QUALIFYING EXAM IN NUMERICAL ANALYSIS

08-22-2005

There are **10 problems** in total in **two sections**. To pass the exam you have to solve **5** out of the **ten** problems below and you should solve at least **two** from each section. Partial credit will be given in borderline cases.

1. Section A

1. Consider the midpoint quadrature rule on the unit interval

$$\int_0^1 f(x)\, dx \approx f(\frac{1}{2}),$$

where $f(\cdot)$ is a smooth function.

a. Find the Peano kernel representation of the error when the integral is approximated by the above rule.

b. Write down the composite midpoint rule on a given finite interval $[a, b]$, with step size $h = (b - a)/N$, ($N > 1$ is an integer). Using the result from the previous item, prove an error bound (in terms of h, a, b and derivatives of $f(\cdot)$) for the composite rule.

2.
- State Jackson's theorem for the approximation of C^1 periodic functions by trigonometric polynomials.
- State Jackson's theorem for the approximation of C^1 functions on the unit interval by ordinary polynomials.
- Prove the latter using the former.

3. Consider the reversed Lax-Friedrichs scheme:

$$\frac{U_{n+1}^{j+1} - 2U_n^j + U_{n-1}^{j+1}}{2k} + c\frac{U_{n+1}^{j+1} - U_{n-1}^{j+1}}{2h} = 0.$$

for the advection equation ($c > 0$) with periodic conditions (for $t > 0$):

$$u_t + cu_x = 0; \qquad u(x, 0) = u_0(x); \qquad u(x + 1, t) = u(x, t).$$

Here U_j^n approximates the value of the solution $u(jk, nh)$, where k is the time step and h is the discretization step in space.

a. Show that if $\lambda = ck/h$ then this scheme can be written as:

$$(1 + \lambda)U_{n+1}^{j+1} + (1 - \lambda)U_{n-1}^{j+1} = 2U_n^j.$$

b. Use von Neumann analysis to investigate the stability of this scheme.

c. Show that this scheme can be viewed as a discretization of the following differential operator:

$$u_t + cu_x + \frac{h^2}{2k}u_{xx}.$$

4. Gaussian elimination and LU factorization:

a. Let A be a symmetric and positive definite matrix of order n, $A = \begin{pmatrix} a_{11} & \mathbf{a}^T \\ \mathbf{a} & A_{n-1} \end{pmatrix}$, where $\mathbf{a}$ is a column vector and A_{n-1} is an $(n-1) \times (n-1)$ matrix. After one step of Gaussian elimination A is converted to a matrix of the form $\begin{pmatrix} a_{11} & \mathbf{a}^T \\ 0 & \tilde{A} \end{pmatrix}$.
Show that the $(n-1) \times (n-1)$ matrix $\tilde{A} := A_{n-1} - (\mathbf{a}\mathbf{a}^T)/a_{11}$ is symmetric and positive definite.

b. Let $(p_1, p_2, \dots, p_n)$ be a permutation of $(1, 2, \dots, n)$. Define the permutation matrix P as

$$P_{ij} = \begin{cases} 1, & i = p_j \\ 0, & i \neq p_j \end{cases}, \quad i = 1, 2, \dots, n.$$

Prove or disprove: if A is nonsingular (not necessarily symmetric and positive definite), then there exists a permutation matrix P, and a pair: *unit* lower triangular matrix L and upper triangular matrix U such that $PA = LU$.

c. Show that if all principal minors of A are nonsingular, P can be taken to be the identity and in such case the pair L, U is unique.

5. Find the quadratic polynomial which is the best approximation to the function $f(x) = x^3$ with respect to the maximum norm on the interval $[0, 1]$. Justify your answer.

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2. Section B

1. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the following boundary value problem:

$$\begin{cases} -\Delta u + u_x &= f, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{cases}$$

a. Write down the variational formulation of the above differential problem: Find $u \in H_0^1(\Omega)$ such that

$$B(u, v) = f(v), \quad \text{for all } v \in H_0^1(\Omega).$$

Show that this variational problem has a unique solution $u \in H_0^1(\Omega)$ for any right hand side $f \in L^2(\Omega)$.

b. Let V_h be a finite dimensional subspace of $H_0^1(\Omega)$. Show that the discrete problem: Find $u_h \in V_h$ such that

$$B(u_h, v_h) = f(v_h), \quad \text{for all } v_h \in V_h,$$

is well posed and that the following quasi-optimal error estimate holds:

$$|u - u_h|_{H_0^1(\Omega)} \leq C \inf_{\chi \in V_h} |u - \chi|_{H_0^1(\Omega)}.$$

2. Consider the conjugate gradient method for the minimization of $\frac{1}{2}(Au,u)-(b,u)$. (A is a symmetric and positive definite matrix) in the form: Starting with $u^0=0$, $r_0=b$ and $p_0=r_0$ the successive approximations to the minimizer are computed by

$$\begin{aligned} u^{k+1} &= u^k+\alpha_k p_k, \qquad r_{k+1}=r_k-\alpha_k A p_k;\\ p_{k+1} &= r_{k+1}-\beta_k p_k \end{aligned}$$

where $\alpha_k=(r_k,p_k)/\|p_k\|_A^2$ and $\beta_k=-(r_{k+1},p_k)_A/\|p_k\|_A^2$.

a. Show that for $k=0,1,2,\ldots$ the following relations are true

$$span\{p_0,p_1,\ldots,p_k\}=span\{r_0,r_1,\ldots,r_k\}=span\{r_0,Ar_0,\ldots,A^k r_0\}.$$

b. Show that if $A\in R^{n\times n}$ then for some $m\le n$, $r_m=0$ (assume that all operations are performed exactly).

3. Let A be a diagonally dominant M-matrix. Prove that the Jacobi and Gauss-Seidel iterative methods both converge to the solution of $Ax=b$, for any initial guess.

4. Consider the implicit scheme

$$y_{n+1}=y_n+\frac{h}{2}(f(t_n,y_n)+f(t_{n+1},y_{n+1})). \tag{1}$$

for the solution of the initial value problem:

$$y'=f(t,y),\quad y(0)=y_0,\quad t\in[0,T]. \tag{2}$$

a. Assume that the right hand side f is Lipschitz continuous, and show that for sufficiently small step size h, (**??**) has a solution y_{n+1} for any y_n.

b. Assume that the solution $y(t)$ to the initial value problem (**??**) is twice continuously differentiable, i.e. $y\in C^2([0,T]))$, and prove that

$$\lim_{h\to 0}|y(t_n)-y_n|=0,$$

where the approximations y_n are obtained via the implicit trapezoid scheme. Here $t_n=nh$, $0\le n\le N$ and $h=T/N-1$, $N>1$.

5. Let $\hat{T}$ and T be the following non-degenerate simplexes,

$$\begin{aligned} \hat{T} &:= \{\hat{\mathbf{x}}\in\mathbb{R}^2 \mid \hat{x}_i\ge 0,\ i=1,2;\quad \hat{x}_1+\hat{x}_2\le 1\},\\ T &:= \{\mathbf{x}\in\mathbb{R}^2,\ \mid \mathbf{x}=B\hat{\mathbf{x}}+\mathbf{x}_0,\ \hat{\mathbf{x}}\in\hat{T}\}, \end{aligned}$$

where B is a nonsingular 2×2 matrix and $\mathbf{x}_0\in\mathbb{R}^2$ is a fixed vector.

a. Find the three linear functions, $\{\hat{\phi}_i(\hat{\mathbf{x}})\}_{i=0}^2$, such that $B\hat{\mathbf{x}}+\mathbf{x}_0=\sum_{i=0}^{2}\hat{\phi}_i(\hat{\mathbf{x}})\mathbf{x}_i$, where $\mathbf{x}_i$ are the vertecies of T.

b. Assume that $f\in H^2(T)$. Give an estimate of $\|f-f_I\|_{L_2}$ and of $\|\nabla(f-f_I)\|_{L_2}$. Justify your answer. Here f_I is the linear interpolant of $f(\mathbf{x})$, defined as

$$f_I(\mathbf{x}) := \sum_{i=0}^{2} f(\mathbf{x}_i)\phi_i(\mathbf{x}), \qquad \phi_i(\mathbf{x})=\hat{\phi}_i\left(B^{-1}(\mathbf{x}-\mathbf{x}_0)\right).$$