Foundations of Computational Math I Exam 1
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators/Computers Allowed
No collaborations with anyone
Due beginning of Class Monday, December 5, 2011

Question	Points Possible	Points Awarded
1. Linear Systems	30	
2. Linear Systems	30	
3. Nonlinear Equations	40	
4. Nonlinear Equations	30	
Total Points	130	

Name:________________________________

Alias: ________________________________

to be used when posting anonymous grade list.

Problem 1

(30 points)
Consider the system of equations

$$Ax = b$$

where A is a nonsingular lower triangular matrix, i.e.,

$$A = D - L$$

where D is diagonal and nonsingular, and L is strictly lower triangular.

(1.a) Show that Forward Gauss-Seidel will converge to $x = A^{-1}b$ in a finite number of steps (in exact arithmetic) for any in initial guess x_0 and give a tight upper bound on the number of steps required.

(1.b) What is the complexity of solving the system using Forward Gauss-Seidel: $O(n)$ or $O(n^2)$ or $O(n^3)$ or more?

(1.c) Show that Backward Gauss-Seidel will converge to $x = A^{-1}b$ in a finite number of steps (in exact arithmetic) for any in initial guess x_0 and give a tight upper bound on the number of steps required or show that it does not converge in a finite number of steps for all x_0.

(1.d) What is the complexity of solving the system using Backward Gauss-Seidel: $O(n)$ or $O(n^2)$ or $O(n^3)$ or more?

(1.e) What is the relationship between Backward Gauss-Seidel and Jacobi iterations for solving this system?

Problem 2

(30 points)

2.a

(10 points)

(2.a.i) Suppose you must solve a linear system $Ax = b$ and all you know about A is that it is nonsingular and well-conditioned. Suppose further you only have a iterative method that is applicable to linear systems where the coefficient matrix is symmetric positive definite. How would you solve $Ax = b$?

(2.a.ii) When attempting to solve $Ax = b$ where A is known to be nonsingular via an iterative method, we have seen various theorems that give sufficient conditions on A to guarantee the convergence of various iterative methods. It is not always easy to verify these conditions for a given matrix A. Let P and Q be two permutation matrices. Rather than solving $Ax = b$ we could solve $(PAQ)(Q^T x) = Pb$ using an iterative method. Sometimes it is possible to examine A and choose P and/or Q so that it is easy to apply one of our sufficient condition theorems.

Can you choose P and/or Q so that the permuted system converges for one or both of Gauss-Seidel and Jacobi with

$$A = \begin{pmatrix} 3 & 7 & -1 \\ 7 & 4 & 1 \\ -1 & 1 & 2 \end{pmatrix}?$$

2.b

(10 points)

Let

$$T = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$

and consider solving a linear system $Tx = b$ with accelerated stationary Richardson's method

$$x_{k+1} = x_k + \alpha r_k$$

(2.b.i) Can α be set so that the iteration will converge for all x_0? Justify your answer.

(2.b.ii) If values of α exist for which the iteration conveges for all x_0 how would you set its value so the convergence rate was optimal or nearly so? Give specific values in your response. If no such α exists indicate what method you would use to solve the system.

2.c

(10 points)
Let

$$T = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & 0 & -1 & -4 \end{pmatrix}$$

and consider solving a linear system $Tx = b$ with accelerated stationary Richardson's method

$$x_{k+1} = x_k + \alpha r_k$$

(2.c.i) Can α be set so that the iteration will converge for all x_0? Justify your answer.

(2.c.ii) If values of α exist for which the iteration conveges for all x_0 how would you set its value so the convergence rate was optimal or nearly so? Give specific values in your response. If no such α exists indicate what method you would use to solve the system.

Problem 3

(40 points)

3.a

(30 points)

Let $f(x) = x + \ln x$ on $x > 0$ and define

$$\phi_1(x) = -\ln x$$

$$\phi_2(x) = e^{-x}$$

$$\phi_3(x) = \frac{\alpha x + e^{-x}}{\alpha + 1} \qquad \alpha \neq -1$$

(3.a.i) Which of iteration functions $\phi_1(x)$ and $\phi_2(x)$ would you recommend to solve $f(x) = 0$ on $x > 0$? Justify your answer.

(3.a.ii) Suppose you have, β, an approximation to the root x_*, i.e., $\beta \approx x_*$. How would you set the parameter α in $\phi_3(x)$ so that the resulting iteration was better than your choice between $\phi_1(x)$ and $\phi_2(x)$?

(3.a.iii) Show that the closer β is to the root x_* the faster the resulting convergence of $\phi_3(x)$ using the associated α.

(3.a.iv) Determine the iteration defined by Newton's method to solve $f(x) = 0$ and comment on its convergence compared to $\phi_3(x)$ using β and the associated α.

(3.a.v) Use Newton's method and one of the others to determine the root x^*. You need not turn in any code simply report the computed value of x^* and indicate something about the observed rate of convergence.

3.b

(10 points)
Discuss the main differences in convergence and complexity when solving a nonlinear equation using

(3.b.i) Newton's method

(3.b.ii) Secant method

(3.b.iii) Regula Falsi method

Problem 4

(30 points)

Suppose you are asked to design an algortihm that could be used as an intrinsic library function to compute $\sqrt[3]{\alpha}$ for any $\alpha \in \mathbb{R}$ and $\alpha > 0$.

(4.a) Give an efficient algorithm in the form of $x = \phi(x)$ that solves the problem.

(4.b) Determine the restrictions, if any, on your choice of x_0. Also indicate how you would generate x_0 given α.

(4.c) Determine the order of convergence for your method and the number of computations needed to get single precision accuracy given the initial accuracy for x_0.

(4.d) Use your algorithm to determine $\sqrt[3]{69}$. Give the value of x_0 used and describe the observed convergence.