Set 11: Orthogonality and Approximation-Part 2

Kyle A. Gallivan

Department of Mathematics

Florida State University

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Practical Situations

- In $\mathcal{L}^2_\omega$, $\gamma_i = (f, \phi_i)_\omega$ must be computed either analytically or numerically.
- Various numerical quadrature methods will be discussed later and we will return to the Generalized Fourier Series (GFS)
- power series are often used and truncated to finite degree polynomials

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots \rightarrow p_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

- For each orthogonal polynomial family GFS yields optimality with respect to a specific norm.
- For a particular function, economization reduces the number of terms used to achieving similar accuracy in some norm independent of the various families of polynomials.
- Economization: change of basis plus truncation

Basis Change

- The GFS can be used to change the basis from monomials to an orthogonal basis or from one orthogonal to another.
- All inner products involve weighted integration of polynomials – analytically tractable.
- Coefficients can also be created via incremental algebraic manipulation
- When approximating $f(x)$ on $a \leq x \leq b$, a change of variables to the interval related to the inner product is done before economization and undone to get the final form of the approximation.

Basis Change

Consider monomials, Chebyshev and Legendre bases:

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

$$= \frac{(p, T_0)_\omega}{(T_0, T_0)_\omega} T_0(x) + \frac{(p, T_1)_\omega}{(T_1, T_1)_\omega} T_1(x) + \frac{(p, T_2)_\omega}{(T_2, T_2)_\omega} T_2(x) + \frac{(p, T_3)_\omega}{(T_3, T_3)_\omega} T_3(x)$$

$$= \frac{(p, P_0)_1}{(P_0, P_0)_1} P_0(x) + \frac{(p, P_1)_1}{(P_1, P_1)_1} P_1(x) + \frac{(p, P_2)_1}{(P_2, P_2)_1} P_2(x) + \frac{(p, P_3)_1}{(P_3, P_3)_1} P_3(x)$$

Monomial to Legendre: inner products

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$(P_0, P_0) = 2, \quad (P_1, P_1) = \frac{2}{3}, \quad (P_2, P_2) = \frac{2}{5}, \quad (P_3, P_3) = \frac{2}{7}$$

$$(1, P_0) = 2, \quad (1, P_1) = (1, P_2) = (1, P_3) = 0$$

$$(x, P_0) = 0, \quad (x, P_1) = \frac{2}{3}, \quad (x, P_2) = (x, P_3) = 0$$

$$(x^2, P_0) = \frac{2}{3}, \quad (x^2, P_1) = 0, \quad (x^2, P_2) = \frac{4}{15}, \quad (x^2, P_3) = 0$$

$$(x^3, P_0) = 0, \quad (x^3, P_1) = \frac{2}{5}, \quad (x^3, P_2) = 0, \quad (x^3, P_3) = \frac{4}{35}$$

Monomial to Legendre: inner products

Note $j > k \to (x^k, P_j) = 0$. The other 0's come from even/odd structure.

$$\gamma_0 = \frac{(p, P_0)}{(P_0, P_0)} = \frac{1}{2}\left(\alpha_0(1, P_0) + \alpha_1(x, P_0) + \alpha_2(x^2, P_0) + \alpha_3(x^3, P_0)\right)$$
$$\gamma_1 = \frac{(p, P_1)}{(P_1, P_1)} = \frac{3}{2}\left(\alpha_0(1, P_1) + \alpha_1(x, P_1) + \alpha_2(x^2, P_1) + \alpha_3(x^3, P_1)\right)$$
$$\gamma_2 = \frac{(p, P_2)}{(P_2, P_2)} = \frac{5}{2}\left(\alpha_0(1, P_2) + \alpha_1(x, P_2) + \alpha_2(x^2, P_2) + \alpha_3(x^3, P_2)\right)$$
$$\gamma_3 = \frac{(p, P_3)}{(P_3, P_3)} = \frac{7}{2}\left(\alpha_0(1, P_3) + \alpha_1(x, P_3) + \alpha_2(x^2, P_3) + \alpha_3(x^3, P_3)\right)$$

Monomial to Legendre: inner products

$$\gamma_0 = \frac{1}{2}\left(2\alpha_0 + 0\alpha_1 + \frac{2}{3}\alpha_2 + 0\alpha_3\right) = \alpha_0 + \frac{1}{3}\alpha_2$$

$$\gamma_1 = \frac{3}{2}\left(0\alpha_0 + \frac{2}{3}\alpha_1 + 0\alpha_2 + \frac{2}{5}\alpha_3\right) = \alpha_1 + \frac{3}{5}\alpha_3$$

$$\gamma_2 = \frac{5}{2}\left(0\alpha_0 + 0\alpha_1 + \frac{4}{15}\alpha_2 + 0\alpha_3\right) = \frac{2}{3}\alpha_2$$

$$\gamma_3 = \frac{7}{2}\left(0\alpha_0 + 0\alpha_1 + 0\alpha_2 + \frac{4}{35}\alpha_3\right) = \frac{2}{5}\alpha_3$$

Monomial to Legendre: incremental substitution

$$x^0 = P_0(x), \quad x = P_1(x)$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \to x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \to x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

$$= \alpha_0 P_0(x) + \alpha_1 P_1(x) + \alpha_2 \left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right) + \alpha_3 \left(\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right)$$

$$= \left(\alpha_0 + \frac{1}{3}\alpha_2\right) P_0(x) + \left(\alpha_1 + \frac{3}{5}\alpha_3\right) P_1(x) + \left(\frac{2}{3}\alpha_2\right) P_2(x) + \left(\frac{2}{5}\alpha_3\right) P_3(x)$$

$$p(x) = \gamma_0 P_0(x) + \gamma_1 P_1(x) + \gamma_2 P_2(x) + \gamma_3 P_3(x)$$

Economization of Power Series

Consider the Taylor series expansion approximating $\cos x$ on $[-1, 1]$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots \pm \frac{x^n}{n!} \mp \frac{x^{n+2}}{(n+2)!} \pm \cdots$$

$$|\cos x - p_n(x)| = |R_n(x)| = |\frac{x^{n+2}}{(n+2)!} \cos(\xi)|$$

$$|\cos x - p_n(x)| \leq \frac{1}{(n+2)!}$$

Truncate to get a polynomial approximation.

Economization of Power Series

Taylor series truncation yields:

$$|\cos x - p_2(x)| = |R_2(x)| \le \frac{1}{4!} \approx 0.04167$$
$$|\cos x - p_4(x)| = |R_4(x)| \le \frac{1}{6!} \approx 0.00139$$
$$|\cos x - p_8(x)| = |R_8(x)| \le \frac{1}{10!} \approx 2.76 \times 10^{-7}$$

We will start with $p_8(x)$ and attempt to get an error around 10^{-5} with fewer terms than the truncated Taylor approximations.

For Chebyshev Economization there are two equivalent views:

- Change of basis followed by truncation.
- Repeated minimax approximation of monomials.

Change of Basis to Chebyshev Polynomials

$$T_0 = 1, \quad T_1 = x$$

$$T_2 = 2x^2 - 1$$

$$T_3 = 4x^3 - 3x$$

$$T_4 = 8x^4 - 8x^2 + 1$$

$$T_5 = 16x^5 - 20x^3 + 5x$$

$$T_6 = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_7 = 64x^7 - 112x^5 + 56x^3 - 7x$$

$$T_8 = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

$$T_{n+1} = 2xT_n - T_{n-1}, \quad \|T_n(x)\|_\infty = 1, \;\; -1 \leq x \leq 1$$

Change of Basis to Chebyshev Polynomials

Using this basis to rewrite the monomials yields:

$$x^0 = T_0, \quad x^1 = T_1, \quad x^2 = \frac{1}{2}T_2 + \frac{1}{2}T_0, \quad x^3 = \frac{1}{4}T_3 + \frac{3}{4}T_1,$$

$$x^4 = \frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0, \quad x^5 = \frac{1}{16}T_5 + \frac{5}{16}T_3 + \frac{5}{8}T_1$$

$$x^6 = \frac{1}{32}T_6 + \frac{3}{16}T_4 + \frac{15}{32}T_2 + \frac{5}{16}T_0$$

$$x^7 = \frac{1}{64}T_7 + \frac{7}{64}T_5 + \frac{7}{18}T_3 + \frac{35}{64}T_0$$

$$x^8 = \frac{1}{128}T_8 + \frac{1}{16}T_6 + \frac{7}{32}T_4 + \frac{7}{16}T_2 + \frac{35}{128}T_0$$

Note. As before, this can also be done via the GFS coefficients.

Change of Basis to Chebyshev Polynomials

$$
\begin{aligned}
p_8(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \frac{x^8}{8!} \\
&= T_0 - \frac{1}{2}\left(\frac{1}{2}T_2 + \frac{1}{2}T_0\right) + \frac{1}{24}\left(\frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0\right) \\
&\quad - \frac{1}{6!}\left(\frac{1}{32}T_6 + \frac{3}{16}T_4 + \frac{15}{32}T_2 + \frac{5}{16}T_0\right) \\
&\quad + \frac{1}{8!}\left(\frac{1}{128}T_8 + \frac{1}{16}T_6 + \frac{7}{32}T_4 + \frac{7}{16}T_2 + \frac{35}{128}T_0\right) \\
&= 0.76519775T_0 - 0.22980686T_2 + 0.0049533419T_4 \\
&\quad -4.185265 \times 10^{-5}T_6 + 1.937624 \times 10^{-7}T_8
\end{aligned}
$$

Truncation and Error Bounds

We have in terms of a polynomial of degree 4 and an error:

$$\cos x = 0.76519775 T_0 - 0.22980686 T_2 + 0.0049533419 T_4 + E(x)$$

$$= C_4(x) + E(x)$$

$$E(x) = R_8(x) - 4.185265 \times 10^{-5} T_6(x) + 1.937624 \times 10^{-7} T_8(x)$$

Truncation and Error Bounds

Since we know a bound on the Taylor truncation error for $p_8(x)$ and $\|T_n(x)\|_\infty = 1$ on $-1 \le x \le 1$ we have the error bound:

$$|E(x)| = |R_8(x) - 4.185265 \times 10^{-5} T_6(x) + 1.937624 \times 10^{-7} T_8(x)|$$

$$|E(x)| \le 2.76 \times 10^{-7} + 4.185265 \times 10^{-5} + 1.937624 \times 10^{-7}$$

$$|E(x)| \le 5.0 \times 10^{-5} \quad -1 \le x \le 1$$

Error Comparison

$$\cos x \approx C_4(x)$$

$$C_4(x) = 0.76519775T_0(x) - 0.22980686T_2(x) + 0.0049533419T_4(x)$$

$$= 0.99995795 - 0.49924045x^2 + 0.03962674x^4$$

- Either form of $C_4(x)$ can be used.
- Efficient algorithms, e.g., Clenshaw's recurrence, exist for evaluating linear combinations of orthogonal polynomials at a given value of x.

Error Comparison

Recall, truncation of the Taylor series yielded

$$p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$|\cos x - p_4(x)| = |R_4(x)| \le \frac{1}{6!} \approx 1.4 \times 10^{-3}$$

$$|\cos x - C_4(x)| = |E(x)| \le 5.0 \times 10^{-5}$$

So Chebyshev Economization yields a quartic polynomial that has two orders of magnitude better error than the quartic Taylor in the ∞ norm on $-1 \le x \le 1$.

Chebyshev Economization Redux

Recall, that the monic forms, $t_n(x)$ of $T_n(x)$ have the minimum deviation from $g(x) \equiv 0$ on $[-1, 1]$.

Writing the monic form as

$$t_{n+1}(x) = x^{n+1} - q_n(x) \to q_n(x) = x^{n+1} - t_{n+1}(x)$$

where $q_n(x)$ is a good approximation of x^{n+1} by a polynomial of degree n or less and $t_{n+1}(x)$ is the error with minimal maginitude by construction.

Chebyshev Economization Redux

Consider the expansion approximating $\cos x$ on $[-1, 1]$

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$$p_2(x) = 1 - \frac{x^2}{2} \to \|\cos x - p_2(x)\|_\infty = |R_2(x)| \leq 0.042$$

$$p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} \to \|\cos x - p_4(x)\|_\infty = |R_4(x)| \leq 0.0014$$

Find a quadratic approximation that is better than $p_2(x)$.

Chebyshev Economization Redux

$$t_4(x) = \frac{1}{8}T_4(x) = x^4 - x^2 + \frac{1}{8}$$

$$q_2(x) = x^4 - t_4(x) = x^2 - \frac{1}{8} \approx x^4$$

$$p_4(x) = \tilde{p}_2(x) + \text{error} = 1 - \frac{x^2}{2} + \left[\frac{x^2}{24} - \frac{1}{192}\right] + \frac{1}{24}t_4(x)$$

$$\tilde{p}_2(x) = (1 - \frac{1}{192}) - x^2(\frac{1}{2} - \frac{1}{24}) = \frac{191}{192} - \frac{11}{24}x^2$$

$$= 0.99479 - 0.45833x^2$$

Error Comparison

$$\|\cos x - p_2(x)\|_\infty < 0.042$$

$$\|\cos x - p_4(x)\|_\infty < 0.0014$$

$$\|\cos x - \tilde{p}_2(x)\|_\infty < \|\cos x - p_4(x)\|_\infty + \frac{1}{24}\|t_4(x)\|_\infty$$

$$= 0.0014 + \frac{1}{24} \times \frac{1}{8} \approx 0.007$$

- A quadratic with an order of magnitude improvement over the quadratic $p_2(x)$
- Within a factor of ≈ 2 of the quartic $p_4(x)$.

Equivalence

We can verify for this example the equivalence of the two approaches:

$$
\begin{aligned}
p_4(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \\
&= T_0 - \frac{1}{2}\left(\frac{1}{2}T_2 + \frac{1}{2}T_0\right) \\
&+ \frac{1}{24}\left(\frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0\right) \\
\tilde{p}_2(x) &= \frac{49}{64} - \frac{11}{48}T_2 \\
&= \frac{49}{64} - \frac{11}{48}(2x^2 - 1) \\
&= \frac{49}{64} + \frac{11}{48} - \frac{11}{24}x^2 \\
&= \frac{191}{192} - \frac{11}{24}x^2.
\end{aligned}
$$

Economization Summary

- Start with a power series or some other basis expansion on $[a, b]$:

$$f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$$
$$f(x) = \beta_0 B_0(x) + \beta_1 B_1(x) + \beta_2 B_2(x) + \dots$$

- Choose the family of orthogonal polynomials and note the associated inner product and interval $[r, s]$.
- Change variables

$$p_n(x) = q_n(z) = \nu_0 + \nu_1 z + \cdots + \nu_n z^n$$
$$p_n(x) = q_n(z) = \tau_0 B_0(z) + \tau_1 B_1(z) + \cdots + \tau_n B_n(z)$$
$$a \le x \le b \quad \text{and} \quad r \le z \le s$$

Economization Summary

- Truncate using a large enough number of terms so that the remainder is acceptably small – this initial error bound term will remain after economization.

$$f(z) = p_n(z) + R(z)$$

- Change the basis to the chosen orthogonal polynomials.

$$q_n(z) = \gamma_0 P_0(z) + \gamma_1 P_1(z) + \cdots + \gamma_n P_n(z)$$

Economization Summary

- Truncate, bound error and add to initial error for final error bound.

$$q_n(z) = \gamma_0 P_0(z) + \gamma_1 P_1(z) + \cdots + \gamma_n P_n(z)$$

$$q_d(z) = \gamma_0 P_0(z) + \gamma_1 P_1(z) + \cdots + \gamma_d P_d(z), \quad d < n$$

$$q_n(z) = q_d(z) + E(z)$$

- Change variables back to original $z \rightarrow x$ for final approximation

$$q_d(z) \rightarrow f_d(x) \approx f(x)$$