2.2: The Limit of a Function

One-sided Limits

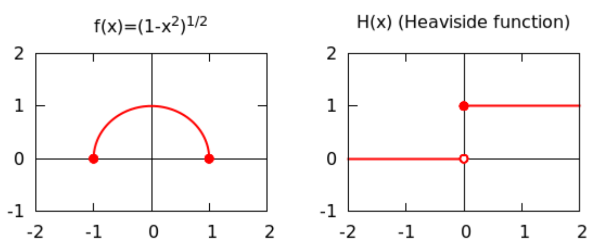
If $f(x)$ approaches L as x approaches a from the left ($x < a$), then

$$\lim_{x\to a^-} f(x) = L.$$

If $f(x)$ approaches L as x approaches a from the right ($x > a$), then

$$\lim_{x\to a^+} f(x) = L.$$

Example 1. *(a)* $f(x) = \sqrt{1-x^2}$ *(b)* $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$



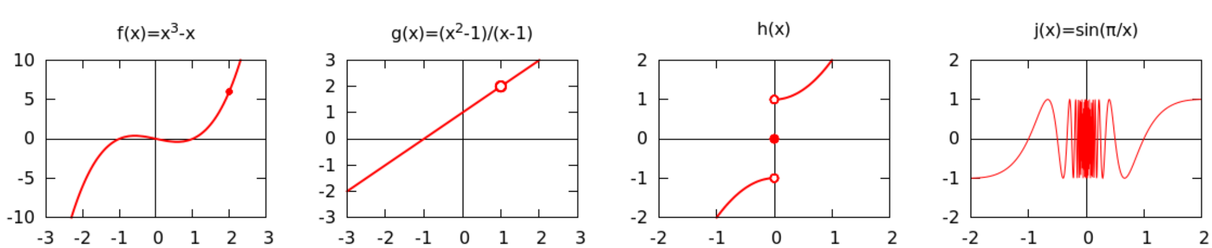
Limit of a Function

If $f(x)$ approaches L as x approaches a from both the left and right ($x \neq a$), then

$$\lim_{x\to a} f(x) = L.$$

More precisely, this means that the value of $f(x)$ can be made as close to L as we like by taking x sufficiently close to a. Notice that $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. Also, notice that $f(a)$ need not equal $\lim_{x\to a} f(x)$ nor even be defined for $\lim_{x\to a} f(x)$ to exist.

Example 2. *(a)* $f(x) = x^3 - x$ *(b)* $g(x) = \frac{x^2-1}{x-1}$ *(c)* $h(x) = \begin{cases} -x^2-1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^2+1 & \text{if } x > 0 \end{cases}$ *(d)* $j(x) = \sin\frac{\pi}{x}$



Example 3. *Sketch the graph of* $f(x) = \begin{cases} 1+x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ 2-x & \text{if } x \geq 1 \end{cases}$ *and determine each of*

(a) $\displaystyle\lim_{x \to \frac{1}{2}} f(x) =$

(b) $\displaystyle\lim_{x \to -1^-} f(x) =$

(c) $\displaystyle\lim_{x \to -1^+} f(x) =$

(d) $\displaystyle\lim_{x \to -1} f(x) =$

(e) $f(-1) =$

(f) $\displaystyle\lim_{x \to 1^-} f(x) =$

(g) $\displaystyle\lim_{x \to 1^+} f(x) =$

(h) $\displaystyle\lim_{x \to 1} f(x) =$

(i) $f(1) =$

Infinite Limits

If $f(x)$ takes arbitrarily large *positive* values as x approaches a (from both the left and right), then

$$\lim_{x \to a} f(x) = \infty.$$

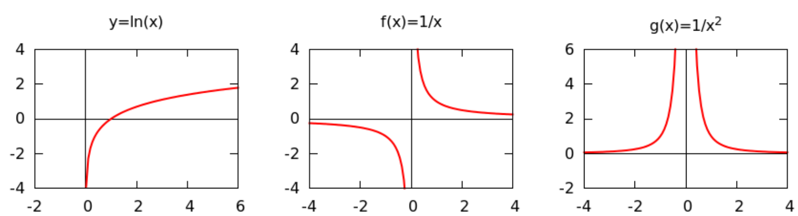
Similarly, if $f(x)$ takes arbitrarily large *negative* values as x approaches a (from both the left and right), then

$$\lim_{x \to a} f(x) = -\infty.$$

$f(x)$ has a **vertical asymptote** at $x = a$ if one of the following are true

- $\displaystyle\lim_{x \to a^-} f(x) = \infty$
- $\displaystyle\lim_{x \to a^+} f(x) = \infty$
- $\displaystyle\lim_{x \to a^-} f(x) = -\infty$
- $\displaystyle\lim_{x \to a^+} f(x) = -\infty.$

Example 4. *(a)* $y = ln(x)$ *(b)* $f(x) = \frac{1}{x}$ *(c)* $g(x) = \frac{1}{x^2}$



Example 5. *Sketch the graph of* $f(x) = \frac{x^2-2x-8}{x^2-5x+6}$ *and determine each of the limits*

(a) $\lim_{x \to 2^-} f(x) =$

(b) $\lim_{x \to 2^+} f(x) =$

(c) $\lim_{x \to 2} f(x) =$

(d) $\lim_{x \to 3^-} f(x) =$

(e) $\lim_{x \to 2^+} f(x) =$

(f) $\lim_{x \to 3} f(x) =$

(g) $\lim_{x \to -\infty} f(x) =$

(h) $\lim_{x \to \infty} f(x) =$