Answers and Hints to Selected Odd-Numbered Exercises in Chapters 14-16

Chapter 14

14.1 (a) **Proof.** Let $a, b \in k\mathbf{Z}$. Then $a = kx$ and $b = ky$ for some $x, y \in \mathbf{Z}$. Note that $a + b = kx + ky = k(x + y)$ and $ab = (kx)(ky) = k(kxy)$. Since $x + y, kxy \in \mathbf{Z}$, it follows that $a + b, ab \in k\mathbf{Z}$; so the addition and multiplication defined are binary operations on $k\mathbf{Z}$. Since $k\mathbf{Z} \subseteq \mathbf{Z}$ and the binary operations in $k\mathbf{Z}$ are the same as those in $\mathbf{Z}$, properties R1, R2, R5, and R6 are automatically satisfied. Moreover, since $0 = k \cdot 0$ and $0 \in \mathbf{Z}$, it follows that $k\mathbf{Z}$ has an additive identity. To show that property R4 is also satisfied, let $a \in k\mathbf{Z}$. So $a = kx$, where $x \in \mathbf{Z}$. Then $-a = -(kx) = k(-x)$. Since $-x \in \mathbf{Z}$, it follows that $-a \in k\mathbf{Z}$. ■

14.3 (a) **Solution** We show that $(S, *, \circ)$ is not a ring. Certainly, $*$ and $\circ$ are binary operations on S. However, property R6 is not satisfied. To see this, let $a = b = c = 0$. Then $a \circ (b * c) = 0 \circ 1 = 0$ and $(a \circ b) * (a \circ c) = 0 * 0 = 1$. ♢

14.7 **Proof.** Let $a \in \mathbf{R}$. Then $a^2 = a$. Thus $(a + a)^2 = (a + a)(a + a) = a(a + a) + a(a + a) = (a^2 + a^2) + (a^2 + a^2) = (a + a) + (a + a)$. Since $(a + a)^2 = a + a$, it follows that $(a + a) + (a + a) = (a + a) + 0$. Applying the Cancellation Law of Addition (Theorem 14.10), we obtain $a + a = 0$. Therefore, $-a = a$.

14.9 (a) Since the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ belongs to S, it follows that $S \neq \emptyset$. Let $M_1, M_2 \in S$. Thus $M_1 = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$, where $a_i, b_i \in \mathbf{R}$ for $i = 1, 2$. Then $M_1 - M_2 = \begin{bmatrix} a_1 - a_2 & 0 \\ 0 & b_1 - b_2 \end{bmatrix}$ and $M_1M_2 = \begin{bmatrix} a_1a_2 & 0 \\ 0 & b_1b_2 \end{bmatrix}$ belong to S. By the Subring Test, S is a subring of $M_2(\mathbf{R})$.

14.11 **Solution** The set $2G$ of even Gaussian integers is a subring of G.

Proof. Since $0 \in 2\mathbf{Z}$, it follows that $0 = 0 + 0i \in 2G$ and so $2G \neq \emptyset$. Let $x, y \in 2G$. Then $x = a_1 + b_1 i$ and $y = a_2 + b_2 i$, where $a_i, b_i \in 2\mathbf{Z}$ for $i = 1, 2$. Then $x - y = (a_1 - a_2) + (b_1 - b_2)i$ and $xy = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$. Since $a_1 - a_2, b_1 - b_2, a_1a_2 - b_1b_2, a_1b_2 + a_2b_1 \in 2\mathbf{Z}$, it follows by the Subring Test that $2G$ is a subring of G. ■

14.13 (a) Since the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ belongs to S, it follows that $S \neq \emptyset$. Let $M_1, M_2 \in S$. Thus $M_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$, where $a_i, b_i \in \mathbf{R}$ for $1 \leq i \leq 2$. Then $M_1 - M_2 = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & 0 \end{bmatrix}$ and $M_1M_2 = \begin{bmatrix} a_1a_2 & a_1b_2 \\ 0 & 0 \end{bmatrix}$ belong to S. By the Subring Test, S is a subring of $M_2(\mathbf{R})$.

(b) Let $E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and let $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ be an arbitrary element of S. Then $EA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$. Let $C = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \in S$. Then $CE = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \neq C$.

14.15 **Proof.** First we show that $(2\mathbf{Z}, +, \circ)$ is a ring. Certainly, $2\mathbf{Z}$ is closed under addition. Let $a, b, c \in 2\mathbf{Z}$. Then $a = 2x$, $b = 2y$, and $c = 2z$, where $x, y, z \in 2\mathbf{Z}$. So $a \circ b = (2x)(2y)/2 = 2(xy)$. Since xy is an integer, $2\mathbf{Z}$ is closed under this multiplication. Because $(2\mathbf{Z}, +, \cdot)$ is a ring, where $\cdot$ is ordinary multiplication, $(2\mathbf{Z}, +, \circ)$ satisfies properties R1–R4 and the integer 0 is the zero element. Now $a \circ (b \circ c) = a \circ (bc/2) = a(bc)/4 = (ab)c/4 = (ab/2) \circ c = (a \circ b) \circ c$; so $(2\mathbf{Z}, +, \circ)$ satisfies property R5. Finally, $a \circ (b + c) = a(b + c)/2 = (ab/2) + (ac/2) = (a \circ b) + (a \circ c)$, and so $(2\mathbf{Z}, +, \circ)$ satisfies property R6. Therefore, $(2\mathbf{Z}, +, \circ)$ is a ring. Since $a \circ b = ab/2 = ba/2 = b \circ a$, the ring $(2\mathbf{Z}, +, \circ)$ is commutative. Because $a \circ 2 = (a \cdot 2)/2 = a$ and $2 \in 2\mathbf{Z}$, the integer 2 is a unity for $(2\mathbf{Z}, +, \circ)$. Next, suppose that $a \circ b = 0$, where $a, b \in \mathbf{Z}$. Then $ab/2 = 0$ and so $ab = 0$, implying that $a = 0$ or $b = 0$. Therefore, $(2\mathbf{Z}, +, \circ)$ is an integral domain. ∎

14.19 Hint: Consider the following rings R and subrings S:

(a) $R = M_2(\mathbf{R})$; $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbf{R} \right\}$.

(b) $R = \mathbf{R}[x]$; $S = \{f \in \mathbf{R}[x] : f$ is a constant function$\}$.

(c) $R = \mathbf{Q} \times \mathbf{Z}$; $S = \mathbf{Q} \times \{0\}$.

14.21 Hint: First show that $\mathbf{Q}[i]$ is a subring of $\mathcal{C}$. Then show that every nonzero element of $\mathbf{Q}[i]$ is a unit.

14.23 (a) $\mathbf{Z}_n$ $(n \geq 2)$

(b) $\mathbf{Z}$

(c) $M_2(\mathbf{Z}_2)$

(d) $M_2(\mathbf{R})$

14.25 (3) occurs. Now explain your answer with justification.

Chapter 15

15.1 **Proof.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ and $\alpha, \beta \in \mathbf{R}$. Then $\mathbf{u} = a + bi$ and $\mathbf{v} = c + di$, where $a, b, c, d \in \mathbf{R}$. Then $\mathbf{u} + \mathbf{v} = (a + bi) + (c + di) = (a + c) + (b + d)i$ and $\alpha\mathbf{u} = \alpha(a + bi) = \alpha a + \alpha bi$. Since $a + c, b + d, \alpha a, \alpha b \in \mathbf{R}$, it follows that $\mathbf{u} + \mathbf{v} \in \mathcal{C}$ and $\alpha\mathbf{u} \in \mathcal{C}$. Now $\mathbf{u} + \mathbf{v} = (a + c) + (b + d)i = (c + a) + (d + b)i = \mathbf{v} + \mathbf{u}$, and property 1 is satisfied. Let $\mathbf{w} = e + fi$, where $e, f \in \mathbf{R}$. Then $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = [(a + c) + (b + d)i] + (e + fi) = [(a + c) + e] + [(b + d) + f]i = [a + (c + e)] + [b + (d + f)]i = (a + bi) + [(c + e) + (d + f)i] = (a + bi) + [(c + di) + (e + fi)] = \mathbf{u} + (\mathbf{v} + \mathbf{w})$; so property 2 is satisfied.

Let $\mathbf{z} = 0 + 0i$. Since $\mathbf{u} + \mathbf{z} = (a + bi) + (0 + 0i) = a + bi = \mathbf{u}$, property 3 is satisfied. Let $-\mathbf{u} = (-a) + (-b)i$. Then $\mathbf{u} + (-\mathbf{u}) = (a + bi) + [(-a) + (-b)i] = 0 + 0i = \mathbf{z}$, and property 4 is satisfied. Because $\alpha(\mathbf{u} + \mathbf{v}) = \alpha[(a + bi) + (c + di)] = \alpha[(a + c) + (b + d)i] = (\alpha a + \alpha c) + (\alpha b + \alpha d)i = (\alpha a + \alpha bi) + (\alpha c + \alpha di) = \alpha(a + bi) + \alpha(c + di) = \alpha\mathbf{u} + \alpha\mathbf{v}$, property 5 is satisfied. Now $(\alpha + \beta)\mathbf{u} = (\alpha + \beta)(a + bi) = (\alpha + \beta)a + (\alpha + \beta)bi = \alpha a + \beta a + \alpha bi + \beta bi = (\alpha a + \alpha bi) + (\beta a + \beta bi) = \alpha(a + bi) + \beta(a + bi) = \alpha\mathbf{u} + \beta\mathbf{u}$. Thus property 6 is satisfied. Since $(\alpha\beta)\mathbf{u} = (\alpha\beta)(a + bi) = (\alpha\beta)a + (\alpha\beta)bi = \alpha(\beta a) + \alpha(\beta bi) = \alpha(\beta a + \beta bi) = \alpha(\beta(a + bi)) = \alpha(\beta\mathbf{u})$, property 7 is satisfied. Finally, $1 \cdot \mathbf{u} = 1(a + bi) = 1 \cdot a + 1 \cdot bi = a + bi = \mathbf{u}$, and so property 8 is satisfied. ∎

15.3 (a) Since $(1,0,0)+(0,1,0)=(1,0,0)$ and $(0,1,0)+(1,0,0)=(0,1,0)$, property 1 is not satisfied and so $\mathbf{R}^3$ is not a vector space.

(c) Let $\mathbf{v}=(1,0,0)$ and let $\mathbf{z}=(a,b,c)$ be the zero vector, where $a,b,c\in\mathbf{R}$. Then $\mathbf{v}+\mathbf{z}=(0,0,0)\neq\mathbf{v}$; so property 3 is not satisfied and $\mathbf{R}^3$ is not a vector space.

(e) Let $\mathbf{v}=(1,0,0)$. Since $1\mathbf{v}=(0,0,1)\neq\mathbf{v}$, property 8 is not satisfied and $\mathbf{R}^3$ is not a vector space.

15.5 **Proof.** Observe that $\alpha(-\mathbf{v})=\alpha((-1)\mathbf{v})=(\alpha(-1))\mathbf{v}=(-\alpha)\mathbf{v}=((-1)\alpha)\mathbf{v}=(-1)(\alpha\mathbf{v})=-(\alpha\mathbf{v})$. ■

15.7 (a) The statement is false. Since $\mathbf{z}+\mathbf{z}=\mathbf{z}$, it follows that $-\mathbf{z}=\mathbf{z}$. ♢

15.9 (a) The set W_1 is a subspace of $\mathbf{R}^4$. **Proof.** Since $(0,0,0,0)\in W_1$, it follows that $W_1\neq\emptyset$. Let $\mathbf{u},\mathbf{v}\in W_1$ and $\alpha\in\mathbf{R}$. Then $\mathbf{u}=(a,a,a,a)$ and $\mathbf{v}=(b,b,b,b)$ for some $a,b\in\mathbf{R}$. Then $\mathbf{u}+\mathbf{v}=(a+b,a+b,a+b,a+b)$ and $\alpha\mathbf{u}=(\alpha a,\alpha a,\alpha a,\alpha a)$. Because $\mathbf{u}+\mathbf{v},\alpha\mathbf{u}\in W_1$, it follows that W_1 is a subspace of $\mathbf{R}^4$ by the Subspace Test. ■

(c) Since $(0,0,0,1)\in W_3$ but $2(0,0,0,1)\notin W_3$, it follows that W_1 is not closed under scalar multiplication and so W_3 is not a subspace of $\mathbf{R}^4$. ♢

15.11 (a) The set U_1 is a subspace of $\mathbf{R}[x]$. **Proof.** Since the zero function f_0 defined by $f_0(x)=0$ for all $x\in\mathbf{R}$ belongs to $\mathbf{R}[x]$, it follows that $U_1\neq\emptyset$. Let $f,g\in U_1$ and $\alpha\in\mathbf{R}$. Then there exist constants a and b such that $f(x)=a$ and $g(x)=b$ for all $x\in\mathbf{R}$. Then $(f+g)(x)=f(x)+g(x)=a+b$ and $(\alpha f)(x)=\alpha f(x)=\alpha a$. Since $f+g,\alpha f\in U_1$, it follows by the Subspace Test that U_1 is a subspace of $\mathbf{R}[x]$. ■

(b) Since the function h defined by $h(x)=x^3$ for all $x\in\mathbf{R}$ belongs to U_2, but $(0\cdot h)(x)=0\cdot h(x)=0\cdot x^3=0$ does not belong to U_2, it follows that U_2 is not closed under scalar multiplication and so U_2 is not a subspace of $\mathbf{R}[x]$. ♢

15.15 **Proof.** Since $(0,0)$, that is, $x=0$ and $y=0$, is a solution of the equation, $(0,0)\in S$ and so $S\neq\emptyset$. Let $(x_1,y_1),(x_2,y_2)\in S$ and $\alpha\in\mathbf{R}$. Then $3x_1-5y_1=0$ and $3x_2-5y_2=0$. However, $3(x_1+x_2)-5(y_1+y_2)=(3x_1-5y_1)+(3x_2-5y_2)=0$. Thus $(x_1+x_2,y_1+y_2)\in S$. Furthermore, $3(\alpha x_1)-5(\alpha y_1)=\alpha(3x_1-5y_1)=\alpha\cdot 0=0$, and so $\alpha(x_1,y_1)=(\alpha x_1,\alpha y_1)\in S$. Therefore, S is a subspace of $\mathbf{R}^2$ by the Subspace Test. ■

15.17 $i=-\frac{1}{2}\mathbf{u}_1+\frac{1}{2}\mathbf{u}_2+\frac{1}{2}\mathbf{u}_3$.

15.19 **Proof.** Let $\mathbf{v}\in W$. Thus $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$, where $c_i\in\mathbf{R}$ for $1\leq i\leq n$. Furthermore, let $\mathbf{v}_i=a_{i1}\mathbf{w}_1+a_{i2}\mathbf{w}_2+\ldots+a_{im}\mathbf{w}_m$, where $a_{ij}\in\mathbf{R}$ for $1\leq i\leq n$ and $1\leq j\leq m$. Then

$$\mathbf{v}=[c_1\ c_2\ \ldots\ c_n]\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \vdots\\ \mathbf{v}_n\end{bmatrix}=[c_1\ c_2\ \ldots\ c_n]\ A\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\\ \vdots\\ \mathbf{w}_m\end{bmatrix},$$

where $A=\begin{bmatrix}a_{11}&a_{12}&\ldots&a_{1m}\\ a_{21}&a_{22}&\ldots&a_{2m}\\ \vdots&\vdots&\vdots&\vdots\\ a_{n1}&a_{n2}&\ldots&a_{nm}\end{bmatrix}$. Hence $\mathbf{v}$ is a linear combination of $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_m$ and so $\mathbf{v}\in W'$.

15.21 **Proof.** We first show that $\langle \mathbf{u}, \mathbf{v} \rangle \subseteq \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$. Observe that $\mathbf{u} \in \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$ and $\mathbf{v} = (-2)\mathbf{u} + 1 \cdot (2\mathbf{u} + \mathbf{v}) \in \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$. By Exercise 15.19, $\langle \mathbf{u}, \mathbf{v} \rangle \subseteq \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$.

Next, we show that $\langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle \subseteq \langle \mathbf{u}, \mathbf{v} \rangle$. Since $\mathbf{u} \in \langle \mathbf{u}, \mathbf{v} \rangle$ and $2\mathbf{u}+\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, it follows that $\langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle \subseteq \langle \mathbf{u}, \mathbf{v} \rangle$, again by Exercise 15.19. ■

15.23 Hint: One possibility is to choose $\mathbf{w} = (1, 0, 0)$. Now consider $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = (0, 0, 0)$, where $a, b, c \in \mathbf{R}$.

15.25 (a) The set S is not linearly independent since $1 + (-1)\sin^2 x + (-1)\cos^2 x = 0$ for all $x \in \mathbf{R}$.

(b) The set S is linearly independent. **Proof.** Let $a, b, c \in \mathbf{R}$ such that $a \cdot 1 + b \cdot \sin x + c \cdot \cos x = 0$. We show that $a = b = c = 0$. Letting $x = 0$, $x = \pi/2$, and $x = -\pi/2$, we obtain $a + c = 0$, $a + b = 0$, and $a - b = 0$, respectively. Solving these equations simultaneously, we obtain $a = b = c = 0$. ■

15.27 Hint: Consider a proof by mathematical induction.

15.29 **Proof.** Define the mapping $T : \mathbf{R}^2 \to \mathcal{C}$ by $T(a, b) = a + bi$. Assume that $T(a, b) = T(c, d)$. Then $a + bi = c + di$, which implies that $a = c$ and $b = d$. Thus $(a, b) = (c, d)$. Hence T is one-to-one. Next let $a + bi \in \mathcal{C}$. Since $T(a, b) = a + bi$, the mapping T is onto. Therefore, T is bijective. Let $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$ be vectors in $\mathbf{R}^2$ and let $\alpha \in \mathbf{R}$. Then $T(\mathbf{u} + \mathbf{v}) = T(a + c,\ b + d) = (a + c) + (b + d)i = (a + bi) + (c + di) = T(\mathbf{u}) + T(\mathbf{v})$. Also, $T(\alpha\mathbf{u}) = T(\alpha a, \alpha b) = (\alpha a) + (\alpha b)i = \alpha(a + bi) = \alpha T(\mathbf{u})$. Since T preserves both addition and scalar multiplication, T is a linear transformation. ■

15.33 (a) $D(W) = \mathbf{R}$

(b) $D(W) = \{0\}$

(c) $\ker(T) = \mathbf{R}$.

15.35 (1) occurs.

Chapter 16

16.1 (a) property (1) is not satisfied. For example, $d(2, 1) = -1 < 0$.
property (2) is satisfied since $d(x, y) = y - x = 0$ if and only if $x = y$.
property (3) is not satisfied. For example, $d(2, 1) = -1$ and $d(1, 2) = 1$.
property (4) is satisfied since $d(x, y) + d(y, z) = (y - x) + (z - y) = z - x = d(x, z)$.

(b) Since $d(x, y) = (x - y) + (y - x) = 0$, property (1) is satisfied.
Since $d(1, 2) = 0$ and $1 \neq 2$, property (2) is not satisfied.
Since $d(x, y) = d(y, x) = 0$, property (3) is satisfied.
Since $d(x, y) + d(z, x) = d(x, z) = 0$, property (4) is satisfied.

16.3 Hint: For $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, $d'(P_1, P_2) = d(x_1, x_2) + d(y_1, y_2) \geq 0 + 0 = 0$, so (1) is satisfied. If $P_1 = P_2$, then $x_1 = x_2$ and $y_1 = y_2$. Thus $d'(P_1, P_2) = d(x_1, x_2) + d(y_1, y_2) = 0 + 0 = 0$. Conversely, if $d'(P_1, P_2) = 0$, then $d(x_1, x_2) + d(y_1, y_2) = 0$. Since $d(x_1, x_2) \geq 0$ and $d(y_1, y_2) \geq 0$, it follows that $d(x_1, x_2) = 0$ and $d(y_1, y_2) = 0$. So $x_1 = x_2$ and $y_1 = y_2$, it follows that $P_1 = P_2$, and so (2) is satisfied. Now properties (3) and (4) remain to be considered.

16.5 Hint: It is straightforward to show that properties (1)-(3) are satisfied. So only property (4) needs to be investigated. Consider $d(x, y)$ for various pairs x, y of elements of A.

16.7 (a) Let $P_1 = (1, 2)$ and $P_2 = (1, 3)$. Since $d(P_1, P_2) = 0$ and $P_1 \neq P_2$, it follows that $(\mathbf{R}^2, d)$ is not a metric space.

16.9 (a) Hint: Consider beginning a proof as follows: Let O be an open set in $(\mathbf{R}^2, d)$. To show that O is open in $(\mathbf{R}^2, d')$, we show that every point $P_0 = (x_0, y_0)$ is the center of an open sphere in $(\mathbf{R}^2, d')$ that is contained in O. Since O is open in $(\mathbf{R}^2, d)$, there exists a real number $r > 0$ such that $S_r(P_0) \subseteq O$. It remains to show that $S'_r(P_0) = \{P \in \mathbf{R}^2 : d'(P, P_0) < r\}$ is contained in $S_r(P_0)$.

16.11 (a) Hint: Consider beginning a proof as follows: Let $P_0 = (x_0, y_0) \in \mathbf{R}^2$, and let $\epsilon > 0$ be given. We show that there exists $\delta > 0$ such that if $d(P, P_0) < \delta$, where $P = (x, y)$, then $d'(f(P), f(P_0)) < \epsilon$. Notice that $d(P, P_0) = |x - x_0| + |y - y_0|$ and $d'(f(P), f(P_0)) = \left|\frac{1}{2}(x - y) - \frac{1}{2}(x_0 - y_0)\right|$.

16.13 (a) No, since $X \notin S_1$.

(b) No, since $\{a, b\} \cap \{a, c\} = \{a\} \notin S_2$.

(c) Yes.

16.17 Hint: Consider beginning a proof as follows: Observe that the result is true if $|X| \leq 1$. So we may assume that $|X| \geq 2$. Assume that (X, τ) is a metric space, say (X, d). First we show that $\{a\}$ is open for every $a \in X$. Let $r = \min\{d(x, a) : x \in X - \{a\}\}$. Since $X - \{a\}$ is finite, $r > 0$. Then $S_r(a) = \{a\}$ is open. Now complete the proof of this implication.

For the converse, assume that (X, τ) is a discrete topological space. Then $\tau = \mathcal{P}(X)$. We define the "discrete" metric d on X by $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. It remains to show that every subset of X is open in (X, d).

16.19 Hint: It is useful to prove the **Lemma**: If $O_1, O_1, \ldots, O_n$ are countable sets, where $n \in \mathbf{N}$, then $\cup_{i=1}^{n} O_i$ is countable.

16.21 Let a and b be distinct real numbers, where, say $a < b$, and let O_a and O_b be open sets containing a and b, respectively. Since $(a, \infty) \subset O_a$ and $(b, \infty) \subset O_b$, it follows that $(b, \infty) \subset O_a \cap O_b$. So $O_a \cap O_b \neq \emptyset$.

16.23 We claim that $f(1) = 1$.

Proof. Let $f(a) = 1$. Since $\{1\}$ is an open set and f is continuous, it follows that $f^{-1}(\{1\}) = \{a\}$ is open. Since $1 \in \{a\}$, it follows that $a = 1$. Thus $f(1) = 1$. ■

16.27 (3) occurs. The fact that $O_1, O_2, \ldots, O_n$ $(n \in \mathbf{N})$ are infinite sets does not imply that $\cap_{\alpha \in I} O_\alpha$ is infinite. For example, let $X = \mathbf{Z}$, $n = 2$, $O_1 = \{k \in \mathbf{Z} : k \geq 0\}$ and $O_2 = \{k \in \mathbf{Z} : k \leq 0\}$. Then O_1 and O_2 are infinite, but $O_1 \cap O_2 = \{0\}$.

16.29 (a). **Solution**: The statement is true.

Proof. Let $b \in X - \{a\}$, and let $d(b, a) = r$. Then the open sphere $S_r(b)$ is contained in $X - \{a\}$. ■

16.31 The statement is false. Now a counterexample must be found.