Homework 9 Foundations of Computational Math 1 Fall 2012

Problem 9.1

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix, $C \in \mathbb{R}^{n \times n}$ be a symmetric nonsingular matrix, and $b \in \mathbb{R}^n$ be a vector. The matrix $M = C^2$ is therefore symmetric positive definite. Also, let $\tilde{A} = C^{-1}AC^{-1}$ and $\tilde{b} = C^{-1}b$.

The preconditioned Steepest Descent algorithm to solve $Ax = b$ is:

A, M are symmetric positive definite
x_0 arbitrary; $r_0 = b - Ax_0$; solve $Mz_0 = r_0$

do $k = 0, 1, \ldots$ until convergence

$$
\begin{aligned}
w_k &= Az_k \\
\alpha_k &= \frac{z_k^T r_k}{r_k^T w_k} \\
x_{k+1} &\leftarrow x_k + z_k \alpha_k \\
r_{k+1} &\leftarrow r_k - w_k \alpha_k \\
&\text{solve } Mz_{k+1} = r_{k+1}
\end{aligned}
$$

end

The Steepest Descent algorithm to solve $\tilde{A}\tilde{x} = \tilde{b}$ is:

$\tilde{A}$ is symmetric positive definite
$\tilde{x}_0$ arbitrary; $\tilde{r}_0 = \tilde{b} - \tilde{A}\tilde{x}_0$; $\tilde{v}_0 = \tilde{A}\tilde{r}_0$

do $k = 0, 1, \ldots$ until convergence

$$
\begin{aligned}
\tilde{\alpha}_k &= \frac{\tilde{r}_k^T \tilde{r}_k}{\tilde{r}_k^T \tilde{v}_k} \\
\tilde{x}_{k+1} &\leftarrow \tilde{x}_k + \tilde{r}_k \tilde{\alpha}_k \\
\tilde{r}_{k+1} &\leftarrow \tilde{r}_k - \tilde{v}_k \tilde{\alpha}_k \\
\tilde{v}_{k+1} &\leftarrow \tilde{A}\tilde{r}_{k+1}
\end{aligned}
$$

end

Show that given the appropriate consistency between initial guesses the preconditioned steepest descent recurrences to solve $Ax = b$ can be derived from the steepest descent recurrences to solve $\tilde{A}\tilde{x} = \tilde{b}$.

Problem 9.2

Consider the generic Conjugate Direction algorithm for solving the minimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f(x) = x^T Ax - x^T b$, $b \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite.

Denote the A-orthogonal direction vectors $d_0, d_1, \ldots$ and let $r_k = b - Ax_k$. Show that

$$\frac{d_{k-1}^T r_0}{d_{k-1}^T A d_{k-1}} = \frac{d_{k-1}^T r_{k-1}}{d_{k-1}^T A d_{k-1}}$$

Problem 9.3

When solving $Ax = b$ or equivalently the associated quadratic definite minimization problem using CG, we have

$$x_{k+1} = x_0 + \alpha_0 p_0 + \cdots + \alpha_k p_k$$

where the p_j are $A-$orthogonal vectors. It can be shown that

$$span[p_0, \ldots, p_k] = span[r_0,\ Ar_0, \ldots, A^k r_0]$$

where $r_0 = b - Ax_0$ and x_0 is the initial guess at the solution $x^* = A^{-1}b$. Therefore,

$$x_{k+1} = x_0 + \gamma_0 r_0 + \gamma_1 Ar_0 + \cdots + \gamma_k A^k r_0 = x_0 - P_k(A) r_0$$

where $P_k(A) = \gamma_0 I + \gamma_1 A + \cdots + \gamma_k A^k$ is a matrix that is called a matrix polynomial of evaluated at A. (A space whose span can be defined by a matrix polynomial is called a Krylov space.)

Denote $d_j = A^j r_0$ for $j = 0,\ 1, \ldots$ and determine the relationship between the coefficients $\alpha_0, \ldots, \alpha_k$ and the coefficients $\gamma_0, \ldots, \gamma_k$.

Problem 9.4

Recall the basic CD/CG properties given that at step k it is assumed CG has not converged,

- $x_k = \alpha_0 d_0 + \cdots + \alpha_{k-1} d_{k-1}$ is optimal (inherited from CD), i.e.,

$$\forall x \in x_0 + span[d_0, d_1, \ldots, d_{k-1}],\ \ \|x_k - A^{-1}b\|_A \leq \|x - A^{-1}b\|_A$$

- $< d_k,\ d_j >_A = 0$ $i \neq j$ for $0 \leq i, j \leq k-1$ (inherited from CD).

- $< r_k,\ d_j >= 0$ for $0 \leq j \leq k-1$ (inherited from CD).

- $< r_k,\ r_j >= 0$ for $0 \leq j \leq k-1$ (CG-specific).
- $span[d_0, d_1, \ldots, d_k] = span[r_0, r_1, \ldots, r_k]$ (CG-specific).
- $span[r_0, r_1, \ldots, r_k] = span[r_0, Ar_0, \ldots, A^k r_0]$ (CG-specific).

Given the inherited properties prove the three CG-specific properties.