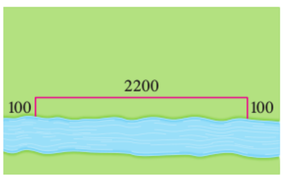
4.7: Optimization Problems

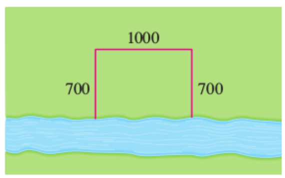
The methods from Sections 4.1 and 4.3 for finding extreme values can be used to find *optimal* solutions to many real-world problems. We will see a number of these as well as some abstract examples in this section. A general approach for solving optimization problems is to

1. Identify given and unkown quantities and draw a diagram if possible,
2. Write an equation for the quantity to be optimized in terms of a single variable,
3. Use methods from Sections 4.1 and 4.3 to find the absolute maximum or minimum.

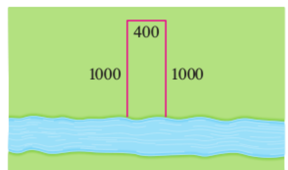
Example 1. *A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?*



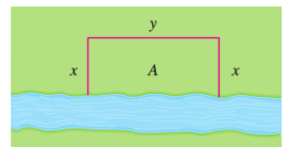
Area = 100 · 2200 = 220,000 ft²



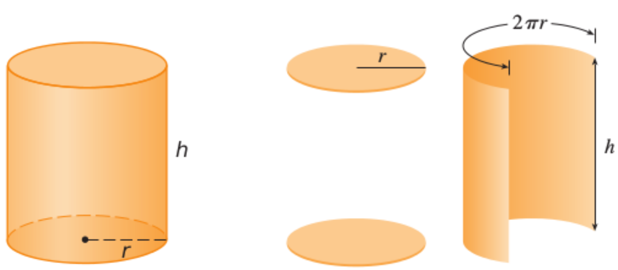
Area = 700 · 1000 = 700,000 ft²



Area = 1000 · 400 = 400,000 ft²



Example 2. *A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.*



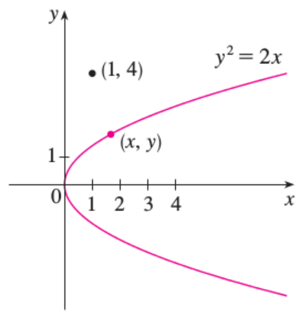
Area $2(\pi r^2)$ Area $(2\pi r)h$

First Derivative Test for Absolute Extreme Values: Suppose that c is a critical number of a continuous function f defined on an interval.

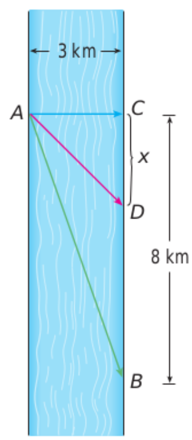
(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

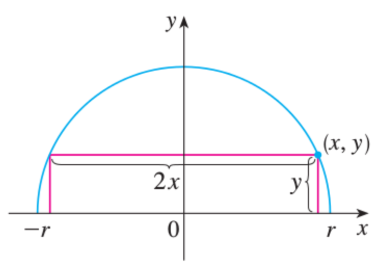
Example 3. *Find the point on the parabola* $y^2 = 2x$ *that is closest to the point (1,4).*



Example 4. *A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure below). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)*



Example 5. *Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.*



Example 6. *A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?*

Example 7. *A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the* *largest volume that such a box can have.*

Example 8. *Find the points on the ellipse* $4x^2 + y^2 = 4$ *that are farthest away from the point* $(1, 0)$.

Example 9. *Find the area of the largest rectangle that can be inscribed in the ellipse* $x^2/a^2+y^2/b^2 = 1$.

Example 10. *A poster is to have an area of 180 in with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?*

Example 11. *A right circular cylinder is inscribed in a sphere of radius* r. *Find the largest possible volume of such a cylinder.*

Example 12. *Find an equation of the line through the point* $(3, 5)$ *that cuts off the least area from the first quadrant.*