11.5: The Bionomial Theorem

Supplementary Notes

Factorial function: For positive integer n,

$$\begin{aligned} 0! &= 1 \\ n! &= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \quad \text{for } n \geq 2 \end{aligned}$$

Binomial Coeffiecients: For positive integers n and k, the number of ways of choosing a subset of k elements from a set of n elements, pronounced n *choose* k is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where with $k \leq n$.

Pascal's Triangle:

$$\begin{array}{lccccccccccccccccccccccc}
\text{Row 0:} & & & & & & \binom{0}{0} & & & & & & & & & & & & 1 & & & & & \\
\text{Row 1:} & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & & & & & 1 & & 1 & & & & \\
\text{Row 2:} & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & & & & 1 & & 2 & & 1 & & & \\
\text{Row 3:} & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & & & 1 & & 3 & & 3 & & 1 & & \\
\text{Row 4:} & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & & 1 & & 4 & & 6 & & 4 & & 1 & \\
\text{Row 5:} & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & & \vdots & & & & & & & & & & & & \vdots & & & & &
\end{array}$$

The Binomial Theorem: For real numbers a, b and positive integer n,

$$\begin{aligned} (a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{k} a^{n-k} b^k + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n \end{aligned}$$

Exercises

1. Evaluate the binary coefficients $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{n-1}$, and $\binom{n}{n}$.

2. Find the middle term of the expansion of $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^8$, if the terms are arranged in decreasing powers of the first term.

3. Find the sixth term of the expansion of $(3a^2 - \sqrt{b})^9$, if the terms are arranged in decreasing powers of the first term.

4. Find the sixth term of the expansion $(\frac{3}{c}+\frac{c^2}{4})^7$, if the terms are arranged in decreasing powers of the first term.
5. Find the term that does not contain x in the expansion $(3x-\frac{1}{4x})^6$.
6. Find the term that does not contain y in the expansion $(xy-3y^{-3})^8$.