Foundations of Computational Math I Exam 2
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Due beginning of Class Wednesday, December 1, 2010

Question	Points Possible	Points Awarded
1. Iterative Methods for $Ax = b$	25	
2. Iterative Methods for $Ax = b$	30	
3. Nonlinear Equations	30	
4. Nonlinear Equations	25	
Total Points	110	

Name:____

Alias: ____

to be used when posting anonymous grade list.

Problem 1

(25 points)
Suppose $B \in \mathbb{R}^{n \times n}$ is a symmetric positive definite tridiagonal matrix of the form

$$B = \begin{pmatrix} D_r & L \\ L^T & D_b \end{pmatrix}$$

where $n = 2k$, D_r and D_b are diagonal matrices of order $n/2$, L is a lower triangular matrix with nonzeros restricted to its main diagonal and its first subdiagonal, and U is upper triangular matrix with nonzeros restricted to its main diagonal and its first superdiagonal.

Assume that $Ax = b$ can be solved using Jacobi's method, i.e., the iteration converges acceptably fast. Partition each iterate x_i into the top half and bottom half, i.e.,

$$x_i = \begin{pmatrix} x_i^{(top)} \\ x_i^{(bot)} \end{pmatrix}$$

1.a. Assume an initial guess x_0 is given and identify what information, i.e., the pieces of x_i for $0 \le i \le j$, determines the values found in the vectors $x_j^{(top)}$ and $x_j^{(bot)}$ for any $j > 0$.

1.b. Can the relationships from 1.a be used to design an iteration that approximates the solution essentially as well but only requires half of the work of Jacobi's method?

1.c. Relate your new method from 1.b to applying Gauss-Seidel to solve $Bx = b$ starting from the same initial guess x_0.

Problem 2

(30 points)

2.a

(10 points)

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \mu_1 \\ -1 & 1 & 0 & \mu_2 \\ -1 & -1 & 1 & \mu_3 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Suppose Gauss-Seidel is to be used to solve $Ax = b$. Find positive real values $\mu_1 > 0$, $\mu_2 > 0$, and $\mu_3 > 0$ such that Gauss-Seidel converges.

2.b

(10 points)

Prove the following:

Lemma 2.1. *If A can be written $A = I - P$ where $P \geq 0$ and $\rho(P) < 1$ then A is an M-matrix.*

2.c

(10 points)
Consider simple accelerated Richardson's method to solve $Ax = b$

$$\begin{gathered}\text{Given,} \quad x_0 \\ x_{k+1} = x_k + \alpha r_k \\ r_k = b - Ax_k \\ \alpha > 0\end{gathered}$$

Consider the matrix

$$A = \begin{pmatrix} 6 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

What value would choose for $\alpha > 0$ so that simple accelerated Richardson's method will converge for any x_0? Justify your answer.

Problem 3

(30 points)

Let k be a positive integer and $\alpha > 0$ be a positive real number. Consider the fixed point iteration by the function

$$\phi(x) = (1 - k^{-1})x + k^{-1}\alpha x^{-(k-1)}$$

Provide justification to all of your answers for the following:

(i) **(5 points)** Find a function $f(x)$ such that $f(x_*) = 0$ if and only if $x_* = \phi(x_*)$.

(ii) **(10 points)** Let $k = 2$. Find two intervals on which $\phi(x)$ is guaranteed to be a contraction mapping and for each interval find the fixed point to which $x_{i+1} = \phi(x_i)$ converges if x_0 is in the interval.

(iii) **(5 points)** Is the order of convergence on these intervals linear ($p = 1$) or higher ($p \geq 2$)?

(iv) **(10 points)** Let $k = 2$ and $\alpha = 2$. What happens to the iteration if $x_0 = 0.5$ and how does this relate to the intervals you found above?

Problem 4

(25 points)

4.a

(15 points)

Consider the polynomial

$$p(x) = x^3 - x - 5$$

The following three iterations can be derived from $p(x)$.

$$\phi_1(x) = x^3 - 5$$
$$\phi_2(x) = \sqrt[3]{x+5}$$
$$\phi_3(x) = 5/(x^2 - 1)$$

The polynomial $p(x)$ has a root $r > 1.5$. Determine which of the $\phi_i(x)$ produce an iteration that converges to r.

4.b

(10 points)

Consider the function

$$f(x) = xe^{-x} - 0.06064$$

4.a. Write the update formula for Netwon's method to find the root of $f(x)$.

4.b. $f(x)$ has a root of $\alpha = 0.06469263\ldots$. If you run Newton's method with $x_0 = 0$ convergence occurs very quickly, e.g., in double precision $|f(x_4)| \approx 10^{-10}$. However, if x_0 is large and negative or if x_0 is near 1 convergence is much slower until there is a very rapid improvement in accuracy in the last one or two steps. For example, if $x_0 = 0.98$ then

$$|f(x_{47})| \approx 0.6 \times 10^{-2}$$
$$|f(x_{49})| \approx 1.5 \times 10^{-5}$$
$$|f(x_{50})| \approx 2.8 \times 10^{-10}$$

Explain this behavior. You might find it useful to plot $f(x)$ and run a few examples of Newton's method.