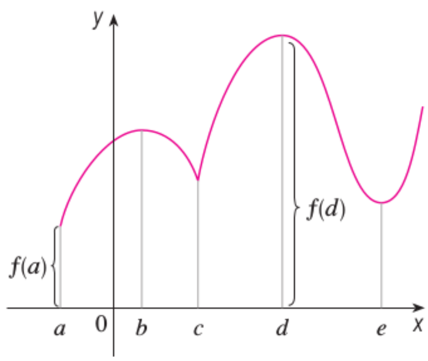
4.1: Minimum and Maximum Values

Let c be a number in the domain D of f. Then $f(c)$ is the

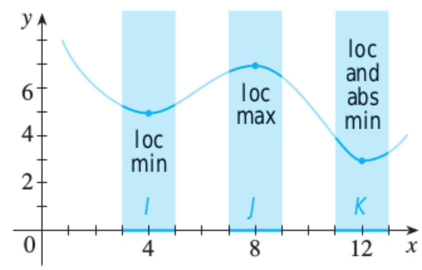
- **absolute minimum** value of f if $f(c) \leq f(x)$ for all x in D.
- **absolute maximum** value of f if $f(c) \geq f(x)$ for all x in D.

The absolute minimum and maximum are also refered to as the **global** minimum and maximum. Together, they are called **extreme values** of f. The number $f(c)$ is a

- **local minimum** value of f if $f(c) \leq f(x)$ for all x near c.
- **local maximum** value of f if $f(c) \geq f(x)$ for all x near c.



Abs min $f(a)$, abs max $f(d)$,
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$



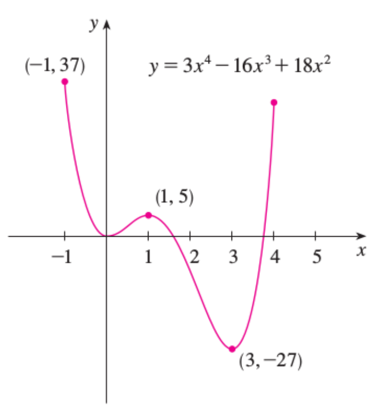
Example 1. *Find the local and absolute extreme values of*

(a) $f(x) = \cos x$

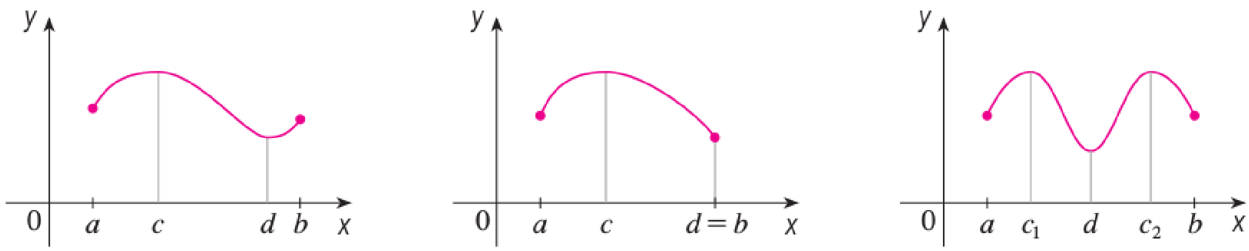
(b) $f(x) = x^2$

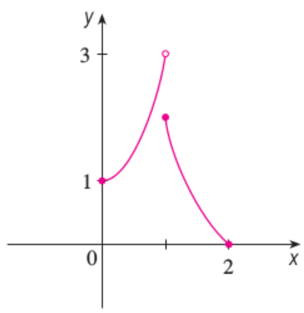
(c) $f(x) = x^3$

Example 2. *Find the local and absolute extreme values of* $f(x) = 3x^4 - 16x^3 + 18x^2$ *for* $-1 \leq x \leq 4$

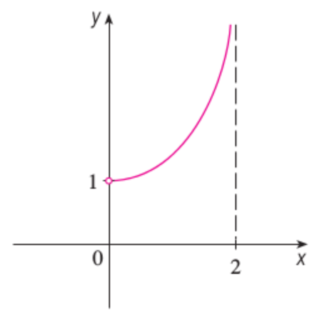


The Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains an absolute minimum value and an absolute maximum value at some numbers in $[a, b]$.





This function has minimum value $f(2) = 0$, but no maximum value.



This continuous function g has no maximum or minimum.

Fermat's Theorem: If f has a local minimum or maximum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Notice this theorem states that if f is differentiable, then $f' = 0$ where f has an extreme value. The converse, "If $f'(x) = 0$, then f has an extreme value at x", is not true as the following example illustrates.

Example 3. *Find the x-values at which $f' = 0$ if $f(x) = x^3$. Does f have any extreme values?*

Also, notice that a function need not satisfy the assumptions of Fermat's Theorem since it may not be differentiable at an extreme value as the following example illustrates.

Example 4. *Find the x-value at which $f(x) = |x|$ has an extreme value. Is f differentiable at this x-value?*

Despite these subtleties, identifying the x-values at which $f' = 0$ is a useful approach to locate the extreme values of f.

A **critical number** of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

Example 5. *Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.*

To find the absolute minimum and maximum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers in (a, b)
2. Find the values of f at the endpoints of the interval.
3. The smallest value from Steps 1 and 2 is the absolute minimum, and the largest value is the absolute maximum.

Example 6. *Find the absolute minimum and maximum values of*

$$f(x) = x^3 - 3x^2 + 1 \qquad \textit{for } -\frac{1}{2} \leq x \leq 4$$

Example 7. *Find the absolute minimum and maximum values of* $f(x) = x - 2\sin x$ *for* $0 \leq x \leq 2\pi$.

Example 8. *Sketch the graph of a function* f *that is continuous on* $[1, 5]$ *and has the given properties*

(a) Absolute minimum at 2, absolute maximum at 3, local minimum at 4.

(b) Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4.

(c) f *has no local maximum or minimum, but 2 and 4 are critical numbers.*

Example 9. *Find the critical numbers of the function.*

(a) $f(x) = 2x^3 - 3x^2 - 36x$

(b) $f(x) = 2x^3 + x^2 + 2x$

(c) $g(t) = |3t - 4|$

(d) $h(t) = 3t - \arcsin t$

Example 10. *Find the absolute maximum and minimum values of* f *on the given interval.*

(a) $f(x) = (x^2 - 1)^3$, $[-1, 2]$

(b) $f(x) = x + \frac{1}{x}$, $[0.2, 4]$

(c) $f(x) = x - \ln x$, $[\frac{1}{2}, 2]$