Homework 8 Foundations of Computational Math 2 Spring 2012

Solutions will be posted Monday, 3/12/12

Problem 8.1

This is not a programming assignment and you need not turn in any code. This problem considers the used of discrete least squares for approximation by a polynomial. Recall, the distinct points $x_0 < x_1 < \cdots < x_m$ are given and the metric

$$\sum_{i=0}^{m} \omega_i (f(x_i) - p_n(x_i))^2$$

with $\omega_i > 0$ is used to determine the polynomial, $p_n^*(x)$, of degree n that achieves the minimal value. We will assume $\omega_i = 1$ for this exercise. Typically, $m \gg n$. If $m = n$ then the unique interpolating polynomial is the solution.

If we let

$$p_n(x) = \sum_{j=0}^{n} \phi_j(x)\gamma_j$$

then the conditions are

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{pmatrix} - \begin{pmatrix} \phi_0(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \dots & \phi_n(x_1) \\ \vdots & & \vdots \\ \phi_0(x_m) & \dots & \phi_n(x_m) \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$
$$r = (b - Ag)$$

Use the Chebyshev polynomials to form an orthonormal basis, i.e.,

$$\phi_i(x) = \alpha_i T_i(x)$$

and the roots of $T_{m+1}(x)$ as the x_i.

1. Consider the i-th row of A. Show that row i can be determined by solving an $n+1 \times n+1$ system of linear equations. Also show that the matrix that determines this system has structure such that the system can be solved in $O(n)$ comptuations.

2. Use your solution to implement a code that assembles the least squares problem and make sure to exploit the algebraic properties of the matrix A to have an efficient solution.

3. Apply your code to several $f(x)$ choices and use multiple n and m values to explore the accuracy of the approximation. Approximate $\|f - p_n^*\|_\infty$ by sampling the difference between f and the polynomial at a large number of points in the interval and taking the maximum magnitude.

Problem 8.2

Consider the two quadrature formulas

$$I_2(f) = \frac{2}{3}\left[2f(-1/2) - f(0) + 2f(1/2)\right]$$
$$I_4(f) = \frac{1}{4}\left[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)\right]$$

- What is the degree of exactness when $I_2(f)$ is used to approximate $I(f)$ on $[-1, 1]$?
- What is the degree of exactness when $I_2(f)$ is used to approximate $I(f)$ on $[-1/2, 1/2]$?
- What is the degree of exactness when $I_4(f)$ is used to approximate $I(f)$ on $[-1, 1]$?

Problem 8.3

Consider the quadrature formula

$$I_0(f) = (b-a)f(a) \approx \int_a^b f(x)dx$$

- What is the degree of exactness?
- What is the order of infinitesimal?

Problem 8.4

Consider

$$\int_a^b \omega(x)f(x)dx \approx \alpha f(x_0)$$

where $\omega(x) = \sqrt{x}$ and $0 \le a < b$.

Determine α and x_0 such that the degree of exactness is maximized.