Qualifying Exam

Computational Mathematics

August 2009

Do all six problems. Each problem is worth 20 points.

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1. (20 points) Consider the 2D Poisson problem

$$-\bigtriangledown^2 u = f,$$

on the unit square with $u = 0$ on the boundary.

(a) (10 points) Discretize the equation on a uniform Cartesian mesh with grid spacing h using the 9-point Laplacian. Modify the right-hand side so that the method has a local truncation error that is fourth-order accurate. In particular, which of the following is the largest class of f for which this will hold: analytic, constant, harmonic, polynomial, differentiable.

(b) (10 points) Prove that the method from part (1a) converges to the exact solution at fourth-order in the L_2 norm.

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2. (20 points) Consider the following reaction-diffusion equation

$$\frac{\partial u}{\partial t} = (Du_x)_x + Ru$$

on the interval $[0, 1]$, with $u(0, t) = u(1, t) = 0$ for all t.

(a) (5 points) Assume that D takes on two different values, $D_1 > 0$ on $[0, 0.5)$ and $D_2 > 0$ on $[0.5, 1]$. Adapt the standard second order approximation for u_{xx} in order to calculate $(Du_x)_x$ correctly.

(b) (5 points) Using the forward Euler method, write a finite difference method for the above equation in the form $U^{n+1} = BU^n$. Assume $[0, 1]$ is divided using space step h and that a time step k is used (you may assume for simplicity that $1/h$ is an odd integer.)

(c) (5 points) Estimate under what conditions on D, R, h and k, if any, the method in (2b) will converge. Hint: consider the cases $D \equiv D_1$ and $D \equiv D_2$, and argue whether one of them provides the sharp bound or whether the answer lies between the bounds for these two.

(d) (5 points) Repeat parts (2b) and (2c) for the backwards Euler and Trapezoid methods, where $U^{n+1} = BU^n$ is replaced by the appropriate equation in each case.

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3. (20 points)

   (a) (8 points) Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

      i. (2 points) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$

      ii. (2 points) $U^{n+1} = U^n$

      iii. (2 points) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$

      iv. (2 points) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

   (b) (12 points) For the following schemes, determine if they are consistent with the PDE given. If so, apply von Neumann analysis to determine conditions under which they are stable. Assume the PDE is defined using periodic boundary conditions on the interval $[0, 1]$.

      i. (6 points) $U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1})$ $\qquad u_t = 2u_{xx}$

      ii. (6 points) $U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1})$ $\qquad u_t + au_x = 0 \ (a > 0)$

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4. (20 points) Consider the following weak formulation of Poisson's equation in 2D on the domain $\Omega \subset \mathbb{R}^2$ with boundary condition $u = 0$ on $\partial\Omega$:

   Find $u \in \mathcal{V}$ such that for all $v \in \mathcal{V}$:

$$\int_\Omega \nabla u \cdot \nabla v \, d\vec{x} = \int_\Omega f \, v \, d\vec{x} \qquad \Big(\text{compact form:} \quad \langle \nabla u, \, \nabla v \rangle = \langle f, \, v \rangle\Big),$$

   where $\mathcal{V} := \big\{ v(\vec{x}) : \ \langle v, v \rangle < \infty, \ \langle \nabla v, \, \nabla v \rangle < \infty, \ v\big|_{\partial\Omega} = 0 \big\}$.

   (a) (5 points) Prove that the continuous Galerkin finite element method of order 1 produces an approximation u_h that is optimal in the sense that

$$\|u - u_h\| \le \|u - v\| \quad \text{for all} \quad v \in \mathcal{V}_h,$$

   where $\|u\| := \sqrt{\langle u, \, u \rangle}$.

   (b) (5 points) Use your result from part (a) to show that the error $e := u - u_h$ satisfies:

$$\|\nabla e\|^2 = \langle f, \, e - \pi_h e \rangle - \langle \nabla u_h, \, \nabla \left(e - \pi_h e \right) \rangle.$$

   (c) (10 points) Use your result from part (b) to prove the following *a posteriori* error estimate:

$$\|\nabla e\| \le C_1 \, \|h R_1(u_h)\| + C_2 \, \|\sqrt{h} R_2(u_h)\|,$$

   where

$$R_1(u_h) := f + \triangle u_h \quad \text{and} \quad R_2(u_h) := \max_{S \subset \partial K} |\big[\partial_S \, u_h\big]|.$$

   Hint:

$$\|h^{-1} \left(e - \pi_h e \right) \| \le C \|\nabla e\| \quad \text{and} \quad \sum_K \int_{\partial K} h_K^{-1/2} \left(e - \pi_h e \right) \, d\vec{x} \le C \|\nabla e\|.$$

5. (20 points) Consider the following 1D advection equation on $[0,1]$:

$$\begin{aligned} \textbf{PDE}: &\quad \frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = 0, \\ \textbf{BCs}: &\quad u(0,t) = u(1,t), \\ \textbf{IC}: &\quad u(x,0) = f(x), \end{aligned}$$

where $a > 0$ is a constant.

(a) (5 points) Discretize time using the forward Euler method with constant step size k. Discretize space on a uniform mesh with grid spacing h using a continuous Galerkin method of order 1.

(**NOTE:** you can approximate all integrals with the trapezoidal rule.)

(b) (5 points) Determine the maximum CFL number, $\frac{ak}{h}$, for which the discretization from part (a) is stable.

(c) (5 points) Discretize time using the forward Euler method with constant step size k. Discretize space on a uniform mesh with grid spacing h using a discontinuous Galerkin method of order 0. In particular, take the *trial functions* from the space of piecewise constant functions and the *test functions* from the space of constant functions.

(**NOTE 1:** this means that the trial functions are discontinuous, while the test functions are continuous.)

(**NOTE 2:** define the solution at each node to be the solution from the upwind side (i.e., the left side).)

(d) (5 points) Determine the maximum CFL number, $\frac{ak}{h}$, for which the discretization from part (c) is stable.

6. (20 points) This question concerns finite volume schemes for scalar conservation laws of the form

$$q_t + f(q)_x = 0.$$

(a) (10 points) Prove the following theorem.

Theorem. *Consider the following 3-point finite volume method:*

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n \left(Q_i^n - Q_{i-1}^n\right) + D_i^n \left(Q_{i+1}^n - Q_i^n\right),$$

where C_{i-1}^n *and* D_i^n *are coefficients that may depend on* Q^n*. Then*

$$TV(Q^{n+1}) \leq TV(Q^n),$$

where

$$TV(Q^n) := \sum_{i=-\infty}^{\infty} |Q_i^n - Q_{i-1}^n|,$$

provided that the following conditions are satisfied $\forall i$*:*

$$C_{i-1}^n \geq 0, \qquad D_i^n \geq 0, \qquad C_i^n + D_i^n \leq 1.$$

(b) (10 points) An *E-scheme* is a 1st order finite volume method of the form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2}^n - F_{i-1/2}^n\right],$$

where the numerical flux, $F_{i-1/2}^n$, satisfies

$$\text{sign}\left(Q_i^n - Q_{i-1}^n\right)\left[F_{i-1/2}^n - f(q)\right] \leq 0,$$

for all q between Q_{i-1} and Q_i.

Use the theorem from part (a) to prove that any E-scheme is TVD (i.e., $TV(Q^{n+1}) \leq TV(Q^n)$) if the Courant number is sufficiently small.