5.4: Indefinite Integrals and the Net Change Theorem

Indefinite Integrals

Both parts of the Fundamental Theorem of Calculus establish connections between antiderivatives and definite integrals. Part 1 says that if f is continuous, then $\int_a^x f(t)\ dt$ is an antiderivative of f, and Part 2 says that $\int_a^b f(x)\ dx$ can be evaluated by $F(b) - F(a)$, where F is an antiderivative of f.

The notation $\int f(x)\ dx$, called the **indefinite integral** of f, is used to denote an antiderivate of f. Thus,

$$\int f(x)\ dx = F(x) \qquad \text{means that} \qquad F'(x) = f(x).$$

For example,

$$\int x^2\ dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

and

$$\int \sec^2 x\ dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x.$$

Note that the *definite integral* $\int_a^b f(x)\ dx$ is a number whereas the *indefinite integral* $\int f(x)\ dx$ is a function, and the connection between the two is given by Part 2 of the Fundamental Theorem of Calculus

$$\int_a^b f(x)\ dx = \int f(x)\ dx \bigg|_{x=a}^{b}.$$

Below is a table of indefinite integrals.

1 Table of Indefinite Integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \dfrac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \dfrac{1}{x^2+1}\,dx = \tan^{-1} x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

Example 1. *Find the general indefinite integral*

$$\int (10x^4 - 2\sec^2 x)\ dx.$$

Example 2. *Evaluate* $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\ d\theta.$

Example 3. *Evaluate* $\displaystyle\int_0^3 (x^3 - 6x)\ dx.$

Example 4. *Find* $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\ dx.$

Example 5. *Evaluate* $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\ dt.$

Applications

Part 2 of the Fundamental Theorem of Calculus says that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is an antiderivative of f so the equation can be written

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

$y = F'(x)$ represents the rate of change of F with respect to x and $F(b) - F(a)$ is the net change in y from a to b, so we can reformulate Part 2 of the Fundamental Theorem of Calculus is words as follows.

Theorem 1. The Net Change Theorem: *The integral of a rate of change is the net change:*

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

This principle can be applied to any rate of change in the natural or social sciences. For example,

- If the rate of growth of a population is $\frac{dn}{dt}$, then

  $$\int_{t_1}^{t_2} \frac{dn}{dt}\ dt = n(t_2) - n(t_1)$$

  is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$. So

$$\int_{x_1}^{x_2} C'(x)\ dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from x_1 units to x_2 units.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

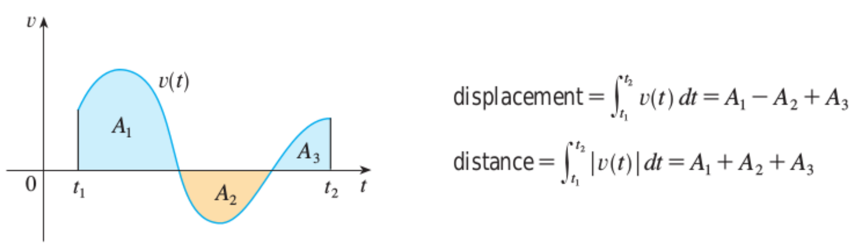
$$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1)$$

is the net change of position, or displacement, of the particle during the time period from t_1 to t_2.

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v'(t) > 0$ (the particle moves to the right) and also the intervals when $v'(t) < 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore,

$$\int_{t_1}^{t_2} |v(t)|\ dt = \text{ total distance traveled.}$$

The figure below shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\ dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

Example 6. *A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).*

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

(b) Find the distance traveled during this time period.