Foundations of Computational Math II Exam 1
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Wednesday February 22, 2012

Question	Points Possible	Points Awarded
1. Basics	20	
2. Splines	30	
3. Interpolation	35	
4. Piecewise Interpolation	30	
Total Points	115	

Name: **Alias:**

Problem 1

(20 points)

1.a

(5 points)

Briefly describe the main advantage of using a piecewise polynomial interpolant rather than a single global interpolating polynomial when approximating a function $f(x)$ using data points (x_i, f_i) for $0 \leq i \leq n$ for various values of n.

1.b

(5 points)

Briefly describe the main differences and similarities between a piecewise cubic Hermite interpolating polynomial and a cubic spline to approximate a function $f(x)$ using data points (x_i, f_i) for $0 \leq i \leq n$.

1.c

(5 points) Let the interpolating polynomial, $p_n(x)$, be defined by the points (x_i, y_i), $0 \leq i \leq n$. What is used as the condition number for the sensitivity of $p_n(x)$ to perturbations in the values of the y_i?

1.d

(5 points) Briefly explain what an interpolatory strategy for approximating a function $f(x)$ with polynomials of degree $n = 0, 1, 2, \ldots$ is and give sufficient conditions such that the strategy yields uniform convergence to f as $n \to \infty$.

Problem 2

(30 points)

Assume x_i, for $0 \le i \le 2$ are uniformly spaced and you are given the data pairs function values:

$$(x_0, f(x_0)) = (1, 11), \quad (x_1, f(x_1)) = (2, 12), \quad (x_2, f(x_2)) = (3, 11)$$

second derivative values:

$$(x_0, f''(x_0)) = (1, -6), \quad (x_2, f''(x_2)) = (3, -6)$$

We know a cubic spline $s(x)$ that interpolates these values can be defined as a linear combination of the cubic B splines defined in the notes and text:

$$s(x) = \alpha_{-1}B_{-1}(x) + \alpha_0 B_0(x) + \alpha_1 B_1(x) + \alpha_2 B_2(x) + \alpha_3 B_3(x)$$

(2.a) (10 points)

Determine a system of equations that define $s(x)$ and solve the system to determine the coefficients α_i.

(2.b) (10 points)

Recall, the spline can also be defined in terms of parameters that are the values of $s''(x_i)$, i.e., $Ts'' = d$. Write the system of equations that determine this parameterization and solve it to find the value of the parameters.

(2.c) (10 points)

Choose either form above and determine the two cubic polynomials

$$s(x) = \begin{cases} p_1(x) & \text{if } 1 \le x \le 2 \\ p_2(x) & \text{if } 2 \le x \le 3 \end{cases}$$

and verify that all required interpolation and continuity conditions are satisfied.

Problem 3

(35 points)

3.a

(10 points)

Suppose you have the Lagrange form of the unique interpolating polynomial of degree n through points (x_i, f_i)

$$p_n(x) = \sum_{i=0}^{n} f_i \ell_i(x)$$

Is it true or false that

$$\sum_{i=0}^{n} \ell_i'(x) = 1$$

Justify your answer.

3.b

(10 points)

Given the data points

$$(x_0, f(x_0)) = (1, 10), \quad (x_1, f(x_1)) = (3, 65)$$
$$(x_2, f(x_2)) = (4, 150), \quad (x_3, f(x_3)) = (6, 425)$$

find the unique cubic interpolating polynomial, $p_3(x)$ and evaluate it at $x = 0$ and $x = 10$.

3.c

(15 points)

Consider the interpolation conditions

$$p_2(a) = f(a)$$
$$p_2'(a) = f'(a)$$
$$p_2'(b) = f'(b)$$

where $a,\ b \in \mathbb{R}$. Show that if $a \neq b$ then there is a unique quadratic polynomial, $p_2(x)$, satisfying the conditions.

Problem 4

(30 points)

Let $f(x) = 2/(1 + 10x^2)$ on $-1 \le x \le 1$. Suppose $f(x)$ is to be approximated by a piecewise linear interpolating function, $g_1(x)$. The accuracy required is

$$\forall 0 \le x \le 1, \quad |f(x) - g_1(x)| \le 10^{-6}$$

Determine a bound on $h = x_i - x_{i-1}$ for uniformly spaced points that satisfies the required accuracy.