Homework 5 Foundations of Computational Math 2 Spring 2012

Problem 5.1

Recall that we have derived different sets of linear equations for the coefficients of an interpolating cubic spline.

Assume that $f(x) = x^3$ and analyze the equations and boundary conditions that define $s(x)$ in the forms below and determine what can be said about the relationship between $s(x)$ and $f(x)$.

5.1.a. $s(x)$ is determined by $Ts'' = d$ where s'' is a vector containing s''_i $1 \le i \le n-1$ and boundary conditions $s''_0 = f''(x_0)$ and $s''_n = f''(x_n)$.

5.1.b. $s(x)$ is determined by $\tilde{T}s' = \tilde{d}$ where s' is a vector containing s'_i $1 \le i \le n-1$ and boundary conditions $s'_0 = f'(x_0)$ and $s'_n = f'(x_n)$.

Solution:

Consider first when $s(x)$ is determined by $Ts'' = d$ where s'' is a vector containing s''_i $1 \le i \le n-1$ and boundary conditions $s''_0 = f''(x_0)$ and $s''_n = f''(x_n)$.

We have $s''_i = 6x_i$, $x_{i+1} = x_i + h_{i+1}$, $x_{i-1} = x_i - h_i$, and $f_i = x_i^3$. Given that we have set the boundary conditions to the acutal values of f'' we need only show that the basic equation is satisfied by $s(x) = x^3$. We have

$$\mu_i s''_{i-1} + 2s''_i + \lambda_i s''_{i+1} = d_i$$

$$\mu_i = \frac{h_i}{h_i + h_{i+1}}, \quad \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}$$

$$d_i = \frac{6}{h_i + h_{i+1}}[\frac{(f_{i+1} - f_i)}{h_{i+1}} - \frac{(f_i - f_{i-1})}{h_i}]$$

If we substitute in for s''_i, s''_{i+1}, s''_{i-1} and multiply both sides by $(h_i + h_{i+1})/6$ we get

$$\begin{aligned}
h_i(x_i - h_i) + 2x_i(h_i + h_{i+1}) + h_{i+1}(x_i + h_{i+1}) &= [\frac{(f_{i+1} - f_i)}{h_{i+1}} - \frac{(f_i - f_{i-1})}{h_i}] \\
&= [\frac{(x_i + h_{i+1})^3 - x_i^3}{h_{i+1}} - \frac{x_i^3 - (x_i - h_i)^3}{h_i}] \\
&= \frac{1}{h_{i+1}}(3h_{i+1}x_i^2 + 3h_{i+1}^2 x_i + h_{i+1}^3) - \frac{1}{h_i}(3h_i x_i^2 - 3h_i^2 x_i + h_i^3) \\
&= 3x_i^2 + 3h_{i+1}x_i + h_{i+1}^2 - 3x_i^2 + 3h_i x_i - h_i^2 \\
&= 3h_{i+1}x_i + h_{i+1}^2 + 3h_i x_i - h_i^2
\end{aligned}$$

So now simplifying the left side yields the desired result

$$h_i(x_i - h_i) + 2x_i(h_i + h_{i+1}) + h_{i+1}(x_i + h_{i+1}) = 3h_{i+1}x_i + h_{i+1}^2 + 3h_i x_i - h_i^2$$
$$3h_{i+1}x_i + h_{i+1}^2 + 3h_i x_i - h_i^2 = 3h_{i+1}x_i + h_{i+1}^2 + 3h_i x_i - h_i^2$$

So $s(x) = x^3$ is the spline produced by this choice of equations and boundary conditions given $f(x) = x^3$.

We can repeat the exercise for when $s(x)$ is determined by $\tilde{T}s' = \tilde{d}$ where s' is a vector containing s'_i $1 \leq i \leq n-1$ and boundary conditions $s'_0 = f'(x_0)$ and $s'_n = f'(x_n)$.

We have

$$\lambda_i s'_{i-1} + 2s'_i + \mu_i s'_{i+1} = \tilde{d}_i$$
$$\mu_i = \frac{h_i}{h_i + h_{i+1}}, \quad \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}$$
$$\tilde{d}_i = 3\big[\lambda_i \frac{(f_i - f_{i-1})}{h_i} + \mu_i \frac{(f_{i+1} - f_i)}{h_{i+1}}\big]$$

If we substitute in for s'_i, s'_{i+1}, s'_{i-1} and multiply both sides by $(h_i + h_{i+1})/3$ we get

$$h_{i+1}(x_i - h_i)^2 + 2(h_i + h_{i+1})(x_i)^2 + h_i(x_i + h_{i+1})^2 = \big[h_{i+1}\frac{(f_i - f_{i-1})}{h_i} + h_i \frac{(f_{i+1} - f_i)}{h_{i+1}}\big]$$

Simplifying the left-hand side yields:

$$h_{i+1}(x_i - h_i)^2 + 2(h_i + h_{i+1})(x_i)^2 + h_i(x_i + h_{i+1})^2 = 3h_{i+1}x_i^2 + 3h_i x_i^2 + h_i^2 h_{i+1} + h_i h_{i+1}^2$$

Simplifying the right-hand side yields:

$$\begin{aligned}\big[h_{i+1}\frac{(f_i - f_{i-1})}{h_i} + h_i \frac{(f_{i+1} - f_i)}{h_{i+1}}\big] &= \frac{h_{i+1}}{h_i}\big(3h_i x_i^2 - 3h_i^2 x_i + h_i^3\big)\frac{h_i}{h_{i+1}}\big(3h_{i+1}x_i^2 - 3h_{i+1}^2 x_i - h_{i+1}^3\big) \\ &= 3h_{i+1}x_i^2 + {h_i}^2 h_{i+1} + h_i h_{i+1}^2 + 3h_i x_i^2\end{aligned}$$

and the desired equality is seen. So $s(x) = x^3$ is the spline produced by this choice of equations and boundary conditions given $f(x) = x^3$.

Problem 5.2

Assuming that the nodes are uniformly spaced, we have derived the form of the cubic B-spline $B_{3,i}(t)$ and determined its values and the values of $B'_{3,i}(t)$ and $B''_{3,i}(t)$ at the nodes t_{i-2}, t_{i-1}, t_{i-1}, t_i, t_{i+1}, and t_{i+2}. We also derived $B_{1,i}(t)$ and saw that it was the familiar hat function.

5.2.a. Derive the formula of the quadratic B-spline $B_{2,i}(t)$ and determine its values and the values of $B'_{2,i}(t)$ and $B''_{2,i}(t)$ at the appropriate nodes.

5.2.b. Derive the formula of the quintic B-spline $B_{5,i}(t)$ and determine its values and the values of $B'_{5,i}(t)$ and $B''_{5,i}(t)$ at the appropriate nodes.

Solution:

Consider the quadratic B-spline. The generalization to m degree B-spline requires the use of $F_t(x) = (x-t)_+^2$ and $K(t) = \Delta^3 F_t(x_0)$. This in turn implies the use of the points x_0, x_1, x_2, x_3 to define each B-spline.

We have

$$\begin{gathered} K(t) = F_t(x_3) - 3F_t(x_2) + 3F_t(x_1) - F_t(x_0) \\ = (x_3-t)_+^2 - 3(x_2-t)_+^2 + 3(x_1-t)_+^2 - (x_0-t)_+^2 \\ t \le x_0 \to \Delta^3 F_t(x_0) \equiv 0 \\ x_0 \le t \le x_1 \to (x_3-t)^2 - 3(x_2-t)^2 + 3(x_1-t)^2 - 0 \\ x_1 \le t \le x_2 \to (x_3-t)^2 - 3(x_2-t)^2 + 0 - 0 \\ x_2 \le t \le x_3 \to (x_3-t)^2 - 0 + 0 - 0 \\ x_3 \le t \to 0 \end{gathered}$$

So we define

$$B_{2,i}(t) = \frac{1}{h^2}\Delta^3 F_t(x_i)$$

which is 0 for $t \le t_i$ and $t \ge t_{i+3}$ and piecewise quadratic in between with the maximum achieved at $t_{i+1} + (h/2)$ given that we have assumed uniform stepsize h. We have

$$\begin{gathered} t \le t_i \to B_{2,i}(t) \equiv 0 \\ t_i \le t \le t_{i+1} \to h^2 B_{2,i}(t) = (t_{i+3}-t)^2 - 3(t_{i+2}-t)^2 + 3(t_{i+1}-t)^2 \\ t_{i+1} \le t \le t_{i+2} \to h^2 B_{2,i}(t) = (t_{i+3}-t)^2 - 3(t_{i+2}-t)^2 \\ t_{i+2} \le t \le t_{i+3} \to h^2 B_{2,i}(t) = (t_{i+3}-t)^2 \\ t_{i+3} \le t \to B_{2,i}(t) \equiv 0 \end{gathered}$$

We can write this in terms of h and chosen endpoints of each interval. Note that there is no standard form for this choice. Here we follow the form we presented for the cubic B-spline.

$$\begin{gathered} t \le t_i \to B_{2,i}(t) \equiv 0 \\ t_i \le t \le t_{i+1} \to h^2 B_{2,i}(t) = h^2 - 2h(t_{i+1}-t) + (t_{i+1}-t)^2 \\ t_{i+1} \le t \le t_{i+2} \to h^2 B_{2,i}(t) = h^2 + 2h(t_{i+2}-t) - 2(t_{i+2}-t)^2 \\ t_{i+2} \le t \le t_{i+3} \to h^2 B_{2,i}(t) = (t_{i+3}-t)^2 \\ t_{i+3} \le t \to B_{2,i}(t) \equiv 0 \end{gathered}$$

We have the following values at the node points:

	t_i	t_{i+1}	t_{i+2}	t_{i+3}
$B_{2,i}(t)$	0	1	1	0
$B'_{2,i}(t)$	0	$2/h$	$-2/h$	0

It is useful to note the shape and location of the maximum of this curve. Roots are at t_i and t_{i+3}. It has the value 1 at t_{i+1} and t_{i+2}. Its maximum value is 3/2 at the midpoint $t_{i+1} + 0.5h$.

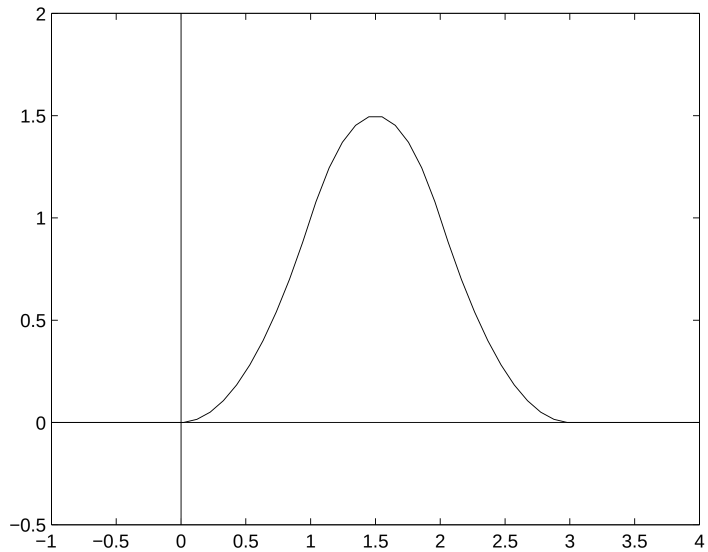

Figure 1: Quadratic B-spline basis function.

Since this is a quadratic spline, $B''_{2,i}(t)$ is not guaranteed to be continuous and therefore is not well-defined at these nodes. This is easily verified from the formulas.

The quintic B-spline requires the use of $F_t(x) = (x-t)^5_+$ and $K(t) = \Delta^6 F_t(x_0)$. This in turn implies the use of the points $x_0, x_1, x_2, x_3, x_4, x_5, x_6$ to define each B-spline.

We have

$$K(t) = F_t(x_6) - 6F_t(x_5) + 15F_t(x_4) - 20F_t(x_3) + 15F_t(x_2) - 6F_t(x_1) + F_t(x_0)$$

We define $B_{5,i}(t)$ by shifting the formulas to place $t_{i-3} \to x_0$, $t_{i-2} \to x_1$, $t_{i-1} \to x_2$, $t_i \to x_3$, $t_{i+1} \to x_4$, $t_{i+2} \to x_5$, $t_{i+3} \to x_6$ and scaling by h^{-5}.

$$h^5 B_{5,i}(t) = (t_{i+3}-t)_+^5 - 6(t_{i+2}-t)_+^5 + 15(t_{i+1}-t)_+^5 - 20(t_i-t)_+^5$$
$$+15(t_{i-1}-t)_+^5 - 6(t_{i-2}-t)_+^5 + (t_{i-3}-t)_+^5$$
$$t_{i-3} \le t \le t_{i-2} \to h^5 B_{5,i}(t) = (t_{i+3}-t)^5 - 6(t_{i+2}-t)^5 + 15(t_{i+1}-t)^5 - 20(t_i-t)^5$$
$$+15(t_{i-1}-t)^5 - 6(t_{i-2}-t)^5 + 0$$
$$t_{i-2} \le t \le t_{i-1} \to h^5 B_{5,i}(t) = (t_{i+3}-t)^5 - 6(t_{i+2}-t)^5 + 15(t_{i+1}-t)^5 - 20(t_i-t)^5 + 15(t_{i-1}-t)^5 + 0 + 0$$
$$t_{i-1} \le t \le t_i \to h^5 B_{5,i}(t) = (t_{i+3}-t)^5 - 6(t_{i+2}-t)^5 + 15(t_{i+1}-t)^5 - 20(t_i-t)^5 + 0 + 0 + 0$$
$$t_i \le t \le t_{i+1} \to h^5 B_{5,i}(t) = (t_{i+3}-t)^5 - 6(t_{i+2}-t)^5 + 15(t_{i+1}-t)^5 + 0 + 0 + 0 + 0$$
$$t_{i+1} \le t \le t_{i+2} \to h^5 B_{5,i}(t) = (t_{i+3}-t)^5 - 6(t_{i+2}-t)^5 + 0 + 0 + 0 + 0 + 0$$
$$t_{i+2} \le t \le t_{i+3} \to h^5 B_{5,i}(t) = (t_{i+3}-t)^5 + 0 + 0 + 0 + 0 + 0 + 0$$
$$\text{otherwise } h^5 B_{5,i}(t) = 0$$

This, of course, can be written in terms of h and chosen endpoints of each interval.
We have the following values at the node points:

	t_{i-3}	t_{i-2}	t_{i-3}	t_i	t_{i+1}	t_{i+2}	t_{i+3}
$B_{5,i}(t)$	0	1	26	66	26	1	0
$B'_{5,i}(t)$	0	$5/h$	$50/h$	0	$-50/h$	$-5/h$	0
$B''_{5,i}(t)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0

Problem 5.3

Consider a set of equidistant mesh points, $x_k = x_0 + kh$, $0 \le k \le m$.

5.3.a. Determine a cubic spline $b_i(x)$ that satisfies the following conditions:

$$b_i(x_j) = \begin{cases} 0 & \text{if } j < i-1 \text{ or } j > i+1 \\ 1 & \text{if } j = i \end{cases}$$
$$b_i'(x) = b_i''(x) = 0 \quad \text{for } x = x_{i-2} \text{ and } x = x_{i+2}$$

(For simplicity, you may assume that $2 < i < m-2$.)

5.3.b. Show that $b_i(x) = 0$ when $|x - x_i| \ge 2h$.

5.3.c. Show that $b_i(x) > 0$ when $|x - x_i| < 2h$.

5.3.d. What is the relationship between the spline, $b_i(x)$, and a B-spline?

Solution:

There are various ways to solve this problem. We follow an argument similar to that in the text by Prenter cited in the notes. First, however, consider the number of conditions imposed to verify that we have the correct number for an interpolatory cubic spline. We have

$$\begin{aligned} b_i(x_i) &= 1 \\ b_i(x_j) &= 0, \quad 0 \leq j \leq i-2 \\ b_i(x_j) &= 0, \quad i+2 \leq j \leq n \end{aligned}$$

Note **we do not require** $b_i(x_j) = 0$ for $j = i-1$ and $j = i+1$. Also, we have not specified two boundary conditions. So we have 4 conditions free. The four conditions that complete the definition are:

$$\begin{aligned} b_i'(x_{i-2}) &= 0 \\ b_i''(x_{i-2}) &= 0 \\ b_i'(x_{i+2}) &= 0 \\ b_i''(x_{i+2}) &= 0 \end{aligned}$$

To see that the resulting spline has local support, specifically,

$$b_i(x) = \begin{cases} 0 & \text{if } x \leq x_{i-2} \\ 0 & \text{if } x \geq x_{i+2} \end{cases}$$

we can consider the cubic polynomial, $p_{i-2}(x)$, that defines the spline on $[x_{i-3}, x_{i-2}]$. We have the following constraints

$$\begin{aligned} b_i(x_{i-2}) &= p_{i-2}(x_{i-2}) = 0 \\ b_i'(x_{i-2}) &= p_{i-2}'(x_{i-2})0 \\ b_i''(x_{i-2}) &= p_{i-2}''(x_{i-2}) = 0 \\ b_i(x_{i-3}) &= p_{i-2}(x_{i-3}) = 0 \end{aligned}$$

$p_{i-2}(x)$ has a second order root at x_{i-2} due to the first two conditions and a simple root at x_{i-3} due to the fourth condition. This means that we can express $p_{i-2}(x)$ in the form

$$p_{i-2}(x) = \rho(x - x_{i-2})^2(x - x_{i-3})$$

So to satisfy the final condition, $p_{i-2}''(x_{i-2}) = 0$, we must either increase the order of the polynomial by creating a third order root at x_{i-2} or set the remaining parameter ρ to 0 thereby making $p_{i-2}(x) = 0$ on the entire interval $[x_{i-3}, x_{i-2}]$. Since $p_{i-2}(x) = 0$ must be a cubic we must make it identically 0 on $[x_{i-3}, x_{i-2}]$. Given this it follows that

$$\begin{aligned} b_i(x_{i-3}) &= p_{i-3}(x_{i-3}) = 0 \\ b_i'(x_{i-3}) &= p_{i-3}'(x_{i-3})0 \\ b_i''(x_{i-3}) &= p_{i-3}''(x_{i-3}) = 0 \\ b_i(x_{i-4}) &= p_{i-3}(x_{i-4}) = 0 \end{aligned}$$

are constraints on the cubic $p_{i-3}(x_{i-4})$ that defines the spline on $[x_{i-4}, x_{i-3}]$. Repeating the same argument as above show that the polynomial and therefore the spline is identically 0 on the interval. This can be repeated on all intervals less than x_{i-2} and similarly for all intervals greater than x_{i+2} to show $b_i(x) = 0$ when $|x - x_i| \geq 2h$.

We can therefore simplify the problem very significantly by immediately defining

$$b_i(x) = \begin{cases} 0 & \text{if } x \leq x_{i-2} \\ 0 & \text{if } x \geq x_{i+2} \end{cases}$$

So **by construction** we have $b_i(x)$ identically 0 outside the interval (x_{i-2}, x_{i+2}) while the conditions above guarantee $b_i(x) \in \mathcal{C}^{(2)}$. The problem therefore simplifies to considering the definition of $b_i(x)$ on 4 intervals as cubic polynomials

$$b_i(x) = \begin{cases} p_{i-1}(x) & \text{if } x_{i-2} \leq x \leq x_{i-1} \\ p_i(x) & \text{if } x_{i-1} \leq x \leq x_i \\ p_{i+1}(x) & \text{if } x_i \leq x \leq x_{i+1} \\ p_{i+2}(x) & \text{if } x_{i+1} \leq x \leq x_{i+2} \end{cases}$$

The 4 cubic polynomials have 16 degrees of freedom, i.e., parameters, and we must therefore find 16 constraints from which we will derive 16 equations.

- The constraints that $b_i(x) \in \mathcal{C}^{(0)}$ and $b_i(x_i) = 1$ yield 2 equations:

$$p_i(x_i) = 1$$
$$p_{i+1}(x_i) = 1$$

- The constraints $b_i(x_{i-2}) = b_i'(x_{i-2}) = b_i''(x_{i-2}) = 0$ yield the 3 equations:

$$p_{i-1}(x_{i-2}) = 0$$
$$p_{i-1}'(x_{i-2}) = 0$$
$$p_{i-1}''(x_{i-2}) = 0$$

- The constraints $b_i(x_{i+2}) = b_i'(x_{i+2}) = b_i''(x_{i+2}) = 0$ yield the 3 equations:

$$p_{i+2}(x_{i+2}) = 0$$
$$p_{i+2}'(x_{i+2}) = 0$$
$$p_{i+2}''(x_{i+2}) = 0$$

The interpolation conditions therefore give us 8 equations. The continuity constraints must be used to generate the remaining 8 equations.

- The continuity of $b_i(x)$, $b_i'(x)$ and $b_i''(x)$ at x_{i-1} generates the 3 equations:

$$\begin{aligned} p_{i-1}(x_{i-1}) &= p_i(x_{i-1}) \\ p'_{i-1}(x_{i-1}) &= p'_i(x_{i-1}) \\ p''_{i-1}(x_{i-1}) &= p''_i(x_{i-1}) \end{aligned}$$

- The continuity of $b_i(x)$, $b_i'(x)$ and $b_i''(x)$ at x_{i+1} generates the 3 equations:

$$\begin{aligned} p_{i+1}(x_{i+1}) &= p_{i+2}(x_{i+1}) \\ p'_{i+1}(x_{i+1}) &= p'_{i+2}(x_{i+1}) \\ p''_{i+1}(x_{i+1}) &= p''_{i+2}(x_{i+1}) \end{aligned}$$

- The continuity of $b_i'(x)$ and $b_i''(x)$ at x_i generates the 2 equations:

$$\begin{aligned} p'_i(x_i) &= p'_{i+1}(x_i) \\ p''_i(x_i) &= p''_{i+1}(x_i) \end{aligned}$$

Therefore we have 16 equations and 16 unknowns.

We must choose a form of the cubic polynomials in terms of particular parameters. Inspired by the form of the B-spline given in the notes, we choose the following shifted monomial basis forms:

$$\begin{aligned} p_{i-1}(x) &= \alpha_{i-1}(x - x_{i-2})^3 + \beta_{i-1}(x - x_{i-2})^2 + \gamma_{i-1}(x - x_{i-2}) + \delta_{i-1} \\ p_i(x) &= \alpha_i(x - x_{i-1})^3 + \beta_i(x - x_{i-1})^2 + \gamma_i(x - x_{i-1}) + \delta_i \\ p_{i+1}(x) &= \alpha_{i+1}(x_{i+1} - x)^3 + \beta_{i+1}(x_{i+1} - x)^2 + \gamma_{i+1}(x_{i+1} - x) + \delta_{i+1} \\ p_{i+2}(x) &= \alpha_{i+2}(x_{i+2} - x)^3 + \beta_{i+2}(x_{i+2} - x)^2 + \gamma_{i+2}(x_{i+2} - x) + \delta_{i+2} \end{aligned}$$

Substituting the forms into the equations yields the system that must be solved to specify $b_i(x)$. Fortunately, 6 of the equations yield immediate simplification.

$$\begin{gathered} p_{i-1}(x_{i-2}) = 0 \rightarrow \delta_{i-1} = 0 \\ p_{i+2}(x_{i+2}) = 0 \rightarrow \delta_{i+2} = 0 \\ p'_{i-1}(x_{i-2}) = 0 \rightarrow \gamma_{i-1} = 0 \\ p'_{i+2}(x_{i+2}) = 0 \rightarrow \gamma_{i+2} = 0 \\ p''_{i-1}(x_{i-2}) = 0 \rightarrow \beta_{i-1} = 0 \\ p''_{i+2}(x_{i+2}) = 0 \rightarrow \beta_{i+2} = 0 \\ \Downarrow \\ p_{i-1}(x) = \alpha_{i-1}(x - x_{i-2})^3 \\ p_{i+2}(x) = \alpha_{i+2}(x_{i+2} - x)^3 \end{gathered}$$

The remaining 10 equations in 10 unknowns are:

$$
\begin{aligned}
p_i(x_i) = 1 &\rightarrow \boxed{\alpha_i h^3 + \beta_i h^2 + \gamma_i h + \delta_i = 1} \\
p_{i+1}(x_i) = 1 &\rightarrow \boxed{\alpha_{i+1} h^3 + \beta_{i+1} h^2 + \gamma_{i+1} h + \delta_{i+1} = 1} \\
p_{i-1}(x_{i-1}) = p_i(x_{i-1}) &\rightarrow \boxed{\alpha_{i-1} h^3 = \delta_i} \\
p_{i+1}(x_{i+1}) = p_{i+2}(x_{i+1}) &\rightarrow \boxed{\delta_{i+1} = \alpha_{i+2} h^3} \\
p'_{i-1}(x_{i-1}) = p'_i(x_{i-1}) &\rightarrow \boxed{3\alpha_{i-1} h^2 = \gamma_i} \\
p'_{i+1}(x_{i+1}) = p'_{i+2}(x_{i+1}) &\rightarrow \boxed{\gamma_{i+1} = 3\alpha_{i+2} h^2} \\
p''_{i-1}(x_{i-1}) = p''_i(x_{i-1}) &\rightarrow \boxed{6\alpha_{i-1} h = 2\beta_i} \\
p''_{i+1}(x_{i+1}) = p''_{i+2}(x_{i+1}) &\rightarrow \boxed{2\beta_{i+1} = 6\alpha_{i+2} h} \\
p'_i(x_i) = p'_{i+1}(x_i) &\rightarrow \boxed{3\alpha_i h^2 + 2\beta_i h + \gamma_i = -3\alpha_{i+1} h^2 - 2\beta_{i+1} h - \gamma_{i+1}} \\
p''_i(x_i) = p''_{i+1}(x_i) &\rightarrow \boxed{6\alpha_i h + 2\beta_i = 6\alpha_{i+1} h + 2\beta_{i+1}}
\end{aligned}
$$

Again using the B-spline as inspiration to guess a solution, it is easily verified that we have the following:

$$
\begin{aligned}
\alpha_{i-1} &= \frac{1}{4h^3}, \quad \beta_{i-1} = \gamma_{i-1} = \delta_{i-1} = 0 \\
\alpha_{i+2} &= \frac{1}{4h^3}, \quad \beta_{i+2} = \gamma_{i+2} = \delta_{i+2} = 0 \\
\alpha_{i+1} &= \frac{-3}{4h^3}, \quad \beta_{i+1} = \frac{3}{4h^2}, \quad \gamma_{i+1} = \frac{3}{4h^2}, \quad \delta_{i+1} = \frac{1}{4} \\
\alpha_i &= \frac{-3}{4h^3}, \quad \beta_i = \frac{3}{4h^2}, \quad \gamma_i = \frac{3}{4h^2}, \quad \delta_i = \frac{1}{4}
\end{aligned}
$$

The cubic spline $b_i(x)$ is $0.25B_i(x)$. This can also be seen to be consistent with the table of values for $B_i(x), B'_i(x), B''_i(x)$ given in the notes. The interpolation conditions are simply the table divided by 4.

It also follows from the definition on each interval that $b_i(x) > 0$ when $|x - x_i| < 2h$.