Qualifying Exam
Computational Mathematics
January 2010

Do all six problems. Each problem is worth 20 points.

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1. (20 points) Consider the diffusion equation in 1D on $0 \le x \le 1$ and $t \ge 0$:

$$\begin{aligned} \textbf{PDE}: &\quad u_t = \beta\, u_{xx}, \quad \kappa > 0, \\ \textbf{IC}: &\quad u(x,0) = f(x), \\ \textbf{BCs}: &\quad u(0,t) = u(1,t) \quad \text{and} \quad u'(0,t) = u'(1,t), \end{aligned}$$

and the numerical method:

$$\frac{U_i^{n+1} - U_i^{n-1}}{2k} = \beta \left(\frac{U_{i+1}^n - \left(U_i^{n+1} + U_i^{n-1}\right) + U_{i-1}^n}{h^2} \right).$$

(a) (5 points) Determine the local truncation error of this method.

(b) (15 points) Find conditions on the time k under which this method is stable.

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2. (20 points) Consider the following system of ODEs:

$$\vec{U}'(t) = A\,\vec{U}(t) + B\,\vec{U}(t),$$

where A and B are time-independent matrices, and the the following operator split method is used to compute the solution from t^n to t^{n+1}:

Step 1: Solve over $[t^n, t^{n+1}]$: $\vec{U}'(t) = A\,\vec{U}(t)$, $\vec{U}(t^n) = \vec{U}^n \implies$ Produces $\vec{U}^\star$,

Step 2: Solve over $[t^n, t^{n+1}]$: $\vec{U}'(t) = B\,\vec{U}(t)$, $\vec{U}(t^n) = U^\star \implies$ Produces $\vec{U}^{\star\star}$,

Step 3: Solve over $[t^n, t^{n+1}]$: $\vec{U}'(t) = B\,\vec{U}(t)$, $\vec{U}(t^n) = \vec{U}^n \implies$ Produces $\vec{U}^\dagger$,

Step 4: Solve over $[t^n, t^{n+1}]$: $\vec{U}'(t) = A\,\vec{U}(t)$, $\vec{U}(t^n) = \vec{U}^\dagger \implies$ Produces $\vec{U}^{\dagger\dagger}$,

Step 5: Set $\vec{U}^{n+1} = \frac{1}{2}\left(\vec{U}^{\star\star} + \vec{U}^{\dagger\dagger}\right)$,

where $t^{n+1} = t^n + k$.

(a) (10 points) Compute the local truncation for this method assuming that each subproblem is solved exactly.

(b) (5 points) Assume that A and B are both symmetric negative definite matrices. Under what conditions on the time-step k is the above operator split method L_2-stable?

(c) (5 points) Write down the Strang splitting method for the above problem. Under the assumption that A and B are symmetric negative definite matrices, again compute conditions on k for L_2-stability.

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3. (20 points) Consider the 1D hyper-diffusion equation on $0 \le x \le 1$ and $t \ge 0$:

$$\begin{aligned}
\textbf{PDE}: &\quad u_t = -u_{xxxx},\\
\textbf{IC}: &\quad u(x,0) = f(x),\\
\textbf{BCs}: &\quad u(0,t) = u(1,t), \quad u'(0,t) = u'(1,t), \quad u''(0,t) = u''(1,t), \quad u'''(0,t) = u'''(1,t).
\end{aligned}$$

(a) (5 points) Derive a $\mathcal{O}(h^2)$ central finite difference formula for u_{xxxx}.

(b) (5 points) In part (a) you found a difference formula that can be written as

$$u_{xxxx}(x_i) = \sum_{j=-N}^{N} a_j\, u(x_i + jh) + \tau_i,$$

where $2N+1$ are the number of points in your finite difference stencil, a_j are the weights, and τ_i is the truncation error.

In this problem I want you to consider the effect of round-off error. Let

$$u(x_i + jh) = \tilde{u}(x_i + jh) + \varepsilon_j,$$

where ε_j is the round-off error.

Find an upper bound on the error: $|E_i| := |u_{xxxx}(x_i) - \sum_{j=-N}^{N} a_j\, \tilde{u}(x_i + jh)|$.

(c) (5 points) Based on your result from part (b), find the optimal h that minimizes the upper bound on the error $|E_i|$.

(d) (5 points) Discretize the hyper-diffusion equation using your finite difference method from part(a) and the forward Euler method. Find conditions on the time-step k such that this method is stable.

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4. (20 points) Consider the following third-order linear boundary value problem:

$$\textbf{ODE}: \quad \begin{cases} v''(x) + u(x) = f(x) \\ v(x) = u'(x) \end{cases}, \quad 0 \le x \le 1,$$

$$\textbf{BCs}: \quad u(0) = \alpha, \quad v(0) = \beta, \quad u(1) = \gamma.$$

(a) (5 points) Write down the weak form of the above equation, clearly indicating the spaces from which each of the functions is drawn.

(b) (5 points) Prove that under suitable conditions on $f(x)$ the original problem and the weak form are equivalent.

(c) (10 points) Discretize your result from part (a) via a cG(1) method. Write your final method as a linear algebra problem, clearly defining the coefficient matrix, the vector of unknowns, and the right hand side vector.

5. (20 points) Consider the 2D poisson equation:

$$\textbf{PDE}: \quad -\nabla^2 u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2,$$
$$\textbf{BC}: \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

(a) (5 points) Recast this problem in weak form. Clearly explain the spaces from which test and trial functions are drawn.

(b) (5 points) Prove that this weak form has a unique solution.

(c) (10 points) Discretize the weak form via a cG(1) method on a mesh $\mathcal{T}$ that is made up of triangles. Assume that such a mesh has been created and that the only thing we know about this mesh are the following pieces of information:

$$\begin{aligned}
\texttt{p}(\texttt{1}:\texttt{m},\texttt{1}:\texttt{2}) &:= \text{list of } \texttt{m} \text{ interior nodes},\\
\texttt{p}(\texttt{m}+\texttt{1}:\texttt{M},\texttt{1}:\texttt{2}) &:= \text{list of } (\texttt{M}-\texttt{m}) \text{ boundary nodes},\\
\texttt{t}(\texttt{1}:\texttt{k},\texttt{1}:\texttt{3}) &:= \text{list of } \texttt{k} \text{ elements}.
\end{aligned}$$

NOTE 1: $\texttt{p}(\texttt{i},\texttt{1})$ and $\texttt{p}(\texttt{i},\texttt{2})$ refer to the x and y-coordinates of the i^{th} node.

NOTE 2: $\texttt{t}(\texttt{j},\texttt{1})$, $\texttt{t}(\texttt{j},\texttt{2})$, and $\texttt{t}(\texttt{j},\texttt{3})$ are all integers and refer to the 3 nodes that define the element j. For example, the x and y coordinates of the 3 nodes that define the j^{th} element are:

$$(x_1,y_1) := \texttt{p}(\texttt{t}(\texttt{j},\texttt{1}),\texttt{1}:\texttt{2}), \quad (x_2,y_2) := \texttt{p}(\texttt{t}(\texttt{j},\texttt{2}),\texttt{1}:\texttt{2}), \quad (x_3,y_3) := \texttt{p}(\texttt{t}(\texttt{j},\texttt{3}),\texttt{1}:\texttt{2}).$$

Write a MATLAB-type code that uses the arrays **p** and **t** as defined above to construct the stiffness matrix A *(just write the code on paper, you don't need a computer).*

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6. (20 points) Let $x_j = jh$ for $j = 1, 2, \ldots, N$ where N is an even integer and $h = 2\pi/N$. Consider the forward and inverse DFTs:

$$\hat{v}_k := h \sum_{j=1}^{N} e^{-ikjh}\, v_j, \quad \text{for} \quad k = -\frac{N}{2}, \ldots, \frac{N}{2},$$
$$v_j := \frac{1}{2\pi} \sum_{k=-N/2}^{N/2} a_k\, e^{ikjh}\, \hat{v}_k, \quad \text{for} \quad j = 1, \ldots, N,$$

where $a_{N/2} = a_{-N/2} = 1/2$ and $a_k = 1$ for all other k.

(a) (5 points) Find the band-limited interpolant, $p(x)$, of the Kronecker delta:

$$\delta_j = \begin{cases} 1, & j = 0 \,(\mathrm{mod}\, N), \\ 0, & j \neq 0 \,(\mathrm{mod}\, N). \end{cases}$$

(b) (5 points) Use the result from part (a) to find the band-limited interpolant, $p'(x)$, of any grid function v_j.

(c) (10 points) Compute $w_k = p'(x_k)$ using your result from part (b).