Foundations of Computational Math II Exam 2
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Due beginning of Class Wednesday, 10 April, 2013

Question	Points Possible	Points Awarded
1. Quadrature	25	
2. Orthogonal Polynomials	25	
3. Economization	25	
4. DFT	25	
Total Points	100	

Name: **Alias:**

Problem 1 (25 points)

1.a

Consider approximating the definite intergral

$$\int_0^6 f(x)dx$$

via Simpson's rule, I_2, Simpson's second rule, I_3, and an extrapolated rule

$$I_{3,2} = \frac{9}{5}I_3 - \frac{4}{5}I_2$$

Is $I_{3,2}$ a Newton-Cotes quadrature method?

1.b

Show that $I_{3,2}$ satisfies

$$I_{3,2} = \int_0^6 p_4(x)dx = \gamma_0^{(4)}f(0) + \gamma_1^{(4)}f(2) + \gamma_2^{(4)}f(3) + \gamma_3^{(4)}f(4) + \gamma_4^{(4)}f(6)$$

where $p_4(x)$ is the polynomial of degree 4 that interpolates $f(0)$, $f(2)$, $f(3)$, $f(4)$, $f(6)$ and has the form

$$p_4(x) = \ell_0^{(4)}(x)f(0) + \ell_1^{(4)}(x)f(2) + \ell_2^{(4)}(x)f(3) + \ell_3^{(4)}(x)f(4) + \ell_4^{(4)}(x)f(6)$$

where the $\ell_i^{(4)}(x)$ are the Lagrange basis functions defined by the interpolation points.

1.c

For this integral, we have

$$I_2 = f(0) + 4f(3) + f(6) = \int_0^6 p_2(x)dx$$

$$p_2(x) = \ell_0^{(2)}(x)f(0) + \ell_1^{(2)}(x)f(3) + \ell_2^{(2)}(x)f(6)$$

$$I_3 = \frac{3}{4}f(0) + \frac{9}{4}f(2) + \frac{9}{4}f(4) + \frac{3}{4}f(6) = \int_0^6 p_3(x)dx$$

$$p_3(x) = \ell_0^{(3)}(x)f(0) + \ell_1^{(3)}(x)f(2) + \ell_2^{(3)}(x)f(4) + \ell_3^{(3)}(x)f(6)$$

where the $\ell_i^{(2)}(x)$ and $\ell_i^{(3)}(x)$ are the quadratic and cubic Lagrange basis functions defined by the interpolation points respectively.

Therefore, we have

$$I_{3,2} = \frac{9}{5}I_3 - \frac{4}{5}I_2$$

$$= \frac{9}{5}\int_0^6 p_3(x)dx - \frac{4}{5}\int_0^6 p_2(x)dx = \int_0^6 \left(\frac{9}{5}p_3(x) - \frac{4}{5}p_2(x)\right) dx$$

$$= f(0)\int_0^6 \left(\frac{9}{5}\ell_0^{(3)}(x) - \frac{4}{5}\ell_0^{(2)}(x)\right) dx + f(6)\int_0^6 \left(\frac{9}{5}\ell_3^{(3)}(x) - \frac{4}{5}\ell_2^{(2)}(x)\right) dx$$

$$f(2)\int_0^6 \frac{9}{5}\ell_1^{(3)}(x)dx - f(3)\int_0^6 \frac{4}{5}\ell_1^{(2)}(x)dx + f(4)\int_0^6 \frac{9}{5}\ell_2^{(3)}(x)dx$$

$$= \gamma_0^{(3,2)}f(0) + \gamma_1^{(3,2)}f(2) + \gamma_2^{(3,2)}f(3) + \gamma_3^{(3,2)}f(4) + \gamma_4^{(3,2)}f(6)$$

Note, however, that the $\gamma_i^{(3,2)}$ are the intergrals of polynomials of degree 2 or degree 3 while the $\gamma_i^{(4)}$ are the intergrals of polynomials of degree 4.

Verify the equivalence of the two expressions for $I_{3,2}$ and explain the apparent contradiction.

Problem 2 (25 points)

2.a

Let $\mathcal{P} = \{P_n\}, n = 0,\ 1,\ 2, \ldots$ be a complete orthogonormal set of polynomials that form a basis for $\mathcal{L}^2_\omega[a,b]$ with inner product and associated norm

$$(f,g)_\omega = \int_a^b \omega(x) f(x) g(x) dx, \quad \|f\|^2_\omega = (f,f)_\omega$$

Given a particular value of n, define $\mathcal{M}_n$ to be the set of polynomials with degree n whose coefficient of x^n is identical to the coefficient of x^n in $P_n(x) \in \mathcal{P}$. Show that $P_n(x)$ solves the minimization problem

$$\min_{q_n \in \mathcal{M}_n} \|q_n\|^2_\omega$$

2.b

A semigroup is a set $\mathcal{S}$ and binary operation $*$ such that

- $\mathcal{S}$ is closed under $*$, i.e., $\forall a,\ b \in \mathcal{S}, \quad a * b \in \mathcal{S}$
- $*$ is associative, i.e., $\forall a,\ b \in \mathcal{S} \quad (a * b) * c = a * (b * c)$.

Note that $*$ need not be commutative.

(i) Let $\mathcal{S}$ be the set of Legendre polynomials and let $*$ be polynomial multiplication, i.e., $P_n * P_m = P_n(x)\ P_m(x)$. Is $\mathcal{S},\ *$ a semigroup?

(ii) Let $\mathcal{S}$ be the set of Chebyshev polynomials and let $*$ be polynomial multiplication, i.e., $T_n * T_m = T_n(x)\ T_m(x)$. Is $\mathcal{S},\ *$ a semigroup?

(iii) Let $\mathcal{S}$ be the set of Chebysheve polynomials and let $*$ be function composition, i.e., $T_n * T_m = T_n(T_m(x))$. Is $\mathcal{S},\ *$ a semigroup?

Problem 3 (25 points)

The probability density function for a Gaussian random variable with mean 0 and variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5x^2/\sigma^2}$$

3.a. Use Chebyshev Economization to derive a polynomial approximation to $f(x)$ on $[-1, 1]$ and derive an expression for the error that can be used to determine the degree of the polynomial required to give

$$|f(x) - p_n(x)| \leq \tau, \quad -1 \leq x \leq 1$$

given the variance σ^2.

3.b. Given $\sigma^2 = 1$, find the polynomial that satisfies the error bound with $\tau = 10^{-6}$.

3.c. Explain what happens to the degree n as $\sigma^2 \to 0$ for a fixed error tolerance τ.

3.d. Explain what happens to the degree n as $\sigma^2 \to \infty$ for a fixed error tolerance τ.

Problem 4 (25 points)

4.a

Recall that evaluating a DFT coefficient could be viewed as applying a composite left end-point composite rectangle rule, i.e.,

$$(f, \phi_k) = \int_0^{2\pi} f(x)\bar{\phi}_k(x)dx \approx (f, \phi_k)_n = \frac{2\pi}{n}\sum_{j=0}^{n-1} f(x_j)e^{-i\theta j(k-n/2)}$$

where f is an element of the space spanned by the Fourier polynomials, $h = \theta = 2\pi/n$, and $x_j = j\theta$.

(i) Determine the error expression for the composite rectangle rule and the order of convergence to the exact integral.

(ii) Suppose that $f(x)$ is periodic on $[0, 2\pi]$ with period 2π. What happens to the order of convergence of the quadrature method? Justify your answer.

4.b

Let x and y be two infinite sequences, i.e.,

$$x = \{\ldots \xi_{-4},\ \xi_{-3},\ \xi_{-2},\ \xi_{-1},\ \xi_0,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \ldots\}$$

$$y = \{\ldots \eta_{-4},\ \eta_{-3},\ \eta_{-2},\ \eta_{-1},\ \eta_0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \ldots\}$$

The convolution $z = x * y$ is an infinite sequence with elements

$$\zeta_k = \sum_{i=-\infty}^{\infty} \eta_i \xi_{i+k}$$

Note ζ_k lines up η_0 with ξ_k and then takes the sum of pairwise products.

Now consider the structured sequences x and y where x is periodic with period n and y is nonzero only in n elements starting at $i = 0$. For example, for $n = 4$ we have

$$\begin{aligned} x &= \{\ldots \xi_{-4},\ \xi_{-3},\ \xi_{-2},\ \xi_{-1},\ \xi_0,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \ldots\} \\ &= \{\ldots \mu_0,\ \mu_1,\ \mu_2,\ \mu_3,\ \mu_0,\ \mu_1,\ \mu_2,\ \mu_3,\ \mu_0,\ \ldots\} \end{aligned}$$

$$\begin{aligned} y &= \{\ldots \eta_{-4},\ \eta_{-3},\ \eta_{-2},\ \eta_{-1},\ \eta_0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \ldots\} \\ &= \{\ldots 0,\ 0,\ 0,\ 0,\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ 0,\ \ldots\} \end{aligned}$$

(i) Show that for the structured x and y sequences the convolution $z = x * y$ is also specificed by only n values and identify its struture.

(ii) Determine the complexity in terms of n required to compute the parameters that specify z for the structured x and y sequences and describe the algorithm that would achieve this complexity.