WSU Economics PhD Mathcamp Notes

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July 25, 2019

These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Set Theory

1.1 Sets

A **set** is a collection of objects. These objects, which we call **elements**, could be anything including but not limited to numbers, letters, words, or even other sets! Below is an example of a set:

$$A = \{1, 2, \text{horse}, \{5, 7\}\}$$

Notice that we use curly brackets to denote a set. Everything within the curly brackets are elements, and each element is separated by a comma. In this particular set, there are four elements: one integers, one word, and one set ($\{5, 7\}$). To denote that an element or elements are in a set, we use the $\in$ symbol. For example, $1, 2 \in A$. We can also use the $\notin$ symbol to denote that an element or elements are not in a set. For example, $3 \notin A$.
Here are some neat properties of sets:

- The order of elements in a set does not matter. The set $\{a, b, c\}$ is equal to the set $\{c, b, a\}$.
- A set does not include duplicate elements. The set $\{a, b, c, d, d\}$ should only contain one d element.
- We do not need to explicitly list out each element of a set (although, a set should be described in a manner such that it is clear which elements are in the set). Below are two examples of how we would go about doing that:
  - **Extrapolation**: use an ellipsis (...) at the end of a set to indicate that the set keeps going given the pattern of the elements before the ellipsis. $S = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$
  - **Interpolation**: use an ellipsis (...) in between elements to indicate that the set maintains the pattern of the elements before the ellipsis up until the element after the ellipsis. $S = \{1, 2, 3, \ldots, 9, 10\}$
- $|A|$ denotes the cardinality of the set A, or number of elements belonging to set A. If $|A| < \infty$, then it A is finite, else it is infinite.

1.2 Set Builder Notation

We can use set builder notation for sets that satisfy some conditions. The general notation is:

$$S = \{f(x) : p(x)\}$$

where S is the set that contains all elements of the form $f(x)$ where the condition $p(x)$ is satisfied.

Example

Let $B = \{6, 8, 10, \ldots\}$. Using set builder notation, we can redefine the set as:

$$B = \{2x : x \geq 3\}$$

Notice that there is usually more than one way to define a set using set builder notation. Other ways to define set B are:

$$B = \{2x : 2x \geq 6\}$$
$$B = \{x : x \text{ is even and } x \geq 6\}$$

Practice

List out the elements of the following sets:

1. $A = \{xy : 0 < x < 3 \text{ and } y \in \{10, 100\}\}$
2. $B = \{x^3 : |x| < 4\}$

There are a few sets that are so common, they get their own symbol:

symbol	set
R	real numbers
Z	integers
N	natural numbers (positive numbers)
Q	rational numbers
I	irrational numbers
$\emptyset$	empty set $\{\}$

1.3 Subsets

A set S is **subset** of a set T if all elements in set S are elements in T. If this is the case, we write $S \subseteq T$ (which reads as S is a subset of T or S is contained in T). Notice that by definition, $S \subseteq S$. Two sets, S and T, are **equal** if $S \subseteq T$ and $T \subseteq S$. If we know that $S \subseteq T$ but $S \neq T$, then we say that S is a **proper subset** of T, or $S \subset T$. Another thing to be aware of is that the empty set, $\emptyset$, is always a subset of a set.

When we talk about sets, usually we need to be aware of what the universal set, U, is. In some cases, U could be the be the set of all real numbers, or in other cases, it could be the set of all intergers. Even if not explicitly defined, it is usually fairly easy to know what it is.

Practice

Show that if $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

The **power set** of S, denoted $\mathcal{P}(S)$, is the set of all subsets of S. If there are n elements in the S, then $\mathcal{P}(S)$ has 2^n elements.

Practice

Find the power set for the following sets:

1. $A = \{1, 2, 3\}$
2. $B = \{0, \emptyset\}$

1.4 Intervals

Often, we can use intervals to define subsets of $\mathbb{R}$. The following table gives a description of interval types:

Interval	Notation	Definition
Open	(a, b)	$\{x \in \mathbb{R} : a < x < b\}$
Closed	$[a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$
Half-Open or Half-Closed	$(a, b]$	$\{x \in \mathbb{R} : a < x \leq b\}$
Half-Open or Half-Closed	$[a, b)$	$\{x \in \mathbb{R} : a \leq x < b\}$

1.5 Set operations

The following table gives a description of common set operations:

Operator	Notation	Definition
Union	$A \cup B$	$\{x : x \in A \text{ or } x \in B\}$
Intersection	$A \cap B$	$\{x : x \in A \text{ and } x \in B\}$
Difference	$A - B$	$\{x : x \in A \text{ and } x \notin B\}$
Complement	$\overline{A}$	$\{x : X \in U \text{ and } x \notin A\}$

Two sets are **disjoint** if $A \cup B = \emptyset$.

1.6 Extending Set Operations

The following table gives a description of common set operations:

Operator	Notation	Definition
Union	$\bigcup_{i=1}^{n} A_i$	$\{x : x \in A_i \text{ for some i, } 1 \le i \le n\}$
Intersection	$\bigcap_{i=1}^{n} A_i$	$\{x : x \in A_i \text{ for all i, } 1 \le i \le n\}$

Sometimes we want to define the union or intersection over a certain set of sets. We can do this by defining an **index set** I to be used to denote which sets we want to take the union or the intersection over. The following table gives a description of set operations using index sets:

Operator	Notation	Definition
Union	$\bigcup_{\alpha \in I} A_\alpha$	$\{x : x \in A_\alpha \text{ for some } \alpha \in I\}$
Intersection	$\bigcap_{\alpha \in I} A_\alpha$	$\{x : x \in A_\alpha \text{ for all } \alpha \in I\}$

Practice

Let $A_1 = \{1, 2, 3\}$, $A_2 = \{\{1, 2\}, 3, 4\}$, $A_3 = \{4, 5, 6\}$. Find the following:

1. $\bigcup_{\alpha \in I} A_\alpha$ where $I = \{1, 2\}$
2. $\bigcup_{\alpha \in I} A_\alpha$ where $I = \{1, 2, 3\}$
3. $\bigcap_{\alpha \in I} A_\alpha$ where $I = \{1, 2\}$
4. $\bigcap_{\alpha \in I} A_\alpha$ where $I = \{1, 2, 3\}$

1.7 Partitions of Sets

The partition of a set A is a collection $\mathcal{S}$ of nonempty subsets of A such that

1. the union of all the sets in $\mathcal{S}$ is equal to A
2. for every two sets $X, Y \in \mathcal{S}$ (such that $X \neq Y$), $X \cap Y = \emptyset$

Example

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The following sets are valid partitions of A:

$$\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}\}$$
$$\{\{4\}, \{2\}, \{1, 5, 6\}, \{3, 7, 8\}\}$$
$$\{\{1, 2, 4, 5, 6, 7, 8\}\}$$

The following are not valid partitions:

$$\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}, \emptyset\}$$
$$\{\{1, 2, 3, 4\}, \{4, 5, 6\}, \{6, 7, 8\}\}$$
$$\{\{2\}, \{1, 5, 6\}, \{3, 7, 8\}\}$$

1.8 Cartesian Products of Sets

The **Cartesian product** of two set A and B, denoted $A \times B$ is defined as:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Notice that each element in $A \times B$ is an ordered pair, (a, b). In other words, if $(a, b) \in A \times B$, it does not mean that $(b, a) \in A \times B$.

1.9 De Morgan's Laws

The following laws constitute De Morgan's laws:

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

The generalized laws are:

1. $\overline{\bigcap_{\alpha \in I} A_\alpha} = \bigcup_{\alpha \in I} \overline{A}_\alpha$
2. $\overline{\bigcup_{\alpha \in I} A_\alpha} = \bigcap_{\alpha \in I} \overline{A}_\alpha$

Exercises

1. List out all the elements of each set, and put the elements within curly brackets { and }.

   (a) $A = \{n \in \mathbb{N} : 5 < n < 13\}$

   (b) $B = \{n \in \mathbb{Z} : |n^3| < 10\}$

   (c) $C = \{x \in \mathbb{R} : x^2 + 1 = 0\}$

2. Put the following sets in set builder notation. In other words, write each set in the form $\{f(x) \in \mathbb{Z} : p(x)\}$, where $f(x)$ is a function of x, and $p(x)$ is a condition of x.

   (a) $D = \{5, 6, 7, \ldots\}$

   (b) $E = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$

   (c) $F = \{-1, 0, 1, 16\}$

3. Let U be a universal set, and let A and B be subsets of U. Draw a venn diagram for the following sets:

   (a) $\overline{A \cap B}$

   (b) $\overline{A} \cap \overline{B}$