Real Analysis Practice Solutions

Joe Patten

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1 Relations and Functions

1.1 Properties of Relations

Practice

Let $S = \{a, b, c\}$. Which of the properties reflexive, transitive, and symmetric do the relations below possess if the relations are from S to S?

1. $R_1 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a), (b, b), (c, c)\}$
2. $R_2 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a)\}$
3. $R_3 = \{(b, c), (c, b), (a, a), (b, b), (c, c)\}$

1. R_1 is reflexive, transitive, and symmetric.
2. R_2 is only symmetric.
3. R_3 is only reflexive.

Practice

Consider $S \in \mathbb{R}$. Let the following be relations from S to S. Show that the following relations are reflexive, transitive, and symmetric. If a property does not hold, provide a counterexample to show that that property does not hold.

1. $R_5 = \{(a, b) \in S \times S : a \geq b\}$
   Let $x, y, z \in S$.
   - $(x, x) \in R_5$ since $x \geq x$, thus R_5 is reflexive.
   - $(x, y) \in R_5 \not\Rightarrow (y, x) \in R_5$. Counterexample: Suppose $x = 5$ and $y = 4$. Thus R_6 is not symmetric.
   - Suppose $(x, y), (y, z) \in R_5$. Thus $x \geq y$ and $y \geq z$. By transitivity of $\mathbb{R}$, it follow that $x \geq z$, thus $(x, z) \in R_5$. Therefore R_5 is transitive.

2. $R_6 = \{(a, b) \in S \times S : a > b\}$
   Let $x, y, z \in S$.
   - $(x, x) \notin R_6$ since $x \not> x$, thus R_6 is not reflexive.
   - $(x, y) \in R_6 \not\Rightarrow (y, x) \in R_6$. Counterexample: Suppose $x = 5$ and $y = 4$. Thus R_6 is not symmetric.
   - Suppose $(x, y), (y, z) \in R_6$. Thus $x > y$ and $y > z$. By transitivity of $\mathbb{R}$, it follow that $x > z$, thus $(x, z) \in R_6$. Therefore R_6 is transitive.

3. $R_7 = \{(a,b) \in S \times S : ab \geq 0\}$
Let $x, y, z \in S$.

- $(x,x) \in R_7$ since $xx \geq 0$ for all $x \in \mathbb{R}$, thus R_7 is reflexive.
- Suppose $(x,y) \in R_7$, then $xy \geq 0$. Notice that $xy = yx$. Thus $yx \geq 0$. So $(y,x) \in R_7$. Thus R_7 is symmetric.
- Counterexample: Notice that $(-1,0) \in R_7$, and $(0,5) \in R_7$. However, $(-1,5) \notin R_7$. Thus R_7 is not transitive.

1.2 Monotonic Functions

Practice

Show that the function $f(x) = log(x)$ is strictly increasing for all $x \in \mathbb{R}_{++}$, where $\mathbb{R}_{++}$ is defined as: $\mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$
Let $x, y \in \mathbb{R}_{++}$. Without loss of generality, assume $x > y$. Since $x > y$, there exists an $\alpha \in (0,1)$ such that $x = \alpha y$. We are required to prove that $\log(x) > \log(y)$. Notice:

$$\begin{aligned} \log(x) &= \log(\alpha y) \\ &= \log(\alpha) + \log(y) \\ &> \log(y) \end{aligned}$$

Thus $f(x) = log(x)$ is a strictly increasing function. Notice that we can also look at the derivative of $f(x)$ and see if it is positive over the whole domain of $\mathbb{R}_{++}$ to see if it is a strictly increasing function.

2 Metric Spaces

Practice

1. Show that $(\mathbb{R}, d_1)$ is a valid metric space.
   Let $x, y, z \in \mathbb{R}$. where $x \neq y \neq z$

   (a) Notice that $|x - x| = 0$ and $|x - y| > 0$.

   (b) Notice that $|x - y| = |y - x|$.

   (c) Notice that $|x - z| = |(x - y) + (y - z)| \Rightarrow |x - z| \leq |x - y| + |y - z|$ since $(x - y) \leq |x - y|$ and $(y - z) \leq |y - z|$.

2. Show that $(\mathbb{R}^2, d_2)$ is a valid metric space.
   Let $x, y, z \in \mathbb{R}^2$. where $x \neq y \neq z$

   (a) If $x \neq y$, and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then either $x_1 \neq y_1$ or $x_2 \neq y_2$. Thus $(x_1 - y_1)^2 + (x_2 - y_2)^2 > 0$. Therefore $||x - x|| = 0$ and $||x - y|| > 0$.

   (b) Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Notice that $|x - y| = |y - x| = \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix}$.

   (c) Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then $||x - z|| = ||(x - y) + (y - z)|| \Rightarrow ||x - z|| \leq ||x - y|| + ||y - z||$ since $(x - y) \leq ||x - y||$ and $(y - z) \leq ||y - z||$.

3. Show that $(\mathbb{R}^n, d_3)$ is a valid metric space.

   (a) If $x \neq y$, then $\exists i$ such that $x_i \neq y_i$. Thus $d_3(x, y) \geq |x_1 - y_1| > 0$.

   (b) Notice that $d_3(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} = \max\{|y_1 - x_1|, \ldots, |y_n - x_n|\} = d_3(y, x)$.

   (c) Suppose $d_3(x, y) = |x_i - y_i|$, $d_3(y, z) = |y_j - z_j|$, and $d_3(x, z) = |x_k - z_k|$ for $i, j, k \in \mathbb{N}$. Notice $|x_k - z_k| \leq |x_i - y_i| + |y_j - z_j|$. (Further clearification is left up to the reader).

3 More Set Theory

Practice

Determine the supremum and infimum for each of the following sets in $\mathbb{R}$. Also determine if the supremum and infimum are equal to the maximum and minimum respectively of each set:

1. $[0, 2]$
   Supremum = maximum = 2
   Infimum = minimum = 0

2. $(0, 2)$
   Supremum = $2 \neq$ maximum
   Infimum = $0 \neq$ minimum

3. $[0, 2] \cup \{3\}$
   Supremum = maximum = 3
   Infimum = minimum = 0

4 Sequences

4.1 Sequence Convergence

Practice

Show (via proof) that:

1. $\lim \frac{2}{\sqrt{2n+4}} = 0$

   Let $\varepsilon > 0$ be arbitrary. Choose N such that $N > \frac{4}{\varepsilon^2}$. Let $n \geq N$. Then:

$$\left| \frac{2}{\sqrt{2n+4}} - 0 \right| < \varepsilon$$

2. $\lim \frac{4n+1}{2n+4} = 2$

   Let $\varepsilon > 0$ be arbitrary. Choose N such that $N > \frac{5}{2\varepsilon}$. Let $n \geq N$. Then:

$$\left| \frac{4n+1}{2n+4} - 2 \right| < \varepsilon$$

4.2 Cauchy Criterion

Practice

Consider the metric space $(\mathbb{R}, d_1)$. Show that every convergent sequence is a Cauchy sequence.

See Assignment 3 solutions

5 Topology

5.1 Open Sets

Practice

Using the definition above and assuming the metric space is $(\mathbb{R}, d_1)$:

1. Show that $B_\varepsilon(a)$ is an open set.

   From the definition of open set, we need to show that for every point in $B_\varepsilon(a)$, we can make an open ball around any point, and that open ball must be contained in $B_\varepsilon(a)$. Consider $y \in B_\varepsilon(a)$. If we define an open ball around y as $B_{\varepsilon_1}(y)$, where $\varepsilon_1 = \varepsilon - |y - a|$, you'll see that $B_{\varepsilon_1}(y) \subseteq B_\varepsilon(a)$. Thus $\subseteq B_\varepsilon(a)$. Hence $B_\varepsilon(a)$ is open.

2. Show that $\mathbb{R}$ is an open set.

   Pick a $y \in \mathbb{R}$, and put a ball of radius $\varepsilon \in \mathbb{R}$ around y. Notice that since $\varepsilon \in \mathbb{R}$, it is always that case that $B_\varepsilon(y) \in \mathbb{R}$. Thus $\mathbb{R}$ is open.

3. Show that $(0, 1)$ is an open set.

   See Assignment 3 solutions.

5.2 Closed Sets

Practice

The set of natural numbers, $\mathbb{N}$, can be written in the form: $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup ...$ where $\{n\}$ is said to be an isolated point. Is $\{n\}$ a limit point? What does that tell us about the set $\mathbb{N}$, is it open, closed, or neither.

$\{n\}$ where $n \in \mathbb{N}$ is not a limit point as we can easily find an $\varepsilon > 0$ such that a ball around every point in the set contains only that point. Thus there are no limit points in the set $\mathbb{N}$. Notice that, trivially, $\mathbb{N}$ contains all of its limit points, so $\mathbb{N}$ is closed.

5.3 Open and Closed Sets

Practice

1. Show that the empty set, $\emptyset$, is both closed and open.

   $\emptyset$ has no limit points. Thus, trivially, it contains all of its limit points. Thus $\emptyset$ is closed. Notice that $\overline{\mathbb{R}} = \emptyset$. We see that $\mathbb{R}$ contains all of its limit points, thus it is closed. Then $\emptyset$ is open.

2. Determine if $[0, 1] \cup \{2\}$ is open, closed, or neither.

   Notice that the set of limit points for $[0, 1] \cup \{2\}$ is $[0, 1]$. Since $[0, 1] \subseteq [0, 1] \cup \{2\}$, then $[0, 1] \cup \{2\}$ is closed.

5.4 Compact Sets

Practice

Show that for a compact set $S \subseteq \mathbb{R}$, the supremum and infimum or S are elements of S.

Let $S \subseteq R$. Suppose that S be bounded and let $b = \sup S$. For every $\varepsilon > 0$, there exists an $s \in S$ such that $b - \varepsilon < s$. Notice that we have defined an open ball $B_\varepsilon(b)$, and we see that $\exists s \in B_\varepsilon(b)$ for any $\varepsilon > 0$. Thus b is a limit point of S. Since S is closed, S must contain all of its limit points. Therefore $b \in S$. Or in other words, $\sup S \in S$.

A similar argument can be used to show that $\inf S \in S$.

6 Advanced Theorems

6.1 Brouwer's Fixed Point Theorem

Practice

Using the Intermediate Value Theorem, prove the Brouwer's Fixed Point Theorem in the metric space $(\mathbb{R}, d_1)$

Assume that $f : [a, b] \to [a, b]$ is a continuous function. Notice that since both the domain and codomain are $[a, b]$, then $f(a), f(b) \in [a, b]$. If $f(a) = a$ or $f(b) = b$, then we are done (since both are fixed points). We now need to consider the case when $f(a) \in (a, b]$ and $f(b) \in [a, b)$.

Case 1 Suppose $f(a) = f(b)$
By intermediate value theorem $\exists x \in (a, b)$ such that $f(x) = x$

Case 2 Suppose WLOG $f(a) > f(b)$
By intermediate value theorem, there exists an $\varepsilon > 0$ such that $r = f(a) - \varepsilon$ where $r \in (f(b), f(a))$ and $r = f(r)$.

Drawing a graph helps for intution. I encourage you to draw one to fully understand what's going on.