Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Tuesday January 15, 2007 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Root finding:

1. Consider the fixed point problem $x = g(x)$, where $g : D \to \Re^2$ is a mapping from the set $D \subset \Re^2$ into the plane, $\Re^2$. Specify whatever assumptions you need on g, its Jacobian, and D to conclude that there exists a unique fixed point in D and that fixed point iteration converges to it linearly in the infinity norm for any initial guess in D. Prove that the fixed point is unique (Do not prove existence or convergence).

Numerical quadrature:

2. a. Consider the usual inner product on [-1,1]

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx \tag{1}$$

and the subspace of polynomials with real coefficients and degree less or equal to $n-1$, P_{n-1}. Construct a discrete inner product of the form

$$\langle f, g \rangle_d = \sum_{j=1}^{n} w_j\, f(x_j)\, g(x_j),$$

that coincides with (1) on P_{n-1}, by providing equations for both the nodes x_j and the weights w_j. State properties of these nodes and weights. Prove that the two inner products are identical on P_{n-1}.

   b. Show that the Lagrange interpolating polynomials constructed using these nodes are orthogonal with respect to the inner product (1) and form a basis of P_{n-1}. Relate the norm of these polynomials with the weights to show that the weights are positive.

Interpolation / Approximation:

3. The following are six plausible approximations to the function e^x over the interval [-1,1]:

(A) $e^x \approx \sum_{k=0}^{n} \frac{x^k}{k!}$,

(B) $e^x \approx 1 / \sum_{k=0}^{n} \frac{(-x)^k}{k!}$

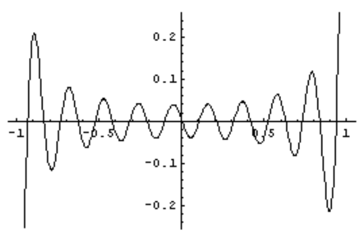
(C) $e^x \approx \left(1+\frac{x}{n}\right)^n$

(D) $e^x \approx \frac{c_0}{2} + \sum_{k=1}^{n} c_k T_k(x)$ where c_k are the Chebyshev coefficients $c_k = \frac{2}{\pi} \int_0^{\pi} e^{\cos t} \cos kt \, dt$

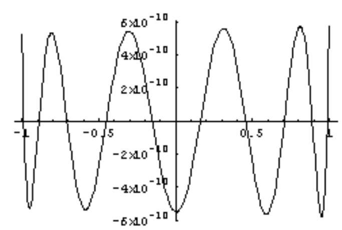
(E) $e^x \approx$ {polynomial obtained by interpolation at the nodes $x_k = -1 + \frac{2k}{n}$, $k = 0, 1, ..., n$}

(F) $e^x \approx \sum_{k=-n}^{n} d_k e^{i\pi k x}$ where $d_k = \frac{1}{2} \int_{-1}^{1} e^x e^{-i\pi k x} dx$

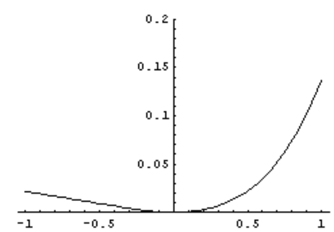
The six figures below show the errors (exact minus approximation) for the six cases over the interval [-1,1] in the case of n = 9. However, the order of the figures (i)-(vi) is different from the order of the approximations (A)-(F):



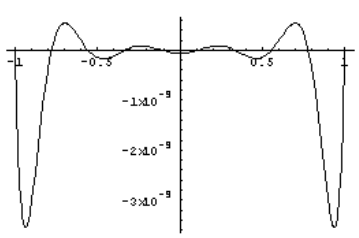
(i)



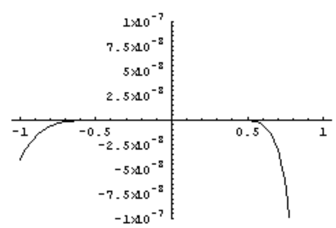
(ii)



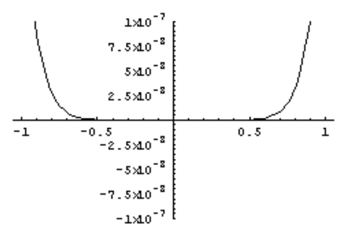
(iii)



(iv)



(v)



(vi)

Determine which error picture corresponds to which approximation. Give convincing explanations (2-3 sentences are sufficient) for how you arrive at each of the answers.

Linear algebra:

4. a. Suppose that $A \in \Re^{n \times n}$ is a symmetric positive definite matrix. Define the Gauss-Seidel iterative method for solving $A\boldsymbol{x} = \boldsymbol{b}$ for given $\boldsymbol{b} \in \Re^n$.

   b. Let $\in_i$ denote the i^{th} coordinate vector that is 0 everywhere except for a 1 in the i^{th} entry and denote the A-norm by $\|\boldsymbol{x}\|_A \equiv (\boldsymbol{x}^T A \boldsymbol{x})^{1/2}$ and the current error by $\boldsymbol{e} \equiv \boldsymbol{x}^* - \boldsymbol{x}$. Show that the i^{th} step of one Gauss-Seidel iteration can be written as $\boldsymbol{x} \leftarrow \boldsymbol{x} + s^* \in_i$, where s^* is the step size that minimizes $\|\boldsymbol{e} - s\in_i\|_A$ over $s \in \Re$.

   c. Use (b) to prove that Gauss-Seidel converges in the A-norm for any initial guess.

Numerical ODE:

5. Consider a first order system of ODEs

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$$

with the initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$. Consider a multistep method of the form

$$a_2 \boldsymbol{y}_{n+2} + a_1 \boldsymbol{y}_{n+1} + a_0 \boldsymbol{y}_n = h(b_2 \boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2}) + b_1 \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) + b_0 \boldsymbol{f}(t_n, \boldsymbol{y}_n))$$

with coefficients a_j, b_j, $j = 0, 1, 2$.

   a. Give the definition of order of such scheme and derive appropriate conditions on its coefficients for the scheme to have order p.

   b. Give a scheme of this type for which the region of absolute stability is precisely the left half plane. Show that it indeed has this property, and tell its order.

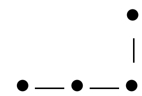
   c. The region of absolute stability can be even larger than the full left half plane. Give an example of such scheme, find its stability domain, and tell its order.

Numerical PDE:

6. Consider the following finite difference scheme:

$$\frac{u(x,t+k)-u(x,t)}{k}+\frac{\frac{3}{2}u(x,t)-2u(x-h,t)+\frac{1}{2}u(x-2h,t)}{h}=0.$$

Graphically, we illustrate its stencil as shown to the right:

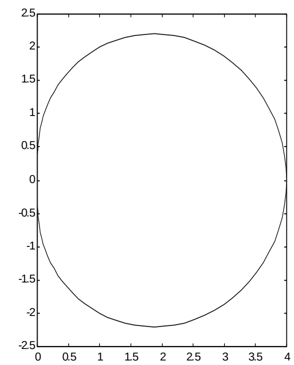


a. Determine which PDE the scheme is consistent with.

b. Determine its order of accuracy in time and space.

c. Determine the stability restriction that the CFL condition imposes on the scheme.

d. Use von Neumann analysis to show that the scheme is unconditionally unstable.



Hint to part d: The bottom figure to the right shows the curve traced out in the complex plane by $f(s)=\frac{3}{2}-2\,e^{-is}+\frac{1}{2}\,e^{-2is}$ for $-\pi\le s\le\pi$..