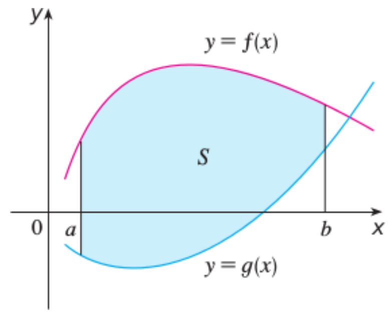
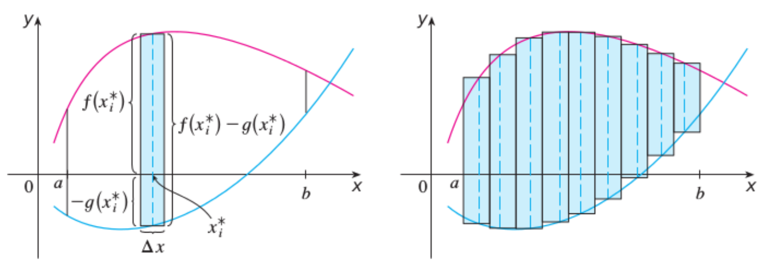
6.1: Areas Between Curves

In Chapter 5 we calculated the areas of regions that lie underneath the graphs of functions. Here we calculate the areas of regions that lie between the graphs of two functions. That is, consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines depicted below.



Just as we did for areas under curves, we divide the area up into n rectangles each of width $\Delta x = \frac{b-a}{n}$ on the subintervals $[x_k, x_{k+1}]$ where $x_k = a + k\Delta x$ for $k = 0, \ldots, n$, but now with heights $f(x_k^*) - g(x_k^*)$ where x_k^* is a sample point in $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$.



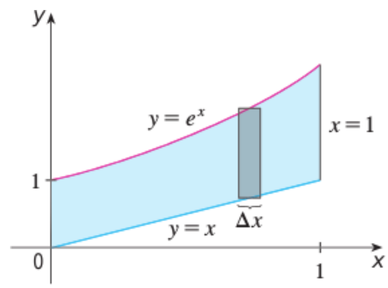
And just as before, we take the area A to be the limit of the Riemann sums as the number of rectangles increases to infinity

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x.$$

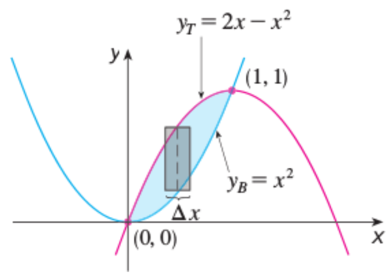
The area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$ is

$$\int_a^b [f(x) - g(x)]\, dx.$$

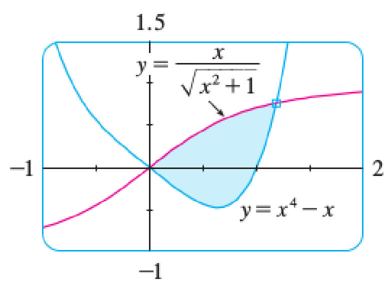
Example 1. *Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.*



Example 2. *Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$*

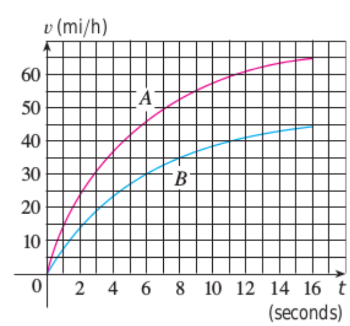


Example 3. *Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.*

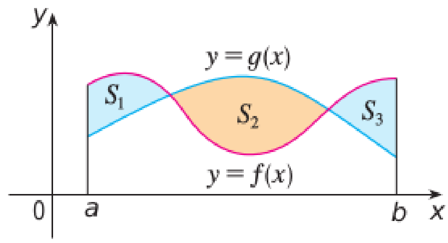


Example 4. *The table and figure below shows the velocities of two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the midpoint rule to estimate it.*

t	0	2	4	6	8	10	12	14	16
v_A	0	34	54	67	76	84	89	92	95
v_B	0	21	34	44	51	56	60	63	65
$v_A - v_B$	0	13	20	23	25	28	29	29	30



If we want to find the area A between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ on some intervals and $g(x) \geq f(x)$ on some intervals, then we split up the region S into several regions $s_1, S_2, \ldots$ with areas $a_1, a_2, \ldots$ as depicted below and sum the areas to obtain $A = A_1 + A_2 + \cdots$.



Since

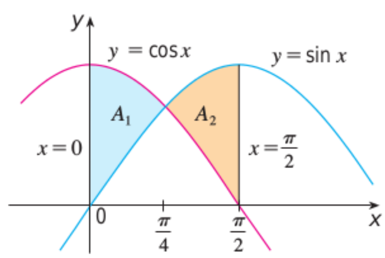
$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

we have the following expression for A.

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

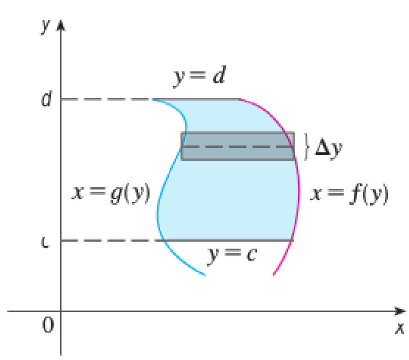
$$\int_a^b |f(x) - g(x)| \, dx.$$

Example 5. *Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.*



Some regions are best treated by regarding x as a function of y. The area of the region bounded by the curves $x = f(y)$ and $x = g(y)$ and the horizontal lines $y = c$ and $y = d$ where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ is

$$\int_c^d [f(y) - g(y)] \, dy.$$



Example 6. *Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.*