Information on the Qualifying Examination for Foundations of Computational Mathematics 1

1 Basic Information

The Foundations of Computational Mathematics I Qualifying Examination is offered each year in January and August. It is a closed book examination. No reference books or notes are allowed. A calculator is permitted. The duration of the examination is typically two hours.

2 Related Texts

In addition to class notes and any readings recommended by the instructor of the course, these textbooks contain material related to the examination:

- Numerical Mathematics, A. Quarteroni, R. Sacco, and F. Saleri, Springer Texts in Applied Mathematics 37, Second Edition.
- Numerical Mathematics and Computing, E. Ward Cheney and David R. Kincaid, Brooks Cole Publishing, Sixth Edition
- Iterative Methods for Sparse Linear Systems, Yousef Saad, SIAM Press, Second Edition.
- Analysis of Numerical Methods, E. Isaacson and H. Keller, Wiley.
- Accuracy and Stability of Numerical Algorithms, N. Higham, SIAM, Second Edition.
- Numerical Methods, Prentice-Hall, G. Dahlquist and A. Bjorck.
- Numerical Computation, Volumes 1 and 2, C. W. Ueberhuber, Springer.
- Matrix Iterative Analysis, Richard Varga, Prentice Hall.
- Applied Iterative Methods, L. A. Hageman and D. M. Young, Academic Press.
- Iterative Solution Methods, O. Axelsson, Cambridge University Press.

3 Topics

The topics include those covered in the Foundations of Mathematics I Course. However, since the emphasis placed on the topics may vary with instructors, the list below, while representative, should not be considered definitive. Students are encouraged to discuss the topic list with recent instructors.

3.1 Finite Precision Arithmetic

1. Floating Point Number Systems
   - Floating point representation of real numbers
   - Floating point arithmetic
   - Overflow/underflow
   - Scaling
   - Using terms of infinite sequences to approximate a value
   - Using infinite series to approximate a value
   - Cancellation
2. Analysis of Numerical Computation in Finite Precision
   - Conditioning of a problem and the condition number
   - Stability of an algorithm
   - Backward error
   - Foward, weak, and backward stability

3.2 Finite Dimensional Vector Spaces

1. Vectors, Matrices, and Vector Spaces
   - Vectors, their operations, and a vector space
   - Linear combination, independence and dependence
   - Bases of subspaces of $\mathbb{R}^n$ and $\mathbb{C}^n$
   - Linear functions between spaces and matrices
   - Subspaces: $\mathcal{R}(A)$, $span[v_1, \ldots, v_k]$
2. Distance, Angle, and Matrices
   - Vector and matrix norms and the relationships to each other
   - Inner products, norms and angles
   - Matrix rank, nonsingular matrices, orthogonal/unitary matrices, isometries
   - Orthonormal bases of subspaces

3.3 Solving Linear Systems of Equations

1. Factorization Methods

   - Linear systems of equations
   - Systems that are easily solved
   - Operations on equations and matrix operations
   - Gauss transforms and their algebraic and computational properties
   - LU factorization via Gauss transforms
   - Pivoting, existence, stability, elementary permutations
   - Data structures, computations, and LU factorization

2. Numerical Analysis of Solving Linear Systems via LU Factorization

   - Conditioning of system solving
   - Backward error of factorization and complete solution algorithm
   - Growth factor and backward stability

3. Linear Stationary Methods

   - Fixed point iterations
   - Richardson's, Jacobi, Gauss-Seidel, SOR
   - Forward, backward, line, block, and symmetric iteration forms
   - Convergence analysis: error and residual behavior
   - Sufficient conditions for convergence for the various methods

4. Nonstationary Methods and Optimization

   - Optmization and system solving
   - Level curves, gradients and the steepest descent method
   - Conjugacy, conjugate directions, and incremental optimization
   - The conjugate gradient method
   - Preconditioning

3.4 Solving Nonlinear Equations

1. Scalar Nonlinear Equation Methods

   - Bisection method
   - Secant method

- Regula falsi method
- Newton's method

2. Fixed Point Analysis for Scalar Nonlinear Equations
    - Contraction mappings
    - Order of convergence
    - Multiplicity of root and convergence rate
    - Sufficient conditions
    - Computational cost, convergence rate and total work

3. Sytems of Nonlinear Equation Methods
    - Generalized linear methods
    - Netwon's method and Newton-like methods
    - The secant condition and Quasi-Newton methods
    - Convergence analysis
    - Computational cost, convergence rate and total work

3.5 Optimization

1. Linear Least Squares Problems
    - The full-rank linear least squares problem
    - Norm invariance and orthogonal transformation-based methods
    - Householder reflectors and solving linear least squares problems
    - Geometry of least squares: subspaces, orthogonal complements and projections
    - The generalized inverse

2. Unconstrained Nonlinear Optimization
    - First and second order necessary and sufficient conditions for an optimal point
    - Global vs. local convergence of a method
    - Line Search Methods
        - Steepest descent, Newton, Inexact Netwon
        - Quasi-Newton
        - Wolfe conditions: sufficient decrease condition and curvature condition
        - Convergence analysis
    - Nonlinear Least Squares Problems
    - Nonlinear Conjugate Gradient Methods
    - Trust Region Idea