Qualifying Exam in Numerical Analysis
August 19, 2002

There are ten problems. Six problems fully and correctly solved will guarantee a pass.

(1) For a given vector b and a given nonsingular upper triangular matrix A, verify that the Jacobi iteration for solving $Ax = b$ is always convergent.

(2) Let $u, v \in \mathbf{R}^m$ and let $\sigma \in R^1$. Define

$$H(u, v, \sigma) = I - \sigma u v^T ,$$

where I represents the $m \times m$ identity matrix.

a. Determine $\sigma \neq 0$ such that $H(u, u, \sigma)$ is orthogonal. For such σ, determine all the eigenvalues and the corresponding eigenvectors of $H(u, u, \sigma) = I - \sigma u u^T$.

b. Let $x \in \mathbf{R}^m$, $x \neq 0$. Show how to choose a vector u such that $H = H(u, u, \sigma)$ has the property that Hx is a multiple of $e^{(1)} = (1, 0, 0, \ldots, 0)^T$ where σ is defined in (a.).

c. Show that one can construct orthogonal transformations $H^{(k)}$, $k = 1. \ldots, \ell$ such that

$$A^{(\ell+1)} = H^{(\ell)} H^{(\ell-1)} \cdots H^{(1)} A \,, \ell \leq \min(m-1, n)$$

has row echelon structure.

(3) Given an arbitrary real matrix A,

a. Describe the singular value decomposition for A.

b. If A is a Hermitian matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, what are the singular values of A? Justify your answer.

c. Describe how the singular value decomposition can be used to compute the rank of A.

(4) Given the following recurrence relation on $[-1, 1]$:

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \quad \forall n \geq 1.$$

a. Find the general solution.

b. Show that if $T_0(x) = 1$, $T_1(x) = x$, then $T_n(x) = \cos(n \arccos x)$ (the nth Chebyshev polynomial).

c. Prove that $T_{nm}(x) = T_n(T_m(x))$ for all integers $n, m > 0$.

(5) Consider the function $f(x) = \cos x + 2x$ on the interval $[a, b]$.

a. If $[a, b] = [-\pi/2, \pi/2]$, what is the best uniform approximation to f among all polynomials of degree at most 1?

b. If $[a, b] = [-\pi/2, \pi/2]$, is there a best uniform approximation to f among all functions of the form $c_1 \sin x + c_2 x^2$ where c_1, c_2 are any two real constants? If so, is such a best approximation unique?

(6) Let $h > 0$ be a small parameter

a. Find w_1, w_2 and x_1, x_2 such that the weighted quadrature rule

$$\int_{-h}^{h} f(x)\,dx \doteq w_1 f(x_1) + w_2 f(x_2)$$

is exact when f is any cubic polynomial.

b. Give an error formula for the quadrature rule in a) in terms of the derivatives of f and the power of h.

(7) Consider the nonlinear equation $F(x) = 0$, where $F : \Omega \mapsto \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a C^1 function.

a. Derive the Newton's method, namely for a given initial guess x_0 derive the formula for x_{k+1} in terms of x_k if Newton's method is used for the approximate solution of $F(x) = 0$.

b. Assume that $F \in C^3$ and $F'(x_*)$ is non-singular, where x_* is a solution of $F(x) = 0$. Prove that the Newton's method is well defined if x_0 is sufficiently close to x_* and that the sequence of Newton iterates converges quadratically to the solution.

(8) Consider the Runge-Kutta method for $y' = f(x, y)$ with f being smooth:

$$y_{n+1} = y_n + \alpha h f(x_n, y_n) + \frac{h}{2} f(x_n + \beta h, y_n + \beta h f(x_n, y_n)) \ , \ n = 0, 1, ...$$

a. For what values of $\{\alpha, \beta\}$, the method is consistent?

b. For what values of $\{\alpha, \beta\}$, the method is stable?

c. For what values of $\{\alpha, \beta\}$, the method is most accurate?

(9) Consider the following differential equation:

$$\begin{cases} -u''(x) + u'(x) - u(x) = f(x), & x \in (0, 1), \\ \hfill u(0) = u(1) = 0 \ . \end{cases}$$

a. Write down the variational formulation of the above differential problem: Find $u \in H_0^1(0, 1)$ such that

$$B(u, v) = f(v), \quad \text{for all } v \in H_0^1(0, 1).$$

Prove a Poincare type inequality in $H_0^1(0, 1)$ and use it to show that the above variational problem has a unique solution $u \in H_0^1(0, 1)$ for any right hand side $f \in L^2(0, 1)$.

b. Let V_h be a finite dimensional subspace of $H_0^1(0, 1)$. Show that the discrete problem: Find $u_h \in V_h$ such that

$$B(u_h, v_h) = f(v_h), \quad \text{for all } v_h \in V_h,$$

is well posed and that the following quasi-optimal error estimate holds:

$$|u - u_h|_{H_0^1(0,1)} \le C \inf_{\chi \in V_h} |u - \chi|_{H_0^1(0,1)}.$$

(10) Given a constant $a > 0$ and the parabolic partial differential equation $u_t - a u_{xx} = 0$ for $x \in (0,1)$, and $t \in [0,\infty)$ with initial condition $u(x,0) = u^0(x)$, and periodic boundary condition in space variable x, consider its finite difference discretization on a uniform mesh with steps $h = 1/(N-1)$ in space and $\tau > 0$ in time:

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\tau} + a\frac{2u_i^n - u_{i-1}^n - u_{i+1}^n}{h^2} &= 0, \quad 2 \le i \le N-1, \\ u_i^0 &= u_0(ih), \\ u_1^{n+1} &= u_{N-1}^{n+1}, \\ u_N^{n+1} &= u_2^{n+1}. \end{aligned}$$

a) Find a constant $c > 0$ such that if $\tau \le ch^2$, then

$$\sum_i |u_i^{n+1}|^2 \le \sum_i |u_i^n|^2 \ .$$

b) Verify that under the condition in a), we also have

$$\max_i u_i^{n+1} \le \max_i u_i^n \ .$$