Foundations of Computational Math II Exam 2
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Due beginning of Class Wednesday, 10 April, 2013

Question	Points Possible	Points Awarded
1. Quadrature	25	
2. Orthogonal Polynomials	25	
3. Economization	25	
4. DFT	25	
Total Points	100	

Name: **Alias:**

Problem 1

1.a

Consider approximating the definite intergral

$$\int_0^6 f(x)dx$$

via Simpson's rule, I_2, Simpson's second rule, I_3, and an extrapolated rule

$$I_{3,2} = \frac{9}{5}I_3 - \frac{4}{5}I_2$$

Is $I_{3,2}$ a Newton-Cotes quadrature method?

1.b

Show that $I_{3,2}$ satisfies

$$I_{3,2} = \int_0^6 p_4(x)dx = \gamma_0^{(4)}f(0) + \gamma_1^{(4)}f(2) + \gamma_2^{(4)}f(3) + \gamma_3^{(4)}f(4) + \gamma_4^{(4)}f(6)$$

where $p_4(x)$ is the polynomial of degree 4 that interpolates $f(0)$, $f(2)$, $f(3)$, $f(4)$, $f(6)$ and has the form

$$p_4(x) = \ell_0^{(4)}(x)f(0) + \ell_1^{(4)}(x)f(2) + \ell_2^{(4)}(x)f(3) + \ell_3^{(4)}(x)f(4) + \ell_4^{(4)}(x)f(6)$$

where the $\ell_i^{(4)}(x)$ are the Lagrange basis functions defined by the interpolation points.

1.c

For this integral, we have

$$I_2 = f(0) + 4f(3) + f(6) = \int_0^6 p_2(x)dx$$

$$p_2(x) = \ell_0^{(2)}(x)f(0) + \ell_1^{(2)}(x)f(3) + \ell_2^{(2)}(x)f(6)$$

$$I_3 = \frac{3}{4}f(0) + \frac{9}{4}f(2) + \frac{9}{4}f(4) + \frac{3}{4}f(6) = \int_0^6 p_3(x)dx$$

$$p_3(x) = \ell_0^{(3)}(x)f(0) + \ell_1^{(3)}(x)f(2) + \ell_2^{(3)}(x)f(4) + \ell_3^{(3)}(x)f(6)$$

where the $\ell_i^{(2)}(x)$ and $\ell_i^{(3)}(x)$ are the quadratic and cubic Lagrange basis functions defined by the interpolation points respectively.

Therefore, we have

$$I_{3,2} = \frac{9}{5}I_3 - \frac{4}{5}I_2$$

$$= \frac{9}{5}\int_0^6 p_3(x)dx - \frac{4}{5}\int_0^6 p_2(x)dx = \int_0^6 \left(\frac{9}{5}p_3(x) - \frac{4}{5}p_2(x)\right) dx$$

$$= f(0)\int_0^6 \left(\frac{9}{5}\ell_0^{(3)}(x) - \frac{4}{5}\ell_0^{(2)}(x)\right) dx + f(6)\int_0^6 \left(\frac{9}{5}\ell_3^{(3)}(x) - \frac{4}{5}\ell_2^{(2)}(x)\right) dx$$

$$f(2)\int_0^6 \frac{9}{5}\ell_1^{(3)}(x)dx - f(3)\int_0^6 \frac{4}{5}\ell_1^{(2)}(x)dx + f(4)\int_0^6 \frac{9}{5}\ell_2^{(3)}(x)dx$$

$$= \gamma_0^{(3,2)}f(0) + \gamma_1^{(3,2)}f(2) + \gamma_2^{(3,2)}f(3) + \gamma_3^{(3,2)}f(4) + \gamma_4^{(3,2)}f(6)$$

Note, however, that the $\gamma_i^{(3,2)}$ are the intergrals of polynomials of degree 2 or degree 3 while the $\gamma_i^{(4)}$ are the intergrals of polynomials of degree 4.

Verify the equivalence of the two expressions for $I_{3,2}$ and explain the apparent contradiction.

Solution:

Using the definition of the the I_2 and I_3 quadrature rules and the interval of the definite integral yields

$$I_{3,2} = \frac{9}{5}I_3 - \frac{4}{5}I_2$$

$$= \frac{11}{20}f(0) + \frac{81}{20}f(2) - \frac{16}{5}f(3) + \frac{81}{20}f(4) + \frac{11}{20}f(6)$$
$$= \gamma_0^{(3,2)}f(0) + \gamma_1^{(3,2)}f(2) + \gamma_2^{(3,2)}f(3) + \gamma_3^{(3,2)}f(4) + \gamma_4^{(3,2)}f(6).$$

It is easily verified that

$$\int_0^6 \ell_0^{(2)}(x)dx = \int_0^6 \frac{(x-3)(x-6)}{18}dx = 1$$
$$\int_0^6 \ell_1^{(2)}(x)dx = -\int_0^6 \frac{(x)(x-6)}{9}dx = 4$$
$$\int_0^6 \ell_2^{(2)}(x)dx = \int_0^6 \frac{(x)(x-3)}{18}dx = 1$$

$$\therefore I_2 = \int_0^6 p_2(x)dx = f(0) + 4f(3) + f(6)$$

Similarly,

$$\int_0^6 \ell_0^{(3)}(x)dx = -\int_0^6 \frac{(x-2)(x-4)(x-6)}{48}dx = \frac{3}{4}$$

$$\int_0^6 \ell_1^{(3)}(x)dx = \int_0^6 \frac{(x)(x-4)(x-6)}{16}dx = \frac{9}{4}$$

$$\int_0^6 \ell_2^{(3)}(x)dx = -\int_0^6 \frac{(x)(x-2)(x-6)}{16}dx = \frac{9}{4}$$

$$\int_0^6 \ell_3^{(3)}(x)dx = \int_0^6 \frac{(x)(x-2)(x-4)}{48}dx = \frac{3}{4}$$

$$\therefore I_3 = \int_0^6 p_3(x)dx = \frac{3}{4}f(0) + \frac{9}{4}f(2) + \frac{9}{4}f(4) + \frac{3}{4}f(6)$$

Using these two sums in the extrapolation formula yields as desired

$$I_{3,2} = \frac{9}{5}I_3 - \frac{4}{5}I_2$$

$$= \frac{11}{20}f(0) + \frac{81}{20}f(2) - \frac{16}{5}f(3) + \frac{81}{20}f(4) + \frac{11}{20}f(6)$$
$$= \gamma_0^{(3,2)}f(0) + \gamma_1^{(3,2)}f(2) + \gamma_2^{(3,2)}f(3) + \gamma_3^{(3,2)}f(4) + \gamma_4^{(3,2)}f(6)$$

We also have

$$I_{3,2} = \int_0^6 p_4(x)dx = \gamma_0^{(4)}f(0) + \gamma_1^{(4)}f(2) + \gamma_2^{(4)}f(3) + \gamma_3^{(4)}f(4) + \gamma_4^{(4)}f(6)$$

The coefficients satisfy

$$p_4(x) = \ell_0^{(4)}(x)f(0) + \ell_1^{(4)}(x)f(2) + \ell_2^{(4)}(x)f(3) + \ell_3^{(4)}(x)f(4) + \ell_4^{(4)}(x)f(6)$$

$$\begin{aligned}
\ell_0^{(4)}(x) &= \frac{(x-2)(x-3)(x-4)(x-6)}{144} \\
\ell_1^{(4)}(x) &= -\frac{(x)(x-3)(x-4)(x-6)}{16} \\
\ell_2^{(4)}(x) &= \frac{(x)(x-2)(x-4)(x-6)}{9} \\
\ell_3^{(4)}(x) &= -\frac{(x)(x-2)(x-3)(x-6)}{16} \\
\ell_4^{(4)}(x) &= \frac{x(x-2)(x-3)(x-4)}{144}
\end{aligned}$$

It is easily verified that

$$\gamma_0^{(4)} = \int_0^6 \ell_0^{(4)}(x)dx = \int_0^6 \frac{(x-2)(x-3)(x-4)(x-6)}{144}dx = \frac{11}{20}$$

$$\gamma_1^{(4)} = \int_0^6 \ell_1^{(4)}(x)dx = -\int_0^6 \frac{(x)(x-3)(x-4)(x-6)}{16}dx = \frac{81}{20}$$

$$\gamma_2^{(4)} = \int_0^6 \ell_2^{(4)}(x)dx = \int_0^6 \frac{(x)(x-2)(x-4)(x-6)}{9}dx = -\frac{16}{5}$$

$$\gamma_3^{(4)} = \int_0^6 \ell_3^{(4)}(x)dx = -\int_0^6 \frac{(x)(x-2)(x-3)(x-6)}{16}dx = \frac{81}{20}$$

$$\gamma_4^{(4)} = \int_0^6 \ell_4^{(4)}(x)dx = \int_0^6 \frac{x(x-2)(x-3)(x-4)}{144}dx = \frac{11}{20}$$

Therefore

$$I_{3,2} = \int_0^6 p_4(x)dx = \frac{9}{5}I_3 - \frac{4}{5}I_2$$

(This is not a Newton Cotes method since is verified that it is based on interpolation at nonuniform points.)

To reconcile the apparent contradiction we must explain how the following hold:

$$\gamma_0^{(4)} = \int_0^6 \ell_0^{(4)}(x)dx = \int_0^6 \left(\frac{9}{5}\ell_0^{(3)}(x) - \frac{4}{5}\ell_0^{(2)}(x)\right) dx = \gamma_0^{(3,2)}$$

$$\gamma_1^{(4)} = \int_0^6 \ell_1^{(4)}(x)dx = \int_0^6 \frac{9}{5}\ell_1^{(3)}(x)dx = \gamma_1^{(3,2)}$$

$$\gamma_2^{(4)} = \int_0^6 \ell_2^{(4)}(x)dx = -\int_0^6 \frac{4}{5}\ell_1^{(2)}(x)dx = \gamma_2^{(3,2)}$$

$$\gamma_3^{(4)} = \int_0^6 \ell_3^{(4)}(x)dx = \int_0^6 \frac{9}{5}\ell_2^{(3)}(x)dx = \gamma_3^{(3,2)}$$

$$\gamma_4^{(4)} = \int_0^6 \ell_4^{(4)}(x)dx = \int_0^6 \left(\frac{9}{5}\ell_3^{(3)}(x) - \frac{4}{5}\ell_2^{(2)}(x)\right) dx = \gamma_4^{(3,2)}$$

even though

$$\begin{aligned}
\ell_0^{(4)}(x) - \left(\frac{9}{5}\ell_0^{(3)}(x) - \frac{4}{5}\ell_0^{(2)}(x)\right) &= r_0(x) \neq 0 \\
\ell_1^{(4)}(x) - \frac{9}{5}\ell_1^{(3)}(x) &= r_1(x) \neq 0 \\
\ell_2^{(4)}(x) + \frac{4}{5}\ell_1^{(2)}(x) &= r_2(x) \neq 0 \\
\ell_3^{(4)}(x) - \frac{9}{5}\ell_2^{(3)}(x) &= r_3(x) \neq 0 \\
\ell_4^{(4)}(x) - \left(\frac{9}{5}\ell_3^{(3)}(x) - \frac{4}{5}\ell_2^{(2)}(x)\right) &= r_4(x) \neq 0
\end{aligned}$$

The answer, of course, follows from the fact that we are integrating the $r_i(x)$ to give the differences

$$\int_0^6 r_i(x)dx = \gamma_i^{(4)} - \gamma_i^{(3,2)}$$

This is easily verified. Scaling and integrating the $r_i(x)$ yields

$$144\int_0^6 r_0(x)dx = \int_0^6 [(x-2)(x-3)(x-4)(x-6) + \frac{27}{5}(x-2)(x-4)(x-6) + \frac{32}{5}(x-3)(x-6)]dx = 0$$

$$-90\int_0^6 r_1(x)dx = \int_0^6 x(x-4)(x-6)(5x-6)dx = 0$$

$$45\int_0^6 r_2(x)dx = \int_0^6 x(x-6)(5x^2-30x+36)dx = 0$$

$$90\int_0^6 r_3(x)dx = \int_0^6 x(x-2)(x-6)(24-5x)dx = 0$$

$$144\int_0^6 r_4(x)dx = \int_0^6 \left(x(x-2)(x-3)(x-4) - \frac{27}{5}x(x-2)(x-4) + \frac{32}{5}x(x-3)\right) dx = 0$$

Problem 2 (25 points)

2.a

Let $\mathcal{P} = \{P_n\}, n = 0,\ 1,\ 2, \dots$ be a complete orthogonormal set of polynomials that form a basis for $\mathcal{L}^2_\omega[a,b]$ with inner product and associated norm

$$(f,g)_\omega = \int_a^b \omega(x)f(x)g(x)dx, \quad \|f\|^2_\omega = (f,f)_\omega$$

Given a particular value of n, define $\mathcal{M}_n$ to be the set of polynomials with degree n whose coefficient of x^n is identical to the coefficient of x^n in $P_n(x) \in \mathcal{P}$. Show that $P_n(x)$ solves the minimization problem

$$\min_{q_n \in \mathcal{M}_n} \|q_n\|^2_\omega$$

Solution:

There are two basic ways to show this result. The first is based on the expansion of any $q_n \in \mathcal{M}_n$ in terms of the orthonormal polynomials. The second uses orthogonality of the residual.

For the first we have

$$\forall q_n \in \mathcal{M}_n q_n = \sum_{k=0}^{n} \alpha_k P_k$$

$$\alpha_n = 1 \quad \text{since the coefficient of } x^n \text{ in } q_n \text{ matches that in } P_n$$

$$\|q_n\|^2_\omega = (q_n, q_n)_\omega = \sum_{k=0}^{n} \alpha_k (q_n, P_k)_\omega = 1 + \sum_{k=0}^{n-1} \alpha_k^2$$

$$\text{minimum at } \alpha_k = 0, 0 \le k \le n-1 \qquad \therefore q_n^* = \sum_{k=0}^{n} \alpha_k P_k = \alpha_n P_n = P_n \qquad \square$$

For the second approach note that since the coefficient of x^n in q_n matches that in P_n, $q_n - P_n$ is a polynomial of degree at most $n-1$ and so $(P_n, q_n - P_n)_\omega = 0$. We then have

$$\begin{aligned}
\|q_n\|^2 - \|P_n\|^2 &= (q_n, q_n)_\omega - (P_n, P_n)_\omega \\
&= (q_n - P_n, q_n - P_n)_\omega + 2(P_n, q_n - P_n)_\omega \\
&= \|q_n - P_n\|^2 \begin{cases} > 0 & q_n \ne P_n \\ = 0 & q_n = P_n \end{cases} \qquad \square
\end{aligned}$$

2.b

A semigroup is a set $\mathcal{S}$ and binary operation $*$ such that

- $\mathcal{S}$ is closed under $*$, i.e., $\forall a,\ b \in \mathcal{S}, \quad a * b \in \mathcal{S}$
- $*$ is associative, i.e., $\forall a,\ b,\ c \in \mathcal{S} \quad (a * b) * c = a * (b * c)$.

Note that $*$ need not be commutative.

(i) Let $\mathcal{S}$ be the set of Legendre polynomials and let $*$ be polynomial multiplication, i.e., $P_n * P_m = P_n(x)\ P_m(x)$. Is $\mathcal{S},\ *$ a semigroup?

(ii) Let $\mathcal{S}$ be the set of Chebyshev polynomials and let $*$ be polynomial multiplication, i.e., $T_n * T_m = T_n(x)\ T_m(x)$. Is $\mathcal{S},\ *$ a semigroup?

(iii) Let $\mathcal{S}$ be the set of Chebysheve polynomials and let $*$ be function composition, i.e., $T_n * T_m = T_n(T_m(x))$. Is $\mathcal{S},\ *$ a semigroup?

Solution:

The first two situations fail because $*$ as defined is not closed on the associated set. This leaves the case where

- $\mathcal{S}$ is the set of Chebyshev polynomials
- and $*$ is function composition, i.e., $T_n * T_m = T_n(T_m(x))$.

We have

$$T_n(x) = \cos n \arccos x$$
$$T_n(T_m(x)) = \cos n \arccos \cos m \arccos x = \cos nm \arccos x = T_{nm}(x) \quad \therefore \quad \text{closed}$$

$$T_n * (T_m * T_r) = T_n * T_{mr} = T_n(T_{mr}(x)) = T_{nmr}$$

$$(T_n * T_m) * T_r = T_n(T_m(x)) * T_r = T_{nm} * T_r = T_{nm}(T_r(x)) = T_{nmr} \quad \therefore \quad \text{associative}$$

Problem 3 (25 points)

The probability density function for a Gaussian random variable with mean 0 and variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5x^2/\sigma^2}$$

3.a. Use Chebyshev Economization to derive a polynomial approximation to $f(x)$ on $[-1, 1]$ and derive an expression for the error that can be used to determine the degree of the polynomial required to give

$$|f(x) - p_n(x)| \leq \tau, \quad -1 \leq x \leq 1$$

given the variance σ^2.

3.b. Given $\sigma^2 = 1$, find the polynomial that satisfies the error bound with $\tau = 10^{-6}$.

3.c. Explain what happens to the degree n as $\sigma^2 \to 0$ for a fixed error tolerance τ.

3.d. Explain what happens to the degree n as $\sigma^2 \to \infty$ for a fixed error tolerance τ.

Solution: We have

$$\begin{aligned} f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5x^2/\sigma^2} &= \frac{1}{\sigma\sqrt{2\pi}} e^{\alpha x^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}} y(x) \end{aligned}$$

Let $\mu = 2\alpha$ and $p(x)$ be a polynomial we have the simple recurrence

$$(p(x)y(x))' = p'(x)y(x) + p(x)\mu x y(x) = (p'(x) + p(x)\mu x)y(x)$$

which is easily applied to produce

$$
\begin{aligned}
y &= e^{\alpha x^2} \\
y' &= \mu x y \\
y'' &= (\mu + \mu^2 x^2) y \\
y''' &= (3\mu^2 x + \mu^3 x^3) y \\
y^{(4)} &= (3\mu^2 + 6\mu^3 x^2 + \mu^4 x^4) y \\
y^{(5)} &= (15\mu^3 x + 10\mu^4 x^3 + \mu^5 x^5) y \\
y^{(6)} &= (15\mu^3 + 45\mu^4 x^2 + 15\mu^5 x^4 + \mu^6 x^6) y \\
y^{(7)} &= (105\mu^4 x + 105\mu^5 x^3 + 21\mu^6 x^5 + \mu^7 x^7) y \\
y^{(8)} &= (105\mu^4 + 420\mu^5 x^2 + 210\mu^6 x^4 + 28\mu^7 x^6 + \mu^8 x^8) y \\
y^{(9)} &= (945\mu^5 x + 1260\mu^6 x^3 + 378\mu^7 x^5 + 36\mu^8 x^7 + \mu^9 x^9) y \\
y^{(10)} &= (945\mu^5 + 4725\mu^6 x^2 + 3150\mu^7 x^4 + 630\mu^8 x^6 + 45\mu^9 x^8 + \mu^{10} x^{10}) y \\
y^{(11)} &= (10395\mu^6 x + 17325\mu^7 x^3 + 6930\mu^8 x^5 + 990\mu^9 x^7 + 55\mu^{10} x^9 + \mu^{11} x^{11}) y \\
y^{(12)} &= (10395\mu^6 + 62370\mu^7 x^2 + 51975\mu^8 x^4 + 13860\mu^9 x^6 + 1485\mu^{10} x^8 + 66\mu^{11} x^{10} + \mu^{12} x^{12}) y
\end{aligned}
$$

To expand about $x = 0$ with $\sigma^2 = 1$ we have $\mu = -1$, $f^{(k)} = y^{(k)}/\sqrt{2\pi} \approx 0.4 y^{(k)}$, and

$$
\begin{aligned}
y(x) &\approx 1 - \frac{x^2}{2} + \frac{3}{24}x^4 - \frac{15}{720}x^6 + \frac{105}{40320}x^8 - \frac{945}{3628800}x^{10} + \frac{10395}{479001600}x^{12} \\
f(x) &\approx \frac{1}{\sqrt{2\pi}}\left(1 - \frac{x^2}{2} + \frac{3}{24}x^4 - \frac{15}{720}x^6 + \frac{105}{40320}x^8 - \frac{945}{3628800}x^{10} + \frac{10395}{479001600}x^{12}\right)
\end{aligned}
$$

The remainder term involves $y^{(14)}$.

The error for the degree 12 Taylor polynomial is $\approx 2 \times 10^{-7}$. The error for the degree 10 Taylor polynomial is $\approx 3 \times 10^{-6}$. So we see that the Taylor series polynomial converges slowly to the 10^{-6} absolute error requested. As a result, one would not expect Chebyshev ecomomization of this polynomial to reduce the degree significantly.

The even/odd degree disparity makes sense since f is an even function. Therefore, since the Chebyshev polynomials are also even and odd based on degree and the weight function in the inner product associated with the Chebyshev polynomials is an even function we must have

$$
\begin{aligned}
(f, T_{2k+1})_\omega &= 0 \\
(f, T_{2k})_\omega &\neq 0
\end{aligned}
$$

(The second inner product unfortunately does not have a nice closed form – try integration by parts). Therefore, since the GFS in terms of Chebyshev will have only even powers and any other polynomial expression for f must simply linearly recombine the Chebyshev polynomials the powers will stay even. So in general expanding about $x = 0$, letting $y_{2k}(x)$ denote the degree $2k$ truncation of the Taylor series of y,

$$f(x) = \frac{1}{\sqrt{2\pi}} \left(y_{2k}(x) + \frac{x^{2k+2}}{(2k+2)!} y^{(2k+2)}(\xi) \right) = \frac{1}{\sqrt{2\pi}} \left(\sum_{j=0}^{k} T_{2j}(x) + \frac{x^{2k+2}}{(2k+2)!} y^{(2k+2)}(\xi) \right)$$

So if economization reduces the polynomial to degree $2s$ from degree $2k$, the error form is

$$E_{2s} = \frac{1}{\sqrt{2\pi}} \left(\sum_{j=s+1}^{k} T_{2j}(x) + \frac{x^{2k+2}}{(2k+2)!} y^{(2k+2)}(\xi) \right)$$

and given a bound on the Taylor remainder and $\|T_j\|_\infty = 1$ the error is easily bounded.

Note also that as the variance increases the derivatives get smaller since μ is proportional to $1/\sigma^2$, and $f(x)$ gets flatter and therefore easier to approximate on the interval with a polynomial. As variance decreases, $f(x)$ narrows to a peaked function and polynomials do not approximate it well.

To see the effectiveness (or lack of it) for Chebyshev economization start with the degree 10 Taylor polynomial for which the Taylor remainder error is $\approx 3 \times 10^{-6}$.

$$f(x) \approx p_{10}(x) = \frac{1}{\sqrt{2\pi}} \left(1 - \frac{x^2}{2} + \frac{3}{24}x^4 - \frac{15}{720}x^6 + \frac{105}{40320}x^8 - \frac{945}{3628800}x^{10} \right)$$

Recall we have

$$x^0 = T_0, \quad x^1 = T_1, \quad x^2 = \frac{1}{2}T_2 + \frac{1}{2}T_0, \quad x^3 = \frac{1}{4}T_3 + \frac{3}{4}T_1,$$
$$x^4 = \frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0, \quad x^5 = \frac{1}{16}T_5 + \frac{5}{16}T_3 + \frac{5}{8}T_1$$
$$x^6 = \frac{1}{32}T_6 + \frac{3}{16}T_4 + \frac{15}{32}T_2 + \frac{5}{16}T_0$$
$$x^7 = \frac{1}{64}T_7 + \frac{7}{64}T_5 + \frac{7}{18}T_3 + \frac{35}{64}T_0$$
$$x^8 = \frac{1}{128}T_8 + \frac{1}{16}T_6 + \frac{7}{32}T_4 + \frac{7}{16}T_2 + \frac{35}{128}T_0$$

We can get x^{10} from T_{10}

$$512x^{10} = T_{10} + 1286x^8 - 1120x^6 + 400x^4 - 50x^2 + 1$$

$$x^{10} = \frac{1}{512}T_{10} + \frac{643}{32678}T_8 + \frac{363}{4096}T_6 + \frac{1941}{8192}T_4 + \frac{1701}{4096}T_2 + \frac{8169}{32678}T_0$$

Transforming to the Chebyshev basis yields the following coefficients for each T_8 and T_{10} which are the only two we need to see that economization does not work well beyond reducing from degree 10 to degree 8.

$$T_{10}: \quad \left| -\frac{945}{3628800} \times \frac{1}{512} \right| \approx 5 \times 10^{-7}$$
$$T_8: \quad \left| \frac{105}{40320} \times \frac{105}{40320} - \frac{945}{3628800} \times \frac{643}{32678} \right| \approx 2.5 \times 10^{-5}$$

Recalling that the Taylor remainder error is $\approx 3 \times 10^{-6}$ we see that we can remove the T_{10} term an maintain the $\approx 3 \times 10^{-6}$, however, removing the T_8 term introduces an error at the 10^{-5} level.

Problem 4 (25 points)

4.a

Recall that evaluating a DFT coefficient could be viewed as applying a composite left end-point composite rectangle rule, i.e.,

$$(f,\phi_k) = \int_0^{2\pi} f(x)\bar{\phi}_k(x)dx \approx (f,\phi_k)_n = \frac{2\pi}{n}\sum_{j=0}^{n-1} f(x_j)e^{-i\theta j(k-n/2)}$$

where f is an element of the space spanned by the Fourier polynomials, $h=\theta=2\pi/n$, and $x_j = j\theta$.

(i) Determine the error expression for the composite rectangle rule and the order of convergence to the exact integral.

(ii) Suppose that $f(x)$ is periodic on $[0,2\pi]$ with period 2π. What happens to the order of convergence of the quadrature method? Justify your answer.

Solution:

The left rectangle rule is given by

$$\int_a^b f(x)dx \approx (b-a)f(a) = (b-a)f_0$$

with error

$$\begin{aligned}\int_a^b f(x)dx = \int_a^b f(a) + f'(\xi)(x-a)dx &= (b-a)f_0 \int_a^b f'(\xi)(x-a)dx \\ &= (b-a)f_0 + \frac{(b-a)^2}{2}f'(\mu)\end{aligned}$$

The composite left rectangle rule and error are therefore

$$\begin{aligned}h &= \frac{(b-a)}{n} \\ I_{cr} &= h\sum_{i=0}^{n-1} f(a+ih) \\ E_{cr} &= \sum_{i=0}^{n-1}\frac{h^2}{2}f'(\eta_i) \\ &= (b-a)\frac{h}{2}f'(\zeta)\end{aligned}$$

If $f(0) = f(2\pi)$ then, taking $a = 0$ and $b = 2\pi$,the composite rectangle rule becomes the composite trapezoidal rule

$$h = \frac{(b-a)}{n}$$

$$\begin{aligned} I_{cr} &= h \sum_{i=0}^{n-1} f(a+ih) = hf(a) + h \sum_{i=1}^{n-1} f(a+ih) \\ &= \frac{h}{2} f(a) + \frac{h}{2} f(b) + h \sum_{i=1}^{n-1} f(a+ih) \\ &= I_{ct} \end{aligned}$$

The composite error for the composite trapezoidal rule is

$$E_{ct} = -(b-a)\frac{h^2}{12} f''(\gamma), \quad h = \frac{(b-a)}{n}$$

so we see the method is more accurate for periodic functions. There is extensive literature on this topic for various situations.

4.b

Let x and y be two infinite sequences, i.e.,

$$x = \{\ldots \xi_{-4},\ \xi_{-3},\ \xi_{-2},\ \xi_{-1},\ \xi_0,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \ldots\}$$

$$y = \{\ldots \eta_{-4},\ \eta_{-3},\ \eta_{-2},\ \eta_{-1},\ \eta_0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \ldots\}$$

The convolution $z = x * y$ is an infinite sequence with elements

$$\zeta_k = \sum_{i=-\infty}^{\infty} \eta_i \xi_{i+k}$$

Note ζ_k lines up η_0 with ξ_k and then takes the sum of pairwise products.

Now consider the structured sequences x and y where x is periodic with period n and y is nonzero only in n elements starting at $i = 0$. For example, for $n = 4$ we have

$$\begin{aligned} x &= \{\ldots \xi_{-4},\ \xi_{-3},\ \xi_{-2},\ \xi_{-1},\ \xi_0,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \ldots\} \\ &= \{\ldots \mu_0,\ \mu_1,\ \mu_2,\ \mu_3,\ \mu_0,\ \mu_1,\ \mu_2,\ \mu_3,\ \mu_0,\ \ldots\} \end{aligned}$$

$$\begin{aligned} y &= \{\ldots \eta_{-4},\ \eta_{-3},\ \eta_{-2},\ \eta_{-1},\ \eta_0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \ldots\} \\ &= \{\ldots 0,\ 0,\ 0,\ 0,\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ 0,\ \ldots\} \end{aligned}$$

(i) Show that for the structured x and y sequences the convolution $z = x * y$ is also specificed by only n values and identify its struture.

(ii) Determine the complexity in terms of n required to compute the parameters that specify z for the structured x and y sequences and describe the algorithm that would achieve this complexity.

Solution:

The fact that z is periodic with period n is easily seen by lining up the terms for an example with $n = 4$

$$\begin{aligned} \zeta_0 &= \alpha_0\mu_0 + \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3 \\ \zeta_1 &= \alpha_0\mu_1 + \alpha_1\mu_2 + \alpha_2\mu_3 + \alpha_3\mu_0 \\ \zeta_2 &= \alpha_0\mu_2 + \alpha_1\mu_3 + \alpha_2\mu_0 + \alpha_3\mu_1 \\ \zeta_3 &= \alpha_0\mu_3 + \alpha_1\mu_0 + \alpha_2\mu_1 + \alpha_3\mu_2 \\ \zeta_4 &= \alpha_0\mu_0 + \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3 = \zeta_0 \\ \zeta_5 &= \zeta_1 \end{aligned}$$

Therefore, $\zeta_0,\ \zeta_1,\ \zeta_2,\ \zeta_3$ determines all of z. Continuing with the example we also see that

$$\begin{pmatrix}\zeta_0\\ \zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix} = \begin{pmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3\\ \alpha_3 & \alpha_0 & \alpha_1 & \alpha_2\\ \alpha_2 & \alpha_3 & \alpha_0 & \alpha_1\\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_3\end{pmatrix}\begin{pmatrix}\mu_0\\ \mu_1\\ \mu_2\\ \mu_3\end{pmatrix}$$

The matrix is ciculant and the matrix vector multiplication can be done via the known decomposition

$$C = F^H \Gamma F$$

where the application of F or F^H is $O(n \log n)$ for a period of n and Γ can be determined via a matrix vector product of F^H and the vector containing the n α_i's using the FFT and IFFT.