Foundations of Computational Math II Exam 1
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Wednesday February 22, 2012

Question	Points Possible	Points Awarded
1. Basics	20	
2. Splines	30	
3. Interpolation	35	
4. Piecewise Interpolation	30	
Total Points	115	

Name: **Alias:**

Problem 1

(20 points)

1.a

(5 points)

Briefly describe the main advantage of using a piecewise polynomial interpolant rather than a single global interpolating polynomial when approximating a function $f(x)$ using data points (x_i, f_i) for $0 \leq i \leq n$ for various values of n.

Solution: The main advantage is the fact that when the error $\|f - p_n(x)\|_\infty$ is not acceptable increasing the the number of points allows the error to be reduced for a piecewise polynomial interpolant. If the degree of the polynomial is increased when n increases as happens with a single global interpolating polynomial then the error is not necessarily reduced and divergence may, in general, occur, e.g., Runge phenomenon.

1.b

(5 points)

Briefly describe the main differences and similarities between a piecewise cubic Hermite interpolating polynomial and a cubic spline to approximate a function $f(x)$ using data points (x_i, f_i) for $0 \leq i \leq n$.

Solution:

Both are piecewise cubic polynomials.

The piecewise cubic Hermite interpolating polynomial attempts to improve accuracy and smoothness compared to a Lagrange piecewise or global interpolant by also interpolating $f'(x)$. It therefore requires the values $f'(x_i)$.

The cubic spline introduces smoothness as a constraint and guarantees continuity of $s(x)$, $s'(x)$ and $s''(x)$. It only interpolates $f(x)$.

1.c

(5 points) Let the interpolating polynomial, $p_n(x)$, be defined by the points (x_i, y_i), $0 \le i \le n$. What is used as the condition number for the sensitivity of $p_n(x)$ to perturbations in the values of the y_i?

Solution:

The Lebesgue constant gives the conditioning of the interpolating polynomial and the conditioning of the Lagrange form of a polynomial.

1.d

(5 points) Briefly explain what an interpolatory strategy for approximating a function $f(x)$ with polynomials of degree $n = 0, 1, 2, \ldots$ is and give sufficient conditions such that the strategy yields uniform convergence to f as $n \to \infty$.

Solution:

An interpolatory strategy is a sequence of sets of points $X^{(n)} = \{x_0, \ldots, x_n\}$ for $n = 0, 1, \ldots$. Each set $X^{(n)}$ defines $p_n(x)$, a unique interpolating polynomial of degree n. The points are therefore independent of $f(x)$. Uniform convergence requires

$$\|f - p_n\|_\infty \to 0$$

as $n \to \infty$. This does not happen in general for continuous f. However, the strategy defined by Chebyshev points of the first kind and f either Lipschitz continuous or $\mathcal{C}^{(2)}$ yields uniform convergence.

Problem 2

(30 points)

Assume x_i, for $0 \leq i \leq 2$ are uniformly spaced and you are given the data pairs function values:

$$(x_0, f(x_0)) = (1, 11), \quad (x_1, f(x_1)) = (2, 12), \quad (x_2, f(x_2)) = (3, 11)$$

second derivative values:

$$(x_0, f''(x_0)) = (1, -6), \quad (x_2, f''(x_2)) = (3, -6)$$

We know a cubic spline $s(x)$ that interpolates these values can be defined as a linear combination of the cubic B splines defined in the notes and text:

$$s(x) = \alpha_{-1}B_{-1}(x) + \alpha_0 B_0(x) + \alpha_1 B_1(x) + \alpha_2 B_2(x) + \alpha_3 B_3(x)$$

(2.a) (10 points)

Determine a system of equations that define $s(x)$ and solve the system to determine the coefficients α_i.

(2.b) (10 points)

Recall, the spline can also be defined in terms of parameters that are the values of $s''(x_i)$, i.e., $Ts'' = d$. Write the system of equations that determine this parameterization and solve it to find the value of the parameters.

(2.c) (10 points)

Choose either form above and determine the two cubic polynomials

$$s(x) = \begin{cases} p_1(x) & \text{if } 1 \leq x \leq 2 \\ p_2(x) & \text{if } 2 \leq x \leq 3 \end{cases}$$

and verify that all required interpolation and continuity conditions are satisfied.

Solution:

This follows trivially from what we know of $B_i(x)$,$B_i'(x)$,$B_i''(x)$, i.e.,

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i'(x)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
$B_i''(x)$	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

We need only create the equations for the boundary conditions

$$(x_0, f''(x_0)) \quad \text{and} \quad (x_2, f''(x_2))$$

We have

$$\begin{aligned}
s(x) &= \alpha_{-1}B_{-1}(x) + \alpha_0 B_0(x) + \alpha_1 B_1(x) + \alpha_2 B_2(x) + \alpha_3 B_3(x) \\
s''(x) &= \alpha_{-1}B''_{-1}(x) + \alpha_0 B''_0(x) + \alpha_1 B''_1(x) + \alpha_2 B''_2(x) + \alpha_3 B''_3(x) \\
s''(x_0) &= \alpha_{-1}B''_{-1}(x_0) + \alpha_0 B''_0(x_0) + \alpha_1 B''_1(x_0) + \alpha_2 B''_2(x_0) + \alpha_3 B''_3(x_0) \\
&= \alpha_{-1}B''_{-1}(x_0) + \alpha_0 B''_0(x_0) + \alpha_1 B''_1(x_0) \\
&= \frac{6}{h^2}\alpha_{-1} - \frac{12}{h^2}\alpha_0 + \frac{6}{h^2}\alpha_1 \\
s''(x_2) &= \frac{6}{h^2}\alpha_1 - \frac{12}{h^2}\alpha_2 + \frac{6}{h^2}\alpha_3
\end{aligned}$$

We therefore have the following linear system.

$$\begin{pmatrix} \frac{6}{h^2} & -\frac{12}{h^2} & \frac{6}{h^2} & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & \frac{6}{h^2} & -\frac{12}{h^2} & \frac{6}{h^2} \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} f''_0 \\ f_0 \\ f_1 \\ f_2 \\ f''_n \end{pmatrix}$$

Substituting $h = 1$ and the given data values yields

$$\begin{pmatrix} 6 & -12 & 6 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 6 & -12 & 6 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -6 \\ 11 \\ 12 \\ 11 \\ -6 \end{pmatrix}$$

The solution is easily seen to be

$$\begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

The system $Ts'' = d$ is much simpler since $s''_0 = -6$ and $s''_2 = -6$ are given. Therefore, only s''_1 must be determined implying a 1×1 linear system. Using the form of the coefficients in the notes we have

$$2s''_1 = d_1 - \mu_1 s''_0 - \lambda_1 s''_2$$

The required parameters have the following values

$$\mu_i = \frac{h_i}{h_i + h_{i+1}} = \frac{1}{2}$$
$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}} = \frac{1}{2}$$
$$d_1 = \frac{6}{h_1 + h_2}(f_2 - 2f_1 + f_0) = -6$$

$$\therefore d_1 - \mu_1 s_0'' - \lambda_1 s_2'' = 0 \to s_1'' = 0$$

We can check the required conditions using the definition of $p_i(x)$ in the notes for the parameterization using s_i''. Specifically, we have

$$\begin{aligned}
p_1(x) &= -(2-x)^3 + 12 \quad 1 \le x \le 2 \\
&= (x-2)^3 + 12 = x^3 - 6x^2 + 12x + 4 \\
p_1'(x) &= 3(2-x)^2 \\
p_1''(x) &= -6(2-x) \\
p_2(x) &= -(x-2)^3 + 12 \quad 2 \le x \le 3 \\
&= (2-x)^3 + 12 = -x^3 + 6x^2 - 12x + 20 \\
p_2'(x) &= -3(x-2)^2 \\
p_2''(x) &= -6(x-2)
\end{aligned}$$

$$\begin{aligned}
p_1(1) &= 11 = f_0 \\
p_1(2) &= 12 = f_1 \\
p_2(2) &= 12 = f_1 \\
p_2(3) &= 11 = f_2 \\
p_1'(2) &= p_2'(2) = 0 \\
p_1''(1) &= -6 \\
p_1''(2) &= p_2''(2) = 0 \\
p_2''(3) &= -6
\end{aligned}$$

Therefore all required interpolation and continuity conditions are satisfied.

Problem 3

(35 points)

3.a

(10 points)

Suppose you have the Lagrange form of the unique interpolating polynomial of degree n through points (x_i, f_i)

$$p_n(x) = \sum_{i=0}^{n} f_i \ell_i(x)$$

Is it true or false that

$$\sum_{i=0}^{n} \ell_i'(x) = 1$$

Justify your answer.

Solution:

The statement is not true.

A counter example is easily seen by considering linear interpolation between two points. We have

$$\ell_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} \quad \ell_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$

$$\ell_(x) + \ell_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} + \frac{(x - x_0)}{(x_1 - x_0)}$$

$$= \frac{(x - x_1)}{(x_0 - x_1)} - \frac{(x - x_0)}{(x_0 - x_1)} = 1$$

$$\ell_0'(x) = \frac{(1)}{(x_0 - x_1)} \quad \ell_1'(x) = \frac{(1)}{(x_1 - x_0)}$$

$$\ell_0'(x) + \ell_1'(x) = \frac{(1)}{(x_0 - x_1)} + \frac{(1)}{(x_1 - x_0)} = 0$$

The summation to 0 is true in general for the derivatives of the Lagrange characteristic functions.. Recall, by interpolating $f(x) = 1$ we have

$$\sum_{i=0}^{n} \ell_i(x) = 1$$

and differentiating both sides yields the correct general result

$$\sum_{i=0}^{n} \ell_i'(x) = 0$$

3.b

(10 points)

Given the data points

$$(x_0, f(x_0)) = (1, 10), \quad (x_1, f(x_1)) = (3, 65)$$
$$(x_2, f(x_2)) = (4, 150), \quad (x_3, f(x_3)) = (6, 425)$$

find the unique cubic interpolating polynomial, $p_3(x)$ and evaluate it at $x = 0$ and $x = 10$.

Solution: You may determine the polynomial in any manner you choose.

The Lagrange form is

$$p_3(x) = -\frac{10}{30}(x-3)(x-4)(x-6)+\frac{65}{6}(x-1)(x-4)(x-6)-\frac{150}{6}(x-1)(x-3)(x-6)+\frac{425}{30}(x-1)(x-3)(x-4)$$

The divided difference table is

i	x_i	$f(x_i)$	$f_1[*,*]$	$f_2[*,*,*]$	$f_3[*,*,*,*]$
0	1	10			
			55/2		
1	3	65		115/6	
			85		-1/3
2	4	150		105/6	
			275/2		
3	6	425			

$$p_3(x) = 10 + \frac{55}{2}(x-1) + \frac{115}{6}(x-1)(x-3) - \frac{1}{3}(x-1)(x-3)(x-4)$$

$$p_3(x) = 425 + \frac{275}{2}(x-6) + \frac{105}{6}(x-6)(x-4) - \frac{1}{3}(x-6)(x-4)(x-3)$$

$$p_3(x) = \frac{1}{6}\left(264 - 333x + 131x^2 - 2x^3\right)$$

$$p_3(1) = 10$$
$$p_3(3) = 65$$
$$p_3(4)) = 150$$
$$p_3(6) = 425$$

$$p_3(0) = 44$$
$$p_3(10) = 1339$$

3.c

(15 points)

Consider the interpolation conditions

$$p_2(a) = f(a)$$
$$p_2'(a) = f'(a)$$
$$p_2'(b) = f'(b)$$

where $a,\ b \in \mathbb{R}$. Show that if $a \neq b$ then there is a unique quadratic polynomial, $p_2(x)$, satisfying the conditions.

Solution:

Note that the conditions do not allow the use of the usual divided difference table approach to Hermite-Birkhoff interpolation since $f(b)$ is not known, i.e., we have two disjoint tables. Therefore, the simplest way to approach the problem is to examine the conditions directly as a system of equations.

Specifically, we have

$$p_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$
$$p_2'(x) = \alpha_1 + 2\alpha_2 x$$

$$p_2(a) = f(a)$$
$$p_2'(a) = f'(a)$$
$$p_2'(b) = f'(b)$$
$$\alpha_0 + \alpha_1 a + \alpha_2 a^2 = f(a)$$
$$\alpha_1 + 2\alpha_2 a = f'(a)$$
$$\alpha_1 + 2\alpha_2 b = f'(b)$$

$$\begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 1 & 2b \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} f(a) \\ f'(a) \\ f'(b) \end{pmatrix}$$

The matrix is block upper triangular and the determinant of the 2×2 matrix in the $(2, 2)$ position is $2(a - b)$ which is nonzero if $a \neq b$ and therefore $p_2(x)$ is well-defined and unique when $a \neq b$ for any values $f(a),\ f'(a),\ f'(b)$. Note that if $a = b$ and therefore $f'(a) = f'(b)$ the matrix is singular but the righthand side is in the range so there is an infinite number of quadratics that interpolate $f(a)$ and $f'(a)$.

The case when $a \neq b$ can also be approached by using $f'(a)$ and $f'(b)$ to define a unique linear function $p_2'(x)$. $p_2(x)$ is derived via integration and setting the constant of integration so that $p_2(a) = f(a)$. This is always a unique setting.

The case when $a = b$ using this approach shows that any term of the form $C(x - a)^2$ can be added to the linear function $p_1(x) = f(a) + (x - a)f'(a)$ to get a quadatratic that satisfies the degenerate conditions.

Problem 4

(30 points)

Let $f(x) = 2/(1+10x^2)$ on $-1 \le x \le 1$. Suppose $f(x)$ is to be approximated by a piecewise linear interpolating function, $g_1(x)$. The accuracy required is

$$\forall 0 \le x \le 1, \quad |f(x) - g_1(x)| \le 10^{-6}$$

Determine a bound on $h = x_i - x_{i-1}$ for uniformly spaced points that satisfies the required accuracy.

Solution:

$$\forall x_{i-1} \le x \le x_i, \quad |f(x) - g_1(x)| = \frac{|\omega_2^{(i)}(x)|}{2!}|f^{(n+1)}(\xi(x))|$$
$$\text{where} \quad \omega_2^{(i)}(x) = (x - x_{i-1})(x - x_i) = s(s-1)h, \quad 0 \le s \le 1$$
$$\forall i, \quad \|\omega_2^{(i)}(s)\|_\infty \le \frac{1}{4}h^2$$

We have

$$f(x) = \frac{2}{(1+10x^2)}$$
$$f'(x) = \frac{-40x}{(1+10x^2)^2}$$
$$f''(x) = \frac{1200x^2 - 40}{(1+10x^2)^3}$$

The second derivative is plotted in Figure 1. We see that there is a maximum magnitude of 40 at $x = 0.$ and two local extrema that have much smaller magnitude near ± 0.1. This is easily verified by noting that the denominator of $f^{(3)}$ has the form

$$(1+10x^2)^2\{2400x(1+10x^2) - 60x(1200x^2 - 40)\}$$

which has real roots at 0 and $\pm\sqrt{1/145} \approx \pm 0.83$.

We therefore have

$$|f(x) - g_1(x)| = \frac{|\omega_2^{(i)}(x)|}{2!}|f^{(2)}(\xi(x))|$$
$$\le \frac{1}{2} \times \frac{1}{4} \times 40 \times h^2 = 5h^2$$
$$h^2 \le 0.2 \times 10^{-6}$$
$$h \lesssim 0.5 \times 10^{-3}$$

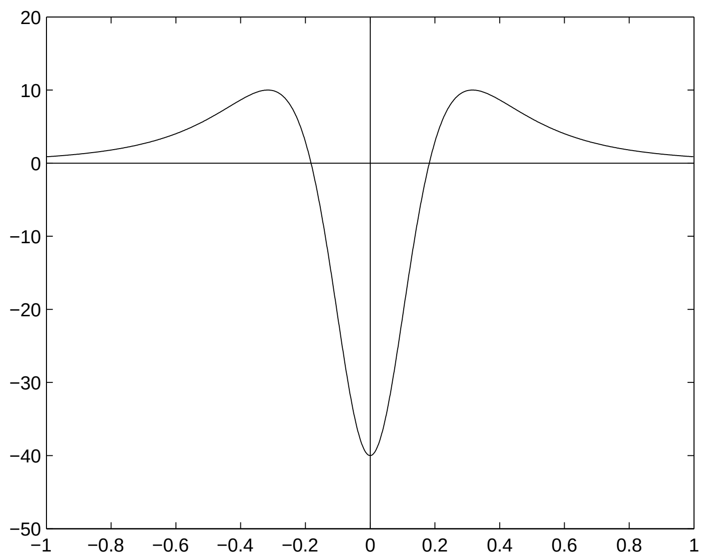

Figure 1: $f^{(2)}$ on $[-1, 1]$