Unit III

Supplementary Notes

10.1 Systems of Linear Equations: Two Equations

$$\begin{cases} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{cases} \xrightarrow{\text{row reduce}} \begin{cases} c_{11}x + c_{12}y &= d_1 \\ \phantom{c_{11}x +} c_{22}y &= d_2 \end{cases}$$

- $c_{22} \neq 0 \Rightarrow$ unique solution
- $c_{22} = 0$ and $d_2 \neq 0 \Rightarrow$ no solution
- $c_{22} = d_2 = 0 \Rightarrow$ infinitely many solutions

10.2 Systems of Linear Equations: Three Equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{cases} \xrightarrow{\text{row reduce}} \begin{cases} c_{11}x + c_{12}y + c_{13}z &= d_1 \\ c_{22}y + c_{23}z &= d_2 \\ c_{33}z &= d_3 \end{cases}$$

- $c_{33} \neq 0 \Rightarrow$ unique solution
- $c_{33} = 0$ and $d_3 \neq 0 \Rightarrow$ no solution
- $c_{33} = d_3 = 0 \Rightarrow$ infinitely many solutions

10.3 Matrices

Valid row operations:

- Interchange two rows
- Multiply row by non-zero constant
- Replace row by non-zero multiple of itself plus multiple of another row

Reduced row echelon form:

- Leading non-zero entry in each row is 1, above and below leading 1, in the same column, is all 0's
- Leading 1 in a row is to the right of any leading 1 in above rows
- Any row of all 0's is at the bottom

10.4 Determinants

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} \end{array}$$

10.4 Determinants (ctd.)

Cramer's Rule: For 3×3 matrix A, if $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $|A| \neq 0$, then $x = \frac{|A_x|}{|A|}$, $y = \frac{|A_y|}{|A|}$, and $z = \frac{|A_z|}{|A|}$

For a square matrix A

- $|A|$ switches sign if two rows are interchanged
- $|A|$ is multiplied by a nonzero constant if a row is multiplied by that constant
- $|A|$ does not change if a row is replaced by itself plus a non-zero multiple of another row

10.5 Matrix Algebra

For $m \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ and real numbers c, c_1, and c_2

- $A = B \Leftrightarrow a_{ij} = b_{ij}$
- $A+B = B+A = (a_{ij}+b_{ij})$
- $(A+B)+C = A+(B+C)$
- $cA = (ca_{ij})$
- $c_1(c_2A) = (c_1c_2)A$
- $c(A+B) = cA + cB$
- $(c_1 + c_2)A = c_1A + c_2A$

For $m \times n$ matrix A, $n \times p$ matrices B, B_1, B_2 and $p \times q$ matrix C

- ij^{th} entry of $m \times p$ matrix $AB = (\text{row } i \text{ of } A) \times (\text{col. } j \text{ of } B)$
- $(AB)C = A(BC)$
- $A(B_1 + B_2) = AB_1 + AB_2$

Inversion: $\left[\, A \mid I \,\right] \xrightarrow{\text{row reduce}} \left[\, I \mid A^{-1} \,\right]$

11.1 Sequences

recursive:
$a_1 = a$ for real a
$a_n = f(a_{n-1})$ for $n \geq 2$

explicit:
$a_n = f(n)$ for $n \geq 1$

sum of first n terms:
$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

11.2 Arithmetic Sequences

recursive:
$a_1 = a$ for real a
$a_n = a_{n-1} + d$ for $n \geq 2$

explicit:
$a_n = a_1 + (n-1)d$ for $n \geq 1$

$$d = \frac{a_m - a_n}{m - n}$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

11.3 Geometric Sequences and Series

recursive:
$a_1 = a$ for real a
$a_n = a_{n-1}r$ for $n \geq 2$

explicit:
$a_n = a_1 r^{n-1}$ for $n \geq 1$

$$r = \sqrt[m-n]{\frac{a_m}{a_n}}$$

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

geometric series:
$$S = \sum_{k=1}^{\infty} a_k = \frac{a_1}{1 - r}$$

Exercises

1. Select the type of solution for the following system

$$\begin{cases} 2x+7y &= 4 \\ -4x-14y &= 2 \end{cases}$$

   A. Infinitely many solutions
   B. No solution
   C. None of these
   D. A unique solution
   E. Exactly two solutions

2. Choose the correct x, y, or z value for the solution to the system

$$\begin{cases} 3x+2y-z &= -7 \\ 2x-y-z &= 3 \\ x+3y-2z &= -6 \end{cases}$$

   A. $x=-4$
   B. $z=-2$
   C. $z=-1$
   D. $y=2$
   E. $y=1$

3. Select the matrix which is in reduced row echelon form

   A. None of these

   B. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   C. $\begin{bmatrix} 1 & -5 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   D. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

   E. $\begin{bmatrix} 1 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. Find the reduced row echelon form of $\begin{bmatrix} -4 & 1 & 2 \\ 12 & -3 & 6 \end{bmatrix}$

5. Given that $\begin{vmatrix} -1 & -8 & 4 \\ 4 & -12 & -1 \\ 1 & 16 & 5 \end{vmatrix} = -92$, use Cramer's rule to find the numeric solution for y to the following system

$$\begin{cases} -x+2y-4z &= -8 \\ 4x-y-z &= -12 \\ x+3y+5z &= 16 \end{cases}$$

6. Find z in the solution of the system

$$\begin{cases} a_1+b_1y+c_1z &= \frac{1}{4} \\ a_2+b_2y+c_2z &= -2 \\ a_3x+b_3y+c_3z &= -1 \end{cases}$$

   if the inverse of $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is $\begin{bmatrix} -2 & 1 & -\frac{1}{2} \\ -1 & -1 & \frac{1}{4} \\ 2 & -\frac{1}{4} & -1 \end{bmatrix}$

7. Select the entry in the first row and second column of the inverse matrix for

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

8. Find the sum $\sum_{n=1}^{5}(-1)^{n-1}\ln\frac{2n-1}{n+2}$

9. Select the statement that is true

   A. $\sum_{m=0}^{n}\frac{(m+2)(m+1)}{3^{m+2}} = \sum_{m=2}^{n+2}\frac{(m+4)(m+3)}{3^{m+2}}$
   B. $\sum_{m=0}^{n}\frac{(m+2)(m+1)}{3^{m+2}} = \sum_{m=2}^{n+2}\frac{m(m-1)}{3^{m}}$
   C. $\sum_{m=0}^{n}\frac{(m+2)(m+1)}{3^{m+2}} = \sum_{m=-2}^{n-2}\frac{(m+4)(m+3)}{3^{m+2}}$
   D. $\sum_{m=0}^{n}\frac{(m+2)(m+1)}{3^{m+2}} = \sum_{m=-2}^{n-2}\frac{m(m-1)}{3^{m}}$

10. Given an arithmetic sequence with $a_{21}=140$ and $a_{46}=65$, find the term a_{60}.

11. Find the sum $(-7)+(-10)+(-13)+(-16)+\cdots+(-106)$

12. Find the 5^{th} term of the geometric sequence whose 2^{nd} term is -4 and whose 7^{th} term is $\frac{1}{8}$.

13. Find the sum $\frac{1}{2}+\frac{1}{4}+\cdots+(\frac{1}{2})^n$

14. Find the sum of the alternating geometric series $\frac{1}{25}-\frac{1}{125}+\cdots+(-1)^{n+1}\frac{1}{5^{n+1}}+\cdots$

15. If the repeating decimal $2.818181\ldots$ is written as $\frac{m}{n}$ in reduced form where m and n and integers, then $m=$