# MAC2311: Calculus 1 - Section 1

## Test 2

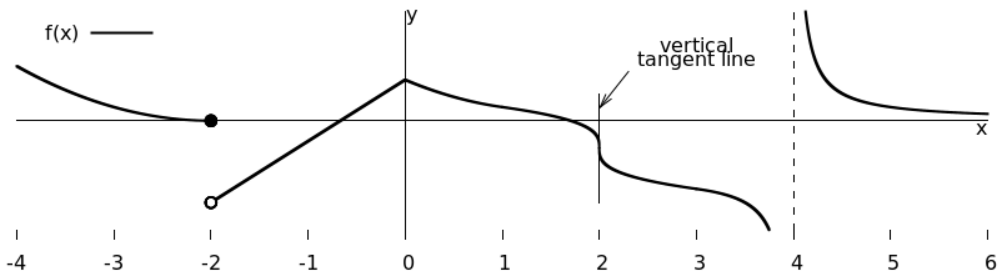
February 19, 2015

Name: ______________________________

Answer each question in the space provided on the question sheets. If you run out of space for an answer, continue on the back of the page. Credit will only be given if you clearly show all of your work. Calculators may be used for this test.

| **Question** | **Points** | **Score** |
|---|---|---|
| 1 | 8 | |
| 2 | 3 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 5 | |
| 6 | 4 | |
| 7 | 4 | |
| 8 | 5 | |
| 9 | 13 | |
| Extra Credit | – | |
| Total: | 52 | |

1. [8 points] Use the following graph of $f$ to answer the questions below it.

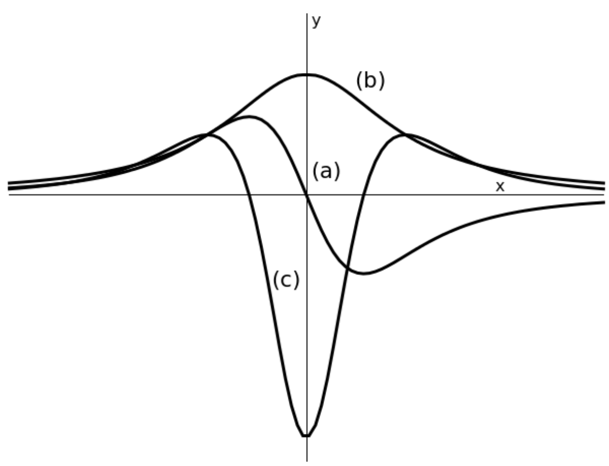



(a) [4 points] List all $x$-values at which $f$ is not differentiable: $x = -2$, 0, 2, and 4

(b) [4 points] For each $x$-value that you listed in part (a), state why $f$ is not differentiable.
$f$ is not differentiable at

- $x = -2$ because $f$ is not continous at $x = -2$
- $x = 0$ because $f$ has a "corner" or "kink" at $x = 0$
- $x = 2$ because $f$ has a vertical tangent line at $x = 2$
- $x = 4$ because $f$ is not continous at $x = 4$

2. [3 points] The following figure shows the graphs of $f$, $f'$, and $f''$.




Complete the following statements by filling in one of $f$, $f'$, or $f''$ in each blank.

(a) is the graph of $f'$

(b) is the graph of $f$

(c) is the graph of $f''$

3. [5 points] Let $f$ and $g$ be differentiable functions. Complete each of the following equations using differentiation rules.

(a) $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

(b) $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

(c) $\frac{d}{dx}\left(2^x\right) = 2^x \ln 2$

(d) $\frac{d}{dx}\left(\cos x\right) = -\sin x$

(e) $\frac{d}{dx}\left(\log_2 x\right) = \frac{1}{2^x \ln 2}$

4. [5 points] Let $f(x) = (x^2+1)e^x$

(a) [3 points] Differentiate $f(x)$.

Using the product rule, $\frac{d}{dx}(e^x) = e^x$, and the power rule,

$$\begin{aligned}\frac{d}{dx}f(x) &= (x^2+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2+1)\\ &= (x^2+1)e^x + e^x(2x)\\ &= (x^2+2x+1)e^x = (x+1)^2e^x\end{aligned}$$

(b) [2 points] Find the $x$-value(s) at which $f$ has a horizontal tangent line.

$f$ has a horizontal tangent line for all $x$ such that $f'(x) = 0$.

$$\begin{aligned}f'(x) = (x+1)^2e^x &= 0\\ (x+1)^2 &= 0\\ x &= -1.\end{aligned}$$

5. [5 points] Consider taking the derivative of $\cot x$ by first expressing $\frac{d}{dx}(\cot x)$ as the derivative of a quotient of trigonometric functions, and then using the quotient rule.

(a) [1 point] Select the equation that correctly expresses $\frac{d}{dx}(\cot x)$ as the derivative of a quotient of trigonometric functions.

A. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$

B. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

**C.** $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

D. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\sec x}{\csc x}\right)$

(b) [4 points] Prove that $\frac{d}{dx}(\cot x) = -\csc^2 x$ by using the quotient rule to evaluate the righthand side of your answer to part (a).

From part (a), $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$. Using the quotient rule, $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, and the identity $\cos^2 x + \sin^2 x + 1$,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) &= \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x} \\
&= \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x
\end{aligned}$$

6. [4 points] Differentiate $e^{\cot \sqrt{x}}$.

Using the chain rule twice, $\frac{d}{dx}(\cot x) = -\csc^2 x$, and the power rule,

$$\begin{aligned}
\frac{d}{dx}\left(e^{\cot \sqrt{x}}\right) &= e^{\cot \sqrt{x}} \frac{d}{dx}(\cot \sqrt{x}) \\
&= e^{\cot \sqrt{x}}(-\csc^2 \sqrt{x}) \frac{d}{dx}(\sqrt{x}) \\
&= e^{\cot \sqrt{x}}(-\csc^2 \sqrt{x}) \left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\
&= -\frac{e^{\cot \sqrt{x}} \csc^2 \sqrt{x}}{2\sqrt{x}}
\end{aligned}$$

7. [4 points] Use implicit differentiation to find $y'$ if $x^2 + y^2 + xy = 1$.

$$\begin{aligned}
\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(xy) &= \frac{d}{dx}(1) \\
2x + 2yy' + (x \cdot y' + y \cdot 1) &= 0 \\
2x + 2yy' + xy' + y &= 0 \\
2yy' + xy' &= -2x - y \\
y'(2y + x) &= -2x - y \\
y' &= -\frac{2x + y}{2y + x}
\end{aligned}$$

8. [5 points] Use logarithmic differentiation to find a formula for $y'$ in terms of $x$ if $y = x^x$.

Using logarithm rules, $\frac{d}{dx} \ln x = \frac{1}{x}$, and the product rule,

$$\begin{aligned}
y &= x^x \\
\ln y &= \ln x^x = x \ln x \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x \ln x) \\
\frac{1}{y} \cdot y' &= \left( x \cdot \frac{1}{x} + 1 \cdot \ln x \right) \\
\frac{y'}{y} &= 1 + \ln x \\
y' &= y(1 + \ln x) \\
y' &= x^x(1 + \ln x)
\end{aligned}$$

9. [13 points] For each part of the following question, use correct units in your final answer.

The position of a particle is given by the equation

$$s(t) = t^3 - 9t^2 + 15t$$

where $s$ is measured in meters and $t \geq 0$ is measured in seconds.

(a) [4 points] Find the velocity at time $t$.

Since $s$ is the equation of motion, the velocity is $v(t) = s'(t)$ Using the power rule,

$$v(t) = s'(t) = 3t^2 - 18t + 15 \text{ m/s}$$

(b) [3 points] When is the particle at rest?

The particle is at rest when $v(t) = 0$.

$$\begin{aligned} v(t) = 3t^2 - 18t + 15 &= 0 \\ 3(t-1)(t-5) &= 0 \\ t = 1 \text{ s and } t &= 5 \text{ s} \end{aligned}$$

(c) [3 points] When is the particle moving forward?

The particle is moving forward when $v(t) > 0$.

$$\begin{aligned} v(t) = 3(t-1)(t-5) &> 0 \\ t < 1 \text{ s and } t &> 5 \text{ s} \end{aligned}$$

(d) [3 points] Find the acceleration at time $t$.

Since $s$ is the equation of motion, the acceleration is $a(t) = v'(t) = s''(t)$ Using the power rule,

$$a(t) = v'(t) = 6t - 18 \text{ m/s}^2$$

**Extra Credit** [2 points]: Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

(a) Show that $f'(x) = ADX$ where

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \text{ and } X = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}.$$

By the power rule,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2$$

And, by matrix multiplication,

$$\begin{aligned} ADX &= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2x \\ 3x^2 \end{pmatrix} = a_1 + 2a_2 x + 3a_3 x^2 \end{aligned}$$

Therefore, $f'(x) = ADX$.

(b) What is $AD^k X$ for $k > 3$?

In general, $f^{(k)}(x) = AD^k X$. Since $f$ is a polynomial of degree 3, $f^{(k)} = AD^k X = 0$ for $k > 3$.