Homework 4 Foundations of Computational Math 2 Spring 2012

Problem 4.1

Suppose we want to approximate a function$f(x)$ on the interval $[a, b]$ with a piecewise quadratic interpolating polynomial with a constant spacing, h, of the interpolation points $a = x_0 < x_1 \ldots < x_n = b$. That is, for any $a \leq x \leq b$, the value of $f(x)$ is approximated by evaluating the quadratic polynomial that interpolates f at x_{i-1}, x_i, and x_{i+1} for some i with $x = x_i + sh$, $x_{i-1} = x_i - h$, $x_{i+1} = x_i + h$ and $-1 \leq s \leq 1$. (How i is chosen given a particular value of x is not important for this problem. All that is needed is the condition $x_{i-1} \leq x \leq x_{i+1}$.)

Suppose we want to guarantee that the **relative error** of the approximation is less than 10^{-d}, i.e., d digits of accuracy. Specifically,

$$\frac{|f(x) - p(x)|}{|f(x)|} \leq 10^{-d}.$$

(It is assumed that $|f(x)|$ is sufficiently far from 0 on the interval $[a, b]$ for relative accuracy to be a useful value.) Derive a bound on h that guarantees the desired accuracy and apply it to interpolating $f(x) = e^x \sin x$ on the interval $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ with relative accuracy of 10^{-4}. (The sin is bounded away from 0 on this interval.)

Solution: On each interval we have

$$\begin{aligned} f(x) - p(x) &= \frac{(x - x_{i-1})(x - x_i)(x - x_{i+1})}{6} f'''(\xi) \\ &= h^3 \frac{s(s+1)(s-1)}{6} f'''(\xi) = h^3 \frac{q(s)}{6} f'''(\xi) \end{aligned}$$

To bound the term with $q(s)$ we note that

$$q(s) = s^3 - s \rightarrow q'(s) = 3s^2 - 1$$
$$\text{extrema are} \quad s_{\pm} = \pm\sqrt{1/3}$$
$$\frac{q(s_{\pm})}{6} = \pm\frac{1}{9}\sqrt{\frac{1}{3}} \approx 0.06416 < \frac{2}{3}\frac{1}{10} \approx 0.067$$

We must bound $|f'''(x)|$ from above and $|f(x)|$ from below to set an upper bound on h

that guarantees the desired accuracy. We have on $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

$$
\begin{gathered}
f'''(x) = 2e^x(\cos x - \sin x) \\
\frac{\pi}{4} \le x \le \frac{\pi}{2} \to |\cos x - \sin x| \le 1 \\
\frac{\pi}{2} \le x \le \frac{3\pi}{4} \to |\cos x - \sin x| \le 2 \\
e^x \le e^{3\pi/4} < 10.6 \\
\therefore |f'''(x)| < 43 \\
|f(x)| \ge f(\pi/4) = \frac{e^{\pi/4}}{\sqrt{2}} \ge 1.6
\end{gathered}
$$

Therefore we can put all of these bounds together to get

$$
\begin{gathered}
h^3 \frac{|q(s)|}{6} \frac{|f'''(x)|}{|f(x)|} < \frac{0.067 \times 43}{1.6} h^3 \le 2h^3 \le 10^{-4} \\
h \le \sqrt[3]{0.00005} \approx 0.04
\end{gathered}
$$

Given that $\pi/4 \approx 0.8$ and $3\pi/4 \approx 2.35$ $h = 0.04$ implies $n \approx 38$.

Problem 4.2

4.2.a

(i) Find the cubic polynomial $p_3(x)$ that interpolates a function $f(x)$ at the values:

$$
\begin{aligned}
f(0) = 0, \quad f'(0) = 1 \\
f(1) = 3, \quad f'(1) = 6
\end{aligned}
$$

(ii) Find the quartic polynomial $p_4(x)$ that interpolates a function $f(x)$ at the values:

$$
\begin{gathered}
f(0) = 0, \quad f'(0) = 0 \\
f(1) = 1, \quad f'(1) = 1 \\
f(2) = 1
\end{gathered}
$$

Solutions:

Consider the data:

$$
\begin{aligned}
f(0) = 0, \quad f'(0) = 1 \\
f(1) = 3, \quad f'(1) = 6
\end{aligned}
$$

We can use the Newton form to derive $p_3(x)$ via the table

$$\begin{array}{ccccccc}
x_0 & & x_0 & & x_1 & & x_1 \\
f_0 & & f_0 & & f_1 & & f_1 \\
& f_0' & & f[x_0,x_1] & & f_1' & \\
& & f[x_0,x_0,x_1] & & f[x_0,x_1,x_1] & & \\
& & & f[x_0,x_0,x_1,x_1] & & &
\end{array}$$

The values are

$$\begin{array}{ccccccc}
0 & & 0 & & 1 & & 1 \\
0 & & 0 & & 3 & & 3 \\
& 1 & & 3 & & 6 & \\
& & 2 & & 3 & & \\
& & & 1 & & &
\end{array}$$

Using the left side of the table yields

$$p_3(x) = 0 + x + 2x^2 + x^2(x-1) = x + 2x^2 + x^2(x-1) = x^3 + x^2 + x$$

We could also use the Hermite form:

$$\begin{aligned}
h &= x_1 - x_0 \\
p_3(x) &= \psi_0(x) f_0 + \psi_1(x) f_1 + \Psi_0(x) f_0' + \Psi_1(x) f_1' \\
\psi_0(x) &= \frac{(x-x_1)^2}{h^2}\left[1 + \frac{2}{h^2}(x-x_0)\right] \\
\psi_1(x) &= \frac{(x-x_0)^2}{h^2}\left[1 - \frac{2}{h^2}(x-x_1)\right] \\
\Psi_0(x) &= \frac{(x-x_1)^2}{h^2}(x-x_0) \\
\Psi_1(x) &= \frac{(x-x_0)^2}{h^2}(x-x_1)
\end{aligned}$$

Inserting the data yields:

$$\begin{gathered}
h = 1 \\
\psi_0(x) = (x-1)^2\,[1+2x] \\
\psi_1(x) = x^2\,[1-2(x-1)] \\
\Psi_0(x) = (x-1)^2 x \\
\Psi_1(x) = x^2(x-1) \\
p_3(x) = \psi_0(x) f_0 + \psi_1(x) f_1 + \Psi_0(x) f_0' + \Psi_1(x) f_1' \\
= 0 * (x-1)^2(1+2x) + 3x^2(1-2(x-1)) + (x-1)^2 x + 6x^2(x-1) \\
= 3x^2(3-2x) + (x-1)^2 x + 6x^2(x-1) \\
= x^3 + x^2 + x
\end{gathered}$$

Finally, we could solve for the monomial form directly. We have

$$
\begin{gathered}
p_3(x) = \alpha_0 + \alpha - 1x + \alpha_2 x^2 + \alpha_3 x^3 \\
p_3'(x) = \alpha - 1 + 2\alpha_2 x + 3\alpha_3 x^2 \\
p_3(0) = 0 = \alpha_0 \\
p_3'(0) = 1 = \alpha_1 \\
p3_1(1) = 3 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
p_3'(1) = 6 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \\
\Downarrow \\
\alpha_2 = \alpha_3 = 1 \\
p_3(x) = x + x^2 + x^3
\end{gathered}
$$

Now consider the data

$$
\begin{gathered}
f(0) = 0, \quad f'(0) = 0 \\
f(1) = 1, \quad f'(1) = 1 \\
f(2) = 1
\end{gathered}
$$

We can use the Newton form to derive $p_3(x)$ via the table

$$
\begin{array}{ccccccccc}
x_0 & & x_0 & & x_1 & & x_1 & & x_2 \\
f_0 & & f_0 & & f_1 & & f_1 & & f_2 \\
& f_0' & & f[x_0, x_1] & & f_1' & & f[x_1, x_2] & \\
& & f[x_0, x_0, x_1] & & f[x_0, x_1, x_1] & & f[x_1, x_1, x_2] & & \\
& & & f[x_0, x_0, x_1, x_1] & & f[x_0, x_1, x_1, x_2] & & & \\
& & & & f[x_0, x_0, x_1, x_1, x_2] & & & &
\end{array}
$$

The values are

$$
\begin{array}{ccccccccc}
0 & & 0 & & 1 & & 1 & & 2 \\
0 & & 0 & & 1 & & 1 & & 1 \\
& 0 & & 1 & & 1 & & 0 & \\
& & 1 & & 0 & & -1 & & \\
& & & -1 & & -1/2 & & & \\
& & & & 1/4 & & & &
\end{array}
$$

We then have

$$
\begin{aligned}
p_4(x) &= f_0 + (x - x_0)f[x_0, x_0] + (x - x_0)^2 f[x_0, x_0, x_1] \\
&+(x - x_0)^2(x - x_1)f[x_0, x_0, x_1, x_1] + (x - x_0)^2(x - x_1)^2 f[x_0, x_0, x_1, x_1, x_2] \\
&= x^2 - x^2(x - 1) + \frac{1}{4}x^2(x - 1)^2 \\
&= \frac{1}{4}x^4 - \frac{3}{2}x^3 + \frac{9}{4}x^2 \\
p_4'(x) &= x^3 - \frac{9}{2}x^2 + \frac{9}{2}x
\end{aligned}
$$

The interpolation conditions are easily verified.

4.2.b

Consider the following data

$$
(x_0, f_0) = (1, 0), \quad (x_1, f_1) = (2, 2), \\
(x_2, f_2) = (4, 12), \quad (x_3, f_3) = (5, 21)
$$

i. Estimate $f(3)$ using quadratic interpolation and points $(x_0, f_0), \quad (x_1, f_1), \quad (x_2, f_2)$.

ii. Estimate $f(3)$ using quadratic interpolation and points $(x_1, f_1), \quad (x_2, f_2), \quad (x_3, f_3)$.

iii. Estimate $f(3)$ using cubic interpolation.

iv. Write the piecewise linear interpolant $g_1(x)$ that uses all of the data points in cardinal basis form and estimate $f(3)$. Verify that your cardinal basis form satisfies the interpolation constraints.

Solution:

We can use the Newton form to derive $p_3(x)$ via the table

$$
\begin{array}{ccccccc}
x_0 & & x_1 & & x_2 & & x_3 \\
f_0 & & f_1 & & f_2 & & f_3 \\
 & f[x_0, x_1] & & f[x_1, x_2] & & f[x_2, x_3] & \\
 & & f[x_0, x_1, x_2] & & f[x_1, x_2, x_3] & & \\
 & & & f[x_0, x_1, x_2, x_3] & & &
\end{array}
$$

Inserting the values yields

$$
\begin{array}{ccccccc}
1 & & 2 & & 4 & & 5 \\
0 & & 2 & & 12 & & 21 \\
 & 2 & & 5 & & 9 & \\
 & & 1 & & \frac{4}{3} & & \\
 & & & \frac{1}{12} & & &
\end{array}
$$

We therefore have

$$
\begin{aligned}
p_2(x) &= f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
&= x^2 - x \\
p_2(1) &= 0, \;\; p_2(2) = 2, \;\; p_2(4) = 12 \\
\tilde{p}_2(x) &= f_1 + (x - x_1)f[x_1, x_2] + (x - x_1)(x - x_2)f[x_1, x_2, x_3] \\
&= \frac{4}{3}x^2 - 3x + \frac{8}{3} \\
&= \frac{1}{3}(4x^2 - 9x + 8) \\
\tilde{p}_2(2) &= 2, \;\; \tilde{p}_2(4) = 12, \;\; \tilde{p}_2(5) = 21 \\
p_3(x) &= p_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
&= x^2 - x + \frac{1}{12}(x - 1)(x - 2)(x - 4) \\
&= \frac{1}{12}(x^3 + 5x^2 + 2x - 8)
\end{aligned}
$$

Evaluating at $x = 3$ yields

$$
p_2(3) = 6, \quad \tilde{p}_3(3) = \frac{17}{3} = 5.666\ldots, \quad p_3(3) = \frac{70}{12} = 5.8333\ldots
$$

Note that if we use the difference table to estimate the errors $|p_2(3) - f(3)|$ and $|\tilde{p}_2(3) - f(3)|$ we exploit the error form

$$
\begin{aligned}
|f(3) - p_2(3)| &= |\frac{f^{(3)}(\xi)}{3!}(3 - 1)(3 - 2)(3 - 4)| \\
&= 2|\frac{f^{(3)}(\xi)}{3!}| \\
|f(3) - \tilde{p}_2(3)| &= |\frac{f^{(3)}(\eta)}{3!}(3 - 2)(3 - 4)(3 - 5)| \\
&= 2|\frac{f^{(3)}(\eta)}{3!}|
\end{aligned}
$$

We use the highest order divided difference to estimate the error

$$
\begin{aligned}
f[x_0, x_1, x_2, x] &= \frac{f^{(3)}(\xi)}{3!} \\
f[x_1, x_2, x_3, x] &= \frac{f^{(3)}(\eta)}{3!} \\
f[x_0, x_1, x_2, x_3] &= \frac{1}{12} \approx \frac{f^{(3)}(x)}{3!}
\end{aligned}
$$

So both errors will have the same estimate since ξ and η are unknown (but in the same interval). We have

$$|f(3) - p_2(3)| \approx |f(3) - \tilde{p}_2(3)| \approx \frac{2}{12} = 0.166666\ldots$$

Therefore the intervals associated with the two quadratic polynomials are

$$\begin{aligned} p_2(x) \pm 0.166666\ldots &\rightarrow [5.84, 6.17] \\ \tilde{p}_2(x) \pm 0.166666\ldots &\rightarrow [5.4, 5.84] \end{aligned}$$

Putting the two intervals together and expressing it as mean and width yields the expectations

$$\begin{aligned} f(3) &\in [5.4, 6.17] \\ f(3) &= 5.8333 \pm 0.17 \end{aligned}$$

The value of $p_3(3) = 5.833$ is consistent with these expectations.

The piecewise linear interpolant, $g_1(x)$ can be written,

$$g_1(x) = \begin{cases} \frac{(x-x_1)}{(x_0-x_1)} f_0 + \frac{(x-x_0)}{(x_1-x_0)} f_1 & \text{if } x_0 \le x \le x_1 \\ \frac{(x-x_2)}{(x_1-x_2)} f_1 + \frac{(x-x_1)}{(x_2-x_1)} f_2 & \text{if } x_1 \le x \le x_2 \\ \frac{(x-x_3)}{(x_2-x_3)} f_2 + \frac{(x-x_2)}{(x_3-x_2)} f_3 & \text{if } x_2 \le x \le x_3 \end{cases}$$

Inserting the x_i data yields

$$g_1(x) = \begin{cases} -(x-2) f_0 + (x-1) f_1 & \text{if } 1 \le x \le 2 \\ -\frac{(x-4)}{(2)} f_1 + \frac{(x-2)}{(2)} f_2 & \text{if } 2 \le x \le 4 \\ -(x-5) f_2 + (x-4) f_3 & \text{if } 4 \le x \le 5 \end{cases}$$

Inserting the f_i data yields

$$g_1(x) = \begin{cases} 2(x-1) & \text{if } 1 \le x \le 2 \\ -(x-4) + 6(x-2) & \text{if } 2 \le x \le 4 \\ -12(x-5) + 21(x-4) & \text{if } 4 \le x \le 5 \end{cases}$$

The interpolation conditions are easily verified in this form.

To generate the cardinal basis we start with

$$g_1(x) = \begin{cases} -(x-2) f_0 + (x-1) f_1 & \text{if } 1 \le x \le 2 \\ -\frac{(x-4)}{(2)} f_1 + \frac{(x-2)}{(2)} f_2 & \text{if } 2 \le x \le 4 \\ -(x-5) f_2 + (x-4) f_3 & \text{if } 4 \le x \le 5 \end{cases}$$

We want

$$g_1(x) = f_0\phi_0(x) + f_1\phi_1(x) + f_2\phi_2(x) + f_3\phi_3(x)$$

Isolating each f_i yields the following definitions:

$$\phi_0(x) = \begin{cases} -(x-2) & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_1(x) = \begin{cases} (x-1) & \text{if } 1 \le x \le 2 \\ -\frac{(x-4)}{(2)} & \text{if } 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \begin{cases} \frac{(x-2)}{(2)} & \text{if } 2 \le x \le 4 \\ -(x-5) & \text{if } 4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_3(x) = \begin{cases} (x-4) & \text{if } 4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

It is easily verified that for $0 \le i \le 3$

$$\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

as desired to enforce the interpolation conditions.

Evaluating $g_1(x)$ at $x = 3$ yields

$$g_1(3) = 7$$

Given the error estimates above we see that for linear interpolation more points are required to get similar accuracy.

Problem 4.3

Consider the function $f(x)$ in table form:

x	$f(x)$
0	0
1	1
2	8
3	27
4	64

Suppose you want to estimate a solution to the equation $f(x) = c$. One way is to interpolate $f(x)$ with a polynomial or some other interpolation function and then solve $p_n(x) = c$. This requires finding a root of a polynomial, i.e. solving a nonlinear equation.

Derive a technique that uses polynomial interpolation to get an estimate of the solution to the equation $f(x) = 2$ but does not require finding the roots of a polynomial. Discuss the accuracy of your solution and how it might be improved if not acceptable.

Solution: Note that the $(x, y) = (x, f(x))$ pairs are such that we can view y as the independent variable and x as the dependent variable, i.e., $x = f^{-1}(y)$ and produce a polynomial interpolant $g(y)$. The solution to the equation $f(x) = 2$ can then be approximated $x_* = g(2)$.

It is easily seen that $(x, f(x))$ are points on the cubic $f(x) = x^3$. So in order for this approach to work well, $f^{-1}(y) = \sqrt[3]{y}$ must be approximated well over the interval of interest by the quartic polynomial $g(y)$.

We have

$$(y_0, g_0) = (0, 0), \quad (y_1, g_1) = (1, 1), \quad (y_2, g_2) = (8, 2), \quad (y_3, g_3) = (27, 3), \quad (y_4, g_4) = (64, 4)$$

The divided differences for this data are:

$f[-,-]$	1.0	0.142857	0.052632	0.027027
$f[-,-,-]$		−0.107143	−0.003470	−0.000457
$f[-,-,-]$			0.003840	0.000048
$f[-,-,-,-]$				−0.000059

We have $g(2) = 1.721863$ which is not a good approximation to $\sqrt[3]{2} \approx 1.2599$. The function $f^{-1}(y) = \sqrt[3]{y}$ is not approximated well by a polynomial. For example, evaluate $g(343)$ (which should approximate $\sqrt[3]{343} = 7$) to see how disastrous it can be!

This technique is usually called inverse interpolation. Its advantage is that it only requires evaluation of the interpolating polynomial. So, assuming the construction of the polynomial is cheaper than finding the appropriate root of the polynomial that interpolates with x as the independent variable, it can be a viable technique.

There are various fixes. If one is restricted to interpolation polynomials on function data then piecewise interpolation is a good approach (however, more data may be needed for this problem). Also note that if the $f(x)$ data contains repeated values then looking for the inverse function does not make sense and some modification of the technique must be made.