APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2011 (January)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ____________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: February 2nd, 2011

Time: 9:00 – 1:00 PM

A1.

a) Solve the following initial value problem

$$\begin{cases} u_x + 2u_y = u^2 \\ u(x,0) = h(x) \end{cases}$$

b) Draw the lines of characteristics of the Burger's equation

$$u_t + uu_x = 0$$

with the following initial condition

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}.$$

A2. Let the domain Ω be the unit dist in R^2. Solve the following elliptic boundary problem

$$u_{xx} + u_{yy} = 0$$

with the boundary condition

$$\frac{\partial u}{\partial n} = h$$

where $h = h(x, y)$ is a given continuous function, and n is the normal direction pointing outwards.

A3. Evaluate

$$\int_\gamma \frac{e^z + z}{z - 2} dz, \tag{1}$$

where γ is (i) the unit circle and (ii) a circle with radius 3 centered at 0.

A4. Show that the Riemann ζ function, defined by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \tag{2}$$

is analytic on the region $A = \{z | Re\ z > 1\}$. Write a convergent series for $\zeta'(z)$ on that set.

A5. Evaluate the integral.

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx \tag{3}$$

B6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\boldsymbol{x}) = x_1^2 + x_1 x_2 + 2x_2^2$.

a) Show the first step of the steepest descent method for minimizing $f(x)$ starting from $x_1 = 1$ and $x_2 = 1$.

b) Show the first step of the Newton's method for minimizing $f(x)$ starting from $x_1 = 1$ and $x_2 = 1$.

c) How many steps does it take for Newton's method to converge to machine precision for this problem?

B7. Consider the initial value problem

$$y' = f(t, y),\ y(0) = y_0.$$

a) Write the ODE in integral form and explain how to use the trapezoidal quadrature rule to derive the trapezoidal method with uniform time-step h:

$$y_{n+1} = y_n + \frac{h}{2}(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)),$$

b) Consider the case $f(t, y) = \lambda y$ with a real $\lambda < 0$. Show that the method is unconditionally stable.

B8. Assume that the discrete values of a smooth function $f(x)$ are given on a uniform mesh $x_i = i\Delta x, i = ..., -1, 0, 1, ...$ Denote $f_i = f(x_i) = f(i\Delta x)$.

a) Use the undetermined coefficient method to find the highest order finite difference scheme for approximating the first order derivative $f_i^{'} = f'(x_i) = \frac{df(x_i)}{dx}$ using the three points stencil: f_{i-1},f_i and f_{i+1}:

$$f_i^{'} = af_{i-1} + bf_i + cf_{i+1}$$

b) Use the undetermined coefficient method to find the highest order compact finite difference scheme for approximating the first order derivative $f'(x_i) = \frac{df(x_i)}{dx}$ using the following formula:

$$\alpha f_{i-1}^{'} + f_i^{'} + \beta f_{i+1}^{'} = af_{i-1} + bf_i + cf_{i+1}.$$

You are not required to give the final explicit expression for those parameters.

B9. Solve the following diffusion equation using an implicit scheme with second order accuracy in time and space ($O(\Delta t^2 + \Delta x^2)$):

$$\begin{cases} u_t = u_{xx} + f(x) \\ u(0, x) = u_0(x) \\ u(t, 0) = g(t) \\ u_x(t, 1) = h(t) \end{cases}$$

where $f(x)$,$u_0(x)$, $g(t)$ and $h(t)$ are given functions.

Assume that the computational domain is $x \in [0, 1]$. The grid length $\Delta x = 0.25$. Thus the 5 grid points are $x_i = i\Delta x, i = 0, 1, 2, 3, 4$. Assume that the function value $u_i^n = u(n\Delta t, i\Delta x)$ is given. Write down the formula for solving $u_i^{n+1}, i = 0, 1, 2, 3, 4$. Hint: you would not need $u_0(x)$ in your formula.

B10. Solve the Burger's equation

$$u_t + uu_x = 0$$

using the Godunov scheme. Derive the scheme in as detail as possible using finite volume method; explain what is the advantage of the method.