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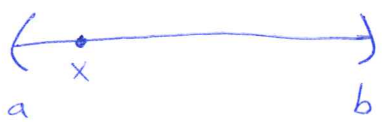
ASSIGNMENT 3

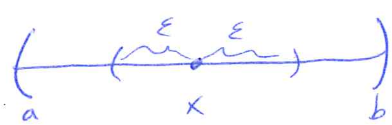
①



I CAN DEFINE AN OPEN BALL $B_\varepsilon(x)$ WHERE

$\varepsilon = \min\left\{ \frac{x-a}{2}, \frac{b-x}{2} \right\}$

IF X IS IN THE MIDDLE OF (a, b), OR IN OTHER WORDS $x = \frac{b-a}{2}$, THEN CHOOSING EITHER $\frac{x-a}{2}$ OR $\frac{b-x}{2}$ WILL WORK OF ε



$B_\varepsilon(x) \subseteq (a, b)$

IF X IS CLOSER TO a, THEN $\varepsilon = \frac{x-a}{2}$ AND $B_\varepsilon(x) \subseteq (a, b)$

AND IF X IS CLOSER TO b, THEN $\varepsilon = \frac{b-x}{2}$ AND $B_\varepsilon(x) \subseteq (a, b)$

② $B = \left\{ \frac{(-1)^n n}{n+1} \,\middle|\, n \in \mathbb{N} \right\}$

a) LIMIT POINTS ARE $\{-1, 1\}$

b) B IS NOT CLOSED AS IT DOES NOT CONTAIN ITS LIMIT POINTS

c) B IS NOT OPEN SINCE IF YOU PUT AN OPEN BALL AROUND ANY POINT IN B, $\exists\ \varepsilon > 0$ such that THE BALL IS NOT CONTAINED IN B.

③ a) WE WANT TO SHOW THAT FOR $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. FOR $m, n \geq N$ IT FOLLOWS THAT:

$$|y_m - y_n| < \varepsilon$$

ASSUME WLOG $n > m$ WHERE $n, m \in \mathbb{N}$

WE SEE
$$\begin{aligned} |y_{m+1} - y_{m+2}| &= |f(y_m) - f(y_{m+1})| \\ &\leq \lambda |y_m - y_{m+1}| \end{aligned}$$

WHERE $\lambda \in (0,1)$, THUS:

$$\begin{aligned} |y_{m+1} - y_{m+2}| &\leq \lambda |y_m - y_{m+1}| \\ &\leq \lambda^2 |y_{m-1} - y_m| \\ &\vdots \\ &\leq \lambda^m |y_1 - y_2| \end{aligned}$$

$$\Rightarrow |y_{m+1} - y_{m+2}| \leq \lambda^m |y_1 - y_2|$$

THEREFORE:

$$\begin{aligned} |y_m - y_n| &= |y_m - y_{m+1} + y_{m+1} - y_{m+2} + y_{m+2} \dots + y_{n-1} - y_n| \\ &\leq |y_m - y_{m+1}| + |y_{m+1} - y_{m+2}| + \dots + |y_{n-1} - y_n| \\ &\leq \lambda^{m-1} |y_1 - y_2| + \lambda^m |y_1 - y_2| + \dots + \lambda^{n-2} |y_1 - y_2| \\ &= \lambda^{m-1} (1 + \lambda + \lambda^2 + \dots + \lambda^{n-m-1}) |y_1 - y_2| \\ &< \lambda^{m-1} \left(\frac{1}{1-\lambda}\right) |y_1 - y_2| \end{aligned}$$

LET $\varepsilon > 0$ AND CHOOSE $N \in \mathbb{N}$ s.t.:

$$\lambda^{N-1} < \frac{\varepsilon(1-\lambda)}{|y_1 - y_2|}$$

THEN, FOR ~~[illegible]~~ $n > m \geq N$:

$$|y_1 - y_2| < \varepsilon$$

$\Rightarrow \{y_n\}$ IS CAUCHY

b) NOTICE $\lim_{n\to\infty} y_n = y$ AND $\lim_{n\to\infty} y_{n+1} = y$

SINCE $y_{n+1} = f(y_n) \Rightarrow \lim_{n\to\infty} f(y_n) = y$

$$\Rightarrow \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_n = y$$

IN OTHER WORDS, $f(y) = y$, SO y IS FIXED POINT.

(4) THE MATRIX IS <u>NOT</u> POSITIVE DEFINITE OR POSITIVE SEMI-DEFINITE AS THE FIRST ORDER LEADING PRINCIPAL MINOR (-1) IS < 0.

<u>1st ORDER LEADING PRINCIPAL MINOR (OLPM)</u>

$\det(-1) = -1$

<u>2nd OLPM</u>

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$$

SO MATRIX IS <u>NOT</u> NEGATIVE DEFINITE. NOW WE CHECK ALL PRINCIPAL MINORS.

1st ORDER PRINCIPAL MINORS (OPM)

$-1 \qquad -1 \qquad -2$ ALL ≤ 0 ✓

2nd OPM

$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0 \qquad \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 \qquad \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2$

ALL ≥ 0 ✓

3rd OPM

$\begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -2 + 2 = 0 \qquad \leq 0$ ✓

SO MATRIX IS NEGATIVE SEMI-DEFINITE

(9) $f = (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$

LET $h = x^2 + y^2 + z^2$

F.O.C.

$\frac{\partial f}{\partial x}$: $2xe^{-h} + x^2(-2x)e^{-h} + 2y^2(-2x)e^{-h} + 3z^2(-2x)e^{-h} = 0$

$\Rightarrow (-2x)e^{-(x^2+y^2+z^2)}(x^2 + 2y^2 + 3z^2 - 1) = 0$

$\frac{\partial f}{\partial y}$: $(-2y)e^{-(x^2+y^2+z^2)}(x^2 + 2y^2 + 3z^2 - 2) = 0$

$\frac{\partial f}{\partial z}$: $(-2z)e^{-(x^2+y^2+z^2)}(x^2 + 2y^2 + 3z^2 - 3) = 0$

LET'S EXAMINE $\frac{\partial f}{\partial z} = 0$. THE FOC TELLS US

THAT EITHER $-2z = 0$, $e^{-(x^2+y^2+z^2)} = 0$, OR

$x^2 + 2y^2 + 3z^2 - 3 = 0$. WE CAN SAY SIMILAR THINGS

ABOUT THE OTHER FOCs.

CRITICAL POINTS	TYPE
$(0,0,0)$	LOCAL MIN
$(\pm 1, 0, 0)$	SADDLE POINTS
$(0, \pm 1, 0)$	SADDLE POINTS
$(0, 0, \pm 1)$	LOCAL MAXES

10 $\mathcal{L} = k x_1^{\alpha} x_2^{1-\alpha} - \lambda \left[p_1 x_1 + p_2 x_2 - I \right]$

a) KKT CONDITIONS.

$\frac{\partial \mathcal{L}}{\partial x_1}$: $\alpha k x_1^{\alpha - 1} x_2^{1-\alpha} - \lambda p_1 = 0$ (1)

$\frac{\partial \mathcal{L}}{\partial x_2}$: $(1-\alpha) k x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0$ (2)

$\lambda \left[p_1 x_1 + p_2 x_2 - I \right] = 0$

$\lambda \geq 0$

$p_1 x_1 + p_2 x_2 \leq I$

b) COMBINE (1) & (2)

$$\frac{\alpha \cancel{K} x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) \cancel{K} x_1^{\alpha} x_2^{-\alpha}} = \frac{\cancel{\lambda} p_1}{\cancel{\lambda} p_2}$$

$$\Rightarrow \frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

$$\Rightarrow x_2 = \left(\frac{p_1}{p_2}\right) x_1 \left(\frac{1-\alpha}{\alpha}\right)$$

PLUG INTO BUDGET CONSTRAINT:

$$p_1 x_1 + p_2 \left(\frac{p_1}{p_2}\right) x_1 \left(\frac{1-\alpha}{\alpha}\right) = I$$

$$\Rightarrow x_1^* = \frac{\alpha I}{p_1}$$

$$x_2^* = \frac{(1-\alpha) I}{p_2}$$

11 NOTICE $f(x_1, x_2, \dots, x_n)$ IS CONCAVE, AND

$-\sum_{i=1}^{n} w_i x_i$ IS CONCAVE, SO $p f(\vec{x}) - \sum w_i x_i$ IS CONCAVE.

SO WE ONLY NEED TO CONSIDER F.O.C.S:

$$\frac{\partial f}{\partial x_i}: \quad p \frac{\alpha_i}{x_i} f(\vec{x}) = w_i \qquad \text{FOR } i = 1, 2, \dots, n$$

DIVIDE $\frac{\partial f}{\partial x_i}$ BY $\frac{\partial f}{\partial x_1}$

$$\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_1}}: \quad \frac{\cancel{p} \frac{\alpha_i}{x_i} \cancel{f(\vec{x})}}{\cancel{p} \frac{\alpha_1}{x_1} \cancel{f(\vec{x})}} = \frac{w_i}{w_1}$$

$$\Rightarrow x_i^* = \frac{\alpha_i}{w_i} \frac{w_1}{\alpha_1} x_1^* \qquad \text{FOR } i = 2, 3, \dots, n \qquad (1)$$

NOTICE, IF WE PLUG (1) INTO THE $f(\vec{x})$:

$$f(\vec{x}) = \prod_{i=1}^{n} \left(\frac{\alpha_i}{w_i}\right)^{\alpha_i} \left(\frac{w_1}{\alpha_1}\right) x_1^*$$

NOW TAKING FOC WRT x_1 OF $p f(\vec{x}) - \sum_{i=1}^{n} w_i x_i$

GIVES US: $$\frac{p}{\alpha_1} = \prod_{i=1}^{n} \left(\frac{w_i}{\alpha_i}\right)^{\alpha_i}$$

NOTICE, THIS ISN'T A FUNCTION OF x_i. THUS ANY $x_i > 0$ IS A SOLUTION (AND WE CAN FIND ANY OTHER x_i WHERE $i = 2, 3, \dots, n$ BY EQUATION (2)