**APPLIED MATHEMATICS and STATISTICS**

**DOCTORAL QUALIFYING EXAMINATION**

**in COMPUTATIONAL APPLIED MATHEMATICS**

**Spring 2012 (January)**

**(CLOSED BOOK EXAM)**

**This is a two part exam.**
**In part A, solve 4 out of 5 problems for full credit.**
**In part B, solve 4 out of 5 problems for full credit.**

Indicate below which problems you have attempted by circling the appropriate numbers:

| | | | | | |
|---|---|---|---|---|---|
| **Part A:** | 1 | 2 | 3 | 4 | 5 |
| **Part B:** | 6 | 7 | 8 | 9 | 10 |

**NAME** ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 25th, 2012

Time: 9:00 – 1:00 PM

**A1.** Draw the lines of characteristics for the following equations using the Cauchy data:

$$u(x,0) = \begin{cases} -3, & x \leq 0 \\ -2 & x > 0 \end{cases}$$

and then the Cauchy data:

$$\tilde{u}(x,0) = \begin{cases} 3, & x \leq 0 \\ 2 & x > 0 \end{cases}$$

a) $u_t - 2u_x = 0.$

b) $u_t - xu_x = 0.$

c) $u_t - 2uu_x = 0.$

**A2.** Solve the following boundary value problem:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = f(x, y, z), & 0 < x, y, z < 0; \\ u(0, y, z) = u(1, y, z) = 0, & 0 \leq y, z \leq 1; \\ u(x, 0, z) = u(x, 1, z) = 0, & 0 \leq x, z \leq 1; \\ u_z(x, y, 0) = u_z(x, y, 1) = 0, & 0 \leq x, y \leq 1. \end{cases}$$

**A3.** Evaluate

$$\int_\gamma \frac{|z|e^z dz}{z^2 - 2z - 3}$$

using the Cauchy integral formula if $\gamma$ is the circle of radius 2 centered in the origin.

**A4.** Find the principal value of the improper integral

$$\int_{-\infty}^{\infty} \frac{x}{x^4 - 1} dx.$$

using the residue theory.

**A5.** A point charge is located at the point x = 1, y = 0, and the y-axis is grounded (the potential is equal to zero). Find the potential in the right-half plane by constructing a fractional linear transformation (conformal mapping) from the right-half plane to a unit disk that maps point (1,0) into the disk center, point (0,0) into (-1,0), and points $(0, \pm 1)$ into $(0, \pm 1)$.

Recall that the solution of the Green's function problem for the Laplace equation in a unit disk with singularity in the center and zero boundary condition on the unit circle is

$$\frac{1}{2\pi} \log(r).$$

**B6.** The $k$th moment is defined as $\int x^k \, dx$.

a) How many quadrature points (a.k.a. nodes) do you need for a Newton-Cotes rule to integrate all $k$th moments exactly over an interval $[a, b]$ for $k = 0, 1, \ldots, n$? How many quadrature points do you need for a Gaussian quadrature rule?

b) Derive a Newton-Cotes rule that can integrate the $k$th moments $\int_a^b x^k \, dx$ over an interval $[a, b]$ exactly for $k$ up to 3. What is the degree of this rule?

**B7.** Let $y(t)$ be the solution of the scalar initial value problem $y' = f(t, y)$ with $y(0) = y_o$. Consider the following Runge-Kutta method

$$
\begin{aligned}
k_1 &= f(t_n, y_n), \\
k_2 &= f(t_n + h, y_n + hk_1), \\
y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2).
\end{aligned}
$$

Analyze its local truncation error provided that $f$ has continuous second derivatives. What is the order of accuracy of this method?

**B8.** Consider the nonlinear conservation law

$$u_t + (u^4)_x = 0.$$

Write the following schemes for the equation in the form of $u_j^{n+1} = F(u_{j-1}^n, u_j^n, u_{j+1}^n)$.

a) Lax-Friedrich scheme.

b) Lax-Wendroff scheme.

c) Godunov scheme.

**B9.** A generalized numerical scheme for the parabolic equation $u_t = \nu u_{xx}$ has the following form

$$(1 - \alpha r\delta^2)u_j^{n+1} = (1 + \beta r\delta^2)u_j^n,$$

where $r = \nu\frac{\Delta t}{\Delta x^2}$ and $\delta^2$ is the second order central difference operator.

a) What is the condition $\alpha$ and $\beta$ must satisfy to make the finite difference scheme consistent with the PDE?

b) For what values of $\alpha$ and $\beta$ the numerical scheme has the highest order?

c) Analyze the numerical stability of the scheme for $\alpha = 0, 0.5, 1$ respectively.

d) When $\alpha = 0.5$, write the finite difference scheme in the form of system of linear equations $Au^{n+1} = b$, where $u^{n+1} = \{u_1^{n+1}, u_2^{n+1}, \cdots, u_N^{n+1}\}$. Write down the coefficient matrix $A$ and the $RHS$ vector $b$.

e) What is the most efficient way to solve this system of equations? Elaborate.

**B10.** A generalized numerical scheme for the linear hyperbolic equation $u_t + au_x = 0$ has the following form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \chi\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0.$$

a) Find the values of $\chi$ so that the above scheme is the central-explicit, forward, backward, Lax-Friedrich and Lax-Wendroff scheme.

b) What is the order of truncation error for each scheme?

c) What is the stability condition for each of these schemes?

d) In what range of $\chi$ the scheme is unconditionally unstable?

e) Classify the schemes in (a) as (i) dissipative, and (ii) dispersive.