2.7: Derivatives and Rates of Change

Derivatives

The **derivative** of a function f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limits exists and is finite.

Tangents

The **tangent line** to $f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists. Equivalently, letting $x = a + h$, $m = f'(a)$.

Example 1. *Find the derivative of the following functions at the number a. Use the derivative to find an equation of the tangent line to the graph of the function at the given point P.*

(a) $f(x) = x^2, \quad P(1, 1)$

(b) $f(x) = x^2 - 8x + 9, \quad P(3, -6)$

Rates of Change

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The **instantaneous rate of change** of $y = f(x)$ with respect to x at $x = x_1$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Equivalently, letting $x_1 = a$ and $x_2 = a + h$, the instantaneous rate of change of $y = f(x)$ with respect to x at $x_1 = a$ is $f'(a)$.

Example 2. *A manufacturer produces bolts of fabric with fixed width. The cost of producing x yards of this fabric is* $C = f(x)$ *dollars.*

(a) What is the meaning of $f'(x)$*? What are its units?*

(b) In practical terms, what does it mean to say that $f'(1000) = 9$*?*

(c) Which would you expect to be greater? $f'(50)$ *or* $f'(500)$*?*

Example 3. *If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by* $y = 40t - 16t^2$*. Find the velocity at* $t = 2$.

Example 4. *Find the derivative at* $x = a$ *for the following functions* *(a)* $f(x) = x^3$ *(b)* $f(x) = \sqrt{x}$ *(c)* $f(x) = \frac{1}{x}$ *(d)* $f(x) = \frac{1}{x^2}$