Numerical Analysis Qualifier

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INSTRUCTIONS: Do any (and only) 8 of the following 10 problems.

1. (Mathematical Conditioning)
   Consider the approximate evaluation of the function $\tan x$ in finite precision.
   (a) Calculate the *condition number* (amplification factor with respect to relative error) of this function.
   (b) For what values of x is this function "ill conditioned?"
   (c) Near which of the values below would you expect the finite-precision approximate evaluation of $\tan x$ to be close to full machine-precision accuracy?
      i. $x = 0$
      ii. $x = (2n+1)\pi/2$
      iii. $x = n\pi\,, \quad n \neq 0$

      Here n is an integer.

2. (Asymptotic Expansions, Numerical Benchmarking)
   Many numerical procedures (such as quadrature rules, numerical solutions of differential equations, ...) admit asymptotic error expansions of the form

$$T(h) = \tau_0 + \tau_1 h^{p_1} + \tau_2 h^{p_2} + \cdots\,, \quad \text{as } h \to 0\,.$$

   Here $T(h)$ denotes an approximation depending on h (the "discretization parameter"); τ_0 is the true/exact/limiting value; τ_1, τ_2, ... are constants that do *not* depend on h; and the exponents satisfy $0 < p_1 < p_2 < \cdots$. One can attempt to verify or determine the "order of convergence" p_1 by *numerical experiment.*
   (a) Assuming that you have a test problem for which you *know* τ_0, how could you determine p_1 from a numerical experiment?

(b) Could you similarly determine p_1 for a situation in which you *don't* know an exact value for τ_0? If so, how; if not, why not.

3. (Quadrature Rules, Change of Interval, Composite Rules)
   Consider a basic n-point quadrature rule defined for the standard interval $[-1, 1]$:

$$\int_{-1}^{1} f \approx \sum_{i=1}^{n} w_i f(x_i) . \tag{1}$$

   (a) Determine the *weights* and *abscissae* (or *nodes* or *knots*) of an associated quadrature rule for the general interval $[a, b]$:

$$\int_{a}^{b} f \approx \sum_{i=1}^{n} \widetilde{w}_i f(\tilde{x}_i) .$$

   (b) Does the quadrature rule on $[a, b]$ have the same (or possibly different) "polynomial order" (highest degree polynomial family for which the rule is *exact*) as the original one on $[-1, 1]$? Justify your answer.

   (c) Construct the associated *composite quadrature rule* based upon a uniform partition of the general interval $[a, b]$ into N equal-length sub-intervals.

4. (Gaussian Quadrature Rules)
   The basic n-point Gaussian quadrature rule on the standard interval $[-1, 1]$ is of the form (**??**) above. It can be viewed as an "interpolatory quadrature rule," with abscissae specified as the zeros of the n-th *Legendre polynomial*, constructed to be exact on Π_{n-1} (the family of polynomials of degree at most $n - 1$). Given this, prove that in fact the rule is exact on Π_{2n-1}.

5. (Polynomial Interpolation)
   We wish to tabulate equally spaced values of $f(x) = \cos x$ on the interval $[0, \pi/2]$ so that *local linear interpolation* gives 3 accurate decimal places.

   (a) What is the minimum number of entries needed?

   (b) How accurate must the entries in the table be?

6. (LLS Problems)
   Describe and explain

(a) the normal equations,

(b) the orthogonalization method

for solving the linear least square problem

$$\min_{x \in \mathrm{R}^n} \|y - Ax\|_2, \quad A \in \mathrm{R}^{m \times n}, \quad y \in \mathrm{R}^m.$$

Briefly discuss (dis)advantages of the two methods.

7. (Matrices and Matrix Norms)
Let $A = [a_{ij}] \in \mathrm{C}^{n \times n}$ and let $\alpha > 0$ and $\beta > 0$, where

$$\alpha = \min_{1 \le k \le n} (|a_{kk}| - \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|),$$

$$\beta = \min_{1 \le k \le n} (|a_{kk}| - \sum_{\substack{i=1 \\ i \neq k}}^{n} |a_{ik}|).$$

Show that

(a) A is nonsingular,

(b) $$\|A^{-1}\|_\infty^{-1} = \inf_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty},$$

(c) $$\|A^{-1}\|_\infty \le \frac{1}{\alpha},$$

(d) $$\|A^{-1}\|_1 \le \frac{1}{\beta}.$$

8. (Approximate Inverse)
Let A and B be two nonsingular matrices, such that B approximates A^{-1}. Let $\|\cdot\|$ be a subordinate matrix norm.

(a) If $0 < \|I - AB\| < 1$, show that $C := B + B(I - AB)$ is a "better" approximation to A^{-1} in the sense that

$$\|I - AC\| < \|I - AB\|.$$

(b) Show that

$$\frac{\|B - A^{-1}\|}{\|B\|} \leq \text{cond}(A)\frac{\|A - B^{-1}\|}{\|A\|}.$$

9. (Eigenvalue Computation)
Let A be a matrix whose eigenvalues satisfy

$$\lambda_1 = -\lambda_2, \quad |\lambda_1| > |\lambda_3| \geq \cdots \geq |\lambda_n|$$

and let $y^k = A^k y^0$, $k = 1, 2, \ldots$, for some vector y^0. Show that with a large k, y_i^{2k+2}/y_i^{2k} can be used to find λ_1, provided certain assumptions hold true (state those assumptions). Then show that $y^k \pm \lambda_1 y^{k-1}$ approximate the eigenvectors corresponding to $\pm\lambda_1$.

10. (Gershgorin Circles)
Let A and B be two $n \times n$ matrices and let λ be an eigenvalue of A, which is not an eigenvalue of B. Prove that

$$1 \leq \|(\lambda I - B)^{-1}(A - B)\| \leq \|(\lambda I - B)^{-1}\| \|A - B\|.$$

Then use this result with a special choice of B and $\|\cdot\|$ to derive the Gershgorin statement about the row–circles.