3.8: Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. In biology, animal or bacteria populations under ideal conditions grow proportional to the population. In nuclear physics, the mass of a radioactive substance decays proportional to its mass. In chemistry, the rate of some chemical reactions occurs proportional to the concentration of reactant. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to the present value.

In general, if $y(t)$ is the value of a quantity at time t and if the rate of change of y with respect to t is proportional to $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant. This equation is sometimes called the **law of uninhibited growth** (if $k > 0$) or the **law of uninhibited decay** (if $k < 0$). This equation is also an example of a **differential equation** because it involves an unkown function $y(t)$ and its derivative $\frac{dy}{dt}$.

The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}.$$

Example 1. *A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour, the population has increased to 420.*

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 3 hours.

(c) Find the rate of growth after 3 hours.

(d) When will the population reach 20,000 cells?

Example 2. *How long will it take an investment to double in value if the interest rate is 6% compounded continuously?*