, x_4, x_3, x_2, x_1] \\
p_{1,5}(x) &= f_5 + (x - x_5)f[x_4, x_5] + (x - x_5)(x - x_4)f[x_3, x_4, x_5] \\
&+ (x - x_5)(x - x_4)(x - x_3)f[x_2, x_3, x_4, x_5] \\
&+ (x - x_5)(x - x_4)(x - x_3)(x - x_2)f[x_1, x_2, x_3, x_4, x_5]
\end{aligned}$$

due to the equivalence of divided differences independent of ordering. We therefore have

$$\begin{aligned}
p_{1,5}(x) &= f_5 + (x - x_5)f[x_4, x_5] + (x - x_5)(x - x_4)f[x_3, x_4, x_5] \\
&+ (x - x_5)(x - x_4)(x - x_3)f[x_2, x_3, x_4, x_5] \\
&+ (x - x_5)(x - x_4)(x - x_3)(x - x_2)f[x_1, x_2, x_3, x_4, x_5] \\
&= 91 + 24(x - 6) - \frac{25}{2}(x - 6)(x - 5) \\
&- \frac{47}{8}(x - 6)(x - 5)(x - 4) - \frac{55}{48}(x - 6)(x - 5)(x - 4)(x - 2)
\end{aligned}$$

Note that

$$p_{1,5}(0) = 91 - 144 - 375 + 705 - 275 = 2$$

and the other points can also be verified.

There are of course many other paths connecting the pairs to $f[x_1, x_2, x_3, x_4, x_5]$ that select a single divided difference from each row required. All define different expressions for $p_{1,5}(x)$.

1.2.b

What is the significance of the value of 0 for $f[x_0, x_1, x_2, x_3, x_4]$?

Solution: Any path yielding a divided difference form of $p_{0,4}(x)$ must have $f[x_0, x_1, x_2, x_3, x_4]$ in its last term. Since this is 0, $p_{0,4}(x)$ must be of degree 3 rather than degree 4 as the number of points would lead one to expect. This does not say that the data in $(x_0, f_0), \ldots, (x_5, f_5)$ can be interpolated by a lower degree polynomial. Nor does it say anything about $f(x)$ overall.

1.2.c

Denote by $p_{0,4}(x)$, the unique polynomial, that interpolates the data given by pairs (x_0, f_0), (x_1, f_1), (x_2, f_2), (x_3, f_3), and (x_4, f_4) and recall the definition of $p_{1,5}(x)$ from part (a). Use the divided difference information about the unknown function $f(x)$ to derive error estimates for $f(x) - p_{1,5}(x)$ and $f(x) - p_{0,4}(x)$ for any $x_0 \leq x \leq x_5$.

Solution:

We have for any x

$$f(x) - p_{0,4}(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) f[x_0, x_1, x_2, x_3, x_4, x]$$
$$f(x) - p_{1,5}(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) f[x_1, x_2, x_3, x_4, x_5, x]$$

The products can be evaluated for any x but we have no way of determining the required divided differences. So use the information remaining in the table, i.e., $f[x_0, x_1, x_2, x_3, x_4, x_5] = -55/336$ for the estimates

$$\begin{aligned}
f(x) - p_{0,4}(x) &\approx (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) f[x_0, x_1, x_2, x_3, x_4, x_5] \\
&= -\frac{55}{336}(x + 1)x(x - 2)(x - 4)(x - 5) \\
f(x) - p_{1,5}(x) &\approx (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) f[x_0, x_1, x_2, x_3, x_4, x_5] \\
&= -\frac{55}{336} x(x - 2)(x - 4)(x - 5)(x - 6)
\end{aligned}$$

Problem 1.3

Assume you are given distinct points $x_0, \ldots, x_n$ and, $p_n(x)$, the interpolating polynomial defined by those points for a function f.

1.3.a. If $p_n(x) = \sum_{i=0}^n f(x_i)\ell_i(x)$ is the Lagrange form show that

$$\sum_{i=0}^n \ell_i(x) = 1$$

1.3.b. Assume $x \neq x_i$ for $0 \leq i \leq n$ and show that the divided difference $f[x_0, \ldots, x_n, x]$ satisfies

$$f[x_0, \ldots, x_n, x] = \sum_{i=0}^n \frac{f[x, x_i]}{\prod_{j=0, j\neq i}^n (x_i - x_j)}$$

Solution:

Let $p_n(x) = \sum_{i=0}^n f(x_i)\ell_i(x)$ be the Lagrange interpolating polynomial for a given function $f(x)$. We know that a polynomial of degree n is uniquely determined by $n+1$ points. Therefore, if $f(x)$ is a polynomial of degree $m \leq n$ we must have $p_n(x) = f(x)$. This is seen easily to be consistent with the pointwise error formula that dependes on the $n+1$-st derivative of $f(x)$. This is identically 0 if $f(x)$ is a polynomial of degree $m \leq n$.

So if we take for example $f(x) = x^m$ with $m \leq n$ we have

$$x^m = \sum_{i=0}^n x_i^m \ell_i(x)$$

for any $n+1$ distinct points $x_0 < x_1 < \cdots < x_n$. The result follows from taking $m = 0$, i.e., $f(x) \equiv 1$, since

$$x^0 = 1 = \sum_{i=0}^n 1 \times \ell_i(x) = \sum_{i=0}^n \ell_i(x)$$

For the second part we first note that we can now write

$$\sum_{i=0}^n \ell_i(x) = \sum_{i=0}^n \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)} = 1$$

$$\therefore \frac{1}{\omega_{n+1}(x)} = \sum_{i=0}^n \frac{1}{(x - x_i)\omega'_{n+1}(x_i)}$$

We also have from our notes that

$$f[x_0, \ldots, x_n, x] = \frac{f(x) - p_n(x)}{\omega_{n+1}(x)}$$

$$p_n(x) = \sum_{i=0}^n \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)} f(x_i)$$

We can prove the result as follows:

$$\begin{aligned} f[x_0,\dots,x_n,x] &= \frac{f(x)-p_n(x)}{\omega_{n+1}(x)} \\ &= \frac{f(x)}{\omega_{n+1}(x)} - \frac{p_n(x)}{\omega_{n+1}(x)} \end{aligned}$$

$$= \frac{f(x)}{\omega_{n+1}(x)} - \sum_{i=0}^{n} \frac{f(x_i)}{(x-x_i)\omega'_{n+1}(x_i)} = \sum_{i=0}^{n} \frac{f(x)}{(x-x_i)\omega'_{n+1}(x_i)} - \sum_{i=0}^{n} \frac{f(x_i)}{(x-x_i)\omega'_{n+1}(x_i)}$$

$$\begin{aligned} &= \sum_{i=0}^{n} \frac{f(x)-f(x_i)}{(x-x_i)\omega'_{n+1}(x_i)} \\ &= \sum_{i=0}^{n} \frac{f[x,x_i]}{\omega'_{n+1}(x_i)} \quad \square \end{aligned}$$

Problem 1.4

Text exercise 8.10.1 on page 375
Solution:

Given $n+1$ distinct points $x_0 < x_1 < \cdots < x_n$ we can define the Lagrange characteristic polynomials

$$\ell_i(x) = \frac{\prod_{j=0,j\neq i}^{n}(x-x_j)}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)}$$

Let $\mathbb{P}_n$ be the space of polynomials of degree at most n. (Note $\mathbb{P}_n$ is closed under linear combination.) There are various ways to show that the $\ell_i(x)$ for $0 \le i \le n$ form a basis for $\mathbb{P}_n$.

For example, we can show that we have a nonsingular matrix relating the monomial basis, x^m, for $0 \le m \le n$ to the Lagrange characteristic functions $\ell_i(x)$ for the given points x_i, $0 \le i \le n$.

Any polynomial is uniquely defined by the coefficients of

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

i.e., if any α_i is changed a new polynomial is defined. We must show that there is a set of coefficients β_i for $0 \le i \le n$ uniquely corresponding to the α_i where

$$p(x) = \beta_0\ell_0(x) + \beta_1\ell_1(x) + \cdots + \beta_n\ell_n(x)$$

i.e., $a^T = \begin{pmatrix}\alpha_0 & \dots & \alpha_n\end{pmatrix}$ and $b^T = \begin{pmatrix}\beta_0 & \dots & \beta_n\end{pmatrix}$ are related by a nonsingular matrix defined by the x_i.

To relate the elements of the two potential bases we can exploit that for $0 \le m \le n$

$$x^m = \sum_{i=0}^{n} x_i^m \ell_i(x)$$

for any $n+1$ distinct points $x_0 < x_1 < \cdots < x_n$. Substitution and a bit of algebra yields

$$\begin{aligned}
p(x) &= \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n \\
&= \alpha_0 \sum_{i=0}^{n} \ell_i(x) + \alpha_1 \sum_{i=0}^{n} x_i \ell_i(x) + \cdots + \alpha_n \sum_{i=0}^{n} x_i^n \ell_i(x) \\
&= \sum_{i=0}^{n} \ell_i(x)(\alpha_0 + \alpha_1 x_i + \cdots + \alpha_n x_i^n) \\
&= \beta_0 \ell_0(x) + \beta_1 \ell_1(x) + \cdots + \beta_n \ell_n(x)
\end{aligned}$$

Equating coefficients gives the linear system

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$V^T a = b$$

The matrix V is a Vandermonde matrix and we know V and V^T are nonsingular for any $n+1$ distinct points $x_0 < x_1 < \cdots < x_n$. Therefore, a and b are unique to $p(x)$ and the $\ell_i(x)$ are a basis with the same space as the x^m.

An alternate and more elegant proof starts from the fact that the space $\mathbb{P}_n$ has dimension $n+1$ (easily seen from the monomial basis above). Since we have $n+1$ functions $\ell_i(x) \in \mathbb{P}_n$ if they are linearly independent then they must be a basis.

Suppose they are dependent. There must exist $\beta_0, \ldots, \beta_n$ that are not all 0 such that

$$p(x) = \beta_0 \ell_0(x) + \beta_1 \ell_1(x) + \cdots + \beta_n \ell_n(x) \equiv 0$$

We must therefore have $p(x_i) = 0$ for the distinct x_i, $0 \le i \le n$ that define the $\ell_i(x)$. This can be written

$$\begin{pmatrix} \ell_0(x_0) & \ell_1(x_0) & \cdots & \ell_n(x_0) \\ \vdots & & & \vdots \\ \ell_0(x_n) & \ell_1(x_n) & \cdots & \ell_n(x_n) \end{pmatrix} \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

But, we have $\ell_i(x_j) = \delta_{ij}$ and therefore the matrix is the identity, i.e.,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$I_{n+1} b = 0$$

Since I_{n+1} is a nonsingular matrix $\beta_i = 0$ for $0 \le i \le n$ and therefore we have a contradiction. The $\ell_i(x)$ must therefore be linearly independent and a basis.

Problem 1.5

Text exercise 8.10.3 on page 376

Solution:

We have

$$\ell_i(x) = \frac{\prod_{j=0,j\neq i}^n (x - x_j)}{\prod_{j=0,j\neq i}^n (x_i - x_j)}$$
$$\omega_{n+1} = (x - x_0)(x - x_1)\cdots(x - x_n)$$

The result follows easily from the definition of the derivative:

$$\begin{gathered}
\text{Let } \omega_{-i}(x) = (x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n) \\
\text{then } \omega_{n+1}(x) = (x - x_i)\omega_{-i}(x) \\
\omega'_{n+1}(x) = (x - x_i)\omega'_{-i}(x) + \omega_{-i}(x) \\
\omega'_{n+1}(x_i) = (x_i - x_i)\omega'_{-i}(x) + \omega_{-i}(x_i) = \omega_{-i}(x_i) \\
= \prod_{j=0,j\neq i}^n (x_i - x_j) \\
\ell_i(x) = \frac{\omega_{n+1}(x)}{(x - x_i)\prod_{j=0,j\neq i}^n (x_i - x_j)} \\
= \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)} \\
p(x) = \sum_{i=0}^n f(x_i)\frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)}
\end{gathered}$$

Problem 1.6

Text exercise 8.10.4 on page 376

Solution:

We have $\omega_{n+1} = (x - x_0)(x - x_1)\cdots(x - x_n)$. Let $x_i = x_0 + ih$ and $x = x_0 + sh$ with integers $0 \le i \le n$ and $s \in \mathbb{R}, \ \ 0 \le s \le n$.

Therefore,

$$\omega_{n+1} = (x - x_0)(x - x_1)\cdots(x - x_n) = h^{n+1}\prod_{i=0}^n (s - i)$$

Since $\|f(x)\|_\infty$ is the maximum magnitude of $f(x)$ on a given interval, we must look for the maximum magnitude of a polynomial on an interval defined by n.

For $n = 1$ we have

$$\omega_2(s) = s(s-1)h^2 \quad 0 \le s \le 1$$
$$0 \le s \le 1 \to |s(s-1)| \le \frac{1}{4}$$
$$\therefore \|\omega_2(s)\|_\infty = \frac{h^2}{4}$$

For $n = 2$ we have

$$\omega_3(s) = s(s-1)(s-2)h^3 \quad 0 \le s \le 2$$
$$0 \le s \le 2 \to |s(s-1)(s-2)| \le |s(s-1)(s-2)|_{1\pm\frac{\sqrt{12}}{6}}$$
$$\therefore \|\omega_3(s)\|_\infty \le \omega_3(1 \pm \frac{\sqrt{12}}{6}) \approx 0.385h^3$$