MAC2312: Calculus 2 - Section 3

Test 3 July 16, 2015

Name: ____________________

Answer each question in the space provided on the question sheets. If you run out of space for an answer, continue on the back of the page. Credit will only be given if you clearly show all of your work. Calculators may not be used for this test.

Question	**Points**	**Score**
1	8	
2	8	
3	8	
4	8	
5	8	
6	12	
7	12	
8	16	
9	10	
10	10	
11 (bonus)	–	
Total:	100	

1. (a) [4 points] Convert the point $(4, \pi/6)$ from polar coordinates to Cartesian coordinates.

$$x = r\cos\theta = 4\cos\frac{\pi}{6} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$$
$$y = r\sin\theta = 4\sin\frac{\pi}{6} = 4\cdot\frac{1}{2} = 2$$

$(x, y) = (2\sqrt{3}, 2)$

(b) [4 points] Represent using polar coordinates the point whose Cartesian coordinates are $(1, -\sqrt{3})$.

$$r^2 = x^2 + y^2 = 1^2 + (-\sqrt{3})^2 = 4$$
$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

Since $(1, -\sqrt{3})$ is in the fourth quadrant, $(r, \theta) = (2, -\pi/3 + 2n\pi)$ or $(-2, 2\pi/3 + 2n\pi)$ for any integer n.

2. (a) [4 points] Find a general formula for a_n, the n^{th} term, in the sequence

$$\{a_n\}_{n=1}^{\infty} = \left\{-\frac{1}{4}, \frac{3}{16}, -\frac{5}{64}, \frac{7}{256}, -\frac{9}{1024}, \dots\right\}$$

$$a_n = (-1)^n\frac{2n-1}{4^n}$$

(b) [4 points] Find the sum of the series

$$2 + \frac{6}{5} + \frac{18}{25} + \frac{54}{125} + \frac{162}{625} + \cdots$$

This is a geometric series with first term $a = 2$ and common ratio $r = \frac{3}{5}$. Since $|r| < 1$, the sum exists and is

$$\frac{a}{1-r} = \frac{2}{1-3/5} = \frac{2}{2/5} = 5$$

3. [8 points] Match the graphs of the parametric equations $x = f(t)$ and $y = g(t)$ in (a)–(d) with the parametric curves labeled I–IV by filling I, II, III, and IV in the following blanks.

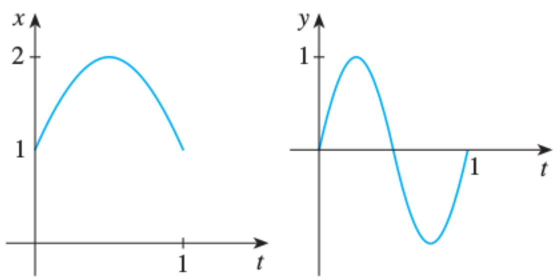
a) ______I______

b) ______III______

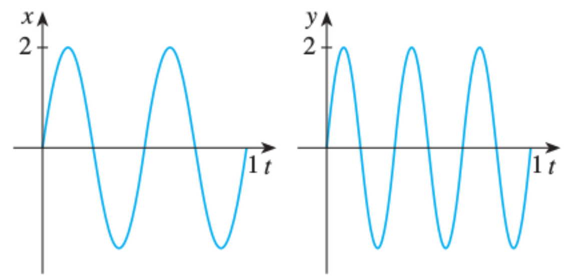
c) ______II______

d) ______IV______

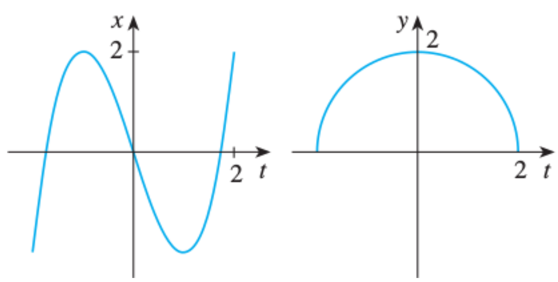
(a)



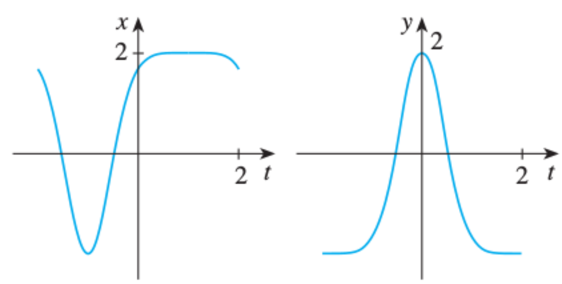
(b)



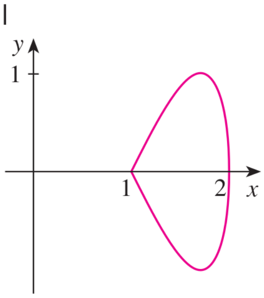
(c)



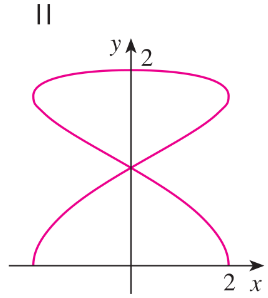
(d)



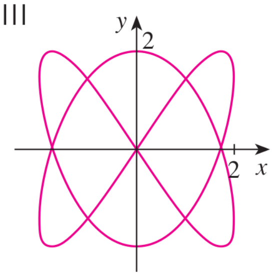
I



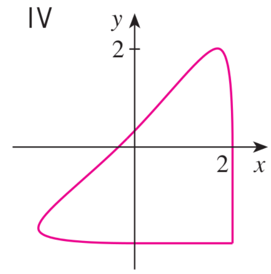
II



III



IV



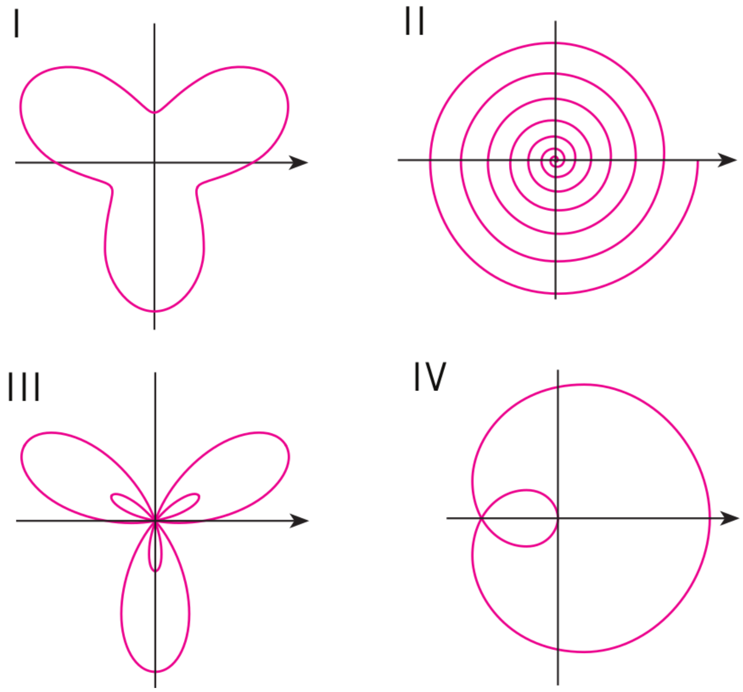
4. [8 points] Match the polar equations in (a)–(d) with the graphs labeled I–IV by filling I, II, III and IV in the following blanks.

a) $r = \theta^2$ ______II______

b) $r = \cos(\theta/3)$ ______IV______

c) $r = 2 + \sin 3\theta$ ______I______

d) $r = 1 + 2\sin 3\theta$ ______III______



5. [8 points] Set up an integral that represents the length of the curve with polar equation $r = \theta^2$ for $0 \le \theta \le 8\pi$.

$$L = \int ds = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{8\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

6. [12 points] Set up an integral that represents the area of the surface obtained by rotating one arch of the cycloid with parametric equations $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ about the x-axis.

One arch of the cycloid is traced out for $0 \le \theta \le 2\pi$, so the surface area is

$$S = \int 2\pi y\, ds = \int y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$
$$= \int_0^{2\pi} 2\pi(1 - \cos\theta)\sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta$$

7. [12 points] Set up an integral that represents the area of the region enclosed by one loop of the curve with polar equation $r = \sin(2\theta)$.

The curve forms a loop for $0 \le \theta \le \frac{\pi}{2}$, so the area is

$$A = \int \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{1}{2} \sin^2(2\theta) \, d\theta$$

8. Consider the curve defined by the parametric equations $x = 3t^2$, $y = 6t - 2t^3$.

(a) [4 points] Find all points where the tangent is either horizontal or vertical.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6 - 6t^2}{6t} = \frac{1 - t^2}{t} = \frac{(1-t)(1+t)}{t}$$

The tangent is horizontal when $t = \pm 1 \Rightarrow (x, y) = (3, \pm 4)$ and vertical when $t = 0 \Rightarrow (x, y) = (0, 0)$.

(b) [4 points] For what values of t is the curve concave upward?

Since $\frac{dy}{dx} = \frac{1-t^2}{t} = t^{-1} - t$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^{-1} - t)}{6t} \\ &= \frac{-t^{-2} - 1}{6t} = -\frac{t^2 + 1}{6t^3} \end{aligned}$$

The curve is concave upward when $d^2y/dx^2 > 0 \Rightarrow t < 0$.

(c) [8 points] Find the area bounded by the y-axis and the curve for $-1 \le t \le 1$.

$$\begin{aligned} A &= \int x \, dy = \int_{-1}^{1} 3t^2 \cdot (6 - 6t^2) \, dt = 18 \int_{-1}^{1} (t^2 - t^4) \, dt \\ &= 18 \cdot 2 \int_0^1 (t^2 - t^4) \, dt && \text{by symmetry} \\ &= 36 \left[\frac{t^3}{3} - \frac{t^5}{5} \right]_{t=0}^{1} = \frac{24}{5} \end{aligned}$$

9. [10 points] Determine if the series $\sum_{n=1}^{\infty} \sqrt[n]{2}$ converges or diverges.

$$\lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^{\lim_{n\to\infty} 1/n} = 2^0 = 1 \neq 0$$

By the divergence test, the series diverges.

10. [10 points] Given that the following sequence is increasing and bounded, find its limit.

$$\{a_n\}_{n=1}^{\infty} = \left\{1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$$

The sequence is defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n}$ for $n = 1, 2, 3, \ldots$. Let L be the limit of the sequence, then

$$\begin{aligned}
\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{2a_n} \\
\lim_{n\to\infty} a_{n+1} &= \sqrt{2 \lim_{n\to\infty} a_n} \\
L &= \sqrt{2L} \\
L^2 &= 2L \\
L(L-2) &= 0 \Rightarrow L = 0, 2
\end{aligned}$$

The first term in the sequence is $1 > 0$ and the sequence is increasing so $L = 0$ cannot be the limit, thus, the limit is $L = 2$.

11. [5 points (bonus)] Evaluate $\int_0^1 f(x)\, dx$ if

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x = 1/n \text{ for some integer } n \\ 1 & \text{otherwise.} \end{cases}$$

Since f is discontinuous at $x = 0$ and $x = 1/n$ for $n = 2, 3, 4, \ldots$

$$\begin{aligned} \int_0^1 f(x)\, dx &= \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} f(x)\, dx = \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} dx \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1 \end{aligned}$$