Chapter 16

Proofs in Topology

Recall from calculus that a function $f : X \to \mathbf{R}$, where $X \subseteq \mathbf{R}$, is **continuous** at $a \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. When we write $|x - a|$, we are referring to how far apart x and a are, that is, the distance between them. Similarly, $|f(x) - f(a)|$ is the distance between $f(x)$ and $f(a)$. It is not surprising that distance enters the picture here since when we say that f is continuous at a, we mean that if x is a number that is close to a, then $f(x)$ is close to $f(a)$. The term "close" only has meaning once we understand how we are measuring the distances between the two pairs of numbers involved. It might seem obvious that the distance between two real numbers x and y is $|x-y|$; however, it turns out that the distance between x and y need not be defined as $|x-y|$, although it is certainly the most common definition. Furthermore, when the continuity of a function $f : A \to B$ is being considered, it is not essential that A and B be sets of real numbers. That is, it is possible to place these concepts of calculus in a more general setting. The area of mathematics that deals with this is topology.

16.1 Metric Spaces

We have already mentioned that the distance between two real numbers x and y is given by $|x - y|$. There are four properties that this distance has, which will turn out to be especially interesting to us:

$$\begin{array}{ll} (1) & |x-y| \geq 0 \text{ for all } x, y \in \mathbf{R}; \\ (2) & |x-y| = 0 \text{ if and only if } x = y \text{ for all } x, y \in \mathbf{R}; \\ (3) & |x-y| = |y-x| \text{ for all } x, y \in \mathbf{R}; \\ (4) & |x-z| \leq |x-y| + |y-z| \text{ for all } x, y, z \in \mathbf{R}. \end{array} \tag{16.1}$$

Many of the fundamental results from calculus depend on these four properties. Using these properties as our guide, we now define distance in a more general manner.

Let X be a nonempty set and let $d : X \times X \to \mathbf{R}$ be a function from the Cartesian product $X \times X$ to the set $\mathbf{R}$ of real numbers. Hence for each ordered pair $(x, y) \in X \times X$, it follows that $d((x, y))$ is a real number. For simplicity, we write $d(x, y)$ rather than $d((x, y))$ and refer to $d(x, y)$ as the **distance** from x to y. The distance d is called a **metric** on X if it satisfies the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$;

(2) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in \mathbf{R}$;

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (**symmetric property**);

(4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (**triangle inequality**).

A set X together with a metric d defined on X is called a **metric space** and is denoted by (X, d). Since the set $\mathbf{R}$ of real numbers together with the distance d defined on $\mathbf{R}$ by $d(x, y) = |x - y|$ satisfies the properties listed in (16.1), it follows that $(\mathbf{R}, d)$ is a metric space.

We now consider two other ways of defining the distance between two real numbers.

Example 16.1 *For $X = \mathbf{R}$, let the distance $d : X \times X \to \mathbf{R}$ be defined by $d(x, y) = x - y$. Determine which of the four properties of a metric are satisfied by this distance.*

Solution. Since $d(1, 2) = -1$, property 1 is not satisfied. On the other hand, since $d(x, y) = x - y = 0$ if and only if $x = y$, property 2 is satisfied. Because $d(2, 1) = 1$, it follows that $d(1, 2) \neq d(2, 1)$ and so the symmetric property (property 3) is not satisfied. Finally,

$$d(x, z) = x - z = (x - y) + (y - z) = d(x, y) + d(y, z),$$

and the triangle inequality (property 4) holds. $\diamondsuit$

In our next example, we present a distance function that is actually a metric on $\mathbf{R}$.

Result 16.2 *For $X = \mathbf{R}$, let $d : X \times X \to \mathbf{R}$ be defined by*

$$d(x, y) = |2^x - 2^y|.$$

Then (X, d) is a metric space.

Proof. Clearly, $d(x, y) = |2^x - 2^y| \geq 0$ and $d(x, y) = 0$ if and only if $2^x = 2^y$. Certainly, if $x = y$, then $2^x = 2^y$. Assume next that $2^x = 2^y$. If we take logarithms to the base 2 of both 2^x and 2^y, then we have $x = y$. Thus, $d(x, y) = 0$ if and only if $x = y$. Since $d(x, y) = |2^x - 2^y| = |2^y - 2^x| = d(y, x)$, it follows that d satisfies the symmetric property. Finally, by property 4 in (16.1),

$$d(x, z) = |2^x - 2^z| = |(2^x - 2^y) + (2^y - 2^z)| \leq |2^x - 2^y| + |2^y - 2^z| = d(x, y) + d(y, z)$$

and the triangle inequality holds. $\blacksquare$

Another set on which you have undoubtedly seen a distance defined is $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$. Hence an element $P \in \mathbf{R}^2$ can be expressed as (x, y), where $x, y \in \mathbf{R}$. Here we are discussing points in the Cartesian plane, as you saw in the study of analytic geometry. There the (Euclidean) distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

This distance is actually a metric on $\mathbf{R}^2$. That the first three properties are satisfied depends only on the following facts for real numbers a and b: (1) $a^2 \geq 0$, (2) $a^2 + b^2 = 0$ if and only if $a = b = 0$, (3) $a^2 = (-a)^2$. The triangle inequality is more difficult to verify, however, and its proof depends on the following lemma, which is a special case of a result commonly called **Schwarz's Inequality**.

Lemma 16.3 *If $a, b, c, d \in \mathbf{R}$, then*

$$ab + cd \leq \sqrt{(a^2 + c^2)(b^2 + d^2)}.$$

Proof. Certainly, $(ab + cd)^2 + (ad - bc)^2 \geq (ab + cd)^2$. Since

$$\begin{aligned}
(ab + cd)^2 + (ad - bc)^2 &= (a^2b^2 + 2abcd + c^2d^2) + (a^2d^2 - 2abcd + b^2c^2) \\
&= a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 \\
&= (a^2 + c^2)(b^2 + d^2),
\end{aligned}$$

the desired inequality follows. ■

We can now show that this distance is a metric.

Result 16.4 *For $X = \mathbf{R}^2$, let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in $\mathbf{R}^2$ and let $d : X \times X \to \mathbf{R}$ be defined by*

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Then (X, d) is a metric space.

Proof. We have already mentioned that the first three properties of a metric are satisfied, so only the triangle inequality remains to be verified. Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$. Thus using Lemma 16.3, where $a = x_1 - x_2$, $b = x_2 - x_3$, $c = y_1 - y_2$, and $d = y_2 - y_3$, we have

$$\begin{aligned}
[d(P_1, P_3)]^2 &= (x_1 - x_3)^2 + (y_1 - y_3)^2 \\
&= [(x_1 - x_2) + (x_2 - x_3)]^2 + [(y_1 - y_2) + (y_2 - y_3)]^2 \\
&= (x_1 - x_2)^2 + (x_2 - x_3)^2 + 2(x_1 - x_2)(x_2 - x_3) + \\
&\quad\ 2(y_1 - y_2)(y_2 - y_3) + (y_1 - y_2)^2 + (y_2 - y_3)^2 \\
&\leq (x_1 - x_2)^2 + (x_2 - x_3)^2 + \\
&\quad\ 2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} + \\
&\quad\ (y_1 - y_2)^2 + (y_2 - y_3)^2 \\
&= \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}\right)^2 \\
&= [d(P_1, P_2) + d(P_2, P_3)]^2,
\end{aligned}$$

which gives us the desired result. ■

There is a metric defined on $\mathbf{N} \times \mathbf{N} = \mathbf{N}^2$ which goes by the name of the **Manhattan metric** or **taxicab metric**. For points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{N}^2$, the distance $d(P_1, P_2)$ is defined by

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|.$$

For example, consider the points $P_1 = (2, 2)$ and $P_2 = (4, 6)$ shown in Figure 16.1 (a). The taxicab distance between these two points is $d(P_1, P_2) = |2 - 4| + |2 - 6| = 6$. Thinking of the points (x, y) as street intersections in a certain city (Manhattan), we have a minimum of 6 blocks to travel (by taxicab). Two such routes are shown in Figure 16.1 (b), (c).

Not only is the Manhattan metric a metric on $\mathbf{N}^2$, it is also a metric on $\mathbf{Z}^2$ and on $\mathbf{R}^2$. A proof of the following result is left as an exercise (Exercise 16.2).

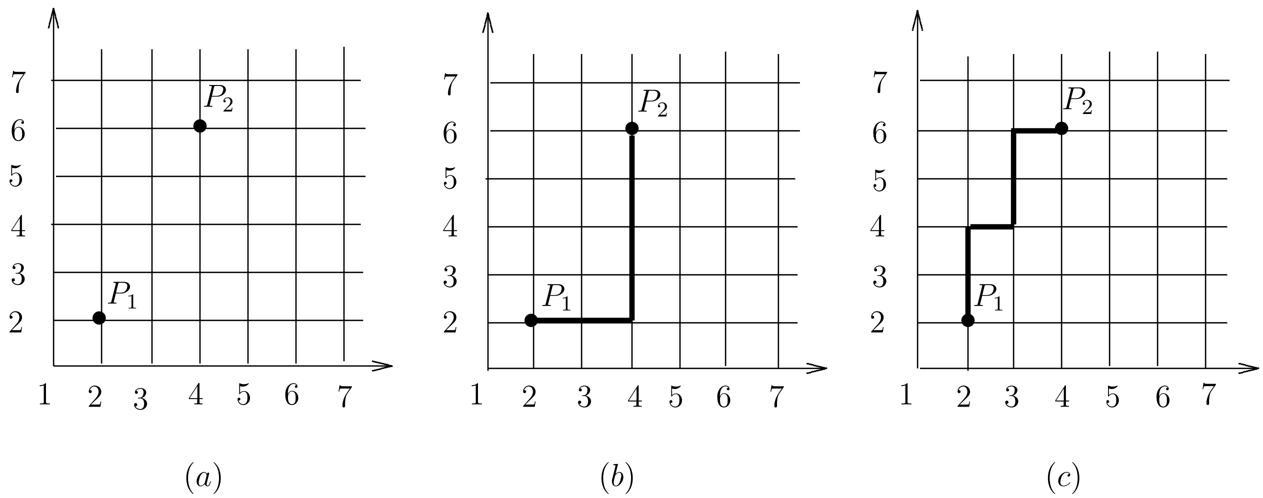

Figure 16.1: The Manhattan metric

Result 16.5 *For points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$, the distance $d(P_1, P_2)$ defined by*

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

is a metric on $\mathbf{R}^2$ (the Manhattan metric).

We have seen that there is more than one metric on both $\mathbf{R}$ and $\mathbf{R}^2$. The metric spaces $(\mathbf{R}, d)$, where $d(x, y) = |x - y|$, and $(\mathbf{R}^2, d)$, where $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, are called **Euclidean spaces** and the associated metrics are the **Euclidean metrics**. These are certainly the most familiar metrics on $\mathbf{R}$ and $\mathbf{R}^2$.

For every nonempty set A, it is always possible to define a distance $d : A \times A \to \mathbf{R}$ that is a metric. For $x, y \in A$, the distance

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is called the **discrete metric** on A.

Result 16.6 *The discrete metric d defined on a nonempty set A is a metric.*

Proof. By definition, $d(x, y) \geq 0$ for all $x, y \in A$ and $d(x, y) = 0$ if and only if $x = y$. Also, by the definition of this distance, $d(x, y) = d(y, x)$ for all $x, y \in A$. Now let $x, y, z \in A$. If $x = z$, then certainly $0 = d(x, z) \leq d(x, y) + d(y, z)$. If $x \neq z$, then $d(x, z) = 1$. Since $x \neq y$ or $y \neq z$, it follows that $d(x, y) + d(y, z) \geq 1 = d(x, z)$. In any case, the triangle inequality holds. ■

16.2 Open Sets in Metric Spaces

Returning to our discussion of a real-valued function f from calculus, we said that f is continuous at a real number a in the domain of f if for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. This, of course, is what led us to rethink what we meant by distance and which then led us to metric spaces. However, continuity itself can be described in a somewhat different manner. A function f is continuous at a if for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if x is a number in the open interval $(a - \delta, a + \delta)$, then $f(x)$ is a number in the open interval $(f(a) - \epsilon, f(a) + \epsilon)$. That is, continuity can be defined in terms of open intervals. What are some properties of open intervals? Of course, an open interval is a certain kind of subset of the set of real numbers. But each open interval has a property that can be generalized in a very useful manner. An open interval I of real numbers has the property that for every $x \in I$, there exists a real number $r > 0$ such that $(x - r, x + r) \subseteq I$, that is, for every $x \in I$, there is an open interval I_1 centered at x that is contained in I.

Let (X, d) be a metric space. Also, let $a \in X$ and let a real number $r > 0$ be given. The subset of X consisting of those points (elements) $x \in X$ such that $d(x, a) < r$ is called the **open sphere with center a and radius r** and is denoted by $S_r(a)$. Thus $x \in S_r(a)$ if and only if $d(x, a) < r$. For example, the open sphere $S_r(a)$ in the Euclidean space $(\mathbf{R}, d)$ is the open interval $(a - r, a + r)$ with mid-point a and length $2r$. Conversely, each open interval in $(\mathbf{R}, d)$ is an open sphere according to this definition. So the open spheres in $(\mathbf{R}, d)$ are precisely the open intervals of the form (a, b), where $a < b$ and $a, b \in \mathbf{R}$. In the Euclidean space $(\mathbf{R}^2, d)$, the open sphere $S_r(P)$ is the interior of the circle with center P and radius r. In the Manhattan metric space $(\mathbf{R}^2, d)$, where the distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$, the open sphere $S_3(P)$ for $P = (5, 4)$ is the interior of the square shown in Figure 16.2.

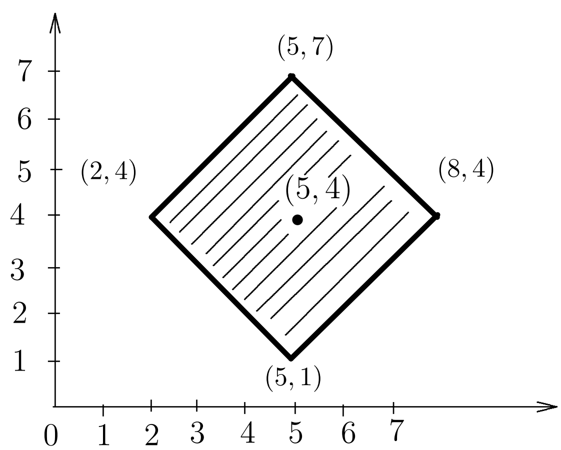

Figure 16.2: An open sphere $S_3(P)$ for $P = (5, 4)$

Since every point in a metric space (X, d) belongs to an open sphere in X (indeed, it is the center of an open sphere), it is immediate that every two distinct points of X belong to distinct open spheres. In fact, they belong to disjoint open spheres.

Theorem 16.7 *Every two distinct points in a metric space belong to disjoint open spheres.*

Proof. Let a and b be distinct points in a metric space (X, d) and suppose that $d(a, b) = r$. Necessarily, $r > 0$. Consider the open spheres $S_{\frac{r}{2}}(a)$ and $S_{\frac{r}{2}}(b)$ having radius $r/2$ centered at a and b, respectively. We claim that $S_{\frac{r}{2}}(a) \cap S_{\frac{r}{2}}(b) = \emptyset$. Assume, to the contrary, that $S_{\frac{r}{2}}(a) \cap S_{\frac{r}{2}}(b) \neq \emptyset$. Then there exists $c \in S_{\frac{r}{2}}(a) \cap S_{\frac{r}{2}}(b)$. Thus $d(c, a) < r/2$ and $d(c, b) < r/2$. By the triangle inequality, $r = d(a, b) \leq d(a, c) + d(c, b) < r/2 + r/2 = r$, which is a contradiction. ∎

A subset O of a metric space (X, d) is defined to be **open** if for every point a of O, there exists a positive real number r such that $S_r(a) \subseteq O$, that is, each point of O is the center of an open sphere contained in O. In the Euclidean space $(\mathbf{R}, d)$, each open interval (a, b), where

$a < b$, is an open set. To see this, for each $x \in (a, b)$, let $r = \min(x - a, b - x)$. Then the open sphere $S_r(x) = (x - r, x + r)$ is contained in (a, b). In fact, the set $(-\infty, a) \cup (a, \infty)$ is open in $(\mathbf{R}, d)$ for each $a \in \mathbf{R}$. On the other hand, the half-open set (or half-closed set) $(a, b]$ is not open since there exists no open sphere centered at b and contained in $(a, b]$. Similarly, the sets $[a, b]$, $[a, b)$, $(-\infty, a]$, and $[a, \infty)$ are not open in $(\mathbf{R}, d)$.

Every metric space contains some open sets, as we now show.

Theorem to Prove In a metric space (X, d),

(i) the empty set $\emptyset$ and the set X are open, and

(ii) every open sphere is an open set.

Proof Strategy To show that a subset A of X is open, it is required to show that if a is a point of A, then a is the center of an open sphere contained in A. The empty set satisfies this condition vacuously and X satisfies this condition trivially; so we concentrate on verifying (ii).

We begin with an open sphere $S_r(a)$ having center a and radius r. For an arbitrary element $x \in S_r(a)$, we need to show that there is an open sphere centered at x and with an appropriate radius that is contained in $S_r(a)$. Since the theorem concerns an arbitrary metric space (X, d), there is not necessarily any geometric appearance to the open sphere $S_r(a)$. On the other hand, it is helpful to visualize $S_r(a)$ as the interior of circle (see Figure 16.3).

Since $d(x, a) < r$, it follows that $r' = r - d(x, a)$ is a positive real number. It appears likely that $S_{r'}(x) \subseteq S_r(a)$. To show this, it remains to show that if $y \in S_{r'}(x)$, then $y \in S_r(a)$; that is, if $d(y, x) < r'$, then $d(y, a) < r$. It is natural to use the triangle inequality in an attempt to verify this. $\diamondsuit$

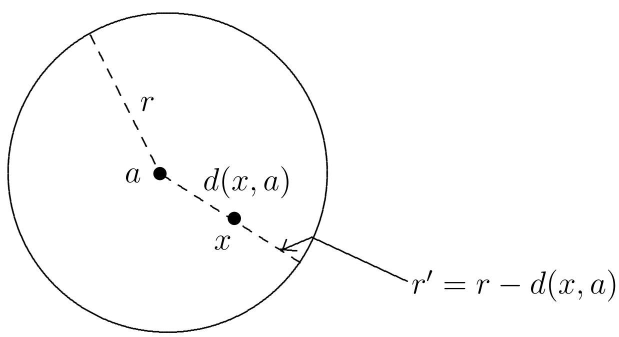

Figure 16.3: A diagram indicating an open sphere $S_r(a)$ in a metric space (X, d)

Theorem 16.8 *In a metric space* (X, d),

(i) *the empty set* $\emptyset$ *and the set* X *are open, and*

(ii) *every open sphere is an open set.*

Proof. Since there is no point in $\emptyset$, the statement that $\emptyset$ is open is true vacuously. For each point $a \in X$, every open sphere centered at a is contained in X. Thus X is open and (i) is verified.

To verify (ii), let $S_r(a)$ be an open sphere in (X, d) and let $x \in S_r(a)$. We show that there exists an open sphere centered at x and contained in $S_r(a)$. Since $d(x, a) < r$, it follows that

$r' = r - d(x,a) > 0$. We show that $S_{r'}(x) \subseteq S_r(a)$. Let $y \in S_{r'}(x)$. Since $d(y,x) < r'$ and $d(x,a) = r - r'$, it follows by the triangle inequality that

$$d(y,a) \leq d(y,x) + d(x,a) < r' + (r - r') = r.$$

Therefore, $y \in S_r(a)$ and so $S_{r'}(x) \subseteq S_r(a)$. ∎

To illustrate Theorem 16.8, we return to the metric space (X,d) described in Result 16.2, namely, $X = \mathbf{R}$ with $d(x,y) = |2^x - 2^y|$ for $x, y \in \mathbf{R}$. Thus $\emptyset$ and $X = \mathbf{R}$ are open sets as are all open spheres $S_r(a)$, where $a \in \mathbf{R}$ and $r > 0$. One such open sphere is $S_1(0) = \{x \in \mathbf{R} : |2^x - 2^0| < 1\}$. The inequality $|2^x - 2^0| < 1$ is equivalent to the inequalities $-1 < 2^x - 1 < 1$ and $0 < 2^x < 2$. Since $2^x > 0$ for all x, it follows that $0 < 2^x < 2$ is satisfied for all real numbers in the infinite interval $(-\infty, 1)$, and so $(-\infty, 1)$ is the open sphere with center 0 and radius 1 (according to the given metric). We also consider the open sphere $S_6(1) = \{x \in \mathbf{R} :\ |2^x - 2^1| < 6\}$. Here $|2^x - 2^1| < 6$ is equivalent to the inequalities $-6 < 2^x - 2 < 6$ and $-4 < 2^x < 8$ and so $S_6(1)$ is the open sphere $(-\infty, 3)$ with center 1 and radius 6.

We are now prepared to present a characterization of open sets in any metric space.

Theorem 16.9 *A subset O of a metric space is open if and only if it is a (finite or infinite) union of open spheres.*

Proof. Let (X,d) be a metric space. First let O be an open set in (X,d). We show that O is a union of open spheres. If $O = \emptyset$, then O is the union of zero open spheres. So we may assume that $O \neq \emptyset$. Let $x \in O$. Since O is open, there exists a positive number r_x such that $S_{r_x}(x) \subseteq O$. This implies that $\bigcup_{x \in O} S_{r_x}(x) \subseteq O$. On the other hand, if $x \in O$, then $x \in S_{r_x}(x) \subseteq \bigcup_{x \in O} S_{r_x}(x)$, implying that $O \subseteq \bigcup_{x \in O} S_{r_x}(x)$. Therefore, $O = \bigcup_{x \in O} S_{r_x}(x)$.

Next we show that if O is a subset of (X,d) that is a union of open spheres, then O is open. If $O = \emptyset$, then O is open. Hence we may assume that $O \neq \emptyset$. Let $x \in O$. Since O is a union of open spheres, x belongs to some open sphere, say $S_r(a)$. Since $S_r(a)$ is open, there exists $r' > 0$ (as we saw in the proof of Theorem 16.8) such that $S_{r'}(x) \subseteq S_r(a) \subseteq O$. Therefore, O is open. ∎

Two important properties of open sets are established in the next theorem.

Theorem 16.10 *Let (X,d) be a metric space. Then*

(i) *the intersection of any finite number of open sets in X is open, and*

(ii) *the union of any number of open sets in X is open.*

Proof. We first verify (i). Let $O_1, O_2, \cdots, O_k$ be k open sets in X, and let $O = \cap_{i=1}^k O_i$. If O is empty, then O is open by Theorem 16.8(i). Thus, we may assume that O is nonempty and let $x \in O$. We show that x is the center of an open sphere that is contained in O. Since $x \in O$, it follows that $x \in O_i$ for all i $(1 \leq i \leq k)$. Because each set O_i is open, there exists an open sphere $S_{r_i}(x) \subseteq O_i$, where $1 \leq i \leq k$. Let $r = \min\{r_1, r_2, \cdots, r_k\}$. Then $r > 0$ and $S_r(x) \subseteq S_{r_i}(x) \subseteq O_i$ for each i $(1 \leq i \leq k)$. Therefore, $S_r(x) \subseteq \cap_{i=1}^k O_i = O$. Thus O is open.

Next we verify (ii). Let $\{O_\alpha\}_{\alpha \in I}$ be an indexed collection of open sets in X, and let $O = \cup_{\alpha \in I} O_\alpha$. We show that O is open. If $O = \emptyset$, then again O is open. So we assume that $O \neq \emptyset$. By Theorem 16.9, each open set O_α $(\alpha \in I)$ is the union of open spheres. Thus, O is a union of open spheres. It again follows by Theorem 16.9 that O is open. ∎

For the Euclidean space $(\mathbf{R}, d)$, each open interval $I_n = \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right), n \in \mathbf{N}$, is an open set. By Theorem 16.10, $\bigcup_{n=1}^{\infty} I_n = (-2, 2)$ is an open set, as is $\bigcap_{n=1}^{100} I_n = \left(-\frac{101}{100}, \frac{101}{100}\right)$. However, Theorem 16.10 does not guarantee that $\bigcap_{n=1}^{\infty} I_n$ is open. Indeed, $\bigcap_{n=1}^{\infty} I_n$ is the closed interval $[-1, 1]$, which is not an open set. The open interval $J_n = \left(0, \frac{1}{n}\right), n \in \mathbf{N}$, is an open set as well. Thus $\bigcup_{n=1}^{\infty} J_n = (0, 1)$ is an open set. In this case, $\bigcap_{n=1}^{\infty} J_n = \emptyset$, which is also an open set.

We now turn to the Euclidean space $(\mathbf{R}^2, d)$. Let $P_0 = (0, 0)$. For $n \in \mathbf{N}$, the open sphere $S_n(P_0)$ centered at $(0, 0)$ and having radius n is an open set. Here $\bigcup_{n=1}^{\infty} S_n(P_0) = \mathbf{R}^2$, which is open; while $\bigcap_{n=1}^{\infty} S_n(P_0) = S_1(P_0)$, which is open. In $(\mathbf{R}^2, d)$, where d is the discrete metric, $S_1(P_0) = \{P_0\}$, while $S_2(P_0) = \mathbf{R}^2$. Of course, all sets are open in a discrete metric space.

There is another important class of sets in metric spaces that arise naturally from open sets. Let (X, d) be a metric space. A subset F of X is called **closed** if its complement $\overline{F}$ is open. For example, in the Euclidean space $(\mathbf{R},\ d)$, each closed interval $[a, b]$ where $a < b$, is closed since its complement $(-\infty, a) \cup (b, \infty)$ is open. Let a be a point in a metric space and let $S_r[a]$ consist of those points $x \in X$ such that $d(x, a) \leq r$. The set $S_r[a]$ is called a **closed sphere** with center a and radius r. Not surprisingly, $S_r[a]$ is closed, as we show next. Moreover, $\emptyset$ and X are both open and closed.

Theorem 16.11 *In a metric space (X, d),*

(i) $\emptyset$ and X are closed, and

(ii) every closed sphere is closed.

Proof. Since $\emptyset$ and X are complements of each other and each is open, it follows that each is closed. To verify (ii), let $S_r[a]$ be a closed sphere in (X, d), where $a \in X$. We show that its complement $\overline{S_r[a]}$ is open. We may assume that $\overline{S_r[a]}$ is nonempty and a proper subset of X. Let $x \in \overline{S_r[a]}$. Thus $d(x, a) > r$ and $r^* = d(x, a) - r > 0$. We show that $S_{r^*}(x) \subseteq \overline{S_r[a]}$, that is, if $y \in S_{r^*}(x)$, then $y \notin S_r[a]$. Let $y \in S_{r^*}(x)$. Since $d(x, y) < r^* = d(x, a) - r$, it then follows by the triangle inequality that

$$d(y, a) \geq d(x, a) - d(x, y) > d(x, a) - r^* = r$$

and so $d(y, a) > r$. Hence $y \in \overline{S_r[a]}$, which implies that $S_{r^*}(x) \subseteq \overline{S_r[a]}$. ∎

Some other useful facts about closed sets follow immediately from Theorem 16.10. First, it is useful to recall from Result 9.15 and Exercise 9.24 that if $A_1, A_2, \ldots, A_n$ are $n \geq 2$ sets, then

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i} \quad \text{and} \quad \overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}.$$

These are DeMorgan's Laws for any finite number of sets. There is a more general form of DeMorgan's Laws.

Theorem 16.12 (*Extended DeMorgan Laws*) *For an indexed collection $\{A_\alpha\}_{\alpha\in I}$ of sets,*

$$(a)\ \ \overline{\bigcup_{\alpha\in I} A_\alpha} = \bigcap_{\alpha\in I} \overline{A_\alpha} \qquad \textit{and} \qquad (b)\ \ \overline{\bigcap_{\alpha\in I} A_\alpha} = \bigcup_{\alpha\in I} \overline{A_\alpha}.$$

We present the proof of (a) only, leaving the proof of (b) as an exercise (Exercise 16.14).

Proof of Theorem 16.12 (a). First we show that $\overline{\bigcup_{\alpha\in I} A_\alpha} \subseteq \bigcap_{\alpha\in I} \overline{A_\alpha}$. Let $x \in \overline{\bigcup_{\alpha\in I} A_\alpha}$. Then $x \notin \bigcup_{\alpha\in I} A_\alpha$. Hence $x \notin A_\alpha$ for each $\alpha \in I$, which implies that $x \in \overline{A_\alpha}$ for all $\alpha \in I$. Consequently, $x \in \bigcap_{\alpha\in I} \overline{A_\alpha}$ and so $\overline{\bigcup_{\alpha\in I} A_\alpha} \subseteq \bigcap_{\alpha\in I} \overline{A_\alpha}$.

Next we show that $\bigcap_{\alpha\in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha\in I} A_\alpha}$. Let $x \in \bigcap_{\alpha\in I} \overline{A_\alpha}$. Then $x \in \overline{A_\alpha}$ for each $\alpha \in I$. Thus $x \notin A_\alpha$ for all $\alpha \in I$. This implies, however, that $x \notin \bigcup_{\alpha\in I} A_\alpha$ and hence that $x \in \overline{\bigcup_{\alpha\in I} A_\alpha}$. Therefore, $\bigcap_{\alpha\in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha\in I} A_\alpha}$. ■

Corollary 16.13 *Let (X, d) be a metric space. Then*

(i) the union of any finite number of closed sets in X is closed, and

(ii) the intersection of any number of closed sets in X is closed.

Proof. Let $F_1, F_2, \cdots, F_k$ be k closed sets in X and let $F = \bigcup_{i=1}^{k} F_i$. Then $\overline{F} = \overline{\bigcup_{i=1}^{k} F_i} = \bigcap_{i=1}^{k} \overline{F_i}$. Since each set F_i $(1 \le i \le k)$ is closed, each set $\overline{F_i}$ is open. By Theorem 16.10, $\overline{F}$ is open and so F is closed. This verifies (i).

Next we verify (ii). Let $\{F_\alpha\}_{\alpha\in I}$ be an indexed collection of closed sets in X, and let $F = \bigcap_{\alpha\in I} F_\alpha$. Then $\overline{F} = \overline{\bigcap_{\alpha\in I} F_\alpha} = \bigcup_{\alpha\in I} \overline{F_\alpha}$ by Theorem 16.12. Since each set F_α $(\alpha \in I)$ is closed, each set $\overline{F_\alpha}$ is open. By Theorem 16.10, $\overline{F}$ is open and so F is closed. ■

16.3 Continuity in Metric Spaces

We have seen then in calculus that defining a function f to be continuous at a real number can be formulated in terms of distance or in terms of open intervals, each of which can be generalized. Now we generalize the concept of continuity itself.

Let (X, d) and (Y, d') be metric spaces, and let $a \in X$. A function $f : X \to Y$ is said to be **continuous at the point** a if for every positive real number ϵ, there exists a positive real number δ such that if $x \in X$ and $d(x, a) < \delta$, then $d'(f(x), f(a)) < \epsilon$. The function $f : X \to Y$ is **continuous** on X if it is continuous at each point of X. If $X = Y = \mathbf{R}$ and $d = d'$ is defined by $d(x, y) = |x - y|$ for all $x, y \in \mathbf{R}$, then we are giving the standard definition of continuity in calculus.

We now consider some examples of continuous functions in this more general setting.

Result 16.14 *Let $(\mathbf{R}^2, d)$ be the Manhattan metric space whose distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$ is defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$, and let $(\mathbf{R}, d')$ be the Euclidean space, where $d'(a, b) = |a - b|$. Then* (*i*) *the function* $f : \mathbf{R}^2 \to \mathbf{R}$ *defined by* $f((x, y)) = f(x, y) = x + y$ *is continuous.*

(*ii*) *the function* $g : \mathbf{R}^2 \to \mathbf{R}$ *defined by* $g(x, y) = d'(x, y) = |x - y|$ *is continuous.*

Proof. We first verify (*i*). Let $\epsilon > 0$ be given and let $P_0 = (x_0, y_0) \in \mathbf{R}^2$. We choose $\delta = \epsilon$. Now let $P = (x, y) \in \mathbf{R}^2$ such that $d(P, P_0) = |x - x_0| + |y - y_0| < \delta$. Then

$$\begin{aligned} d'(f(x, y), f(x_0, y_0)) &= d'(x + y, x_0 + y_0) = |(x + y) - (x_0 + y_0)| \\ &= |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| < \delta = \epsilon. \end{aligned}$$

Therefore, f is continuous.

We now verify (*ii*). Again, let $\epsilon > 0$ be given and let $P_0 = (x_0, y_0) \in \mathbf{R}^2$. For a given $\epsilon > 0$, choose $\delta = \epsilon$. Let $P = (x, y) \in \mathbf{R}^2$ such that $d(P, P_0) = |x - x_0| + |y - y_0| < \delta$. We show that

$$d'(g(P), g(P_0)) = d'(|x - y|, |x_0 - y_0|) = ||x - y| - |x_0 - y_0|| < \epsilon,$$

which is equivalent to $-\epsilon < |x - y| - |x_0 - y_0| < \epsilon$. Observe, by the triangle inequality, that

$$\begin{aligned} |x - y| - |x_0 - y_0| &= |(x - x_0) + (x_0 - y_0) + (y_0 - y)| - |x_0 - y_0| \\ &\leq |x - x_0| + |x_0 - y_0| + |y_0 - y| - |x_0 - y_0| \\ &= |x - x_0| + |y_0 - y| < \delta = \epsilon. \end{aligned}$$

Similarly, $|x_0 - y_0| - |x - y| \leq |x - x_0| + |x - y| + |y_0 - y| - |x - y| = |x - x_0| + |y_0 - y| < \epsilon$. ■

Proof Analysis Let's review how Theorem 16.14(*ii*) was proved. The main goal was to show that $||x-y|-|x_0-y_0|| < \epsilon$ given that $|x-x_0|+|y_0-y| < \delta$. Letting $a = |x-y|$ and $b = |x_0-y_0|$, we have the inequality $|a - b| < \epsilon$ to verify, which is equivalent to $-\epsilon < a - b < \epsilon$, which, in turn, is equivalent to

$$a - b < \epsilon \text{ and } b - a < \epsilon.$$

Thus one of the inequalities we wish to establish is $|x - y| - |x_0 - y_0| < \epsilon$. Since we know that $|x-x_0|+|y_0-y| < \delta$, this suggests working the expression $|x-x_0|+|y_0-y|$ into the expression $|x-y|-|x_0-y_0|$. This can be accomplished by adding and subtracting the appropriate quantities. Observe that

$$\begin{aligned} |x - y| - |x_0 - y_0| &= |(x - x_0) + (x_0 - y_0) + (y_0 - y)| - |x_0 - y_0| \\ &\leq |x - x_0| + |x_0 - y_0| + |y_0 - y| - |x_0 - y_0| \\ &= |x - x_0| + |y_0 - y| < \delta. \end{aligned}$$

This suggests choosing $\delta = \epsilon$. Of course, we must be certain that with this choice of δ, we can also show that $|x_0 - y_0| - |x - y| < \epsilon$. ♢

The function $i : \mathbf{R} \to \mathbf{R}$ defined by $i(x) = x$ for all $x \in \mathbf{R}$ is, of course, the identity function. It would probably seem that this function must surely be continuous. However, this depends on the metrics being used.

Example 16.15 *Let* $(\mathbf{R}, d)$ *be the discrete metric space and* $(\mathbf{R}, d')$ *the Euclidean space with* $d'(x, y) = |x - y|$ *for all* $x, y \in \mathbf{R}$*. Then*

(*i*) *the function* $f : (\mathbf{R}, d) \to (\mathbf{R}, d')$ *defined by* $f(x) = x$ *for all* $x \in \mathbf{R}$ *is continuous, and* *(ii) the function $g : (\mathbf{R}, d') \to (\mathbf{R}, d)$ defined by $g(x) = x$ for all $x \in \mathbf{R}$ is not continuous.*

Solution. First we verify (i). Let $a \in \mathbf{R}$ and let $\epsilon > 0$ be given. Choose $\delta = 1/2$. Let $x \in \mathbf{R}$ such that $d(x, a) < \delta = 1/2$. We show that $d'(f(x), f(a)) < \epsilon$. Since d is the discrete metric and $d(x, a) < 1/2$, it follows that $x = a$. Hence $d'(f(x), f(a)) = |f(x) - f(a)| = |x - a| = |a - a| = 0 < \epsilon$.

Next we verify (ii). Let $a \in \mathbf{R}$ and choose $\epsilon = 1/2$. Let δ be any positive real number. Let $x = a + \delta/2 \in \mathbf{R}$. Then $d'(x, a) = |x - a| = |(a + \delta/2) - a| = \delta/2 < \delta$. Since $x \neq a$, $d(g(x), g(a)) = d(x, a) = 1 > \epsilon$. Hence for $\epsilon = 1/2$, there is no $\delta > 0$ such that if $d'(x, a) < \delta$, then $d(g(x), g(a)) < \epsilon$. Therefore, g is not continuous at a. $\diamondsuit$

Continuity of functions defined from one metric space to another can also be described by means of open sets. To do this, we need additional definitions and notation. Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. If A is a subset of X, then its **image** $f(A)$ is that subset of Y defined by

$$f(A) = \{f(x) \ : \ x \in A\}.$$

Similarly, if B is a subset of Y, then its **inverse image** $f^{-1}(B)$ is defined by

$$f^{-1}(B) = \{x \in X \ : \ f(x) \in B\}.$$

To illustrate these concepts, consider a function $f : \mathbf{R} \to \mathbf{R}$, for some metric d on $\mathbf{R}$, where f is defined by $f(x) = x^2$ for all $x \in \mathbf{R}$. Then $f(x)$ is a polynomial (whose graph is a parabola). Let $A = (-1, 2]$, $B = [-2, 2]$, and $C = [0, 4]$. Then $f(A) = C$, while $f^{-1}(C) = B$.

Now let (X, d) and (Y, d') be metric spaces, let $f : X \to Y$, and let $a \in X$. Suppose that for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $d(x, a) < \delta$, then $d'(f(x), f(a)) < \epsilon$. Then f is continuous at a. Equivalently, f is continuous at a if whenever $x \in S_\delta(a)$, then $f(x) \in S_\epsilon(f(a))$. Hence f is continuous at a if for each $\epsilon > 0$, there exists $\delta > 0$ such that $f(S_\delta(a)) \subseteq S_\epsilon(f(a))$. We now present a characterization of those functions f that are continuous on the entire set X.

Theorem to Prove Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. Then f is continuous on X if and only if for each open set O in Y, the inverse image $f^{-1}(O)$ is an open set in X.

Proof Strategy Let's begin with the implication: If f is continuous on X, then for each open set O in Y, the inverse image $f^{-1}(O)$ is an open set in X. Using a direct proof, we would assume that f is continuous and that O is an open set in Y. If $f^{-1}(O) = \emptyset$, then $f^{-1}(O)$ is an open set in X; while if $f^{-1}(O) \neq \emptyset$, then we are required to show that every element $x \in f^{-1}(O)$ is the center of an open sphere contained in $f^{-1}(O)$. So let $x \in f^{-1}(O)$. Therefore, $f(x) \in O$. We know that O is open; so there is some open sphere $S_\epsilon(f(x))$ contained in O. However, f is continuous at x; so there exists $\delta > 0$ such that $f(S_\delta(x)) \subseteq S_\epsilon(f(x))$. Hence $S_\delta(x) \subseteq f^{-1}(O)$.

We also attempt a direct proof to verify the converse. We begin then by assuming that for each open set O in Y, the set $f^{-1}(O)$ is open in X. Our goal is to show that f is continuous on X. We let $a \in X$ and $\epsilon > 0$ be given. The open sphere $S_\epsilon(f(a))$ is an open set in Y. By hypothesis, $f^{-1}(S_\epsilon(f(a)))$ is an open set in X. Furthermore, $a \in f^{-1}(S_\epsilon(f(a)))$. Therefore, there exists $\delta > 0$ such that $f(S_\delta(a)) \subseteq S_\epsilon(f(a))$ and f is continuous on X. $\diamondsuit$

We now give a more concise proof.

Theorem 16.16 *Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. Then f is continuous on X if and only if for each open set O in Y, the inverse image $f^{-1}(O)$ is an open set in X.*

Proof. Assume first that f is continuous on X. Let O be an open set in Y. We show that $f^{-1}(O)$ is open in X. If $f^{-1}(O) = \emptyset$, then $f^{-1}(O)$ is open; so we may assume that $f^{-1}(O) \neq \emptyset$. Let $x \in f^{-1}(O)$. Since $x \in f^{-1}(O)$, it follows that $f(x) \in O$. Because O is open, there exists an open sphere $S_\epsilon(f(x))$ that is contained in O. Since f is continuous at x, there exists $\delta > 0$ such that $f(S_\delta(x)) \subseteq S_\epsilon(f(x)) \subseteq O$. Thus, $S_\delta(x) \subseteq f^{-1}(O)$, as desired.

For the converse, assume that for each open set O of Y, the inverse image $f^{-1}(O)$ is an open set of X. We show that f is continuous on X. Let a be an arbitrary point in X. Let $\epsilon > 0$ be given. The set $S_\epsilon(f(a))$ is open in Y and so its inverse image $f^{-1}(S_\epsilon(f(a)))$ is open in X and contains a. Then there exists $\delta > 0$ such that the open sphere $S_\delta(a) \subseteq f^{-1}(S_\epsilon(f(a)))$. Therefore, $f(S_\delta(a)) \subseteq S_\epsilon(f(a))$ and so f is continuous at a. Hence f is continuous on X. ∎

With the aid of Theorem 16.16, it can now be shown that any constant function from one metric space to another is continuous.

Result 16.17 *Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$ be a constant function, that is, $f(x) = c$ for some $c \in Y$. Then f is continuous.*

Proof. Let O be an open set in Y. Then $f^{-1}(O) = \emptyset$ if $c \notin O$; otherwise $f^{-1}(O) = X$. In any case, $f^{-1}(O)$ is open. By Theorem 16.16, f is continuous on X. ∎

16.4 Topological Spaces

In the previous section, we introduced the concept of a continuous function from one metric space to another, and the definition was formulated in terms of the metrics on the spaces involved. However, Theorem 16.16 shows that the continuity of a function on a metric space can be established in terms of open sets only, without any direct reference to metrics. This suggests the possibility of discarding metrics altogether, replacing them by open sets, and describing continuity in an even more general setting. This gives rise to another mathematical structure, called a topological space.

Let X be a nonempty set, and let τ (the Greek letter "tau") be a collection of subsets of X. Then (X, τ) is called a **topological space**, and τ itself is called a **topology** on X, if the following properties are satisfied:

(1) $X \in \tau$ and $\emptyset \in \tau$.

(2) If $O_1, O_2, \cdots, O_n \in \tau$, where $n \in \mathbf{N}$, then $\cap_{i=1}^n O_i \in \tau$.

(3) If, for an index set I, $O_\alpha \in \tau$ for each $\alpha \in I$, then $\cup_{\alpha \in I} O_\alpha \in \tau$.

In a topological space (X, τ), we refer to each element of τ as an **open set** of X. Property (1) states that X and the empty set are open. Property (2) states that the intersection of any finite number of open sets is open; while property (3) states the union of any number of open sets is open. For example, for a nonempty set X, let $\tau_1 = \{\emptyset, X\}$ and $\tau_2 = \mathcal{P}(X)$, the set of all subsets of X. Then for $i = 1, 2$, (X, τ_i) is a topological space. The topology τ_1 is called the **trivial topology** on X, while τ_2 is the **discrete topology** on X. In (X, τ_1), the only open sets are X and $\emptyset$; while in (X, τ_2), every subset of X is open.

It follows immediately from the definition of a topological space and the properties of open sets in a metric space that every metric space is a topological space. The converse is not true however. When we say that a topological space (X, τ) is a metric space, we mean that it is possible to define a metric d on X such that the set of open sets of (X, d) is τ.

Example 16.18 *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is a topological space that is not a metric space.*

Solution. To see that (X, τ) is a topological space, it suffices to observe that the union or intersection of any elements of τ also belongs to τ.

We now show that (X, τ) is not a metric space, that is, there is no way to define a metric on X such that the resulting open sets are precisely the elements of τ. We verify this by contradiction. Assume, to the contrary, that there exists a metric d such that the open sets in (X, d) are the elements of τ. Let $r = \min\{d(a, b), d(b, c)\}$. Necessarily, $r > 0$. Then

$$S_r(b) = \{x \in X \ : \ d(x, b) < r\} = \{b\},$$

which, however, does not belong to τ, a contradiction. $\diamondsuit$

We now present two other examples of topological spaces, the first of which is suggested by the preceding result.

Result 16.19 *Let X be a nonempty set. For $a \in X$, let τ consist of $\emptyset$ and each subset of X containing a. Then (X, τ) is a topological space.*

Proof. Since $a \in X$, it follows that $X \in \tau$. Furthermore, $\emptyset \in \tau$; so property (1) is satisfied. Let $O_1, O_2, \cdots, O_n$ be n elements of τ. If $O_i = \emptyset$ for some i $(1 \le i \le n)$, then $\cap_{i=1}^n O_i = \emptyset$ and so $\cap_{i=1}^n O_i \in \tau$. Otherwise, $a \in O_i$ for all i with $1 \le i \le n$. Thus $a \in \cap_{i=1}^n O_i$, implying that $\cap_{i=1}^n O_i \in \tau$. Finally, for an index set I, let $\{O_\alpha\}_{\alpha \in I}$ be a collection of elements of τ. If $O_\alpha = \emptyset$ for all $\alpha \in I$, then $\cup_{\alpha \in I} O_\alpha = \emptyset$ and so $\cup_{\alpha \in I} O_\alpha \in \tau$. Otherwise, $a \in O_\alpha$ for some $\alpha \in I$ and so $a \in \cup_{\alpha \in I} O_\alpha$. Therefore, $\cup_{\alpha \in I} O_\alpha \in \tau$. Hence (X, τ) is a topological space. ■

Our next example of a topological space uses the Extended DeMorgan Laws (Theorem 16.12).

Result to Prove Let X be a nonempty set, and let τ be the set consisting of $\emptyset$ and each subset of X whose complement is finite. Then (X, τ) is a topological space.

Proof Strategy If X is a finite set, then τ consists of all subsets of X. In this case, τ is the discrete topology on X, and (X, τ) is a topological space. Hence we need only be concerned with the case when X is infinite. We already known that $\emptyset \in \tau$. Also $\overline{X} = \emptyset$, which is finite; so $X \in \tau$ as well. So (X, τ) satisfies property (1) required of a topological space.

In order to show that (X, τ) satisfies property (2), we let $O_1, O_2, \cdots, O_n \in \tau$ for $n \in \mathbf{N}$. We are required to show that $\cap_{i=1}^n O_i \in \tau$. If any of the open sets $O_1, O_2, \cdots, O_n$ is empty, then $\cap_{i=1}^n O_i = \emptyset$ and so $\cap_{i=1}^n O_i$ belongs to τ. Hence it suffices to assume that $O_i \ne \emptyset$ for all i $(1 \le i \le n)$. It is necessary to show that $\overline{\cap_{i=1}^n O_i}$ is finite. However, $\overline{\cap_{i=1}^n O_i} = \cup_{i=1}^n \overline{O_i}$ by DeMorgan's law. Since each set $\overline{O_i}$ is finite $(1 \le i \le n)$, the union of these sets is finite as well. Therefore, $\cap_{i=1}^n O_i \in \tau$ and property (2) is satisfied.

To show that property (3) is satisfied, we begin with an indexed family $\{O_\alpha\}_{a \in I}$ of open sets in X and are required to show that $\cup_{a \in I} O_\alpha \in \tau$. We can proceed in a manner similar to the verification of property (2). $\diamondsuit$

Result 16.20 *Let X be a nonempty set, and let τ be the set consisting of $\emptyset$ and each subset of X whose complement is finite. Then (X, τ) is a topological space.*

Proof. If X is finite, then τ is the discrete topology. Hence we may assume that X is infinite. Since the complement of X is $\emptyset$, it follows that $X \in \tau$. Since $\emptyset \in \tau$ as well, (1) holds. Let $O_1, O_2, \cdots, O_n$ be n elements of τ. If $O_i = \emptyset$ for some i $(1 \le i \le n)$, then $\cap_{i=1}^n O_i = \emptyset \in \tau$. Hence we may assume that $O_i \ne \emptyset$ for all i $(1 \le i \le n)$. Then each set $\overline{O_i}$ is finite. By DeMorgan's law, $\overline{\cap_{i=1}^n O_i} = \cup_{i=1}^n \overline{O_i}$. Since $\overline{\cap_{i=1}^n O_i}$ is a finite union of finite sets, it is finite. Thus $\cap_{i=1}^n O_i \in \tau$ and so (2) is satisfied. To verify (3), let $\{O_\alpha\}_{a \in I}$ be any collection of elements of τ. Again, by DeMorgan's law,

$$\overline{\bigcup_{a \in I} O_\alpha} = \bigcap_{a \in I} \overline{O_\alpha}.$$

If $O_\alpha = \emptyset$ for all $\alpha \in I$, then $\overline{O_\alpha} = X$ and so $\overline{\bigcup_{a \in I} O_\alpha} = X$. Thus we may assume that there is some $\beta \in I$ such that $O_\beta \ne \emptyset$. Hence $\overline{O_\beta}$ is finite and $\bigcap_{a \in I} \overline{O_\alpha} \subseteq \overline{O_\beta}$. So $\bigcap_{a \in I} \overline{O_\alpha}$ is finite as well. Therefore, $\cup_{a \in I} O_\alpha \in \tau$ and (3) is satisfied. ■

We saw in Theorem 16.7 that every two distinct points in a metric space (X, d) belong to disjoint open spheres in X. Since open spheres are open sets in X, it follows that two distinct points in X belong to disjoint open sets. This is often a useful property for a topological space to have.

A topological space (X, τ) is called a **Hausdorff space** (named for the mathematician Felix Hausdorff) if for each pair a, b of distinct points of X, there exist disjoint open sets O_a and O_b of X containing a and b, respectively. The following result is a consequence of Theorem 16.7.

Corollary 16.21 *Every metric space is a Hausdorff space.*

On the other hand, not every topological space is a Hausdorff space and not every Hausdorff space is a metric space. We verify the first of these. The second of these is a deeper question in topology.

Example 16.22 *Let X be an infinite set and let τ be the set consisting of $\emptyset$ and every subset of X whose complement is finite. Then (X, τ) is a topological space that is not a Hausdorff space.*

Solution. We saw in Result 16.20 that (X, τ) is a topological space; so it remains only to show that (X, τ) is not a Hausdorff space. Let a and b be any two distinct elements of X. We claim that there do not exist two disjoint open sets, one containing a and the other b. Assume, to the contrary, that there exist (nonempty) open sets O_a and O_b containing a and b, respectively, such that $O_a \cap O_b = \emptyset$. Then, by DeMorgan's law, $\overline{O_a \cap O_b} = X = \overline{O_a} \cup \overline{O_b}$. Since X is infinite, at least one of $\overline{O_a}$ and $\overline{O_b}$ is infinite. This implies that at least one of O_a and O_b is not open, which is a contradiction. ◇

16.5 Continuity in Topological Spaces

By Theorem 16.16, if (X, d) and (Y, d') are metric spaces, then a function $f : X \to Y$ is continuous if and only if $f^{-1}(O)$ is an open set in X for each open set O in Y. Hence, instead of defining a function f to be continuous in terms of distances in the two metric spaces (as we did), we could have defined f to be continuous in terms of open sets. Since it would be meaningless to define a function from one topological space to another to be continuous in terms of distance, we have a logical alternative.

Let (X, τ) and (Y, τ') be two topological spaces. A function $f : X \to Y$ is defined to be **continuous** if $f^{-1}(O)$ is an open set in X for every open set O in Y. Let's see how this definition works in practice.

Result 16.23 *Let (X, τ) and (Y, τ') be two topological spaces.*

(i) *If τ is the discrete topology on X, then every function $f : X \to Y$ is continuous.*

(ii) *Let τ be the trivial topology on X and let $f : X \to Y$ be a surjective function. Then f is continuous if and only if τ' is the trivial topology on Y.*

Proof. First we verify (i). Let O be an open set in Y. Since $f^{-1}(O)$ is a subset of X, it follows that $f^{-1}(O)$ is an open set in X and so f is continuous.

Next we verify (ii). Assume first that τ' is the trivial topology on Y. Then Y and $\emptyset$ are the only open sets in Y. Since $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$ are open sets in X, it follows that f is continuous. For the converse, assume that τ' is a topology on Y that is not the trivial topology. Then there exists some open set O in Y distinct from Y and $\emptyset$. Since f is surjective, $f^{-1}(O)$ is distinct from X and $\emptyset$. Thus $f^{-1}(O)$ is not an open set in X, implying that f is not continuous. ■

Result 16.24 *Let (X, τ) and (Y, τ') be topological spaces.*

(i) *The identity function $i : X \to X$ (defined by $i(x) = x$ for all $x \in X$) is continuous.*

(ii) *If $g : X \to Y$ is a constant function, that is, if $g(x) = c$ for all $x \in X$, where $c \in Y$, then g is continuous.*

Proof. We first verify (i). Let O be an open set in X. Since $i^{-1}(O) = O$ is an open set in X, the function i is continuous.

Next we verify (ii). Let O be an open set in Y. If $c \in O$, then $g^{-1}(O) = X$; while if $c \notin O$, then $g^{-1}(O) = \emptyset$. In either case, $g^{-1}(O)$ is an open set in X and so g is continuous. ■

Example 16.25 *Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and let $f : X \to X$ be defined by $f(a) = b$, $f(b) = c$, and $f(c) = a$. Determine whether f is continuous.*

Solution. Since $O = \{a\}$ is an open set in X and $f^{-1}(O) = \{b\}$ is not an open set in X, the function f is not continuous. ◇

Based on the definition given of a continuous function from one metric space to another, it might appear more natural, for topological spaces (X, τ) and (Y, τ'), to define a function $f : X \to Y$ to be continuous if, for every $x \in X$ and every open set O of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteq O$. This is equivalent to our definition, as we are about to see. First, a lemma is useful.

Lemma 16.26 *Let X and Y be nonempty sets and let $f : X \to Y$ be a function. For every subset B of Y,*

$$f\left(f^{-1}(B)\right) \subseteq B.$$

Proof. Let $y \in f\left(f^{-1}(B)\right)$. Then there is $x \in f^{-1}(B)$ such that $f(x) = y$. This implies that $y \in B$. ■

Result to Prove Let (X, τ) and (Y, τ') be topological spaces. Then $f : X \to Y$ is continuous if and only if for every $x \in X$ and every open set O of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteq O$.

Proof Strategy Assume first that f is continuous. Let $x \in X$ and let O be an open set in Y containing $y = f(x)$. What we are required to do is to find an open set U of X containing x such that $f(U) \subseteq O$. There is an obvious choice for U, however, namely, $f^{-1}(O)$. An application of Lemma 16.26 will complete the proof of this implication.

Next, we consider the converse. Assume that for every $x \in X$ and every open set O of Y containing $f(x)$, there is an open set U of X containing x such that $f(U) \subseteq O$. Since our goal is to show that f is continuous, we need to show that for every open set B of Y, the set $f^{-1}(B)$ is open in X. Of course, if $f^{-1}(B) = \emptyset$, then $f^{-1}(B)$ is an open set; so we assume that $f^{-1}(B) \neq \emptyset$. If we can show that $f^{-1}(B)$ is the union of open sets, then $f^{-1}(B)$ is open. Let $x \in f^{-1}(B)$. Then $f(x) \in B$. By hypothesis, there is an open set U_x in X containing x such that $f(U_x) \subseteq B$. This implies that $f^{-1}(B)$ is a union of open sets in X. ◇

Result 16.27 *Let (X, τ) and (Y, τ') be topological spaces. Then $f : X \to Y$ is continuous if and only if for every $x \in X$ and every open set O of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteq O$.*

Proof. Assume first that f is continuous. Let $x \in X$ and let O be an open set in Y that contains $f(x)$. Since f is continuous, $f^{-1}(O)$ is an open set in X containing x. Let $U = f^{-1}(O)$. By Lemma 16.26, $f(U) = f\left(f^{-1}(O)\right) \subseteq O$.

For the converse, assume that for every $x \in X$ and every open set O of Y containing $f(x)$, there is an open set U of X containing x such that $f(U) \subseteq O$. Let B be an open set in Y. We show that $f^{-1}(B)$ is an open set in X. If $f^{-1}(B) = \emptyset$, then $f^{-1}(B)$ is open in X. So we may assume that $f^{-1}(B) \neq \emptyset$. For each $x \in f^{-1}(B)$, the set B is an open set in Y containing $f(x)$. By assumption, there is an open set U_x in X containing x such that $f(U_x) \subseteq B$. Thus $U_x \subseteq f^{-1}(B)$. However, then, $f^{-1}(B) = \bigcup_{x \in f^{-1}(B)} U_x$ and so $f^{-1}(B)$ is an open set in X as well. ■

Exercises for Chapter 16

16.1 In each of the following, a distance is defined on the set $\mathbf{R}$ of real numbers. Determine which of the four properties of a metric space are satisfied by d. Verify your answers.

(**a**) $d(x, y) = y - x$	(**b**) $d(x, y) = (x - y) + (y - x)$
(*c*) $d(x, y) = \lvert x - y\rvert + \lvert y - x\rvert$	(*d*) $d(x, y) = x^2 + y^2$
(*e*) $d(x, y) = \lvert x^2 - y^2\rvert$	(*f*) $d(x, y) = \lvert x^3 - y^3\rvert$

16.2 For points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$, the Manhattan metric $d(P_1, P_2)$ is defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$. Prove that the Manhattan metric is, in fact, a metric on $\mathbf{R}^2$.

16.3 Let (X, d) be a metric space. For two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in X^2, define $d' : X \times X \to \mathbf{R}$ by $d'(P_1, P_2) = d(x_1, x_2) + d(y_1, y_2)$. Which of the four properties of a metric space are satisfied by d'?

16.4 Let (X, d) be a metric space. For two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in X^2, define $d^* : X \times X \to \mathbf{R}$ by $d^*(P_1, P_2) = \sqrt{[d(x_1, x_2)]^2 + [d(y_1, y_2)]^2}$. Which of the four properties of a metric space are satisfied by d^*?

16.5 Let A be a set and let a and b be two distinct elements of A. A distance $d : A \times A \to \mathbf{R}$ is defined as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } \{x, y\} = \{a, b\} \\ 2 & \text{if } x \neq y \text{ and } \{x, y\} \neq \{a, b\}. \end{cases}$$

Which of the four properties of a metric space are satisfied by this distance?

16.6 Let (X, d) be a metric space.

(a) Define $d_1(x, y) = d(x, y)/[1 + d(x, y)]$. Prove that d_1 is a metric for X.

(b) Define $d_2(x, y) = \min\{1, d(x, y)\}$. Prove that d_2 is a metric for X.

16.7 In each part that follows, a distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is defined on the Cartesian product $\mathbf{R}^2$. Determine which of the four properties of a metric space is satisfied by each distance d. For those distances that are metrics, describe the associated open spheres.

(a) $d(P_1, P_2) = \min\{|x_1 - x_2|,\ |y_1 - y_2|\}$

(b) $d(P_1, P_2) = \max\{|x_1 - x_2|,\ |y_1 - y_2|\}$

(c) $d(P_1, P_2) = (|x_1 - x_2| + |y_1 - y_2|)/2$

16.8 Let $(\mathbf{R}^2, d)$ be the metric space whose distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Prove that the set $S = \{(x, y)\ :\ -1 < x < 1 \text{ and } -1 < y < 1\}$ is open in $(\mathbf{R}^2, d)$.

16.9 Let $(\mathbf{R}^2, d)$ and $(\mathbf{R}^2, d')$ be metric spaces, where for two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$, $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $d'(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$. Prove each of the following.

(a) Every open set in $(\mathbf{R}^2, d)$ is open in $(\mathbf{R}^2, d')$.

(b) Every open set in $(\mathbf{R}^2, d')$ is open in $(\mathbf{R}^2, d)$.

16.10 In the metric space $(\mathbf{R}, d)$, where $d(x, y) = |x - y|$, determine which of the following sets are open, closed, or neither and verify your answers.

(a) $(0, 1]$	(b) $[0, 1]$	(c) $(-\infty, 1]$
(d) $(0, \infty)$	(e) $(0, 2) - \{1\}$	(f) $\mathbf{Q}$
(g) $\mathbf{I}$	(h) $\{\frac{1}{n} \mid n \in \mathbf{N}\}$	(i) $\{\frac{1}{n} \mid n \in \mathbf{N}\} \cup \{0\}$

16.11 Let $(\mathbf{R}^2, d)$ be the metric space whose distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$ is defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$, and let $(\mathbf{R}, d')$ be the metric space with $d'(a, b) = |a - b|$.

Verify each of the following.

(a) The function $f : (\mathbf{R}^2, d) \to (\mathbf{R}, d')$ defined by $f(x, y) = \frac{1}{2}(x - y)$ is continuous.

(b) The function $g : (\mathbf{R}^2, d) \to (\mathbf{R}, d')$ defined by $g(x, y) = x$ is continuous.

16.12 Let $(\mathbf{R}^2, d)$ be the metric space whose distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$ is defined by $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and let d' be the discrete metric, that is,

$$d'(P_1, P_2) = \begin{cases} 0 & \text{if } P_1 = P_2 \\ 1 & \text{if } P_1 \neq P_2. \end{cases}$$

Verify each of the following.

(a) The function $f : (\mathbf{R}^2, d) \to (\mathbf{R}^2, d')$ defined by $f(x, y) = (x, y)$ is continuous.

(b) The function $g : (\mathbf{R}^2, d') \to (\mathbf{R}^2, d)$ defined by $g(x, y) = (x, y)$ is not continuous.

16.13 Let $X = \{a, b, c, d\}$. Determine which of the following collections of subsets of X are topologies on X. Verify your answers.

(a) $S_1 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$

(b) $S_2 = \{\emptyset, X, \{a, b\}, \{a, c\}\}$

(c) $S_3 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$

16.14 Prove the Extended DeMorgan Law in Theorem 16.12(b).

16.15 let X be a nonempty set and let $S \subseteq X$. Let τ consist of $\emptyset$ and each subset of X containing S. Prove that (X, τ) is a topological space.

16.16 Let (X, τ) be a topological space. Prove that if $\{x\}$ is an open set for every $x \in X$, then τ is the discrete topology.

16.17 Let (X, τ) be a topological space, where X is finite. Prove that (X, τ) is a metric space if and only if τ is the discrete topology on X.

16.18 (a) For a set X with $a \in X$, let τ consists of X together with all sets S such that $a \notin S$. Prove that (X, τ) is a topological space.

(b) State and prove a generalization of the result in (a).

16.19 Let X be a nonempty set and let τ be the set consisting of $\emptyset$ and each subset of X whose complement is countable. Prove that (X, τ) is a topological space.

16.20 Let a, b, c be three distinct elements in a Hausdorff space (X, τ). Prove that there exist pairwise disjoint open sets O_a, O_b, and O_c containing a, b, and c, respectively.

16.21 Let τ be the set consisting of $\emptyset$, $\mathbf{R}$, and each interval (a, ∞), where $a \in \mathbf{R}$. It is known that $(\mathbf{R}, \tau)$ is a topological space. (Don't attempt to prove this.) Show that $(\mathbf{R}, \tau)$ is not a Hausdorff space.

16.22 Prove that if (X, τ) is a topological space with the discrete topology, then (X, τ) is a Hausdorff space.

16.23 Let $(\mathbf{N}, \tau)$ be a topological space, where τ consists of $\emptyset$ and $\{S : S \subseteq \mathbf{N}, 1 \in S\}$, and let $f : \mathbf{N} \to \mathbf{N}$ be a continuous permutation. Determine $f(1)$.

16.24 Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Determine all continuous functions from X to X.

16.25 Let (X, τ_1), (Y, τ_2), and (Z, τ_3) be topological spaces, and let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that if f and g are continuous, then the composition $g \circ f$ is a continuous function from X to Z.

16.26 Let τ be the trivial topology on a nonempty set X. Prove that if $f : X \to X$ is continuous, then f is a constant function.

16.27 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let X be an infinite set and let τ consists of $\emptyset$ and all infinite subsets of X. Then (X, τ) is a topological space.

Proof. Since X is an infinite subset of X, it follows that $X \in \tau$. Since $\emptyset \in \tau$, property (1) of a topological space is satisfied. Let $O_1, O_2, \ldots, O_n$ be elements of τ for $n \in \mathbf{N}$. We show that $\cap_{i=1}^{n} O_i \in \tau$. If $O_i = \emptyset$ for some i with $1 \leq i \leq n$, then $\cap_{i=1}^{n} O_i = \emptyset$ and $\cap_{i=1}^{n} O_i \in \tau$. Otherwise, O_i is infinite for all i $(\leq i \leq n)$. Hence $\cap_{i=1}^{n} O_i$ is infinite and so $\cap_{i=1}^{n} O_i \in \tau$. Thus property (2) is satisfied. Next, let $\{O_\alpha\}_{\alpha \in I}$ be an indexed family of open sets. If $O_\alpha = \emptyset$ for each $\alpha \in I$, then $\cup_{\alpha \in I} O_\alpha = \emptyset$ and so $\cup_{\alpha \in I} O_\alpha \in \tau$. Otherwise, O_α is infinite for some $\alpha \in I$ and so $\cup_{\alpha \in I} O_\alpha$ is infinite. Hence $\cup_{\alpha \in I} O_\alpha \in \tau$. Therefore, τ is a topology on X. ■

16.28 Let (X, τ) and (Y, τ') be two topological spaces. According to Result 16.23(i), if τ is the discrete topology on X, then every function $f : X \to Y$ is continuous. The converse of Result 16.23(i) is stated as follows together with a "proof".

Converse of Result 16.23(i): Let (X, τ) and (Y, τ') be two topological spaces. If every function from X to Y is continuous, then τ is the discrete topology on X.

"**Proof.**" Suppose that every function $f : X \to Y$ is continuous and assume, to the contrary, that τ is not the discrete topology on X. Then there exists some subset S of X such that S is not open in X. So S is distinct from X and $\emptyset$. Let T be an open set in Y and let $a, b \in Y$ such that $a \in T$ and $b \notin T$. Define a function $f : X \to Y$ by

$$f(x) = \begin{cases} a & \text{if } x \in S \\ b & \text{if } x \notin S. \end{cases}$$

Since T is open in Y and $f^{-1}(T) = S$ is not open in X, it follows that f is not continuous, which is a contradiction. ■

(a) Is the proposed proof of the converse correct?

(b) If the answer to (a) is yes, then state Result 16.23(i) and its converse using "if and only if". If the answer to (a) is no, then revise the hypothesis of the converse so that it is true (with attached proof).

16.29 Let X be a set with at least two elements, and let $a \in X$. Prove or disprove:

(a) If (X, d) is a metric space, then $X - \{a\}$ is an open set.

(b) If (X, d) is a topological space, then $X - \{a\}$ is an open set.

16.30 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let (X, d) be a metric space. For every open set O in X such that $O \neq \emptyset$, and every element $b \in \overline{O}$, there exists an open sphere $S_r(b)$ in X such that $S_r(b)$ and O are disjoint.

Proof. Let $r = \min\{d(b, x) : x \in O\}$. Consider the open sphere $S_r(b)$. We claim that $S_r(b) \cap O = \emptyset$. Assume, to the contrary, that $S_r(b) \cap O \neq \emptyset$. Then there exists $y \in S_r(b) \cap O$. Since $y \in S_r(b)$, it follows that $d(b, y) < r$. However, since $y \in O$, this contradicts the fact that r is the minimum distance between b and an element of O. ■

16.31 Prove or disprove: Let (X, d) be a metric space. For every open set O in X such that $O \neq \emptyset$, there exist $b \in \overline{O}$ and an open sphere $S_r(b)$ in X such that $S_r(b)$ and O are disjoint.