Solutions for Homework 10 Foundations of Computational Math 2 Spring 2011

Problem 10.1

Consider the following linear multistep method:

$$y_n = -4y_{n-1} + 5y_{n-2} + h(4f_{n-1} + 2f_{n-2})$$

The method is not 0-stable.

10.1.a. Determine, p, the order of consistency of the method.

10.1.b. Determine the coefficient, C_{p+1}, in the discretization error d_n.

10.1.c. Consider the application of the method to $y' = 0$ with $y_0 = 0$ and $y_1 = \epsilon$, i.e., a perturbed initial condition. Show that $|y_n| \to \infty$ as $n \to \infty$, i.e., the numerical method is unstable.

Solution:

We are given the linear multistep method:

$$y_n = -4y_{n-1} + 5y_{n-2} + h(4f_{n-1} + 2f_{n-2})$$

The method is not 0-stable since

$$\rho(\xi) = \xi^2 + 4\xi - 5 = (\xi - 1)(\xi + 5)$$

which violates the root condition for 0-stability of a linear multistep method.

The order of consistency can be checked either by using the formulae for C_i's given in the notes or by inserting Taylor expansions and simplifying. We have

$$\begin{aligned}
\alpha_0 = 1, \quad \alpha_1 = 4, \quad \alpha_2 = -5 \\
\beta_0 = 0, \quad \beta_1 = 4, \quad \beta_2 = 2 \\
C_0 = C_1 = C_2 = C_3 = 0 \\
C_4 = 4\frac{1}{24} - 5\frac{16}{24} + 4\frac{1}{6} + 2\frac{8}{2} \\
= \frac{1}{6} \\
d_n = \frac{1}{6}h^3 y^{(4)}(t_n) + O(h^4)
\end{aligned}$$

Applying the method to $y' = 0$ with $y_0 = 0$ and $y_1 = \epsilon$ we get the following:

$$\begin{aligned}
y_n &= -4y_{n-1} + 5y_{n-2} \\
&= \gamma_1(1)^n + \gamma_2(-5)^n \\
\gamma_1 &= \frac{\epsilon}{6} \\
\gamma_2 &= -\frac{\epsilon}{6} \\
y_n &= \frac{\epsilon}{6} - \frac{\epsilon}{6}(-5)^n \\
&= \frac{\epsilon}{6}(1 + (-1)^{n+1}(-5)^n) = \mu(n)\epsilon
\end{aligned}$$

Clearly, $|\mu(n)| \to \infty$.

The growth is easily seen in the first few terms:

$$\begin{aligned}
y_0 = 0 \ y_1 &= \epsilon \\
y_2 &= -4\epsilon \\
y_3 &= 21\epsilon
\end{aligned}$$

Problem 10.2

Consider the following linear multistep method:

$$y_n = y_{n-2} + \frac{h}{3}(f_n + 4f_{n-1} + f_{n-2})$$

The method is 0-stable but it is weakly stable.

10.2.a. Determine the discretization error d_n.

10.2.b. Consider the application fo the method to $y' = \lambda y$. Write the recurrence that yields y_n.

10.2.c. Show that $|y_n| \to \infty$ as $n \to \infty$, i.e., the numerical method is unstable.

Solution:

For

$$y_n = y_{n-2} + \frac{h}{3}(f_n + 4f_{n-1} + f_{n-2})$$

we have

$$\begin{aligned}
C_0 = C_1 = C_2 = C_3 = C_4 &= 0 \\
C_5 &= -\frac{1}{90}
\end{aligned}$$

So the method is order $p = 4$.

Applying the method to $y' = \lambda y$ yields

$$
\begin{gathered}
y_n - y_{n-2} - \frac{h\lambda}{3} y_n - \frac{4h\lambda}{3} y_{n-1} - \frac{h\lambda}{3} y_{n-2} = 0 \\
(1 - \frac{h\lambda}{3}) y_n - \frac{4h\lambda}{3} y_{n-1} - (1 + \frac{h\lambda}{3}) y_{n-2} = 0 \\
p(\xi) = (1 - \frac{h\lambda}{3})\xi^2 - \frac{4h\lambda}{3}\xi - (1 + \frac{h\lambda}{3}) = 0 \\
(1-\mu)\xi^2 - 4\mu\xi - (1+\mu) = 0 \\
\mu = \frac{h\lambda}{3}
\end{gathered}
$$

The characteristic polynomial $p(\xi)$ can be used to solve for y_n. The roots and solution are

$$
\begin{gathered}
\eta_\pm = \frac{2\mu \pm \sqrt{1 + 3\mu^2}}{(1-\mu)} \\
y_n = \gamma_1(\eta_+)^n + \gamma_2(\eta_-)^n
\end{gathered}
$$

The γ_i would be set given the initial conditions for the recurrence, i.e., y_0 and y_1.

Since this is examining the absolute stability of the method we know that we are trying to track $e^{\lambda t}$, i.e., $y(t_n) = e^{nh\lambda} = (e^{h\lambda})^n$. Given that the numerical solution is a combination of the roots raised to the power n we would like to have one track the solution and the other decay to 0. So considering the roots as an approximation to $e^{h\lambda}$ can also be used to address this issue.

Note we have done this implicitly before in our analysis of forward and backward Euler. The single roots of those methods applied to $y' = \lambda y$ are

$$
\begin{gathered}
\eta^F = 1 + h\lambda \\
\eta^B = \frac{1}{1 - h\lambda}
\end{gathered}
$$

We have

$$
\begin{gathered}
e^{h\lambda} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + O(h^4) \\
\eta^F = 1 + h\lambda \\
\eta^B = \frac{1}{1 - h\lambda} = 1 + h\lambda + (h\lambda)^2 + (h\lambda)^3 + O(h^4) \\
e^{h\lambda} - \eta^F = \frac{1}{2}(h\lambda)^2 O(h^3) \\
e^{h\lambda} - \eta^B = -\frac{1}{2}(h\lambda)^2 O(h^3)
\end{gathered}
$$

So both roots yield a stable approximation to $y(t_n) = e^{nh\lambda}$ when λ is in the left half plane, i.e., when the analytical solution is stable.

We can perform a similar analysis on the 4-th order method in this problem (usually referred to as the Generalized Milne-Simpson two-step method). Using Taylor expansions yields

$$\eta_+ = e^{h\lambda} + O(h^5)$$
$$\eta_- = -e^{-\frac{h\lambda}{3}} + O(h^3)$$

The first term gives the 4-th order approximation we expect but when λ is anywhere in the left half plane the analytical solution is stable but the numerical solution is dominated by η_-^n and is blowing up, i.e., unstable.

This problem could also have been addressed by drawing the curve

$$h\lambda = \frac{\rho(\xi)}{\sigma(\xi)} = \frac{\xi^2 - 1}{\frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3}}$$

and noting the same lack of stability in the left half plane.

Problem 10.3

Recall, our model problem

$$f = \lambda(y - F(t)) + F'(t) \quad y(0) = y_0$$
$$y(t) = (y_0 - F(0))e^{\lambda t} + F(t)$$

Take $F(t) = \sin t$ and $y(0) = 1$ and consider $y(t)$ on $0 \leq t \leq 1$.

Consider the 4 methods

- Method 1
$$y_n = -4y_{n-1} + 5y_{n-2} + h(4f_{n-1} + 2f_{n-2})$$
- Method 2 – explicit midpoint
$$y_n = y_{n-2} + 2hf_{n-1}$$
- Method 3 – Adams Bashforth two-step
$$y_n = y_{n-1} + \frac{h}{2}(3f_{n-1} - f_{n-2})$$
- Method 4 – BDF two-step
$$y_n = \frac{4}{3}y_{n-1} - \frac{1}{3}y_{n-2} + \frac{2}{3}hf_n$$

Apply the 4 methods to the model problem using exact initial conditions, i.e., $y_0 = y(0)$ and $y_1 = y(h)$. Use $h = 0.01$ for $\lambda = 10$, $\lambda = -10$, and $\lambda = -500$. Explain your results. How would reducing the stepsize h affect the results?

Note it is recommended you implement this as a simple piece of code in, e.g., MATLAB. You need not turn in your code. Simply present the solution values you observe for the required numerical integrations in support of you explanations.

Solution:

We start with the best method for this problem. The BDF2 is an implicit method and so we must deal with the solution of the equation that defines y_n.

We have

$$\begin{aligned} y_n &= \frac{4}{3}y_{n-1} - \frac{1}{3}y_{n-2} + \frac{2}{3}hf_n \\ &= \frac{4}{3}y_{n-1} - \frac{1}{3}y_{n-2} + \frac{2}{3}h\left(\lambda(y_n - F(t_n)) + F'(t_n)\right) \\ (1 - \frac{2}{3}h\lambda)y_n &= \frac{4}{3}y_{n-1} - \frac{1}{3}y_{n-2} + \frac{2}{3}h\left(F'(t_n) - \lambda F(t_n)\right) \\ y_n &= (1 - \frac{2}{3}h\lambda)^{-1}\{\frac{4}{3}y_{n-1} - \frac{1}{3}y_{n-2} + \frac{2}{3}h\left(F'(t_n) - \lambda F(t_n)\right)\} \end{aligned}$$

If we set $y0 = 0$ then we have the solution $y(t) = F(t) = \sin(t)$. For $\lambda = -10$ or $\lambda = -500$ integral curves damp fairly rapidly or very rapidly to $y(t) = F(t) = \sin(t)$. Running BDF2 with $y_0 = 0$, $h = 0.01$ and $\lambda = -10$ yields an error below 3×10^{-6} on the interval $0 \leq t \leq 1$. Running BDF2 with $y_0 = 0$, $h = 0.01$ and $\lambda = -500$ yields an error below 6×10^{-8} on the interval $0 \leq t \leq 1$.

If we set $y0 = 1$ then our solution has an initial transient but damps to $y(t) = F(t) = \sin(t)$ very quickly if $\lambda = -500$. Running BDF2 with $y_0 = 1$, $h = 0.01$ and $\lambda = -500$ yields an error that is noticeable for the first 7 steps but reaches $\approx 10^{-5}$ by step 8 and quickly falls to $\approx 10^{-8}$ for most of the interval $0 \leq t \leq 1$. Note this points out that if we want accuracy in the transient region (where the exponential contributes to the solution) we have to reduce the step size due to the large derivatives. However, once we reach the stiff region for this h the BDF provides very good accuracy and filters the highly damped integral curves effectively. h could be increased significantly for this case and still provide reasonable accuracy.

If we set $y0 = 1$ then our solution has an initial transient but damps to $y(t) = F(t) = \sin(t)$ much less quickly if $\lambda = -10$. Running BDF2 with $y_0 = 1$, $h = 0.01$ and $\lambda = -10$ yields an error of $\approx 10^{-3}$ in the transient region. As the transient exponential becomes less significant in $y(t)$ the error steadily drops to less than 10^{-5} for most of the interval $0 \leq t \leq 1$.

If we set $y0 = 1$ and $\lambda = 10$ then our solution $y(t)$ is growing and we are worried about accuracy over the interval. Our numerical solution should grow at a rate that keeps some number of digits accuracy. For the BDF, we have the added feature that if h is large enough the numerical solution will damp the large positive λ components and settle to a smooth underlying integral curve. However, for $h\lambda = 0.1$ we are still in the unstable region in the right half plane and so we expect to follow the increasing solution accurately. It is observed that BDF2 maintains 2 digits accuracy over the interval $0 \leq t \leq 1$.

The Adams Bashforth 2 step method is a second order method but since it is explicit care must be taken with respect to stiffness and accuracy.

Running AB2 with $y_0 = 0$, $h = 0.01$ and $\lambda = -10$ has $h\lambda = 0.1$ which is in the region of absolute stability. The solution is $y(t) = \sin(t)$. The method yields an error magnitude below 3×10^{-6} on the interval $0 \leq t \leq 1$.

Running AB2 with $y_0 = 1$, $h = 0.01$ and $\lambda = -10$ has the soltion $y(t) = e^{\lambda t} + \sin t$ and the numerical solution provides at least 2 digits of accuracy in the transient region and an steadily decreasing error reaching 10^{-5} for much of the interval. So AB2 can handle this problem successfully for this $h\lambda$.

Running AB2 with $y_0 = 0$, $h = 0.01$ and $\lambda = -500$ has $h\lambda = 5$ which outside the region of absolute stability. Even though the solution is $y(t) = \sin(t)$. The method yields very rapidly blows up yielding an error 10^6 by step 18!

Running AB2 with $y_0 = 1$, $h = 0.01$ and $\lambda = -500$ also blows up almost immediately due to instability.

Running AB2 with $y_0 = 1$, $h = 0.01$ and $\lambda = 10$ yields a numerical solution that grows with the analytical solution. It does not provide the same accuracy as BDF2 however. AB2 yields 2 digits accuaracy on much of the interval but in the latter portion only 1 digit accuracy is achieved.

Testing the explicit midpoint rule under the same series yields

y_0	λ	Result
0	-10	10^{-5} for part 10^{-3} at end
1	-10	10^{-2} for part no accuracy at end
0	-500	immediate instability
1	-500	immediate instability
1	10	10^{-1} for part no accuracy at by $t = .3$

Testing Method 1 under the same series yields to instability in all cases.