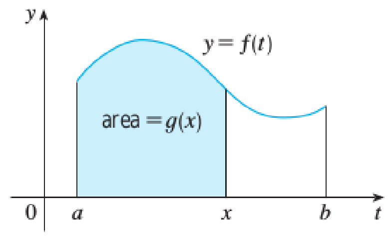
5.3: The Fundamental Theorem of Calculus

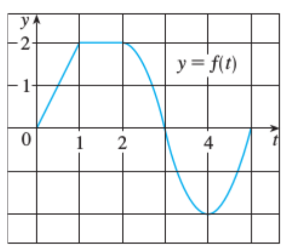
The first part of the Fundamental Theorem of Calculus deals with the function

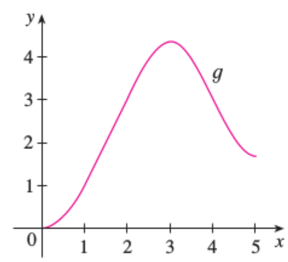
$$g(x) = \int_a^x f(t)\ dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b. Notice that g is a function of the upper limit of integration.

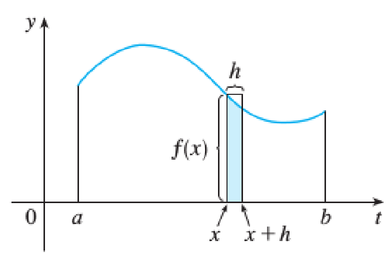


Example 1. *If $g(x) = \int_0^x f(t)\ dt$ and f is the function graphed below, find $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$.*





To see the connection between differentiation and integration, consider taking the derivative of $g(x) = \int_a^x f(t)\ dt$. To compute the derivative of $g(x)$ from the limit definition of derivative, we first observe that $g(x + h) - g(x)$ for $h > 0$ is the area under f on the interval $[x, x + h]$ (depicted below).



This area is approximately the area of the rectangle with width h and height $f(x)$, that is,

$$hf(x) \approx g(x + h) - g(x)$$

$$f(x) \approx \frac{g(x + h) - g(x)}{h}$$

Intuitively, we would expect that $f(x) = \lim\limits_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$. That this is actually true is the first part of **The Fundamental Theorem of Calculus:**

Theorem 1. *If f is continuous on $[a, b]$, then the function $g(x) = \int_a^x f(t)\ dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.*

Example 2. *Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\ dt$.*

Example 3. *Find $\frac{d}{dx} \int_1^{x^4} \sec t\ dt$.*

The second part of **The Fundamental Theorem of Calculus** is

Theorem 2. *If f is continuous on $[a, b]$, then $\int_a^b f(x)\ dx = F(b) - F(a)$ where F is an antiderivative of f, that is, a function such that $F' = f$.*

Example 4. *Evaluate the integral $\int_1^3 e^x\ dx$.*

Example 5. *Find the area under the parabola $y = x^2$ from 0 to 1.*

Example 6. *Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.*

Example 7. *Find the area under the cosine function from 0 to b, when $0 \leq b \leq \pi/2$.*

Differentiation and Integration as Inverse Processes

Theorem 3. The Fundamental Theorem of Calculus: *Suppose f is continuous on $[a, b]$.*

1. *If $g(x) = \int_a^x f(t)\ dt$, then $g'(x) = f(x)$.*
2. *$\int_a^b f(x)\ dx = F(b) - F(a)$, where F is an antiderivative of f, that is, $F' = f$.*

From the first part of the Fundamental Theorem of Calculus

$$\frac{d}{dx} \int_a^x f(t)\ dt = f(x).$$

That is, if we integrate f, then differentiate, we get back the original function f. And by the second part of the Fundamental Theorem of Calculus

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

That is, if we differentiate F, then integrate, we get back the original function F, but in the form $F(b) - F(a)$.