Ph.D. qual. exam. and M.S. comp. exam. on Numerical analysis. Wednesday August 16, 2006.

Answer at least 8 questions with at least 6 having a different number (1-7). Show your calculations and justify your answers.

1a. Consider the linear interpolation polynomial $P_1(x)$ of the function $\ln(x)$ on $[1, e]$ at the points $x_0 = 1, x_1 = e$. Estimate the maximum of the absolute error $|P_1(x) - \ln(x)|$ for $x \in [1, e]$, i.e., give an upper bound for

$$\|P_1 - \ln(.)\|_\infty = \max_{x\in[1,e]} |P_1(x) - \ln(x)| \leq \,?$$

1b. Is the following function on the interval $[1, 3]$ a cubic spline? If yes is it a periodic spline?

$$s(x) = \begin{cases} x^3 - 2x^2 - 2x + 5 & \text{for} \quad x \in [1, 2], \\ -x^3 + 10x^2 - 26x + 21 & \text{for} \quad x \in [2, 3]. \end{cases}$$

2a. To obtain an approximation to

$$\int_{-1}^{1} e^{-x^2} dx$$

with the midpoint rule and using an equidistant subdivision of the interval $[-1, 1]$, how many evaluations of the function e^{-x^2} are sufficient to ensure a total error smaller than 10^{-8}?

2b. We consider a quadrature formula

$$\int_{\alpha}^{\alpha+h} f(x)dx \approx h \sum_{i=1}^{s} b_i f(\alpha + c_i h)$$

given by the coefficients

$$\begin{aligned} (b_1, b_2, b_3, b_4) &= (1/8, 3/8, 3/8, 1/8), \\ (c_1, c_2, c_3, c_4) &= (0, 1/3, 2/3, 1). \end{aligned}$$

(a) What is its order?

(b) Is this quadrature formula symmetric?

3a. What are the roots of the polynomial of degree 9 of the form $p(x) = x^9 + \ldots$ which minimizes

$$\max_{x \in [-3,3]} |p(x)|?$$

3b. What is the polynomial $p_2(t)$ of degree 2 approximating the function $f(t) = |t|$ on the interval $[-1, 1]$ which minimizes

$$\int_{-1}^{1} (f(t) - p(t))^2 dt \ ?$$

4a. To find a zero x^* of

$$xe^x - 1 = 0$$

(a) Start by computing one iterate $x^{(1)}$ of Newton's method from $x^{(0)} = 0$;

(b) Then compute one iterate $x^{(2)}$ of the secant method using $x^{(0)}$ and $x^{(1)}$.

(c) How many iterations of the bisection method are necessary to obtain x^* to a precision 10^{-6} when starting with the interval $[-8, 4]$?

4b. To find a zero of

$$\begin{pmatrix} 4x_1^2 + x_2^2 - 4x_1 \\ x_2 e^{2x_1} - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

compute one iterate $x^{(1)}$ of Newton's method starting from the point $x^{(0)} = (0, 1)^T$.

5a. Compute the Cholesky decomposition of the matrix

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

Then solve the linear system of equation $Ax = b$ where

$$b = \begin{pmatrix} 16 \\ 18 \\ 6 \end{pmatrix}.$$

5b. Find $x \in \mathbb{R}^2$ minimizing $\|Ax - b\|_2$ with

$$A := \begin{pmatrix} 4 & 0 \\ 2 & 0 \\ 4 & 3 \end{pmatrix}, \qquad b := \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}.$$

Hint: use the method of your choice!

6a. Consider the system of ODEs $y' = f(t, y)$ and the following explicit Runge-Kutta method

$$\begin{aligned} Y_1 &= y_0 \\ Y_2 &= y_0 + h\frac{1}{2} f(t_0, Y_1) \\ Y_3 &= y_0 + h\left(-f(t_0, Y_1) + 2f(t_0 + h/2, Y_2)\right) \\ y_1 &= y_0 + h\left(\frac{1}{6} f(t_0, Y_1) + \frac{2}{3} f(t_0 + h/2, Y_2) + \frac{1}{6} f(t_0 + h, Y_3)\right) \end{aligned}$$

(a) What is the local order of this method?

(b) What is the stability function $R(z)$ of this method ($z := h\lambda$ and $y' = \lambda y$)?

6b. We consider the following explicit linear multistep method with step-size h applied to $y' = f(t, y)$ (using the notation $f_j := f(t_j, y_j)$)

$$y_{n+1} = 3y_n - 2y_{n-1} + h\left(\frac{1}{2} f_n - \frac{3}{2} f_{n-1}\right).$$

(a) What is its order?

(b) Is it 0-stable?

(c) Is it convergent?

7a. Consider the matrix

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & 2 & 3 \end{pmatrix}$$

and an approximation μ to the middle eigenvalue λ_2 given by the value $\mu = 2$. Apply one step of the inverse iteration of Wielandt with $\mu = 2$ starting from the vector

$$y_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

to obtain a better approximation to λ_2 and give the value of the new approximation that you have obtained.

7b. For a matrix $A \in \mathbb{R}^{n\times n}$ write down the QR algorithm to find the eigenvalues of a matrix. Prove that the iterates of the QR algorithm are similar, i.e., they have the same eigenvalues.