Set 7: Splines – Part 2

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Splines, Error, and Extrema

The results concerning splines relative to the function $f(x)$ interpolated and to other splines address three main properties:

- energy in a function

$$\int_a^b (g(x))^2\, dx = \|g\|_2^2$$

- curvature and total curvature of a function

$$\kappa(g) = \frac{g''(x)}{(1 + (g(x)')^2)^{3/2}}$$

$$\int_a^b (\kappa(x))^2\, dx$$

- approximation error

$$\|f - s\|_\infty$$

Extremal Property

Theorem 7.1 (Ueberhuber)**.** *Given $f(x)$, let $w(x) \in \mathcal{C}^2[a,b]$ be any interpolating function, i.e., $w(x_i) = f(x_i)$, satisfying any one of the sets of boundary conditions:*

1. $w''(a) = w''(b) = 0$
2. $w'(a) = f'(a)$ *and* $w'(b) = f'(b)$
3. $w(a) = w(b)$,$w'(a) = w'(b)$ *and* $w''(a) = w''(b)$

If $s(x)$ is the interpolating cubic spline of $f(x)$ satisfying the same set of boundary conditions then

$$\|s''\|_2^2 \leq \|w''\|_2^2$$

i.e., the integral is minimal when $w(x) = s(x)$ over all appropriate interpolating $w(x)$.

Curvature

If $|w'(x)| \ll 1$ or if $(w'(x))^2$ is almost constant then

$$\int_a^b (\kappa(x))^2\, dx \approx \int_a^b (w''(x))^2\, dx = \|w''(x)\|_2^2$$

So the spline has minimal curvature.

It may be the case, that the assumptions on w' are not valid. In such a case a cubic interpolating spline **may** have undesirable oscillation. Techniques are available to address this, see, e.g., Ueberhuber.

Boundary Conditions

- Natural boundary condition

$$s''(a) = s''(b) = 0$$

- Periodic boundary condition – assumes $f(a) = f(b)$

$$s''(a) = s''(b) \text{ and } s'(a) = s'(b)$$

- Hermite boundary conditions

$$s' = f'(a) \text{ and } s'(b) = f'(b) \quad (1)$$

$$s''(a) = f''(a) \text{ and } s''(b) = f''(b) \quad (2)$$

Boundary Conditions

- Hermite boundary conditions (derivative-free form)
  - Define two cubic interpolation polynomials, $c_1(x)$ and $c_2(x)$, based on (x_0, x_1, x_2, x_3) and $(x_{n-3}, x_{n-2}, x_{n-1}, x_n)$.
  - Use the value of the first or second derivatives of $c_1(x)$ and $c_2(x)$

$$s' = c_1'(a) \text{ and } s'(b) = c_2'(b) \tag{3}$$

$$s''(a) = c_1''(a) \text{ and } s''(b) = c_2''(b) \tag{4}$$

- Not-a-knot boundary conditions

$$s_-'''(x_1) = s_+'''(x_1) \quad \text{and} \quad s_-'''(x_{n-1}) = s_+'''(x_{n-1}) \tag{5}$$

Error

Hall and Meyer (Jour. Approx. Theory V 16, 1976) show the following result:

Theorem 7.2. *If* $s(x)$ *be a cubic interpolating spline for* $f(x) \in \mathcal{C}^4[a,b]$ *and boundary conditions* (1) *or* (2) *then*

$$\|f^{(j)} - s^{(j)}\|_\infty \le C_j h^{4-j} \|f^{(4)}\|_\infty$$

for $j = 0, 1, 2, 3$ *where*

$$C_0 = 5/384, \;\; C_1 = 1/24, \;\; C_2 = 3/8, \;\; C_3 = (\beta + \beta^{-1})/2$$

where β *is the ratio of the largest interval size,* h*, to the smallest interval size.*

Error

- C_0 and C_1 are optimal over all nonzero $f \in \mathcal{C}^4$ and meshes with distinct points.
- So we have uniform convergence of s, s' and s'' to f, f' and f'' as $h \to 0$ and for s''' when β is uniformly bounded.
- Other degrees of smoothness for f and other boundary conditions have also been analyzed in the literature.

Error

Theorem 7.3. *(Beatson SIAM J. Num. Analysis, V 23, 1986, also see Ueberhuber) Let* $s(x)$ *be a cubic interpolating spline for* $f(x) \in \mathcal{C}^1[a,b]$ *at distinct nodes* x_i *for* $0 \le i \le n$ *with* $n \ge 5$ *for boundary condtions* (3), (5), *and* (4). *Let* $h = \max_i(x_i - x_{i-1})$. *It follows that on* $[x_{i-1},\ x_i]$

$$\|(s-f)^{(k)}\|_\infty \le \begin{cases} C_1 \lambda_1^{-1} h_1^{1-k} \omega(f';h) & i = 1 \\ C_1 h_i^{1-k} \omega(f';h) & 2 \le i \le n-1 \\ C_1 \mu_n^{-1} h_{n-1}^{1-k} \omega(f';h) & i = n \end{cases}$$

where $k = 0, 1$, h *is the largest interval size, and*

$$\omega(f^{(j)};h) = \max(|f^{(j)}(u) - f^{(j)}(v)| \ : \ \ u, v \in [a,b], \ \ |u-v| \le h)$$

is the modulus of continuity of $f^{(j)}$.

Error

Theorem 7.4. *(Beatson SIAM J. Num. Analysis, V 23, 1986, also see Ueberhuber) Let* $s(x)$ *be a cubic interpolating spline for* $f(x) \in \mathcal{C}^j[a,b]$ *at distinct nodes* x_i *for* $0 \le i \le n$ *with* $n \ge 5$ *for boundary condtions* (3), (5)*, and* (4)*. Let* $h = \max_i(x_i - x_{i-1})$*. It follows that on* $[x_{i-1},\ x_i]$

$$\|(s-f)^{(k)}\|_\infty \le C_2 h_i^{2-k} h^{j-2} \omega(f^{(j)};h) \;\; 1 \le i \le n$$

for $j = 2$ *or* $j = 3$ *and* $k = 0, 1, 2$*, where* h *is the largest interval size, and*

$$\omega(f^{(j)};h) = \max(|f^{(j)}(u) - f^{(j)}(v)| \;\;:\;\; u, v \in [a,b], \;\; |u-v| \le h)$$

is the modulus of continuity of $f^{(j)}$.

Summary of Uniform Convergence

Lemma 7.5. *For boundary conditions* (3), (4), (5) *and with* $f \in \mathcal{C}^j[a,\ b]$ *for* $j = 1, 2, 3$

$$\|s - f\|_\infty = O(h^{j+1})$$

For boundary conditions (1), (2), *and with* $f \in \mathcal{C}^4[a,\ b]$

$$\|s - f\|_\infty = O(h^4)$$

Derivation via Basis

- Given a partition π the linear space of cubic splines, $S_3(\pi)$, has dimension $n+3$.
- Find a basis for $S_3(\pi)$ consisting of $n+3$ linearly independent cubic splines.
- Determine interpoloatory spline by computing the $n+3$ coeffiecients of its expansion

  $$s(x) = \sum_{1}^{n+3} \alpha_i \rho_i(x)$$

  given $f_0, \ldots, f_n$.
- Many bases are possible, e.g., cardinal splines (see text)
- B-splines are often used.

B-splines

Assume we have uniformly space nodes $x_i = x_0 + i\frac{(b-a)}{n}$ $0 \leq i \leq n$.
Introduce 4 additional nodes, $x_{-2} < x_{-1} < x_0$, and $x_{n+2} > x_{n+1} > x_n$.

Definition 7.1. Let $h = (b-a)/n$. The function $B_i(t)$ defined by

$$B_i(x) = \frac{1}{h^3}(x - x_{i-2})^3, \text{ if } x_{i-2} \leq x \leq x_{i-1}$$

$$= \frac{1}{h^3}(h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3), \text{ if } x_{i-1} \leq x \leq x_i$$

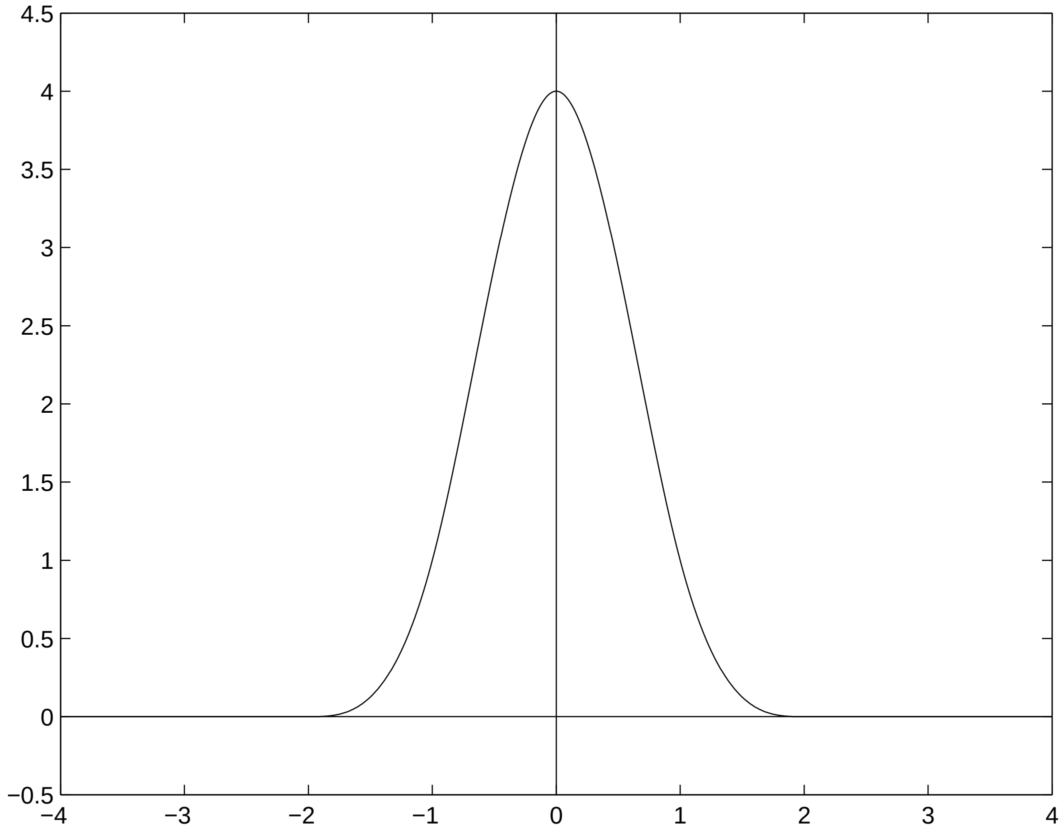
$$= \frac{1}{h^3}(h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3), \text{ if } x_i \leq x \leq x_{i+1}$$

$$= \frac{1}{h^3}(x_{i+2} - x)^3, \text{ if } x_{i+1} \leq x \leq x_{i+2}$$

$$= 0, \text{ otherwise}$$

is a cubic B-spline.

B-spline – Uniform Grid



B-splines

Lemma. *The B-spline $B_i(x)$ is twice continuously differentiable on $\mathbb{R}$, and is identically 0 when $x \geq x_{i+2}$ or $x \leq x_{i-2}$.*

The set $\mathcal{B} = \{B_{-1}, B_0, \ldots, B_{n+1}\}$ is a basis for $S_3(\pi)$.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	*0*	*1*	*4*	*1*	*0*
$B_i'(x)$	*0*	$\frac{3}{h}$	*0*	$-\frac{3}{h}$	*0*
$B_i''(x)$	*0*	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	*0*

Cubic Spline Interpolation via B-splines

Theorem 7.6. *Given,* f_0'*,* f_n'*,* f_i *and* $x_0 < x_1 < \cdots < x_n$ *for* $0 \le i \le n$ *the unique cubic spline,* $s(x)$*, such that*

$$s(x_i) = f_i \quad 0 \le i \le n, \quad s'(x_0) = f_0', \quad s'(x_n) = f_n'$$

is given by $s(x) = \sum_{i=-1}^{n+1} \alpha_i B_i(x)$ *where*

$$\begin{pmatrix} -\frac{3}{h} & 0 & \frac{3}{h} & 0 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & \cdots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{3}{h} & 0 & \frac{3}{h} \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix} = \begin{pmatrix} f_0' \\ f_0 \\ \vdots \\ f_n \\ f_n' \end{pmatrix}$$

Cubic Spline Interpolation via B-splines

Proof. The linear system is simply a statement of the conditions imposed on $s(x)$.

$$\alpha_{-1}B'_{-1}(x_0) + \alpha_0 B'_0(x_0) + \cdots + \alpha_{n+1}B'_{n+1}(x_0) = f'_0$$
$$\alpha_{-1}B_{-1}(x_0) + \alpha_0 B_0(x_0) + \cdots + \alpha_{n+1}B_{n+1}(x_0) = f_0$$
$$\vdots$$
$$\alpha_{-1}B_{-1}(x_n) + \alpha_0 B_0(x_n) + \cdots + \alpha_{n+1}B_{n+1}(x_n) = f_n$$
$$\alpha_{-1}B'_{-1}(x_n) + \alpha_0 B'_0(x_n) + \cdots + \alpha_{n+1}B'_{n+1}(x_n) = f'_n$$

The matrix is irreducibly diagonally dominant and nonsingular therefore the solution and spline is unique. $\square$

B-splines

For a uniform grid the idea behind a B-spline can be explained in terms of forward difference and a basic cubic spline.

Recall the forward difference $\Delta f_i = f_{i+1} - f_i$ and that $\Delta^d p(x_i) \equiv 0$ if $p(x)$ is a polynomial with degree $d - 1$ or less.

We seek a cubic spline with limited support, i.e., that is identically 0 outside an interval.

Since we intend to associate the splines with various points (the nodes) and evaluate them over all of $[x_0, x_n]$ the simplest spline should have two parameters x and t.

B-spline – Uniform Grid

Consider

$$F_t(x) = (x-t)_+^3 = \begin{cases} (x-t)^3, & t \le x \\ 0, & t > x \end{cases}$$

$F_t(x)$ is twice continuously differentiable with respect to x for a fixed t (and vice versa) but is not three times differentiable.

$F_t(x)$ is piecewise cubic with respect to t for a fixed x and vice versa.

Note. $F_t(x)$ has infinite support interval as a function of t with fixed x or vice versa. It also has an unbounded value asymptotically.

B-spline – Uniform Grid

- We would like local support for the basis functions $B_i(x)$.
- Since $F_t(x)$ is piecewise cubic w/r to x, we know Δ^4 should have intervals where it is identically 0.

Define $K(t)$ as below and consider its behavior w/r to x and t:

$$\begin{aligned} K(t) &= \Delta^4 F_t(x_0) \\ &= F_t(x_4) - 4F_t(x_3) + 6F_t(x_2) - 4F_t(x_1) + F_t(x_0) \\ &= (x_4 - t)_+^3 - 4(x_3 - t)_+^3 + 6(x_2 - t)_+^3 - 4(x_1 - t)_+^3 + (x_0 - t)_+^3 \end{aligned}$$

Behavior of $K(t)$

First, we identify the intervals where $K(t) \equiv 0$.

- $x_0 < x_1 < x_2 < x_3 < x_4 \leq t$
  - By definition, for a fixed x, $(x-t)_+^3 = 0$ for all $t \geq x$.
  - $\therefore (x_i - t)_+^3 = 0$ for $0 \leq i \leq 4$ and $t \geq x_4$
  - $\therefore K(t) \equiv 0, \quad t \geq x_4$.
- $t \leq x_0 < x_1 < x_2 < x_3 < x_4$
  - By definition, for a fixed t, $(x-t)_+^3 = (x-t)^3$ for $x \geq t$.
  - For $\forall t \leq x_0$, the $F_t(x_i)$ are values of a cubic polynomial in x.
  - $\therefore \Delta^4 F_t(x_0) = 0, \; t \leq x_0$
  - $\therefore K(t) \equiv 0, \quad t \leq x_0$.

Behavior of $K(t)$

Only nonzero in $x_0 \le t \le x_4$.

$$K(t) = (x_4 - t)_+^3 - 4(x_3 - t)_+^3 + 6(x_2 - t)_+^3 - 4(x_1 - t)_+^3 + (x_0 - t)_+^3$$

$$x_0 \le t \le x_1 \to K(t) = (x_4 - t)^3 - 4(x_3 - t)^3 + 6(x_2 - t)^3 - 4(x_1 - t)^3 + 0$$

$$x_1 < t \le x_2 \to K(t) = (x_4 - t)^3 - 4(x_3 - t)^3 + 6(x_2 - t)^3 - 0 + 0$$

$$x_2 < t \le x_3 \to K(t) = (x_4 - t)^3 - 4(x_3 - t)^3 + 0 - 0 + 0$$

$$x_3 < t \le x_4 \to K(t) = (x_4 - t)^3 - 0 + 0 - 0 + 0$$

Equivalence to B-splines

Let $x_i = x_0 + ih$ and $x_{i+j} = x_i + jh$ and $x_2 < t \leq x_3$. We have

$$\begin{aligned} K(t) &= (x_4 - t)^3 - 4(x_3 - t)^3 \\ &= (x_3 - t + h)^3 - 4(x_3 - t)^3 \\ &= h^3 + 3h^2(x_3 - t) + 3h(x_3 - t)^2 - 3(x_3 - t)^3 \\ &= h^3 B_2(t) \end{aligned}$$

The equality is easily shown on the other intervals defining $B_2(t)$.

Definition 7.2. The cubic B-spline is defined by:

$$B_i(t) = \frac{1}{h^3} \Delta^4 F_t(x_{i-2})$$

Generalization to Higher Degree

Lemma. *Let* $F_{t,m}(x) = (x-t)_+^m$. *If*

$$K_m(t) = \Delta^{m+1} F_{t,m}(x) = \sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i (x_{m+1-i} - t)_+^m$$

then

$$K(t) \equiv 0 \quad t \le x_0 \ \textit{and}\ t \ge x_{m+1}$$

$$K(t) \in \mathcal{C}^{m-1}$$

$$\therefore K(t) \in S_m(\pi)$$

- The text gives the definition of B-splines for nonuniform nodes in terms of divided differences
- Note there is scale of $6 = 3!$ applied to all $B_i(t)$ in the uniform node case in the text.

Example

Let $m = 1$. Define $F_t(x) = (x - t)_+, \quad K(t) = \Delta^2 F_t(x_0)$.

$$K(t) = (x_2 - t)_+ - 2(x_1 - t)_+ + (x_0 - t)_+$$

$$= \begin{cases} (x_2 - t) & x_1 \le t \le x_2 \\ (x_2 - t) - 2(x_1 - t) & x_0 \le t \le x_1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} h + (x_1 - t) & x_1 \le t \le x_2 \\ h - (x_1 - t) & x_0 \le t \le x_1 \\ 0 & \text{otherwise} \end{cases}$$

After scaling with h to get the B-spline, this is the hat function on $[x_0, x_2]$.