Solutions for Homework 11 Foundations of Computational Math 2 Spring 2012

Problem 11.1

Consider the Runge Kutta method called the implicit midpoint rule given by:

$$\hat{y}_1 = y_{n-1} + \frac{h}{2} f_1$$
$$f_1 = f(t_{n-1} + \frac{h}{2}, \hat{y}_1)$$
$$y_n = y_{n-1} + h f_1$$

An alternate form of the the method is given by:

$$y_n = y_{n-1} + h f\big(\frac{t_n + t_{n-1}}{2}, \frac{y_n + y_{n-1}}{2}\big)$$

Show that the two forms are identical.
Solution:

$$t_{n-1} + \frac{h}{2} = t_{n-1} + \frac{t_n - t_{n-1}}{2} = \frac{t_n + t_{n-1}}{2}$$
$$y_n = y_{n-1} + h f_1 \rightarrow f_1 = \frac{y_n - y_{n-1}}{h}$$
$$\hat{y}_1 = y_{n-1} + \frac{h}{2} f_1 = y_{n-1} + \frac{h}{2}\frac{y_n - y_{n-1}}{h} = \frac{y_n + y_{n-1}}{2}$$
$$y_n = y_{n-1} + h f(t_{n-1} + \frac{h}{2}, \hat{y}_1)$$
$$= y_{n-1} + h f(\frac{t_n + t_{n-1}}{2}, \frac{y_n + y_{n-1}}{2})$$

Problem 11.2

Consider the Runge Kutta method called the explicit trapezoidal rule given by:

$$\hat{y}_1 = y_{n-1} + h f(t_{n-1}, y_{n-1})$$
$$y_n = y_{n-1} + \frac{h}{2}\big(f(t_{n-1}, y_{n-1}) + f(t_n, \hat{y}_1)\big)$$

Show that the method has truncation error $O(h^2)$.
Solution:

$$
\begin{aligned}
d_n &= \frac{y(t_n) - y(t_{n-1})}{h} - \frac{1}{2}\big(f(t_{n-1}, y_{n-1}) + f(t_n, y(t_{n-1}) + hf(t_{n-1}, y(t_{n-1})))\big) \\
\frac{y(t_n) - y(t_{n-1})}{h} &= y'(t_{n-1}) + \frac{h}{2}y''(t_{n-1}) + \frac{h^2}{6}y'''(t_{n-1}) + O(h^3) \\
&= y' + \frac{h}{2}y'' + \frac{h^2}{6}y''' + O(h^3)
\end{aligned}
$$

where the argument is dropped any time it is at t_{n-1}. Similarly dropping the argument and letting subscripts indicate partial differentiation, we have

$$
f(t_n, y + hf) = f + f_t h + f_y f h + \frac{1}{2}\big(f_{tt}h^2 + f_{yy}f^2h^2 + 2f_{ty}fh^2\big) + O(h^3)
$$

Combining the two expressions yields

$$
\begin{aligned}
d_n &= y' + \frac{h}{2}y'' + \frac{h^2}{6}y''' - \big[f + f_t h + f_y f h + \frac{1}{2}\big(f_{tt}h^2 + f_{yy}f^2h^2 + 2f_{ty}fh^2\big)\big] + O(h^3) \\
&= y' + \frac{h}{2}y'' + \frac{h^2}{6}y''' - \frac{1}{2}\big[2f + h(f_t + f_y f) + \frac{h^2}{2}\big(f_{tt} + f_{yy}f^2 + 2f_{ty}f\big)\big] + O(h^3) \\
&= (y' - f) + \frac{h}{2}(y'' - f_t - f_y f) + h^2\big(\frac{1}{6}y''' - \frac{1}{4}f_{tt} - \frac{1}{4}f_{yy}f^2 - \frac{1}{2}f_{ty}f\big) + O(h^3) \\
&= h^2\big(\frac{1}{6}y''' - \frac{1}{4}f_{tt} - \frac{1}{4}f_{yy}f^2 - \frac{1}{2}f_{ty}f\big) + O(h^3) \\
&= O(h^2)
\end{aligned}
$$

We have used the identities

$$
\begin{aligned}
y' &= f \\
y'' &= f_t + f_y f \\
y''' &= f_{tt} + 2f_{ty}f + f_y f_t + f_{yy}f^2 + f_y^2 f
\end{aligned}
$$