Qualifying Exam

**Computational Mathematics**

January 2011

**Do all six problems. Each problem is worth 20 points.**

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1. (20 points) Consider the Runge-Kutta schemes for the ODE $u' = f(u)$:

SCHEME 1:
$$\begin{aligned} f_1 &= f(u^n), \\ f_2 &= f(u^n + b_1 k f_1), \\ u^{n+1} &= u^n + k(c_1 f_1 + c_2 f_2). \end{aligned}$$

SCHEME 2:
$$\begin{aligned} f_1 &= f(u^n), \\ f_2 &= f(u^n + b_1 k f_1), \\ f_3 &= f(u^n + b_2 k f_2), \\ u^{n+1} &= u^n + k(c_1 f_1 + c_2 f_2 + c_3 f_3). \end{aligned}$$

   (a) (5 points) Under what conditions is SCHEME 1 second order and SCHEME 2 third order? (Hint: choosing $f(u) = \lambda u$ makes the calculations much easier).

   (b) (5 points) Are the constants $b_i, c_i$ uniquely determined by the conditions found above?

   (c) (5 points) Find the equations defining the stability regions of the above schemes (with $f = \lambda u$). Where do these regions intersect the real and imaginary axis in the complex $k\lambda$ plane?

   (d) (5 points) Which scheme would you choose for a conservative oscillatory problem? Why?

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2. (20 points) Consider a function $u(x)$, periodic with period $2\pi$, with the Fourier transform (series):

$$
\begin{aligned}
\hat{u}_\xi &= \int_{-\pi}^{\pi} u(x) e^{-i\xi x}\, dx, \qquad \xi \in \mathbb{Z} \\
u(x) &= \frac{1}{2\pi} \sum_{\xi=-\infty}^{\infty} \hat{u}_\xi e^{i\xi x},
\end{aligned}
$$

and the periodic discrete function $v_j$, with its discrete Fourier transform

$$
\begin{aligned}
\hat{v}_\xi &= h \sum_{j=-N/2}^{N/2-1} v_j e^{-ij\xi h}, \qquad Nh = 2\pi \\
v_j &= \frac{1}{2\pi} \sum_{\xi=-N/2}^{N/2-1} \hat{v}_\xi e^{ij\xi h}.
\end{aligned}
$$

(a) (5 points) Derive the *aliasing* formula. In other words, suppose you have a periodic function $f(x)$ and discretize it with gridsize $h$ such that $g_j = f(x_j)$. What is the relationship between $\hat{f}$ and $\hat{g}$?

(b) (5 points) Suppose now that $f(x)$ is band-limited on $\xi \in [-N/2, N/2]$ (ie. $\hat{f}$ is zero outside that interval). In this case show that $\hat{f} = \hat{g}$

(c) (5 points) For the band-limited $f(x)$, suppose you compute $h(x) = f^2(x)$ and then compute $\hat{h}_\xi$. What is its support in $\xi$? If $f$ were discretized on the grid as in (a) and $q_j = g_j^2$, is $\hat{q} = \hat{h}$ for $\xi \in [-N/2, N/2]$? Why?

(d) (5 points) In order to compute $\hat{h}$ correctly on $\xi \in [-N/2, N/2]$ using only $g_j$, here is the 3/2 dealiasing procedure:

(1) Take $\hat{g}$ and extend its spectrum with $N/4$ zero modes on each side. That is define $\hat{G} = \hat{g}$ for $\xi \in [-N/2, N/2]$, and $\hat{G} = 0$ for $\xi \in [-3N/4, -N/2] \cup [N/2, 3N/4]$.

(2) Compute $Q_j = G_j^2$.

(3) Compute $\hat{Q}$.

Why does this work? That is, show that $\hat{Q} = \hat{h}$ for $\xi \in [-N/2, N/2]$.

3. (20 points) In this problem you will compare and contrast the Leapfrog and Lax-Friedrichs schemes to solve the advection equation $u_t + au_x = 0$.

$$\begin{array}{lcl} \text{Leapfrog:} & \dfrac{u_j^{n+1} - u_j^{n-1}}{2k} & = \; a\,\dfrac{u_{j+1}^n - u_{j-1}^n}{2h} \\ \text{Lax-Friedrichs:} & \dfrac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{k} & = \; a\,\dfrac{u_{j+1}^n - u_{j-1}^n}{2h} \end{array}$$

(a) (5 points) What is the order of accuracy of each scheme?

(b) (5 points) Find the stability criterion for each scheme.

(c) (5 points) Is the leading order error term dissipative or dispersive in each equation? Write down what (approximating) equation corresponds to the finite difference scheme at this order. How will dissipation and dispersion manifest itself when computing solutions?

(d) (5 points) What would be the simplest Godunov scheme for this equation?

4. (20 points) Consider the following 2-point boundary value problem:

$$\textbf{ODE:} \quad -\frac{d}{dx}\left(a(x)\,\frac{du}{dx}\right) = f(x), \quad x \in [0,1],$$
$$\textbf{BCs:} \quad u(0) = u(1) = 0,$$

where $a(x) > 0$ for all $x \in [0,1]$.

(a) (2 points) Prove that the solution this ODE is unique.

(b) (2 points) Write down the weak formulation for this ODE and prove that the solution to the weak formulation is unique.

(c) (8 points) Under conditions to be determined, prove that the solutions to the ODE and the weak formulation are the same.

(d) (8 points) Discretize the weak formulation using a cG(1) finite element method on a uniform mesh with grid spacing $h$. Prove an *a priori* error estimate using the following norm:

$$\|u\|_E := \left(\int_0^1 a(x)\, u'(x)\, dx\right)^{1/2}.$$

5. (20 points) Consider the following 2D biharmonic equation:

$$\begin{aligned}
\textbf{PDE:} &\quad \nabla^2\nabla^2 u = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2, \\
\textbf{BC:} &\quad u(\mathbf{x}) = 0 \quad \text{on} \quad \Gamma, \\
\textbf{BC:} &\quad \nabla u(\mathbf{x}) \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma,
\end{aligned}$$

where $\Omega$ is a closed subset of $\mathbb{R}^2$, $\Gamma$ is the boundary of $\Omega$, and $\mathbf{n}$ is the outward pointing unit normal to $\Gamma$.

(a) (4 points) Construct a weak formulation for this PDE and prove that this weak formulation has a unique solution. What minimum degree of smoothness on the the test and trial functions does this weak formulation require?

(b) (8 points) Let $\Omega$ be approximated by $M$ triangular elements and a total of $N$ nodes. On the canonical triangular element, construct appropriate basis functions that will reduce the weak formulation from part (a) to finite dimensions.

(c) (8 points) The finite element approximation from part (b) yields a linear algebraic equation of the form:

$$A\vec{U} = \vec{F},$$

where $A \in \mathbb{R}^{N\times N}$, $\vec{U} \in \mathbb{R}^{N\times 1}$, and $\vec{F} \in \mathbb{R}^{N\times 1}$. Using your solution to part (b), write down an algorithm in pseudo-code to build the coefficient matrix $A$.

6. (20 points) Consider the steady advection-diffusion equation

$$\begin{aligned} \textbf{ODE:} \quad & u'(x) - \varepsilon u''(x) = f(x), \quad 0 \le x \le 1, \\ \textbf{BC:} \quad & u(0) = u(1) = 0, \end{aligned}$$

where $\epsilon > 0$ is a constant.

(a) (2 points) Prove the following maximum principle:

$$f(x) \ge 0 \quad \Longrightarrow \quad u(x) \ge 0.$$

(b) (6 points) Discretize this equation via the standard cG(1) method on a mesh with uniform grid spacing. Write out the full linear system:

$$A\vec{U} = \vec{F}.$$

For the next three parts refer to the following definition.

**Definition.** *Matrix $A$ is called an $M$-matrix if it satisfies the following three conditions:*

- *All of the off-diagonal entries of $A$ are non-positive ($a_{ij} \le 0 \quad \forall i \ne j$);*
- *$A$ is invertible;*
- *All of the entries of $A^{-1}$ are non-negative.*

(c) (2 points) Show that if the matrix from part (b) is an $M$-matrix, then the following *discrete* maximum principle is satisfied:

$$F_j \ge 0 \quad \forall j \quad \Longrightarrow \quad U_k \ge 0 \quad \forall k.$$

(d) (6 points) Find conditions under which the matrix from part (b) is an $M$-matrix.

(e) (4 points) Give an intuitive explanation as to what happens in the case when the matrix $A$ from part (b) fails to be an $M$-matrix.