Assignment 3

WSU PhD Economic Math Bootcamp 2016

Problem 1. If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Problem 2. For the function $f(x,y) = 3xy^4 + x^3y^4$; find the partial derivatives f_{xxy} and f_{yyy}.

Problem 3. For $w = \frac{x}{y+2z}$ find the partial derivatives $\frac{\partial^3 w}{\partial z \partial y \partial x}$ and $\frac{\partial^3 w}{\partial x^2 \partial y}$.

Problem 4. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(x = 1, y = 1, z = 3)$.

Problem 5. Find the equation of the tangent plane to the surface $z = \sqrt{xy}$ at the point $(1, 1, 1)$.

Problem 6. Show that $f(x,y) = xe^{xy}$ is differentiable at the point $(1, 0)$. (Hint: use properties of the partial derivatives to make an argument)

Problem 7. Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$.

(i) Express the value of $\frac{\partial u}{\partial t_i}$.

(ii) Assume the partial derivative is evaluated for vectors $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{t}_0 \in \mathbb{R}^m$. Express the partial derivative in part (i) as the dot-product of two vectors and identify and discuss what those vectors are.

(iii) Note that f maps $\mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x} = (x_1(\mathbf{t}), \ldots x_n(\mathbf{t}))$, as a function of $\mathbf{t}$ maps $\mathbb{R}^m$ into $\mathbb{R}^n$. Hence, considering f as a composition it is a map from $\mathbb{R}^m$ into $\mathbb{R}$. As a function $\mathbf{t}$ express the gradient of f denoted ∇f. (The LaTeXcode for ∇ is `\nabla`)

Problem 8. Consider the following equation $F(x,y) = 0, \forall x, y$. We might be interested (*in the future you will be wink, wink, nudge, nudge, know what I mean?*) in knowing if and when there is some function $f : \mathbb{R} \to \mathbb{R}$ that implicitly defines y in terms of x for the equation $F(x,y) = 0$ such that we can re-express the equation as $F(x, f(x)) = 0$.

(i) Use the chain rule to differentiate both sides of equation $F(x, y(x)) = 0$.

(ii) Under what conditions on the partial derivatives can we solve the result of (i) for dy/dx?

(iii) Solve for dy/dx.

Problem 9. Find y' if $x^3 + y^3 - 6xy = 0$.

Problem 10. For a function $f(x, y)$, when is the directional derivative of f at (x_0, y_0) the same as the partial derivatives of f at (x_0, y_0)?

Problem 11. Given the function $f(\mathbf{x})$ for all $x \in \mathbb{R}^n$ under what conditions will $\nabla f(\mathbf{c}) \cdot \mathbf{h}$ be the directional derivative of f in the direction of $\mathbf{h}$?

Problem 12. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined as $z = f(p, q, r)$. Derive the total differential about the point $\mathbf{c} \in \mathbb{R}^3$ and show that the gradient ∇f evaluated at $\mathbf{c}$ is a linear functional of the form $\mathbf{a} \cdot \mathbf{x}$ over the set $\mathbb{R}^3$. What about when ∇f isn't evaluated at a given point? Is it still a mapping $\mathbb{R}^3 \mapsto \mathbb{R}$?

Problem 13. Consider three functions f_1, f_2, f_3 such that for each $i = 1, 2, 3$, $f_i : \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{aligned} z_1 &= f_1(p, q, r) \\ z_2 &= f_2(p, q, r) \\ z_3 &= f_3(p, q, r) \end{aligned}$$

Can you derive the total differential for this "system" of functions at a point $\mathbf{c} \in \mathbb{R}^3$?

Problem 14. Let $C = \mathbb{R}^n$ be the space of consumption bundles and $u : C \to \mathbb{R}$ be a continuously differentiable utility function representing a rational preference relation $\succsim$. If $u_0 \in \mathbb{R}$ we define $L = \{\mathbf{x} \in C : u(\mathbf{x}) = u_0\}$ as an *indifference curve* (level set). Let $\mathbf{r}(t) = \mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ such that $\forall t, \mathbf{r}(t) \in L$. This means that for all t, $\mathbf{r}(t)$ is a point on the indifference curve u_0. Furthermore we call $\mathbf{r}'(t) = (dx_1/dt, \ldots dx_n/dt)$ evaluated at some t_0 the *tangent vector* to the curve at the point t_0.

Show that for any point $\mathbf{x}_0 \in L$, the gradient ∇u at $\mathbf{x}_0$ and the tangent vector of the indifference curve at $\mathbf{x}_0$ are orthogonal, i.e, $\nabla u(\mathbf{x}_0) \cdot \mathbf{r}'(t_0) = 0$ where $\mathbf{r}(t_0) = \mathbf{x}_0$.

Problem 15. Consider the production function $F(K, L) = K^\alpha L^\beta$ and the profit maximization problem

$$\max_{K,L} \; \pi(K, L) = pF(K, L) - wL - rK$$

(i) Find the Jacobian matrix of the profit function $\pi(K, L)$.

(ii) Find the critical points of the profit function $\pi(K, L)$.

(iii) Find the Hessian matrix of the profit function $\pi(K, L)$.

(iv) Find the determinate of the Hessian matrix from part (iii).

(v) If the determinate of the Hessian matrix were evaluated at the critical points from part (ii), what would the value of the determinate tell you about those critical points?