Sketches of solutions, $\boxed{\text{August 97}}$

1 b). The DFT matrix (complex version) is

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} w^0 & w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & w^2 & \cdots & w^{N-1} \\ w^0 & w^2 & w^4 & \cdots & w^{2N-2} \\ \vdots & & & & \vdots \\ w^0 & w^{2N-1} & \cdots & \cdots & \end{bmatrix} \quad \text{where } w = e^{2\pi i/N}.$$

Scaling constant may vary dependent on our definition of DFT.

$$F^2 = \frac{1}{N}\begin{bmatrix} & \end{bmatrix}\begin{bmatrix} & \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \end{bmatrix}$$

Each element in this matrix becomes a finite geometric progression which sums to zero except as shown.

a) The net effect of applying F twice is that it leaves the first vector element unchanged, and reverses the order of the remaining elements.

2)

Hermite Basis:

Lagrange Poly

$$\ell_0(x) = \frac{x(x-1)}{+2} = \frac{x^2-x}{2} \qquad \ell_0'(-1) = -\frac{3}{2}$$

$$\ell_1(x) = \frac{(x-1)(x+1)}{-1} = -(x^2-1) \qquad \ell_1'(0) = 0$$

$$\ell_2(x) = \frac{x^2+x}{2} \qquad \ell_2'(1) = 3/2$$

Hermite Polynomials:

a) $h_0(x) = \ell_0(x)^2 \left[2\ell_i'(x_i)[1-x] + 1\right] = \frac{(x^2-x)^2}{4}\left[3(x-1)+1\right]$

$$\boxed{\frac{(x^2-x)^2}{4}[3x+\cancel{4}]}$$

$$h_1(x) = (x^2-1)^2[1] = \boxed{(x^2-1)^2}$$

$$h_2(x) = \frac{(x^2+x)^2}{4}\left[3(1-x)+1\right] = \boxed{\frac{(x^2+x)^2}{4}[4-3x]}$$

$$\tilde{h}_0(x) = (\ell_0(x))^2(x-x_0) = \boxed{\frac{(x^2-x)^2}{4}(x+1)}$$

$$\tilde{h}_1(x) = \boxed{(x^2-1)^2 x}$$

$$\tilde{h}_2(x) = \boxed{\frac{(x^2+x)^2}{4}(x-1)}$$

b)

$$E(f) = p(x) - f(x) = \underset{(x-1)^2x^2(x+1)^2}{\wedge} f[x_0, x_0, x_1, x_1, x_2, x_2, \cancel{x}] = \frac{f^{6}(\xi)}{6!} \Bigg| \quad \xi \in [-1, 1]$$

$$(x-1)^2 x^2 (x+1)^2$$

: #2 c) Lots of symmetry $\int_{-1}^{1} \text{odd} = 0$

$$\int_{-1}^{1} h_0(x) = \int_{-1}^{1} \frac{\cancel{x^4}-2x^3+\cancel{x^2}}{4}(3x) + \int_{-1}^{1} x^4 - \cancel{2x^3} + x^2 - \frac{1}{2}\int_{-1}^{1} x^4 + \int_{-1}^{1} x^2 = -\frac{2}{10} + \frac{2}{3} = \boxed{\frac{7}{15}}$$

$$\int_{-1}^{1} h_1(x) = \int_{-1}^{1} x^4 - 2x^2 + 1 = \frac{2}{5} - \frac{4}{3} + 2 = \boxed{\frac{16}{15}}$$

$$\int_{-1}^{1} h_2(x) = +\int_{-1}^{1} h_0(x) = \boxed{-\frac{7}{15}}$$

$$\int_{-1}^{1} \tilde{h}_0(x) = \frac{1}{4}\int_{-1}^{1} (x^4 - 2x^3 + x^2)x + \frac{1}{4}\int_{-1}^{1}(x^4 - 2x^3 + x^2) = -\frac{1}{2}\int_{-1}^{1} x^4 + \int_{-1}^{1}\frac{x^4}{4} + \int_{-1}^{1}\frac{x^2}{4}$$

$$= \frac{2}{4}\left(\frac{-1}{5} + \frac{1}{3}\right) = \boxed{\frac{1}{15}}$$

$$\int_{-1}^{1} \tilde{h}_1(x) = \int_{-1}^{1} (x^2-1)^2 x = \int_{-1}^{1} \text{odd} = 0$$

$$\int_{-1}^{1} \tilde{h}_2(x) = -\int_{-1}^{1} \tilde{h}_0(x) = \boxed{-\frac{1}{15}}$$

$$E(f) = \int_{-1}^{1} (f(t) - H(t))\,dt = \int_{-1}^{1} \frac{f^{6}(3(t))}{6!}\,dt$$

If $f(x)$ is a polynomial of degree ≤ 5, then $f^6 = 0$

3.

answers (a) $hD = \ln(I+\Delta_+) = \Delta_+ + O(h^2)$; $hD = -\ln(I+\Delta_-) = \Delta_- + O(h^2)$

$\Rightarrow D = (\Delta_+ + \Delta_-)/2 + O(h)$

(b) $z''(kh) = \frac{1}{h^2}\left(z((k+1)h) - 2z(kh) + z((k-1)h)\right) + O(h^2)$

(c) $\|e\| = \|u - u^*\|$, but $u^{*\prime\prime}(kh) = \Delta^2_h u + O(h^2)$

$\Rightarrow A(u-u^*) = O(h^2) \Rightarrow \|e\| \le \|A^{-1}\| \, \|A(u-u^*)\| = O(h^2)$

symm. & $\lambda_{min} >> 0$

4.

answers (a) $\int_0^1 (-y''+y)z\,dx = \int_0^1 fz\,dx \Rightarrow -y'z\Big|_{x=0}^{1} + \int_0^1 (y'z'+yz)\,dx = \int_0^1 fz\,dx$

$\Rightarrow \int_0^1 (y'z'+yz)\,dx = \int_0^1 fz\,dx$

(b) $a_{kk} = \int_0^1 ({\varphi_k'}^2 + \varphi_k^2)\,dx = 2\int_0^h \frac{dx}{h^2} + 2\int_0^h \left(\frac{x}{h}\right)^2 dx = \frac{2}{h} + \frac{2h}{3}$

$a_{k\,k\pm1} = \int_0^1 (\varphi_k'\varphi_{k\pm1}' + \varphi_k\varphi_{k\pm1})\,dx = \int_0^h \frac{-dx}{h^2} + \int_0^h \frac{x(h-x)}{h^2}\,dx = -\frac{1}{h} + \frac{h}{6}$

$a_{kl} = 0$ for $|k-l| > 1$

(c) bdd: $|\langle \mathcal{L}v, w\rangle| = |\langle \mathcal{L}^{1/2}v, \mathcal{L}^{1/2}w\rangle| \le \|\mathcal{L}^{1/2}v\|\,\|\mathcal{L}^{1/2}w\|$

$= \sqrt{\langle \mathcal{L}v,v\rangle\langle \mathcal{L}w,w\rangle} \le \|v\|_H \, \|w\|_H$

coercive: $\langle \mathcal{L}v, v\rangle = \int_0^1 ({v'}^2 + v^2)\,dx \ge \int_0^1 v^2\,dx = \|v\|^2$

$\Rightarrow \langle \mathcal{L}v, v\rangle = \frac{1}{2}\langle \mathcal{L}v,v\rangle + \frac{1}{2}\langle \mathcal{L}v,v\rangle \ge \frac{1}{2}\langle \mathcal{L}v,v\rangle + \frac{1}{2}\|v\|^2$

$= \frac{1}{2}\|v\|_H^2$.

5 Eigenvalues

i) $\|x_0\| = 1$ Largest eigenvalue (in $|\cdot|$)

$\hat{x}_{i+1} = A x_i$

$x_{i+1} = \hat{x}_{i+1} / \|\hat{x}_{i+1}\|$

ii Inverse Power method Smallest eigenvalue (in $|\cdot|$)

$\|x_0\| = 1$

$\vdots$

$\hat{x}_{i+1} = A^{-1} x_i$

$x_{i+1} = \hat{x}_{i+1} / \|\hat{x}_{i+1}\|$

iii) Shifted inverse Power Method

$\|x_0\| = 1$ eigenvalue that minimizes

$\vdots$

$\hat{x}_{i+1} = (A - \alpha I)^{-1} x_i$ $|\lambda - \alpha|$

$x_{i+1} = \hat{x}_{i+1} / \|\hat{x}_{i+1}\|$

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b) $x_0 = \sum_{i=1}^{N} \alpha_i \underline{v}_i$

$x_\ell = \sum_{i=1}^{N} \alpha_i \frac{1}{\lambda_i} v_i = \sum_{i=1}^{N} \alpha_i \frac{1}{(\lambda_i)^\ell} v_i$

$$x_e = \alpha \frac{1}{(\lambda_1)^e} \underline{V}_1 + \sum_{j=2}^{N} \alpha_j \frac{1}{(\lambda_j)^e} \underline{V}_e$$

$$= \frac{1}{(\lambda_1)^e} \left[\alpha_1 \underline{V}_1 + \sum_{j=2}^{N} \alpha_j \underbrace{\cancel{\left(\frac{\lambda_1}{\lambda_j}\right)^e}}_{\to 0} \underline{V}_e \right]$$

$$\rightarrow \boxed{\frac{1}{(\lambda_1)^e} \alpha_1 \underline{V}_1} \quad \text{Normalized} \Rightarrow \underline{V}_1$$

ii) Rate of convergence : $\left| \frac{\lambda_1}{\lambda_2} \right|$

iii may not converge.

6) a) $\|A\|_1 = \text{Max Column Sum} = \max\{2.2 \quad 4.0 \quad 2.4\} = \boxed{4.0}$

b) $\|A^{-1}\|_1 = \max \; 8, 6, 5 = 8$

$$\text{Cond}_1(A) = \|A\|_1 \|A^{-1}\|_1 = \boxed{32}$$

c) $$\frac{\|x-\hat{x}\|_1}{\|x\|_1} \le \frac{\|A^{-1}E\|_1}{1-\|A^{-1}E\|_1} \le \frac{.08}{.92} = \frac{8}{92} = \frac{2}{23} \cong .087$$

$$\|A^{-1}E\|_1 \le \|A^{-1}\|_1 \|E\|_1 \le (8)(.01) = .08$$

7.

	char. eq.	roots	stab	acc	consist.	lead error term	Converg.
a,	$r^2 - \frac{1}{2}r - \frac{1}{2} = 0$	$1, -\frac{1}{2}$	Y	0	NO	$-\frac{1}{2}hf(\xi)$	NO
b,	$r - 1 = 0$	1	Y	0	NO	$hf(\xi)$	NO
c,	$r^4 - 1 = 0$	$\pm 1, \pm i$	Y	2	Y	$\frac{8}{3!}h^3 f'''(\xi)$	Y
d,	$r^2 - 1 = 0$	± 1	Y	3	Y	$\frac{8}{4!}h^4 f''''(\xi)$	Y
e,	$r^4 - \frac{8}{19}r^3 + \frac{8}{19}r - 1 = 0$	± 1, $\frac{4}{19} \pm \frac{\sqrt{345}}{19}i$ (on unit circle)	Y	6	Y	$-\frac{864}{19 \cdot 7!}h^7 f^{(7)}(\xi)$	Y
f,	$r^3 + r^2 - r - 1 = 0$	$1, -1, -1$	No	2	Y	$\frac{4}{3!}h^3 f'''(\xi)$	No