Proofs

Joe Patten

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1 Statements and Open Sentences

1.1 Statements

A **statement** is a declarative sentence or assertion that is either true or false. They are often labelled with a capital letter (P, Q, R are most commonly used). Below are examples of statements:

P_1: The integer 6 is even.
P_2: A square has 5 sides.

Notice that the first statement is true, whereas the second statement is false.

1.2 Open Sentences

An **open sentence** is similar to a statement, except it contains one or more variables. Below are examples of open sentences:

P_3: The integer k is even.
P_4: A square has j sides.

Notice that in the first open sentence, there exist some values of k where the statement holds true. In the second open sentence, the statement holds true only if $j = 4$.

1.3 Negation

The **negation** of a statement (or proposition) P is denoted by $\sim P$ or $\neg P$, and is pronounced "not P". $\sim P$ is the opposite of P. The example below shows a statement P_5, and its negation $\sim P_5$:

P_5: The integer 7 is odd.
$\sim P_5$: The integer 7 is even.

Recall that a statement or open sentence can only take on one of two values: true or false. Thus, the negation of a statement or open sentence will take on the opposite truth value. This can be seen in the following truth table:

P	$\sim P$
T	F
F	T

Table 1: Truth table for P and $\sim P$.

2 Logical Connectives

2.1 Disjunction

The **disjunction** of the statements P and Q is denoted as $P \vee Q$ is defined as the statement P or Q. $P \vee Q$ is true if either P or Q is true, otherwise it is false. Notice that from the first example, $P_1 \vee P_2$ is true since P_1 is true and P_2 is false. Below is a truth table for $P \vee Q$.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2: Truth table for $P \vee Q$.

2.2 Conjunction

The **conjunction** of the statements P and Q is denoted as $P \wedge Q$ is defined as the statement P and Q. $P \wedge Q$ is true if either P and Q are both true, otherwise it is false. Notice that from the first example, $P_1 \wedge P_2$ is false since P_1 is true and P_2 is false. Below is a truth table for $P \wedge Q$.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 3: Truth table for $P \wedge Q$.

2.3 Implication and Biconditional

An **implication** is usually denoted as $P \Rightarrow Q$, and means either "If P, then Q" or "P implies Q". Below is a truth table for $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 4: Truth table for $P \Rightarrow Q$.

There are multiple ways of expressing $P \Rightarrow Q$:

P implies Q
If P, then Q
P only if Q
P is sufficient for Q
Q if P
Q is necessary for P

$Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. If $P \Rightarrow Q$ is true, it's not necessarily the case that its converse, $Q \Rightarrow P$, is true.

A **biconditional** of P and Q is usually denoted by $P \Leftrightarrow Q$, and means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. There are multiple ways of expressing $P \Rightarrow Q$:

$$P \text{ if and only if } Q$$
$$P \text{ iff } Q$$
$$P \text{ is equivalent to } Q$$

Below is a truth table for $P \Leftrightarrow Q$:

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table 5: Truth table for $P \Leftrightarrow Q$.

2.4 Compound Statements

The operators explained before ($\sim, \vee, \wedge, \Rightarrow, \Leftrightarrow$, and $\Leftrightarrow$) are referred to as logical connectors. The combination of at least one statement and at least one connector is called a **compound statement**. Notice that the following are compound statements:

$$\sim P$$
$$P \vee Q$$
$$P \Rightarrow Q$$
$$(P \wedge Q) \wedge (Q \Rightarrow \sim P)$$

2.5 Tautologies

A compound statement is a **tautology** if all possible truth values are true. An example of a tautology is $P \vee (\sim P)$. The following truth table shows that all the possible truth values are true:

P	$\sim P$	$P \vee (\sim P)$
T	F	T
F	T	T

Table 6: Truth table for $P \vee (\sim P)$.

2.6 Contradictions

A compound statement is a **contradiction** if all possible truth values are false. An example of a contradiction is $P \wedge (\sim P)$. The following truth table shows that all the possible truth values are true:

P	$\sim P$	$P \wedge (\sim P)$
T	F	F
F	T	F

Table 7: Truth table for $P \wedge (\sim P)$.

2.7 Logical Equivalence

Two compound statements R and S are **logically equivalent** if they have the same truth values in a truth table. If R and S are logically equivalent, then we write $R \equiv S$. For example, we see that $P \implies Q$ and $(\sim P) \vee Q$ are logically equivalent as all truth values for $P \implies Q$ and $(\sim P) \vee Q$ are the same:

P	Q	$P \Rightarrow Q$	$\sim P$	$(\sim P) \vee Q$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

Table 8: Truth table for $P \Leftrightarrow Q$.

3 Proofs

Before we discuss proofs, we need to introduce some terminology. An **axiom** is a true statement that is accepted without proof. A **theorem** is a true statement that can be proven. Oftentimes, the term theorem is only used when talking about statements that have some sort of significance or importance. A **corollary** is a result that can be derived or deduced from a previous result. A **lemma** is a result that is used to establish another result.

3.1 Direct Proof

A **direct proof**, sometimes called a constructive proof, is a method of proof that is used to show $P \Rightarrow Q$. In a direct proof, we assume P to be true, and through a number of statements make our way to a statement that shows that Q is true. In order to demonstrate how to go about doing a direct proof, I shall present a few properties about integers:

1. The negative or any integer is also an integer
2. The summation of any two integers results in an integer
3. The product of any two integers results in an integer
4. Any even number can be written in the form: $2k$, where $k \in \mathbb{Z}$
5. Any odd number can be written in the form: $2k + 1$, where $k \in \mathbb{Z}$

Example 1 Let $n \in \mathbb{Z}$. If n is odd, then $5n + 9$ is even.

Result 1 Assume n is odd.
Thus n can be written in the following form: $2k + 1$ where $k \in \mathbb{Z}$.
This means that $5n + 9 = 5(2k + 1) + 9 = 10k + 14 = 2(5k + 7)$.
Notice that since $(5k + 7) \in \mathbb{Z}$, therefore $5n + 9$ is even.

Example 2 Let $n \in \mathbb{Z}$. If n is even, then $-3n - 5$ is odd.

Result 2 Assume n is even.
Thus n can be written in the following form: $2k$ where $k \in \mathbb{Z}$.
This means that $-3n - 5 = -3(2k) - 5 = -6k - 5 = -6k - 5 = -6k - 6 + 1 = 2(-3k - 3) + 1$.
Notice that since $(-3k - 3) \in \mathbb{Z}$, therefore $-3n - 5$ is odd.

3.2 Proof by Contrapositive

The **contrapositive** for an implication $P \Rightarrow Q$ is defined as $(\sim Q) \Rightarrow (\sim P)$. Notice that $(\sim Q) \Rightarrow (\sim P)$ is the logical equivalent of $P \Rightarrow Q$. Proofs by contrapositive are very similar to direct proofs. The only difference is that we start with $\sim Q$ and and through a number of statements make our way to a statement that shows that $\sim P$ is true. Proofs by contrapositive are often used when it is easier to work with $\sim Q$ then it is to work with P.

Example 1 Let $n \in \mathbb{Z}$. If $3n - 9$ is even, then n is odd

Result 1 Assume n is even.

Thus n can be written in the following form: $2k$ where $k \in \mathbb{Z}$
This means that $3n - 9 = 3(2k) - 9 = 6k - 9 = 2(3k - 10) + 1$
Since $(3k - 10) \in \mathbb{Z}$, it follows that $3n - 9$ is odd.

Example 2 Let $x \in \mathbb{Z}$. $9n - 5$ is even if and only if n is odd.

Result 2 When a biconditional is involved, we need to prove both directions. In other words, we need to prove that ($9x - 5$ is even) $\Rightarrow$ (n is odd), and (n is odd) $\Rightarrow$ ($9x - 5$ is even). We will first show that ($9x - 5$ is even) $\Rightarrow$ (n is odd).

$\Rightarrow$) ($9n - 5$ is even) $\Rightarrow$ (n is odd)
Assume n is even.
Thus n can be written in the following form: $2k$ where $k \in \mathbb{Z}$
This means that $9n - 5 = 9(2k) - 5 = 2(9k - 3) + 1$
Since $9k - 3 \in \mathbb{Z}$, it follows that $9n - 5$ is odd.

$\Leftarrow$) (n is odd) $\Rightarrow$ ($9n - 5$ is even)
Assume n is odd.
Thus n can be written in the following form: $2m$ where $m \in \mathbb{Z}$
This means that $9n - 5 = 9(2m + 1) - 5 = 2(9m - 2)$
Since $9m - 2 \in \mathbb{Z}$, it follows that $9n - 5$ is even.

3.3 Cases

Oftentimes, it is easier to break the domain of a premise into subsets. The prover then works through the proof for each subset or **case**. Notice that these subsets need to exhaust the domain, meaning every point or element in the domain needs to be covered by a case. The following are examples of cases that could be used for their domains.

Example 1: $\forall x \in \mathbb{R}$:

Case 1: $x > 0$

Case 2: $x < 0$

Case 3: $x = 0$

Example 2: $\forall n \in \mathbb{Z}$:

Case 1: n is odd

Case 2: n is even

Example 3: Let $m, n \in \mathbb{Z}$. If mn is odd, then m and n are odd.

Result Assume m or n are even. Then, either m is even and n is odd, n is even and m is odd, or both m and n are even.

Case 1: Assume m and n are even.
Thus $m = 2r$ and $n = 2s$ where $r, s \in \mathbb{Z}$.
Therefore $mn = 2r \cdot 2s = 4rs = 2(rs)$.
Since $rs \in \mathbb{Z}$, it follows that mn is even.

Case 2: Assume without loss of generality that m is even and n is odd.
Thus $m = 2t$ and $n = 2u + 1$ where $t, u \in \mathbb{Z}$.
Therefore $mn = 2t \cdot (2u + 1) = 4tu + 2t = 2(2tu + t)$.
Since $2tu + t \in \mathbb{Z}$, it follows that mn is even.

Notice that in the previous example, we had three cases: either m is even and n is odd, n is even and m is odd, or both m and n are even. However, we only walked through 2 cases: both m and n are even, and m is even and n is odd. In case 2, we used the phrase **without loss of generality**, because both the cases of m is even and n is odd, and n is even and m is odd are similar, and so the proof of one case will be sufficient to cover the two cases.

3.4 Proof by Contradiction

If we are trying to prove $P \Rightarrow Q$, we assume both P and $\sim Q$ are true, and then we try to deduce a contradiction $(R \wedge (\sim R)\ )$. We usually start the proof by saying "Assume to the contrary..." or "Assume by contradiction that..." followed by P and $\sim Q$.

Example 1 Let $n \in \mathbb{Z}$. If n is even, then $5n + 3$ is odd.

Result 1 Assume to the contrary that there exists an even integer n such that $5n + 3$ is even.
Since n is even, we can write $n = 2k$ where $k \in \mathbb{Z}$.
Thus, $5n + 3 = 5(2k) + 3 = 10k + 3 = 2(5k + 1) + 1$.
Since $(5k + 1) \in \mathbb{Z}$, then $5n + 3$ is odd, which is a contradiction.

Example 2 Show that 100 cannot be written as a sum of one odd integer and two even integers.

Result 2 Assume to the contrary that 100 can be written as a sum of one odd integer and two even integers.
Thus, $100 = (2k + 1) + (2m) + (2j)$ where $k, n, j \in \mathbb{Z}$.
$100 = (2k + 1) + (2m) + (2j) = 2(k + m + j) + 1$.
Since $k + m + j \in \mathbb{Z}$, we see that 100 is odd. This is a contradiction.

3.5 Counterexample

We have used direct proof, proof by contrapositive, and proof by contradiction to show that $P \Rightarrow Q$. However, it is not always the case that $P \Rightarrow Q$. If we can find an x in the domain of the premise P such that Q is false, then it is *not* the case that $P \Rightarrow Q$.

Example 1 Disprove the following statement:

$$\text{If } x \in \mathbb{Z}, \text{ then } \frac{x^2 + 2x}{x^2 - 3x} = \frac{x + 2}{x - 3}$$

Result To disprove the statement above, we only need to provide a counterexample in the domain of P, in this case, an $x \in \mathbb{R}$ where the expression does not hold.
Consider $x = 0$. We see that $\frac{x^2+2x}{x^2-3x}$ is undefined at $x = 0$, however, $\frac{x+2}{x-3} = -\frac{2}{3}$ when $x = 0$. Thus, $x = 0$ is a counterexample to the statement above.

3.6 Mathematical Induction

Let P(n) be a statement, where $n \in \mathbb{N}$. To prove by induction, we need to prove two things:

1. A base case (usual base case is $n = 1$)
2. The inductive step: $\forall k \in \mathbb{N}$, the implication: $P(k) \Rightarrow P(k + 1)$ is true.

Example 1 Show that the sum of the first n positive integers is $n(n+1)/2$. Or in other words:

$$1 + 2 + 3 + 4 + ... + n = n(n+1)/2$$

Result 1 Let $P(n) : 1 + 2 + 3 + 4 + ... + n = n(n+1)/2$ where $n \in \mathbb{N}$

1. **Base case**: $P(1) : 1 = 1(1+1)/2 = 1$. Thus the base case is true.
2. **Inductive step**: Assume $P(k)$ is true, thus:

$$P(k) : 1 + 2 + 3 + 4 + ... + k = k(k+1)/2$$

Now we show that $P(k+1)$ is true, or that $1 + 2 + 3 + 4 + ... + k + (k+1) = (k+1)(k+2)/2$
$1+2+3+4+...+k+(k+1) = k(k+1)/2+(k+1) = k(k+1)/2+2(k+1)/2 = (k+2)(k+1)/2$
By induction, $P(n)$ is true for every integer n.

4 Real Analysis

Given a point $a \in \mathbb{R}$ and $\varepsilon > 0$, the ε neighborhood of a is the set:

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$

Another way to write this set is using interval notation: $V_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$. Notice that ε could be any positive real number. A set