Qualifying Exam in Numerical Analysis
August 25, 2003

There are ten problems. Solve any seven problems out of the ten problems listed. If you attempt to solve more than seven problems, indicate below which seven you want to be graded

Problems to be graded: ______________________________________

(1) Let H be a real inner product space and V be a closed, convex subset of H. Prove that for any $f \in H$ there exists a unique best approximation by elements in V.

(2) Let $x_0 < x_1 < \cdots < x_n$, and $f \in C^n([x_0, x_n])$. Show that

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some $\xi \in (x_0, x_n)$. Here $f[x_0, \cdots, x_n]$ (called *nth divided difference of*) is defined by the recurrence relations: $f[x_i] := f(x_i)$, and

$$f[x_0, ..., x_n] = \frac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0}.$$

(3) Find a, b, and $\xi \in [-1, 1]$ so that the quadrature rule

$$\int_{-1}^{1} f\ dx \approx a\ f(-\sqrt{3}/3) + b\ f(\xi),$$

on $[-1, 1]$ has the maximum possible degree of precision. Justify your answer by proving that the degree of precision can not be increased.

(4) Prove or disprove: The *secant* method converges to a zero of $F(x) = x^3 + 2x - 12$ for any initial guesses x_0 and x_1, $x_0 \neq x_1$.

(5) Let A be a *symmetric, indefinite and invertible* matrix, $A \in \mathbb{R}^{n\times n}$, $n > 1$. Assume that A has at least one positive and one negative eigenvalue. Consider the following sequence of iterates, defined via the Richardson's method,

$$u_{k+1} = u_k + \frac{1}{2\rho(A)}(f - Au_k),$$

where

$$\rho(A) = \max\{|\lambda|\ :\quad \lambda \text{ is an eigenvalue of } A\}$$

Prove or disprove the following statement: The sequence $\{u_k\}_{k=0}^{\infty}$ converges to the solution of $Au = f$, for any initial guess u_0.

(6) Describe an algorithm for constructing a natural cubic spline for interpolating the data $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

(7) Let u_h be the finite element solution to

$$-u'' = f, \quad u(0) = u(1) = 0, \tag{1}$$

by piecewise linear continuous finite elements over a uniform mesh $\{x_i = ih\}_{i=0}^{N}$ where $h = 1/N$. If u_I is the piecewise, continuous linear interpolant of the solution u to (1) at the points x_i.

a. Show that $u_h = u_I$.

b. Estimate $|u - u_h|_{L_2([0,1])}$ in terms of powers of h.

(8) a. Describe an algorithm for the decomposition of symmetric positive definite matrix A as:

$$A = LL^t,$$

where L is a lower triangular matrix and L^t is the transpose of L.

b. Is this decomposition unique? Justify your answer.

c. Count the total number of long operations (multiplications, and divisions) that are needed to obtain such decomposition, by using your algorithm.

(9) Consider the implicit trapezoid scheme

$$y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1})). \tag{2}$$

for the solution of the initial value problem:

$$y' = f(t, y), \quad y(0) = y_0, \quad t \in [0, T]. \tag{3}$$

Assume that the right hand side f is Lipschitz continuous, and show that for sufficiently small step size h, (2) has a solution y_{n+1} for all n.

(10) Let A be a symmetric, positive definite $n \times n$ matrix. P is an $n \times m$ matrix with $\text{rank}(P) = m \leq n$.

a. Show that $A_C = P^t AP$ is invertible.

b. Define $Q_A = PA_C^{-1}P^t A$. Show that $Q_A u$ is the best approximation of u in Range(P) w.r.t. the A-inner product. The A-inner product is defined as follows:

$$(u, v)_A = (Au, v) \quad \forall u \in \mathbb{R}^n, \quad \forall v \in \mathbb{R}^n,$$

where $(\cdot, \cdot)$ is the standard inner product on $\mathbb{R}^n$, and for $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

its definition is as follows:

$$(u, v) = \sum_{i=1}^{n} u_i v_i.$$