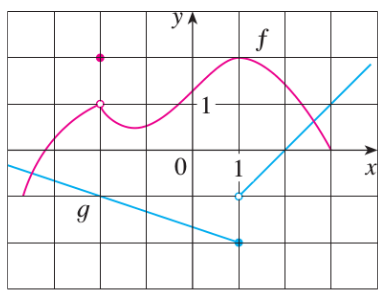
2.3: Calculating Limits Using Limit Laws

Limit Laws

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then the following *limit laws* are true

- $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
- $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \left(\lim_{x\to a} g(x) \neq 0\right)$
- $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer $\left(\lim_{x\to a} f(x) > 0 \text{ if } n \text{ is even}\right)$

Example 1. *Use the limit laws and the graphs of f and g below to evaluate the following limits, if they exist.*



(a) $\lim_{x\to -2}[f(x)+5g(x)]$

(b) $\lim_{x\to 1}[f(x)g(x)]$

(c) $\lim_{x\to 2} \frac{f(x)}{g(x)}$

(d) $\lim_{x\to 3^-} [f(x)+g(x)]^2$

Also, if f is *continuous* at a, that is, if the graph of f has no holes, jumps, essential discontinuities, or vertical asymptotes at a, then $\lim_{x\to a} f(x) = f(a)$.

Example 2. *Evaluate the following limits*

(a) $\lim_{x\to -1}(x^4-3x)(x^2+5x+3)$ *(b)* $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$. *(c)* $\lim_{x\to 2} \sqrt{\frac{2x^2+1}{3x-2}}$.

Example 3. *Evaluate the following limits*

(a) $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$ (b) $\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}$ (c) $\lim_{x\to -1} \frac{x^2+2x+1}{x^4-1}$.

Theorem 1. *$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.*

Example 4. *If*

$$f(x) = \begin{cases} -(x-2)^2+3 & \text{if } x \leq 2 \\ 8-2x & \text{if } 2 < x < 4 \\ \sqrt{x-4} & \text{if } x > 4 \end{cases}$$

evaluate the following limits, if they exist,

(a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 2} f(x)$

Theorem 2. *(Squeeze Theorem) If $f(x) \leq g(x) \leq h(x)$ for x in a neighborrhood of a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example 5. *Prove the following using the Squeeze Theorem.*

(a) $\lim_{x\to 0} x^2 \sin \frac{\pi}{x} = 0$.

(b) *If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x, then $\lim_{x\to 1} g(x) = 2$,*