3.7: Rates of Change in the Natural and Social Sciences

Recall the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_2 - x_1}$$

can be interpreted as the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and the slope of the secant line passing through points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Also, recall the derivative

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

can be interpreted as the **instantaneous rate of change of y with respect to x** at x_1 ($\Delta x \to 0$ as $x_2 \to x_1$) and the slope of the tangent line at $(x_1, f(x_1))$. In this section, we examine some applications of the derivative.

Physics

If $s(t)$ is the position function of a particle, the instantaneous rate of change of position with respect to time is $v = \frac{ds}{dt}$, the **velocity** of the particle. The instantaneous rate of change of velocity with respect to time is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, the **acceleration** of the particle.

Example 1. *The position of a particle is given by the equation*

$$s(t) = t^3 - 6t^2 + 9t$$

where s is measured in meters and t is measured in seconds.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward?

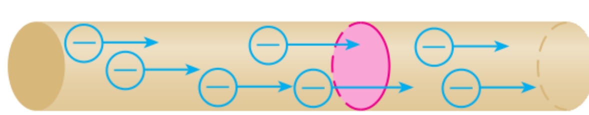
(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \le t \le 5$.

A current exists whenever electric charges move. For example, imagine electrons flowing through a wire depicted below.



If $Q(t)$ is the amount of charge (measured in coulombs) that flows through an area at time t, the instantaneous rate of change of Q with respect to time is $I = \frac{dQ}{dt}$, called **current** (measured in amperes = coulombs/s).

Example 2. *The quantity of charge Q in coulombs (C) that passes through a cross-section of a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 3t^2 + 4t + 2$.*

(a) Find the current when $t = 0.5$ s, $t = 1$ s.

(b) At what time is the current lowest?

Biology

If $n(t)$ is the number of individuals in a population at time t, the instaneous rate of change of n with respect to time is $\frac{dn}{dt}$, the **instantaneous growth rate** of the population.

Example 3. *The number of individuals in a population that doubles every hour is given by the equation*

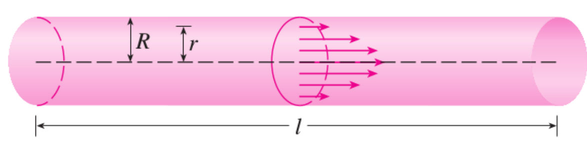
$$n(t) = n_0 \cdot 2^t$$

where n_0 is the initial population at time $t = 0$ and t is measured in hours.

(a) Find the growth rate of the population at time t.

(b) Suppose the initial population is 100. What is the growth rate after 4 hours?

Example 4. *Consider the flow of blood through a blood vessel, such as a vein or artery. We could model the blood vessel as a cylidrical tube with radius R:*



The velocity of blood flow is given by the **law of laminar flow**

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where r is the distance from the central axis, η is the viscosity of the blood, and P is the pressure difference between the ends of the tube.

(a) Find the **velocity gradient** $\frac{dv}{dr}$.

(b) What is the significance of the velocity gradient being negative for all $r > 0$?

Economics

If $C(x)$ is the cost for a particular company to produce x units of a commodity, the instantaneous rate of change of C with respect to x is $\frac{dC}{dx}$, called the **marginal cost**. Since

$$\frac{dC}{dx} = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} \approx C(x+1) - C(x)$$

the marginal cost of producing x units can be interpreted as the cost of producing the next [i.e. the $(x+1)^{st}$] unit.

Example 5. *A company has estimated that the cost (in dollars) of producing x items is*

$$C(x) = 10,000 + 5x + 0.01x^2.$$

(a) Find the marginal cost function.

(b) What is the marginal cost at the production level of 500 items?

(c) What is the cost of producing the 501^{st} item? Compare this to your answer to part (b).

Extra Examples

Example 6. *A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and 5 s. What can you conclude?*

Example 7. *A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius r when (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?*