Numerical Analysis Qualifier

prepared by
Chuck Gartland & Lothar Reichel
Kent State University
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INSTRUCTIONS: Do any (and only) 8 of the following 10 problems.

1. (Floating-Point Arithmetic) By the periodicity of the sine function, the following relation should hold for all integers k:

$$\sin\left(\frac{\pi}{4} + 2k\pi\right) = \frac{\sqrt{2}}{2} \doteq .7071067\,.$$

Yet when this expression is evaluated in single-precision floating-point arithmetic, the following output results:

k	computed result	error
0	.707 106 8	.60(−7)
10^1	.707 107 5	.77(−6)
10^2	.707 116 6	.99(−5)
10^3	.707 105 1	.16(−5)
10^4	.708 629 0	.15(−2)
10^5	.747 494 3	.40(−1)
10^6	.929 413 4	.22

Explain.

2. (Interpolation) Let $P_{i_0 \cdots i_k}$ denote the polynomial of degree k that interpolates to $f(x)$ at $x_{i_0}, \ldots, x_{i_k}$.

   (a) Prove that

$$P_{i_0 \cdots i_{k+1}}(x) = \frac{(x - x_{i_0}) P_{i_1 \cdots i_{k+1}}(x) + (x_{i_{k+1}} - x) P_{i_0 \cdots i_k}(x)}{(x_{i_{k+1}} - x_{i_0})}.$$

   (b) What is the name of the algorithm that recursively uses this relationship to evaluate the interpolating polynomial (without explicitly forming it) at a given point x?

3. (Interpolation) The error formula for polynomial interpolation to a smooth function $f(t)$ by a polynomial of degree at most $n-1$ at n distinct points $t_1 < \cdots < t_n$ is given by

$$f(t) - P_{n-1}(t) = \frac{f^{(n)}(\theta(t))}{n!}(t-t_1)\cdots(t-t_n).$$

Use it to derive the bound

$$\max_{t_1 \le t \le t_n} |f(t) - P_{n-1}(t)| \le \frac{Mh^n}{4n},$$

where M is any upper bound on $f^{(n)}$,

$$|f^{(n)}(t)| \le M, \quad t_1 \le t \le t_n,$$

and $h := \max_{i=1,\dots,n-1} |t_{i+1} - t_i|$.

4. (Trigonometric Interpolation) Suppose that you have available to you a procedure that calculates complex Fourier coefficients along the lines of Matlab's `fft()` routine:

$$c_l = \frac{1}{2J}\sum_{k=0}^{2J-1} f_k\, e^{-i\frac{l\pi k}{J}}, \quad l = 0, \dots, 2J-1.$$

Given data sampled from a real $2L$-periodic function,

$$f(x_k), \quad k = 0, \dots, 2J-1, \qquad x_k = k\Delta x, \quad \Delta x = \frac{L}{J},$$

indicate how you would use the procedure above to determine the coefficients in the trigonometric polynomial

$$\Psi(x) = A_0 + \sum_{l=1}^{J-1}\left(A_l \cos\frac{l\pi x}{L} + B_l \sin\frac{l\pi x}{L}\right) + A_J \cos\frac{J\pi x}{L}$$

that interpolates to this data:

$$\Psi(x_k) = f(x_k), \quad k = 0, \dots, 2J-1.$$

5. (Quadrature Rules) Suppose that a quadrature rule has the following (typical) error/remainder formula:

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} w_k f(x_k) + C(b-a)h^m f^{(m)}(\xi), \quad \text{some } \xi \in (a,b).$$

(a) What can you deduce from this to be the *polynomial degree* of this quadrature rule, and why?

(b) Give a simple way to determine the constant C.

6. (Quadrature Rules) The basic "corrected Trapezoid rule" is given by

$$\int_0^h f(x)\,dx \approx \frac{h}{2}[f(0)+f(h)] + \frac{h^2}{12}[f'(0)-f'(h)].$$

   (a) *Derive* this rule.

   (b) Determine the *polynomial degree* of this rule.

   (c) Give an expression for the associated *composite rule* for the general case of non-uniform cell widths, as well as the simplification for the case of a uniform mesh.

7. (Orthogonal Polynomials) Define the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)w(x)\,dx,$$

where $w(x)$ is a positive weight function, and consider the family of orthogonal polynomials $p_0, p_1, p_2, \ldots$ with respect to this inner product. Here p_j is of degree j and has leading coefficient one.

   (a) Give the form of the three-term recurrence relation that these polynomials satisfy.

   (b) Prove the existence of the three-term recurrence relation. Give expressions for the coefficients in the recurrence relation.

8. (Gram-Schmidt Procedure) Consider the $m \times n$ matrix A with linearly independent columns and $m \geq n$.

   (a) Describe the Gram-Schmidt procedure for orthogonalizing the columns of A.

   (b) Describe the modified Gram-Schmidt procedure for orthogonalizing the columns of A.

   (c) Should you use the Gram-Schmidt procedure or modified Gram-Schmidt procedure in computations in floating point arithmetic. Explain.

9. (Projectors)

   (a) Give the defining equation for a projector P.

   (b) Show that $I - P$ also is a projector.

   (c) Show that $I - P$ projects onto the nullspace of P.

   (d) What is an orthogonal projector? How do the range and nullspace of an orthogonal projector relate?

   (e) Give a formula for the orthogonal projector onto the range of $A \in R^{m \times n}$, $m > n$. The matrix A is assumed to have linearly independent columns.

10. Singular Value Decomposition Let $A \in R^{m \times n}$, $m \geq n$.

   (a) What is the singular value decomposition of A? Define all the matrices involved and discuss their properties.

   (b) Prove the existence of the singular value decomposition of A.