Set 24: Rational Interpolation – Part 2

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Inverse Differences

Definition 24.1. Given points (x_i, f_i) the inverse differences are defined as

$$\phi(x_i, x_j) = \frac{(x_i - x_j)}{f_i - f_j}$$

$$\phi(x_i, x_j, x_k) = \frac{(x_j - x_k)}{\phi(x_i, x_j) - \phi(x_i, x_k)}$$

$$\phi(x_i, \ldots, x_s, x_m, x_n) = \frac{(x_m - x_n)}{\phi(x_i, \ldots, x_s, x_m) - \phi(x_i, \ldots, x_s, x_n)}$$

Note. Values of ∞ can result.

Continued Fraction and Inverse Differences

$$r_{nn}(x) = \frac{p_n(x)}{q_n(x)}, \quad r_{nn}(x_i) = f_i,\ 0 \le i \le 2n$$

$$r_{00}(x) = f_0,\ \ r_{10}(x) = f_0 + \frac{(x - x_0)}{\phi(x_0, x_1)}$$

$$r_{11}(x) = f_0 + \cfrac{(x - x_0)}{\phi(x_0, x_1) + \cfrac{(x - x_1)}{\phi(x_0, x_1, x_2)}}$$

$$r_{21}(x) = f_0 + \cfrac{(x - x_0)}{\phi(x_0, x_1) + \cfrac{(x - x_1)}{\phi(x_0, x_1, x_2) + \cfrac{(x - x_2)}{\phi(x_0, x_1, x_2, x_3)}}}$$

Continued Fraction and Inverse Differences

$$r_{22}(x) = f_0 + \cfrac{(x - x_0)}{\phi(x_0, x_1) + \cfrac{(x - x_1)}{\phi(x_0, x_1, x_2) + \cfrac{(x - x_2)}{\phi(x_0, x_1, x_2, x_3) + \cfrac{(x - x_3)}{\phi(x_0, x_1, x_2, x_3, x_4)}}}}$$

General Form

Given the inverse differences we have:

$$r_{nn}(x) = f_0 + \cfrac{(x - x_0)}{\phi(x_0, x_1) + \cfrac{(x - x_1)}{\phi(x_0, x_1, x_2) + \cfrac{(x - x_2)}{\phi(x_0, x_1, x_2, x_3) + \cdots}}}$$

$$\vdots$$

$$+ \frac{(x - x_{2n-1})}{\phi(x_0, \ldots, x_{2n})}$$

Continued Fraction and Inverse Differences

The expression can be evaluated in a Horner's rule-like fashion, e.g.,

$r_{21}(x)$: initialize $\tau = 0$

$$\tau = \tau + \phi(x_0, x_1, x_2, x_3, x_4) \rightarrow \tau = \frac{(x - x_3)}{\tau}$$

$$\tau = \tau + \phi(x_0, x_1, x_2, x_3) \rightarrow \tau = \frac{(x - x_2)}{\tau}$$

$$\tau = \tau + \phi(x_0, x_1, x_2) \rightarrow \tau = \frac{(x - x_1)}{\tau}$$

$$\tau = \tau + \phi(x_0, x_1) \rightarrow \tau = \frac{(x - x_0)}{\tau}$$

$$r_{21}(x) = f_0 + \tau$$

Inverse Differences

i	x_i	f_i	$\phi(x_0, x_i)$	$\phi(x_0, x_1, x_i)$	$\phi(x_0, x_1, x_2, x_i)$
0	0	$\boxed{0}$			
1	1	-1	$\boxed{-1}$		
2	2	-2/3	-3	$\boxed{-1/2}$	
3	3	9	1/3	3/2	$\boxed{1/2}$

Given this data we can build up to $r_{21}(x)$.

Continued Fraction and Inverse Differences

$$r_{10}(x) = 0 + \frac{(x-0)}{(-1)}$$
$$= -x$$

$$r_{10}(0) = 0, \;\; r_{10}(1) = -1, \;\; r_{10}(2) = -2 \neq -\frac{2}{3}, \;\; r_{10}(3) = -3 \neq 9$$

Continued Fraction and Inverse Differences

$$r_{11}(x) = 0 + \cfrac{(x-0)}{(-1) + \cfrac{(x-1)}{(-1/2)}}$$

$$= \frac{x}{-2x+1}$$

$$r_{11}(0) = 0, \;\; r_{11}(1) = -1, \;\; r_{11}(2) = -\frac{2}{3}, \;\; r_{11}(3) = -\frac{3}{5} \neq 9$$

Continued Fraction and Inverse Differences

$$r_{21}(x) = 0 + \cfrac{(x-0)}{(-1) + \cfrac{(x-1)}{(-1/2) + \cfrac{(x-2)}{(1/2)}}}$$

$$= \frac{4x^2 - 9x}{-2x + 7}$$

$$r_{21}(0) = 0, \;\; r_{21}(1) = -1, \;\; r_{21}(2) = -\frac{2}{3}, \;\; r_{21}(3) = 9$$

Consistency Check

$$\begin{pmatrix} 1 & x_0 & x_0^2 & -f_0 & -f_0 x_0 \\ 1 & x_1 & x_1^2 & -f_1 & -f_1 x_1 \\ 1 & x_2 & x_2^2 & -f_2 & -f_2 x_2 \\ 1 & x_3 & x_3^2 & -f_3 & -f_3 x_3 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2/3 & 4/3 \\ 1 & 3 & 9 & -9 & -27 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \\ 4 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Inverse Differences

i	x_i	f_i	$\phi(x_0, x_i)$	$\phi(x_0, x_1, x_i)$	$\phi(x_0, x_1, x_2, x_i)$
0	0	$\boxed{0}$			
1	1	-1	$\boxed{-1}$		
2	2	-2/3	-3	$\boxed{-1/2}$	
3	3	0	∞	0	$\boxed{2}$

Given this data we can build up to $r_{21}(x)$.

Continued Fraction and Inverse Differences

Only $r_{21}(x)$ changes from the previous example:

$$r_{21}(x) = 0 + \cfrac{(x-0)}{(-1) + \cfrac{(x-1)}{(-1/2) + \cfrac{(x-2)}{(2)}}}$$

$$= \frac{x^2 - 3x}{x+1}$$

$$r_{21}(0) = 0, \;\; r_{21}(1) = -1, \;\; r_{21}(2) = -\frac{2}{3}, \;\; r_{21}(3) = 0$$

Reciprocal Differences

In practice, the related quantities, reciprocal differences often are used along with Thiele's continued fraction representation. (see http://www.eecs.berkeley.edu/ wkahan/Math128/THIELEDF)

Definition 24.2. Given $(x_0, f_0),\ (x_1, f_1) \ldots$, the reciprocal differences are:

$$\rho(x_i) = f_i, \quad \rho(x_i, x_k) = \frac{(x_i - x_k)}{(f_i - f_k)},$$

$$\rho(x_i, x_{i+1}, \ldots, x_{i+k}) = \frac{(x_i - x_{i+k})}{\rho(x_i, x_{i+1}, \ldots, x_{i+k-1}) - \rho(x_{i+1}, \ldots, x_{i+k})}$$

Lemma. *If* $\rho(x_0, x_1, \ldots, x_{p-2}) := 0$ *for* $p = 1$ *then*

$$\phi(x_0, x_1, \ldots, x_p) = \rho(x_0, x_1, \ldots, x_p) - \rho(x_0, x_1, \ldots, x_{p-2})$$

Thiele's Form

Given the reciprocal differences we have:

$$r_{nn}(x) = f_0 + \cfrac{(x - x_0)}{\rho(x_0, x_1) + \cfrac{(x - x_1)}{\rho(x_0, x_1, x_2) - \rho(x_0) + \cfrac{(x - x_2)}{\rho(x_0, x_1, x_2, x_3) - \rho(x_0, x_1) + \cdots}}}$$

$$\vdots$$

$$+ \frac{(x - x_{2n-1})}{\rho(x_0, \ldots, x_{2n}) - \rho(x_0, \ldots, x_{2n-2})}$$

Single Point Rational Approximation

$$f(x) = \phi_0 + \phi_1 x + \phi_2 x^2 + \dots$$

$$r_{nm}(x) = \frac{p_n(x)}{q_m(x)} = \frac{\beta_0 + \beta_1 x + \cdots + \beta_m x^m}{\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n}$$

$$E(x) = f(x) - r_{nm}(x) = \sum_{j=0}^{\infty} \epsilon_j x^j$$

Problem 24.1. Given n and m, let $N = n + m$, find $p_n(x)$ and $q_m(x)$ such that $\epsilon_j = 0$ for $0 \leq j \leq N$.

Called momement matching or more usually Padé approximation.

Single Point Rational Approximation

The solution satisfies:

$$q_m(x)f(x) - p_n(x) = q_m(x)E(x)$$

$$(\beta_0 + \beta_1 x + \cdots + \beta_m x^m)(\phi_0 + \phi_1 x + \phi_2 x^2 + \ldots) - (\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n)$$

$$= (\sum_{j=0}^{m} \beta_j x^j)(\sum_{j=n+m+1}^{\infty} \epsilon_j x^j)$$

$$(\beta_0 + \beta_1 x + \cdots + \beta_m x^m)(\phi_0 + \phi_1 x + \phi_2 x^2 + \ldots) - (\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n)$$

$$= \gamma_{N+1} x^{N+1} + \gamma_{N+2} x^{N+2} + \cdots$$

Single Point Rational Approximation

Equate powers:

$$x^0: \quad \phi_0\beta_0 - \alpha_0 = 0$$

$$x^1: \quad \phi_1\beta_0 + \phi_0\beta_1 - \alpha_1 = 0$$

$$x^2: \quad \phi_2\beta_0 + \phi_1\beta_1 + \phi_0\beta_2 - \alpha_2 = 0$$

$$\vdots$$

$$x^k: \quad \sum_{j=0}^{k} \phi_j\beta_{k-j} - \alpha_k = 0, \quad (0 \leq k \leq N)$$

$$\beta_k \equiv 0,\ k > m \text{ and } \alpha_k \equiv 0,\ k > n$$

Yields an $N+1 \times N+1$ system of equations. Typically $\beta_0 = 1$ chosen.

Example with $n = m = 2$

$$
\begin{aligned}
x^0 : &\quad \phi_0\beta_0 - \alpha_0 = 0 \\
x^1 : &\quad \phi_1\beta_0 + \phi_0\beta_1 - \alpha_1 = 0 \\
x^2 : &\quad \phi_2\beta_0 + \phi_1\beta_1 + \phi_0\beta_2 - \alpha_2 = 0 \\
x^3 : &\quad \phi_3\beta_0 + \phi_2\beta_1 + \phi_1\beta_2 + \phi_0\beta_3 - \alpha_3 = 0 \\
x^4 : &\quad \phi_4\beta_0 + \phi_3\beta_1 + \phi_2\beta_2 + \phi_1\beta_3 + \phi_0\beta_4 - \alpha_4 = 0
\end{aligned}
$$

Example with $n = m = 2$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & \phi_0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \phi_1 & \phi_0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & \phi_2 & \phi_1 & \phi_0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \phi_3 & \phi_2 & \phi_1 & \phi_0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \phi_4 & \phi_3 & \phi_2 & \phi_1 & \phi_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example with $n = m = 2$

Impose $\alpha_3 = \alpha_4 = \beta_3 = \beta_4 = 0$:

$$\begin{bmatrix} -1 & 0 & 0 & \phi_0 & 0 & 0 \\ 0 & -1 & 0 & \phi_1 & \phi_0 & 0 \\ 0 & 0 & -1 & \phi_2 & \phi_1 & \phi_0 \\ 0 & 0 & 0 & \phi_3 & \phi_2 & \phi_1 \\ 0 & 0 & 0 & \phi_4 & \phi_3 & \phi_2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Looking for a vector in the null space.

Example with $n = m = 2$

Impose $\beta_0 = 1$.

$$\left[\begin{array}{ccc|cc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \phi_0 & 0 \\ 0 & 0 & -1 & \phi_1 & \phi_0 \\ \hline 0 & 0 & 0 & \phi_2 & \phi_1 \\ 0 & 0 & 0 & \phi_3 & \phi_2 \end{array}\right] \left[\begin{array}{c} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \hline \beta_1 \\ \beta_2 \end{array}\right] = \left[\begin{array}{c} -\phi_0 \\ -\phi_1 \\ -\phi_2 \\ \hline -\phi_3 \\ -\phi_4 \end{array}\right]$$

Block upper triangular. $(2, 2)$ block is a Toeplitz matrix and the righthand-side vector has related structure. If the vector is in the range of the matrix then an $r_{nm}(x)$ normalized this way exists. If not change normalization.

The Exponential with $n = m = 2$

$$f(x) = e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots$$

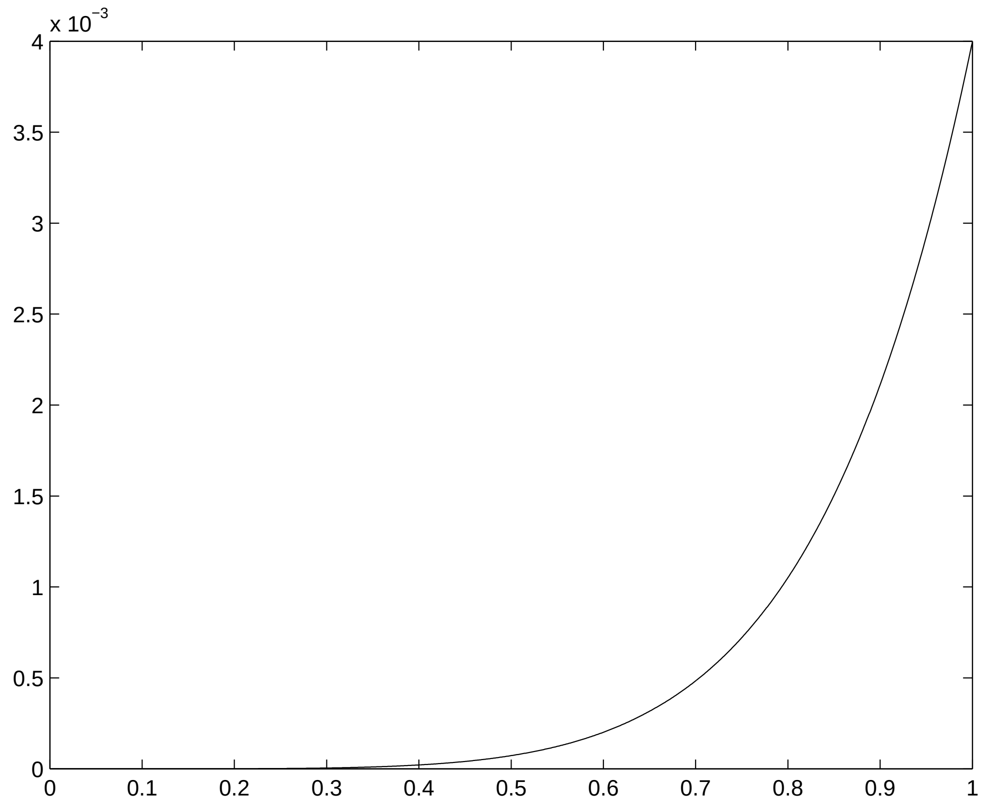
$$\left[\begin{array}{ccc|cc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 1/6 & 1/2 \end{array}\right] \left[\begin{array}{c} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \hline \beta_1 \\ \beta_2 \end{array}\right] = \left[\begin{array}{c} -1 \\ -1 \\ -1/2 \\ \hline -1/6 \\ -1/24 \end{array}\right]$$

The Exponential

We have

$$\alpha_0 = 1, \;\; \alpha_1 = \frac{1}{2}, \;\; \alpha_2 = \frac{1}{12}$$
$$\beta_0 = 1, \;\; \beta_1 = -\frac{1}{2}, \;\; \beta_2 = \frac{1}{12}$$
$$r_{22}(x) = \frac{12 + 6x + x^2}{12 - 6x + x^2}$$

Rational Approximation Error



Algorithms

- computations (depends on form used) exploit matrix structure
- accuracy: rational tends to be better than polynomial
- stability:
  - structured matrix algorithms on often poorly conditioned matrices for large n and m
  - other basis for polynomials $p_n(x)$ and $q_m(x)$ possible
  - representations other than ratio of polynomials often used for dynamical system problems

The General Rational Interpolation Problem

Let $s_i \in \mathbb{C}$ and $f_{ij} \in \mathbb{C}$ and consider the points

$$(s_i, f_{ij}), \;\; 0 \le j \le \ell_i - 1, \;\; 1 \le i \le k$$

$$i \ne j \to s_i \ne s_j, \;\; N = \sum_{i=1}^{k} \ell_i$$

Find all rational relatively prime $r(s)$ that interpolate

$$\frac{d^j r(s)}{ds^j}|_{s=s_i} = f_{ij}$$
$$0 \le j \le \ell_i - 1, \;\; 1 \le i \le k$$

Sometimes called multipoint momment matching or multipoint Padé approximation.

Other Constraints

Many problems are related to linear system realization theory.

- Find the admissible degrees of complexity, i.e., those positive intergers k for which solutions to the interpolation problem exist and $k = \max(n, m)$.
- Nevanlinna-Pick – norm-based
  - Do there exist bounded real interpolating functions?
  - If so what is the minimum norm and how are they constructed?
- Positive real interpolants?

$$s \in \mathbb{C},\ Re(s) \geq 0 \rightarrow Re(r(s)) \geq 0$$

Minimax Rational Approximation

There is theory similar to the best polynomial approximation for rational functions.

Definition 24.3. The set of rational functions $R_m^n[a, b]$ is the set of all ratios of $p(x)/q(x)$ where $p(x)$ is a polynomial with degree less than or equal to n, $q(x)$ is a polynomial with degree less than or equal to m, $q(x) > 0$ on $[a, b]$ and $p(x)$ and $q(x)$ are relatively prime.

Existence

Theorem 24.1. *For every function* $f(x) \in \mathcal{C}[a, b]$ *there is a best (uniform) rational approximation in the class* $R^n_m[a, b]$.

Proof. See Blum. Note that even though the set of rational functions is a linear space, $R^n_m[a, b]$ is not a subspace so any linear theory approach does not apply. $\square$

Characterization

Theorem 24.2. *Let* $r^*(x) = p^*(x)/q^*(x) \in R_m^n[a, b]$ *and let* $f(x) \in [a, b]$. *The error* $e^* = f - r^*$ *assumes its extreme values* $\pm\|f - r^*\|_\infty$ *with successively alternating sign on at least* k *points in* $[a, b]$, *where*

$$k = 2 + \max\{n + deg(q^*), m + deg(p^*)\},$$

if and only if r^* *is the best approximation to* f *in* $R_m^n[a, b]$.

Proof. See Blum. Note that if $deg(q^*) = m$ and $deg(p^*) = n$ then $k = 2 + n + m$. □

Uniqueness

Theorem 24.3. *The best approximation, $r^* \in R_m^n[a,b]$, to a function $f \in \mathcal{C}[a,b]$ is unique.*

Proof. See Blum $\square$