Final Exam 2016

WSU Economics PhD Mathematics Bootcamp

August 12, 2016

Problem 1.

Consider the market for commodity x. The quantity demanded in this market depends on the market price for x, p, and the value of an external unpriced factor, r, such as the "status symbol" value of owning commodity x which increases quantity demanded at any price. Given a known constant b_D we have the demand function:

$$q = b_D - 2p + r$$

The supply of x to the market increases with market price and we assume for some reason that higher values of r, the status symbol value, decrease the willingness to supply at any given price. Important cost factors faced by suppliers result in a known constant b_S and we have the supply function:

$$q = b_S + 2p - r$$

Finally, the status symbol value r increases the more expensive the commodity is and it decreases as more of the commodity is supplied because it becomes more common. Given some known constant value b_r, the status symbol value is determined by the function:

$$r = b_r + 2p - q$$

(a) Using the functions provided, construct a proper system of linear equations to represent the market for commodity x.

(b) Derive a matrix expression for the system in part (a) that takes the form $A\mathbf{x} = \mathbf{b}$.

(c) The system of equations describing the market for commodity x has three equations in three unknowns. The system either has no solution, infinitely many solutions or a unique solution. Perform a test on the matrix A (show your process and results) from part(b) that can tell you whether there is a unique solution and interpret the result.

(d) Find the inverse matrix of A (show it) and use it to solve the system for the equilibrium values of q, p, r when we have the constants

$$\begin{bmatrix} b_D \\ b_S \\ b_r \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 2 \end{bmatrix}$$

Problem 2.

Consider the market for another commodity, y, with the following demand and supply functions where b_D and b_S are constants.

$$q_D = b_D e^p$$
$$q_S = b_S e^{-p}$$

(a) Find the equilibrium price function $p^* : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ such that for some values (b_D, b_S) the equilibrium price for y will be $p^*(b_D, b_S)$.

(b) Derive the gradient function ∇p^*.

(c) We are interested in the behavior of the equilibrium price in the market for commodity y as structural aspects, b_D and b_S change. Suppose through econometric analysis it is determined that the current values are $(b_D, b_S) = (2, 3)$. Calculate the total differential of equilibrium price p^* as the values of b_D and b_S change slightly $(db_D, db_S) = (1/9, -1/7)$.

(d) Using the values from part (c), calculate a linear approximation of the equilibrium market price as the structural parameters change as in part (c).

Problem 3.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Let $V = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| = 1\}$ be the set of all vectors in $\mathbb{R}^n$ of unit length. Also, let $\mathbf{x}_0 \in \mathbb{R}^n$ such that $\nabla F(\mathbf{x}_0) \neq \mathbf{0}$. Prove that over the set V, the direction $\mathbf{v}$ in which F increases most rapidly at the point $\mathbf{x}$ is the direction of $\nabla F(\mathbf{x}_0)$.

Problem 4. Prove or disprove the following statement. *There is no largest integer.*

Problem 5.

Consider two firms engaging in Cournot competition in a market M. Both firms face an inverse demand curve $P(Q, \mu)$ where P is the market price when total market quantity is Q and consumer taste intensity is $\mu > 0$. Furthermore $P(Q, \mu)$ follows the law of demand so that $P_Q(Q, \mu) < 0$ and more intense preferences increase demand $P_\mu(Q, \mu) > 0$.

Firm 1 has total production costs $C_1(q_1)$ when it produces q_1 units. Firm 2 has total production costs $C_2(q_2)$ when it produces q_2 units. For both firms costs increase in q, so that $C_i'(q_i) > 0$. Total market output $Q = q_1 + q_2$. The firm's profits are as follows

$$\pi_1(q_1, q_2, \mu) = P(q_1 + q_2; \mu)q_1 - C_1(q_1)$$
$$\pi_2(q_1, q_2, \mu) = P(q_1 + q_2; \mu)q_2 - C_2(q_2)$$

Each firm wants to choose its output quantity q_i to maximize its profit π_i given the output decision of the other firm. The solution to this maximization problem yields a *best response* function $R_i(q_{-i}, \mu)$ which has the interpretation that, for example, firm 1 will choose its output $q_1 = R_1(q_2, \mu)$. So R is a function that takes firm 2's output as an argument and tells firm 1 what output, q_1 will maximize its own profit. Firm 2 has a similar function $q_2 = R_2(q_1, \mu)$.

Using $q_1 = R_1(q_2, \mu)$ and $q_2 = R_2(q_1, \mu)$ we can create a composite function

$$q' = R_1(R_2(q_1, \mu), \mu) = F(q, \mu)$$

An equilibrium will be a *fixed point* of this composite function, which means that if you put a q in for some fixed μ and you get the same q back out, then that q is a Nash Equilibrium. Let q_e represent such values.

(a) Calculate the partial derivative $\pi_{q_1\mu}$ for firm 1's profit function.

(b) Calculate the partial derivative $\pi_{q_1q_2}$ for firm 1's profit function.

(c) For the identity $q_e = F(q_e, \mu) = R_1(R_2(q_e, \mu), \mu)$, calculate the derivative $\frac{dq_e}{d\mu}$, telling us how the Nash equilibrium quantity changes as consumer taste intensity changes. [Hint: Consider the use of total differentials and um...the chain rule.]

(d) (Bonus) Can you sign the derivative in part (a)? i.e., does the equilibrium output increase or decrease as μ increases?

True, False or Uncertain For the following problems, determine whether the statement, as given, is true, false or if its uncertain. If you determine the statement is false or uncertain, provide an explanation.

Problem 6.

True, False or Uncertain: Suppose $f(x, y)$ is defined on a set D that contains a point (a, b). If the partial derivative functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Problem 7.

True, False or Uncertain: For a function $f(x, y)$ if the partial derivatives f_x and f_y exist near a point (a, b) then f is differentiable at (a, b).

Problem 8.

True, False or Uncertain: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any vector $\mathbf{v}$ and

$$D_v f(x, y) = \nabla f \cdot \mathbf{v}$$