Foundations of Computational Math I Exam 1
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
No collaborations with anyone
Due beginning of Class Wednesday, October 26, 2011

Question	Points Possible	Points Awarded
1. Basics	25	
2. Linear operators	25	
3. Floating point	25	
4. Factorization	25	
5. Orthogonal Factorization	25	
Total Points	125	

Name:________________________________

Alias: ________________________________

to be used when posting anonymous grade list.

Problem 1

(25 points)

1.a

(10 points)

Suppose $A \in \mathbb{R}^{m \times n}$ and consider the matrix 2-norm

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$$

Show that $\|A\|_2 \geq \|A_1\|_2$ where

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

$m = m_1 + m_2$, $A_1 \in \mathbb{R}^{m_1 \times n}$, and $A_2 \in \mathbb{R}^{m_2 \times n}$.

1.b

(15 points)

Let $\mathcal{S}_1 \subset \mathbb{R}^n$ and $\mathcal{S}_2 \subset \mathbb{R}^n$ be two subspaces of $\mathbb{R}^n$.

(i) **(5 points)** – Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Show that x_1 and x_2 are linearly independent.

(ii) **(10 points)** – Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Also, suppose that $x_3 \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $x_3 \neq 0$, i.e., the intersection is not empty. Show that x_1, x_2 and x_3 are linearly independent. (Note the result of the previous part of the problem may be useful.)

Problem 2

(25 points)

2.a

(15 points)

Recall that $\mathcal{P}_n$, the set of polynomials of degree less than or equal to n, and the operation of polynomial addition is equivalent to the vector space $\mathbb{C}^{n+1}$.

(i) **(5 points)** Show that the mapping from a polynomial $p(\tau) \in \mathcal{P}_n$ to its derivative with respect to τ, $p'(\tau) \in \mathcal{P}_n$ can be expressed as an $n+1 \times n+1$ matrix applied to a vector v, i.e., $v' = Dv$, where the vector $v \in \mathbb{C}^{n+1}$ represents $p(\tau)$ and the vector $v' \in \mathbb{C}^{n+1}$ represents $p'(\tau)$.

(ii) **(5 points)** What is the null space $\mathcal{N}(D)$ and how does it relate to the derivatives of the polynomials?

(iii) **(5 points)** Recall that the $n+1-$st derivative of a polynomial of degree less than or equal to n is identically 0. How is this reflected in the algebraic properties of D?

2.b

(10 points)

Consider computing the matrix vector product $y = Tx$, i.e., you are given T and x and you want to compute y. Suppose further that the matrix $T \in \mathbb{R}^{n \times n}$ is tridiagonal with constant values on each diagonal. For example, if $n = 6$ then

$$\begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \alpha & \beta & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta & 0 & 0 \\ 0 & 0 & \gamma & \alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma & \alpha & \beta \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}$$

(i) Write a simple loop-based psuedo-code that computes $y = Tx$ for such a matrix $T \in \mathbb{R}^n$.

(ii) How many operations are required as a function of n?

(iii) How many storage locations are required as a function of n?

Problem 3

(25 points)

3.a

(20 points)

Define the function $f(x) = x - 1$ on the domain $x > 1$. Let $x_0 \in \mathbb{R}$, $x_0 > 2$, and $x_1 = x_0(1+\delta)$ where $\delta \in \mathbb{R}$ with $|\delta| < 1$.

(i) **(10 points)** Determine the relative error between $f(x_1)$ and $f(x_0)$, and the relative condition number $\kappa_{rel}(x_0)$.

(ii) **(10 points)** Suppose $|\delta| < 10^{-7}$. Can we expect that the relative error between $f(x_1)$ and $f(x_0)$ is no more than 10^{-4} for the region of values assumed for x_0?

3.b

(5 points)

Suppose x, y and z are floating point numbers in a standard model floating point arithmetic system. Is it true that

$$(x \boxed{op} (y \boxed{op} z)) = ((x \boxed{op} y) \boxed{op} z) \;?$$

Problem 4

(25 points)

4.a

(15 points)

If $A \in \mathbb{R}^{n\times n}$ is a matrix with rank $1 \le k < n$ then there exists two matrices $X \in \mathbb{R}^{n\times k}$ and $Y \in \mathbb{R}^{n\times k}$ both of which have full column rank k and such that

$$A = XY^T$$

This is called a **full rank factorization** of A.

The reverse is also true, i.e., if there exist two matrices $X \in \mathbb{R}^{n\times k}$ and $Y \in \mathbb{R}^{n\times k}$ both of which have full column rank k such that $A = XY^T$, then the rank of A is k.

(i) **(5 points)** Show that the full rank factorization of A **is not unique**.

(ii) **(5 points)** Find a basis for $\mathcal{R}(A)$, the range of A.

(iii) **(5 points)** Characterize a vector in the null space $\mathcal{N}(A)$.

4.b

(10 points)

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Suppose when computing the Cholesky factorization of A using IEEE floating point arithmetic we encounter at some step a computed Schur complement that is identically 0, i.e., every element in the computed Schur complement has the value of 0. What can we conclude about the original matrix A? Justify your answer.

Problem 5

(25 points)

5.a

(10 points)

Recall, that, given a full column rank matrix $A \in \mathbb{R}^{n\times k}$, we have discussed a reliable algorithm to compute an orthonomal basis of $\mathcal{R}(A)$ by computing the Householder reflectors $H_1^T, \dots, H_k^T$ that transform A

$$H_k^T \dots H_1^T A = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

where R is upper triangular and nonsingular, and then evaluating the computing efficiently the first k columns of

$$H = H_1 \dots H_k$$

to get the k orthonormal columns of Q where $A = QR$.

(i) Suppose you want to compute an orthonormal basis of $\mathcal{R}(A)$ but the H_i were not saved, i.e., we have only R and A. Describe how you would compute Q where $A = QR$. You need not worry about numerical issues.

(ii) How many operations are required as a function of n and k?

5.b

(15 points)

Suppose you are given the nonsingular tridiagonal matrix $T \in \mathbb{R}^{n\times n}$ For example, if $n = 6$ then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}$$

(i) Suppose you use Householder reflectors to transform A to upper triangular, i.e.,

$$H_{n-1}^T \dots H_1^T T = R.$$

What is the zero/nonzero structure of R?

(ii) What is the structure of each of the reflectors H_i?

(iii) What is the computational complexity of the factorization, i.e., what is k in $O(n^k)$? (You do not have to determine the constant in the complexity expression.)