2.1: The Tangent and Velocity Problems

The Tangent Problem

A **secant line** of a curve f passes through two distinct points, P and Q on f. A **tangent line** of a curve f at a point P *touches* but *does not cross* f at P.

The slope $m_{PQ} = \frac{f(x)-f(a)}{x-a}$ of the secant line passing through $P(a, f(a))$ and $Q(x, f(x))$ approaches the slope m of the tangent line to f at P as $x \to a$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Equivalently, letting $x = a + h$,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 1. *Find the equation of the tangent line to the parabola $y = x^2$ at $x = 1$.*

The Velocity Problem

The **average velocity** v_{ave} of an object over a given time interval $[t_1, t_2]$ is the change in position s divided by the change in time,

$$v_{ave} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

The **instantaneous velocity** v of an object at a given time $t = a$ is the limit as t approaches a of the average velocity over the time interval $[a, t]$,

$$v = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

Equivalently, letting $t = a + h$,

$$v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

Example 2. *If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.*

a) Find the average velocity over the given time intervals
i) $[1, 2]$ *ii)* $[1, 1.5]$ *iii)* $[1, 1.1]$ *iv)* $[1, 1.01]$ *v)* $[1, 1.001]$

b) Estimate the instaneous velocity at $t = 1$.

Example 3. *The table shows the position of a cyclist*

t (seconds)	*0*	*1*	*2*	*3*	*4*	*5*
s (meteres)	*0*	*1.4*	*5.1*	*10.7*	*17.7*	*25.8*

a) Find the average velocity for each time period
i) $[1, 3]$ *ii)* $[2, 3]$ *iii)* $[3, 4]$ *iv)* $[3, 5]$

b) Estimate the instantaneous velocity at $t = 3$ by averaging the slopes of the secant lines adjacent to $t = 3$.