Set 10: Orthogonality and Approximation-Part 1

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Additional References

- J. Dettman, *Mathematical Methods in Physics and Engineering*, McGraw Hill, 1969
- D. Luenberger, *Optimization by Vector Space Methods*, Wiley, 1969

Finite Dimensional Spaces

Let $M = M^*$ be positive definite. $\mathbb{R}^n$ and $\mathbb{C}^n$ are finite dimensional Hilbert spaces with

$$< x, y >= y^* M x, \quad \|x\|_M^2 =< x, x >$$

$$< x, y >=< y, x >^*$$

$$< \alpha x + \beta z, y >= \alpha < x, y > + \beta < z, y >$$

$$\forall x \neq 0, \ \ < x, x > > \ 0, \ \ \text{and} \ \ < x, x >= 0 \leftrightarrow x = 0$$

Vectors x and y are orthogonal when $< x, y >= y^* M x = 0$.

$*$ is transpose or Hermitian transpose.

Finite Dimensional Spaces

A set of orthonormal vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ satisfy

$$< q_i, q_j >= \delta_{ij}, \quad 1 \leq i, j \leq k \leq n$$

$$Q_k = \begin{pmatrix} q_1 & q_2 & \dots & q_k \end{pmatrix}$$

$$Q_k^* M Q_k = I_k$$

$\mathcal{R}(Q_k)$ is a k-dimensional subspace with orthonormal basis $\{q_1, q_2, \dots, q_k\}$

$$v \in \mathcal{R}(Q_k) \leftrightarrow v = Q_k c, \quad c = Q_k^* M v$$

$$v = q_1 < v, q_1 > + q_2 < v, q_2 > + \dots + q_k < v, q_k >$$

with c unique.

Finite Dimensional Spaces

- angle is preserved from the subspace to $\mathbb{R}^n$ or $\mathbb{C}^n$

$$< v_1, v_2 >=< Q_k c_1, Q_k c_2 >= c_2^* Q_k^* M Q_k c_1 =< c_1, c_2 >= c_1^* c_2$$

- length is preserved

$$\|v\|^2 =< Q_k c, Q_k c >= c^* Q_k^* M Q_k c =< c, c >= \|c\|_2^2 = \sum_{i=1}^{k} |\gamma_i|^2$$

Finite Dimensional Spaces

- $\forall f \in \mathcal{H}, \quad f = Q_n c$, i.e., if $k = n$ then $f \leftrightarrow c$ uniquely.
- $\hat{f} = q_1 < f, q_1 > + q_2 < f, q_2 > + \cdots + q_k < f, q_k > \in \mathcal{R}(Q_k)$

$$\forall v \in \mathcal{R}(Q_k), \quad \|f - \hat{f}\| \leq \|f - v\|$$

$$f - \hat{f} \perp \mathcal{R}(Q_k)$$

- $\forall f \in \mathcal{H}, \quad \|f\|^2 \geq \sum_{i=1}^{k} |\gamma_i|^2$, where $\gamma_i = < f, q_i >, \quad k \leq n$.

An Hilbert Space

Definition 10.1. A Hilbert space is a vector space that:

- has an inner product, denoted $< x, y >$;
- that induces a norm $\|x\| =< x, x >$,
- and is complete under the norm.

Example:

$\mathbb{R}^n$ and $\mathbb{C}^n$ are finite dimensional Hilbert spaces.

An Infinite Dimensional Hilbert Space

If the space in infinite dimensional we must deal with linear combinations that have an infinite number of terms. Convergence?

Definition 10.2. The set $\{b_1, \ldots, b_i, \ldots\}$ where $b_i \in \mathcal{H}$ is an orthogonal sequence if

$$< b_i, b_j > \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \ . \end{cases}$$

It is an orthonormal sequence if $< b_i, b_j >= \delta_{ij}$.

An Infinite Dimensional Hilbert Space

Definition 10.3. Given a sequence $\{x_1, \dots, x_i, \dots\}$ with $x_i \in \mathcal{H}$, the infinite series $\sum_{i=1}^{\infty} x_i$ converges to $x \in \mathcal{H}$ if

$$\lim_{n \to \infty} \|s_n - x\| = 0$$

where $s_n = \sum_{i=1}^{n} x_i$ and we assume throughout that the norm is induced by the inner product.

An Infinite Dimensional Hilbert Space

Theorem 10.1. *Given an orthonormal sequence* $\{b_1, \ldots, b_i, \ldots\}$ *with* $b_i \in \mathcal{H}$*, the infinite series* $\sum_{i=1}^{\infty} \gamma_i b_i$ *converges to* $x \in \mathcal{H}$ *if and only if* $\sum_{i=1}^{\infty} |\gamma_i|^2 < \infty$ *in which case* $\gamma_i =< x, b_i >$.

So, a square summable sequence maps to an element of $\mathcal{H}$.

An Infinite Dimensional Hilbert Space

Lemma 10.2. *(Bessel's Inequality)*

Given an orthonormal sequence $\{b_1, \ldots, b_i, \ldots\}$ *with* $b_i \in \mathcal{H}$*, we have*

$$\forall x \in \mathcal{H} \quad \sum_{i=1}^{\infty} |< x, b_i >|^2 = \sum_{i=1}^{\infty} |\gamma_i|^2 \leq \|x\|^2$$

So, $x \in \mathcal{H}$ maps to a square summable sequence.

Infinite Dimensional Hilbert Space

- Given an orthonormal sequence $\{b_1, \ldots, b_i, \ldots\}$ with $b_i \in \mathcal{H}$,

$$x = \sum_{i=1}^{\infty} \gamma_i b_i \Rightarrow \gamma_i =< x,\ b_i >$$

$$\forall x \in \mathcal{H},\ \ \sum_{i=1}^{\infty} < x,\ b_i > b_i = \hat{x} \in \mathcal{H}$$

- When $\mathcal{H}$ is finite dimensional $k = n$ or $k \neq n$ is enough to force $x = \hat{x}$ for all $x \in \mathcal{H}$.
- $\mathcal{H}$ can have subspaces with infinite dimension.
- $x = \hat{x}$ vs $x \neq \hat{x}$ depends on another property of the b_i's

Infinite Dimensional Hilbert Space

Lemma 10.3. *Given a Hilbert space $\mathcal{H}$, let $S \subseteq \mathcal{H}$ (not necessarily a subspace). We have*

- $S^\perp$ *is a closed subspace, i.e., the limit of every convergent sequence on $S^\perp$ is in $S^\perp$.*
- $S \subseteq S^{\perp\perp}$
- $S^{\perp\perp}$ *is the smallest closed subspace containing S, denoted $\overline{[S]}$.*

An Infinite Dimensional Hilbert Space

Theorem 10.4 (Luenberger, p. 51 and 60)**.** *Let $\mathcal{H}$ be a Hilbert space consider an element $x \in \mathcal{H}$. Given an orthonormal sequence $\{b_1, \ldots, b_i, \ldots\}$ with $b_i \in \mathcal{H}$, the series*

$$\sum_{i=1}^{\infty} < x,\ b_i > b_i$$

converges to an element $\hat{x}$ in the closed subspace, M, generated by the sequence. Further,

$$x - \hat{x} \perp M$$

$$\forall v \in M, \quad \|x - \hat{x}\| \leq \|x - v\|$$

An Infinite Dimensional Hilbert Space

- So, in general, $\hat{x} \neq x$ but it is an optimal approximation.
- This generalizes the finite dimensional classical projection theorm of least squares.
- We will look at specific cases of Theorem 10.4.
- We are interested in subspaces of an infinite dimensional Hilbert space that have finite dimension or fininte codimension.
- We need $\hat{x} = x$ conditions to yield a more desireable infinite expansion that will be truncated to a finite number of terms.

Example

The set of continuous functions on $[0,\ 2\pi]$ with the following inner product

$$< f, g >= \int_0^{2\pi} f(x)g(x)dx$$

is a Hilbert space, $\mathcal{H}$, when equality is taken to mean equal almost everywhere.

Consider the sequence of elements in $\mathcal{H}$

$$b_n(x) = \frac{1}{\sqrt{\pi}} \sin nx, \quad n = 0,\ 1, \ldots$$

Example

$$< b_s, b_r >= \int_0^{2\pi} (\sin sx)(\sin rx) dx = 0, \quad r \neq s$$

$$f = \cos x \in \mathcal{H}$$

$$< f, b_n >= \int_0^{2\pi} (\sin nx)(\cos x) dx = 0$$

The orthonormal sequence $b_0, b_1, \ldots,$ has countably infinite number of orthogonal directions but there is still a direction, given by f, that is orthogonal to **all of them**.

An Infinite Dimensional Hilbert Space

Definition 10.4. An orthonormal sequence, $\{b_i\}$, in a Hilbert space $\mathcal{H}$ is said to be complete if the closed subspace generated by the b_i's is $\mathcal{H}$.

Lemma 10.5. *An orthonormal sequence* $\{b_1, \ldots, b_i, \ldots\}$ *with* $b_i \in \mathcal{H}$ *is complete (or closed) if*

$$\nexists f \in \mathcal{H} \quad | \quad \|f\| = 1 \quad \textit{and} \quad \forall b_i, \quad < f, b_i >= 0$$

Or equivalently,

$$\forall b_i, \quad < f, b_i >= 0 \rightarrow f = 0$$

An Infinite Dimensional Hilbert Space

Theorem 10.6. *Given a complete orthonormal sequence* $\{b_1, \ldots, b_i, \ldots\}$ *with* $b_i \in \mathcal{H}$*, we have*

$$\forall f \in \mathcal{H}, \quad \lim_{n \to \infty} \|f - \sum_{i=1}^{n} \gamma_i b_i\|^2 = 0$$

where $\gamma_i =< f, b_i >$.

Additionally, we have the completeness relation or Parseval's equality

$$\|f\|^2 = \sum_{i=1}^{\infty} |\gamma_i|^2$$

Parseval's Equality

Theorem 10.7. *Given a complete orthonormal sequence* $\{b_1, \ldots, b_i, \ldots\}$ *with* $b_i \in \mathcal{H}$*,*

$$\forall f, g \in \mathcal{H} \quad \textit{let} \quad \gamma_i =< f, b_i > \quad \textit{and} \quad \mu_i =< g, b_i >$$

$$\textit{then} \quad f = \sum_{i=1}^{\infty} \gamma_i b_i, \quad g = \sum_{i=1}^{\infty} \mu_i b_i, \quad \textit{and} \quad < f, g >= \sum_{i=1}^{\infty} \mu_i^* \gamma_i$$

Note. The form in Theorem 10.6 results by taking $f = g$.

Recall, the finite dimensional result

$$< v_1, v_2 >=< Q_k c_1, Q_k c_2 >= c_2^* Q_k^* M Q_k c_1 =< c_1, c_2 >= c_1^* c_2$$

An Infinite Dimensional Hilbert Space

Definition 10.5. Given $\omega(x) : [a, b] \rightarrow \mathbb{R}$, a nonnegative integrable function, the space $\mathcal{L}^2_\omega[a, b]$ is the set

$$\left\{ f \mid f : [a, b] \rightarrow \mathcal{S}, \quad \int_a^b |f(x)|^2 \omega(x) dx < \infty \right\}$$

$$\mathcal{S} = \mathbb{R} \quad \text{or} \quad \mathcal{S} = \mathbb{C}$$

The inner product and induced norm on the space are

$$(f, g)_\omega = \int_a^b g(x)^* f(x) \omega(x) dx \quad \text{and} \quad \|f\|^2_\omega = \int_a^b |f(x)|^2 \omega(x) dx$$

An Infinite Dimensional Hilbert Space

- The function $\omega(x)$ is analogous to the positive definite matrix M.
- The integration here is in general Lebesgue.
- Riemann integration can be used when f is taken to be piecewise continuous on $[a,b]$ with, e.g., a finite number of discontinuities.
- Equality is "equal almost everywhere" so the 0 element is an equivalence class of functions that are 0 everywhere but, e.g., a finite number of points for Riemannian integration and more generally a set of measure 0 for Riemannian and Lebesgue integration.
- Convergence in $\|f\|_\omega^2$ is called convergence in the mean.
- Convergence in the mean $\not\Longrightarrow$ pointwise convergence

An Infinite Dimensional Hilbert Space

- When $\mathcal{S} = \mathbb{C}$ care must be taken to be consistent in the inner product and use of the complex conjugate,

$$\forall f, g \in \mathcal{L}^2_\omega[a,b], \;\; (f,g)_\omega = (g,f)^*_\omega$$
$$\forall \alpha \in \mathbb{C}, \quad |\alpha|^2 = \alpha^* \alpha = \alpha \alpha^*$$

- The text considers $\mathcal{S} = \mathbb{R}$, i.e., real-valued functions on the real interval $[a,b]$, so we have

$$\forall f, g \in \mathcal{L}^2_\omega[a,b], \;\; (f,g)_\omega = (g,f)_\omega = \int_a^b f(x) g(x) \omega(x) dx$$
$$\forall \alpha \in \mathbb{R}, \quad |\alpha|^2 = \alpha^2$$

Representation on the Space

Functions in the space $\mathcal{L}^2_\omega[a,b]$ can be represented via bases with a countably infinite number of elements:

$\forall f \in \mathcal{L}^2_\omega[a,b]$ we have

$$f(x) = \sum_{i=0}^{\infty} \alpha_i \phi_i(x)$$

where

$$\{\phi_0(x), \phi_1(x), \ldots\}$$

is a complete set of orthogonal functions, i.e., a basis for $\mathcal{L}^2_\omega[a,b]$.

Orthogonal Basis and the Representation

Definition 10.6. Let $\{\phi_0(x), \phi_1(x), \ldots\}$ be a complete set of orthogonal functions in $\mathcal{L}^2_\omega[a,b]$. $\forall f \in \mathcal{L}^2_\omega[a,b]$ the series

$$Sf = \sum_{i=0}^{\infty} \alpha_i \phi_i(x) \text{ where } \alpha_i = \frac{(f, \phi_i)_\omega}{(\phi_i, \phi_i)_\omega}$$

is called the generalized Fourier series of f.

In terms of an orthonormal sequence $\{\tilde{\phi}_0(x), \ldots\}$,

$$Sf = \sum_{i=0}^{\infty} \tilde{\alpha}_i \tilde{\phi}_i(x), \quad \tilde{\phi}_i = \frac{\phi_i}{\|\phi_i\|}, \quad \tilde{\alpha} = (f, \tilde{\phi}_i)_\omega = \alpha\|\phi_i\|$$

Orthogonal Polynomials

- choose $[a, b]$ and $\omega(x)$
- Gram-Schmidt process applied to $\{1, x, x^2, \dots\}$
- low-order recurrence relation from basic properties
- orthogonality and inner product values
- Rodrigues' form – derivative of a polynomial
- some $\phi_i(x)$ are eigenfunctions of Sturm-Liouville differential equation for a specific set of coefficients $\rightarrow$ they form an orthogonal basis, e.g. Jacobi polynomials.

Sturm-Liouville Theory

Definition 10.7. The regular Sturm-Liouville differential equation on $a \le x \le b$ is

$$-(p(x)u'(x))' + q(x)u(x) = \lambda\omega(x)u(x)$$

$$\alpha_0 u'(a) - \alpha_1 u(a) = 0$$

$$\beta_0 u'(b) - \beta_1 u(b) = 0$$

$$p(x) \in \mathcal{C}^1[a,b], \quad w(x), q(x) \in \mathcal{C}^0[a,b]$$

$$a < x < b, \quad p(x) > 0, \;\; \omega(x) > 0, \;\; q(x) \ge 0$$

where at least one of the α pair and at least one of the β pair are nonzero. $(\lambda, u(x))$ pairs are an eigenvalue and its associated eigenfunction.

Sturm-Liouville Theory

Definition 10.8. The singular Sturm-Liouville differential equation on $a \le x \le b$ is

$$-(p(x)u'(x))' + q(x)u(x) = \lambda\omega(x)u(x)$$

$$p(x) \in \mathcal{C}^1[a,b], \quad w(x), q(x) \in \mathcal{C}^0[a,b]$$

$$a < x < b, \quad p(x) > 0, \;\; \omega(x) > 0, \;\; q(x) \ge 0$$

with $p(a) = p(b) = 0$ and $p(x)u'(x) \to 0$ as $x \to a$ or as $x \to b$. In other words, u' cannot grow faster than p goes to 0 at the boundary.

Sturm-Liouville Theory

Theorem 10.8. *Given the Sturm-Liouville differential equation on* $a \leq x \leq b$*, there exists a countably infinite set of eigenvalues and associated eigenfunctions* $(\lambda_i, u_i(x))$*,* $i = 1, 2, \ldots$ *such that*

- $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots$
- $\forall i \neq j, \;\; (u_i, u_j)_\omega = 0$
- $(u_i, u_i)_\omega \neq 0$
- $\{u_1(x), u_2(x), \ldots\}$ *is a complete orthogonal set*
- $u_i(x)$ *has* $i - 1$ *distinct* 0*'s in* $a < x < b$

Legendre Polynomials

- $[a, b] = [-1, 1]$ and $\omega(x) = 1$
- $P_0(x) = 1$, $P_1(x) = x$ and

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

- $P_n(1) = 1$ and $P_n(-x) = (-1)^n P_n(x)$
- Rodrigues' form

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n]$$

- orthogonality: $(P_n, P_m) = 0$ for $m \neq n$ and

$$(P_n, P_n) = \frac{2}{2n+1}$$

Legendre and the Singular Sturm-Liouville Equation

$$-1 \leq x \leq 1, \quad p(x) = 1 - x^2, \quad q(x) = 0, \quad \omega(x) = 1$$

$$p(1) = p(-1) = 0, \quad \lambda = n(n+1)$$

$$\left[(1 - x^2)y'\right]' + \lambda y = 0$$

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

Easy to verify the S-L problem is satisfied for

$$P_0(x) = 1, \;\; P_1(x) = x, \;\; P_2(x) = \frac{1}{2}(3x^2 - 1), \;\; \ldots$$

Chebyshev Polynomials

- $[a, b] = [-1, 1]$ and $\omega(x) = 1/\sqrt{1 - x^2}$
- $T_n(x) = \cos(n \arccos x)$
- $T_0(x) = 1$, $T_1(x) = x$ and

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x)$$

- $T_n(1) = 1$ and $T_n(-x) = (-1)^n T_n(x)$
- Rodrigues' form $n \geq 1$

$$T_n(x) = \frac{\sqrt{1 - x^2}}{(-1)^n (2n-1)(2n-3)\cdots 1} \frac{d^n}{dx^n}[(1 - x^2)^{n-1/2}]$$

- orthogonality: $(T_n, T_m) = 0$ for $m \neq n$ and

$$(T_0, T_0) = \pi \text{ and } (T_n, T_n) = \frac{\pi}{2},\ n \geq 1$$

Chebyshev and the Singular Sturm-Liouville Equation

$$-1 \le x \le 1, \quad p(x) = \sqrt{1-x^2}, \quad q(x) = 0, \quad \omega(x) = 1/\sqrt{1-x^2}$$

$$p(1) = p(-1) = 0, \quad \lambda = n^2$$

$$\left[\sqrt{1-x^2}\ \ y'\right]' + \frac{n^2}{\sqrt{1-x^2}} y = 0$$

$$(1-x^2)y'' - xy' + n^2 y = 0$$

Easy to verify the S-L problem is satisfied for

$$T_0 = 1, \quad T_1 = x, \quad , T_2 = 2x^2 - 1 \ldots$$

Jacobi Polynomials

The algebraic polynomials that are eigenfunctions of the singular Sturm-Liouville equation are the two-parameter Jacobi polynomials:

$$J^{\alpha\beta}(x), \quad \alpha > -1, \quad \beta > -1$$

$$p(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}, \;\; q(x) = 0, \;\; \omega(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

$$p(1) = p(-1) = 0, \quad \lambda = n(n+\alpha+\beta+1)$$

$$(p(x)y'(x))' + \lambda\omega(x)y(x) = 0$$

$$\alpha = \beta = 0 \rightarrow \quad \text{Legendre polynomials}$$

$$\alpha = \beta = -\frac{1}{2} \rightarrow \quad \text{Chebyshev polynomials}$$

Laguerre Polynomials

- normalized form
- $[a,b] = [0,\infty)$ and $\omega(x) = e^{-x}$
- $L_0(x) = 1$, $L_1(x) = -x+1$ and

$$(n+1)L_{n+1} = (2n+1-x)L_n(x) - nL_{n-1}(x)$$

- Rodrigues' form $n \geq 1$

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}[x^n e^{-x}]$$

- orthogonality: $(L_n, L_m) = 0$ for $m \neq n$ and

$$(L_n, L_n) = 1$$

Laguerre and the Singular Sturm-Liouville Equation

$$0 \le x \le \infty, \quad p(x) = xe^{-x}, \quad q(x) = 0, \quad \omega(x) = e^{-x}$$

$$p(0) = p(\infty) = 0, \quad \lambda = n$$

$$\left[xe^{-x} \ \ y' \right]' + ne^{-x} y = 0$$

$$xy'' + (1-x)y' + ny = 0$$

Easy to verify the S-L problem is satisfied for

$$L_0 = 1, \quad L_1 = 1 - x, \ \ , L_2 = \frac{1}{2}(x^2 - 4x + 2) \ldots$$

Hermite Polynomials

- $[a, b] = (-\infty, \infty)$ and $\omega(x) = e^{-\sigma x^2}$
- $\sigma = 1$ is physics form and $\sigma = 1/2$ is probability form
- $H_0(x) = 1$, $H_1(x) = 2\sigma x$ and

$$H_{n+1} = 2\sigma x H_n(x) - 2n\sigma H_{n-1}(x)$$

- $H_n(-x) = (-1)^n H(x)$
- Rodrigues' form $n \geq 0$, $\sigma = 1$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}[e^{-x^2}]$$

- orthogonality: $\sigma = 1$, $(H_n, H_m) = 0$ for $m \neq n$ and $(H_n, H_n) = 2^n n! \sqrt{\pi}$.

Hermite and the Singular Sturm-Liouville Equation

$$\sigma = 1, \quad -\infty \le x \le \infty, \quad p(x) = e^{-x^2}, \quad q(x) = 0, \quad \omega(x) = e^{-x^2}$$

$$p(-\infty) = p(\infty) = 0, \quad \lambda = 2n$$

$$\left[e^{-x^2} \quad y'\right]' + 2ne^{-x^2} y = 0$$

$$y'' - 2xy' + 2ny = 0$$

Easy to verify the S-L problem is satisfied for

$$H_0 = 1, \quad H_1 = 2x, \quad , H_2 = 4x^2 - 2 \ldots$$

Fourier Polynomials

- $[a, b] = (0, 2\pi)$ and $\omega(x) = 1$
- complex-valued $f(x)$ used to define space:

$$\mathcal{L}^2_\omega[a, b] = \left\{ f \mid f : [a, b] \to \mathbb{C}, \quad \int_a^b |f(x)|^2 dx < \infty \right\}$$

- The inner product and induced norm on the space are

$$(f, g)_\omega = \int_a^b g(x)^* f(x) dx \text{ and } \|f\|^2_\omega = \int_a^b |f(x)|^2 dx$$

- $\phi_k(x) = e^{ikx}$ where $i = \sqrt{-1}$ for $k = 0, \pm 1, \pm 2, \ldots$
- orthogonality: $(\phi_n, \phi_m) = 0$ for $m \neq n$ and $(\phi_n, \phi_n) = 2\pi$
- trigonometric polynomials: $e^{ikx} = \cos kx + i \sin kx$
- related to regular Sturm-Liouville ODE

Orthogonal Polynomials

Suppose $\{\phi_0(x), \phi_1(x), \ldots\}$ is a complete set of orthogonal polynomials in $\mathcal{L}^2_\omega[a, b]$.

Theorem 10.9. *The roots, $x_1, \ldots, x_n$ of $\phi_n(x)$ are real, simple and lie in the interval $a < x < b$, i.e., the interior of $[a, b]$.*

This was mentioned with the Sturm-Liouville Theory characterization of complete orthogonal eigenfunctions.

It can be proven directly based on orthogonality.

Orthogonal Polynomials

Theorem 10.10. *Suppose* $\{\phi_0(x), \phi_1(x), \ldots\}$ *is a complete set of orthogonal polynomials in* $\mathcal{L}^2_\omega[a,b]$ *with* $(\phi_i, \phi_i) = 1$. *If* α_k *and* β_k *are the coefficients of of* x^k *and* x^{k-1} *respectively in* $\phi_k(x)$ *then*

$$\phi_{n+1}(x) = (a_n x + b_n)\phi_n(x) - c_n \phi_{n-1}(x)$$

$$a_n = \frac{\alpha_{n+1}}{\alpha_n}, \quad b_n = \frac{\alpha_{n+1}}{\alpha_n}\left(\frac{\beta_{n+1}}{\alpha_{n+1}} - \frac{\beta_n}{\alpha_n}\right), \quad c_n = \frac{\alpha_{n+1}\alpha_{n-1}}{\alpha_n^2}$$

Orthogonal Polynomials

The recurrence can be initialized with

- $\phi_{-1}(x) = 0$ and $\phi_0(x) = \alpha_0$
- or with $\phi_0(x)$ and $\phi_1(x)$ specific polynomials.

In the latter case take note of the normalization of the norm of $\phi_i(x)$, e.g., it is often made 1 but not necessarily.

The textbook has a form of the recurrence that assumes monic $\phi_i(x)$.

Truncated Generalized Fourier Series

Lemma. *Let* $f \in \mathcal{L}^2_\omega[a,b]$ *and define the generalized Fourier series truncated after* $n+1$ *terms*

$$f_n(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x).$$

We have convergence in the mean, i.e., in the $\mathcal{L}^2_\omega[a,b]$ *sense,*

$$\lim_{n\to\infty} \|f - f_n\|_\omega = 0.$$

Further,

$$\|f\|^2_\omega = \sum_{i=0}^{\infty} \alpha_i^2 \|\phi_i(x)\|^2_\omega, \quad \|\hat{f}\|^2_\omega = \sum_{i=0}^{n} \alpha_i^2 \|\phi_i(x)\|^2_\omega$$

$$\|f - f_n\|^2_\omega = \sum_{i=n+1}^{\infty} \alpha_i^2 \|\phi_i(x)\|^2_\omega$$

Optimality

Theorem 10.11. *Let* $\{\phi_0(x), \phi_1(x), \ldots\}$ *be a complete set of orthogonal functions in* $\mathcal{L}^2_\omega[a,b]$*,* $f \in \mathcal{L}^2_\omega[a,b]$ *and* $f_n(x) = \sum_{i=0}^n \alpha_i \phi_i(x)$*.*

If $\mathcal{S}_n = span[\phi_0(x), \phi_1(x), \ldots, \phi_n(x)]$ *then*

$$\|f - f_n\|_\omega = \min_{q \in \mathcal{S}_n} \|f - q\|_\omega$$

Optimality

Proof. We have

$$r_n = f - f_n = \sum_{i=n+1}^{\infty} \alpha_i \phi_i(x)$$

$$\therefore \quad r_n \perp \phi_j, \quad 0 \le j \le n \quad \text{and} \quad \forall q \in \mathcal{S}_n \quad r_n \perp q$$

We have $\forall q \in \mathcal{S}_n$,

$$\|r_n\|_2^2 = (r_n, f - f_n + q - q)_\omega = (r_n, f - q)_\omega + (r_n, q - f_n)_\omega$$
$$= (r_n, f - q)_\omega + (r_n, \tilde{q})_\omega = (r_n, f - q)_\omega \le \|r_n\|_\omega \|f - q\|_\omega$$
$$\therefore \|r_n\|_\omega \le \|f - q\|_\omega$$

$\square$

Optimality

- f_n is a continuous weighted least-squares approximation to f.
- Need $\{\phi_0(x), \phi_1(x), \ldots\}$ a complete set of orthogonal functions in $\mathcal{L}^2_\omega[a, b]$,
- to approximate $f \in \mathcal{L}^2_\omega[a, b]$ over $\mathcal{S}_n$ we must compute the $\alpha_i = (f, \phi_i)_\omega / (\phi_i, \phi_i)_\omega$ and define $f_n(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x)$.
- α_i are typically approximated numerically.
- Note that f_n is easily incremented to f_{n+1} with a single additional coefficient. This is a crucial property in many efficient algorithms that exploit orthogonality, e.g., conjugate directions and conjugate gradient.

Pointwise Error and Convergence

We have convergence in the mean, for $f \in \mathcal{L}^2_\omega[a,b]$

$$\lim_{n\to\infty} \|f - f_n\|_\omega = 0$$

where $f_n(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x)$.

This does not say anything about pointwise error or convergence, i.e.,

$$\lim_{n\to\infty} |f(x) - f_n(x)|$$

Orthogonal Polynomials

Theorem 10.12. *(Christoffel-Darboux) Suppose* $\{\phi_0(x), \phi_1(x), \ldots\}$ *is a complete set of orthogonal polynomials in* $\mathcal{L}^2_\omega[a,b]$ *with* $(\phi_i, \phi_i) = 1$.

$$(x-\xi)G_n(x,\xi) = (x-\xi)\sum_{i=0}^{n}\phi_i(x)\phi_i(\xi)$$

$$= \frac{\alpha_n}{\alpha_{n+1}}\left[\phi_{n+1}(x)\phi_n(\xi) - \phi_{n+1}(\xi)\phi_n(x)\right].$$

$G_n(x,\xi)$ *is called the kernel of the set of orthogonal polynomials.*

Pointwise Error and Convergence

Lemma. *Since it can be shown that*

$$\int_a^b \omega(\xi) G_n(x,\xi) d\xi = 1, \ \textit{where } G_n(x,\xi) = \sum_{i=0}^{n} \phi_i(x)\phi_i(\xi)$$

and $(\phi_i, \phi_i)_\omega = 1$, *the pointwise error has the form*

$$R_n(x) = f(x) - f_n(x) = f(x) - \sum_{i=0}^{n} \alpha_i \phi_i(x)$$

$$= f(x) - \sum_{i=0}^{n} \phi_i(x) \int_a^b \omega(\xi) f(\xi) \phi_i(\xi) d\xi$$

$$= f(x) - \int_a^b \omega(\xi) f(\xi) \sum_{i=0}^{n} \phi_i(x)\phi_i(\xi) d\xi$$

$$= f(x) - \int_a^b \omega(\xi) G_n(x,\xi) f(\xi) d\xi = \int_a^b \omega(\xi) G_n(x,\xi)(f(x) - f(\xi)) d\xi$$

Pointwise Error and Convergence

Need extra smoothness to state uniform convergence results.

Theorem 10.13. *Suppose* $\{P_0(x), P_1(x), \ldots\}$ *are the Legendre polynomials in* $\mathcal{L}^2_\omega[-1,1]$. *Let* $f \in \mathcal{L}^2_\omega[-1,1]$ *have continuous first and second derivatives. If* $f_n(x)$ *is the optimal polynomial of degree* n *approximating* $f(x)$ *with* $\omega(x) = 1$ *then for any* $\epsilon > 0, \exists n > 0$ *such that* $\forall -1 \leq x \leq 1$

$$|f(x) - f_n(x)| \leq \frac{\epsilon}{\sqrt{n}}$$

$$|f(x) - f_n(x)| = O\big(n^{-1/2}\big)$$

Pointwise Error and Convergence

Theorem 10.14. *Suppose* $\{T_0(x), T_1(x), \ldots\}$ *are the Chebyshev polynomials in* $\mathcal{L}^2_\omega[-1,1]$. *Let* $f \in \mathcal{L}^2_\omega[-1,1]$ *have continuous first and second derivatives. If* $f_n(x)$ *is the optimal polynomial of degree* n *approximating* $f(x)$ *with* $\omega(x) = 1/\sqrt{1-x^2}$ *then for any* $\epsilon > 0, \exists n > 0$ *such that* $\forall -1 \le x \le 1$

$$|f(x) - f_n(x)| = O\big(n^{-1}\big)$$

Note. $|f(x) - f_n(x)| = O\big(n^{-1/2}\big)$ can be shown for any of the families of orthogonal polynomials under mild assumptions and a continuous second derivative.