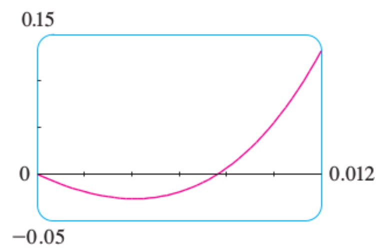
4.8: Newton's Method

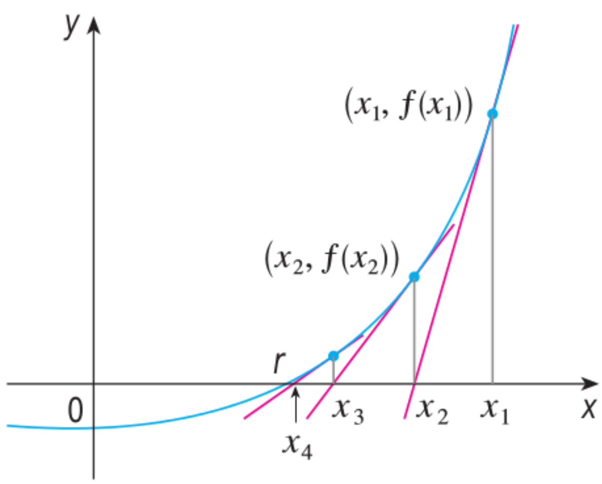
Suppose you need find the root of an equation $f(x) = 0$. If f is a polynomial and $n = 2, 3$, or 4, there are formulas (the "quadratic formula" for $n = 2$) that will give answers, but if $n \geq 5$, there is no such formula. When confronted with the latter scenario, we can use methods to approximate the solution. For example, consider solving

$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$$

whose lefthand side is graphed below.



From the graph we see there are at least two roots of the equation: one at $x = 0$ and one near $x = 0.007$. If you ask WolframAlpha or your calculator for the answer, you get the root near $x = 0.007$ correct to seven decimal places is $x \approx 0.0076286$. How do WolframAlpha or your calculator find this answer? They use a *numerical rootfinding* method such as **Newton's Method.** The geometry of Newton's Method is illustrated in the figure below.



The idea is that we iteratively find the roots of tangent lines to f. With this method we take a guess x_1 at the root r and refine it with each step of the method. Hence, we generate a sequence of approximations $x_1, x_2, x_3, \ldots$ to the root r. This sequence is given by the recursion

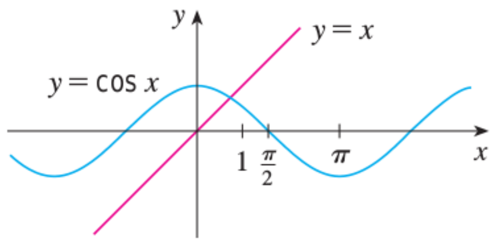
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{for } n \geq 1.$$

If $\lim\limits_{x\to\infty} x_n = r$, then the sequence *converges* to the root, and Newton's Method is successful. For some initial guesses, Newton's Method will not converge to the desired root.

Example 1. *Starting with* $x_1 = 2$, *find the third approximation* x_3 *to the root of the equation* $x^3 - 2x - 5 = 0$.

Example 2. *Use Newton's method to find* $\sqrt[6]{2}$ *correct to eight decimal places.*

Example 3. *Find the root of the equation* $\cos x = x$ *correct to six decimal places.*



Example 4. *Newton's Method does not always converge as this example illustrates. Let* $f(x) = x^{1/3}$ *(a) What is the root* r *of* $f(x) = 0$? *(b) Find Newton's Method recursion for* f. *(c) For which initial guesses* x_1 *does Newton's Method converge to* r?.