Homework 2 Foundations of Computational Math 1 Fall 2012

Problem 2.1

Let $n = 4$ and consider the lower triangular system $Lx = f$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_{21} & 1 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 1 & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

Recall, that it was shown in class that the column-oriented algorithm could be derived from a factorization $L = L_1 L_2 L_3$ where L_i was an elementary unit lower triangular matrix associated with the i-th column of L.

Show that the row-oriented algorithm can be derived from a factorization of L of the form

$$L = R_2 R_3 R_4$$

where R_i is associated with the i-th row of L.

Problem 2.2

A first order linear recurrence is defined as follows:

$$\begin{aligned} \alpha_0 &= \gamma_0 \\ \alpha_i &= \beta_i \alpha_{i-1} + \gamma_i \\ i &= 1, \cdots, n \end{aligned}$$

where α_i,γ_i, β_i are all scalars.

2.2.a. Show how this can be written as a system of equations.

2.2.b. Comment on any structural properties of the matrix and how they might be exploited to solve the recurrence.

2.2.c. How many operations are required to solve the system?

Problem 2.3

Consider the matrix vector product $x = Lb$ where L is an $n \times n$ unit lower triangular matrix with **all** of its nonzero elements equal to 1. For example, if $n = 4$ then

$$x = Lb$$
$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

The vector x is called the scan of b. Show that x can be computed in $O(n)$ computations rather than the $O(n^2)$ typically required by a matrix vector product. Express your solution in terms of matrices and vectors.

Problem 2.4

Consider an $n \times n$ real matrix where

- $\alpha_{ij} = e_i^T A e_j = -1$ when $i > j$, i.e., all elements strictly below the diagonal are -1;
- $\alpha_{ii} = e_i^T A e_i = 1$, i.e., all elements on the diagonal are 1;
- $\alpha_{in} = e_i^T A e_n = 1$, i.e., all elements in the last column of the matrix are 1;
- all other elements are 0

For $n = 4$ we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

2.4.a. Compute the factorization $A = LU$ for $n = 4$ where L is unit lower triangular and U is upper triangular.

2.4.b. What is the pattern of element values in L and U for any n?