Foundations of Computational Math II Exam 2
Take-home Exam
Open Notes, Textbook, Homework Solutions Only
Calculators Allowed
Friday 13 April, 2012

Question	Points Possible	Points Awarded
1. Approximation	25	
2. Quadrature	25	
3. GFS	25	
4. LMS Methods	25	
Total Points	100	

Name: **Alias:**

Problem 1

(25 points)

Suppose you are given the function $f(x)$ on $[0, 2]$:

$$f(x) = \sqrt{x}$$

1.a

Find, $p_1(x)$, the linear polynomial that is the near-minimax approximation to $f(x)$ on the interval $[0, 2]$.

Solution:

The linear near minimax approximation to $f(z)$ is the interpolating polynomial with z_0 and z_1 taken as the roots of the quadratic Chebyshev polynomial on $[-1, 1]$. The monic version $t_2(z) = z^2 - 1/2$ is relevant to the error and so we have $z_0 = -\sqrt{1/2}$, $z_1 = \sqrt{1/2}$.

$$p_1(z) = f(z_0) + (z - z_0)f[z_0, z_1]$$

So we must convert $f(x)$ to $f(z)$ via $x = z + 1$

$$f(z) = \sqrt{z+1}$$

$$p_1(z) = \sqrt{1 - \sqrt{1/2}} + (z + \sqrt{1/2}))f[-\sqrt{1/2}, \sqrt{1/2}]$$

$$f[-\sqrt{1/2}, \sqrt{1/2}] = \frac{\sqrt{1+\sqrt{1/2}} - \sqrt{1-\sqrt{1/2}}}{\sqrt{2}} = \frac{1}{2}\left(\sqrt{2+\sqrt{2}} - \sqrt{2-\sqrt{2}}\right)$$

It is easy to verify that $p_1(z - 1) = f(x)$ at x_0 and x_1.

1.b

Find, $q_1(x)$, the linear polynomial that is the minimax (best) approximation to $f(x)$ on the interval $[0, 2]$.

Solution:

We must find 3 points that have equal by alternating sign of error. Let $q_1(x) = \alpha x + \beta$ and $e(x) = f(x) - \alpha x - \beta$

$f(x)$ is monotonically increasing, i.e., $f'(x) > 0$ on $[-1, 1]$ and therefore we look for an interior point c such that

$$e'(c) = f'(c) - \alpha = 0 \rightarrow \alpha = \frac{1}{2\sqrt{c}}$$

Now we impose the maximum error conditions

$$e(0) = e(2) \rightarrow 0 = \sqrt{2} - 2\alpha = \sqrt{2} - 2\frac{1}{2\sqrt{c}} \therefore \sqrt{c} = \frac{1}{\sqrt{2}} \rightarrow c = 1/2 \rightarrow \alpha = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

β can now be recovered easily

$$\begin{aligned} e(0) &= -e(c) \\ -\beta &= -\sqrt{c} + \alpha c + \beta \\ 2\beta &= \frac{1}{2\sqrt{2}} \\ \beta &= \frac{\sqrt{2}}{8} \end{aligned}$$

The best linear approximation to $f(x)$ is therefore

$$q_1(x) = \alpha x + \beta = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{8}$$

1.c

Give a bound for the error $|f(x) - p_1(x)|$ on the interval $[0, 2]$.
Solution:

A form of the bound follows trivially from the Chebyshev interpolation.

On the z interval $[-1, 1]$, $|t_2(z)| \leq 1/2$. Since the near minimax polynomial is an interpolant, we have the standard error formula that exploits knowledge of the derivative,

$$
\begin{aligned}
|f(z) - p_1(z)| &= |\frac{t_2(z)}{2}\frac{d^2 f}{dz^2}(\zeta))| \\
&\leq \frac{1}{4}|\frac{d^2 f}{dz^2}(\zeta))| \\
x = z + 1, \;\; dx/dz = 1 \rightarrow \frac{d^2 f}{dz^2}(z) &= \frac{1}{4\sqrt{(z+1)^3}}
\end{aligned}
$$

But $f(z) = \sqrt{z+1}$ so at $x = 0$ and $z = -1$ we have a problem of the unbounded derivative with respect to either variable. So the bound cannot be used directly on the entire interval. Even with $x > 1$ and $z > 0$ the bound is still 0.25. Indeed, sampling the error shows errors on the order of 0.5 in maginitude near $x = 0$.

A more rigourous way to do this is to compute the error

$$f(x) - p_1(x)$$

and examine the roots of its derivative and the value of the error at the boundary points since it is optimization on a closed interval. The resulting bound is consistent with the approximation of 0.5.

Another way is to use the relationship of the best approximation and the Chebyshev interpolating polynomial, i.e., the near minimax. This can be applied since you have computed the error bound for the best polynomial. We have

$$\|f(x) - p_1(x)\|\infty \leq 4E_n^*(f) = 4\sqrt{2}/8 = \sqrt{2}/2$$

This is not surprising since $f(x) = \sqrt{x}$ as a function near 0 does not resemble a polynomial at all.

1.d

Give a bound for the error $|f(x) - q_1(x)|$ on the interval $[0, 2]$.
Solution:

This follows immediately from the derivation since the max error is given with alternating sign at 0, 1/2 and 2 by construction. We have

$$e(0) = -\beta = -\frac{\sqrt{2}}{8} \approx -0.17$$

so $|f(x) - q_1(x)| \leq \sqrt{2}/8$. Again we see that a linear polynomial, in this case the best one, is not a very good approximation for the $f(x)$.

Problem 2

(25 points)

Approximate the integral

$$\int_0^2 e^x \, dx$$

using Gauss-Legendre quadrature method $I_4(f)$, i.e., using 5 points. Compare the result to using an open Newton-Cotes formula and a closed Newton-Cotes formula with the same number of points.

Solution:

Gauss-Legendre quadrature is defined on the interval $[-1, 1]$ with the points and coeffiecients given by:

$$\begin{aligned}
&I_4(f):\\
&\qquad z_0 = 0.000000000, \quad \gamma_0 = 0.56888889\\
&\qquad z_1 = +0.53846931, \quad \gamma_1 = 0.47862867\\
&\qquad z_2 = -0.53846931, \quad \gamma_2 = 0.47862867\\
&\qquad z_3 = +0.90617985, \quad \gamma_3 = 0.23692689\\
&\qquad z_4 = -0.90617985, \quad \gamma_4 = 0.23692689
\end{aligned}$$

We must change intervals in order to apply the method. We have

$$\begin{gathered}
a \le x \le b \text{ and } -1 \le z \le 1\\
x = \frac{z(b-a)+b+a}{2} \rightarrow dx = \frac{b-a}{2} dz\\
\int_a^b f(x)dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{z(b-a)+b+a}{2}\right) dz\\
I_n(f) = \frac{b-a}{2} \sum_{i=0}^{n} \gamma_i f(x_i)\\
x_i = \frac{z_i(b-a)+b+a}{2} \text{ and } P_{n+1}(z_i) = 0
\end{gathered}$$

In this case since $a = 0$ and $b = 2$ we have

$$\begin{gathered}
x_i = z_i + 1\\
I_n(f) = \sum_{i=0}^{n} \gamma_i f(x_i)
\end{gathered}$$

So we have the sum

$$
\begin{gathered}
I_4(f) = \gamma_0 f(x_0) + \gamma_1 f(x_1) + \gamma_2 f(x_2) + \gamma_3 f(x_3) + \gamma_4 f(x_4) \\
x_0 = 1 + 0.000000000, \quad \gamma_0 = 0.56888889 \\
x_1 = 1 + 0.53846931, \quad \gamma_1 = 0.47862867 \\
x_2 = 1 - 0.53846931, \quad \gamma_2 = 0.47862867 \\
x_3 = 1 + 0.90617985, \quad \gamma_3 = 0.23692689 \\
x_4 = 1 - 0.90617985, \quad \gamma_4 = 0.23692689
\end{gathered}
$$

This yields

$$
I(f) = +6.38905609893e{+}00, \quad I_4(f) = +6.38905614054e{+}00, \quad I(f){-}I_4(f) = -4.16101952894e{-}08
$$

Newton-Cotes close with 5 points uses

$$
h = b - a/n, \;\; x_0 = 0, \;\; x_1 = 0.5, \;\; x_2 = 1 \;\; x_3 = 1.5, \;\; x_4 = 2
$$

$$
I_4 = \frac{2h}{45}\left(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\right)
$$

This yields

$$
I_4(f) = +6.38924234549e + 00, \;\; I(f) - I_4(f) = -1.86246563689e - 04
$$

Newton-Cotes open with 5 points uses

$$
h = b - a/(n+2), \;\; x_0 = 1/3, \;\; x_1 = 2/3, \;\; x_2 = 1 \;\; x_3 = 4/3, \;\; x_4 = 5/3
$$

$$
I_4 = \frac{3h}{10}\left(11f_0 - 14f_1 + 26f_2 - 14f_3 + 11f_4\right)
$$

This yields

$$
I_4(f) = +6.38868276707e + 00, \;\; I(f) - I_4(f) = +3.73331858396e - 04
$$

So we see as expected that Gauss-Legendre has the square of the error of Newton-Cotes.

Problem 3

(25 points)

3.a

Consider $f(x) = \sin x$ on $[-1, 1]$. Determine the economized power series of degree 2 for Legendre polynomials, $\{P_i(x)\}$, and Chebyshev polynomials, $\{T_i(x)\}$, for the Taylor series of degree 4 of $f(x)$.

Solution:

We consider the Taylor series expansion up to the degree 4 with remainder expanded around 0, i.e.,

$$\begin{aligned} sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots \\ sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \sin(\xi) \\ sin(x) &= p_3(x) + R(x), \quad \|R(x)\| \leq \frac{1}{120} \\ p_3(x) &= x - \frac{x^3}{3!} \end{aligned}$$

Since we are on the interval $[-1, 1]$ no change of variables is needed, So we proceed by substituting in expressions for the powers of x in terms of $P_i(x)$ and $T_i(x)$. We have

$$\begin{aligned} 1 &= P_0 = T_0 \\ x &= P_1 = T_1 \\ x^2 &= \frac{2}{3}P_2 + \frac{1}{3}P_0 = \frac{1}{2}T_2 + \frac{1}{2}T_0 \\ x^3 &= \frac{2}{5}P_3 + \frac{3}{5}P_1 = \frac{1}{4}T_3 + \frac{3}{4}T_1. \end{aligned}$$

Transforming and truncating P_3 and T_3 to get the economizations yields

$$\begin{aligned} p_3(x) &= x - \frac{x^3}{3!} \\ p_3(x) &= P_1(x) - \frac{1}{3!}\left(\frac{2}{5}P_3 + \frac{3}{5}P_1\right) \\ &= \frac{9}{10}P_1(x) - \frac{1}{15}P_3(x) \\ &\approx \frac{9}{10}x \\ \sin x &= \frac{9}{10}x - \frac{1}{15}P_3(x) + R(x) \end{aligned}$$

for Legendre and

$$\begin{aligned}
p_3(x) &= T_1(x) - \frac{1}{3!}\left(\frac{1}{4}T_3 + \frac{3}{4}T_1\right) \\
&= \frac{7}{8}T_1(x) - \frac{1}{24}T_3 \\
&\approx= \frac{7}{8}x \\
\sin x &= \frac{7}{8}x - \frac{1}{24}T_3(x) + R(x)
\end{aligned}$$

for Chebyshev.

3.b

i. Consider the space of polynomials of degree n or less, $\mathbb{P}_n$, and the subspaces $span[P_0(x),\ P_1(x), \dots, P_d(x)]$ and $span[T_0(x),\ T_1(x), \dots, T_d(x)]$ where $d \leq n$. Is there any relationship between the two subspaces of $\mathbb{P}_n$?

ii. Consider an arbitrary smooth function, $f(x)$, and its Generalized Fourier Series in terms of the Legendre polynomials, $\{P_i(x)\}$, and its Generalized Fourier Series in terms of the Chebyshev polynomials, $\{T_i(x)\}$, i.e.,

$$f(x) = \sum_{i=0}^{\infty} \alpha_i P_i(x) = \sum_{i=0}^{\infty} \beta_i T_i(x).$$

Suppose you truncate each series at degree n, defining

$$f(x) \approx f_P(x) = \sum_{i=0}^{n} \alpha_i P_i(x) \quad f(x) \approx f_T(x) = \sum_{i=0}^{n} \beta_i T_i(x).$$

Are the two truncations the same? If so prove it, if not how is it possible?

Solution:

The subspaces $span[P_0(x),\ P_1(x), \dots, P_d(x)]$ and $span[T_0(x),\ T_1(x), \dots, T_d(x)]$ of $\mathbb{P}_n$ are equivalent. Both are sets are bases for $\mathbb{P}_d \subseteq \mathbb{P}_n$. This follows from the fact that for any $P_j(x)$, $0 \leq j \leq d$ there is a unique linear combination of the $T_i(x)$, $0 \leq i \leq d$ that is equal to $P_j(x)$ and vice versa.

Let

$$f(x) \approx f_P(x) = \sum_{i=0}^{n} \alpha_i P_i(x) \quad f(x) \approx f_T(x) = \sum_{i=0}^{n} \beta_i T_i(x).$$

The two polynomials $f_P(x)$ and $f_T(x)$ are both elements of $\mathbb{P}_d$ however they are not in general the same. Each are the approximation that is closest to $f(x)$ on $\mathbb{P}_d$. However, they are not the closest with respect to the same norm. Recall, $\mathbb{P}_d \subseteq \mathbb{P}_n \subseteq \mathcal{L}^2_\omega[-1,1]$ where $\omega(x)$ can be chosen in various ways to define the inner product and the associated norm.

We have

$$f_P(x) = \underset{p \in \mathbb{P}_d}{\text{argmin}} \int_{-1}^{1} (f(\tau) - p(\tau))^2 d\tau$$

$$f_T(x) = \underset{p \in \mathbb{P}_d}{\text{argmin}} \int_{-1}^{1} \frac{(f(\tau) - p(\tau))^2}{\sqrt{1-\tau^2}} d\tau$$

$$\forall q \in \mathbb{P}_d, \quad (f(x) - f_P(x), q(x))_\omega = 0, \omega(x) = 1$$

$$\forall q \in \mathbb{P}_d, \quad (f(x) - f_T(x), q(x))_\omega = 0, \omega(x) = \frac{1}{\sqrt{1-x^2}}$$

Problem 4

(25 points)

Consider an explicit linear multistep method of the form

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = h f_{n-1}$$

4.a. Is there a consistent method of this form with order at least 2? Is there more than one such method? Justify your answer.

4.b. If one or more such methods exists, choose one and determine if it is 0-stable and find the expression for its local truncation error. If there is no such method indicate how you would change the form so that one does exist.

Solution:

We have $\beta_0 = \beta_2 = 0$ and $\beta_1 = 1$ and we want

$$\begin{aligned} \alpha_0 + \alpha_1 + \alpha_2 &= C_0 = 0 \\ -\alpha_1 - 2\alpha_2 - \beta_1 &= C_1 = 0 \\ 0.5\alpha_1 + 2\alpha_1 + \beta_1 &= C_2 = 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0.5 & 2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

So we have

$$\rho(\xi) = \frac{1}{2}\xi^2 - \frac{1}{2}$$

with roots 1 and -1. The method is therefore consistent and $0-$stable. Note however it is weakly stable and therefore not recommended for use as a linear multistep method.

To determine its truncation error we note

$$\begin{aligned} C_3 &= -\frac{1}{6}\alpha_1 - \frac{8}{6}\alpha_2 - \frac{1}{2}\beta_1 - \frac{4}{2}\beta_2 \\ &= \frac{8}{12} - \frac{1}{2} = \frac{1}{6} \end{aligned}$$