Numerical Analysis Qualifier

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INSTRUCTIONS: Do any (and only) 8 of the following 10 problems.

Throughout this exam $\|\cdot\|$ denotes the Euclidean vector norm or the associated induced matrix norm.

1. Floating-point arithmetic: By the periodicity of the cosine function, the following relation should hold for all integers k:
$$\cos\left(\frac{\pi}{4}+2k\pi\right)=\frac{1}{\sqrt{2}}.$$
Yet when the left-hand side is evaluated in MATLAB with k=10^8, we obtain
$$\left|\cos\left(\frac{\pi}{4}+2\cdot 10^8\pi\right)-\frac{1}{\sqrt{2}}\right|=8.2\cdot 10^{-8}.$$
Explain.

2. Stability: Let X and Y be normed vector spaces. Let $\tilde{f}: X\to Y$ denote an algorithm for the solution of the problem $f: X\to Y$.
   (a) What does it mean that the algorithm $\tilde{f}$ is *stable* for each $x\in X$?
   (b) What does it mean that the algorithm $\tilde{f}$ is *backward stable* for each $x\in X$?

3. Gaussian elimination: Consider the problem of solving $Ax=b$, where $A\in\mathbb{R}^{n\times n}$ is nonsingular and $x,b\in\mathbb{R}^n$, by Gaussian elimination with partial pivoting (GEPP).
   (a) Let $PA=LU$, where P is a permutation matrix. Let $\tilde{L}$ and $\tilde{U}$ be the computed factors by GEPP. Then
$$\tilde{L}\tilde{U}=PA+E,$$
where the matrix E satisfies
$$\frac{\|E\|}{\|L\|\|U\|}=\mathcal{O}(\varepsilon_{\text{machine}})$$
and $\varepsilon_{\text{machine}}$ denotes machine epsilon. Is GEPP backward stable? Justify your answer.

(b) Let $x = A^{-1}b$ and let $\tilde{x}$ be the approximate solution computed with GEPP. Give a bound for $\|x - \tilde{x}\|$. Justify your bound.

4. The singular value decomposition:

   (a) What is the singular value decomposition of a matrix $A \in \mathbb{R}^{m\times n}$, $m \geq n$? Describe the matrices involved.

   (b) Express the norm of A and the condition number of A in terms of the singular values of A.

   (c) Give the matrix of rank one that best approximates A.

5. Interpolation: The error formula for polynomial interpolation to a smooth function $f(t)$ by a polynomial of degree at most $n-1$ at n distinct points $t_1 < \cdots < t_n$ is given by

$$f(t) - P_{n-1}(t) = \frac{f^{(n)}(\theta(t))}{n!}(t - t_1)\cdots(t - t_n).$$

   (a) Prove the above formula.

   (b) What are the zeros of the Chebyshev polynomial T_n and what is their significance in the context of polynomial interpolation?

   (c) Let all nodes t_j live in the interval $[-1, 1]$ and let $f(t) = \exp(t)$. Will the interpolating polynomials P_{n-1} converge to f on the interval $[-1, 1]$ as the number of interpolation points is increased? Justify your answer.

6. Orthogonal Polynomials: Define the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)w(x)\,dx, \tag{1}$$

   where $w(x)$ is a positive weight function, and consider the family of orthogonal polynomials $p_0, p_1, p_2, \ldots$ with respect to this inner product. Here p_j is of degree j and has leading coefficient one.

   (a) Give the form of the three-term recurrence relation that these polynomials satisfy.

   (b) Prove the existence of the three-term recurrence relation. Give expressions for the coefficients in the recurrence relation.

7. Gauss quadrature: Let w be the weight function of the inner product (1).

(a) Describe the Gauss quadrature rule for approximating the integral

$$\int_a^b f(x)w(x)dx \approx \sum_{k=1}^{n} w_k f(x_k).$$

How are the nodes related to the orthogonal polynomials of problem 6?

(b) Prove that this rule is exact for all polynomials of degree strictly less than $2n$.

8. Newton's method: Let $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear differentiable function. The equation $f(x) = 0$ can be solved by Newton's method.

   (a) Describe Newton's method for the solution of $f(x) = 0$.

   (b) Define quadratic convergence of an iterative method.

   (c) Show that Newton's method yields quadratic convergence.

9. Gram-Schmidt Orthogonalization: Consider the $m \times n$ matrix A with linearly independent columns and $m \geq n$. Describe the modified Gram-Schmidt procedure for orthogonalizing the columns of A.

10. Projectors:

    (a) Give the defining equation for a projector P.

    (b) Show that $I - P$ also is a projector.

    (c) Show that $I - P$ projects onto the nullspace of P.

    (d) What is an orthogonal projector? How do the range and nullspace of an orthogonal projector relate?

    (e) Give a formula for the orthogonal projector onto the range of $A \in R^{m \times n}$, $m > n$. The matrix A is assumed to have linearly independent columns.