MAC2312: Calculus 2 - Section 3

Test 1 June 4, 2015

Name: ______________________________

> Answer each question in the space provided on the question sheets. If you run out of space for an answer, continue on the back of the page. Credit will only be given if you clearly show all of your work. Calculators may not be used for this test.

Question	**Points**	**Score**
1	20	
2	20	
3	10	
4	10	
5	24	
6	16	
7 (bonus)	–	
Total:	100	

1. [20 points] Evaluate **ONE** of the following integrals using integration by parts. You may use other methods if you'd like. Only one of part (a) or (b) will be graded. State which part you are attempting, a or b: ___________

(a) $\displaystyle\int (x^2+2x)\cos x\ dx$

$$= (x^2+2x)\sin x - \int (2x+2)\sin x\ dx + C_1 \qquad \begin{bmatrix} u = x^2+2x & v = \sin x \\ du = (2x+2)\ dx & dv = \cos x\ dx \end{bmatrix}$$

$$= (x^2+2x)\sin x - \left[(2x+2)(-\cos x) - 2\int(-\cos x)\ dx\right] + C_2 \qquad \begin{bmatrix} u_1 = 2x+2 & v_1 = -\cos x \\ du_1 = 2\ dx & dv_1 = \sin x\ dx \end{bmatrix}$$

$$= (x^2+2x)\sin x - [(2x+2)(-\cos x) + 2\sin x] + C$$
$$= (x^2+2x-2)\sin x + 2(x+1)\cos x + C$$

(b) $\displaystyle\int \arcsin x\ dx$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\ dx + C_1 \qquad \begin{bmatrix} u = \arcsin x & v = x \\ du = \frac{1}{\sqrt{1-x^2}}\ dx & dv = dx \end{bmatrix}$$

$$= x\arcsin x + \frac{1}{2}\int u^{-1/2}\ du + C_1 \qquad \begin{bmatrix} u = 1-x^2 \\ -\frac{1}{2}du = x\ dx \end{bmatrix}$$

$$= x\arcsin x + \frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

2. [20 points] Evaluate **ONE** of the following trigonometric integrals. Only one of part (a) or (b) will be graded. State which part you are attempting, a or b: ___________

(a) $\displaystyle\int_0^{\pi/2} \sin^2 x \ \cos^2 x \ dx$

$=\int_0^{\pi/2} \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\ dx$

$= \frac{1}{4}\int_0^{\pi/2}(1-\cos^2 2x)\ dx$

$= \frac{1}{4}\int_0^{\pi/2}\sin^2 2x\ dx$

$= \frac{1}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4x)\ dx$

$= \frac{1}{8}\left[x-\frac{1}{4}\sin 4x\right]_{x=0}^{\pi/2}$

$= \frac{1}{8}\cdot\frac{\pi}{2} = \frac{\pi}{16}$

(b) $\displaystyle\int \tan^4 x \ \sec^6 x \ dx$

$$= \int \tan^4 x(\sec^2 x)^2 \sec^2 x\ dx$$

$$= \int \tan^4 x(\tan^2 x + 1)^2 \sec^2 x\ dx$$

$$= \int u^4(u^2+1)^2\ du \qquad \begin{bmatrix} u = \tan x \\ du = \sec^2 x\ dx \end{bmatrix}$$

$$= \int (u^8 + 2u^6 + u^4)\ du$$

$$\frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + C$$

$$\frac{1}{9}\tan^9 x + \frac{2}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$$

3. [10 points] Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-16}}$ using trigonometric substitution.

$$\left[\begin{array}{c} x = 4\sec\theta \\ dx = 4\sec\theta\tan\theta \; d\theta \end{array}\right]$$

$$\begin{aligned}
&= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta - 16}}\; d\theta \\
&= \frac{1}{16}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}\; d\theta \\
&= \frac{1}{16}\int \cos\theta \; d\theta \\
&= \frac{1}{16}\sin\theta + C \\
&= \frac{\sqrt{x^2-16}}{16x} + C
\end{aligned}$$

$$\left[\sec\theta = \frac{x}{4} \Rightarrow \sin\theta = \frac{\sqrt{x^2-16}}{x}\right]$$

4. [10 points] Use the formula

$$\int \frac{du}{u^2\sqrt{u^2+a^2}} = -\frac{\sqrt{u^2+a^2}}{a^2u} + C$$

to evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4x^2+9}}$.

$$\left[\begin{array}{ll} u = 2x & \Rightarrow x = \frac{u}{2} \\ \frac{1}{2}du = dx & \end{array}\right]$$

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4x^2+9}} &= \int \frac{\frac{1}{2}du}{\frac{u^2}{4}\sqrt{u^2+9}} \\
&= 2\int \frac{du}{u^2\sqrt{u^2+9}} \\
&= -\frac{2\sqrt{u^2+9}}{9u} + C \\
&= -\frac{\sqrt{4x^2+9}}{9x} + C
\end{aligned}$$

5. Consider the rational function $\dfrac{x^2 - x + 6}{x(x^2+3)}$.

(a) [2 points] State the linear factor of the denominator and its multiplicity.

Factor: $\underline{\quad x \quad}$ Multiplicity: $\underline{\quad 1 \quad}$

(b) [2 points] State the irreducible quadratic factor of the denominator and its multiplicity.

Factor: $\underline{\quad x^2+3 \quad}$ Multiplicity: $\underline{\quad 1 \quad}$

(c) [10 points] Write $\dfrac{x^2 - x + 6}{x(x^2+3)}$ as a sum of partial fractions.

$$\frac{x^2 - x + 6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx + C}{x^2+3}$$

$$x^2 - x + 6 = A(x^2+3) + (Bx+C)x$$

$$x^2 - x + 6 = Ax^2 + 3A + Bx^2 + Cx$$

$$x^2 - x + 6 = (A+B)x^2 + Cx + 3A$$

$$\Rightarrow A = 2, B = -1, C = -1$$

so

$$\frac{x^2 - x + 6}{x(x^2+3)} = \frac{2}{x} - \frac{x+1}{x^2+3}$$

(d) [10 points] Evaluate $\displaystyle\int \left(\frac{2}{x} - \frac{x+1}{x^2+3}\right)\, dx$.

$$\begin{aligned}
&= \int \left(\frac{2}{x} - \frac{x+1}{x^2+3}\right)\, dx \\
&= \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right)\, dx \\
&= 2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C
\end{aligned}$$

6. Consider the definite integral $\int_0^3 x^2\ dx$.

(a) [10 points] Use the Trapezoidal Rule with $n = 3$ subintervals to approximate $\int_0^3 x^2\ dx$

$f(x) = x^2$, $\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$

$$\begin{aligned}\int_0^3 x^2\ dx \approx T_3 &= \frac{1}{2}\left[f(0) + 2f(1) + 2f(2) + f(3)\right] \\ &= \frac{1}{2}\left(0 + 2\cdot 1 + 2\cdot 4 + 9\right) = \frac{19}{2}\end{aligned}$$

(b) [6 points] What is the exact value of $\int_0^3 x^2\ dx$, and what is the error in using the Trapezoidal Rule with $n = 3$ to approximate $\int_0^3 x^2\ dx$?

The exact value is

$$\int_0^3 x^2\ dx = \left[\frac{x^3}{3}\right]_{x=0}^{3} = 9.$$

The error is

$$\int_0^3 x^2\ dx - T_3 = 9 - \frac{19}{2} = -\frac{1}{2}.$$

Exact Value: $\frac{19}{2}$ Error: $-\frac{1}{2}$

7. [5 points (bonus)] Show that

$$\frac{2}{\pi}\int_0^\pi \sin(mx)\ \sin(nx)\ dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

for nonzero integers m, n.

If $m = n$

$$\begin{aligned}
\frac{2}{\pi}\int_0^\pi \sin(mx)\ \sin(nx)\ dx &= \frac{2}{\pi}\int \sin^2(nx)\ dx \\
&= \frac{1}{\pi}\int [1 - \cos(2nx)]\ \ dx \\
&= \frac{1}{\pi}\left[x - \frac{\sin(2nx)}{2n}\right]_{x=0}^{\pi} \\
&= \frac{1}{\pi}\left[\pi - \frac{\sin(2n\pi)}{2n}\right] \\
&= 1
\end{aligned}$$

If $m \neq n$

$$\begin{aligned}
\frac{2}{\pi}\int_0^\pi \sin(mx)\ \sin(nx)\ dx &= \frac{1}{\pi}\int \{\cos[(m-n)x] - \cos[(m+n)x]\}\ \ dx \\
&= \frac{1}{\pi}\left\{\frac{\sin[(m-n)x]}{m-n} - \frac{\sin[(m+n)x]}{m+n}\right\}_{x=0}^{\pi} \\
&= \frac{1}{\pi}\left\{\frac{\sin[(m-n)\pi]}{m-n} - \frac{\sin[(m+n)\pi]}{m+n}\right\} \\
&= \frac{1}{\pi}(0 - 0) = 0
\end{aligned}$$

Formula Sheet

Trigonometric Identities

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$
- $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$
- $\sin x_1 \sin x_2 = \frac{1}{2}[\cos(x_1 - x_2) - \cos(x_1 + x_2)]$
- $\cos x_1 \cos x_2 = \frac{1}{2}[\cos(x_1 - x_2) + \cos(x_1 + x_2)]$
- $\sin x_1 \cos x_2 = \frac{1}{2}[\sin(x_1 - x_2) + \sin(x_1 + x_2)]$
- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x$

Trigonometric Integrals

- $\displaystyle\int \tan x \; dx = \ln|\sec x| + C$
- $\displaystyle\int \csc x \; dx = -\ln|\csc x + \cot x| + C$
- $\displaystyle\int \sec x \; dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int \cot x \; dx = \ln|\sin x| + C$
- $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$