MAC2312: Calculus 2 - Section 3

Test 4 August 6, 2015

Name: ______________________________

Answer each question in the space provided on the question sheets. If you run out of space for an answer, continue on the back of the page. Credit will only be given if you clearly show all of your work. Calculators may not be used for this test.

Question	**Points**	**Score**
1	10	
2	15	
3	15	
4	15	
5	20	
6	25	
7 (bonus)	–	
Total:	100	

1. [10 points] Determine whether the following statements are true or false.

(a) ___False___ If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) ___True___ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

(c) ___True___ A conditionally convergent series is convergent but not absolutely convergent.

(d) ___True___ The Maclaurin series of a function f is the Taylor series of f centered at 0.

(e) ___False___ The interval of convergence of a power series could be $[0, \infty)$.

2. [15 points] Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ is convergent or divergent.

The Ratio and Root tests are inconclusive, but the Comparison Test, Limit Comparison Test, or Integral Test can be used.

Comparison Test solution:

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1} > \sum_{n=1}^{\infty} \frac{n}{n^2+1} > \sum_{n=1}^{\infty} \frac{n}{n^2+n^2} = \sum_{n=1}^{\infty} \frac{n}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is a p-series with $p = 1$. Therefore, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ is divergent.

Limit Comparison Test solution:

For $a_n = \frac{n+1}{n^2+1}$ and $b_n = \frac{n}{n^2} = \frac{1}{n}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n+1}{n^2+1} \cdot \frac{n}{1} \right) = \lim_{n\to\infty} \frac{n^2+n}{n^2+1} = 1 > 0.$$

So, by the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge, but $\sum_{n=1}^{\infty} 1/n$ diverges since it is a p-series with $p = 1$. Therefore, $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ is divergent.

Integral Test solution:

$f(x) = \frac{x+1}{x^2+1}$ is continuous on $(-\infty, \infty)$ and positive on $(-1, \infty)$. f is decreasing on $(\sqrt{2}-1, \infty)$ since

$$f'(x) = \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2} < 0 \Rightarrow x \text{ is in } (-\infty, -1-\sqrt{2}) \cup (\sqrt{2}-1, \infty)$$

$$\begin{aligned}
\int_1^{\infty} \frac{x+1}{x^2+1}\, dx &= \lim_{t\to\infty} \int_1^t \frac{x+1}{x^2+1}\, dx = \lim_{t\to\infty} \left(\int_1^t \frac{x}{x^2+1}\, dx + \int_1^t \frac{1}{x^2+1}\, dx \right) \\
&= \lim_{t\to\infty} \left[\frac{1}{2}\ln(x^2+1) + \arctan x \right]_{x=1}^{t} \\
&= \lim_{t\to\infty} \left[\frac{1}{2}\ln(t^2+1) + \arctan t \right] - \left(\frac{1}{2}\ln 2 + \arctan 1 \right) \\
&= \lim_{t\to\infty} \frac{1}{2}\ln(t^2+1) + \frac{\pi}{2} - \frac{1}{2}\ln 2 - \frac{\pi}{4} = \infty
\end{aligned}$$

Therefore, by the Integral Test, $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ is divergent.

3. [15 points] Determine whether $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$ is convergent or divergent.

For $b_n = \frac{\ln n}{n}$,

(i) $b_{n+1} \le b_n$ for $n \ge 3$ since $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x\cdot(1/x) - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$ for $x > e$, and

(ii) $b_n \to 0$ as $n \to \infty$ since $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$ by L'Hôspital's Rule.

Therefore, by the Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$ is convergent. Additionally, the series is conditionally convergent, since $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ is divergent (by the Comparison Test or Integral Test).

4. [15 points] Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n}{n!}$ is convergent or divergent.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)}{(n+1)!} \cdot \frac{n!}{2^n n} \right| = 2 \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

Therefore, by the Ratio Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n}{n!}$ is convergent.

5. [20 points] Find a power series representation of $f(x) = \dfrac{x}{2-x}$ and determine the interval of convergence.

$$\begin{aligned}
\frac{x}{2-x} = \frac{x}{2} \cdot \frac{1}{1-\frac{x}{2}} &= \frac{x}{2}\left[1 + \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\right] = \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n && \left|\frac{x}{2}\right| < 1 \\
&= \frac{x}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}} && |x| < 2 \\
&= \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{2^n}
\end{aligned}$$

The interval of convergence is $(-2, 2)$.

6. (a) [20 points] Find the Taylor series of $f(x) = \ln x$ centered at $a = 1$.

$$\begin{aligned}
f(x) &= \ln x & f(1) &= 0 \\
f'(x) &= x^{-1} & f'(1) &= 1 \\
f''(x) &= -x^{-2} & f''(1) &= -1 \\
f^{(3)}(x) &= 2 \cdot x^{-3} & f^{(3)}(1) &= 2 \\
f^{(4)}(x) &= -2 \cdot 3 \cdot x^{-4} & f^{(4)}(1) &= -2 \cdot 3 \\
f^{(5)}(x) &= 2 \cdot 3 \cdot 4 \cdot x^{-5} & f^{(5)}(1) &= 2 \cdot 3 \cdot 4 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^{n+1}(n-1)!\; x^{-n} & f^{(n)}(1) &= (-1)^{n+1}(n-1)! \qquad (n \geq 1)
\end{aligned}$$

Therefore, the Taylor series of f at 1 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$$

(b) [5 points] Given that the Taylor series of $f(x) = \ln x$ centered at $a = 1$ is equal to $f(x) = \ln x$ for $|x-1| < 1$, use part (a) to find the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\, 3^n}$$

Plugging $x = 4/3$ into the Taylor series found in part (a) gives

$$\ln\left(\frac{4}{3}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4/3-1)^n}{n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(1/3)^n}{n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n\, 3^n}$$

7. [5 points (bonus)] Find a formula for c_m, the m^{th} coefficient in the following series expansion of f

$$f(x) = c_1 \sin x + c_2 \sin(2x) + c_3 \sin(3x) + c_4 \sin(4x) + c_5 \sin(5x) + \cdots = \sum_{m=1}^{\infty} c_m \sin(mx)$$

Hint: $\int_0^{2\pi} \sin(mx)\ \sin(nx)\ dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$ for nonzero integers m, n.

Multiply both sides of $f(x) = \sum_{m=1}^{\infty} c_m \sin(mx)$ by $\sin(nx)$ for $n \geq 1$ then integrate over the interval $[0, 2\pi]$ to get

$$\begin{aligned} \int_0^{2\pi} f(x)\ \sin(nx)\ dx &= \int_0^{2\pi} \sum_{m=1}^{\infty} c_m \sin(mx)\ \sin(nx)\ dx \\ &= \sum_{m=1}^{\infty} c_m \int_0^{2\pi} \sin(mx)\ \sin(nx)\ dx = c_m \pi \\ c_m &= \frac{1}{\pi} \int_0^{2\pi} f(x)\ \sin(mx)\ dx \end{aligned}$$

The hint above is very similar to the bonus question on Test 1. The series expansion above is sometimes called the Fourier sine series. There is a similar but more general expansion called just the Fourier series.