2.5: Continuity

Continuity at a Point

A function f is **continuous from the left** at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

A function f is **continuous from the right** at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

Example 1. *(a)* $f(x) = \lfloor x \rfloor$ *(b)* $g(x) = \lceil x \rceil$

A function f is **continuous** at a if

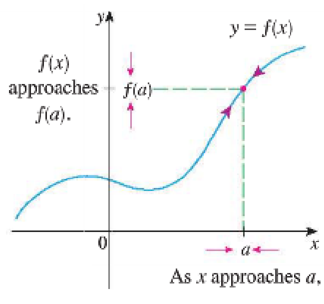
$$\lim_{x \to a} f(x) = f(a).$$

Notice this definition implicitly assumes that $\lim_{x \to a} f(x)$ and $f(a)$ exist.

Therfore, a function f is **discontinuous** at a if at least one of the following is true

- $\lim_{x \to a} f(x)$ does not exist,
- $f(a)$ does not exist, or
- $\lim_{x \to a} f(x) \neq f(a)$.

Simply, the graph of a continuous function can be drawn without lifting your pencil.



Example 2. *Find the discontinuities of the following functions, and state why each is discontinuous.*

(a) $f(x) = \tan(x)$ *(b)* $g(x) = \frac{1}{x^2}$ *(c)* $h(x) = \frac{(x+1)^2(x-1)}{x+1}$

Example 3. *Find each x-value at which f is dicontinuous and for each x-value, determine whether f is continuous from the right, from the left or neither.*

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

Continuity on an Interval

A function f is **continuous** on an *open* interval (a, b) where $a < b$, if f is continuous at every $c \in (a, b)$. Furthermore, f is continuous on the *closed* interval $[a, b]$ if f is continuous on (a, b), continuous from the right at a, and continuous from the left at b.

Example 4. *Show that* $f(x) = 1 - \sqrt{1 - x^2}$ *is continuous on the interval* $[-1, 1]$.

Example 5. *For what value of the constant* c *is the function* f *continuous on* $(-\infty, \infty)$*?*

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Example 6. *Find the intervals on which the following functions are continuous*

(a) $f(x) = \frac{\ln x + \arctan x}{x^2 - 1}$

(b) $g(x) = \ln(\sin^2 x)$

Theorem 1. *If* f *and* g *are continuous at* a *and* c *is a constant, then the following functions are continous at* a*:*

- $f + g$
- $f - g$
- cf
- fg
- $\frac{f}{g}$ $\quad (g(a) \neq 0)$

Example 7. *Suppose that* f *and* g *are continuous functions such that* $\lim\limits_{x \to 2} [3f(x) + f(x)g(x)] = 36$ *and* $g(2) = 6$. *Find* $f(2)$.

Theorem 2. *Any polynomial function is continuous on its domain* $\mathbb{R} = (-\infty, \infty)$. *Any rational function* $f = \frac{P}{Q}$, *for polynomials* P, Q, *is continuous on its domain* $\{x \in \mathbb{R} \mid Q(x) \neq 0\}$.

Additionally, the following types of functions are continuous on their domains

- root functions
- trigonometric
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Theorem 3. *If* $\lim\limits_{x \to a} g(x) = L$ *and* f *is continuous at* L, *then*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

Theorem 4. *If* g *is continuous at* a *and* f *is continuous at* $g(a)$, *then* $f \circ g$ *is continuous at* a. *That is,*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(g(a)).$$

Example 8. *Use continuity to evaluate the limits*

(a) $\lim\limits_{x \to \pi} \cos\left(x + \cos \frac{x}{2}\right)$

(b) $\lim\limits_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$

Theorem 5. *(Intermediate Value Theorem) If f is continuous on $[a, b]$, then for every $N \in (f(a), f(b))$ there exists $c \in (a, b)$ such that $f(c) = N$.*

Example 9. *Use the Intermediate Value Theorem to show that there is a root of the equation $\sqrt[3]{x} = 1 - x$ in the interval $(0, 1)$.*

Example 10. *Is there a number that is exactly 1 more than its cube?*