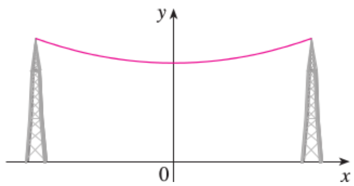
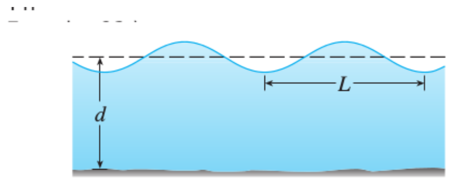
3.11: Hyperbolic Functions

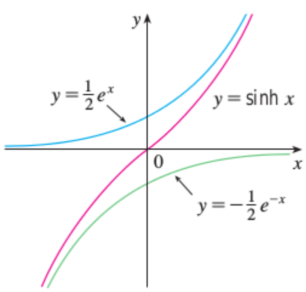
In many applications, a set of functions called the **hyperbolic functions** analogous to the trigonometric functions are used. For example, the shape of a hanging wire suspended between two points is modeled by the **hyperbolic cosine** function. In another example, the velocity of a water wave of length L moving across water with depth d is modeled in terms of the **hyperbolic tangent** function.

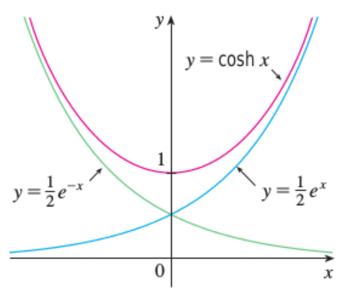


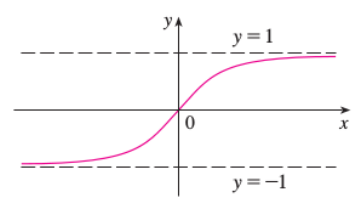


The **hyperbolic functions** are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$



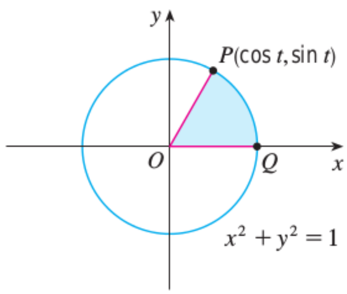


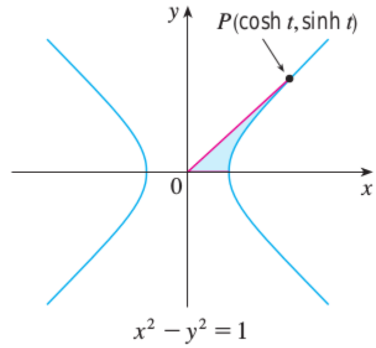


The hyperbolic functions satisfy the following identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Example 1. *Show that (a)* $\cosh^2 x - \sinh^2 x = 1$ *and (b)* $1 - \tanh^2 x = \operatorname{sech}^2 x$.





The derivatives of the hyperbolic functions are

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = \operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example 2. *Show that (a)* $\frac{d}{dx}(\sinh x) = \cosh x$ *and (b)* $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$.