Homework 4 Foundations of Computational Math 2 Spring 2012

Solutions will be posted Monday, 2/10/12

Problem 4.1

Suppose we want to approximate a function$f(x)$ on the interval $[a, b]$ with a piecewise quadratic interpolating polynomial with a constant spacing, h, of the interpolation points $a = x_0 < x_1 \ldots < x_n = b$. That is, for any $a \leq x \leq b$, the value of $f(x)$ is approximated by evaluating the quadratic polynomial that interpolates f at x_{i-1}, x_i, and x_{i+1} for some i with $x = x_i + sh$, $x_{i-1} = x_i - h$, $x_{i+1} = x_i + h$ and $-1 \leq s \leq 1$. (How i is chosen given a particular value of x is not important for this problem. All that is needed is the condition $x_{i-1} \leq x \leq x_{i+1}$.)

Suppose we want to guarantee that the **relative error** of the approximation is less than 10^{-d}, i.e., d digits of accuracy. Specifically,

$$\frac{|f(x) - p(x)|}{|f(x)|} \leq 10^{-d}.$$

(It is assumed that $|f(x)|$ is sufficiently far from 0 on the interval $[a, b]$ for relative accuracy to be a useful value.) Derive a bound on h that guarantees the desired accuracy and apply it to interpolating $f(x) = e^x \sin x$ on the interval $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ with relative accuracy of 10^{-4}. (The sin is bounded away from 0 on this interval.)

Problem 4.2

4.2.a

(i) Find the cubic polynomial $p_3(x)$ that interpolates a function $f(x)$ at the values:

$$\begin{aligned} f(0) &= 0, \quad f'(0) = 1 \\ f(1) &= 3, \quad f'(1) = 6 \end{aligned}$$

(ii) Find the quartic polynomial $p_4(x)$ that interpolates a function $f(x)$ at the values:

$$\begin{gathered} f(0) = 0, \quad f'(0) = 0 \\ f(1) = 1, \quad f'(1) = 1 \\ f(2) = 1 \end{gathered}$$

4.2.b

Consider the following data

$$(x_0, f_0) = (1, 0), \quad (x_1, f_1) = (2, 2),$$
$$(x_2, f_2) = (4, 12), \quad (x_3, f_3) = (5, 21)$$

i. Determine the quadratic interpolating polynomial, $p_2(x)$, for points (x_0, f_0), (x_1, f_1), (x_2, f_2). Estimate $f(3)$ using $p_2(x)$.

ii. Determine the quadratic interpolating polynomial, $\tilde{p}_2(x)$, for points (x_1, f_1), (x_2, f_2), (x_3, f_3). Estimate $f(3)$ using $\tilde{p}_2(x)$.

iii. Estimate $f(3)$ using a cubic interpolating polynomial $p_3(x)$.

iv. Estimate the errors $|f(3) - p_2(x)|$ and $|f(3) - \tilde{p}_2(x)|$ an use the estimates to determine an interval in which you expect $f(3)$ to reside. How does the value of $p_3(3)$ relate to this interval?

v. Write the piecewise linear interpolant $g_1(x)$ that uses all of the data points in cardinal basis form and estimate $f(3)$. Verify that your cardinal basis form satisfies the interpolation constraints.

Problem 4.3

Consider the function $f(x)$ in table form:

x	$f(x)$
0	0
1	1
2	8
3	27
4	64

Suppose you want to estimate a solution to the equation $f(x) = c$. One way is to interpolate $f(x)$ with a polynomial or some other interpolation function and then solve $p_n(x) = c$. This requires finding a root of a polynomial, i.e. solving a nonlinear equation.

Derive a technique that uses polynomial interpolation to get an estimate of the solution to the equation $f(x) = 2$ but does not require finding the roots of a polynomial. Discuss the accuracy of your solution and how it might be improved if not acceptable.