Information on the Qualifying Examination for Foundations of Computational Mathematics II

1 Basic Information

The Foundations of Computational Mathematics II Qualifying Examination is offered each year in January and August. It is a closed book examination. No reference books or notes are allowed. A calculator is permitted. The duration of the examination is typically two hours.

2 Related Texts

In addition to class notes and any readings recommended by the instructor of the course, these textbooks contain material related to the examination:

- Numerical Mathematics, A. Quarteroni, R. Sacco, and F. Saleri, Springer Texts in Applied Mathematics 37, Second Edition.
- Numerical Mathematics and Computing, E. Ward Cheney and David R. Kincaid, Brooks Cole Publishing, Sixth Edition
- Analysis of Numerical Methods, E. Isaacson and H. Keller, Wiley.
- Accuracy and Stability of Numerical Algorithms, N. Higham, SIAM, Second Edition.
- Introduction to Numerical Analysis, J. Stoer and R. Bulirsch, Springer.
- Numerical Methods, G. Dahlquist and A. Bjorck, Prentice-Hall.
- Numerical Computation, Volumes 1 and 2, C. W. Ueberhuber, Springer.
- Splines and Variational Methods, P. M. Prenter, Wiley.
- The Elements of Real Analysis, R. Bartle, Wiley.
- Mathematical Methods in Physics and Engineering, J. Dettman, McGraw Hill
- Optimization by Vector Space Methods, D. Luenberger, Wiley.

- Computer Methods for Ordinary Differential Equations and Differential-algebraic Equations, U. Ascher and L. Petzold, SIAM.
- Numerical Initial Value Problems in Ordinary Differential Equations, C. W. Gear, Prentice Hall.
- Computational Frameworks for the Fast Fourier Transform, C. Van Loan, SIAM.
- The DFT: An Owners Manual for the Disrete Fourier Transform, W. L. Briggs and V. E. Henson, SIAM.

3 Topics

The topics include those covered in the Foundations of Mathematics II Course. However, since the emphasis placed on the topics may vary with instructors, the list below, while representative, should not be considered definitive. Students are encouraged to discuss the topic list with recent instructors.

3.1 Polynomial Interpolation

1. Interpolation Overview
   - Interpolation vs. Approximation
   - Existence and uniqueness of polynomial interpolant
2. Forms of Interpolating Polynomials
   - Lagrange form
   - Newton form
   - Barycentric form
   - Equidistant point forms
   - Computational complexity of formation, evaluation, updating
3. Polynomial Interpolant Error
   - Pointwise error
   - Conditioning with repspect to function values
   - Conditioning with respect to representation
   - Stability and practical limitations
   - Uniform convergence of interpolating stragtegies
   - Bernstein's Theorm and polynomials

- Runge's phenomenon

4. Hermite Interpolation
    - Monomial form, existence and uniqueness
    - Lagrange form
    - Newton form
    - Osculating polynomial and its relationship to other interpolants
    - Pointwise error

5. Piecewise Interpolation
    - Forms of polynomial within an interval:
        - Monomial
        - Lagrange/Baycentric
        - Newton
        - Hermite
        - Basis forms
    - Pointwise error
    - Achieving a prescribed error via interval size selection

6. Splines
    - Smoothness as a constraint in polynomial splines
    - Cubic splines
    - Choice of local form
    - Boundary conditions and system of equations defining coefficients
    - Approximation error in splines
    - Spline bases and B-splines

7. Multidimensional Interpolation
    - Dimension of general problem
    - Special cases of 2-dimensional interpolation: mesh and triangulation
    - Basis forms

3.2 Approximation

1. Parametric Curves
   - Bernstein polynomials and Bezier curves
   - B-spline parametric curves
   - De Casteljau's Algorithm
2. Best (Minimax) Polynomial Approximation
   - Characterization of minimax polynomial
   - Analytical solutions for special problems
3. Chebyshev (Near Minimax) Approximation
   - Minimal ∞-norm monic polynomial
   - Relationship to pointwise interpolation error
   - Near-minimax polynomials
   - Relationship in error to minimax polynomial
4. $\mathcal{L}_\omega^2$ Approximation
   - The vector space $\mathcal{L}_\omega^2$ and associated norms
   - Complete orthogonal sequences and bases
   - Orthogonal polynomials and their basic properties
   - Generalized Fourier Series
   - Least squares approximation on a finite dimensional subspace of $\mathcal{L}_\omega^2$
5. Economization of power series using orthogonal polynomials
6. Discrete least squares approximation using orthogonal polynomials
7. Rational Interpolation and Approximation
   - Necessary and sufficient conditions for the existence of a rational interpolant
   - Inverse and reciprocal difference forms
   - Pade approximation

3.3 Numerical Quadrature

1. Newton-Cotes Quadrature
   - Interpolatory quadrature
   - Interpolatory quadrature error
   - Newton-Cotes closed and open formulas
   - Degree of exactness, order of infinitesimal, and error
2. Composite Newton-Cotes Quadrature
   - Composite Newton-Cotes open and closed formulas
   - Error for Composite Newton-Cotes open and closed formulas
   - Achieving a prescribed error via interval size selection
3. Adaptive Quadrature
   - Step halving
   - Error estimation and correction
   - Efficient Newton-Cotes adaptive quadrature
   - Recursive adaptive quadrature
4. Gauss Quadrature
   - Maximum degree of exactness and orthogonality
   - Gauss Legendre quadrature open formulas
   - Gauss Labatto quadrature closed formulas
   - Gauss Radau quadrature formulas
   - Gauss Legendre quadrature error
   - Alternate weight functions and Gauss quadrature

3.4 Discrete Fourier Transform

1. Trigonometric approximation and interpolation
2. Transforms, Gauss quadrature and the Generalized Fourier series
3. Discrete Fourier transform and unitary matrices
4. FFT
   - Matrix structure and the FFT

   - Polynomial structure and the FFT
   - Cooley-Tukey FFT

5. DFT aliasing

3.5 Numerical Integration of Ordinary Differential Equations

1. Differential operators and difference operators
   - Discretization error (local truncation error) and consistency
   - Convergence and consistency
   - Order of convergence and order of discretization error
   - Local error and discretization error

2. Stability
   - 0-stability
   - Absolute stability
   - Stability, consistency and convergence

3. Linear multistep methods
   - Derivation of methods: Adams-Bashforth, Adams-Moulton, BDFs
   - Consistency of linear multistep methods
   - 0-stability, weak, strong and absolute stability of linear multistep methods
   - Convergence of linear multistep methods
   - Predictor-Corrector methods
   - Error estimation and stepsize control

4. Runge Kutta methods
   - One-step r-stage methods general form
   - Butcher array form
   - Consistency analysis of standard Runge Kutta methods
   - Achievable order for explicit methods
   - Embedded pair methods
   - Derivation of implicit Runge Kutta methods based on collocation and Gauss quadrature
   - 0-stability and absolute stability of Runge Kutta methods
   - Computational complexity of explicit, implicit and SDIRK Runge Kutta methods

- Computational complexity compared to linear multistep methods

5. Stiffness

   - Definition of stiffness
   - Absolute stability is not enough.
   - A-stability is not enough.
   - Stiff decay
   - Methods appropriate for stiff situations.