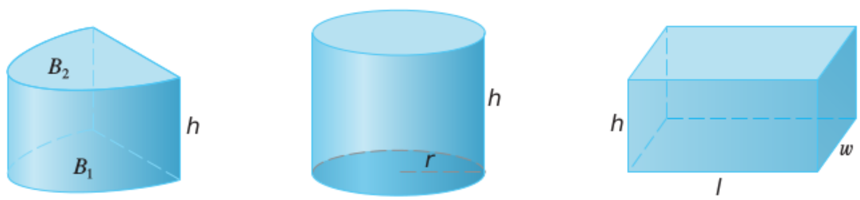
6.2: Volumes

In addition to calculating *areas*, integration is used to calculate *volumes.* Recall that we used a number of *rectangles* to approximate the *area* of a two-dimensional region. Similarly, we use a number of *cylinders* to approximate the *volume* of a three-dimensional region.

A **cylinder** is bounded by a plane region B_1, called the base, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and that join B_1 and B_2. If the area of the base is A, and the height (perpendicular distance from the plane containing B_1 to the plane containing B_2) is h, then the volume of the cylinder is

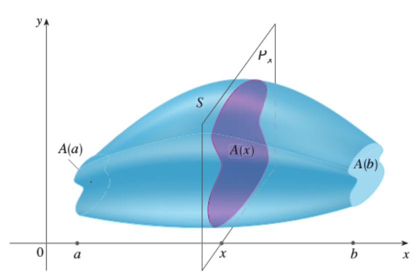
$$V = Ah.$$



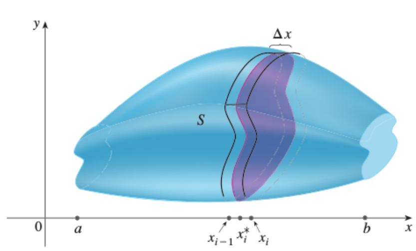
(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

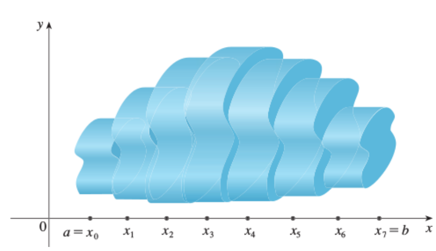
To calculate the volume V of a solid S that is not a cylinder, we "cut" S into pieces and approximate each piece by a cylinder. We estimate V by adding the volumes of the cylinders. We find the exact value of V by a limitng process in which the number of pieces becomes large.

First notice that by intersecting a solid S with a plane P_x perpendicular to the x-axis, we obatain a region called a *cross-section* of S. Let $A(x)$ denote the area of this cross-section for $a \leq x \leq b$, and notice that $A(x)$ varies as x varies between a and b.



To approximate the volume V of S, divide the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ each of width $\Delta x = \frac{b-a}{n}$ for $x_k = a + k\Delta x$ and $k = 0, \ldots, n$. Now approximate the volume on each subinterval $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$ with the volume of the cylinder with "height" Δx and base area $A(x_k^*)$ where x_k^* is a sample point in the subinterval $[x_{k-1}, x_k]$.





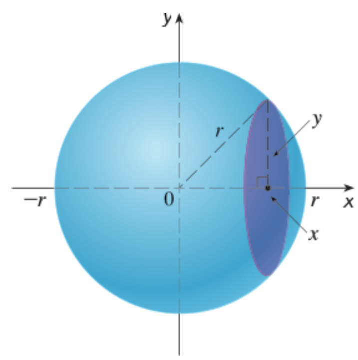
The sum volume of these cylinders is approximately the volume of S:

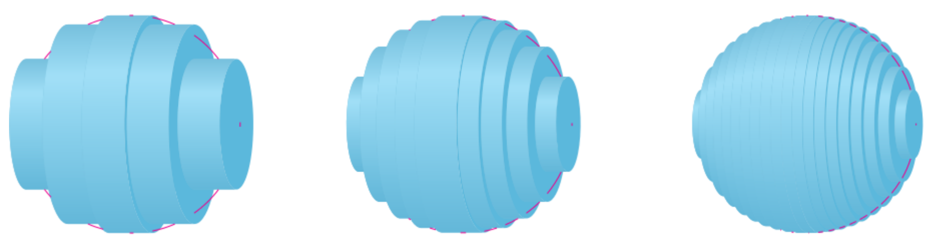
$$V \approx \sum_{k=1}^{n} A(x_k)\Delta x.$$

The exact volume of S is calculated by increasing the number of subintervals to infinity:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x = \int_a^b A(x)\ dx.$$

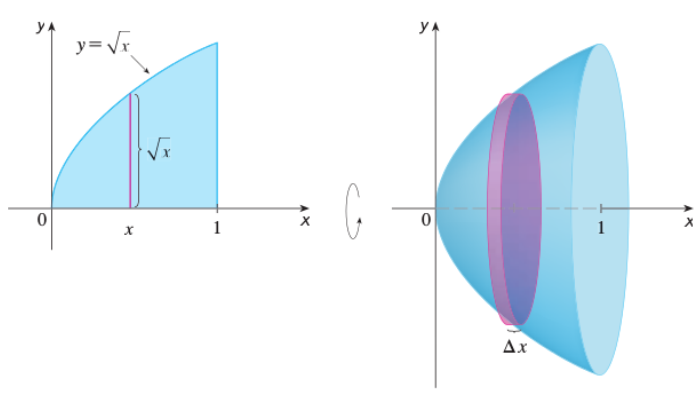
Example 1. *Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.*



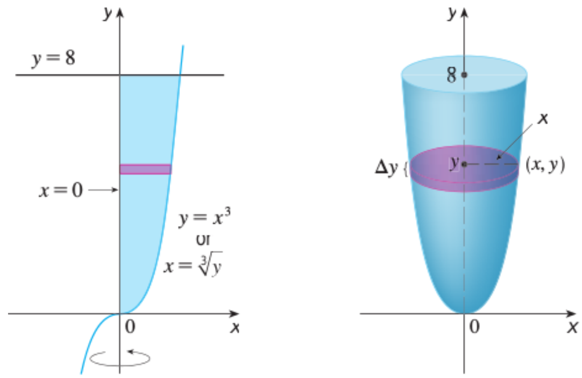


(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

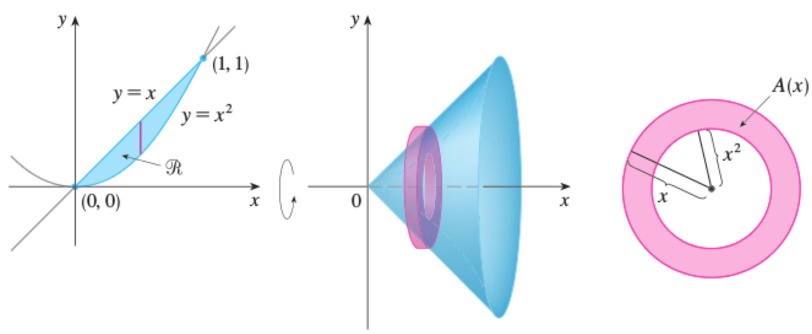
Example 2. *Find the volume of the solid obtained by rotating the region under the curve $y = \sqrt{x}$ from $x = 0$ to 1 about the x-axis.*



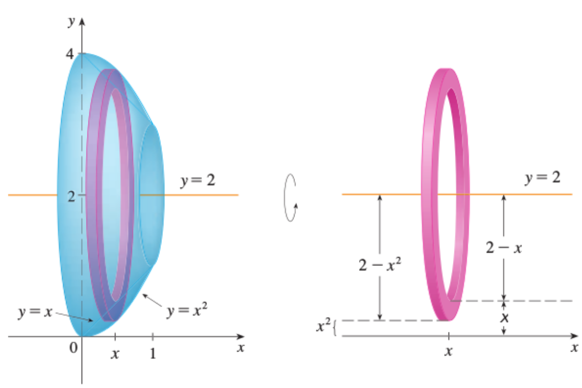
Example 3. *Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.*



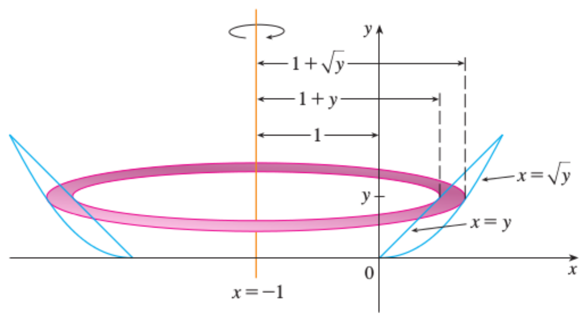
Example 4. *Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the x-axis.*



Example 5. *Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.*



Example 6. *Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the line $x = -1$.*



Example 7. *Find the volume of a pyramid whose base is a square with side length L and whose height is h.*