# Answers to Odd-Numbered Section Exercises

## EXERCISES FOR CHAPTER 1

### Section 1.1: Describing a Set

**1.1** Only (d) and (e).

**1.3** **(a)** $|A| = 5$, **(b)** $|B| = 11$, **(c)** $|C| = 51$, **(d)** $|D| = 2$, **(e)** $|E| = 1$, **(f)** $|F| = 2$

**1.5** **(a)** $A = \{-1, -2, -3, \ldots\} = \{x \in \mathbf{Z} : x \leq -1\}$
**(b)** $B = \{-3, -2, \ldots, 3\} = \{x \in \mathbf{Z} : -3 \leq x \leq 3\} = \{x \in \mathbf{Z} : |x| \leq 3\}$
**(c)** $C = \{-2, -1, 1, 2\} = \{x \in \mathbf{Z} : -2 \leq x \leq 2, x \neq 0\} = \{x \in \mathbf{Z} : 0 < |x| \leq 2\}$

**1.7** **(a)** $A = \{\cdots, -4, -1, 2, 5, 8, \cdots\} = \{3x + 2 : x \in \mathbf{Z}\}$
**(b)** $B = \{\cdots, -10, -5, 0, 5, 10, \cdots\} = \{5x : x \in \mathbf{Z}\}$
**(c)** $C = \{1, 8, 27, 64, 125, \cdots\} = \{x^3 : x \in \mathbf{N}\}$

**1.9** $A = \{2, 3, 5, 7, 8, 10, 13\}$
$B = \{x \in A : x = y + z, \text{ where } y, z \in A\} = \{5, 7, 8, 10, 13\}$
$C = \{r \in B : r + s \in B \text{ for some } s \in B\} = \{5, 8\}$

### Section 1.2: Subsets

**1.11** Let $r = \min(c - a, b - c)$ and let $I = (c - r, c + r)$. Then $I$ is centered at $c$ and $I \subseteq (a, b)$.

**1.13** See Figure 1.

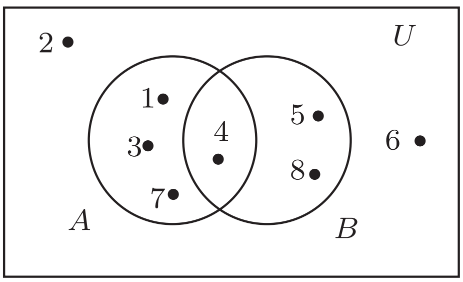


**Figure 1** Answer for Exercise 1.13

**1.15** $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, A\}$

**1.17** $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$; $|\mathcal{P}(A)| = 8$

**1.19** **(a)** $S = \{\emptyset, \{1\}\}$. **(b)** $S = \{1\}$.
**(c)** $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4, 5\}\}$. **(d)** $S = \{1, 2, 3, 4, 5\}$.

**1.21** $B = \{1, 4, 5\}$.

## Section 1.3: Set Operations

**1.23** Let $A = \{1, 2, \ldots, 6\}$ and $B = \{4, 5, \ldots, 9\}$. Then $A - B = \{1, 2, 3\}$, $B - A = \{7, 8, 9\}$ and $A \cap B = \{4, 5, 6\}$. Thus $|A - B| = |A \cap B| = |B - A| = 3$. See Figure 2.

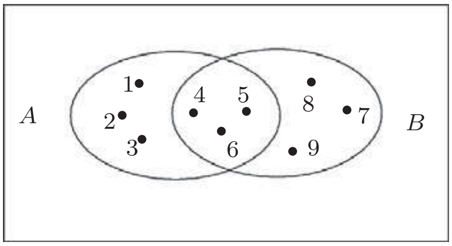


**Figure 2** Answer for Exercise 1.23

**1.25** **(a)** $A = \{1\}, B = \{\{1\}\}, C = \{1, 2\}$.
**(b)** $A = \{\{1\}, 1\}, B = \{1\}, C = \{1, 2\}$.
**(c)** $A = \{1\}, B = \{\{1\}\}, C = \{\{1\}, 2\}$.

**1.27** Let $U = \{1, 2, \ldots, 8\}$ be a universal set, $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then $A - B = \{1, 2\}$, $B - A = \{5, 6\}$, $A \cap B = \{3, 4\}$ and $\overline{A \cup B} = \{7, 8\}$. See Figure 3.

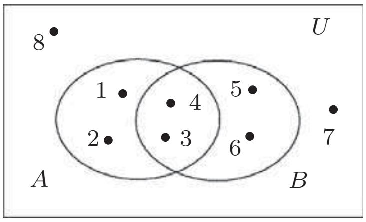


**Figure 3** Answer for Exercise 1.27

**1.29** **(a)** The sets $\emptyset$ and $\{\emptyset\}$ are elements of $A$. **(b)** $|A| = 3$.
**(c)** All of $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are subsets of $A$. **(d)** $\emptyset \cap A = \emptyset$.
**(e)** $\{\emptyset\} \cap A = \{\emptyset\}$. **(f)** $\{\emptyset, \{\emptyset\}\} \cap A = \{\emptyset, \{\emptyset\}\}$.
**(g)** $\emptyset \cup A = A$. **(h)** $\{\emptyset\} \cup A = A$. **(i)** $\{\emptyset, \{\emptyset\}\} \cup A = A$.

**1.31** $A = \{1, 2\}, B = \{2\}, C = \{1, 2, 3\}, D = \{2, 3\}$.

**1.33** $A = \{1\}, B = \{2\}$. Then $\{A \cup B, A \cap B, A - B, B - A\}$ is the power set of $\{1, 2\}$.

**1.35** Let $U = \{1, 2, \ldots, 8\}$, $A = \{1, 2, 3, 5\}$, $B = \{1, 2, 4, 6\}$ and $C = \{1, 3, 4, 7\}$. See Figure 4.

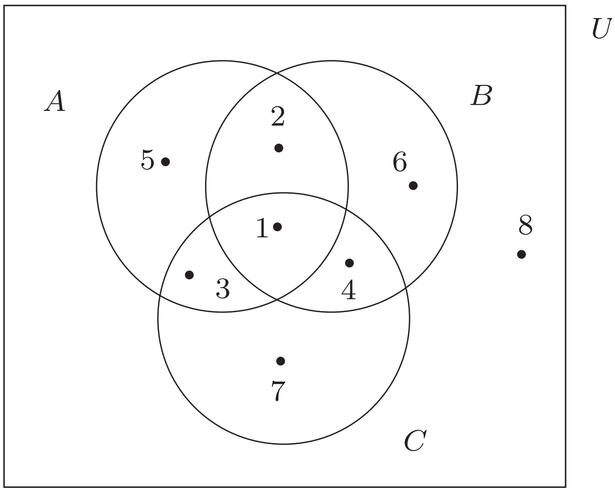


**Figure 4** Answer for Exercise 1.35

## Section 1.4: Indexed Collections of Sets

**1.37** $\bigcup_{X\in S} X = A \cup B \cup C = \{0, 1, 2, \ldots, 5\}$ and $\bigcap_{X\in S} X = A \cap B \cap C = \{2\}$.

**1.39** Since $|A| = 26$ and $|A_\alpha| = 3$ for each $\alpha \in A$, we need to have at least nine sets of cardinality 3 for their union to be $A$; that is, in order for $\bigcup_{\alpha\in S} A_\alpha = A$, we must have $|S| \geq 9$. However, if we let $S = \{a, d, g, j, m, p, s, v, y\}$, then $\bigcup_{\alpha\in S} A_\alpha = A$. Hence the smallest cardinality of a set $S$ with $\bigcup_{\alpha\in S} A_\alpha = A$ is 9.

**1.41** **(a)** $\{A_n\}_{n\in\mathbf{N}}$, where $A_n = \{x \in \mathbf{R} : 0 \leq x \leq 1/n\} = [0, 1/n]$.
**(b)** $\{A_n\}_{n\in\mathbf{N}}$, where $A_n = \{a \in \mathbf{Z} : |a| \leq n\} = \{-n, -(n-1), \ldots, (n-1), n\}$.

**1.43** $\bigcup_{r\in\mathbf{R}^+} A_r = \bigcup_{r\in\mathbf{R}^+}(-r, r) = \mathbf{R}$; $\bigcap_{r\in\mathbf{R}^+} A_r = \bigcap_{r\in\mathbf{R}^+}(-r, r) = \{0\}$.

**1.45** $\bigcup_{n\in\mathbf{N}} A_n = \bigcup_{n\in\mathbf{N}}(-\frac{1}{n}, 2 - \frac{1}{n}) = (-1, 2)$; $\bigcap_{n\in\mathbf{N}} A_n = \bigcap_{n\in\mathbf{N}}(-\frac{1}{n}, 2 - \frac{1}{n}) = [0, 1]$.

## Section 1.5: Partitions of Sets

**1.47** **(a)** $S_1$ is not a partition of $A$ since 4 belongs to no element of $S_1$.
**(b)** $S_2$ is a partition of $A$.
**(c)** $S_3$ is not a partition of $A$ because 2, for example, belongs to two elements of $S_3$.
**(d)** $S_4$ is not a partition of $A$ since $S_4$ is not a set of subsets of $A$.

**1.49** $A = \{1, 2, 3, 4\}$. $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$ and $S_2 = \{\{1, 2\}, \{3\}, \{4\}\}$.

**1.51** Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbf{Q} : x > 1\}$, $A_2 = \{x \in \mathbf{Q} : x < 1\}$ and $A_3 = \{1\}$.

**1.53** Let $S = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is positive}\}$,
$A_2 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is negative}\}$, $A_3 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is nonnegative}\}$,
$A_4 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is negative}\}$.

**1.55** $|\mathcal{P}_1| = 2$, $|\mathcal{P}_2| = 3$, $|\mathcal{P}_3| = 5$, $|\mathcal{P}_4| = 8$, $|\mathcal{P}_5| = 13$, $|\mathcal{P}_6| = 21$.

## Section 1.6: Cartesian Products of Sets

**1.57** $A \times B = \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$.

**1.59** $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$, $A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, A), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, A)\}$.

**1.61** $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$, $\mathcal{P}(B) = \{\emptyset, B\}$, $A \times B = \{(1, \emptyset), (2, \emptyset)\}$,
$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset), (\emptyset, B), (\{1\}, \emptyset), (\{1\}, B), (\{2\}, \emptyset), (\{2\}, B), (A, \emptyset), (A, B)\}$.

**1.63** $S = \{(3, 0), (2, 1), (1, 2), (0, 3), (-3, 0), (-2, 1), (-1, 2), (2, -1), (1, -2), (0, -3), (-2, -1), (-1, -2)\}$.
See Figure 5.

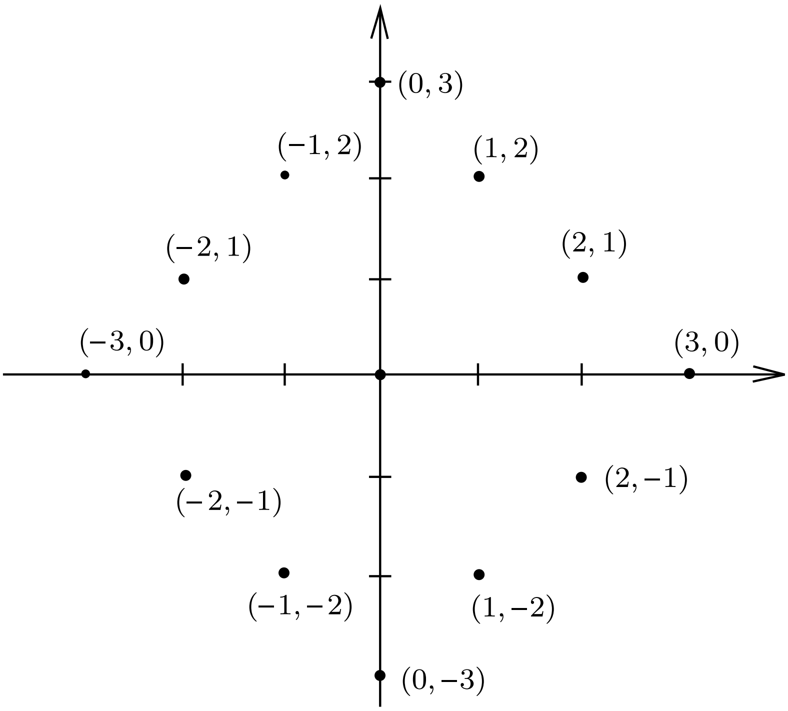


**Figure 5** Answer for Exercise 1.63

**1.65** $A \times B = [-1, 3] \times [2, 6]$, which is the set of all points on and within the square bounded by $x = -1$, $x = 3$, $y = 2$ and $y = 6$.

# EXERCISES FOR CHAPTER 2

## Section 2.1: Statements

**2.1** **(a)** A false statement **(b)** A true statement **(c)** Not a statement **(d)** Not a statement (an open sentence) **(e)** Not a statement **(f)** Not a statement (an open sentence) **(g)** Not a statement

**2.3** **(a)** False. $\emptyset$ has no elements. **(b)** True **(c)** True
**(d)** False. $\{\emptyset\}$ has $\emptyset$ as its only element. **(e)** True **(f)** False. 1 is not a set.

**2.5** **(a)** $\{x \in \mathbf{Z} : x > 2\}$ **(b)** $\{x \in \mathbf{Z} : x \leq 2\}$

**2.7** 3, 5, 11, 17, 41, 59

**2.9** $P(n) : \frac{n-1}{2}$ is even. $P(n)$ is true only for $n = 5$ and $n = 9$.

## Section 2.2: The Negation of a Statement

**2.11** **(a)** $\sqrt{2}$ is not a rational number.
**(b)** 0 is a negative integer.
**(c)** 111 is not a prime number.

**2.13** **(a)** The real number $r$ is greater than $\sqrt{2}$.
**(b)** The absolute value of the real number $a$ is at least 3.
**(c)** At most one angle of the triangle is $45^o$.
**(d)** The area of the circle is less than $9\pi$.
**(e)** The sides of the triangle have different lengths.
**(f)** The point $P$ lies on or within the circle $C$.

## Section 2.3: The Disjunction and Conjunction of Statements

**2.15** See Figure 6.

| $P$ | $Q$ | $\sim Q$ | $P \wedge (\sim Q)$ |
|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ |

**Figure 6** Answer for Exercise 2.15

**2.17** **(a)** $P \vee Q$: 15 is odd or 21 is prime. (True)
**(b)** $P \wedge Q$: 15 is odd and 21 is prime. (False)
**(c)** $(\sim P) \vee Q$: 15 is not odd or 21 is prime. (False)
**(d)** $P \wedge (\sim Q)$: 15 is odd and 21 is not prime. (True)

## Section 2.4: The Implication

**2.19** **(a)** $\sim P$: 17 is not even (or 17 is odd). (True)
**(b)** $P \vee Q$: 17 is even or 19 is prime. (True)
**(c)** $P \wedge Q$: 17 is even and 19 is prime. (False)
**(d)** $P \Rightarrow Q$: If 17 is even, then 19 is prime. (True)

**2.21** **(a)** $P \Rightarrow Q$: If $\sqrt{2}$ is rational, then 22/7 is rational. (True)
**(b)** $Q \Rightarrow P$: If 22/7 is rational, then $\sqrt{2}$ is rational. (False)
**(c)** $(\sim P) \Rightarrow (\sim Q)$: If $\sqrt{2}$ is not rational, then 22/7 is not rational. (False)
**(d)** $(\sim Q) \Rightarrow (\sim P)$: If 22/7 is not rational, then $\sqrt{2}$ is not rational. (True)
**2.23** (a), (c), (d) are true.
**2.25** **(a)** true. **(b)** false. **(c)** true. **(d)** true. **(e)** true.
**2.27** Cindy and Don attended the talk.
**2.29** Only (c) implies that $P \vee Q$ is false.

## Section 2.5: More on Implications

**2.31** **(a)** $P(x) \Rightarrow Q(x)$: If $|x| = 4$, then $x = 4$.
$P(-4) \Rightarrow Q(-4)$ is false. $P(-3) \Rightarrow Q(-3)$ is true.
$P(1) \Rightarrow Q(1)$ is true. $P(4) \Rightarrow Q(4)$ is true. $P(5) \Rightarrow Q(5)$ is true.
**(b)** $P(x) \Rightarrow Q(x)$: If $x^2 = 16$, then $|x| = 4$. True for all $x \in S$.
**(c)** $P(x) \Rightarrow Q(x)$: If $x > 3$, then $4x - 1 > 12$. True for all $x \in S$.
**2.33** **(a)** True for $(x, y) = (3, 4)$ and $(x, y) = (5, 5)$, false for $(x, y) = (1, -1)$.
**(b)** True for $(x, y) = (1, 2)$ and $(x, y) = (6, 6)$, false for $(x, y) = (2, -2)$.
**(c)** True for $(x, y) \in \{(1, -1), (-3, 4), (1, 0)\}$ and false for $(x, y) = (0, -1)$.

## Section 2.6: The Biconditional

**2.35** $P \Leftrightarrow Q$: The integer 18 is odd if and only if 25 is even. (True)
**2.37** The real number $|x - 3| < 1$ if and only if $x \in (2, 4)$.
The condition $|x - 3| < 1$ is necessary and sufficient for $x \in (2, 4)$.
**2.39** **(a)** True for all $x \in S - \{-4\}$. **(b)** True for $x \in S - \{3\}$. **(c)** True for $x \in S - \{-4, 0\}$.
**2.41** True if $n = 3$.
**2.43** $P(1) \Rightarrow Q(1)$ is false (since $P(1)$ is true and $Q(1)$ is false).
$Q(3) \Rightarrow P(3)$ is false (since $Q(3)$ is true and $P(3)$ is false).
$P(2) \Leftrightarrow Q(2)$ is true (since $P(2)$ and $Q(2)$ are both true).
**2.45** True for all $n \in S - \{11\}$.

## Section 2.7: Tautologies and Contradictions

**2.47** The compound statement $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ is a contradiction since it is false for all combinations of truth values for the component statements $P$ and $Q$. See the truth table below.

| $P$ | $Q$ | $\sim Q$ | $P \wedge Q$ | $P \wedge (\sim Q)$ | $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ |
|---|---|---|---|---|---|
| T | T | F | T | F | **F** |
| T | F | T | F | T | **F** |
| F | T | F | F | F | **F** |
| F | F | T | F | F | **F** |

**2.49** The compound statement $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology since it is true for all combinations of truth values for the component statements $P$, $Q$, and $R$. See the truth table below.

| $P$ | $Q$ | $R$ | $P \Rightarrow Q$ | $Q \Rightarrow R$ | $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ | $P \Rightarrow R$ | $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | **T** |
| T | F | T | F | T | F | T | **T** |
| F | T | T | T | T | T | T | **T** |
| F | F | T | T | T | T | T | **T** |
| T | T | F | T | F | F | F | **T** |
| T | F | F | F | T | F | F | **T** |
| F | T | F | T | F | F | T | **T** |
| F | F | F | T | T | T | T | **T** |

$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$: If $P$ implies $Q$ and $Q$ implies $R$, then $P$ implies $R$.

## Section 2.8: Logical Equivalence

**2.51** **(a)** See the truth table below.

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $P \Rightarrow Q$ | $(\sim P) \Rightarrow (\sim Q)$ |
|---|---|---|---|---|---|
| T | T | F | F | **T** | **T** |
| T | F | F | T | **F** | **T** |
| F | T | T | F | **T** | **F** |
| F | F | T | T | **T** | **T** |

Since $P \Rightarrow Q$ and $(\sim P) \Rightarrow (\sim Q)$ do not have the same truth values for all combinations of truth values for the component statements $P$ and $Q$, the compound statements $P \Rightarrow Q$ and $(\sim P) \Rightarrow (\sim Q)$ are not logically equivalent. Note that the last two columns in the truth table are not the same.

**(b)** The implication $Q \Rightarrow P$ is logically equivalent to $(\sim P) \Rightarrow (\sim Q)$.

**2.53** **(a)** The statements $P \Rightarrow Q$ and $(P \wedge Q) \Leftrightarrow P$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements $P$ and $Q$. See the truth table.

| $P$ | $Q$ | $P \Rightarrow Q$ | $P \wedge Q$ | $(P \wedge Q) \Leftrightarrow P$ |
|---|---|---|---|---|
| T | T | **T** | T | **T** |
| T | F | **F** | F | **F** |
| F | T | **T** | F | **T** |
| F | F | **T** | F | **T** |

**(b)** The statements $P \Rightarrow (Q \vee R)$ and $(\sim Q) \Rightarrow ((\sim P) \vee R)$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements $P$, $Q$ and $R$. See the truth table.

| $P$ | $Q$ | $R$ | $\sim P$ | $\sim Q$ | $Q \vee R$ | $P \Rightarrow (Q \vee R)$ | $(\sim P) \vee R$ | $(\sim Q) \Rightarrow ((\sim P) \vee R)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | **T** | T | **T** |
| T | F | T | F | T | T | **T** | T | **T** |
| F | T | T | T | F | T | **T** | T | **T** |
| F | F | T | T | T | T | **T** | T | **T** |
| T | T | F | F | F | T | **T** | F | **T** |
| T | F | F | F | T | F | **F** | F | **F** |
| F | T | F | T | F | T | **T** | T | **T** |
| F | F | F | T | T | F | **T** | T | **T** |

**2.55** The statements $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements $P$, $Q$ and $R$. See the truth table.

| $P$ | $Q$ | $R$ | $P \vee Q$ | $(P \vee Q) \Rightarrow R$ | $P \Rightarrow R$ | $Q \Rightarrow R$ | $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | **T** | T | T | **T** |
| T | F | T | T | **T** | T | T | **T** |
| F | T | T | T | **T** | T | T | **T** |
| F | F | T | F | **T** | T | T | **T** |
| T | T | F | T | **F** | F | F | **F** |
| T | F | F | T | **F** | F | T | **F** |
| F | T | F | T | **F** | T | F | **F** |
| F | F | F | F | **T** | T | T | **T** |

**2.57** Since there are only four different combinations of truth values of $P$ and $Q$ for the second and third rows of the statements $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$, at least two of these must have identical truth tables and so are logically equivalent.

## Section 2.9: Some Fundamental Properties of Logical Equivalence

**2.59** **(a)** Both $x \neq 0$ and $y \neq 0$.

**(b)** Either the integer $a$ is odd or the integer $b$ is odd.

**2.61** Either $x^2 = 2$ and $x \neq \sqrt{2}$ or $x = \sqrt{2}$ and $x^2 \neq 2$.

**2.63** If $3n + 4$ is odd, then $5n - 6$ is odd.

## Section 2.10: Quantified Statements

**2.65** $\forall x \in S, P(x)$: For every odd integer $x$, the integer $x^2 + 1$ is even.
$\exists x \in S, Q(x)$: There exists an odd integer $x$ such that $x^2$ is even.

**2.67** **(a)** There exists a set $A$ such that $A \cap \overline{A} \neq \emptyset$.
**(b)** For every set $A$, we have $\overline{A} \not\subseteq A$.

**2.69** **(a)** False, since $P(1)$ is false. **(b)** True, for example, $P(3)$ is true.

**2.71** **(a)** $\exists a, b \in \mathbf{Z}, ab < 0$ and $a + b > 0$.
**(b)** $\forall x, y \in \mathbf{R}, x \neq y$ implies that $x^2 + y^2 > 0$.
**(c)** For all integers $a$ and $b$ either $ab \geq 0$ or $a + b \leq 0$.
There exist real numbers $x$ and $y$ such that $x \neq y$ and $x^2 + y^2 \leq 0$.
**(d)** $\forall a, b \in \mathbf{Z}, ab \geq 0$ or $a + b \leq 0$. $\exists x, y \in \mathbf{R}, x \neq y$ and $x^2 + y^2 \leq 0$.

**2.73** (b) and (c) imply that $P(x) \Rightarrow Q(x)$ is true for all $x \in T$.

**2.75** Let $S = \{3, 5, 11\}$ and $P(s, t) : st - 2$ is prime.
**(a)** $\forall s, t \in S, P(s, t)$.
**(b)** False since $P(11, 11)$ is false.
**(c)** $\exists s, t \in S, \sim P(s, t)$.
**(d)** There exist $s, t \in S$ such that $st - 2$ is not prime.
**(e)** True since the statement in (a) is false.

**2.77** **(a)** There exists a triangle $T_1$ such that for every triangle $T_2, r(T_2) \geq r(T_1)$.
**(b)** $\forall T_1 \in A, \exists T_2 \in B, \sim P(T_1, T_2)$.
**(c)** For every triangle $T_1$, there exists a triangle $T_2$ such that $r(T_2) < r(T_1)$.

**2.79** **(a)** There exists $b \in B$ such that for every $a \in A, a - b < 0$.
**(b)** Let $b = 10$. Then $3 - 10 = -7 < 0, 5 - 10 = -5 < 0$ and $8 - 10 = -2 < 0$.

## Section 2.11: Characterizations of Statements

**2.81** An integer $n$ is odd if and only if $n^2$ is odd.

**2.83** **(a)** a characterization. **(b)** a characterization. **(c)** a characterization.
**(d)** a characterization. (Pythagorean theorem) **(e)** not a characterization. (Every positive number is the area of some rectangle.)

# EXERCISES FOR CHAPTER 3

## Section 3.1: Trivial and Vacuous Proofs

**3.1** ***Proof*** Since $x^2 - 2x + 2 = (x - 1)^2 + 1 \geq 1$, it follows that $x^2 - 2x + 2 \neq 0$ for all $x \in \mathbf{R}$. Hence the statement is true trivially. ■

**3.3** ***Proof*** Note that $\frac{r^2+1}{r} = r + \frac{1}{r}$. If $r \geq 1$, then $r + \frac{1}{r} > 1$; while if $0 < r < 1$, then $\frac{1}{r} > 1$ and so $r + \frac{1}{r} > 1$. Thus $\frac{r^2+1}{r} \leq 1$ is false for all $r \in \mathbf{Q}^+$ and so the statement is true vacuously. ■

**3.5** ***Proof*** Since $n^2 - 2n + 1 = (n - 1)^2 \geq 0$, it follows that $n^2 + 1 \geq 2n$ and so $n + \frac{1}{n} \geq 2$. Thus the statement is true vacuously. ■

**3.7** ***Proof*** Since $(x - y)^2 + (x - z)^2 + (y - z)^2 \geq 0$, it follows that $2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz \geq 0$ and so $x^2 + y^2 + z^2 \geq xy + xz + yz$. Thus, the statement is true vacuously. ■

## Section 3.2: Direct Proofs

**3.9** ***Proof*** Let $x$ be an even integer. Then $x = 2a$ for some integer $a$. Thus

$$5x - 3 = 5(2a) - 3 = 10a - 4 + 1 = 2(5a - 2) + 1.$$

Since $5a - 2$ is an integer, $5x - 3$ is odd. ■