Foundations of Computational Math II Exam 1
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Due beginning of Class Monday, 4 March, 2013

Question	Points Possible	Points Awarded
1. Interpolation	30	
2. Piecewise Linear Interpolation	25	
3. Minimax	25	
4. Splines	35	
5. Optimization via Approximation	25	
Total Points	140	

Name: **Alias:**

Problem 1 (30 points)

1.a

Consider, $r_4(x)$, the unique polynomial of degree 4 that interpolates the data:

$$(x_0, 0),\ (x_1, 0),\ (x_2, 0),\ (x_3, z_3),\ (x_4, z_4)$$

where the x_i are distinct. Let $\omega_{i:i+k}(x) = (x - x_i) \dots (x - x_{i+k})$.

(i) Mark the positions in the divided difference table with 0 if the entry is guaranteed to be 0 and $*$ if it may be nonzero.

i	0		1		2		3		4
x_i	x_0		x_1		x_2		x_3		x_4
z_i	0		0		0		z_3		z_4
$z[-,-]$		___		___		___		___	
$z[-,-,-]$			___		___		___		
$z[-,-,-,-]$				___		___			
$z[-,-,-,-,-]$					___				

(ii) Give the Newton form of $r_4(x)$ using the divided differences $z[x_0, \dots, x_i]$.

(iii) Use the appropriate identities to rewrite the divided differences into a form that makes it clear that $r_4(x)$ satisfies the required interpolation conditions.

1.b

Consider the data:

$$(x_0, f_0),\ (x_1, f_1),\ (x_2, f_2),\ (x_3, f_3),\ (x_4, f_4)$$

where the x_i are distinct. Let $p_4(x)$ be the unique interpolating polynomial of degree 4 that interpolates these 5 data points. Let $p_2(x)$ be the unique interpolating polynomial of degree 2 that interpolates the first 3 data points

$$(x_0, f_0),\ (x_1, f_1),\ (x_2, f_2).$$

Let $\omega_{i:i+k}(x) = (x - x_i)\dots(x - x_{i+k})$.

(i) Let $a_4(x)$ be the polynomial of degree 4 such that

$$p_4(x) = p_2(x) + a_4(x).$$

We know $a_4(x)$ can be expressed as

$$a_4(x) = f[x_0, x_1, x_2, x_3]\omega_{0:2}(x) + f[x_0, x_1, x_2, x_3, x_4]\omega_{0:3}(x)$$

Find values of z_3 and z_4 that show that $a_4(x)$ can also be expressed as a interpolating polynomial with interpolating conditions like those imposed on $r_4(x)$ in the first part of the problem.

(ii) Show the relationships between the coefficients of $a_4(x)$ expressed in terms of the z_i, $0 \le i \le 4$, and the divided differences of the f_i, $0 \le i \le 4$. What derivation of the divided differences $f[x_0, \dots, x_i]$ does this exercise generalize?

Problem 2 (25 points)

Let $f(x) = \sin x$ and consider using a piecewise linear interpolating polynomial$g_1(x)$ to approximate $f(x)$ on $-\pi \leq x \leq \pi$. In Set 5 of the class notes a sufficent bound on a uniform separation $h = 2\pi/n$ between the x_i was derived to guarantee that

$$\|f(x) - g_1(x)\|_\infty \leq 10^{-d}.$$

If that bound is applied with $d = 6$ the resulting bound is $h \leq 0.0028$ and the number of points required is over 2200.

Show that by careful consideration of the structure of the problem and removing the restriction of uniform spacing the number of points required for a piecewise linear interpolating polynomial can be reduced substantially while still achieving

$$\|f(x) - g_1(x)\|_\infty \leq 10^{-6}.$$

Problem 3 (25 points)

For this problem let $f(x) = \sin x$.

3.a. Derive the linear minimax approximation, $p_1(x) = \alpha x + \beta$, to $f(x)$ on $0 \le x \le \pi$.

3.b. Derive the linear near-minimax polynomial approximation, $c_1(x)$, to $f(x)$ on $0 \le x \le \pi$.

3.c. Compare $c_1(x)$ and $p_1(x)$.

Problem 4 (35 points)

Consider a interpolatory quadratic spline, $s(x)$, that satisfies the following interpolation conditions and single boundary condition:

$$s(x_i) = f(x_i) = f_i, \;\; 0 \leq i \leq n$$

$$s'(x_0) = f'(x_0) = f'_0$$

where the x_i are distinct.

4.a. Derive a linear system of equations that yields the values

$$s'(x_i) = s'_i \;\; 0 \leq i \leq n$$

that are used as parameters to define the quadratic spline $s(x)$.

4.b. Identify important structure in the linear system and show that it defines a uniqiue quadratic spline.

4.c. Use the structure of the system to show that if $f(x)$ is a quadratic polynomial then $s(x) = f(x)$.

Problem 5 (25 points)

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function with at least 4 continuous derivatives and with a unique minimizer, x^*. Assume that you do not have $f(x)$ or any of its derivatives analytically but you do have a routine that allows you to get values of f for any value of x. You may assume that the computational cost of the evaluation of $f(x)$ is small.

Consider solving the problem

$$\min_{x \in \mathbb{R}} f(x)$$

numerically using Newton's method.

5.a. Clearly, since by assumption, $f(x)$ and its derivatives are not available some method that approximates Newton must be used. Describe a method that uses techniques discussed in class to approximates Newton's method to solve the unconstrained optimization problem.

5.b. Show that the method is parameterized so that the method must approach the performance of Newton's method as the parameter is moved toward a limit. Your argument need not be a formal proof but it must contain all the essential facts. You may assume that you have an initial guess $x^{(0)}$ that is sufficiently close to x^*.