

Real Analysis Practice Solutions

Joe Patten

August 12, 2018

1 Relations and Functions

1.1 Properties of Relations

Practice

Let $S = \{a, b, c\}$. Which of the properties reflexive, transitive, and symmetric do the relations below possess if the relations are from S to S ?

1. $R_1 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a), (b, b), (c, c)\}$
2. $R_2 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a)\}$
3. $R_3 = \{(b, c), (c, b), (a, a), (b, b), (c, c)\}$

1. R_1 is reflexive, transitive, and symmetric.
2. R_2 is only symmetric.
3. R_3 is only reflexive.

Practice

Consider $S \in \mathbb{R}$. Let the following be relations from S to S . Show that the following relations are reflexive, transitive, and symmetric. If a property does not hold, provide a counterexample to show that that property does not hold.

1. $R_5 = \{(a, b) \in S \times S : a \geq b\}$
Let $x, y, z \in S$.
 - $(x, x) \in R_5$ since $x \geq x$, thus R_5 is reflexive.
 - $(x, y) \in R_5 \not\Rightarrow (y, x) \in R_5$. Counterexample: Suppose $x = 5$ and $y = 4$. Thus R_6 is not symmetric.
 - Suppose $(x, y), (y, z) \in R_5$. Thus $x \geq y$ and $y \geq z$. By transitivity of \mathbb{R} , it follow that $x \geq z$, thus $(x, z) \in R_5$. Therefore R_5 is transitive.
2. $R_6 = \{(a, b) \in S \times S : a > b\}$
Let $x, y, z \in S$.
 - $(x, x) \notin R_6$ since $x \not> x$, thus R_6 is not reflexive.
 - $(x, y) \in R_6 \not\Rightarrow (y, x) \in R_6$. Counterexample: Suppose $x = 5$ and $y = 4$. Thus R_6 is not symmetric.
 - Suppose $(x, y), (y, z) \in R_6$. Thus $x > y$ and $y > z$. By transitivity of \mathbb{R} , it follow that $x > z$, thus $(x, z) \in R_6$. Therefore R_6 is transitive.

3. $R_7 = \{(a, b) \in S \times S : ab \geq 0\}$

Let $x, y, z \in S$.

- $(x, x) \in R_7$ since $xx \geq 0$ for all $x \in \mathbb{R}$, thus R_7 is reflexive.
- Suppose $(x, y) \in R_7$, then $xy \geq 0$. Notice that $xy = yx$. Thus $yx \geq 0$. So $(y, x) \in R_7$. Thus R_7 is symmetric.
- Counterexample: Notice that $(-1, 0) \in R_7$, and $(0, 5) \in R_7$. However, $(-1, 5) \notin R_7$. Thus R_7 is not transitive.

1.2 Monotonic Functions

Practice

Show that the function $f(x) = \log(x)$ is strictly increasing for all $x \in \mathbb{R}_{++}$, where \mathbb{R}_{++} is defined as: $\mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$

Let $x, y \in \mathbb{R}_{++}$. Without loss of generality, assume $x > y$. Since $x > y$, there exists an $\alpha \in (0, 1)$ such that $x = \alpha y$. We are required to prove that $\log(x) > \log(y)$. Notice:

$$\begin{aligned}\log(x) &= \log(\alpha y) \\ &= \log(\alpha) + \log(y) \\ &> \log(y)\end{aligned}$$

Thus $f(x) = \log(x)$ is a strictly increasing function. Notice that we can also look at the derivative of $f(x)$ and see if it is positive over the whole domain of \mathbb{R}_{++} to see if it is a strictly increasing function.

2 Metric Spaces

Practice

1. Show that (\mathbb{R}, d_1) is a valid metric space.

Let $x, y, z \in \mathbb{R}$. where $x \neq y \neq z$

- (a) Notice that $|x - x| = 0$ and $|x - y| > 0$.
- (b) Notice that $|x - y| = |y - x|$.
- (c) Notice that $|x - z| = |(x - y) + (y - z)| \Rightarrow |x - z| \leq |x - y| + |y - z|$ since $(x - y) \leq |x - y|$ and $(y - z) \leq |y - z|$.

2. Show that (\mathbb{R}^2, d_2) is a valid metric space.

Let $x, y, z \in \mathbb{R}^2$. where $x \neq y \neq z$

- (a) If $x \neq y$, and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then either $x_1 \neq y_1$ or $x_2 \neq y_2$. Thus $(x_1 - y_1)^2 + (x_2 - y_2)^2 > 0$. Therefore $\|x - x\| = 0$ and $\|x - y\| > 0$.
- (b) Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Notice that $|x - y| = |y - x| = \sqrt{\begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix}^T \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix}}$.
- (c) Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then $\|x - z\| = \|(x - y) + (y - z)\| \Rightarrow \|x - z\| \leq \|x - y\| + \|y - z\|$ since $(x - y) \leq \|x - y\|$ and $(y - z) \leq \|y - z\|$.

3. Show that (\mathbb{R}^n, d_3) is a valid metric space.

- (a) If $x \neq y$, then $\exists i$ such that $x_i \neq y_i$. Thus $d_3(x, y) \geq |x_1 - y_1| > 0$.
- (b) Notice that $d_3(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\} = \max\{|y_1 - x_1|, \dots, |y_n - x_n|\} = d_3(y, x)$.
- (c) Suppose $d_3(x, y) = |x_i - y_i|$, $d_3(y, z) = |y_j - z_j|$, and $d_3(x, z) = |x_k - z_k|$ for $i, j, k \in \mathbb{N}$. Notice $|x_k - z_k| \leq |x_i - y_i| + |y_j - z_j|$. (Further clarification is left up to the reader).

3 More Set Theory

Practice

Determine the supremum and infimum for each of the following sets in \mathbb{R} . Also determine if the supremum and infimum are equal to the maximum and minimum respectively of each set:

1. $[0, 2]$
Supremum = maximum = 2
Infimum = minimum = 0
2. $(0, 2)$
Supremum = 2 \neq maximum
Infimum = 0 \neq minimum
3. $[0, 2] \cup \{3\}$
Supremum = maximum = 3
Infimum = minimum = 0

4 Sequences

4.1 Sequence Convergence

Practice

Show (via proof) that:

1. $\lim_{n \rightarrow \infty} \frac{2}{\sqrt{2n+4}} = 0$

Let $\varepsilon > 0$ be arbitrary. Choose N such that $N > \frac{4}{\varepsilon^2}$. Let $n \geq N$. Then:

$$\left| \frac{2}{\sqrt{2n+4}} - 0 \right| < \varepsilon$$

2. $\lim_{n \rightarrow \infty} \frac{4n+1}{2n+4} = 2$

Let $\varepsilon > 0$ be arbitrary. Choose N such that $N > \frac{5}{2\varepsilon}$. Let $n \geq N$. Then:

$$\left| \frac{4n+1}{2n+4} - 2 \right| < \varepsilon$$

4.2 Cauchy Criterion

Practice

Consider the metric space (\mathbb{R}, d_1) . Show that every convergent sequence is a Cauchy sequence.

[See Assignment 3 solutions](#)

5 Topology

5.1 Open Sets

Practice

Using the definition above and assuming the metric space is (\mathbb{R}, d_1) :

1. Show that $B_\varepsilon(a)$ is an open set.
2. Show that \mathbb{R} is an open set.
3. Show that $(0, 1)$ is an open set.

[See Assignment 3 solutions.](#)

5.2 Closed Sets

Practice

The set of natural numbers, \mathbb{N} , can be written in the form: $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \dots$ where $\{n\}$ is said to be an isolated point. Is $\{n\}$ a limit point? What does that tell us about the set \mathbb{N} , is it open, closed, or neither.

5.3 Open and Closed Sets

Practice

1. Show that the empty set, \emptyset , is both closed and open.
2. Determine if $[0, 1] \cup \{2\}$ is open, closed, or neither.

5.4 Compact Sets

Practice

Show that for a compact set $S \in \mathbb{R}$, the supremum and infimum of S are elements of S .

6 Advanced Theorems

6.1 Brouwer's Fixed Point Theorem

Practice

Using the Intermediate Value Theorem, prove the Brouwer's Fixed Point Theorem in the metric space (\mathbb{R}, d_1)