3.6: Derivatives of Logarithmic Functions

Implicit differentiation can be used to find that

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

For example, if $a = e$, then

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example 1. *Differentiate* $y = \ln(x^3 + 1)$.

Derivative of the natural logarithm function combined with the chain rule: In general, if g is a differentiable function, then

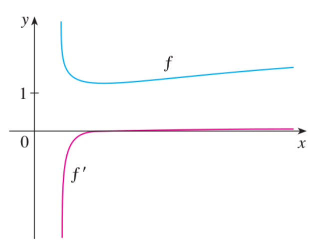
$$\frac{d}{dx}\left[\ln g(x)\right] = \frac{g'(x)}{g(x)}$$

Example 2. *Find* $\frac{d}{dx} \ln(\sin x)$.

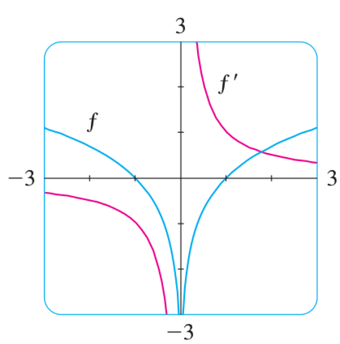
Example 3. *Differentiate* $f(x) = \sqrt{\ln x}$.

Example 4. *Differentiate* $f(x) = \log_{10}(2 + \sin x)$

Example 5. *Find* $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$.



Example 6. *Find* $f'(x)$ *if* $f(x) = \ln|x|$.



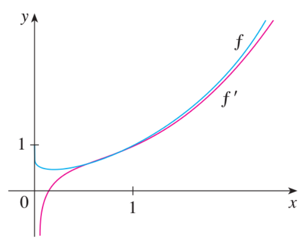
Logarithmic Differentiation

Example 7. *Differentiate* $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Steps in Logarithmic Differentiation:

1. Take natural logarithms of both sides of the equation $y = f(x)$ and use laws of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve for y'.

Example 8. *Differentiate* $x^{\sqrt{x}}$.



Example 9. *Differentiate* $(\sin x)^x$.

The number e as a limit

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Then, $f'(1) = 1$, and also, by the limit definition of the derivative,

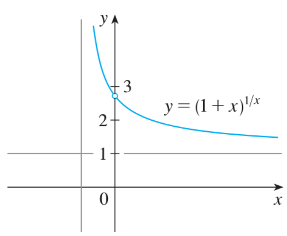
$$f'(1) = \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h} = \lim_{h \to 0} \frac{\ln(1+h)}{h} = \lim_{h \to 0} \ln(1+h)^{1/h}$$

So, using limit laws,

$$e = e^{\lim\limits_{h \to 0} \ln(1+h)^{1/h}} = \lim_{h \to 0} e^{\ln(1+h)^{1/h}} = \lim_{h \to 0} (1+h)^{1/h}.$$

Thus,

$$e = \lim_{x \to 0} (1+x)^{1/x}.$$



x	$(1+x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181