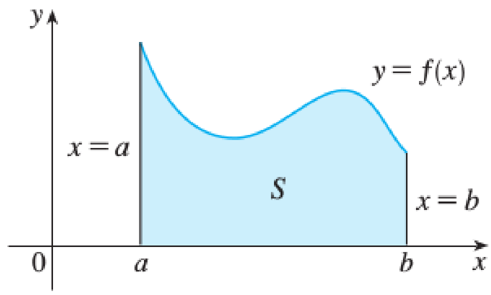
5.1: Areas and Distances

In Chapter 2, we considered tangent lines and rates of change to motivate one of the two main branches of calculus, *differential calculus.* In this chapter, we consider areas and distances to motivate the other complementary branch, *integral calculus.*

The Area Problem

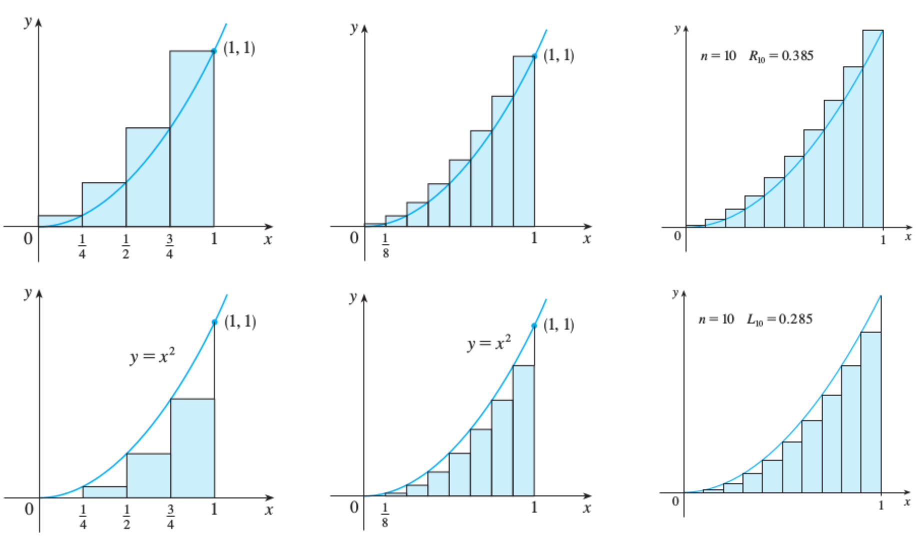
How would you solve the following problem? Find the area of the region S that lies under a curve $y = f(x)$ (where $f(x) \geq 0$) from a to b. That is, find the area bounded by the graph of the function f, the vertical lines $x = a$ and $x = b$, and the x-axis depicted below.



$$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$$

Unlike calculating the area of a rectangle, a difficulty in finding the area of S is that one of its sides is not straight. One approach is to approximate the area of S with a number of rectangles, then calculate the limit as the number of rectangles increases. This approach should be reminiscent of the approach we took to calculate the slope of a tangent line: taking the limit of slopes of secant lines. The following example illustrates how to answer the question posed at the beginning of this section.

Example 1. *Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.*



n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

Example 2. *For the region in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,*

$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$

For this example, we need the formula

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

In general, the **area** A of the region that lies under the graph of a continuous function f from a to b is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] \text{ or}$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

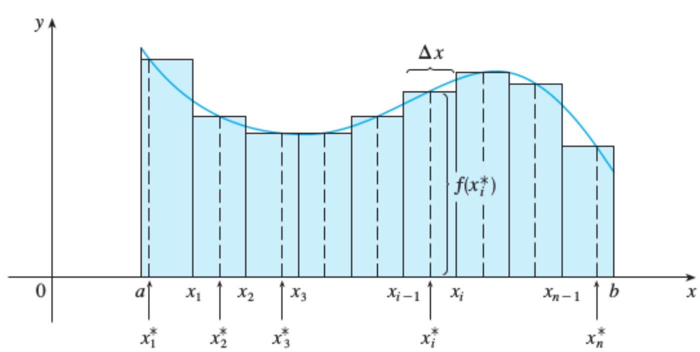
where $[a, b]$ is partitioned into n subintervals

$$[x_0, x_1], [x_1, x_2,], \ldots, [x_{n-1}, x_n]$$

each of width $\Delta x = (b - a)/n$ so that $x_k = a + k\Delta x$ for $k = 1, \ldots, n$. In fact,

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

where x_k^* is any *sample point* in $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$.



Example 3. *Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.*

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

The Distance Problem

Example 4. *Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:*

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, lets convert the velocity readings to feet per second (1 mi/hr = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41