Solutions for Homework 7 Foundations of Computational Math 1 Fall 2012

Problem 7.1

7.1.a

Let $f(x) = x^3 - 3x + 1$. This polynomial has three distinct roots.

(i) Consider using the iteration function

$$\phi_1(x) = \frac{1}{3}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_1(x)$ and how would you choose $x^{(0)}$ for each computable root?

(ii) Consider using the iteration function

$$\phi_2(x) = \frac{3}{2}x - \frac{1}{6}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_2(x)$ and how would you choose $x^{(0)}$ for each computable root?

(iii) For each of the roots you identified as computable using either $\phi_1(x)$ or $\phi_2(x)$, apply the iteration to find the values of the roots. (You need not turn in any code, but using a simple program to do this is recommended.)

Solution: We can evaluate $f(x) = x^3 - 3x + 1$ to get a broad idea of where the roots are located if we find an appropriate sign pattern to the values of f.

We have

$$\begin{aligned} f(-2) &= -8 + 6 + 1 = -1 \\ f(-1) &= -1 + 3 + 1 = 3 \\ f(0) &= 1 \\ f(1) &= 1 - 3 + 1 = -1 \\ f(2) &= 8 - 6 + 1 = 3 \end{aligned}$$

So we have 3 distinct roots and the open intervals that contain them are

$$\begin{aligned} -2 < \alpha_0 < -1 \\ -1 < \alpha_1 < 1 \\ 1 < \alpha_2 < 2 \end{aligned}$$

Consider $\phi_1(x) = \frac{1}{3}(x^3 + 1)$. We have

$$\phi_1'(x) = x^2$$
$$|x| < 1 \to |\phi_1'(x)| < 1$$

and so $\phi_1(x)$ is a contraction mapping for $|x| < 1$ and converges to a unique fixed point. We must verify that this fixed point is a root of $f(x)$. We have

$$f(x) = x^3 - 3x + 1$$
$$f(x^*) = 0 \to 0 = (x^*)^3 - 3x^* + 1 \to 3x^* = (x^*)^3 + 1 \to x^* = \frac{1}{3}\left((x^*)^3 + 1\right)$$
$$\therefore \phi_1(x^*) = x^*$$

for any root, x^* of $f(x)$.

However, since $|x| > 1 \to |\phi_1'(x)| > 1$ we know that there must exist neighborhoods around α_0 and α_2 on which $\phi_1(x)$ does not converge to α_0 and α_2 respectively. We also know this because $\phi_1(x)$ cannot be a contraction mapping on an interval containing more than one fixed point.

To see that we can extend the interval of convergence beyond $-1 < x^{(0)} < 1$ note that

$$\phi_1(1) = \frac{2}{3} \quad \text{and} \quad \phi_1(-1) = 0$$

So after one application of ϕ_1 we are in the interval that we know converges to α_1 and therefore have

If $-1 \le x^{(0)} \le 1$ then $x^{(k+1)} = \phi_1(x^{(k)}) \to \alpha_1$

This interval can be extended on the rightside by noting that if $1 < x^{(0)} < \sqrt[3]{2}$ then $x^{(1)} < 1$ and therefore convergence to α_1 would follow. Similar extension could be considered on the left. The reasoning can be extended further by finding values of $x^{(0)}$ where $-1 \le x^{(2)} \le 1$ etc. These extensions on left and right are essentially trying to iterate to get the bound on the interval around α_1 on which $\phi_1(x)$ is a contraction mapping.

Now consider $\phi_2(x) = \frac{3}{2}x - \frac{1}{6}(x^3 + 1)$ We have

$$f(x^*) = 0 \to 0 = (x^*)^3 - 3x^* + 1 \to 0 = \frac{(x^*)^3}{6} - \frac{1}{2}x^* + \frac{1}{6}$$
$$-\frac{1}{2}x^* = -\frac{(x^*)^3}{6} - \frac{1}{6}$$
$$x^* - \frac{3}{2}x^* = -\frac{(x^*)^3}{6} - \frac{1}{6}$$
$$x^* = \frac{3}{2}x^* - \frac{1}{6}((x^*)^3 + 1)$$
$$\therefore \phi_2(x^*) = x^*$$

for any root, x^* of $f(x)$.

Analyzing the derivative to get a sufficient condition on convergence, we have

$$\phi_2'(x) = \frac{3}{2} - \frac{1}{2}x^2$$
$$1 < |x| < \sqrt{5} \rightarrow |\phi_2'(x)| < 1$$

and so $\phi_2(x)$ is a contraction mapping for $-2 < x^{(0)} < -1$ on which $x^{(k)} \rightarrow \alpha_0$ and for $1 < x^{(0)} < 2$ on which $x^{(k)} \rightarrow \alpha_2$.

We have the following results for varying $x^{(0)}$ for $\phi_1(x)$ and $\phi_2(x)$ to support these conclusions.

$\phi_1(x) = \frac{1}{3}(x^3 + 1).\phi_1(x)$

k	$x^{(k)}$	$f(x^{(k)})$
0	0.5000000000000	-0.3750000000000
1	0.3750000000000	-0.0722656250000
2	0.3509114583333	-0.0095235410112
3	0.3477369446629	-0.0011621409248
4	0.3473495643547	-0.0001403707329
5	0.3473027741104	-0.0000169336891
6	0.3472971295473	-0.0000020424951
7	0.3472964487156	-0.0000002463557

$\phi_1(x) = \frac{1}{3}(x^3 + 1)$

k	$x^{(k)}$	$f(x^{(k)})$
0	0.5000000000000	2.3750000000000
1	0.2916666666667	0.1498119212963
2	0.3416039737654	0.0150509651186
3	0.3466209621383	0.0017822673811
4	0.3472150512653	0.0002144996805
5	0.3472865511588	0.0000258650406
6	0.3472951728390	0.0000031196069
7	0.3472962127080	0.0000003762692

$\phi_1(x) = \frac{1}{3}(x^3 + 1)$

k	$x^{(k)}$	$f(x^{(k)})$
0	1.5000000000000	-0.1250000000000
1	1.4583333333333	-0.2735098379630
2	1.3671633873457	-0.5460762291479
3	1.1851379776297	-0.8908259852958
4	0.8881959825311	-0.9638971507168
5	0.5668969322922	-0.5185059212110
6	0.3940616252219	-0.1209931878177
7	0.3537305626160	-0.0169310412156
8	0.3480868822108	-0.0020848814012
9	0.3473919217437	-0.0002521095548
10	0.3473078852254	-0.0000304175109
11	0.3472977460551	-0.0000036689372
12	0.3472965230760	-0.0000004425300

So we see that $\alpha_1 \approx 0.347296$.

$\phi_2(x) = \frac{3}{2}x - \frac{1}{6}(x^3+1)$

k	$x^{(k)}$	$f(x^{(k)})$
0	1.5000000000000	-0.1250000000000
1	1.5208333333333	-0.0449128327546
2	1.5283188054591	-0.0151729727862
3	1.5308476342568	-0.0050099232254
4	1.5316826214610	-0.0016413186081
5	1.5319561745624	-0.0005363249216
6	1.5320455620493	-0.0001751029374
7	1.5320747458722	-0.0000571528707
8	1.5320842713507	-0.0000186527646
9	1.5320873801448	-0.0000060874512
10	1.5320883947200	-0.0000019866601
11	1.5320887258300	-0.0000006483511

So we see that $\alpha_2 \approx 1.5320887$

$\phi_2(x) = \frac{3}{2}x - \frac{1}{6}(x^3+1)$

k	$x^{(k)}$	$f(x^{(k)})$
0	-1.5000000000000	2.1250000000000
1	-1.8541666666667	0.1879973234954
2	-1.8854995539159	-0.0466569569939
3	-1.8777233944169	0.0126082676268
4	-1.8798247723547	-0.0033398823367
5	-1.8792681252986	0.0008895691088
6	-1.8794163868167	-0.0002365930548
7	-1.8793769546409	0.0000629493495
8	-1.8793874461992	-0.0000167469648
9	-1.8793846550384	0.0000044554624
10	-1.8793853976155	-0.0000011853492
11	-1.8793852000573	0.0000003153557

So we see that $\alpha_0 \approx -1.8793852$.

7.1.b

Let $\phi(x) : [a,b] \to [a,b]$ be a continuous function. Show that if $\phi(x)$ is a contraction mapping on $[a,b]$ then the sequence $\{x^{(k)}\}$ defined by $x^{(k+1)} = \phi(x^{(k)})$ is a Cauchy sequence.

Solution:

Recall the two relevant definitions.

Definition 7.1.1. A real sequence $\{x^{(k)}\}$ is a Cauchy sequence if $\forall \epsilon > 0$ and integer $n > 0$, $\exists M > 0$ such that

$$\forall m > M \quad |x^{(m)} - x^{(m+n)}| < \epsilon$$

Definition 7.1.2. A continuous function $\phi(x) : [a, b] \to \mathbb{R}$ is a contraction mapping on $[a, b]$ if $\exists 0 < L < 1$ such that $\forall x,\ y \in [a, b]$

$$|\phi(x) - \phi(y)| \le L|x - y|$$

i.e., $\phi(x)$ is Lipschitz continuous with constant strictly less than 1.

Since $\phi(x)$ is a contraction we have

$$|x^{(k+1)} - x^{(k)}| = |\phi(x^{(k)}) - \phi(x^{(k-1)})| \le L|x^{(k)} - x^{(k-1)}|$$

Applying this idea repeatedly gives

$$|x^{(k+1)} - x^{(k)}| \le L^k|x^{(1)} - x^{(0)}| \tag{1}$$

Now consider the term that must be bounded in order to be a Cauchy sequence.

$$\begin{gathered}
|x^{(m)} - x^{(m+n)}| = |(x^{(m)} - x^{(m+1)}) + (x^{(m+1)} - x^{(m+2)}) + \cdots + (x^{(m+n-1)} - x^{(m+n)})| \\
\le |(x^{(m)} - x^{(m+1)})| + |(x^{(m+1)} - x^{(m+2)})| + \cdots + |(x^{(m+n-2)} - x^{(m+n-1)})| + |(x^{(m+n-1)} - x^{(m+n)})| \\
\le (L^m + L^{m+1} + \cdots + L^{m+n-1})|x^{(1)} - x^{(0)}|
\end{gathered}$$

where the last step is due to 1. We therefore have

$$|x^{(m)} - x^{(m+n)}| \le L^m \frac{1 - L^n}{1 - L}|x^{(1)} - x^{(0)}|$$

For any $\epsilon > 0$ we can shoose M large enough so $|x^{(m)} - x^{(m+n)}| < \epsilon$ since $L < 1$. We therefore have a Cauchy sequence as desired.

Problem 7.2

Textbook, p. 283, Problem 2
Solution:
Since α is an order m root of $f(x)$ we have

$$\begin{gathered}
f(x) = (x - \alpha)^m h(x) \\
h(\alpha) \ne 0 \\
f^{(i)}(\alpha) = 0, \quad 1 \le i \le m - 1 \\
f'(x) = m(x - \alpha)^{m-1}h(x) + (x - \alpha)^m h'(x)
\end{gathered}$$

Modified Newton is defined by the iteration

$$\phi(x) = x - m\frac{f(x)}{f'(x)}$$

where m is assumed known. We must therefore examine $\phi'(\alpha)$ and $\phi''(\alpha)$ to demonstrate quadratic convergence.

We have $\phi(\alpha) = \alpha$ by inspection and

$$\begin{aligned}
\phi(x) &= x - m\frac{f(x)}{f'(x)} \\
&= x - m\left[\frac{(x-\alpha)^m h(x)}{m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x)}\right] \\
&= x - m\left[\frac{(x-\alpha)h(x)}{mh(x) + (x-\alpha)h'(x)}\right]
\end{aligned}$$

The derivative is

$$\begin{aligned}
\phi'(x) &= 1 - m\left[\frac{(x-\alpha)^m h(x)}{m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x)}\right]' \\
&= 1 - m\left[\frac{(x-\alpha)^m}{m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x)}\right]' h(x) \\
&\quad - m\left[\frac{(x-\alpha)^m}{m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x)}\right] h'(x)
\end{aligned}$$

We have

$$\left[\frac{(x-\alpha)^m}{m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x)}\right]' = \frac{mh(x) - (x-\alpha)h'(x) - (x-\alpha)^2 h''(x)}{(mh(x) + (x-\alpha)h'(x))^2}$$

So

$$\phi'(\alpha) = 1 - m\left[\frac{mh(\alpha)}{m^2h^2(\alpha)}\right] h(\alpha) = 0$$

as desired and the method is at least quadratic in convergence.

$\phi''(x)$ can be computed and evaluated at α to see that $\phi''(\alpha) \neq 0$ and therefore the method is quadratically convergent.

Note also that the analysis above yields that if $\phi_N(x)$ is the iteration for Netwon's method then

$$\phi_N'(\alpha) = 1 - \left[\frac{mh(\alpha)}{m^2h^2(\alpha)}\right] h(\alpha) = 1 - \frac{1}{m}$$

and we have linear convergence for any $m \geq 2$.

Problem 7.3

Textbook, p. 283, Problem 5

Solution:

The iteration is defined to be

$$x^{(k+1)} = \Psi(x^{(k)}) = x^{(k)} - \frac{f(x^{(k)})}{\phi(x^{(k)})}$$

$$\phi(x) = \frac{f(x + f(x)) - f(x)}{f(x)}$$

Steffensen's method is important because it demonstrates an alternative to Newton's method for gaining quadratic convergence. Note that it does not require f' but it uses an extra function evaluation per step. The secant method achieves superlinear convergence but only requires one function evaluation per step since the slope uses a function value computed on the previous step. It does not achieve quadratic convergence however. In order for that to happen the method must either use derivative information or additional function evaluations. Steffensen's method can be analyzed from several points of view. It is an acceleration of a simpler iteration. It is an instance of a much larger class of methods for which there are convergence theorems that cover Steffensen's method. See Ortega and Rheinboldt for a complete and rigorous analysis. However, there is a simpler asymptotic argument given by Dahlquist and Bjorck in their Numerical Methods text. It exploits the method's relationship with Newton's method.

Let $\beta_k = f(x^{(k)})$ and expand $\phi(x)$ around $x^{(k)}$ This yields

$$\begin{aligned}\phi(x^{(k)}) &= \frac{f(x^{(k)} + \beta_k) - f(x^{(k)})}{f(x^{(k)})} \\ &= \frac{\beta_k f'(x^{(k)}) + 0.5\beta_k^2 f''(x^{(k)}) + O(\beta_k^3)}{\beta_k} \\ &= f'(x^{(k)}) + 0.5\beta_k f''(x^{(k)}) + O(\beta_k^2) \\ &= f'(x^{(k)})(1 - 0.5h_k f''(x^{(k)}) + O(\beta_k^2))\end{aligned}$$

where

$$h_k = -\frac{f(x^{(k)})}{f'(x^{(k)})}$$

which is the Newton's method update.

So we can substitute this into the iteration and use Taylor's expansion to get

$$
\begin{aligned}
x^{(k+1)} = \Psi(x^{(k)}) = x^{(k)} - \frac{\beta_k}{\phi(x^{(k)})} \\
= x^{(k)} - \frac{\beta_k}{f'(x^{(k)})(1 - 0.5h_k f''(x^{(k)}) + O(\beta_k^2))} \\
= x^{(k)} - \frac{\beta_k}{f'(x^{(k)})} \frac{1}{(1 - 0.5h_k f''(x^{(k)}) + O(\beta_k^2))} \\
= x^{(k)} - \frac{\beta_k}{f'(x^{(k)})}(1 + 0.5h_k f''(x^{(k)}) + O(\beta_k^2)) \\
= x^{(k)} + h_k(1 + 0.5h_k f''(x^{(k)}) + O(\beta_k^2))
\end{aligned}
$$

This shows the relationship between Newton's and Steffensen's methods.

It can be shown using Taylor's expansion that if $\epsilon_k = x^{(k)} - \alpha$ is the error on the k-th step then

$$h_k = -\epsilon_k + 0.5\epsilon_k^2 \frac{f''(\xi)}{f'(x^{(k)})}$$

For the error we have

$$
\begin{aligned}
x^{(k+1)} = x^{(k)} + h_k(1 + 0.5h_k f''(x^{(k)}) + O(\beta_k^2)) \\
= x^{(k)} + (h_k + 0.5h_k^2 f''(x^{(k)}) + O(\epsilon_k \beta_k^2)) \\
x^{(k+1)} - \alpha = x^{(k)} - \alpha + (h_k + 0.5h_k^2 f''(x^{(k)}) + O(\epsilon_k \beta_k^2)) \\
\epsilon_{k+1} = \epsilon_k + (h_k + 0.5h_k^2 f''(x^{(k)}) + O(\epsilon_k \beta_k^2))
\end{aligned}
$$

Substituting this into the error recurrence yields

$$
\begin{aligned}
\epsilon_{k+1} = \epsilon_k + (h_k + 0.5h_k^2 f''(x^{(k)}) + O(\epsilon_k \beta_k^2)) \\
\epsilon_{k+1} = \epsilon_k + \left\{ -\epsilon_k + 0.5\epsilon_k^2 \frac{f''(\xi)}{f'(x^{(k)})} + 0.5\left[-\epsilon_k + 0.5\epsilon_k^2 \frac{f''(\xi)}{f'(x^{(k)})}\right]^2 f''(x^{(k)}) + O(\epsilon_k \beta_k^2) \right\} \\
\epsilon_{k+1} = \epsilon_k - \epsilon_k + 0.5\epsilon_k^2 \frac{f''(\xi)}{f'(x^{(k)})} + 0.5\epsilon_k^2 f''(x^{(k)}) + O(\epsilon_k^3) + O(\epsilon_k \beta_k^2) \\
\epsilon_{k+1} = 0.5\epsilon_k^2 \left(\frac{f''(\xi)}{f'(x^{(k)})} + f''(x^{(k)}) \right) + O(\epsilon_k^3) + O(\epsilon_k \beta_k^2)
\end{aligned}
$$

k	x_k	$f(x_k)$
1	0.5000000000000	0.1271051211084
2	0.5898675629074	-0.0018502675763
3	0.5885330194929	-0.0000003821054
1	0.1000000000000	0.8050040013891
2	0.6459905289067	-0.0778465688779
3	0.5890102321102	-0.0006621004360
4	0.5885327792712	-0.0000000489427
1	1.0000000000000	-0.4735915436365
2	0.6008262651347	-0.0169658671530
3	0.5885558206853	-0.0000320047227
4	0.5885327440643	-0.0000000001144

Table 1: Steffensen's method to solve $f(x) = e^{-x} - \sin x$.

Using Taylor's expansion around α yields $\beta_k = O(\epsilon_k)$ as $x^{(k)} \to \alpha$ so we have

$$\epsilon_{k+1} = 0.5\epsilon_k^2 \left(\frac{f''(\xi)}{f'(x^{(k)})} + f''(x^{(k)}) \right) + O(\epsilon_k^3) + O(\epsilon_k \beta_k^2)$$
$$\frac{\epsilon_{k+1}}{\epsilon_k^2} = 0.5 \left(\frac{f''(\xi)}{f'(x^{(k)})} + f''(x^{(k)}) \right) + O(\epsilon_k)$$
$$\lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k^2} = 0.5 \frac{f''(\alpha)}{f'(\alpha)} \left(1 + f''(\alpha)\right)$$

which is quadratic convergence.

Applying the method to find the roots of

$$f(x) = e^{-x} - \sin x$$

in Textbook, p. 284, Problem 8 yields results competitive with Newton.

The values for three different initial conditions are given in Table 1.

Problem 7.4

Textbook, p. 283, Problem 6

Solution:

The function $f(x) = x^2 - x - 2$ has two roots $\alpha_1 = -1$ and $\alpha_2 = 2$.

Analysis of $\phi_1(x)$: The iteration is defined by $\phi_1(x) = x^2 - 2$. We have

$$\phi_1(\alpha_1) = \phi_1(-1) = -1$$
$$\phi_1(\alpha_2) = \phi_1(2) = 2$$

Therefore, both roots are fixed points of $\phi_1(x)$.

We have

$$\phi_1'(x) = 2x$$
$$\phi_1'(\alpha_1) = \phi_1'(-1) = -2$$
$$\phi_1'(\alpha_2) = \phi_1'(2) = 4$$
$$\therefore |\phi_1'(\alpha_1)| > 1 \quad \text{and} \quad |\phi_1'(\alpha_2)| > 1$$

Both roots are unstable fixed points, i.e., they both repel the iteration. So the iteration defined by $\phi_1(x)$ does not converge to either root.

Analysis of $\phi_2(x)$: The iteration is defined by $\phi_2(x) = \sqrt{2+x}$. We have

$$\phi_2(\alpha_1) = \phi_2(-1) = 1$$
$$\phi_2(\alpha_2) = \phi_2(2) = 2$$

So α_1 is not a fixed point of $\phi_2(x)$ and therefore the iteration will not converge to it. α_2 is a fixed point of $\phi_2(x)$. Therefore we need only consider α_2.

We have

$$\phi_2'(x) = \frac{1}{2\sqrt{2+x}}$$
$$\phi_2'(\alpha_2) = \phi_2'(2) = 0.25$$

$\therefore |\phi_2'(\alpha_2)| < 1$ and there is an interval around α_2 on which $\phi_2(x)$ is a contraction mapping. We also have

$$x > -\frac{7}{4} \rightarrow 0 < \phi_2'(x) < 1$$

so $\phi_2(x)$ is a contraction mapping on this interval and the iteration will converge to α_2.

If $x < -2$ then $\phi_2(x) \notin \mathbb{R}$ so the iteration is not defined on this interval and will therefore not converge to α_2.

This leaves the interval $-2 \leq x < -7/4$. If $-2 \leq x_0 < -7/4$ then $\phi_2(x_0) > 0$ which says that the iteration will then converge to α_2.

So we have

- $x_0 < -2$ the iteration does not converge due to being undefined.
- $x_0 > -7/4$ converges to α_2 due to being a contraction mapping.
- $-2 \leq x_0 < -7/4$ the iteration converges to α_2 by moving into the region of contraction within one iteration.

Analysis of $\phi_3(x)$: The iteration is defined by $\phi_3(x) = -\sqrt{2+x}$. We have

$$\begin{gathered}\phi_3(\alpha_1) = \phi_3(-1) = -1\\ \phi_3(\alpha_2) = \phi_3(2) = -2\end{gathered}$$

So α_1 is a fixed point and α_2 is not a fixed point. Therefore we need only consider α_1.

$x > -2$ is required for the value $\phi_3(x)$ to be defined in $\mathbb{R}$. Since the negative square root is used we must also have $x < 2$ so that $-2 < \phi_3(x) < 0$ and the iteration can continue.

We also have

$$\phi_3'(x) = -\frac{1}{2\sqrt{2+x}}$$

Therefore $-7/4 < x < 2$ satisfies the requirements above and also guarantees that $\phi_3(x)$ is a contraction mapping. On this interval, the iteration will converge to α_1. If $-2 < x < -7/4$ then $-1/4 \leq \phi_3(x) < 0$ which is in the contraction interval and convergence to α_1 will result. So if $-2 < x_0 < 2$ guarantees convergence to $\alpha_1 = -1$.

Analysis of $\phi_4(x)$: The iteration is defined by $\phi_4(x) = 1 + 2x^{-1}$ with $x \neq 0$. We have

$$\begin{gathered}\phi_4(\alpha_1) = \phi_4(-1) = -1\\ \phi_4(\alpha_2) = \phi_4(2) = 2\end{gathered}$$

So α_1 and α_2 are fixed points.

We have

$$\begin{gathered}\phi_4'(x) = -2x^{-2}\\ |\phi_4'(\alpha_1)| = |\phi_4'(-1)| = |-2| > 1\\ |\phi_4'(\alpha_2)| = |\phi_4'(2)| = |-0.5| < 1\\ x > 2 \to |\phi_4'(x)| < 1\end{gathered}$$

Therefore, α_1 repels the iteration and convergence to α_1 will not occur.

$\phi_4(x)$ is a contraction mapping on $x > 2$ and for x_0 the iteration converges to α_2. If $0 < x_0 < 2$ then $x_1 = \phi_4(x_0) > 2$ which is in the region of convergence. So we have convergence to α_2 for $x > 0$.

Next consider $x < 0$. If $x_0 = -2$ then $x_1 = 0$ and $x_2 = \infty$ so there is no convergence. If $x_0 < -2$ then $x_1 > 0$ and we have convergence from there. If $-2/3 < x_0 < 0$ then $x_1 < -2$ and we have convergence from there.

On the interval $-2 < x_0 < -2/3$ various things can happen. We know $x_0 = -1$ is a fixed point so convergence to α_2 does not occur. There are no other points that will eventually get to $x_k = -1$. However, there are many points in the interval $-2 < x_0 < -2/3$ that eventually get to $x_k = -2$. For example, $x_0 = -6/5$ yields $x_1 = -2/3$, $x_2 = -2$, $x_3 = 0$ and $x_4 = \infty$. Clearly this can be extended to other points by reversing the iteration:

$$x_0 = -2, \quad x_{-k-1} = \frac{2}{x_{-k} - 1}$$

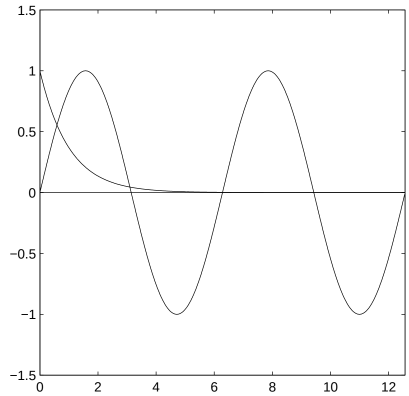

Figure 1: e^{-x} and $\sin x$

Problem 7.5

Textbook, p. 284, Problem 8

Solution:

We have $f(x) = e^{-x} - \sin x$ which has a root $\alpha \approx 0.5885$. To construct iterations that have the roots of $f(x)$ as fixed points we take

$$f(x) = e^{-x} - \sin x = 0$$
$$e^{-x} - \sin x = 0 \rightarrow e^{-x} = \sin x$$

$$e^{-x_{k+1}} = \sin x_k \rightarrow \phi_1(x) = -\ln(\sin x)$$
$$\sin(x_{k+1}) = e^{-x_k} \rightarrow \phi_2(x) = \arcsin(e^{-x})$$

The quadratically convergent Newton iteration is

$$x_{k+1} = \phi_N(x_k) = x_k - \frac{e^{-x_k} - \sin x_k}{-e^{-x_k} - \cos x_k}$$

The quadratically convergent Steffensen's iteration could also be used.

The functions e^{-x} and $\sin x$ are plotted in Figure 1. The intersection points are roots of $f(x)$. The one of interest is the leftmost.

The functions $\phi_1(x)$. $\phi_1(x)$ and $y = x$ are plotted in Figure 2 over the interval $0 < x < \pi/2$. Note the steep profile of $\phi_1(x)$ around α. This hints that $\phi_1'(\alpha) > 1$ and the iteration might be repelled by α. Since $\sin x > 0$ on this interval we have $\phi_1'(x) = -1/\sin x$ Evalutating at $\phi_1'(0.5) \approx 2.08$ and $\phi_1'(0.6) \approx 1.78$ and by continuity and monotonicity we have $\phi_1'(\alpha) > 1$. The fact that the iteration is repelled by α is easily demonstrated. Starting close to α yields the values in Table 2.

Consider $\phi_2(x)$. $\phi_2(x)$ and $y = x$ are plotted in Figure 3 over the interval $0 < x < \pi/2$. The profile is flatter than $\phi_1(x)$ and therefore this iteration may converge. Running with the

k	x_k	$f(x_k)$
1	0.5000000000000	0.1271051211084
2	0.7351666863853	-0.1912852608040
3	0.3994172343090	0.2818292859421
4	0.9444805743936	-0.4213110611480
5	0.2104833131357	0.6012600039455
6	1.5657437074527	-0.7910546648774
7	0.0000127645354	0.9999744710106
8	11.2688399028214	0.9629074812613
9	0.0378112015867	-1.4010959662886

Table 2: $\phi_1(x)$ iteration diverging.

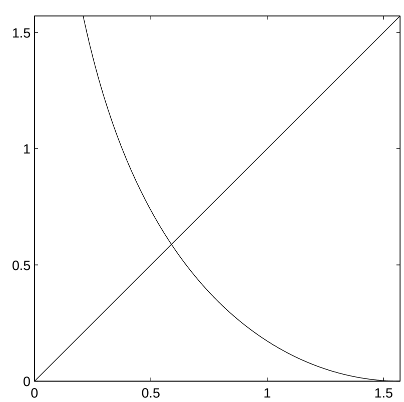

Figure 2: $\phi_1(x)$ and $y = x$

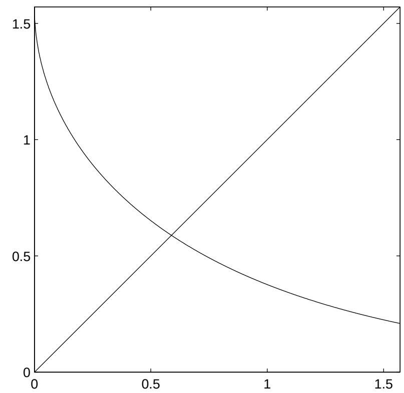

Figure 3: $\phi_2(x)$ and $y = x$

same initial condition as the divergent $\phi_1(x)$ iteration shows convergence is possible. The values are shown in Table . We have

$$\phi_2^(x) = \arcsin(e^{-x}) \rightarrow \phi_2'(x) = -\frac{e^{-x}}{\sqrt{1 - e^{-2x}}}$$

Evalutating at $|\phi_2'(0.5)| \approx 0.76$ and $|\phi_2'(0.6)| \approx 0.66$ and by continuity and monotonicity we have $|\phi_2'(\alpha)| < 1$. So there is an interval around α on which $\phi_2(x)$ is a contraction mapping. Examining $\phi_2(x)$ on the interval shows that it falls below 1 in magnitude between $x = 0.345$ and $x = 0.347$ and then stays below 1 for the rest of the interval. We therefore have that $\phi_2(x)$ defines an iteration that converges to α for $0.347 < x_0 < \pi//2$.

Finally consider Newton's method. Starting at the same initial condition as the other iterations, Newton's method converges very rapidly. The values are shown in Table . Iterations with $x_0 = 1.0$ and $x_0 = 0.1$ converge within 5 steps.

k	x_k	$f(x_k)$
1	0.5000000000000	0.1271051211084
2	0.6516896695013	-0.0853662229825
3	0.5482147609280	0.0568162869340
4	0.6162520655384	-0.0380163225925
5	0.5703948141343	0.0253378042622
6	0.6007995686699	-0.0169292064460
7	0.5804173645256	0.0112917347199
8	0.5939811849703	-0.0075399411541
9	0.5849105354616	0.0050309128069
10	0.5909566712586	-0.0033584759223
11	0.5869177644895	0.0022412584484
12	0.5896118937957	-0.0014960234401
13	0.5878130453983	0.0009984355712
14	0.5890133443421	-0.0006664151267
15	0.5882120869637	0.0004447754411
16	0.5887468105707	-0.0002968629410
17	0.5883898910423	0.0001981336891
18	0.5886280985493	-0.0001322419474
19	0.5884691056115	0.0000882621342
20	0.5885752202646	-0.0000589092415

Table 3: $\phi_2(x)$ convergent iteration.

k	x_k	$f(x_k)$
1	0.5000000000000	0.1271051211084
2	0.5856438169664	0.0040112772060
3	0.5885294126264	0.0000046202542
4	0.5885327439774	0.0000000000062

Table 4: $\phi_N(x)$ convergent iteration.