Assignment 3

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1. Using the definition of convergence of a sequence prove that the following sequence converges to the proposed limit in $\mathbb{R}$:

$$\lim \frac{2}{\sqrt{n+3}} = 0$$

Let $\varepsilon > 0$ be arbitrary. We need to show that there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\frac{2}{\sqrt{n+3}} - 0\right| < \varepsilon$. Notice that:

$$\left|\frac{2}{\sqrt{n+3}} - 0\right| = \frac{2}{\sqrt{n+3}}$$

If we pick an N such that $N > \sqrt{\frac{1}{2\varepsilon}}$. Then it follows that if $n \geq N$, $\left|\frac{2}{\sqrt{n+3}} - 0\right| < \varepsilon$.

2. Consider the metric space $(\mathbb{R}, |\cdot|)$[1]. Prove that convergent sequence is a Cauchy sequence.
Let (x_n) be a convergent sequence, thus $x_n \to x$. Thus, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, n \geq N$, it follows that:

$$|x_n - x| < \frac{\varepsilon}{2}$$

and

$$|x_m - x| < \frac{\varepsilon}{2}$$

Using the triangle inequality (property 3 from metric spaces), we see that:

$$\begin{aligned} |x_m - x_n| &= |x_m - x + x - x_n| \\ &\leq |x_n - x| + |x_m - x| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} & = \varepsilon \end{aligned}$$

Thus $|x_m - x_n| < \varepsilon$ when $m, n \geq N$. Therefore (x_n) is a Cauchy sequence.

3. Consider the metric space $(\mathbb{R}, |\cdot|)$. Using the definition of open ball (or ε-neighborhood), prove that the interval $(0, 1)$ is open.
I am not looking for anything super formal (although I have added a formal proof), I just want you to show that you understand the problem intuitively.

From the definition of open set, we need to show that for every point in $(0, 1)$, we can make an open ball around any point, and that open ball must be contained in $(0, 1)$. Consider $y \in (0, 1)$. If we define an open ball around y as $B_\varepsilon(y)$, where $\varepsilon = \frac{1}{2} - \left|y - \frac{1}{2}\right|$, you'll see that $B_\varepsilon(y) \subseteq (0, 1)$. Thus $(0, 1)$ is open.

[1] The metric $|\cdot|$ is defined as the absolute value of the difference between two points. This metric is described in the notes as d_1

4. Consider the metric space $(\mathbb{R}, |\cdot|)$. Let:

$$B = \left\{ \frac{(-1)^n n}{n+1} : n \in \mathbb{N} \right\}$$

   (a) Find the limit points of B. The limit points for B are $\{-1, 1\}$
   (b) Is B a closed set? B is not a closed set as it does not contain its limit points.
   (c) Is B an open set? B is not a open set since if you put an open ball (or in this case an ε-neighborhood) around any point in B, that ball will not be contained in B.
   (d) Does B contain any isolated points? Yes, every point in B is an isolated point.

5. Find the total differential for the following function:

$$z = 2x \sin y - 3x^2 y^2$$

   Total Differential:

$$dz = (2x \sin y - 6xy^2)dx + (2x \cos y - 6x^2 y)dy$$

6. Let $w = x^2 y - y^2$ where $x = \sin t$ and $y = e^t$.

   (a) Find $\frac{dw}{dt}$.

$$\begin{aligned} \frac{dw}{dt} &= 2xy(\cos t) + (x^2 - 2y)e^t \\ &= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t} \end{aligned}$$

   (b) Evaluate $\frac{dw}{dt}$ at $t = 0$.

$$2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}\Big|_{t=0} = -2$$

7. Consider the following coefficient matrix:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

   (a) Determine the definiteness of the matrix above.
   First we can look at the leading principal minors.

$$\begin{aligned} |A_1| &= -1 \\ |A_2| &= 0 \\ |A_3| &= 0 \end{aligned}$$

   The matrix is not positive or negative definite, nor is it positive semidefinite. We now need to check the other principal minors to check if the matrix is negative semidefinite:

   First-order Principal Minors:

$$\begin{aligned} a_{11} &= -1 \\ a_{22} &= -1 \\ a_{33} &= -2 \end{aligned}$$

Second-order Principal Minors:

$$|A_2| = 0$$
$$\begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2$$
$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$$

Third-order Principal Minor:

$$|A_3| = 0$$

Thus our matrix is negative semidefinite

(b) Convert the coefficient matrix into quadratic form.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -x_1^2 - x_2^2 - 2x_3^2 + 2x_1x_2$$

(c) Is the function convex or concave?
Since the matrix is negative semidefinite, the function is concave.

8. Consider the function $f(x) = \ln(1 + x)$.

(a) Calculate $f(.5)$.

$$\begin{aligned} f(.5) &= \ln(1.5) \\ &= 0.4055 \end{aligned}$$

(b) Using a first order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

$$\begin{aligned} f(.5) &\approx f(0) + f'(0)(.5 - 0) \\ &= \ln(1 + 0) + \frac{1}{1}(.5) \\ &= 0 + .5 \\ &= .5 \end{aligned}$$

(c) Using a second order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

$$\begin{aligned} f(.5) &\approx f(0) + f'(0)(.5 - 0) + \frac{f''(0)}{2!}(.5 - 0)^2 \\ &= \ln(1 + 0) + \frac{1}{1}(.5) + \frac{-1}{2}(.25) \\ &= 0 + .5 - .125 \\ &= .375 \end{aligned}$$

(d) Using a third order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

$$\begin{aligned} f(.5) &\approx f(0) + f'(0)(.5 - 0) + \frac{f''(0)}{2!}(.5 - 0)^2 + \frac{f'''(0)}{3!}(.5 - 0)^3 \\ &= \ln(1 + 0) + \frac{1}{1}(.5) + \frac{-1}{2}(.25) + \frac{2}{6}(.125) \\ &= 0 + .5 - .125 + .04167 \\ &= .4167 \end{aligned}$$

9. Differentiate implicitly to find $\frac{dy}{dx}$:

$$x^2 - 3xy + y^2 - 2x + y - 5 = 0$$

Let $F(x, y) = x^2 - 3xy + y^2 - 2x + y - 5$. Then using the implicit function yields:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\frac{\partial F(x,y)}{\partial x}}{\frac{\partial F(x,y)}{\partial y}} \\ &= -\frac{2x - 3y - 2}{2y - 3x + 1} \\ &= \frac{-2x + 3y + 2}{2y - 3x + 1} \end{aligned}$$

10. Use integration by parts to to evaluate the following integrals:

(a)

$$\int x \cos(x) dx$$

Let $u = x$, $\frac{dv}{dx} = \cos x$, then $\frac{du}{dx} = 1$ and $v = \sin x$. Thus by the using the following equation, we can integrate by parts:

$$\begin{aligned} \int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C \end{aligned}$$

where $C \in \mathbb{R}$

(b)

$$\int x e^{x^2} dx$$

This integration is easily done using u substitution (so I apologize as we did not review this). Let $u = x^2$, thus $\frac{du}{dx} = 2x \Rightarrow du = (2x)dx$. Plugging this into our integral, we get:

$$\begin{aligned} \int x e^{x^2} dx &= \int \frac{e^u}{2} du \\ &= \frac{e^u}{2} + C \\ &= \frac{e^{x^2}}{2} + C \end{aligned}$$

11. Consider the Cobb-Douglas production function: $f(K, L) = AK^a L^b$ where $K, L \geq 0$, and $A > 0$.

(a) What conditions on a and b must be true in order for the function to be (weakly) concave (*Hint*: consider the Hessian matrix)?

First we need to take the Jacobian of our function:

$$\begin{aligned} Df(K, L) &= \begin{bmatrix} \frac{\partial f(K,L)}{\partial K} & \frac{\partial f(K,L)}{\partial L} \end{bmatrix} \\ &= \begin{bmatrix} aAK^{a-1}L^b & bAK^a L^{b-1} \end{bmatrix} \end{aligned} \tag{1}$$

Now we can find the Hessian by taking the derivative of the Jacobian:

$$D^2 f(K,L) = \begin{bmatrix} \frac{\partial^2 f(K,L)}{\partial K^2} & \frac{\partial f^2(K,L)}{\partial K \partial L} \\ \frac{\partial f^2(K,L)}{\partial L \partial K} & \frac{\partial f^2(K,L)}{\partial L^2} \end{bmatrix}$$

$$D^2 f(K,L) = \begin{bmatrix} a(a-1)AK^{a-2}L^b & abAK^{a-1}L^{b-1} \\ abAK^{a-1}L^{b-1} & b(b-1)AK^aL^{b-2} \end{bmatrix} \tag{2}$$

For the function to be concave, the Hessian must be negative semidefinite. In other words, the even principals minors must all be ≥ 0, and the odd principal minors must all be ≤ 0

First order Principal minors:

$$\begin{aligned} a(a-1)AK^{a-2}L^b &\leq 0 \\ \Rightarrow a(a-1) &\leq 0 \\ \Rightarrow a &\geq 0 \end{aligned} \tag{3}$$

$$\begin{aligned} b(b-1)AK^aL^{b-2} &\leq 0 \\ \Rightarrow b(b-1) &\leq 0 \\ \Rightarrow b &\geq 0 \end{aligned} \tag{4}$$

Second order Principal minor:

$$\begin{aligned} |D^2 f(K,L)| &\geq 0 \\ |D^2 f(K,L)| &= a(a-1)AK^{a-2}L^b b(b-1)AK^aL^{b-2} - abAK^{a-1}L^{b-1}abAK^{a-1}L^{b-1} \\ &= a(a-1)b(b-1)A^2K^{a-2+a}L^{b+b-2} - a^2b^2AK^{2a-2}L^{2b-2} \\ &= a(a-1)b(b-1)A^2K^{2a-2}L^{2b-2} - a^2b^2AK^{2a-2}L^{2b-2} \\ &= (a(a-1)b(b-1) - a^2b^2)A^2K^{2a-2}L^{2b-2} \end{aligned}$$

$$\begin{aligned} \Rightarrow a^2b^2 - a^2b - ab^2 + ab - a^2b^2 &\geq 0 \\ \Rightarrow -a^2b - ab^2 + ab &\geq 0 \\ \Rightarrow ab(1-a-b) &\geq 0 \\ \Rightarrow (1-a-b) &\geq 0 \\ \Rightarrow a+b &\leq 1 \end{aligned} \tag{5}$$

The conditions can be seen by inequalities (3), (4), and (5).[2]

(b) What conditions on a and b must be true in order for the function to be strictly concave?
Now, we need to ensure the even leading principal minors are strictly positive and the odd leading principal minors are strictly negative in order for the Hessian to be negative definite:
First order Leading Principal minor:

$$\begin{aligned} a(a-1)AK^{a-2}L^b &< 0 \\ \Rightarrow a(a-1) &< 0 \\ \Rightarrow a &> 0 \end{aligned} \tag{6}$$

Second order Leading Principal minor:

$$\begin{aligned} a^2b^2 - a^2b - ab^2 + ab - a^2b^2 &< 0 \\ \Rightarrow -a^2b - ab^2 + ab &< 0 \\ \Rightarrow ab(1-a-b) &< 0 \quad (7) \\ \Rightarrow (1-a-b) &< 0 \\ \Rightarrow a+b &< 1 \quad (8) \end{aligned}$$

The conditions can be seen by inequalities (6) and (8). Notice that $b > 0$ is also implied by inequality (7).[3].

[2]Notice that also $a, b \leq 1$ from our inequalities.
[3]Notice that also $a, b < 1$ from our inequalities.

12. In microeconomic theory, a budget set or opportunity set, is the set of all possible consumption bundles that an individual can afford given the prices of goods, $\mathbf{p}$, and that individual's income, y. The $n \times 1$ commodity vector, $\mathbf{x}$, is a list of amounts of different commodities. The price vector, $\mathbf{p}$, is an $n \times 1$ vector that tells the price for each commodity. The budget set B is defined as:

$$B = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}^T\mathbf{x} \leq y\}$$

Show B is convex (using the definition of set convexity).[4]

To show that B is convex, we must show that the convex combination of any two points in B is also in B.

Let $x, x' \in B$. In other words,

$$p^T x \leq y \tag{9}$$

and

$$p^T x' \leq y \tag{10}$$

We will show that the convex combination of x and x', in other words $\lambda x + (1-\lambda)x'$ where $\lambda \in [0,1]$, is in B. Notice that from equation (9) and (10), we find that:

$$\lambda p^T x \leq \lambda y \tag{11}$$

and

$$(1-\lambda)p^T x' \leq (1-\lambda)y \tag{12}$$

Combining equations (11) and (12), we find that:

$$\begin{aligned}
\lambda p^T x + (1-\lambda)p^T x' &\leq \lambda y + (1-\lambda)y \\
\lambda p^T x + (1-\lambda)p^T x' &\leq y \\
p^T(\lambda x + (1-\lambda)x') &\leq y \qquad (13)
\end{aligned}$$

By definition of set B, we see that $\lambda x + (1-\lambda)x' \in B$. Thus B is a convex set.[5]

[4] $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, ..., n\}$

[5] Notice that $\lambda p^T x = p^T \lambda x$.