Numerical Analysis Qualifier

prepared by
Jing Li & Lothar Reichel
Kent State University
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INSTRUCTIONS: Do any (and only) 8 of the following 10 problems.

1. (Rounding-Error Analysis) Consider a computer with four digits precision where any real number $x = 1.d_1d_2d_3d_4d_5d_6 \cdots \times 10^E$ is represented by its floating point number $fl(x) = 1.d_1d_2d_3d_4 \times 10^E$.

   (a) Determine the value $100.0 \oplus 0.001$, where $\oplus$ represents the addition implemented on this computer.

   (b) The exact solution of

   $$\begin{bmatrix} 0.001 & 100.0 \\ 100.0 & 100.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 100.0 \\ 0.0 \end{bmatrix} \qquad (1)$$

   is $[x_1, x_2] = [-1.0, 1.0]$ (up to the machine epsilon).
   Solve (1) by implementing *Gaussian Elimination* without pivoting on the above mentioned computer. What is the error of the solution in Euclidean norm? What is its relative error?
   Solve (1) by implementing *Gaussian Elimination* with pivoting.

2. (Singular Value Decomposition) Given a matrix $A \in \mathbb{R}^{n\times n}$,

   (a) Describe the Singular Value Decomposition of A.

   (b) Let σ_1 be the largest singular value of A. Prove that $\|A\|_2 = \sigma_1$, where $\|A\|_2$ is defined by

   $$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

3. (Rayleigh Quotient) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. The *Rayleigh Quotient* associated with A is the function defined by

   $$r(x) := \frac{x^TAx}{x^Tx}, \quad x \in \mathbb{R}^n.$$

   (a) Prove that $\|Ax - r(x)x\|_2 = \min_{\mu\in\mathbb{R}} \|Ax - \mu x\|_2$.

(b) Assume that A is positive definite. Denote its minimum and maximum eigenvalues by $\lambda_{\min}$ and $\lambda_{\max}$. Prove that

$$\lambda_{\min} = \min_{x \neq 0} \frac{x^T A x}{x^T x}, \qquad \lambda_{\max} = \max_{x \neq 0} \frac{x^T A x}{x^T x}.$$

4. (Arnoldi Iteration) Given a matrix $A \in \mathbb{R}^{m \times m}$ and a column vector $q_1 \in \mathbb{R}^m$ with $\|q_1\|_2 = 1$,

   (a) Present an algorithm which generates a sequence of orthonormal vectors $q_1, q_2, \ldots$ and an upper Hessenberg matrix $\widetilde{H}_n \in \mathbb{R}^{(n+1) \times n}$ (where $n < m$) such that

$$A \left[\begin{array}{c|c|c} q_1 & \cdots & q_n \end{array} \right] = \left[\begin{array}{c|c|c} q_1 & \cdots & q_{n+1} \end{array} \right] \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ h_{21} & & h_{2n} \\ & \ddots & \vdots \\ 0 & & h_{n+1,n} \end{bmatrix}.$$

   (b) Prove that $\langle q_1, Aq_1, \ldots, A^n q_1 \rangle = \langle q_1, q_2, \ldots, q_{n+1} \rangle$.

5. (GMRES Algorithm) Given a matrix $A \in \mathbb{R}^{m \times m}$ and a column vector $b \in \mathbb{R}^m$, using the result in Question 5, present an iteration which solves

$$\min_{x \in \langle b, Ab, \ldots, A^{n-1} b \rangle} \|Ax - b\|_2,$$

   at the n-th iteration.

6. (Polynomial Interpolation) Given a set of points (x_i, f_i), $i = 0, 1, \ldots, n$, where $x_0 < x_1 < \ldots < x_n$, and $f_i = f(x_i)$, let $p(x)$ be the interpolation polynomial of degree less or equal than n, such that $p(x_i) = f_i$, $i = 0, 1, \ldots, n$. Assume that $f^{(n+1)}(x)$ is continuous. Prove that for any $x \in [x_0, x_n]$, there exists $\xi \in [x_0, x_n]$ such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)(x - x_1) \ldots (x - x_n).$$

7. (Trigonometric Interpolation) Given a sequence of points (x_k, f_k), $k = 0, 1, \ldots, N-1$, where $x_k = 2\pi k / N$,

   (a) Prove that there exists a unique *phase polynomial* of the form

$$p(x) = \sum_{j=0}^{N-1} \beta_j e^{ijx},$$

   where i denotes the imaginary unit, such that $p(x_k) = f_k$, for $k = 0, 1, \ldots, N-1$.

(b) Prove that the coefficients β_j in (a) are determined by

$$\beta_j = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-2\pi i jk/N}, \quad j = 0, 1, ..., N-1.$$

8. (Peano Kernel Theorem) The Peano Kernal Theorem states that if a functional $R(f)$ satisfies $R(P) = 0$ for all polynomials P of degree less or equal than n, then for all functions $f \in C^{n+1}[a, b]$,

$$R(f) = \int_a^b f^{(n+1)}(t)K(t)dt,$$

where $K(t) = R_x[(x-t)_+^n]/\text{n}!$ and $R_x[(x-t)_+^n]$ represents the application of R on $(x-t)_+^n$ considered as a function of x.

Using the Peano Kernel Theorem, prove that for any $f(x) \in C^2[a, b]$, there exists $\xi \in [a, b]$, such that

$$\frac{b-a}{2}(f(a) + f(b)) - \int_a^b f(x)dx = \frac{(b-a)^3}{12} f''(\xi).$$

(*Hint: Take* $R(f) = \frac{b-a}{2}(f(a) + f(b)) - \int_a^b f(x)dx$ *in the proof.*)

9. (Gauss Quadrature) Let $p_j(x) \in \{p \mid p(x) = x^j + a_1 x^{j-1} + \cdots + a_j\}$, $j = 0, 1, ..., n$, be a set of orthogonal polynomials with respect to the inner product

$$(f, g) = \int_a^b \omega(x) f(x) g(x) dx,$$

where $\omega(x)$ is a nonnegative smooth function. Let $x_1, x_2, ..., x_n$ be the distinct roots of $p_n(x)$, and $w_1, w_2, ..., w_n$ be determined by

$$\begin{bmatrix} p_0(x_1) & p_0(x_2) & \dots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \dots & p_1(x_n) \\ & & \dots & \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \dots & p_{n-1}(x_n) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} (p_0, p_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Prove that

$$\int_a^b \omega(x) p(x) dx = \sum_{i=1}^n w_i p(x_i)$$

hold for all polynomials $p(x)$ of degree less or equal than $2n-1$.

10. (Nonlinear Equations) Describe the Newton's iteration for solving

$$e^{-x} = x.$$

Prove that it is convergent starting from $x_0 = 1$, and determine its rate of convergence.