Homework 3 Foundations of Computational Math 2 Spring 2012

Solutions will be posted Friday, 2/3/12

Problem 3.1

Show that given a set of points

$$x_0, \quad x_1, \ldots, x_n$$

a Leja ordering can be computed in $O(n^2)$ operations.

Problem 3.2

Consider a polynomial

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$p_n(x)$ can be evaluated using Horner's rule (written here with the dependence on the formal argument x more explicitly shown)

$c_n(x) = \alpha_n$
for $i = n-1 : -1 : 0$
    $c_i(x) = xc_{i+1}(x) + \alpha_i$
end

$p_n(x) = c_0(x)$

If the roots of the polynomial are known we can use a recurrence based on

$$p_n(x) = \alpha_n (x - \rho_1) \cdots (x - \rho_n)$$

given by:

$d_0 = \alpha_n$
for $i = 1 : n$
    $d_i = d_{i-1} * (x - \rho_i)$
end

$p_n(x) = d_n$

This algorithm can be shown to compute $p_n(x)$ to high relative accuracy. Specifically,

$$d_n = p_n(x)(1 + \mu), \qquad |\mu| \leq \gamma_{2n} u$$

where $\gamma_k = ku/(1 - ku)$ and u is the unit roundoff of the floating point system used.

3.2.a

An error analysis of Horner's rule shows that the computed value of the polynomial satisfies

$$\hat{c}_0 = (1+\theta_1)\alpha_0 + (1+\theta_3)\alpha_1 x + \cdots + (1+\theta_{2n-1})\alpha_{n-1}x^{n-1} + (1+\theta_{2n})\alpha_n x^n$$

where $|\theta_k| \leq \gamma_k$. (The pattern on the subscript is odd numbers, i.e., increment of 2, until the last which is even, i.e., last increment is 1.)

Let

$$\tilde{p}_n(x) = |\alpha_0| + |\alpha_1|x + \cdots + |\alpha_n|x^n$$

Show that

$$\frac{|p_n(x) - \hat{c}_0|}{|p_n(x)|} \leq \gamma_{2n}\frac{\tilde{p}_n(|x|)}{|p_n(x)|} \tag{1}$$

and therefore

$$\kappa_{rel} = \frac{\tilde{p}(|x|)}{|p(x)|}$$

is a relative condition number for perturbations to the coefficients bounded by γ_{2n}.

3.2.b

Equation 1 also yields an a priori bound on the forward error

$$|p_n(x) - \hat{c}_0|$$

that can be computed along with evaluating $p_n(x)$ with Horner's rule.

Write a code that evaluates $p_n(x)$ and the forward error bound using Horner's rule and $p_n(x)$ using the product form . Apply the code to the polynomial

$$p_9(x) = (x-2)^9$$
$$= x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$$

to evaluate $p_9(x)$ in both forms and the a priori bound on forward error at several hundred points in the interval $[1.91, 2.1]$.

You may view $p_9(x)$ evaluated using the product form as exact for the purposes of this exercise. Plot the product form values across the interval and use the forward error bound to plot curves above and below the "exact" product form curve to show where the computed values must lie. (Recall, for IEEE single precision $u \approx 5.9 \times 10^{-8}$ and for IEEE double precision $u \approx 1.1 \times 10^{-16}$.) Then plot the values of $p_9(x)$ computed with Horner's rule and verify the correctness of the a priori bounding curves. Comment on the tightness of the bounds and the computed values of $p_9(x)$.

3.2.c

The computed value on step i of Horner's rule satisfies

$$(1+\epsilon_i)\hat{c}_i = x\hat{c}_{i+1}(1+\delta_i) + \alpha_i, \quad |\delta_i| \leq u, \quad |\epsilon_i| \leq u$$

Define $\hat{c}_i = c_i + e_i$ with $e_n = 0$ and c_i the exact value of the parameter in Horner's rule evaluated in exact arithmetic. Show that

$$e_i = xe_{i+1} + x\hat{c}_{i+1}\delta_i - \epsilon_i\hat{c}_i$$

$$|e_i| \leq u\beta_i$$

$$\beta_i = |x|\beta_{i+1} + |x||\hat{c}_{i+1}| + |\hat{c}_i|, \quad \beta_n = 0$$

and therefore we have the bound

$$|p_n(x) - \hat{c}_0| \leq u\beta_0$$

This bound is called a running error bound for Horner's rule and can also be easily incorporated into the code for simulatneous evaluation with the values above. Compare this bounds prediction with those of the a priori bound above.

Problem 3.3

Text exercise 8.10.9 on page 377