Solutions for Homework 2 Foundations of Computational Math 1 Fall 2012

Problem 2.1

Let $n = 4$ and consider the lower triangular system $Lx = f$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_{21} & 1 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 1 & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

Recall, that it was shown in class that the column-oriented algorithm could be derived from a factorization $L = L_1 L_2 L_3$ where L_i was an elementary unit lower triangular matrix associated with the i-th column of L.

Show that the row-oriented algorithm can be derived from a factorization of L of the form

$$L = R_2 R_3 R_4$$

where R_i is associated with the i-th row of L.

Solution: We can define R_i in a manner similar to the column forms used for L_i. Specifically, define for the case $n = 4$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 1 \end{pmatrix}$$

It is straighforward to verify that this satisfies $L = R_2 R_3 R_4$. To see that the pattern holds for any n note that

$$\begin{aligned} R_j &= I + e_j r_j^T \\ r_j^T e_k &= 0, \quad j \leq k \leq n \\ R_i R_j &= R_i(I + e_j r_j^T) = R_i + R_i e_j r_j^T \\ &= R_i + e_j r_j^T, \quad \text{since } i < j \end{aligned}$$

and therefore we simply put the nonzeros of j-th row from R_j into the zero positions in the j-th row of R_i to form the product. This easily generalizes when R_i is replaced with a matrix with a set of rows whose indices are all less than j.

To see that $R_j^{-1} = I - e_j r_j^T$ note that $r_j^T e_j = 0$ and

$$R_j e_j = (I + e_j r_j^T) e_j = e_j + e_j (r_j^T e_j) = e_j$$

$$R_j (I - e_j r_j^T) = R_j - R_j e_j r_j^T = R_j - e_j r_j^T = I + e_j r_j^T - e_j r_j^T = I$$

It is easily seen that applying the R_j^{-1} to a vector b yields a series of inner products that are the same as those defined by the row-oriented algorithm.

Problem 2.2

A first order linear recurrence is defined as follows:

$$\begin{aligned} \alpha_0 &= \gamma_0 \\ \alpha_i &= \beta_i \alpha_{i-1} + \gamma_i \\ i &= 1, \cdots, n \end{aligned}$$

where α_i,γ_i, β_i are all scalars.

2.2.a. Show how this can be written as a system of equations.

2.2.b. Comment on any structural properties of the matrix and how they might be exploited to solve the recurrence.

2.2.c. How many operations are required to solve the system?

Solution:

The first order linear recurrence

$$\begin{aligned} \alpha_0 &= \gamma_0 \\ \alpha_i &= \beta_i \alpha_{i-1} + \gamma_i \\ i &= 1, \cdots, n \end{aligned}$$

can be rewritten as the set of equations

$$\begin{aligned} \alpha_0 &= \gamma_0 \\ -\beta_i \alpha_{i-1} + \alpha_i &= \gamma_i \\ i &= 1, \cdots, n \end{aligned}$$

and then written in matrix-vector form. The structure of the matrix is clear from an example with $n = 3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta_1 & 1 & 0 & 0 \\ 0 & -\beta_2 & 1 & 0 \\ 0 & 0 & -\beta_3 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

The matrix is obviously a banded matrix with a single subdiagonal and main diagonal elements all 1. It is easy to see this can be solved in $2n$ operations by considering the standard unit triangular system solving algorithms and removing unnecessary operations due to 0 and 1 elements.

Problem 2.3

Consider the matrix vector product $x = Lb$ where L is an $n \times n$ unit lower triangular matrix with **all** of its nonzero elements equal to 1. For example, if $n = 4$ then

$$x = Lb$$
$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

The vector x is called the scan of b. Show that x can be computed in $O(n)$ computations rather than the $O(n^2)$ typically required by a matrix vector product. Express your solution in terms of matrices and vectors.

Solution:

The key to the $O(n)$ complexity solution is to note that there are common subexpressions in the values ξ_i. We have

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$
$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 + \beta_1 \\ \beta_3 + \beta_2 + \beta_1 \\ \beta_4 + \beta_3 + \beta_2 + \beta_1 \end{pmatrix}$$

Clearly, $\xi_i = \beta_i + \xi_{i-1}$. So we do not need to reevaluate the summations that are already available as ξ_{i-1} to get ξ_i. Essentially this says that the scan operation can also be written as a linear recurrence. It is easily verified that

$$L^{-1} = M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$Mx = b$$

So, x can be computed as the solution of a banded lower triangular system, i.e., as the solution to a first order linear recurrence, which is known to require only $O(n)$ computations.

Problem 2.4

Consider an $n \times n$ real matrix where

- $\alpha_{ij} = e_i^T A e_j = -1$ when $i > j$, i.e., all elements strictly below the diagonal are -1;
- $\alpha_{ii} = e_i^T A e_i = 1$, i.e., all elements on the diagonal are 1;
- $\alpha_{in} = e_i^T A e_n = 1$, i.e., all elements in the last column of the matrix are 1;
- all other elements are 0

For $n = 4$ we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

2.4.a. Compute the factorization $A = LU$ for $n = 4$ where L is unit lower triangular and U is upper triangular.

2.4.b. What is the pattern of element values in L and U for any n?

Solution: Computing the LU factorization (without pivoting) yields:

$$\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & -1 & 1 & 2\\ 0 & -1 & -1 & 2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 & 1\\ -1 & 1 & 0 & 1\\ -1 & -1 & 1 & 1\\ -1 & -1 & -1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 4\\ 0 & 0 & -1 & 4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & -1 & 1 & 2\\ 0 & -1 & -1 & 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 8\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 4\\ 0 & 0 & -1 & 4\end{pmatrix}$$

$$L = \begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -1 & -1 & 1 & 0\\ -1 & -1 & -1 & 1\end{pmatrix}, \quad U = \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 8\end{pmatrix}$$

Since the λ_{ij} are all -1, i.e., elements are eliminated by simply adding rows together, L will be unit lower triangular and all of the elements below the diagonal will be -1 for all n. U is the identity matrix with its last column replaced by a vector of the form

$$Ue_n = \begin{pmatrix}2^0\\ 2^1\\ \vdots\\ 2^{n-2}\\ 2^{n-1}\end{pmatrix}$$

Hence there is exponential growth in the elements of U as a function of n.