

Numerical Qualifier August 17, 2011

Instructions

- Write your ISU ID number on every page that you turn in. DO not write your name on any sheet that you turn in.
- Work all 6 problems. Start each problem on a separate sheet of paper and clearly indicate the problem number.

Problem 1. In the system $Ax = b$, the $n \times n$ matrix A is symmetric and nonnegative definite, i.e. $(Ax, x) \geq 0$ for all $x \in \mathbb{R}^n$. Here (x, y) denotes the dot product in \mathbb{R}^n . Suppose that the only eigenvector corresponding to the zero eigenvalue is the vector $1 = (1, 1, \dots, 1)^\top$ and the right-hand side b satisfies the condition $(b, 1) = 0$. Consider the iteration

$$x(m+1) = x(m) - \tau(Ax(m) - b).$$

- (a) Give conditions on the initial iterate x_0 and iteration parameter τ so that the above iteration converges to the unique solution x of $Ax = b$ satisfying $(x, 1) = 0$.
(b) What is the optimal choice of the iteration parameter τ ?

Problem 2. Consider the conjugate gradient method for the minimization of

$$\frac{1}{2}(Au, u) - (b, u)$$

where A is a symmetric and positive definite matrix, in the form: Starting with $u^0 = 0$, $r_0 = b$ and $p_0 = r_0$ the successive approximations to the minimizer are computed by

$$\begin{aligned} u^{k+1} &= u^k + \alpha_k p_k, & r_{k+1} &= r_k - \alpha_k A p_k; \\ p_{k+1} &= r_{k+1} - \beta_k p_k, \end{aligned}$$

where $\alpha_k = (r_k, p_k) / \|p_k\|_A^2$ and $\beta_k = -(r_{k+1}, p_k)_A / \|p_k\|_A^2$.

a. Show that for $k = 0, 1, 2, \dots$, the following relations are true

$$\text{span}\{p_0, p_1, \dots, p_k\} = \text{span}\{r_0, r_1, \dots, r_k\} = \text{span}\{r_0, Ar_0, \dots, A^k r_0\}.$$

b. Show that if $A \in \mathbb{R}^{n \times n}$ then for some $m \leq n$, $r_m = 0$ (assume that all operations are performed exactly).

Problem 3. This problem concerns trapezoidal rules of numerical integration.

a. Prove the following equation with an error term for the trapezoidal integration rule:

$$\int_{x_1}^{x_2} f(x) dx = \frac{h}{2} [f(x_1) + f(x_2)] - \frac{h^3}{12} f''(\xi)$$

for some $\xi \in (x_1, x_2)$, where $h = x_2 - x_1$ and $f \in C^2[x_1, x_2]$.

b. Derive the composite trapezoidal rule with error term

$$\int_a^b f(x) dx = \frac{h}{2} \left[f(x_1) + 2 \sum_{i=2}^{n-1} f(x_i) + f(x_n) \right] - \frac{h^2(b-a)}{12} f''(\xi)$$

for some $\xi \in [a, b]$, where $h = (b - a)/(n - 1)$, $x_i = a + (i - 1)h$ and $f \in C^2[a, b]$.

c. Write a pseudo-code (algorithm) that computes a sequence of approximations to $\int_a^b f(x) dx$ by the composite trapezoidal rule on a sequence of partitions of $[a, b]$, assuming each partition in the sequence is a refinement of the previous partition by halving the step size h . Stop the computations when the absolute error between two consecutive approximations is less than a user-specified tolerance.

Problem 4. Assume that the functions f and g in this problem are sufficiently smooth.

a. State without proof a convergence theorem for fixed point iterations: $x_{n+1} = g(x_n)$, $n = 1, 2, \dots$ with initial guess x_1 .

b. Consider Newton's method " $x_{n+1} = x_n - f(x_n)/f'(x_n)$, $n = 1, 2, \dots$ with initial guess x_1 " for solving $f(x) = 0$. Prove the local convergence of Newton's iterations. You are allowed to use the convergence result for fixed point iterations as stated in a).

c. Prove that for a simple zero x^* of f , the rate of convergence for Newton's method is quadratic.

d. If x^* is a zero of f of multiplicity 2, what is the rate of convergence in general and why? Suggest a method related to Newton's method that has a quadratic rate of convergence, and use a simple example to explain why your suggested method converges quadratically.

Problem 5. Let U be an open set in \mathbb{R}^{1+n} , and $f : U \rightarrow \mathbb{R}^n$ a vector field. For the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

with initial point $(x_0, y_0) \in U$, consider the linear multi-step formula with constant step size h :

$$\frac{3}{2}y_{j+1} - 2y_j + \frac{1}{2}y_{j-1} = hf(x_{j+1}, y_{j+1}). \quad (1)$$

a. Find the order of consistency of the formula (1). Is it stable?

b. Since y_{j+1} appears on both sides of (1), the formula is implicit, i.e. it is a "corrector." Devise a predictor formula that uses some or all of the data from the previous step

$$y_j, y_{j-1}, y_{j-2} \text{ and } f(x_j, y_j)$$

to obtain an initial iterate $y_{j+1}^{(0)}$ for y_{j+1} .

c. Assume that the vector field f has continuous derivatives of all orders. Give a necessary condition for the corrector iteration

$$y_{j+1}^{(m+1)} = \frac{4}{3}y_j - \frac{1}{3}y_{j-1} + \frac{2}{3}hf(x_{j+1}, y_{j+1}^{(m)})$$

to converge.

Problem 6. Assume $f \in C^4[0, 2]$, and denote $f_0 = f(0)$, $f_1 = f(1)$, $f_2 = f(2)$, and $f'_1 = f'(1)$. Consider the cubic interpolation problem: find a $p \in P_3$ such that $p(0) = f_0$, $p(1) = f_1$, $p(2) = f_2$, and $p'(1) = f'_1$.

a. Construct p in the following form: $p(x) = f_0p_0(x) + f_1p_1(x) + f_2p_2(x) + f'_1q_1(x)$ by finding explicit expressions for each cubic basis polynomial $p_i(x)$, $i=1,2,3$, and $q_1(x)$.

b. Prove the error formula $f(x) - p(x) = \frac{f^4(\xi)}{24}x(x-1)^2(x-2)$ for some $\xi \in (0, 1)$.

(Hint: let x be fixed and consider the auxiliary function $G(t) = [f(t) - p(t)] - \frac{\psi(t)}{\psi(x)}[f(x) - p(x)]$ where $\psi(x) = x(x-1)^2(x-2)$).