Homework 4 Foundations of Computational Math 1 Fall 2011

Problem 4.1

Recall that an elementary reflector has the form $Q = I + \alpha x x^T \in \mathbb{R}^{n\times n}$ with $\|x\|_2 \neq 0$.

4.1.a. Show that Q is orthogonal if and only if

$$\alpha = \frac{-2}{x^T x} \text{ or } \alpha = 0$$

4.1.b. Given $v \in \mathbb{R}^n$, let $\gamma = \pm\|v\|$ and $x = v + \gamma e_1$. Assuming that $x \neq v$ show that

$$\frac{x^T x}{x^T v} = 2$$

4.1.c. Using the definitions and results above show that $Qv = -\gamma e_1$

Problem 4.2

4.2.a

This part of the problem concerns the computational complexity question of operation count.

For both LU factorization and Householder reflector-based orthogonal factorization, we have used elementary transformations, T_i, that can be characterized as rank-1 updates to the identity matrix, i.e.,

$$T_i = I + x_i y_i^T, \quad x_i \in \mathbb{R}^n \text{ and } y_i \in \mathbb{R}^n$$

Gauss transforms and Householder reflectors differ in the definitions of the vectors x_i and y_i. Maintaining computational efficiency in terms of a reasonable operation count usually implies careful application of associativity and distribution when combining matrices and vectors.

Suppose we are to evaluate

$$z = T_3 T_2 T_1 v = (I + x_3 y_3^T)(I + x_2 y_2^T)(I + x_1 y_1^T)v$$

where $v \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$. Show that by using the properties of matrix-matrix multiplication and matrix-vector multiplication, the vector z can be evaluated in $O(n)$ computations (a good choice of version for an algorithm) or $O(n^3)$ computations (a very bad choice of version for an algorithm).

4.2.b

This part of the problem concerns the computational complexity question of storage space.

Recall, that we discussed and programmed an **in-place** implementation of LU factorization that was very efficient in terms of storage space. An array with n^2 entries initialized with $array(I, J) = \alpha_{ij}$ could be used to store the n^2 entries needed to specify L and U, i.e., λ_{ij} for $j < i$, $2 \leq i \leq n$ and $1 \leq j \leq n-1$, and μ_{ij} for $i < j$, $1 \leq i \leq n$ and $1 \leq j \leq n$.

Let $A \in \mathbb{R}^{n\times k}$, $n \geq k$, and $rank(A) = k$. Consider the use of Householder reflectors, H_i, $1 \leq i \leq k$, to transform A to upper trapezoidal form, i.e.,

$$H_k H_{k-1} \cdots H_2 H_1 A = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$R \in \mathbb{R}^{k\times k} \text{ nonsingular upper triangular}$$

Suppose you are given an array with $n \times k$ entries initialized with $array(I, J) = \alpha_{ij}$ and you are to implement your algorithm using minimal storage.

(i) Are you able to store all of the information needed to specify the H_i, $1 \leq i \leq k$ and R within the array with $n \times k$ entries? Justify your answer.

(ii) If you are not able to store all of the information in the array, how much extra storage do you need and what do you store in it?

Problem 4.3

Let $A \in \mathbb{R}^{n\times k}$ have full column rank. Describe an efficient algorithm based on Householder reflectors, H_i, $1 \leq i \leq k$ that computes a matrix $Q \in \mathbb{R}^{n\times k}$ with orthonormal columns such that

$$\mathcal{R}(A) = \mathcal{R}(Q)$$

i.e., A and Q have the same range space.

Problem 4.4

Consider a Householder reflector, H, in $\mathbb{R}^2$. Show that

$$H = \begin{pmatrix} -\cos(\phi) & -\sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

where ϕ is some angle.