Qualifying Exam in Numerical Analysis
August 17, 2001

There are ten problems. Six problems fully and correctly solved will guarantee a pass.

(1) Let $T_n(x)$ denote the nth Chebyshev polynomial on the interval $[-1,1]$ defined by using the following recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

a. Show that $T_n(x) = \cos(n \arccos x)$.

b. Prove that

$$\int_{-1}^{1} T_n T_m \frac{1}{\sqrt{1-x^2}}\, dx = 0 \quad \text{for all integers } n \text{ and } m, \text{ such that } n \neq m, \quad n, m > 0.$$

c. Prove that $T_{nm}(x) = T_n(T_m(x))$ for all integers $n, m > 0$.

a. Let $\alpha = \arccos x$. It is sufficient to show that $\cos(n \arccos x)$ satisfies the same recurrence relation as T_n. For $n = 0, 1$ the result is trivial. Then a. follows from the fact that

$$\cos(n+1)\alpha + \cos(n-1)\alpha = 2\cos\alpha \cos n\alpha.$$

b. and c. are easy and straightforward applications of a.

(2) Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the following partial differential equation:

$$\begin{cases} -\Delta u + u_x &= f, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{cases}$$

a. Write down the variational formulation of the above differential problem: Find $u \in H_0^1(\Omega)$ such that

$$B(u, v) = f(v), \quad \text{for all } v \in H_0^1(\Omega).$$

Show that this variational problem has a unique solution $u \in H_0^1(\Omega)$ for any right hand side $f \in L^2(\Omega)$.

b. Let V_h be a finite dimensional subspace of $H_0^1(\Omega)$. Show that the discrete problem: Find $u_h \in V_h$ such that

$$B(u_h, v_h) = f(v_h), \quad \text{for all } v_h \in V_h,$$

is well posed and that the following quasi-optimal error estimate holds:

$$|u - u_h|_{H_0^1(\Omega)} \leq C \inf_{\chi \in V_h} |u - \chi|_{H_0^1(\Omega)}.$$

a. Variational form is: Find $u \in H_0^1(\Omega)$ such that

$$B(u, v) = f(v), \quad \forall v \in H_0^1(\Omega),$$

where as usual

$$B(u, v) = \int_\Omega \nabla u \nabla v + u_x v\, dx, \quad f(v) = \int_\Omega f v\, dx.$$

Simple integration by parts leads to

$$B(u, u) = |u|^2_{H_0^1(\Omega)}.$$

and from this equality the Lax-Milgram lemma gives the result from (a.)

b. The discrete problem is well posed by the same token as in a. To prove the bound, let $\chi \in V_h$ be arbitrary. Note that

$$B(u - u_h, v_h) = 0, \quad \forall v_h \in V_h,$$

and also by Schwarz inequality and Poincare inequality

$$B(u, v) \leq C|u|_{H_0^1(\Omega)}|v|_{H_0^1(\Omega)}.$$

Combining the above two results we obtain that

$$|u - u_h|^2_{H_0^1(\Omega)} = B(u - u_h, u - u_h) = B(u - u_h, u - \chi) \leq C|u - u_h|_{H_0^1(\Omega)}|u - \chi|_{H_0^1(\Omega)},$$

and the proof of (b.) is completed by taking the infimum over $\chi \in V_h$.

(3) Consider the nonlinear equation $F(x) = 0$, where $F : \Omega \mapsto \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a C^1 function.

a. Derive the Newton's method, namely for a given initial guess x_0 derive the formula for x_{k+1} in terms of x_k if Newton's method is used for the approximate solution of $F(x) = 0$.

b. Assume that $F \in C^3$ and $F'(x_*)$ is non-singular, where x_* is a solution of $F(x) = 0$. Prove that the Newton's method is well defined if x_0 is sufficiently close to x_* and that the sequence of Newton iterates converges quadratically to the solution.

a. By Taylor's formula we have that

$$F(x) \approx F(x_0) + [F'(x_0)](x - x_0).$$

In Newton's method an approximation to the root is obtained by solving the approximate equation, which is linear with respect to x. So given x_k we have that the next iterate x_{k+1} is obtained by

$$x_{k+1} = x_k - [F'(x_k)]^{-1}F(x_k).$$

b. Clearly, if x_k is sufficiently close to x_*, we have that $F'(x_k)$ is non-singular (because is continuous and non-singular at x_*). So we have to prove that if x_k is in a small neighborhood of x_*, then x_{k+1} will stay in the same neighborhood. Let $G(x) := x - [F'(x)]^{-1}F(x)$. Clearly x_* is a fixed point of G. A simple calculation gives

$$G'(x) = I - K(x)F(x) - [F'(x)]^{-1}F'(x) = -K(x)F(x),$$

where

$$K(x) = \left([F'(x)]^{-1}\right)' = -[F'(x)]^{-1}[F''(x)][F'(x)]^{-1}.$$

Note also that for $x, y \in \mathbb{R}^n$

$$G(y) - G(x) - G'(x)(y - x) = \left(\int_0^1 [G'(x + t(y - x)) - G'(x)]\, dt\right)(y - x)$$

We have that $G'(x)$ is Lipschitz (it is even differentiable, because $F \in C^3$) and this gives the following estimate:

$$\|G(y) - G(x) - G'(x)(y - x)\| \leq \frac{C}{2}\|x - y\|^2,$$

where C is the Lipschitz constant (or a bound on the second derivative of G in case when $F(x) \in C^3$). Taking $x = x_*$, $y = x_k$ and using that $G'(x_*) = 0$ we obtain

$$\|x_{k+1} - x_*\| \leq \frac{C}{2}\|x_k - x_*\|^2.$$

(4) Consider the initial value problem

$$y' = f(t, y), \quad y(0) = y_0.$$

a. Derive an explicit, two-stage, second order Runge-Kutta method for the approximate solution of this problem of the form

$$y_{n+1} = y_n + h[\alpha_1 f(t_n, y_n) + \alpha_2 f(t_n + \theta h, y_n + k_n)].$$

Justify your answer.

Let us set $k_n = \beta h f(t_n, y_n)$. We compare

$$y_{n+1} = y_n + h[\alpha_1 f(t_n, y_n) + \alpha_2 f(t_n + \theta h, y_n + k_n)]. \tag{1}$$

with

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + \ldots \tag{2}$$

trying to match the coefficients in front of equal powers of h. Applying Taylor formula for $f(t_n + \theta h, y_n + k_n)$ then gives

$$f(t_n + \theta h, y_n + \beta h f) = f + \theta h f_t + \beta h f f_y + \mathcal{O}(h^2),$$

where $f = f(t_n)$. After substitution in (1) we get

$$y_{n+1} = y_n + (\alpha_1 + \alpha_2)hf + \theta\alpha_2 h^2 f_t + \beta\alpha_2 h^2 f f_y + \mathcal{O}(h^3).$$

Note that $y' = f$ gives $y'' = f_t + f f_y$. Therefore 2 takes the form:

$$y_{n+1} = y_n + hf + \frac{h^2}{2}(f_t + f f_y) + \mathcal{O}(h^3).$$

This leads to the following equations for α_i, β and θ.

$$\alpha_1 + \alpha_2 = 1, \quad \theta\alpha_2 = \frac{1}{2}, \quad \beta\alpha_2 = \frac{1}{2}.$$

There are many solutions to these equations. A popular one is obtained when $\alpha_1 = \alpha_2$ and the corresponding method is given below.

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + k_n)], \quad k_n = hf(t_n, y_n)$$

(5) a. Find α and β such that the weighted quadrature rule

$$\int_0^1 \frac{f(x)}{\sqrt{x}} dx \doteq \alpha f(0) + \beta f(1)$$

is exact when f is linear.

b. Give the Peano kernel error formula for quadrature rule from (a.)

a.

$$\int_0^1 \frac{f(x)}{\sqrt{x}} dx \doteq \frac{4}{3} f(0) + \frac{2}{3} f(1).$$

b. We note that the above rule is exact if $f \in \mathcal{P}_1$. The Peano kernel theorem then gives:

$$\int_0^1 \frac{f(x)}{\sqrt{x}} dx - \left[\frac{4}{3} f(0) + \frac{2}{3} f(1)\right] = \int_0^1 f''(t)K(t)dt,$$

where $K(t)$ is the error in approximating $(x-t)_+$ by the above quadrature rule (the integration is done with respect to x. This gives the following expression for $K(t)$:

$$\int_0^1 \frac{(x-t)_+}{\sqrt{x}}dx - \frac{2}{3}(1-t) = \frac{4}{3}t(\sqrt{t}-1).$$

(6) Let A be the following 2×2 matrix

$$A = \begin{pmatrix} a & -b \\ -a & a \end{pmatrix},$$

where a and b are real numbers, satisfying $a > 0$, $b > 0$ and $a > b$. Show that Gauss-Seidel iteration is convergent for this type of matrices.

The Gauss-Seidel iteration for the matrix A will be convergent iff $\rho(I - BA) < 1$, where

$$B = \begin{pmatrix} a & 0 \\ -a & a \end{pmatrix}^{-1} = \frac{1}{a}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

This gives

$$I - BA = \frac{b}{a}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

From this equation it is straightforward to find that $\rho(I - BA) = \frac{b}{a} < 1$.

(7) Consider the space $\mathcal{P}_2$ of all quadratic polynomials on $[0, 2]$.

a. Prove that the expression

$$\|f\| := |f(0)| + |f(1)| + |f(2)|, \qquad f \in \mathcal{P}_2,$$

defines a norm on $\mathcal{P}_2$.

b. Determine a best approximation to $f(x) = x^2$ by the constant functions with respect to this norm.

c. Is this best constant approximation to $f(x) = x^2$ unique? Justify your answer.

a. The proof that $\|\cdot\|$ is a norm is straightforward. First observe that $f(0) = f(1) = f(2) = 0$ implies that $f \equiv 0$ for a quadratic polynomial f. The other properties easily follow from similar ones for the absolute value. b. Let p be the approximation under question. It follows that p minimizes

$$g(p) = |p| + |p-1| + |p-4|.$$

Evidently g is a piece-wise linear function and its minimal value is achieved at one of the critical points (where $g'(p)$ does not exists). We then easily find that such a point is $p = 1$ and is unique.

(8) Given the following parabolic partial differential equation

$$\begin{aligned} u_t - \Delta u &= 0, \quad x \in \Omega = (0,1)\times(0,1), \quad t \in [0,\infty) \\ u(x,0) &= u^0(x), \\ u(x,t) &= 0, \quad x \in \partial\Omega, t \in [0,\infty), \end{aligned}$$

consider its finite difference discretization on a uniform $N \times N$ mesh with steps $h = \dfrac{1}{N-1}$ in space and $\tau > 0$ in time:

$$\begin{aligned} \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} + \frac{4u_{i,j}^{n+1} - u_{i-1,j}^{n+1} - u_{i+1,j}^{n+1} - u_{i,j-1}^{n+1} - u_{i,j+1}^{n+1}}{h^2} &= 0, \quad 2 \le i,j \le N-1, \\ u_{i,j}^0 &= u_0(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}, \\ u_{i,j}^{n+1} &= 0, \quad (x_i, y_j) \in \partial\Omega, \end{aligned}$$

where

$$u_{i,j}^n = u(x_i, y_j, n\tau), \; x_i = (i-1)h, \; y_j = (j-1)h, \; i = 1,2,\ldots,N, \; j = 1,2,\ldots,N.$$

a. Let $L_h u$ denotes the stationary part of the above finite difference operator, namely:

$$L_h u := \frac{4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}{h^2}, \; 2 \le i,j \le N-1.$$

Show that $I + \tau L_h$ satisfies the following maximum principle:

If $(I + \tau L_h)u \ge 0$ **and** $u_{i,j} \ge 0$ **for** $(x_i, y_j) \in \partial\Omega$**, then** $u_{i,j} \ge 0$**,** $1 \le i,j \le N$

where I denotes the identity operator.

b. Prove that

$$\max_{i,j} u_{i,j}^{n+1} \le \max_{i,j} u_{i,j}^n.$$

To prove a. we will show that if $(I + L_h)u \ge 0$ *and* $u_{i,j}$ *has a local minimum in an internal point* (x_{i_0}, y_{j_0})*, then* $u_{i_0,j_0} \ge 0$*. Let*

$$u_{i_0,j_0} \le u_{k,l}, \quad k = i_0 - 1, i_0 + 1; \; l = j_0 - 1, j_0 + 1.$$

Then

$$\begin{aligned} 0 \le (I + \tau L_h)u &= \\ + \quad & u_{i_0,j_0} + \frac{\tau}{h^2}(4u_{i_0,j_0} - u_{i_0-1,j_0} - u_{i_0+1,j_0} - u_{i_0,j_0-1} - u_{i_0,j_0+1}) \\ \le \quad & u_{i_0,j_0}. \end{aligned}$$

The boundary condition gives that the desired inequality is satisfied on the boundary of the domain and the desired result follows.

b. Let $u_{max}^n := \max_{i,j} u_{i,j}^n$ *and*

$$w_{i,j}^n := u_{max}^n, \text{ for all } i = 1,2,\ldots,N, \; j = 1,2,\ldots,N.$$

To prove b. it is sufficient to show that $u_{max}^n \ge u_{i,j}^{n+1}, \quad \forall\, i,j$*. Note that* u^n *and* u^{n+1} *satisfy the relation*

$$(I + \tau L_h)u^{n+1} = u^n, \quad u_{i,j}^{n+1} = 0 \; (x_i, y_j) \in \partial\Omega.$$

We also have that in the interior of Ω*,*

$$0 \le w^n - u^n = (I + \tau L_h)(w^n - u^{n+1}).$$

Since $(I + \tau L_h)$ *satisfies maximum principle complete the proof by applying (a.)*

(9) Let $A \in \mathbb{R}^{m\times n}$ have rank n, and $b \in \mathbb{R}^m$.

a. Show that the matrix $A^T A$ is invertible.

b. Show that there exists a unique $x \in \mathbb{R}^n$ minimizing $Ax - b$ with respect to the Euclidean norm and $x = (A^T A)^{-1} A^T b$.

Let us first note that $n \leq m$ because A has rank n. Since we consider finite dimensional space we shall prove a. by showing that $\mathrm{Ker}(A^T A) = \{0\}$ or equivalently that $A^T A$ is injective. Assume that there is an $x \in \mathbb{R}^n$ such that $A^T Ax = 0$. We then have

$$0 = (A^T Ax, x) = \|Ax\|^2 \quad \Longrightarrow \quad Ax = 0.$$

But A has rank n which exactly means that there is no $x \neq 0$ for which $Ax = 0$ and therefore $A^T A$ is injection. This in turn implies $A^T A$ is invertible.

b. To prove the statement we first observe that $F(x) = \|Ax - b\|^2$ is a quadratic functional, namely

$$F(x) = (A^T Ax, x) - 2(A^T b, x) + \|b\|^2.$$

Note that F has a unique minimum, because by a. $A^T A$ is invertible and hence positive definite. The Euler-Lagrange equations for such functional then are:

$$A^T Ax = A^T b.$$

and so the solution of $\min\limits_{y\in\mathbb{R}^n} F(y)$ is $x = (A^T A)^{-1} A^T b$.

(10) Let $A = (a_{ij})$ be a symmetric and positive definite matrix of order n.

$$A = \begin{pmatrix} a_{11} & a^T \\ a & A_1 \end{pmatrix}.$$

After one step of Gaussian elimination A is converted to a matrix of the form

$$\begin{pmatrix} a_{11} & a^T \\ 0 & \tilde{A} \end{pmatrix}.$$

a. Find an explicit formula for $\tilde{A}$ in terms of a_{11}, A_1 and a.

b. Show that the $(n-1) \times (n-1)$ matrix $\tilde{A}$ is symmetric and positive definite.

a. $\tilde{A} = A_1 - \dfrac{aa^T}{a_{11}}$.

b. Let $x \in \mathbb{R}^{n-1}$ be arbitrary and $x_1 := -\dfrac{(a,x)}{a_{11}}$. Consider

$$y = \begin{pmatrix} x_1 \\ x \end{pmatrix}$$

Note that $(Ay, y) = (\tilde{A}x, x)$. Since A is symmetric positive definite, we have that there exists a number γ such that $(Ay, y) \geq \gamma\|y\|^2$. Therefore

$$\gamma\|x\|^2 \leq \gamma\|y\|^2 \leq (Ay, y) = (\tilde{A}x, x).$$