4.4: Indeterminate Forms and l'Hôspital's Rule

The limit $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is an

(a) **indeterminate form of the type** $\frac{0}{0}$ if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$.

(b) **indeterminate form of the type** $\frac{\infty}{\infty}$ if $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$.

Example 1. *State the type of indeterminate form of the following limits*

(a) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$

(b) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{x-1}$

(c) $\displaystyle\lim_{x\to 1} \frac{x^2-x}{x-1}$

(d) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

L'Hôspital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). If $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided this limits exists or is $\pm\infty$. L'Hôspital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Example 2. *Evaluate the limits in Example 1.*

Example 3. *Evaluate* $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$.

Example 4. *Evaluate* $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

Example 5. *Evaluate* $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

Example 6. *Evaluate* $\displaystyle\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x}$.

Indeterminate Products

The limit $\lim\limits_{x\to a}[f(x)\cdot g(x)]$ is an **indeterminate form of the type** $0\cdot\infty$ if $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \pm\infty$. This limit can be rewritten in an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by rewriting the product fg as

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}.$$

Example 7. *Evaluate* $\displaystyle\lim_{x\to 0^+} x\ln x$.

Example 8. *Evaluate* $\displaystyle\lim_{x\to\infty} x\sin\frac{\pi}{x}$.

Indeterminate Differences

The limit $\lim_{x\to a}[f(x)-g(x)]$ is an **indeterminate form of the type** $\infty-\infty$ if $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$. This limit sometimes may be rewritten in an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by combining f and g with a common denominator.

Example 9. *Evaluate* $\lim_{x\to(\pi/2)^-}(\sec x-\tan x)$

Example 10. *Evaluate* $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

Indeterminate Powers

The limit $\lim_{x\to a}[f(x)]^{g(x)}$ is an

(a) **indeterminate form of the type** 0^0 if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$.

(b) **indeterminate form of the type** ∞^0 if $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$.

(c) **indeterminate form of the type** 1^∞ if $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\pm\infty$.

These limits can be rewritten in terms of an indeterminate product by rewriting f^g as

$$f^g=e^{g\ln f}.$$

Example 11. *Evaluate* $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Example 12. *Evaluate* $\lim_{x\to 0^+} x^x$.