10.4: Linear Systems of Equations: Determinants

Supplementary Notes

Associated with every square matrix is a value called the *determinant*, which is used for solving systems of linear equations.

The determinant of a 2×2 matrix A with elements a_{ij}, $1 \le i, j \le 2$, is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The determinant of an $n \times n$ matrix A with elements a_{ij}, $1 \le i, j \le n$, is defined recursively as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \begin{array}{l} \text{(expanding across a row):} \\ = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \text{ for any } 1 \le i \le n \\ \\ \text{(expanding down a column):} \\ = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \text{ for any } 1 \le j \le n \end{array}$$

where A_{ij} is the matrix that results from deleting row i and column j from A.

<u>Cramer's Rule:</u> If the coefficient matrix A of the linear system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{cases}$$

has nonzero determinant. that is, if

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then the system has a unique solution

$$x = \frac{|A_x|}{|A|} \qquad y = \frac{|A_y|}{|A|} \qquad z = \frac{|A_z|}{|A|}$$

where

$$|A_x| = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad |A_y| = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad |A_z| = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

<u>Properties of Determinants</u>: For a square matrix A, its determinant $|A|$ has the following properties

- The sign of $|A|$ switches if any two rows (or columns) of A are interchanged.
- If a row (or column) of A gets multiplied by a nonzero number c, so does its determinant.
- If a nonzero multiple of a row (or column) of A is added to another row (or column), $|A|$ remains the same.

Exercises

1. Find $\begin{vmatrix} 1 & -1 & 2 \\ -x & 2y & z \\ 3 & -2 & 4 \end{vmatrix}$.

2. Write the solution given by Cramer's Rule for the following system, where

$$J = \begin{vmatrix} 3 & -4 & 2 \\ 1 & -1 & 2 \\ 2 & 2 & 3 \end{vmatrix}, \quad K = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & -3 & 3 \end{vmatrix}, \quad L = \begin{vmatrix} 1 & -4 & 2 \\ -2 & -1 & 2 \\ -3 & 2 & 3 \end{vmatrix}, \quad M = \begin{vmatrix} 3 & -4 & 1 \\ 1 & -1 & -2 \\ 2 & 2 & -3 \end{vmatrix}$$

$$\begin{cases} 3x - 4y + 2z &= 1 \\ x - y + 2z &= -2 \\ 2x + 2y + 3z &= -3 \end{cases}$$

3. Given that $\begin{vmatrix} -1 & -1 & 9 \\ 3 & 4 & -18 \\ -2 & 1 & -3 \end{vmatrix} = 48$, use Cramer's rule to find the numeric solution for z in the following system.

$$\begin{cases} -x - y - 2z &= 9 \\ 3x + 4y + 3z &= -18 \\ -2x + y - z &= -3 \end{cases}$$

4. Find the following if $\begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ d & e & f \end{vmatrix} = 4$.

$$\begin{vmatrix} d+3 & e+6 & f+9 \\ a & b & c \\ 5 & 10 & 15 \end{vmatrix}$$