

Real Analysis

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1 Relations and Functions

1.1 Relations

A (binary) **relation**, R , from set A to set B is a subset of $A \times B$. Since R is a subset of $A \times B$, it is a set of ordered pairs. If $a \in A$ and $b \in B$, we say $(a, b) \in R$ if a is related to b . We can also write aRb if this holds. If an ordered pair $(c, d) \in A \times B$ is not in the relation R , then we could write either $(c, d) \notin R$ or $c \not R d$.

Example

If $A = \{t, u, v\}$ and $B = \{1, 2\}$, we see that:

$$A \times B = \{(t, 1), (t, 2), (u, 1), (u, 2), (v, 1), (v, 2)\}$$

An example of an relation R would be:

$$R = \{(t, 2), (u, 1), (u, 2)\}$$

Notice that $R \subseteq (A \times B)$

If R is the relation from A to B , then the domain of R is a subset of A defined by:

$$\text{dom} R = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

Likewise, the range is a subset of B defined by:

$$\text{ran} R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$$

The inverse of a relation R from A to B , is denoted R^{-1} , and is defined as:

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$$

Lastly, we can define a relation R from A to A . When we do so, we just call R a relation on A .

1.2 Properties of Relations

Below are some possible properties of a relation R on X :

1. R is **reflexive** $\Leftrightarrow xRx$ for any $x \in X$.
2. R is **transitive** $\Leftrightarrow (xRy \text{ and } yRz \Rightarrow xRz)$ for any $x, y, z \in X$
3. R is **symmetric** $\Leftrightarrow xRy \text{ and } yRx$ for any $x, y \in X$
4. R is **complete** $\Leftrightarrow xRy \text{ or } yRx$ for any $x, y \in X$
5. R is **antisymmetric** $\Leftrightarrow (xRy \text{ and } yRx \Rightarrow x = y)$ for any $x, y \in X$

Practice

Let $S = \{a, b, c\}$. Which of the properties reflexive, transitive, and symmetric do the relations below possess if the relations are from S to S ?

1. $R_1 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a), (b, b), (c, c)\}$
2. $R_2 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a)\}$
3. $R_3 = \{(b, c), (c, b), (a, a), (b, b), (c, c)\}$

Relations can often be defined using set builder notation. Below is an example of a relation from \mathbb{R} to \mathbb{R} :

$$R_4 = \{(a, b) \in \mathbb{R} : a > b\}$$

Notice that $(2, 1), (\sqrt{5}, -3), (\pi, 0) \in R_4$ since $2 > 1, \sqrt{5} > -3$, and $\pi > 0$. However, $(2, 4), (-2, 3.4), (-3, \sqrt{5}) \notin R_4$ since $2 \not> 4, -2 \not> 3.4$, and $-3 \not> \sqrt{5}$.

Practice

Consider $S \in \mathbb{R}$. Let the following be relations from S to S . Show that the following relations are reflexive, transitive, and symmetric. If a property does not hold, provide a counterexample to show that that property does not hold.

1. $R_5 = \{(a, b) \in S \times S : a \geq b\}$
2. $R_6 = \{(a, b) \in S \times S : a > b\}$
3. $R_7 = \{(a, b) \in S \times S : ab \geq 0\}$

1.3 Functions

A relation f from A to B is a **function**, which we write as $f : A \rightarrow B$, iff:

1. for every $a \in A$, there exists a $b \in B$
2. if $(a, b_1) \in f$ and $(a, b_2) \in f$, it must be the case that $b_1 = b_2$

If $(a, b) \in f$, we can write $f(a) = b$. b is called the image of a , and a is referred to as the preimage. When we write $f(a) = b$, we say that f maps a into b .

Example

Let $A = \{a, b, c\}$ and $B = \{3, 6, 7, 8\}$. f_1 and f_2 are examples of functions:

$$\begin{aligned} f_1 &= \{(a, 3), (b, 8), (c, 7)\} \\ f_2 &= \{(a, 8), (b, 7), (c, 8)\} \end{aligned}$$

f_3 and f_4 are examples of relations that are not functions:

$$\begin{aligned} f_3 &= \{(a, 3), (a, 6), (b, 7), (c, 8)\} \\ f_4 &= \{(b, 6), (c, 7)\} \end{aligned}$$

a common function that you have seen before is the function $f(x) = x^2$. We can set f to be a set of all possible ordered pairs for $f(x) = x^2$:

$$f = \{(x, x^2) : x \in \mathbb{R}\}$$

1.4 Set of All Functions

Notice that we can write a number of functions from A and B . We denote the set of all functions from A to B by B^A . More formally, this set is defined as:

$$B^A = \{f : f : A \rightarrow B\}$$

1.5 One-to-One Functions

A function, f , from A to B is said to be **one-to-one** (or **injective**) if every two distinct values of A have distinct images in B . In other words, for every $a, a' \in A$, if $a \neq a'$, then $f(a) \neq f(a')$.

Example

Let $A = \{x, y, z\}$ and $B = \{a, b, c, d\}$. f_1 and f_2 are examples of one-to-one functions from A to B :

$$f_1 = \{(x, b), (y, a), (z, d)\}$$

$$f_2 = \{(x, b), (y, c), (z, d)\}$$

f_3 and f_4 are examples of functions from A to B that are not one-to-one:

$$f_3 = \{(x, b), (y, b), (z, d)\}$$

$$f_4 = \{(x, d), (y, d), (z, d)\}$$

1.6 Onto Functions

A function, f , from A to B is said to be **onto** (or **surjective**) if every element of the codomain (in this case, B) is the image of some element of A .

Example

Let $A = \{e, f, g, h\}$ and $B = \{1, 2, 3\}$. f_1 and f_2 are examples of onto functions from A to B :

$$f_1 = \{(e, 1), (f, 2), (g, 3), (h, 1)\}$$

$$f_2 = \{(e, 3), (f, 2), (g, 2), (h, 1)\}$$

f_3 and f_4 are examples of functions from A to B that are not onto:

$$f_3 = \{(e, 1), (f, 2), (g, 2), (h, 1)\}$$

$$f_4 = \{(e, 1), (f, 1), (g, 1), (h, 1)\}$$

1.7 Bijective Functions

A function, f , from A to B is said to be **bijective** (or a **one-to-one correspondence**) if it is one-to-one and onto.

1.8 Inverse Functions

Let $f : A \rightarrow B$ be a function. Then the **inverse** relation, f^{-1} , is a function from B to A iff f is bijective. Also if f is bijective, then f^{-1} is bijective.

1.9 Function Operations

Let f and g be functions mapping from \mathbb{R} to \mathbb{R} . We can perform the following operations:

1. $(f + g)(x) = f(x) + g(x)$

2. $(fg)(x) = f(x) \cdot g(x)$

$$3. (fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$$4. (g \circ f)(x) = g(f(x))$$

Item 3 comes from the chain rule. Item 4 is called a composition.

1.10 Monotonic Functions

A function $f : A \rightarrow B$ is (weakly) increasing on A if $x < y \Rightarrow f(x) \leq f(y)$ and (weakly) decreasing when $x < y \Rightarrow f(x) \geq f(y)$. A function $f : A \rightarrow B$ is *strictly* increasing on A if $x < y \Rightarrow f(x) < f(y)$ and *strictly* decreasing when $x < y \Rightarrow f(x) > f(y)$. A function is said to be **monotonic** iff it is an increasing or decreasing function, and strictly monotonic iff it is strictly increasing or strictly decreasing.

Example

I will show that $f(x) = x^2 + 1$ is strictly increasing for all $x \in \mathbb{R}_+$, where \mathbb{R}_+ is defined as: $\mathbb{R}_+ = \{y \in \mathbb{R} : y \geq 0\}$

Solution: Take two arbitrary points, $x_1, x_2 \in \mathbb{R}_+$.

Assume without of generality that $0 \leq x_1 < x_2$.

Consider the difference of images:

$$f(x_2) - f(x_1) = (x_2^2 + 1) - (x_1^2 + 1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$$

Notice that $(x_2 - x_1)(x_2 + x_1) > 0$ since $x_1 \geq 0$ and $x_2 > 0$, and by assumption $x_2 > x_1$.

Thus, $f(x) = x^2 + 1$ is strictly increasing over the domain of \mathbb{R}_+

Practice

Show that the function $f(x) = \log(x)$ is strictly increasing for all $x \in \mathbb{R}_{++}$, where \mathbb{R}_{++} is defined as: $\mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$

When a strictly increasing function is applied to a set, we refer to this application as a (positive) **monotonic transformation**. The monotonically transformed set keeps its ordering, in other words, if $a, b \in S$ and $a > b$, and f is a strictly increasing function, then $f(a) > f(b)$.

2 Metric Spaces

Let X be a set. $d : X \times X \rightarrow \mathbb{R}$ is a valid metric or distance function iff:

1. $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$
2. $d(x, y) = d(y, x) \quad \forall x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

If d satisfies the above properties, then (X, d) is said to be a **metric space**.

Example

Let x, y be vectors in X . Some examples of common metrics are:

1. Absolute value metric

$$d_1(x, y) = |x - y|$$

2. Euclidean metric

$$d_2(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$$

3. Square metric

$$d_3(x, y) = \max\{|x_1 - y_1|, \dots, |x_k - y_k|\}$$

Practice

1. Show that (\mathbb{R}, d_1) is a valid metric space.
2. Show that (\mathbb{R}^2, d_2) is a valid metric space.
3. Show that (\mathbb{R}^n, d_3) is a valid metric space.

3 More Set Theory

Consider the metric space (\mathbb{R}, d_1) . A set $S \subseteq \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. We call b an **upper bound** of S . A set $S \subseteq \mathbb{R}$ is **bounded below** if there exists $c \in \mathbb{R}$ such that $a \geq c$ for all $c \in A$. We call C a **lower bound** of S .

The **supremum** s , or least upper bound, for a set $S \subseteq \mathbb{R}$ if:

1. s is an upper bound of S .
2. if d is any upper bound of S , then $s \leq d$.

The **infimum**, or greatest lower bound, for a set $S \subseteq \mathbb{R}$ is defined similarly.

Consider the metric space (\mathbb{R}, d_1) . The number m is the **maximum** of a set S if $m \in S$, and $m \geq a$ for all $a \in S$. The **minimum** of a set can be defined similarly.

Practice

Determine the supremum and infimum for each of the following sets in \mathbb{R} . Also determine if the supremum and infimum are equal to the maximum and minimum respectively of each set:

1. $[0, 2]$
2. $(0, 2)$
3. $[0, 2] \cup \{3\}$

4 Sequences

A **sequence** is a function $x : \mathbb{N} \rightarrow X$, which is either written as (x_n) or $\{x_n\}$. In other words, to define a sequence, we take numbers from the set of natural numbers (starting from 1 and counting up), and using a function, assign them to elements from a set X . Since a sequence is a function, we can write the n^{th} element of a sequence as $x(n)$, however, it is more commonly written as x_n . Recall that order does not matter for sets. This is not the case with sequences. For example, $(1, 1, 2, \dots) \neq (1, 2, 1, \dots)$. You will also notice that elements in a set can be repeated.

4.1 Sequence Convergence

Let (X, d) be a metric space. A sequence (x_n) **converges** to x if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, it follows that $d(x_n, x) < \epsilon$.

If the sequence (x_n) converges to x , then we can either write $\lim x_n = x$ or $(x_n) \rightarrow x$.

Consider the sequence (x_n) , where $x_n = \frac{1}{\sqrt{n}}$. Find what it converges to, and show via proof that it converges to that value.

Proof

Let ϵ be an arbitrary positive number. Choose a natural number such that $N > \frac{1}{\epsilon^2}$. Now, verify that this choice of N has the desired property. Let $n \geq N$. Then:

$$n > \frac{1}{\epsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \epsilon$$

Thus $|x_n - 0| < \epsilon$

Template for $(x_n) \rightarrow x$ proof in metric space (\mathbb{R}, d_1) :

- "Let ϵ be arbitrary."
- Choose an $N \in \mathbb{N}$ (this may take some work).
- Show that your choice of N works.
- "Assume $n \geq N$."
- Now derive the inequality $|x_n - x| < \epsilon$.

A sequence converges to at most one limit.

Proof

We want to show that a sequence converges to at most one limit. I will show that this is the case when (\mathbb{R}, d_1) is the metric space.

Proof: Suppose to the contrary that a sequence converges to both x and x' where $x \neq x'$.

Since $x \neq x'$, we know that $d(x, x') > 0$.

Let $\epsilon = d(x, x')/2$.

By definition of convergence, we see that since $x_n \rightarrow x$, $\exists N_1$ s.t. $d(x_n, x) < \epsilon$.

By definition of convergence, we see that since $x_n \rightarrow x'$, $\exists N_2$ s.t. $d(x_n, x') < \epsilon$.

Now, let $N = \max\{N_1, N_2\}$. If $n > N$, then $|x_n - x| < \epsilon$ and $|x_n - x'| < \epsilon$.

By the triangle inequality: $|x - x'| \leq |x_n - x| + |x' - x_n| < \epsilon + \epsilon = 2\epsilon$.

But, this is a contradiction since $|x - x'| \not< |x - x'|$.

A sequence is said to be bounded if $\exists M > 0$ s.t. $|x_n| < M \quad \forall n \in \mathbb{N}$.

Practice

Show (via proof) that:

1. $\lim_{n \rightarrow \infty} \frac{2}{\sqrt{2n+4}} = 0$
2. $\lim_{n \rightarrow \infty} \frac{4n+1}{2n+4} = 2$

4.2 Algebraic Limit Theorem

Let $\lim x_n = x$ and $\lim y_n = y$. Then:

1. $\lim(cx_n) = cx$
2. $\lim(x_n + y_n) = x + y$
3. $\lim(x_n y_n) = xy$
4. $\lim(x_n/y_n) = x/y$, given $y \neq 0$.

4.3 Squeeze Theorem

If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and $\lim x_n = l$, $\lim z_n = l$, then $\lim y_n = l$.

4.4 Subsequence

A subsequence is derived from a sequence (x_n) by only keeping a subset of the elements while keeping the order of the sequence.

Example

A subsequence of the sequence $(1, 2, 3, 4, 5)$ is $(1, 4, 5)$.

4.5 Cauchy Criterion

A sequence is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|x_n - x_m| < \varepsilon$.

Every convergent sequence is a Cauchy sequence. The converse (i.e. every Cauchy sequence is convergent) is only true for certain metric spaces.

Practice

Consider the metric space (\mathbb{R}, d_1) . Show that every convergent sequence is a Cauchy sequence.

A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X .

4.6 Monotonic Sequences

A sequence is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and is decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotonic** if it is either increasing or decreasing.

5 Topology

5.1 Open Sets

An open ball of radius $\varepsilon > 0$ centered about point $x \in X$ can be defined by:

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$

If we are working in the metric space (\mathbb{R}, d_1) , then the open ball is just an open interval, and is usually referred to as an ε -neighborhood. In the metric space (\mathbb{R}^2, d_2) , the open ball is a circle, and in the metric space (\mathbb{R}^3, d_2) , the open ball is a sphere.

A set $A \subseteq X$ is open if for all points $a \in A$ if there exists an open ball $B_\varepsilon(a) \subseteq A$.

Practice

Using the definition above and assuming the metric space is (\mathbb{R}, d_1) :

1. Show that $B_\varepsilon(a)$ is an open set.
2. Show that \mathbb{R} is an open set.
3. Show that $(0, 1)$ is an open set.

The following theorems hold for open sets:

1. The union of open sets is open.
2. The finite intersection of open sets is open.

5.2 Closed Sets

Let (X, d) be a metric space, and S be a subset of X . A point x is a **limit point** of set S iff $\{B_\varepsilon(x) - \{x\}\} \cap S \neq \emptyset$ where $\varepsilon > 0$. For x to be a limit point of S , if we take an open ball around it, no matter what the size of that open ball, there will exist other points from S other than x in that open ball. Note: for a point x to be a limit point of S , it doesn't necessarily have to be an element of S . It only has to contain an element from S in its open ball for every $\varepsilon > 0$. The set of all limit points of a set S is usually denoted as S' .

We say that a set S is **closed** iff S contains all of its limit points. In other words, $S' \subseteq S$. It does not have to be the case that $S = S'$.

Practice

The set of natural numbers, \mathbb{N} , can be written in the form: $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \dots$ where $\{n\}$ is said to be an isolated point. Is $\{n\}$ a limit point? What does that tell us about the set \mathbb{N} , is it open, closed, or neither.

The following theorems hold for closed sets:

1. The finite union of closed sets is closed.
2. The intersection of closed sets is closed.

5.3 Open and Closed Sets

The following theorem is useful for determining if a set is open or closed: The complement of a closed set is open, and the complement of an open set is closed.

Practice

1. Show that the empty set, \emptyset , is both closed and open.
2. Determine if $[0, 1] \cup \{2\}$ is open, closed, or neither.

5.4 Compact Sets

A subset S in a Euclidean space is said to be compact iff S is closed and bounded.

A subset S is compact if every sequence in S has a subsequence that converges to a point in K .

Practice

Show that for a compact set $S \in \mathbb{R}$, the supremum and infimum of S are elements of S .

6 Advanced Theorems

6.1 Continuous Functions

A function $f : A \rightarrow \mathbb{R}$ is **continuous** at $c \in A$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that when $|x - c| < \delta$, it follows that $|f(x) - f(c)| < \varepsilon$. f is said to be continuous on A if f is continuous at every point in the domain A .

Additional properties: Let $f : A \rightarrow \mathbb{R}$ and $g : A \rightarrow \mathbb{R}$ be continuous at a point $c \in A$. Then:

1. $kf(x)$ is continuous at c for every $k \in \mathbb{R}$
2. $f(x) + g(x)$ is continuous at c .
3. $f(x) \cdot g(x)$ is continuous at c .
4. $\frac{f(x)}{g(x)}$ is continuous at c , given $\frac{f(x)}{g(x)}$ exists.

6.2 Intermediate Value Theorem

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$, and r is a real number such that $f(a) \leq r \leq f(b)$ or $f(z) \geq r \geq f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = r$.

6.3 Fixed Points

Let $f : X \rightarrow X$. A point x^* is a **fixed point** of f iff $f(x^*) = x^*$. Notice that if we apply the function f on a fixed point x^* multiple times, we still get x^* as an output.

6.4 Brouwer's Fixed Point Theorem

Let (\mathbb{R}^n, d) be a metric space. If $f : X \rightarrow X$ be a continuous function, where X is a compact and convex subset of \mathbb{R}^n , then there exists an $x^* \in X$ such that $f(x^*) = x^*$.

Practice

Using the Intermediate Value Theorem, prove the Brouwer's Fixed Point Theorem in the metric space (\mathbb{R}, d_1)

6.5 Contraction

We will use the notation $f^n(x)$ to mean apply the function n times on x . In other words, if $n = 3$, then $f^3(x) = f(f(f(x)))$.

Let (X, d) be a metric space. A function $f : X \rightarrow X$ is said to be a contraction iff there exists a $\lambda < 1$ such that:

$$d(f(x), f(x')) \leq \lambda \cdot d(x, x')$$

for any $x, x' \in X$.

Notice that when a function is a contraction, when we apply the function to two points, the distance between the images are closer than the distance between the preimages.

Example

Let $([\frac{1}{2}, 10], d_1)$ be a metric space, and let f be a function from the set $[\frac{1}{2}, 10]$ into itself. We see that the function as $f(x) = \sqrt{x}$ on this metric space is a contraction. Notice that no matter what x we start with (where $x \in [\frac{1}{2}, 10]$), $\lim_{n \rightarrow \infty} f^n(x) = 1$.

Starting with $x = \frac{1}{2}$:

$$f(\frac{1}{2}) = 0.707107$$

$$f^2(\frac{1}{2}) = 0.84090$$

$$f^3(\frac{1}{2}) = 0.917004$$

\vdots

$$f^{20}(\frac{1}{2}) = 0.999999$$

6.6 Contraction Mapping Theorem

Let (X, d) be a complete metric space, and $f : X \rightarrow X$ be contraction. It follows that there exists a fixed point x^* of f , and for any $x \in X$, $\lim_{n \rightarrow \infty} f^n(x) = x^*$.