MAC2311: Calculus 1 - Section 1

Test 1 January 29, 2015

Name: ______________________________

Answer each question in the space provided on the question sheets. If you run out of space for an answer, continue on the back of the page. Credit will only be given if you clearly show all of your work. Calculators may be used for this test.

Question	**Points**	**Score**
1	8	
2	10	
3	5	
4	13	
5	4	
6	4	
7	5	
8	9	
9	10	
Extra Credit	–	
Total:	68	

1. The height of an object thrown upward with an initial velocity of 10 meters per second and an initial height of 15 meters is given by $y(t) = -5t^2 + 10t + 15$.

   (a) [4 points] Complete the following table by evaluating y at the appropriate times. Round your answer to two decimal places.

t (min)	**y(t) = height (meters)**
2	15
1.5	18.75
1.1	19.95
1	20

   (b) [3 points] Complete the following table by calculating the average velocity of the object for each time interval. Round your answer to two decimal places.

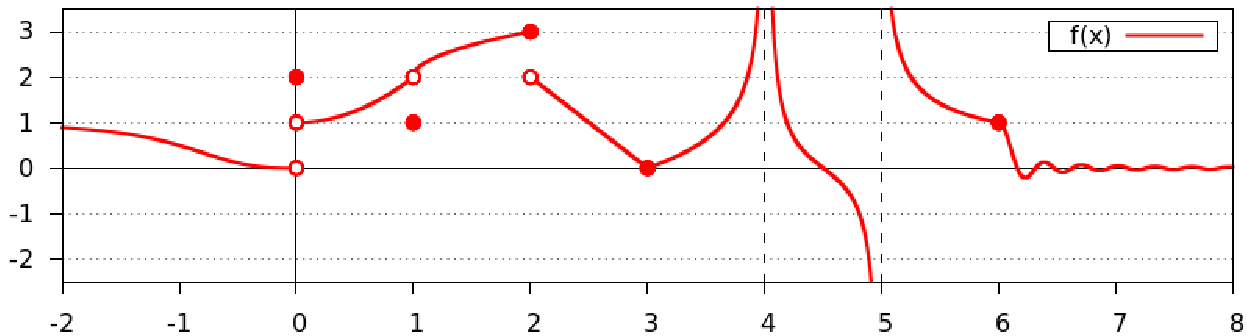
   The average velocity of the object for the time interval $[t_1, t_2]$ is

$$\frac{y(t_2) - y(t_1)}{t_2 - t_1}$$

time period (min)	**average velocity (m/s)**
$[1, 2]$	-5
$[1, 1.5]$	-2.5
$[1, 1.1]$	-0.5

   (c) [1 point] Use part (b) to approximate the instantaneous velocity of the object at $t = 1$: 0 meters/sec

2. [10 points] Use the following graph of $f(x)$ to answer the questions below it.



(a) $\lim_{x\to 0^-} f(x) = 0$

(b) $\lim_{x\to 1} f(x) = 2$

(c) $\lim_{x\to 5^-} f(x) = -\infty$

(d) $\lim_{x\to -\infty} f(x) = 1$

(e) $\lim_{x\to \infty} f(x) = 0$

(f) True or **False**: f is continuous from the right at 2.

(g) True or **False**: f is continuous on $(0, 2]$.

(h) **True** or False: f is continuous at 3.

(i) **True** or False: f is continuous on $\left[\frac{5}{2}, 4\right)$.

(j) **True** or False: f is continuous on $(6, \infty)$.

3. [5 points] Evaluate the following limits for f defined as

$$f(x) = \begin{cases} 2x+1 & x < -2 \\ -x^2+1 & -2 < x \leq 1 \\ \sqrt{x-1} & x > 1 \end{cases}.$$

(a) $\lim_{x\to -2^-} f(x) = 2(-2) + 1 = -3$

(b) $\lim_{x\to -2^+} f(x) = -(-2)^2 + 1 = -3$

(c) $\lim_{x\to -2} f(x) = -3$

(d) $\lim_{x\to 1^+} f(x) = \sqrt{1-1} = 0$

(e) $\lim_{x\to \infty} f(x) = \lim_{x\to\infty} \sqrt{x-1} = \infty$

4. You must clearly show all of your work for full credit. Evaluate the following limits by using limit laws.

(a) [4 points] $\lim_{x\to 1} \sqrt[3]{\frac{4x^3+3x+1}{18x^3+8x+1}} =$

$$\sqrt[3]{\lim_{x\to 1} \frac{4x^3+3x+1}{18x^3+8x+1}}$$
$$= \sqrt[3]{\frac{4(1)^3+3(1)+1}{18(1)^3+8(1)+1}}$$
$$= \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

(b) [4 points] $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} =$

$$\lim_{x\to 2} \frac{(x-1)(x-2)}{(x+3)(x-2)}$$
$$= \lim_{x\to 2} \frac{x-1}{x+3} = \frac{2-1}{2+3} = \frac{1}{5}$$

(c) [5 points] $\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{x+1} =$

$$\lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{x+1}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{\frac{x+1}{x}}$$
$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{1+\frac{1}{x}}$$
$$= \frac{\sqrt{2+\lim_{x\to\infty}\frac{1}{x^2}}}{1+\lim_{x\to\infty}\frac{1}{x}}$$
$$= \frac{\sqrt{2+0}}{1+0} = \sqrt{2}$$

5. Consider the limit $\lim_{x\to 0^-} e^{\frac{1}{x}}$ and let $t = \frac{1}{x}$.

(a) [2 points] Select the choice that correctly completes the following statement.

As x approaches 0 from the left,

A. t approaches ∞, so $\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to -\infty} e^t$.

B. t approaches $-\infty$, so $\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to -\infty} e^t$.

C. t approaches ∞, so $\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to \infty} e^t$.

D. t approaches $-\infty$, so $\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to \infty} e^t$.

(b) [2 points] Evaluate $\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to -\infty} e^t = 0$

6. [4 points] You must clearly show all of your work for full credit.

Use the squeeze theorem to show that $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$. (Hint: $-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$ for all $x > 0$.)

$$\begin{aligned} -\frac{1}{x} \le \frac{\sin x}{x} &\le \frac{1}{x} \quad \text{for } x > 0 \\ \lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} &\le \lim_{x\to\infty} \frac{1}{x} \\ 0 \le \lim_{x\to\infty} \frac{\sin x}{x} &\le 0 \end{aligned}$$

Therefore, $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$.

7. (a) [1 point] Fill in the blank to correctly complete the following definition.

   A function f is continuous at a if $\lim\limits_{x\to a} f(x) = \underline{f(a)}$

   (b) [1 point] Select the choice that correctly completes the following statement.

   The Intermediate Value Theorem states that if f is continuous on the interval $[a, b]$, then

   A. for every N in $(f(a), f(b))$ there exists c in (a, b) such that $f(c) = N$.

   B. for every c in $(f(a), f(b))$ there exists N in (a, b) such that $f(c) = N$.

   C. for every N in $(f(a), f(b))$ there exists c in (a, b) such that $f(N) = c$.

   D. there exists N in $(f(a), f(b))$ such that $f(N) = N$.

   (c) [3 points] Use the Intermediate Value Theorem to show that there is a root of the equation

$$x^3 - 3x^2 + 3x - 1 = 0$$

   on the interval $(0, 2)$.

Let $f(x) = x^3 - 3x^2 + 3x - 1$

$$\begin{aligned} &f(0) = -1 \quad \text{and} \\ &f(2) = 2^3 - 3 \cdot 2^2 + 3 \cdot 2 - 1 = 1. \end{aligned}$$

Since f is continuous, $f(0) < 0$, and $f(2) > 0$, by The Intermediate Value Theorem, there exists c in $(0, 2)$ such that $f(c) = 0$, that is, $x^3 - 3x^2 + 3x - 1 = 0$ has a root on the interval $(0, 2)$.

8. (a) [1 point] Write the limit definition of $f'(x) = \underline{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$

(b) [5 points] Use the limit definition of $f'(x)$ to find the derivative of $f(x) = \frac{1}{x}$.

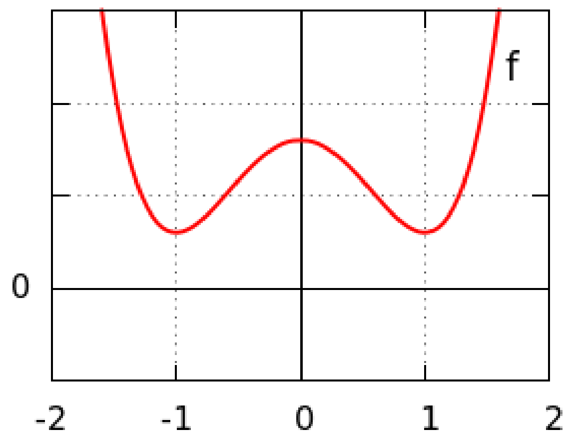
$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}
\end{aligned}$$

(c) [3 points] Find the equation of the line tangent to $f(x) = \frac{1}{x}$ at the point $P(1,1)$.

The slope of the line tangent to $f(x) = \frac{1}{x}$ at $x = 1$ is $f'(1) = -\frac{1}{1^2} = -1$. The equation of the line tangent to $f(x) = \frac{1}{x}$ at $(x_0, y_0) = (1,1)$ is

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - 1 &= -1(x-1) \\
y - 1 &= -x + 1 \\
x + y &= 2.
\end{aligned}$$

9. Use the graph of f below to answer the following questions.



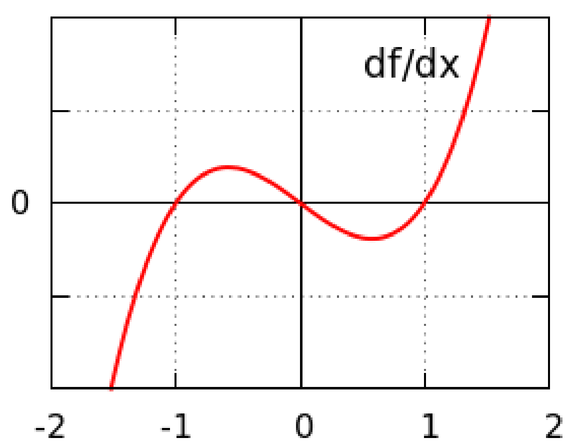
(a) [3 points] List all x-values at which the line tangent to f is horizontal: $\underline{x = -1, 0, 1}$

(b) [2 points] Select the choice that correctly completes the following statement.

The slopes of the lines tangent to f are

A. positive on $(0, \infty)$ and negative on $(-\infty, 0)$.
B. positive on $(-\infty, -1) \cup (1, \infty)$ and negative on $(-1, 1)$.
C. positive on $(-\infty, -1) \cup (0, 1)$ and negative on $(-1, 0) \cup (1, \infty)$.
D. positive on $(-1, 0) \cup (1, \infty)$ and negative on $(-\infty, -1) \cup (0, 1)$.

(c) [5 points] Sketch the graph of f'.



Extra Credit [2 points]: Is the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

continuous anywhere? If so, where? (Hint: Is there a number x such that for any sequence $\{x_n\}$ that approaches x the sequence $\{f(x_n)\}$ approaches $f(x)$?)

f is continous only at $x = 0$. For any other real number x, one can find a rational sequence of numbers $\{y_n\}$ and an irrational sequence of numbers $\{z_n\}$ that both approach x, but that $\{f(y_n)\}$ approaches x and $\{f(z_n)\}$ approaches 0, where $n = 1, 2, 3, \ldots$. For example, if $x \neq 0$ is rational, let $y_n = x + \frac{1}{n}$ and $z_n = x + \frac{\pi}{n}$; and if x is irrational let $\{y_n\} =$ (the decimal representation of x accurate to n digits) and $z_n = x + \frac{1}{n}$.

Another way to see that f is continuous at 0 is to notice $-|x| \leq f(x) \leq |x|$ for all x and apply the squeeze theorem as x approaches 0.