Foundations of Computational Math I Exam 2
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
No collaborations with anyone
Due beginning of Class Wednesday, 5 December, 2012

Question	Points Possible	Points Awarded
1. Linear Iteration	30	
2. Factorization and Preconditioning	30	
3. Nonlinear Iteration	30	
4. Nonlinear Iteration	20	
Total Points	110	

Name:________________________________

Alias: ________________________________

to be used when posting anonymous grade list.

Problem 1 (30 points)

1.a

Given a nonsingular $A \in \mathbb{R}^n$ and a nonsingular $M \in \mathbb{R}^n$, show that for all $b \in \mathbb{R}^n$, $x = A^{-1}b$ is a fixed point of

$$x_{i+1} = M^{-1}Nx_i + M^{-1}b$$

if and only if $A = M - N$.

Solution:

For any fixed point. $v \in \mathbb{R}^n$ we have

$$\begin{aligned} v &= M^{-1}Nv + M^{-1}b \\ Mv &= Nv + b \\ (M-N)v &= b \end{aligned}$$

If, given A, the fixed point is $v = A^{-1}b$

$$b = (M-N)v = (M-N)A^{-1}b$$

Since this is true for any b it follows that $(M-N)A^{-1} = I$ and therefore $(M-N) = A$.

Suppose $M - N = A$ then

$$\begin{aligned} x_{i+1} &= M^{-1}Nx_i + M^{-1}b \\ (A^{-1}b) &= M^{-1}N(A^{-1}b) + M^{-1}b \\ (M-N)(A^{-1}b) &= b \\ b &= b \end{aligned}$$

and therefore $x = A^{-1}b$ is a fixed point for any b.

1.b

Consider the matrix

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 1 & 0 & 1 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

(i) Will Jacobi's method converge when solving $Ax = b$ for all b and x_0? Justify your answer.

(ii) Will Gauss-Seidel converge when solving $Ax = b$ for all b and x_0? Justify your answer.

(iii) Does Gauss-Seidel converge in a finite number of steps when solving $Ax = b$ for all b and x_0? Justify your answer.

(iv) Does a permutation, P, exist such that Gauss-Seidel solving $PAP^TPx = Pb$ for all b and x_0 converges in one step? Justify your answer.

Solution:

Given

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 1 & 0 & 1 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

the Jacobi iteration matrix is

$$G_J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 & 0 & 0.2 \\ 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0.2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Gershgorin disks are all centerered at the orgin and have radii strictly less than 1 therefore $\rho(G_J) < 1$ and Jacobi is convergent.

For Gauss-Seidel we have

$$G_{gs} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow G^2 = 0$$

therefore since $e^{(k)} = G^k e^{(0)}$ we have $e^{(2)} = 0$ for any $x^{(0)}$.

It converges in one step for

$$PAP^T = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 1 & 1 & 0 & 0 & 0 & 5 \end{pmatrix},$$

where P interchanges rows 2 and 6, and rows 3 and 4, since it is lower triangular. This can be determined easily by looking at the graph corresponding to A. It is easily seen that it is acyclic. As discussed in class, this can then be used to generate a permutation (a relabeling of the nodes) that gives upper or lower triangular. In this case since we are considering (forward) Gauss-Seidel. Lower triangular yields one step.

Problem 2 (30 points)

2.a(10 points)

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Suppose when computing the Cholesky factorization of A using IEEE floating point arithmetic at some step we have an active part of the matrix (Schur complement) that is identically 0. Since every element in the computed active part has the value of 0, even if we started complete pivoting at this point it would not progress. What can we conclude about the original matrix A?

Solution:

The Cholesky factorization in finite precision produces a factorization of a perturbed matrix $A + E$. Since the Schur complement is identically zero after, say k, steps we have

$$A + E = L_k L_k^T$$

where $L_k \in \Re^{n \times k}$ is lower trapezoidal with positive elements on the diagonal. This is a rank factorization of $A + E$ which is therefore a singular matrix with rank $k < n$.

We conclude that the original matrix A is near a singular matrix $A + E$, i.e., it is ill-conditioned.

2.b(20 points)

Assume the nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is an M-matrix and partition it

$$A = \begin{pmatrix} \alpha_{11} & a_{12}^T \\ a_{21} & A_{22} \end{pmatrix}, \quad \alpha_{11} \in \mathbb{R} \quad A_{22} \in \mathbb{R}^{n-1\times n-1}$$

The following facts are true (you need not prove them):

- The matrix $S \in \mathbb{R}^{n-1\times n-1}$ defined by

  $$S = A_{22} - a_{21}\alpha_{11}^{-1}a_{12}^T$$

  is also an M-matrix.

- If $C \in \mathbb{R}^{n\times n}$ is an M-matrix, $B \in \mathbb{R}^{n\times n}$ is a matrix whose off-diagonal elements satisfy $\beta_{ij} \leq 0$, $i \neq j$ and

  $$C \leq B$$

  (that is $\alpha_{ij} \leq \beta ij$, $1 \leq i,\ j \leq n$) then B is also an M-matrix.

A very effective way to generate a precondtioner P when using an iterative method to solve $Ax = b$ when A is sparse is to create $P = LU$ where L and U are sparse lower and upper triangular matrices respectively such that $A \approx LU$. This is called an incomplete factorization preconditioner.

This preconditioner can be computed, for example, by modifying LU factorization so that on each step some elements in the portion of the matrix updated are set to 0. By doing this the factors L and U are kept sparse. However, in general, this procedure cannot be guaranteed to complete since the updated matrix after removing the elements might not have an LU factorization.

Use the facts above to show that given that A is an M-matrix the incomplete LU factorization procedure described above must complete and produce L and U. (You need not consider the quality of the approximation to A.)

Solution:

One step of LU on A updates A_{22} to be exactly what is given as S, i.e.,

$$S = A_{22} - a_{21}\alpha_{11}^{-1}a_{12}^T.$$

Since A is an M-matrix, $\alpha_{ii} > 0$ so S is well-defined. It is given that if it exists, S is an M-matrix so it is known that $e_i^T S e_i > 0$ and $e_i^T S e_j \leq 0$.

The dropping strategy changes off-diagonal elements to 0. In some cases the magnitudes of the dropped elements are added to the diagonal. This simply makes them more position. So the matrix after dropping, $\tilde{S}$, can be written

$$\tilde{S} = S + E, \quad e_i^T E e_i \geq 0, \quad e_i^T E e_j \geq 0$$

since to drop a negative element is to add its magnitude to it. Therefore

$$S \leq \tilde{S}, \quad e_i^T \tilde{S} e_j \geq 0$$

and $\tilde{S}$ is an M-matrix by the second given lemma above.

Finally, since LU factorization is repeatedly performing the above two-step procedure of update and drop on M-matrices, the incomplete factorization must exist.

Problem 3 (30 points)

Suppose $g(x)$ be a smooth function, i.e., continuously differentiable to any order, Let $m > 1$ be an integer and consider the modified Newton iteration

$$\phi(x) = x - m\frac{g(x)}{g'(x)}$$

Suppose $\alpha \in \mathbb{R}$ is a root with multiplicity $d \geq 1$, i.e., $g(\alpha) = 0$ and $g^{(j)}(\alpha) = 0$ for $j = 0, \ldots, d-1$.

3.a **(10 points)** Recall that $m = 1$ defines Newton's method which is quadratically convergent for simple roots, i.e., $d = 0$. Under what conditions on m and d can local convergence be guaranteed? Justify your answer.

3.b **(10 points)** For cases the iteration is convergent, determine the associated order of convergence. Justify your answer.

3.c **(5 points)** Under what circumstances, if any, is the iteration divergent? Justify your answer.

3.d **(5 points)** Demonstrate your conclusions by considering $m = 3$ and the two functions $g_1(x) = x^3$ and $g_2(x) = x^3 - 69$. **You do not have to turn in any code. Of course, it is recommended you implement the iteration on whatever machine using whatever language/system that is convenient.**

Solution:

We have

$$\phi(x) = x - m\frac{g(x)}{g'(x)}$$

Recall that we know that $m = 1$ is Newton's method and it has quadratic convergence for a simple root, i.e., $g(\alpha) = 0$ and $g'(\alpha) \neq 0$, and linear convergence for a multiple root.

So consider the iteration for a root of order d. We have

$$g(x) = (x-\alpha)^d h(x), \quad h(\alpha) \neq 0$$
$$g'(x) = (x-\alpha)^{d-1}\left[dh(x) + (x-\alpha)h'(x)\right]$$
$$g''(x) = (x-\alpha)^{d-2}\left[d(d-1)h(x) + 2d(x-\alpha)h'(x) + (x-\alpha)^2 h''(x)\right]$$

We then have, suppressing the argument x for g, h and their derivatives, we have

$$\phi'(x) = 1 - m + \left[\frac{md(d-1)h^2 + 2(x-\alpha)mdhh' - (x-\alpha)^2 mhh''}{(dh + (x-\alpha)h')^2}\right]$$

$$\phi'(\alpha) = \frac{d(d-m)}{d^2} = 1 - \frac{m}{d}$$

We therefore have

$d = m$	$\phi'(\alpha) = 0$	quadratic convergence
$1 \leq m < 2d$	$\lvert\phi'(\alpha)\rvert < 1$	linear convergence
$2d < m$	$\lvert\phi'(\alpha)\rvert > 1$	divergence

Note that for a fixed m as $d \to \infty$ the linear convergence slows since $\phi'(\alpha) \to 1$.

For the two functions we have $g_1(x) = x^3$ and $g_2(x) = x^3 - 69$ are solved via that method then we have

$$g_1(0) = g_1'(0) = g_1''(0) = 0, \quad = g_1'''(0) \neq 0 \to d = 3$$

$$g_2(\sqrt[3]{69}) = 0, \quad g_2'(\sqrt[3]{69}) \neq 0 \to d = 1$$

For $g_1(x) = x^3$ the use of $m = 3$ yields convergence in one step since

$$x^{(1)} = x^{(0)} - 3 * \frac{(x^{(0)})^3}{3(x^{(0)})^2} = 0$$

For $g_2(x) = x^3 - 69$ the use of $m = 3$ with $d = 1$ yields the common wild oscillation of divergence:

k	x	$g(x)$
1	4.2000	5.0880000000000
2	3.9115	-9.1517398111079
3	4.5097	22.7158113982865
4	3.3927	-29.9466002314161
5	5.9943	146.3920045448345
6	1.9202	-61.9191056246131
7	18.712	6482.9570215742933
8	0.1970	-68.9923474673307
9	1776.8	5609664426.1053485870361
10	0.00002	-69.0000000000000
11	1.44×10^{11}	3.0×10^{33}
12	0.00000	-69.000
13	Inf	Inf

Problem 4 (20 points)

Consider the iteration

$$\phi(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

for $a > 0$.

4.a **(5 points)** Determine the fixed points of $\phi(x)$. Justify your answer.

4.b **(15 points)** For each fixed point, determine if $\phi(x)$ converges to it and when convergent determine the associated order of convergence. Justify your answer.

Solution:

We have for $a > 0$

$$\phi(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

The fixed points follow easily

$$x = \phi(x) \rightarrow x^2 = a \rightarrow x = \pm\sqrt{a}$$

Note that if $x > 0$ then $\phi(x) > 0$ so we concentrate on the positive real line. The behavior on the negative real line is simply the mirror image.

We have

$$\phi'(x) = \frac{1}{2}\left(x - \frac{a}{x^2}\right)$$

and

$$x > \frac{\sqrt{a}}{\sqrt{3}} \rightarrow |\phi'(x)| < 1.$$

Therefore, by contraction

$$x^{(0)} > \frac{\sqrt{a}}{\sqrt{3}} \rightarrow \lim_{k\rightarrow\infty} x^{(k)} = \sqrt{a}$$

Now note that if $0 < x^{(0)} \leq \sqrt{a}/\sqrt{3}$ then $x^{(1)} > \sqrt{a}$ and therefore the iteration converges to $\sqrt{a}$. So we have

$$x^{(0)} > 0 \rightarrow \lim_{k\rightarrow\infty} x^{(k)} = \sqrt{a}$$

To determine the order of convergence note that

$$\phi'(\sqrt{\alpha}) = 0$$

and quadratic convergence follows. In fact, it is easy to verify that $\phi(x)$ is the iteration that results from applying Newton's method to find the roots of $g(x) = x^2 - a$.