Problem 4 Hints So for this problem just assume that $X = \mathbb{R}^n$. Then $a \cdot x = a_1 x_1 + \cdots + a_n x_n$. Given two vectors $x, y \in \mathbb{R}^n$ and scalars $\alpha, \beta \in \mathbb{R}$ we have

$$a \cdot (\alpha x + \beta y) = a \cdot (\alpha x) + a \cdot (\beta y)$$
$$= \alpha(a \cdot x) + \beta(a \cdot y)$$

therefore its a linear map.

For part (ii), this is basically the proof that linear transformations between real vector spaces can be represented by real valued matrices. In this case the matrices will all be $n \times 1$. Consider a function $f \in X^*$. By definition f is a linear map from $X = \mathbb{R}^n$ to $\mathbb{R}$. Because X is a linear subspace and $\mathbb{R}$ is a linear subspace and f is a linear map we know that $f(X)$ is a linear subspace in $\mathbb{R}$. Because X and $f(X)$ are subspaces they have a basis. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be the coordinate vectors for $\mathbb{R}^n$ (which are a basis for X). Then for any $x \in X$ we can write $\mathbf{x} = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n$ (a linear combination of the basis vectors). Applying our function f to x gives us

$$f(\mathbf{x}) = f(x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n)$$
$$= x_1 f(\mathbf{e}_1) + \cdots + x_n f(\mathbf{e}_n)$$

Since $\mathbf{e}_i$, for $i = 1, \ldots, n$ is a vector we know $f(\mathbf{e}_i) \in \mathbb{R}$. For all $i = 1, \ldots, n$ define $f(\mathbf{e}_i) = a_i$. Then

$$f(\mathbf{x}) = x_1 a_1 + \cdots + x_n a_n$$
$$= a_1 x_n + \cdots + a_n x_n$$
$$= \mathbf{a} \cdot \mathbf{x}$$

so we can define the $\mathbf{a} = (f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n))$.

For part (iii), you should use the defined operations to show that the vector space axioms hold for this set so that its a vector space. The key operations are that if f and g are linear functions in the set, then these will be our vectors. So $f + g$ is defined as $f(x) + g(x)$ where x is a vector in X. If α is a real number, then we scale the vectors in X^* as $\alpha f = \alpha f(x)$. All the axioms should be tested.

For part (iv) the idea is to use what we just learned in part (ii), that every linear functional $f \in X^*$ can be uniquely determined by a vector $\mathbf{a} = (f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n))$. We can think about finding a set of such $\mathbf{a}$ vectors. For example suppose we have a set of n vectors, say the coordinate vectors again, $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$, do each of these represent their own linear functional? Yes. Now that we are calling them functionals, lets represent the set now as $\{f_1, \ldots, f_n\}$. At this point we establish that any linear functional is just a linear combination of these n linear functionals and that the set is linearly independent. Once you've done that you will have shown that a set of n vectors is a basis for X^* and therefore $\dim(X^*) = n$.

Problem 10 Hints Define the profit function $\pi(p) = \max\{p \cdot y : y \in Y\}$. This function tells you the maximum profit over technology Y when prices are p. Lets suppose we have some fixed $p_0 \in \mathbb{R}^n$. Then what do we know about $\pi(p_0)$? Well we know that for all $y \in Y$ it must be the case that $p_0 \cdot y \leq \pi(p_0)$, right? By definition of the maximum, either y achieves the maximum and $p_0 \cdot y = \pi(p_0)$ or else it must be that $p_0 \cdot y < \pi(p_0)$. So what if we look at all the vectors $y \in \mathbb{R}^n$ (notice not just in Y) that when you take the dot product of it with p_0 it equals $\pi(p_0)$?

$$\{y \in \mathbb{R}^n : p_0 \cdot y = \pi(p_0)\}$$

You should notice from previous exercises that this is a hyperplane in $\mathbb{R}^n$ and we have associated with it two half-spaces.

$$H_{p_0}^{\leq}(\pi(p_0)) = \{y \in \mathbb{R}^n : p_0 \cdot y \leq \pi(p_0)\} \qquad H_{p_0}^{\geq}(\pi(p_0)) = \{y \in \mathbb{R}^n : p_0 \cdot y \geq \pi(p_0)\}$$

The left half-space could be referred to as the "lower" half-space and the right one as the "upper" half-space. Of course we get the hyperplane by simply taking the intersection $H_{p_0}^{\leq}(\pi(p_0)) \cap H_{p_0}^{\geq}(\pi(p_0))$.

If its the case that $Y \subseteq H_{p_0}(\pi(p_0))$ what should we be able to show? As you know from the proofs material the subset relation is an implication of the form $y \in Y \implies y \in H_{p_0}(\pi(p_0))$. As discussed we know that for any $y \in Y$ it must be the case that $p_0 \cdot y \leq \pi(p_0)$ so that rules out the "upper" half-space, right? So we focus on the "lower" half-space. Let $y \in Y$ then by definition of $\pi(p_0)$ we know that $p_0 \cdot y \leq \pi(p_0)$ and since $y \in Y \implies y \in \mathbb{R}^n$ we know that $y \in \{y \in \mathbb{R}^n : p_0 \cdot y \leq \pi(p_0)\} = H_{p_0}^{\leq}(\pi(p_0))$. Since our choice of $y \in Y$ was arbitrary, it holds for all $y \in Y$ and thus $Y \subseteq H_{p_0}^{\leq}(\pi(p_0))$.

Remember that the dot product in this case is a function mapping $\mathbb{R}^n$ into $\mathbb{R}$. Hence, $p \cdot y$ and $\pi(p)$ are both real numbers. The "normed distance" or metric distance in $\mathbb{R}$ is the absolute value, like we used in the calculus proofs on assignment 2. So we will look at the difference $|p \cdot y - \pi(p)|$ over the points in a hyperplane and in Y. As previously discussed we know that it is always true for elements of Y that $p_0 \cdot y \leq \pi(p_0)$ implying that $p_0 \cdot y - \pi(p_0) \leq 0$. The only vectors in Y for which the difference will actually be zero are those vectors y^* that are the profit maximizing production plans. Okay so now consider again the hyperplane $\{y \in \mathbb{R}^n : p_0 \cdot y = \pi(p_0)\}$. If any point in the hyperplane is also an element of Y, then would it be a profit maximizing production plan? It would be. So saying the hyperplane "touches" Y is essentially saying they have an element in common (technically there are topological arguments that slightly loosen this, but we haven't covered those yet, so...). In our case, as long as $\exists y^* \in Y$ such that $p_0 \cdot y^* = \pi(p_0)$ that same point y^* is also in the hyperplane and $|p_0 \cdot y^* - \pi(p_0)| = 0$. As for the question about $F(y)$, just ignore it I realize know I was asking too much on this. The point here is that all points on the "boundary" of Y will satisfy $F(y) = 0$ and that the profit maximizing production plans, if any, will be on the boundary. **So what**? Well we have now established that Y is contained in $H_{p_0}^{\leq}(\pi(p_0))$ and that the hyperplane "touches" the boundary of Y and all this essentially makes the hyperplane *tangent* to the set Y and we call it a **supporting hyperplane**.

For part (iii) remember what you learned about what it means for two sets to be equal

$$A = B \iff [A \subseteq B \land B \subseteq A]$$

Now for all the previous discussion we were working with p_0, was there anything special about p_0? No, we just picked it arbitrarily from the set $\mathbb{R}^n$ which means that the vector doesn't have any zeros in it – no negative or zero prices – sorry for a typo in the assignment. Anyways, since we picked p_0 arbitrarily all our previous results should hold for all elements of the set that we drew p_0 from. So for any $p \in \mathbb{R}^n$ we have a supporting hyperplane that contains Y in its "lower" half-space and "touches" Y on its boundary. Remember that for any p we can define $\pi(p) = \max\{p \cdot y : y \in Y\}$ and we could also create the hyperplane $H(p) = \{y \in \mathbb{R}^n : p \cdot y = \pi(p)\}$ which means we could also create the "lower" half space that we know will contain Y, or $H_p^{\leq}(\pi(p)) = \{y \in \mathbb{R}^n : p \cdot y \leq \pi(p)\}$ Hopefully you remember from your reading on sets that we can represent *indexed collections* of sets (set of sets) so that the set of all "lower" half-spaces could be represented as $\{H_p^{\leq}(\pi(p)) : p \in \mathbb{R}^n\}$ – which is the same as $\{A(p) : p \in \mathbb{R}^n\}$ from the problem.

Okay so we want to show the following

$$Y \subseteq \cap_{p \in \mathbb{R}^n} A(p) \text{ and } \cap_{p \in \mathbb{R}^n} A(p) \subseteq Y$$

Let $y \in Y$. Then since Y is contained in every half-space of the form $A(p)$ we know that it is in the intersection of all half-spaces of the form $A(p)$, right? why? This means that y is also in $\cap_p A(p)$.

The other direction is little more involved. One way to go about this would be proof by contradiction. Essentially, we assume, to the contrary, that $y \notin Y$. If we can show that this implies that $\exists p$ such that $p \cdot y > \pi(p)$ then this means there exists some half-space $A(p)$ that doesn't contain y, which contradicts our primary assumption.

Let $y \in \cap_p A(p)$ and to the contrary let $y \notin Y$; what do we know? Well this means that y is in every half space of the form $\{x \in \mathbb{R}^n : p \cdot x \leq \pi(p)\}$ so we know that $p \cdot y \leq \pi(p)$ for all p. Even though $y \notin Y$ we can a pick a point $z \in Y$ that is "closest" to y in the sense that z minimizes the normed distance $\|y - z\|$. Now the basic idea is that we can form a price vector, $p = y - z$ and think about value $\pi(p)$. Lets create a value $c = p \cdot z$. From this we can think of a hyperplane $H_p(c) = \{x \in \mathbb{R}^n : p \cdot x = c\}$. Because $p \cdot z = c$ and $z \in Y$ we know $z \in H_p(c)$ and that $H_p(c)$ touches Y.

Without going into all the details this ends up being the point. You end up establishing that our vector $z \in Y$ that is closest to $y \notin Y$ ends up achieving the maximum $\pi(p)$ and that for all other vectors $v \in Y$, we have $p \cdot v \leq \pi(p)$ so that Y is contained by the "lower" half-space of the hyperplane we created. However, we find the following holds for the vector $y \in Y$.

$$\begin{aligned} p \cdot y - c &= p \cdot y - p \cdot z \\ &= p \cdot (y - z) \\ &= (y - z) \cdot (y - z) \\ &= \|y - z\|^2 \\ &> 0 \text{ since } y \neq z \end{aligned}$$

What this implies is that $p \cdot y > \pi(p)$. So essentially what you'll end up showing, after more work than I now expect you to do, is that we created a price p with a hyperplane $\{x \in \mathbb{R}^n : p \cdot x = \pi(p)\}$ that **separates** the set Y from the point $y \notin Y$. This implies that y is not in the lower half-space and hence couldn't be in the intersection $\cap_p A(p)$ which is where we get our contradiction.

Duality The idea of duality is that given technology Y you can use the space of prices to create the profit (value) function $\pi(p)$ which tells you the profit of the profit maximizing firm with technology Y and prices p. If instead we know $\pi(p)$, but not Y, we can construct Y (or at least the convex hull of it) using our knowledge of $\pi(p)$ and the half-spaces it generates for us.

You'll encounter hyperplanes, half-spaces, convex sets, duality and so on several times in consumer theory, producer theory and general equilibrium as it plays a large role in convex optimization problems