Assignment 1

July 25, 2019

1. Consider the following system of linear equations:

$$\begin{aligned} -3x_1 - x_2 + 2x_3 &= 7 \\ 2x_2 - 2x_3 &= 8 \\ 6x_1 - 3x_2 + 6x_3 &= -9 \end{aligned}$$

   (a) Put the system of linear equations into an **augmented** matrix.
   (b) Find the reduced row echelon form of the **augmented** matrix.
   (c) What is the rank of the **coefficient** matrix?

2. Consider the following system of linear equations:

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

   (a) Put the system of linear equations into a **coefficient** matrix.
   (b) Find the reduced row echelon form of the **coefficient** matrix.
   (c) What is the dimension of the row space the **coefficient** matrix?

3. What does the rank of a matrix tell us?

4. Let A be a 3×3 matrix with $det(A) = 6$. Find each of the following if possible:
   (a) $\det(A^T)$
   (b) $\det(A + I)$
   (c) $\det(3A)$
   (d) $\det(A^4)$

5. A property of traces is that $tr(AB) = tr(BA)$. Using this property, show that $tr(ABC) = tr(CBA) = tr(ACB)$.

6. *This problem was taken from last year's problem set. It is such a great problem I felt that I needed to include it. Please do not look at last year's solution.*

   Let X be a $n \times k$ real matrix. Define projection matrix $P := X(X'X)^{-1}X'$ and orthogonal matrix $M := I_n - P$. (You can assume $(X'X)^{-1}$ exists.)

   (a) Show that P and M are symmetric and idempotent.
   (b) Show that $tr(P) = k$, $tr(M) = n - k$.

7. Let V be defined as follows:

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

Surprise, surprise, V is not a vector space. Show by counterexample which properties (which are listed in the notes) are violated.

8. Find the (real) eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

9. A diagonal matrix is a square matrix that has only zero value entries on the off-diagonal. Show that the eigenvalues of a diagonal matrix are the values on the diagonal of that matrix.

10. The distance between two $n \times 1$ vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as:

$$dist(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

Redefine this distance formula using the inner product.

11. List out all the elements of each set, and put the elements within curly brackets $\{$ and $\}$.

    (a) $A = \{n \in \mathbb{N} : 5 < n < 13\}$
    (b) $B = \{n \in \mathbb{Z} : |n^3| < 10\}$
    (c) $C = \{x \in \mathbb{R} : x^2 + 1 = 0\}$

12. Put the following sets in set builder notation. In other words, write each set in the form $\{f(x) \in \mathbb{Z} : p(x)\}$, where $f(x)$ is a function of x, and $p(x)$ is a condition of x.

    (a) $D = \{5, 6, 7, \ldots\}$
    (b) $E = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$
    (c) $F = \{-1, 0, 1, 16\}$

13. Let U be a universal set, and let A and B be subsets of U. Draw a venn diagram for the following sets:

    (a) $\overline{A \cap B}$
    (b) $\overline{A} \cap \overline{B}$