Homework 3 Foundations of Computational Math 1 Fall 2012

Problem 3.1

Suppose $A \in \mathbb{R}^{n\times n}$ is a nonsymmetric nonsingular diagonally dominant matrix with the following nonzero pattern (shown for $n = 6$)

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

It is known that a diagonally dominant (row or column dominant) matrix has an LU factorization and that pivoting is not required for numerical reliability.

3.1.a. Describe an algorithm that solves $Ax = b$ as efficiently as possible.

3.1.b. Given that the number of operations in the algorithm is of the form $Cn^k + O(n^{k-1})$, where C is a constant independent of n and $k > 0$, what are C and k?

Problem 3.2

It is known that if partial or complete pivoting is used to compute $PA = LU$ or $PAQ = LU$ of a nonsingular matrix then the elements of L are less than 1 in magnitude, i.e., $|\lambda_{ij}| \le 1$. Now suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, i.e., $A = A^T$ and $x \neq 0 \rightarrow x^T Ax > 0$. It is known that A has a factorization $A = LL^T$ where L is lower triangular with positive elements on the main diagonal (the Cholesky factorization). Does this imply that $|\lambda_{ij}| \le 1$? If so prove it and if not give an $n \times n$ symmetric positive definite matrix with $n > 3$ that is a counterexample and justify that it is indeed a counterexample.

Problem 3.3

Suppose $PAQ = LU$ is computed via Gaussian elimination with complete pivoting. Show that there is no element in $e_i^T U$, i.e., row i of U, whose magnitude is larger than $|\mu_{ii}| = |e_i^T U e_i|$, i.e., the magnitude of the (i, i) diagonal element of U.

Problem 3.4

Suppose you are computing a factorization of the $A \in \mathbb{C}^{n\times n}$ with partial pivoting and at the beginning of step i of the algorithm you encounter the the transformed matrix with the form

$$TA = A^{(i-1)} = \begin{pmatrix} U_{11} & U_{12} \\ 0 & A_{i-1} \end{pmatrix}$$

where $U_{11} \in \mathbb{R}^{i-1\times i-1}$ and nonsingular, and $U_{12} \in \mathbb{R}^{i-1\times n-i+1}$ contain the first $i-1$ rows of U. Show that if the first column of A_{i-1} is all 0 then A must be a singular matrix.

Problem 3.5

Let x and y be two vectors in $\mathbb{R}^n$.

3.5.a. Show that given x and y the value of $\|x - \alpha y\|_2$ is minimized when

$$\alpha_{min} = \frac{x^T y}{y^T y}$$

3.5.b. Show that $x = y\alpha_{min} + z$ where $y^T z = 0$, i.e., x is easily written as the sum of two orthogonal vectors with specifed minimization properties.