# WSU Economics PhD Mathematics Bootcamp

# Assignment 2

**Directions** Problems 1-18 are required for assignment number 1 and are to be typed. The remaining problems are optional. Note: Just because material is not explicitly required on the assignment does not mean that it won't be useful or that you are not responsible for it.

Problems with a (BoP x.yz) are problems from the text *Book of Proof* and should have answers at the end of the book. Please try to avoid cheating yourself of learning by looking at the answers too soon.

I will try and find some material that could be useful for the calculus questions which are not covered in *Book of Proof*. However, using the proof methods we cover and the definition of a sequence plus your previous experience with calculus, you'll have the fundamentals you need to solve those problems.

If you need help, please feel free to email me.

**Problem 1.** *Direct Proof* (BoP 4.1) If $x$ is an even integer, then $x^2$ is even.

**Problem 2.** *Direct Proof* (BoP 4.13) Suppose $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.

**Problem 3.** *Contrapositive Proof* (BoP 5.3) Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then $a$ and $b$ are odd.

**Problem 4.** *Proof by Contradiction* (BoP 6.5) Prove that $\sqrt{3}$ is irrational.

**Problem 5.** *Biconditional* (BoP 7.1) Suppose $x \in \mathbb{Z}$. Then $x$ is even if and only if $3x + 5$ is odd.

**Problem 6.** *Sets* (BoP 8.5) If $p$ and $q$ are positive integers, then $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\} \neq \emptyset$.

**Problem 7.** *Sets* (BoP 8.11) If $A$ and $B$ are sets in a universal set $U$, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

**Problem 8.** *Sets* (BoP 8.23) For each $a \in \mathbb{R}$, let $A_a = \{(x, a(x^2 - 1)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Prove that

$$\bigcap_{a \in \mathbb{R}} A_a = \{(-1, 0), (1, 0)\}$$

**Problem 9. Optional** (BoP 9.1) The following statement is either true or false. If the statement is true, prove it. If the statement is false, disprove it. If $x, y \in \mathbb{R}$, then $|x + y| = |x| + |y|$.

Use the following definitions to help answer Problem 10.

**Definition 1** (Antisymmetric)**.** A relation $R$ on a set $X$ is **antisymmetric** if $\forall x, y \in X$ such that $xRy$ and $yRx$ we have $x = y$.

As an example consider the $\leq$ ordering on $\mathbb{N}$. If for two numbers $a, b \in \mathbb{N}$ we have $a \leq b$ and $b \leq a$ then we know it must be that $a = b$. There does not exist two distinct natural numbers that are both less than or equal to the other. This property is important to ordering the natural numbers, real numbers, etc. In general we can order elements of any set leading intervals and so on if the relation on that set is a *partial order*.

**Definition 2** (Partial Order)**.** Let $X$ be a nonempty set with relation $\preceq \subset X \times X$. Then $\preceq$ is a **partial order** if it is reflexive, antisymmetric and transitive.

**Problem 10.** Let $C$ be a set of consumption bundles. A **preference relation** is a binary relation, $\succsim$, where $x \succsim y$ reads $x$ *is at least as good as* $y$. We say that $\succsim$ is a **rational preference relation** if it satisfies the following properties:

(i) $\forall x, y \in C,\ [x \succsim y\ \vee y \succsim x]$ (Completeness)

(ii) For $x, y, z \in C$ if $[x \succsim y \wedge y \succsim z] \implies x \succsim z$ (Transitivity)

Use the above definitions to complete the following exercises.

(a) Prove that the rational preference relation, $\succsim$, is reflexive.

(b) Defining the **indifference relation**, $\sim$, as $x \sim y$ if $[x \succsim y \wedge y \succsim x]$, prove that $\sim$ is an equivalence relation.

(c) Define the quotient set, call it $\mathcal{I}$, associated with $(C, \sim)$ using set notation and prove that $\mathcal{I}$ is a partition of $C$.

(d) (**Optional**) Interpret the equivalence classes of $\mathcal{I}$ and show that $\succsim$ is a partial order on $\mathcal{I}$.

**Problem 11.** (BoP 12.2.13) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x, y) = (xy, x^3)$. Is $f$ injective? Is it surjective? Bijective? Explain.

**Problem 12.** Let $f$ be a function from $A$ to $B$ and let $E, F \subset A$. Prove statements $a$ through $e$.

(a) If $E \subset F$, then $f(E) \subset f(F)$,

(b) $f(E \cap F) \subset f(E) \cap f(F)$,

(c) $f(E \cup F) = f(E) \cup f(F)$,

(d) $f(E - F) \subset f(E)$

**Problem 13.** **(Level sets of functions)** Let $f : A \to B$ be a function. Define a relation on $A$ denoted $a \sim_f a'$ if $\exists b \in B$ such that $a, a' \in f^{-1}(b)$. Prove the following statements.

(a) Show that $\sim_f$ is an equivalence relation on $A$.

(b) Show that $[a \sim_f a'] \iff [f(a) = f(a')]$.

(c) Prove that the inverse images $f^{-1}(b)$ and $f^{-1}(b')$ are disjoint when $b \neq b'$. (This means that indifference curves never intersect.)

**Problem 14.** Prove the following statements. Let $f$ be a function mapping $A$ to $B$, and let $G$, $H \subset B$. (Reminder: the notation here $f^{-1}(X)$ is the *pre-image* of some set $X$ in the codomain under the function $f$.)

(a) If $G \subset H$, then $f^{-1}(G) \subset f^{-1}(H)$,

(b) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$,

(c) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$, and

(d) $f^{-1}(G - H) = f^{-1}(G) - f^{-1}(H)$.

**Problem 15.** A **sequence** in $A$ is a function $f : \mathbb{N} \to A$ and the range of the sequence is a list $\{f(1), f(2), \dots\} = \{a_1, a_2, \dots\}$. By definition $\lim_{n\to\infty} a_n = L$ if for every positive real number (no matter how small) $\epsilon > 0$, there exists a positive integer $N$ such that if $n$ is an integer with $n > N$, then $|a_n - L| < \epsilon$. By taking the negation of this definition, write out the meaning of $\lim_{n\to\infty} a_n \neq L$ using quantifiers. Then write out the meaning of $\{a_n\}$ diverges using quantifiers. (Hint: the definition of convergence using quantifiers is: $(\forall \epsilon > 0), \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$. Or in terms of an implication $\forall \epsilon > 0, \exists N \in \mathbb{N},\ [n > N \implies |a_n - L| < \epsilon]$)

**Problem 16.** Using the definition of convergence in the previous problem, prove that the sequence $\left\{\frac{1}{2n}\right\}$ converges to 0.

**Definition 3** (Limit of a Function)**.** Let $f$ be a real-valued function defined on a set $X$ of real numbers. We say $L \in \mathbb{R}$ is the **limit** of $f(x)$ as $x$ approaches $a \in \mathbb{R}$ if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every real number $x$ with $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \epsilon$.
( $\forall \epsilon > 0, \exists \delta > 0, \forall x\ [|x - a| < \delta] \implies [|f(x) - L| < \epsilon]$)

**Definition 4** (Epsilon Neighborhood)**.** Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then the set (interval) defined as $B_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$ is called an $\epsilon$-**neighborhood** (or epsilon-ball) of $a$.

The above definition is very important and you will see this used a lot in the future. We can use it to restate the definitions of limits used earlier.

We say the sequence $\{a_n\}$ converges to the limit $L$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n > N,\ a_n \in B_\epsilon(L)$$

Similarly for the limit of a function, we say that the limit of the function $f$ over sequence $\{a_n\}$ is $L$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } x \in B_\delta(L) \implies f(x) \in B_\epsilon(L)$$

**Definition 5** (Continuity)**.** Let $f : X \to \mathbb{R}$. We say that $f$ is **continuous** at a point $\imath n\mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$. Using our previous methods of defining limits this means we can phrase continuity also as

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \text{ such that } |x - a| < \delta, [f(x) - f(a) < \epsilon]$$
$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } x \in B_\delta(a) \implies f(x) \in B_\epsilon(f(a))$$

In general there are three criteria for $f$ to be continuous at point $a$.

(i) $f$ is defined at $a$

(ii) $\lim_{x\to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

**Problem 17.** Prove that the function $f : [1, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x-1}$ is continuous at $x = 10$.

**Problem 18.** Let $f : \mathbb{R} \to \mathbb{R}$. We say the function $f$ is **differentiable** at $a$ and is denoted $f'(a)$ if the following limit exists.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Let $f$ be defined as $f(x) = x^2$. Determine $f'(3)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

# 1 Extra Practice Exercises

## 1.1 Direct Proof

**Problem 19.** (BoP 4.9) Suppose $a$ is an integer. If $7 \mid 4a$ then $7 \mid a$.

## 1.2 Contrapositive Proof

Prove the following using the method of contrapositive proof.

**Problem 20.** (BoP 5.9) Suppose $n \in \mathbb{Z}$. If $3 \nmid n^2$, then $3 \nmid n$.

## 1.3 Either Direct or Contrapositive Proof

Use either direct or the contrapositive method to prove the statements.

**Problem 21.** (BoP 5.15) Suppose $x \in \mathbb{Z}$. If $x^3 - 1$ is even, then $x$ is odd.

**Problem 22.** (BoP 5.21) Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \pmod{n}$, then $a^3 \equiv c^3 \pmod{n}$.

## 1.4 Proof by Contradiction

Use the method of proof by contradiction to prove the following statements.

**Problem 23.** (BoP 6.11) There exist no integers $a$ and $b$ for which $18a + 6b = 1$.

## 1.5 Proving Non-conditional Statements

**Problem 24.** (BoP 7.1) Suppose $x \in \mathbb{Z}$. Then $x$ is even if and only if $3x + 5$ is odd.

**Problem 25.** (BoP 7.21) Every real solution of $x^3 + x + 3 = 0$ is irrational.

## 1.6 Proofs Involving Sets

*Book of Proof* Chapter 8 Exercises: 1,5,7,9,11,21,23 Use any of the methods we've covered to prove the following statements.

**Problem 26.** (BoP 8.1) Prove that $\{12n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

**Problem 27.** (BoP 8.9) If $A$,$B$ and $C$ are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**Problem 28.** (BoP 8.21) Suppose $A$ and $B$ are sets. Prove $A \subseteq B$ if and only if $A - B = \emptyset$.

## 1.7 Disproof

Each of the following statements is either true or false. If a statement is true, prove it. If a statement is false, disprove it.

**Problem 29.** (BoP 9.34) If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

## 1.8 Relations

**Problem 30.** (BoP 11.0.1) Let $A = \{0, 1, 2, 3, 4, 5\}$. Write out the relation $R$ that expresses $>$ on $A$.

**Problem 31.** (BoP 11.1.6) Consider the relation $R = \{(x, x) : x \in \mathbb{Z}\}$ on $\mathbb{Z}$. Is $R$ reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?

**Problem 32.** (BoP 11.1.8) Define a relation on $\mathbb{Z}$ as $xRy$ if $|x - y| < 1$. Is $R$ reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?

## 1.9 Functions

**Problem 33.** (BoP 12.1.7) Consider the set $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 3x + y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

**Problem 34.** (BoP 12.1.9) Consider the set $f = \{(x^2, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

**Problem 35.** (BoP 12.2.5) A function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

**Problem 36.** (BoP 12.5.7) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = ((x^2 + 1)y, x^3)$ is bijective. Then find its inverse.

## 1.10 Calculus Topics

**Problem 37.** Prove that if a sequence $\{s_n\}$ converges to $L$, then the sequence $\{s_{n^2}\}$ also converges to $L$.

**Problem 38.** Give an $\epsilon - \delta$ proof that $\lim_{x \to -1}(3x - 5) = -8$.

**Problem 39.** Show that $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.