Real Analysis

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1 Relations and Functions

1.1 Relations

A (binary) **relation**, R, from set A to set B is a subset of $A \times B$. Since R is a subset of $A \times B$, it is a set of ordered pairs. If $a \in A$ and $b \in B$, we say $(a, b) \in R$ if a is related to b. We can also write aRb if this holds. If an ordered pair $(c, d) \in A \times B$ is not in the relation R, then we could write either $(c, d) \notin R$ or $c \not R\, d$.

Example

If $A\{t, u, v\}$ and $B = \{1, 2\}$, we see that:

$$A \times B = \{(t, 1), (t, 2), (u, 1), (u, 2), (v, 1), (v, 2)\}$$

An example of an relation R would be:

$$R = \{(t, 2), (u, 1), (u, 2)\}$$

Notice that $R \subseteq (A \times B)$

If R is the relation from A to B, then the domain of R is a subset of A defined by:

$$\text{dom}R = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

Likewise, the range is a subset of B defined by:

$$\text{ran}R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$$

The inverse of a relation R from A to B, is denoted R^{-1}, and is defined as:

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$$

Lastly, we can define a relation R from A to A. When we do so, we just call R a relation on A.

1.2 Properties of Relations

Below are some possible properties of a relation R on X:

1. R is **reflexive** $\Leftrightarrow xRx$ for any $x \in X$.
2. R is **transitive** $\Leftrightarrow (xRy$ and $yRz \Rightarrow xRz)$ for any $x, y, z \in X$
3. R is **symmetric** $\Leftrightarrow xRy$ and yRx for any $x, y \in X$
4. R is **complete** $\Leftrightarrow xRy$ or yRx for any $x, y \in X$
5. R is **antisymmetric** $\Leftrightarrow (xRy$ and $yRx \Leftrightarrow x = y)$ for any $x, y \in X$

Practice

Let $S = \{a, b, c\}$. Which of the properties reflexive, transitive, and symmetric do the relations below possess if the relations are from S to S?

1. $R_1 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a), (b, b), (c, c)\}$
2. $R_2 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a)\}$
3. $R_3 = \{(b, c), (c, b), (a, a), (b, b), (c, c)\}$

Relations can often be defined using set builder notation. Below is an example of a relation from $\mathbb{R}$ to $\mathbb{R}$:

$$R_4 = \{(a, b) \in \mathbb{R} : a > b\}$$

Notice that $(2, 1), (\sqrt{5}, -3), (\pi, 0) \in R_4$ since $2 > 1, \sqrt{5} > -3,$ and $\pi > 0$. However, $(2, 4), (-2, 3.4), (-3, \sqrt{5}) \notin R_4$ since $2 \not> 4, -2 \not> 3.4,$ and $-3 \not> \sqrt{5}$.

Practice

Consider $S \in \mathbb{R}$. Let the following be relations from S to S. Show that the following relations are reflexive, transitive, and symmetric. If a property does not hold, provide a counterexample to show that that property does not hold.

1. $R_5 = \{(a, b) \in S \times S : a \geq b\}$
2. $R_6 = \{(a, b) \in S \times S : a > b\}$
3. $R_7 = \{(a, b) \in S \times S : ab \geq 0\}$

1.3 Functions

A relation f from A to B is a **function**, which we write as $f : A \to B$, iff:

1. for every $a \in A$, there exists a $b \in B$
2. if $(a, b_1) \in f$ and $(a, b_2) \in f$, it must be the case that $b_1 = b_2$

If $(a, b) \in f$, we can write $f(a) = b$. b is called the image of a, and a is referred to as the preimage. When we write $f(a) = b$, we say that f maps a into b.

Example

Let $A = \{a, b, c\}$ and $B = \{3, 6, 7, 8\}$. f_1 and f_2 are an examples of functions:

$$f_1 = \{(a, 3), (b, 8), (c, 7)\}$$
$$f_2 = \{(a, 8), (b, 7), (c, 8)\}$$

f_3 and f_4 are examples of relations that are not functions:

$$f_3 = \{(a, 3), (a, 6), (b, 7), (c, 8)\}$$
$$f_4 = \{(b, 6), (c, 7)\}$$

a common function that you have seen before is the function $f(x) = x^2$. We can set f to be a set of all possible ordered pairs for $f(x) = x^2$:

$$f = \{(x, x^2) : x \in \mathbb{R}\}$$

1.4 Set of All Functions

Notice that we can write a number of functions from A and B. We denote the set of all functions from $AtoB$ by B^A. More formally, this set is defined as:

$$B^A = \{f : f : A \to B\}$$

1.5 One-to-One Functions

A function, f, from A to B is said to be **one-to-one** (or **injective**) if every two distinct values of A have distinct images in B. In other words, for every $a, a' \in A$, if $a \neq a'$, then $f(a) \neq f(a')$.

Example

Let A = {x,y,z} and B = {a,b,c,d }. f_1 and f_2 are examples of one-to-one functions from A to B:

$$f_1 = \{(x, b), (y, a), (z, d)\}$$
$$f_2 = \{(x, b), (y, c), (z, d)\}$$

f_3 and f_4 are examples of functions from A to B that are not one-to-one:

$$f_3 = \{(x, b), (y, b), (z, d)\}$$
$$f_4 = \{(x, d), (y, d), (z, d)\}$$

1.6 Onto Functions

A function, f, from A to B is said to be **onto** (or **surjective**) if every element of the codomain (in this case, B) is the image of some element of A.

Example

Let A = { e,f,g,h } and B = {1,2,3}. f_1 and f_2 are examples of onto functions from A to B:

$$f_1 = \{(e, 1), (f, 2), (g, 3), (h, 1)\}$$
$$f_2 = \{(e, 3), (f, 2), (g, 2), (h, 1)\}$$

f_3 and f_4 are examples of functions from A to B that are not onto:

$$f_3 = \{(e, 1), (f, 2), (g, 2), (h, 1)\}$$
$$f_4 = \{(e, 1), (f, 1), (g, 1), (h, 1)\}$$

1.7 Bijective Functions

A function, f, from A to B is said to be **bijective** (or a **one-to-one correspondence**) if it is one-to-one and onto.

1.8 Inverse Functions

Let $f : A \to B$ be a function. Then the **inverse** relation, f^{-1}, is a function from B to A iff f is bijective. Also if f is bijective, then f^{-1} is bijective.

1.9 Function Operations

Let f and g be functions mapping from $\mathbb{R}$ to $\mathbb{R}$. We can perform the following operations:

1. $(f + g)(x) = f(x) + g(x)$
2. $(fg)(x) = f(x) \cdot g(x)$

3. $(fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

4. $(g \circ f)(x) = g(f(x))$

Item 3 comes from the chain rule. Item 4 is called a composition.

1.10 Monotonic Functions

A function $f : A \to B$ is (weakly) increasing on A if $x < y \Rightarrow f(x) \leq f(y)$ and (weakly) decreasing when $x < y \Rightarrow f(x) \geq f(y)$. A function $f : A \to B$ is *strictly* increasing on A if $x < y \Rightarrow f(x) < f(y)$ and *strictly* decreasing when $x < y \Rightarrow f(x) > f(y)$. A function is said to be **monotonic** iff it is an increasing or decreasing function, and strictly monotonic iff it is strictly increasing or strictly decreasing.

Example

I will show that $f(x) = x^2 + 1$ is strictly increasing for all $x \in \mathbb{R}_+$, where $\mathbb{R}_+$ is defined as: $\mathbb{R}_+ = \{y \in \mathbb{R} : y \geq 0\}$

Solution: Take two arbitrary points, $x_1, x_2 \in \mathbb{R}_+$.
Assume without of generality that $0 \leq x_1 < x_2$.
Consider the difference of images:
$f(x_2) - f(x_1) = (x_2^2 + 1) - (x_1^2 + 1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$
Notice that $(x_2 - x_1)(x_2 + x_1) > 0$ since $x_1 \geq 0$ and $x_2 > 0$, and by assumption $x_2 > x_1$.
Thus, $f(x) = x^2 + 1$ is strictly increasing over the domain of $\mathbb{R}_+$

Practice

Show that the function $f(x) = log(x)$ is strictly increasing for all $x \in \mathbb{R}_{++}$, where $\mathbb{R}_{++}$ is defined as: $\mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$

When a strictly increasing function is applied to a set, we refer to this application as a (positive) **monotonic transformation**. The monotonically transformed set keeps it ordering, in other words, if $a, b \in S$ and $a > b$, and f is a strictly increasing function, then $f(a) > f(b)$.

2 Metric Spaces

Let X be a set. $d : X \times X \to \mathbb{R}$ is a valid metric or distance function iff:

1. $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$
2. $d(x, y) = d(y, x) \quad \forall x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

If d satisfies the above properties, then (X, d) is said to be a **metric space**.

Example

Let x, y be vectors in X. Some examples of common metrics are:

1. Absolute value metric

$$d_1(x, y) = |x - y|$$

2. Euclidean metric

$$d_2(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

3. Square metric

$$d_3(x, y) = \max\{|x_1 - y_1|, ..., |x_k - y_k|\}$$

Practice

1. Show that $(\mathbb{R}, d_1)$ is a valid metric space.
2. Show that $(\mathbb{R}^2, d_2)$ is a valid metric space.
3. Show that $(\mathbb{R}^n, d_3)$ is a valid metric space.

3 More Set Theory

Consider the metric space $(\mathbb{R}, d_1)$. A set $S \subseteq \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. We call b an **upper bound** of S. A set $S \in \mathbb{R}$ is **bounded below** if there exists $c \in \mathbb{R}$ such that $a \geq c$ for all $c \in A$. We call C a **lower bound** of S.
The **supremum** s, or least upper bound, for a set $S \subseteq \mathbb{R}$ if:

1. s is an upper bound of S.

2. if d is any upper bound of S, then $s \leq d$.

The **infimum**, or greatest lower bound, for a set $S \subseteq \mathbb{R}$ is defined similarly.

Consider the metric space $(\mathbb{R}, d_1)$. The number m is the **maximum** of a set S if $m \in S$, and $m \geq a$ for all $a \in S$. The **minimum** of a set can be defined similarly.

Practice

Determine the supremum and infimum for each of the following sets in R. Also determine if the supremum and infimum are equal to the maximum and minimum respectively of each set:

1. $[0, 2]$

2. $(0, 2)$

3. $[0, 2] \cup \{3\}$

4 Sequences

A **sequence** is a function $x : \mathbb{N} \to X$, which is either written as (x_n) or $\{x_n\}$. In other words, to define a sequence, we take numbers from the set of natural numbers (starting from 1 and counting up), and using a function, assign them to elements from a set X. Since a sequence is a function, we can write the n^{th} element of a sequence as $x(n)$, however, it is more commonly written as x_n. Recall that order does not matter for sets. This is not the case with sequences. For example, $(1, 1, 2, ...) \neq (1, 2, 1, ...)$. You will also notice that elements in a set can be repeated.

4.1 Sequence Convergence

Let (X, d) be a metric space. A sequence (x_n) **converges** to x if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, it follows that $d(x_n, x) < \epsilon$.

If the sequence (x_n) converges to x, then we can either write $\lim x_n = x$ or $(x_n) \to x$.

Consider the sequence (x_n), where $x_n = \frac{1}{\sqrt{n}}$. Find what it converges to, and show via proof that it converges to that value.

Proof

Let ε be an arbitrary positive number. Choose a natural number such that $N > \frac{1}{\varepsilon^2}$. Now, verify that this choice of N has the desired property. Let $n \geq N$. Then:

$$n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon$$

$$\text{Thus } |x_n - 0| < \varepsilon$$

Template for $(x_n) \to x$ proof in metric space $(\mathbb{R}, d_1)$:

- "Let ε be arbitrary."
- Choose an $N \in \mathbb{N}$ (this may take some work).
- Show that your choice of N works.
- "Assume $n \geq N$."
- Now derive the inequality $|x_n - x| < \varepsilon$.

A sequence converges to at most one limit.

Proof

We want to show that a sequence converges to at most one limit. I will show that this is the case when $(\mathbb{R}, d_1)$ is the metric space.

Proof: Suppose to the contrary that a sequence converges to both x and x' where $x \neq x'$.
Since $x \neq x'$, we know that $d(x, x') > 0$.
Let $\varepsilon = d(x, x')/2$.
By definition of convergence, we see that since $x_n \to x$, $\exists N_1$ s.t. $d(x_n, x) < \varepsilon$.
By definition of convergence, we see that since $x_n \to x'$, $\exists N_2$ s.t. $d(x_n, x') < \varepsilon$.
Now, let $N = \max\{N_1, N_2\}$. If $n > N$, then $|x_n - n| < \varepsilon$ and $|x_n - x'| < \varepsilon$.
By the triangle inequality: $|x - x'| \leq |x_n - x| + |x' - x_n| < |x - x'|$.
But, this is a contradiction since $|x - x'| \not< |x - x'|$.

A sequence is said to be bounded if $\exists M > 0$ s.t. $|x_n| < M \quad \forall n \in \mathbb{N}$.

Practice

Show (via proof) that:

1. $\lim \frac{2}{\sqrt{2n+4}} = 0$
2. $\lim \frac{4n+1}{2n+4} = 2$

4.2 Algebraic Limit Theorem

Let $\lim x_n = x$ and $\lim y_n = y$. Then:

1. $\lim(cx_n)$
2. $\lim(x_n + y_n) = x + y$
3. $\lim(x_n y_n) = xy$
4. $\lim(x_n/y_n) = x/y$, given $y \neq 0$.

4.3 Squeeze Theorem

If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and $\lim x_n = l$, $\lim z_n = l$, then $\lim y_n = l$.

4.4 Subsequence

A subsequence is derived from a sequence (x_n) by only keeping a subset of the elements while keeping the order of the sequence.

Example

A subsequence of the sequence $(1, 2, 3, 4, 5)$ is $(1, 4, 5)$.

4.5 Cauchy Criterion

A sequence is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|x_n - x_m| < \varepsilon$.

Every convergent sequence is a Cauchy sequence. The converse (i.e. every Cauchy sequence is convergent) is only true for certain metric spaces.

Practice

Consider the metric space $(\mathbb{R}, d_1)$. Show that every convergent sequence is a Cauchy sequence.

A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

4.6 Monotonic Sequences

A sequence is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and is decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotonic** if it is either increasing or decreasing.

5 Topology

5.1 Open Sets

An open ball of radius $\varepsilon > 0$ centered about point $x \in X$ can be defined by:

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$

If we a working in the metric space $(\mathbb{R}, d_1)$, then the open ball is just an open interval, and is usually referred to as an ε-neighborhood. In the metric space $(\mathbb{R}^2, d_2)$, the open ball is a circle, and in the metric space $(\mathbb{R}^3, d_2)$, the open ball is a sphere.

A set $A \subseteq X$ is open if for all points $a \in A$ if there exists an open ball $B_\varepsilon(a) \subseteq A$.

Practice

Using the definition above and assuming the metric space is $(\mathbb{R}, d_1)$:

1. Show that $B_\varepsilon(a)$ is an open set.
2. Show that $\mathbb{R}$ is an open set.
3. Show that $(0, 1)$ is an open set.

The following theorems hold for open sets:

1. The union of open sets is open.
2. The finite intersection of open sets is open.

5.2 Closed Sets

Let (X, d) be a metric space, and S be a subset of X. A point x is a **limit point** of set S iff $\{B_\varepsilon(x) - \{x\}\} \cap S \neq \emptyset$ where $\varepsilon > 0$. For x to be a limit point of S, if we take an open ball around it, no matter what the size of that open ball, there will exist other points from S other than x in that open ball. Note: for a point x to be a limit point of S, it doesn't necessarily have to be an element of S. It only has to contain an element from S in its open ball for every $\varepsilon > 0$. The set of all limit points of a set S is usually denoted as S'

We say that a set S is **closed** iff S contains all of its limit points. In other words, $S' \subseteq S$. It does not have to be the case that $S = S'$.

Practice

The set of natural numbers, $\mathbb{N}$, can be written in the form: $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup ...$ where $\{n\}$ is said to be an isolated point. Is $\{n\}$ a limit point? What does that tell us about the set $\mathbb{N}$, is it open, closed, or neither.

The following theorems hold for closed sets:

1. The finite union of closed sets is closed.
2. The intersection of closed sets is closed.

5.3 Open and Closed Sets

The following theorem is useful for determining if a set is open or closed: The compliment of a closed set is open, and the compliment of an open set is closed.

Practice

1. Show that the empty set, $\emptyset$, is both closed and open.
2. Determine if $[0, 1] \cup \{2\}$ is open, closed, or neither.

5.4 Compact Sets

A subset S in a Euclidean space is said to be compact iff S is closed and bounded.

A subset S is compact if every sequence in S has a subsequence that converges to a point in K.

Practice

Show that for a compact set $S \in \mathbb{R}$, the supremum and infimum or S are elements of S.

6 Advanced Theorems

6.1 Continuous Functions

A function $f : A \to \mathbb{R}$ is **continuous** at $c \in A$ if fo all $\varepsilon > 0$, there exists $\delta > 0$ such that when $|x-c|< \delta$, it follows that $|f(x) - f(c)|< \varepsilon$. f is said to be continuous on A if f is continuous at every point in the domain A.

Additional properties: Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be continuous at a point $c \in A$. Then:

1. $kf(x)$ is continuous at c for every $k \in \mathbb{R}$
2. $f(x) + g(x)$ is continuous at c.
3. $f(x) \cdot g(x)$ is continuous at c.
4. $\frac{f(x)}{g(x)}$ is continuous at c, given $\frac{f(x)}{g(x)}$ exists.

6.2 Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and r is a real number such that $f(a) \leq r \leq f(b)$ or $f(z) \geq r \geq f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = r$.

6.3 Fixed Points

Let $f : X \to X$. A point x^* is a **fixed point** of f iff $f(x^*) = x^*$. Notice that if we apply the function f on a fixed point x^* multiple times, we still get x^* as an output.

6.4 Brouwer's Fixed Point Theorem

Let $(\mathbb{R}^n, d)$ be a metric space. If $f : X \to X$ be a continuous function, where X is a compact and convex subset of $\mathbb{R}^n$, then there exists an $x^* \in X$ such that $f(x^*) = x^*$.

Practice

Using the Intermediate Value Theorem, prove the Brouwer's Fixed Point Theorem in the metric space $(\mathbb{R}, d_1)$

6.5 Contraction

We will use the notation $f^n(x)$ to mean apply the function n times on x. In other words, if $n = 3$, then $f^3(x) = f(f(f(x)))$.

Let (X, d) be a metric space. A function $f : X \to X$ is said to be a contraction iff there exists a $\lambda < 1$ such that:

$$d(f(x), f(x')) \leq \lambda \cdot d(x, x')$$

for any $x, x' \in X$.

Notice that when a function is a contraction, when we apply the function to two points, the distance between the images are closer than the distance between the preimages.

Example

Let $([\frac{1}{2}, 10], d_1)$ be a metric space, and let f be a function from the set $[\frac{1}{2}, 10]$ into itself. We see that the function as $f(x) = \sqrt{x}$ on this metric space is a contraction. Notice that no matter what x we start with (where $x \in [\frac{1}{2}, 10]$), $\lim_{n\to\infty} f^n(x) = 1$.

Starting with $x = \frac{1}{2}$:
$f(\frac{1}{2}) = 0.707107$
$f^2(\frac{1}{2}) = 0.84090$
$f^3(\frac{1}{2}) = 0.917004$
$\vdots$
$f^{20}(\frac{1}{2}) = 0.999999$

6.6 Contraction Mapping Theorem

Let (X, d) be a complete metric space, and $f : X \to X$ be contraction. It follows that there exists a fixed point x^* of f, and for any $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.