respectively. The **weak** and **strict lower $\succsim$-contour sets** of x are defined analogously:

$$L_{\succsim}(x) := \{y \in X : x \succsim y\} \quad \text{and} \quad L_{\succ}(x) := \{y \in X : x \succ y\}.$$

4.2 Utility Representation of Complete Preference Relations

While the preference relation of an agent contains all the information that concerns her tastes, this is really not the most convenient way of summarizing this information. Maximizing a binary relation (while a well-defined matter) is a much less friendlier exercise than maximizing a (utility) function. Thus, it would be quite useful if we knew how and when one can find a real function that attaches to an alternative x a (strictly) higher value than an alternative y iff x is ranked (strictly) above y by a given preference relation. As you will surely recall, such a function is called the *utility function* of the individual who possesses this preference relation. A fundamental question in the theory of individual choice is therefore the following: What sort of preference relations can be described by means of a utility function?

We begin by formally defining what it means to "describe a preference relation by means of a utility function."

DEFINITION. Let X be a nonempty set, and $\succsim$ a preference relation on X. For any $\emptyset \neq S \subseteq X$, we say that $u \in \mathbb{R}^S$ **represents** $\succsim$ **on** S, if u is an order-preserving function, that is, if

$$x \succsim y \quad \text{if and only if} \quad u(x) \geq u(y)$$

for any $x, y \in S$. If u represents $\succsim$ on X, we simply say that u **represents** $\succsim$. If such a function exists, then $\succsim$ is said to be **representable**, and u is called a **utility function** for $\succsim$.

Thus, if u represents $\succsim$, and $u(x) > u(y)$, we understand that x is strictly preferred to y by an agent with the preference relation $\succsim$. (Notice that if u represents $\succsim$, then $\succsim$ is complete, and $u(x) > u(y)$ iff $x \succ y$, and $u(x) = u(y)$ iff $x \sim y$, for any $x, y \in X$.) It is commonplace to say in this case that the agent derives more utility from obtaining alternative x than y; hence the term *utility function*. However, one should be careful in adopting this interpretation, for a utility function that represents a preference relation $\succsim$ is *not* unique. Therefore, a utility function cannot be thought of as measuring the "util" content of an alternative. It is rather *ordinal* in the sense that if u represents $\succsim$, then so does $f \circ u$ for any strictly increasing self-map f on $\mathbb{R}$. More formally, we say that *an ordinal utility function is unique up to strictly increasing transformations.*

Proposition 7. *Let X be any nonempty set and $u \in \mathbb{R}^X$ represent the preference relation $\succsim$ on X. Then, $v \in \mathbb{R}^X$ represents $\succsim$ if, and only if, there exists a strictly increasing function $f \in \mathbb{R}^{u(X)}$ such that $v = f \circ u$.*

Exercise 21. Prove Proposition 7.

Exercise 22.[H] Let $\succsim$ be a complete preference relation on a nonempty set X, and let $\emptyset \neq B \subseteq A \subseteq X$. If $u \in [0,1]^A$ represents $\succsim$ on A and $v \in [0,1]^B$ represents $\succsim$ on B, then there exists an extension of v that represents $\succsim$ on A. True or false?

We now proceed to the analysis of preference relations that actually admit a representation by a utility function. It is instructive to begin with the trivial case in which X is a finite set. A moment's reflection will be enough to convince yourself that any complete preference relation $\succsim$ on X is representable in this case. Indeed, if $|X| < \infty$, then all we have to do is to find the set of least preferred elements of X (which exist by finiteness), say S, and assign the utility value 1 to any member of S. Next we choose the least preferred elements in $X \backslash S$, and assign the utility value 2 to any such element. Continuing this way, we eventually exhaust X (since X is finite) and hence obtain a representation of $\succsim$ as we sought. (Hidden in the argument is the Principle of Mathematical Induction, right?)

In fact, Proposition 5 brings us exceedingly close to concluding that any complete preference relation $\succsim$ on X is representable whenever X is a countable set. The only reason why we cannot conclude this immediately is because this proposition is proved for linear orders while a preference relation need not be antisymmetric (that is, its indifference classes need not be singletons). However, this is not a serious difficulty; all we have to do is to make sure we assign the same utility value to all members that belong to the same indifference class. (We have used the same trick when proving Corollary A.1, remember?)

Proposition 8. *Let X be a nonempty countable set, and $\succsim$ a complete preference relation on X. Then $\succsim$ is representable.*

Proof. Recall that $\sim$ is an equivalence relation, so the quotient set $X/_\sim$ is well-defined. Define the linear order $\succsim^*$ on $X/_\sim$ by $[x]_\sim \succsim^* [y]_\sim$ iff $x \succsim y$. (Why is $\succsim^*$ well-defined?) By Proposition 5, there exists a function $f : X/_\sim \to \mathbb{Q}$ that represents $\succsim^*$. But then $u \in \mathbb{R}^X$, defined by $u(x) := f([x]_\sim)$, represents $\succsim$. ■

This proposition also paves way towards the following interesting result that applies to a rich class of preference relations.

Proposition 9. *Let X be a nonempty set, and $\succsim$ a complete preference relation on X. If X contains a countable $\succsim$-dense subset, then $\succsim$ can be represented by a utility function $u \in [0,1]^X$.*

Proof. If $\succ = \emptyset$, then it is enough to take u as any constant function, so we may assume $\succ \neq \emptyset$ to concentrate on the nontrivial case. Assume that there is a countable $\succsim$-dense set Y in X. By Proposition 8, there exists a $w \in \mathbb{R}^Y$ such that $w(x) \geq w(y)$

iff $x \succsim y$ for any $x, y \in Y$. Clearly, the function $v \in [0,1]^Y$ defined by $v(x) := \frac{w(x)}{1+|w(x)|}$, also represents $\succsim$ on Y. (Why?)

Now take any $x \in X$ with $L_{\succ}(x) \neq \emptyset$, and define

$$\alpha_x := \sup\{v(t) : t \in L_{\succ}(x) \cap Y\}.$$

By $\succsim$-denseness of Y and boundedness of v, α_x is well-defined for any $x \in X$. (Why?) Define next the function $u \in [0,1]^X$ by

$$u(x) := \begin{cases} 1, & \text{if } U_{\succ}(x) = \emptyset \\ 0, & \text{if } L_{\succ}(x) = \emptyset \\ \alpha_x, & \text{otherwise} \end{cases}.$$

$\succ \neq \emptyset$ implies that u is well-defined. (Why?) The rest of the proof is to check that u actually represents $\succsim$ on X.[16] We leave verifying this as an exercise, but just to get you going, let's show that $x \succ y$ implies $u(x) > u(y)$ for any $x, y \in X$ with $L_{\succ}(y) \neq \emptyset$. Since Y is $\succsim$-dense in X, there must exist $z_1, z_2 \in Y$ such that $x \succ z_1 \succ z_2 \succ y$. Since $z_1 \in L_{\succ}(x) \cap Y$ and v represents $\succsim$ on Y, we have $\alpha_x \geq v(z_1) > v(z_2)$. On the other hand, since $v(z_2) > v(t)$ for all $t \in L_{\succ}(y) \cap Y$ (why?), we also have $v(z_2) \geq \alpha_y$. Combining these observations yields $u(x) > u(y)$. ■

Exercise 23. Complete the proof of Proposition 9.

The next exercise provides a generalization of Proposition 9 by offering an alternative denseness condition that is actually *necessary* and sufficient for the representability of a linear order.

**Exercise 24.*[H] Let X be a nonempty set, and $\succsim$ a linear order on X. Then, $\succsim$ is representable iff X contains a countable set Y such that, for each $x, y \in X \backslash Y$ with $x \succ y$, there exists a $z \in Y$ such that $x \succ z \succ y$.

The characterization result given in Exercise 24 can be used to identify certain preference relations that are not representable by a utility function. Here is a standard example of such a relation.

Example 1. (*The Lexicographic Preference Relation*) Consider the linear order $\succsim_{\text{lex}}$ on $\mathbb{R}^2$ defined as $x \succsim_{\text{lex}} y$ iff either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. (This relation is called the **lexicographic order**.) We write $x \succ_{\text{lex}} y$ whenever $x \succsim_{\text{lex}} y$ and $x \neq y$. For instance, $(1,3) \succ_{\text{lex}} (0,4)$ and $(1,3) \succ_{\text{lex}} (1,2)$. As you probably recall from a microeconomics class you've taken before, $\succsim_{\text{lex}}$ is not representable by a utility function. We now provide two proofs of this fact.

[16] Alternative representations may be obtained by replacing the role of α_x in this proof with $\lambda\alpha_x + (1-\lambda)\inf\{v(y) : y \in U_{\succ}(x) \cap Y\}$ for any $\lambda \in [0,1]$.

First proof. Suppose that there exists a set $Y \subset \mathbb{R}^2$ such that, for all $x, y \in \mathbb{R}^2 \backslash Y$ with $x \succ_{\text{lex}} y$, we have $x \succ_{\text{lex}} z \succ_{\text{lex}} y$ for some $z \in Y$. We shall show that Y must then be uncountable, which is enough to conclude that $\succsim_{\text{lex}}$ is not representable in view of Exercise 24. Take any real a, and pick any $z^a \in Y$ such that $(a, 1) \succ_{\text{lex}} z^a$ and $z^a \succ_{\text{lex}} (a, 0)$. Clearly, we must have $z_1^a = a$ and $z_2^a \in (0, 1)$. But then $z^a \neq z^b$ for any $a \neq b$, while $\{z^a : a \in \mathbb{R}\} \subseteq Y$. It follows that Y is not countable (Proposition 1).

The following is a more direct proof of the fact that $\succsim_{\text{lex}}$ is not representable.

Second proof. Let $u : \mathbb{R}^2 \to \mathbb{R}$ represent $\succsim_{\text{lex}}$. Then, for any $a \in \mathbb{R}$, we have $u(a, a+1) > u(a, a)$ so that $I(a) := (u(a, a), u(a, a+1))$ is a nondegenerate interval in $\mathbb{R}$. Moreover, $I(a) \cap I(b) = \emptyset$ for any $a \neq b$, for we have

$$u(b, b) > u(a, a+1) \;\text{ whenever }\; b > a,$$

and

$$u(b, b+1) < u(a, a) \;\text{ whenever }\; b < a.$$

Therefore, the map $a \mapsto I(a)$ is an injection from $\mathbb{R}$ into $\{I(a) : a \in \mathbb{R}\}$. But since $\{I(a) : a \in \mathbb{R}\}$ is countable (Proposition 3), this entails that $\mathbb{R}$ is countable (Proposition 1), a contradiction. $\square$

The class of all preference relations that do not possess a utility representation is recently characterized by Beardon et al. (2002). As the next example illustrates, this class includes non-lexicographic preferences as well.[17]

> *Exercise 25.* (Dubra-Echenique) Let X be a nonempty set, and let $\mathcal{P}(X)$ denote the class of all partitions of X. Assume that $\succsim$ is a complete preference relation on $\mathcal{P}(X)$ such that $\mathcal{A} \subset \mathcal{B}$ implies $\mathcal{B} \succ \mathcal{A}$.[18] Prove:
> (a) If $X = [0, 1]$, then $\succsim$ is not representable by a utility function.
> (b) If X is uncountable, then $\succsim$ is not representable by a utility function.

4.3 Utility Representation of Incomplete Preference Relations

As noted earlier, there is a conceptual advantage in taking a (possibly incomplete) preorder as the primitive of analysis in the theory of rational choice. Yet an incomplete preorder cannot be represented by a utility function – if it did, it would not be incomplete. Thus it seems like the analytical scope of adopting such a point of view is, per force, rather limited. Fortunately, however, it is still possible to provide a utility representation for an incomplete preference relation, provided that we suitably generalize the notion of a "utility function."

[17] For another example in an interesting economic context, see Basu and Mitra (2003).
[18] Sziplrajn's Theorem assures us that there exists such a preference relation. (Why?)

To begin with, let us note that while it is obviously not possible to find a $u \in \mathbb{R}^X$ for an incomplete preorder $\succsim$ such that $x \succsim y$ iff $u(x) \geq u(y)$ for any $x, y \in X$, there may nevertheless exist a real function u on X such that

$$x \succ y \text{ implies } u(x) > u(y) \quad \text{and} \quad x \sim y \text{ implies } u(x) = u(y)$$

for any $x, y \in X$. Among others, this approach was explored first by Richter (1966) and Peleg (1970). We shall thus refer to such a real function u as a **Richter-Peleg utility function** for $\succsim$. Recall that Sziplrajn's Theorem guarantees that $\succsim$ can be extended to a complete preference relation (Corollary A.1). If $\succsim$ has a Richter-Peleg utility function u then, and only then, $\succsim$ can be extended to a complete preference relation that is represented by u in the ordinary sense.

The following result shows that Proposition 9 can be extended to the case of incomplete preorders if one is willing to accept this particular notion of utility representation. (You should contrast the associated proofs.)

Lemma 1. (Richter) *Let X be a nonempty set, and $\succsim$ a preference relation on X. If X contains a countable $\succsim$-dense subset, then there exists a Richter-Peleg utility function for $\succsim$.*

Proof. (Peleg) We will prove the claim assuming that $\succsim$ is a partial order; the extension to the case of preorders is carried out as in the proof of Proposition 8. Obviously, if $\succ = \emptyset$, then there is nothing to prove. So let $\succ \neq \emptyset$, and assume that X contains a countable $\succsim$-dense set – let's call this set Y. Clearly, there must then exist $a, b \in Y$ such that $b \succ a$. Thus $\{(a, b)_{\succsim} : a, b \in Y \text{ and } b \succ a\}$ is a countably infinite set, where $(a, b)_{\succsim} := \{x \in X : b \succ x \succ a\}$. We enumerate this set as $\{(a_1, b_1)_{\succsim}, (a_2, b_2)_{\succsim}, ...\}$. Each $(a_i, b_i)_{\succsim} \cap Y$ is partially ordered by $\succsim$ so that, by the Hausdorff Maximal Principle, it contains a $\supseteq$-maximal loset, say $(Z_i, \succsim)$. By $\succsim$-denseness of Y, Z_i has neither a $\succsim$-maximum nor a $\succsim$-minimum. Moreover, by its maximality, it is $\succsim$-dense in itself. By Corollary 2, therefore, there exists a bijection $f_i : Z_i \to (0, 1) \cap \mathbb{Q}$ such that $x \succsim y$ iff $f_i(x) \geq f_i(y)$ for any $x, y \in Z_i$. Now define the map $\varphi_i \in [0, 1]^X$ by

$$\varphi_i(x) := \begin{cases} \sup\{f_i(t) : x \succ t \in Z_i\}, & \text{if } L_{\succ}(x) \cap Z_i \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have $\varphi_i(x) = 0$ for all $x \in L_{\succsim}(a_i)$, and $\varphi_i(x) = 1$ for all $x \in U_{\succsim}(b_i)$. Using this observation and the definition of f_i, one can show that, for any $x, y \in X$ with $x \succ y$ we have $\varphi_i(x) \geq \varphi_i(y)$. (Verify!) To complete the proof, then, define

$$u(x) := \sum_{i=1}^{\infty} \frac{\varphi_i(x)}{2^i} \quad \text{for all } x \in X.$$

(Since the range of each φ_i is contained in $[0, 1]$ and $\sum^{\infty} \frac{1}{2^i} = 1$ (Example 8.[2]), u is well-defined.) Notice that, for any $x, y \in X$ with $x \succ y$, there exists a $j \in \mathbb{N}$ with

$x \succ b_j \succ a_j \succ y$ so that $\varphi_j(x) = 1 > 0 = \varphi_j(y)$. Since $x \succ y$ implies $\varphi_i(x) \geq \varphi_i(y)$ for every i, therefore, we find that $x \succ y$ implies $u(x) > u(y)$, and the proof is complete. ■

Unfortunately, the Richter-Peleg formulation of utility representation has a serious shortcoming in that it may result in a substantial information loss. Indeed, one cannot recover the original preference relation $\succsim$ from a Richter-Peleg utility function u; the information contained in u is strictly less than that contained in $\succsim$. All we can deduce from the statement $u(x) > u(y)$ is that it is not the case that y is strictly better than x for the subject individual. We cannot tell if this agent actually likes x better than y, or that she is unable to rank x and y. (That is, we are unable to capture the "indecisiveness" of a decision maker by using a Richter-Peleg utility function.) The problem is, of course, due to the fact that the range of a real function is completely ordered while its domain is not. One way of overcoming this problem is by using a poset-valued utility function, or better, by using a *set of* real-valued utility functions in the following way.

Definition. Let X be a nonempty set, and $\succsim$ a preference relation on X. We say that the set $\mathcal{U} \subseteq \mathbb{R}^X$ **represents** $\succsim$, if

$$x \succsim y \quad \text{if and only if} \quad u(x) \geq u(y) \text{ for all } u \in \mathcal{U},$$

for any $x, y \in X$.

Here are some immediate examples.

Example 2. [**1**] Let $n \in \{2, 3, ...\}$. Obviously, we cannot represent the partial order $\geq$ on $\mathbb{R}^n$ by a single utility function, but we can represent it by a (finite) set of utility functions. Indeed, defining $u_i(x) := x_i$ for each $x \in \mathbb{R}^n$, $i = 1, ..., n$, we find

$$x \geq y \quad \text{iff} \quad u_i(x) \geq u_i(y) \;\; i = 1, ..., n.$$

for any $x, y \in \mathbb{R}^n$.

[**2**] Let $Z := \{1, ..., m\}$, $m \geq 2$ and let $\mathcal{L}_Z$ stand for the set of all probability distributions (lotteries) on Z, that is, $\mathcal{L}_Z := \{(p_1, ..., p_m) \in \mathbb{R}^m_+ : \sum^m p_i = 1\}$. Then the **first-order stochastic dominance ordering** on $\mathcal{L}_Z$, denoted by $\succcurlyeq_{\mathrm{FSD}}$, is defined as follows:

$$p \succcurlyeq_{\mathrm{FSD}} q \quad \text{iff} \quad \sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \;\; k = 1, ..., m-1.$$

(Here we interpret m as the best prize, $m-1$ as the second best prize, and so on.) The partial order $\succcurlyeq_{\mathrm{FSD}}$ on $\mathcal{L}_Z$ is represented by the set $\{u_1, ..., u_{m-1}\}$ of real functions on $\mathcal{L}_Z$, where $u_k(p) := -\sum^k p_i$, $k = 1, ..., m-1$.

[3] For any nonempty set X, the diagonal relation D_X can be represented by a set of two utility functions. This follows from Exercise 20 and Proposition 8. (*Note.* If X is countable, we do not have to use the Axiom of Choice to prove this fact.) $\square$

Exercise 26. Let $X := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \text{ and } x_1 \neq 0\}$ and define the partial order $\succsim$ on X as

$$x \succsim y \quad \text{iff} \quad x_1 y_1 > 0 \text{ and } x_2 \geq y_2.$$

Show that $\succsim$ can be represented by a set of two continuous real functions on X.

Exercise 27. Prove: If there exists a countable set of bounded utility functions that represent a preference relation, then there is a Richter-Peleg utility function for that preference relation.

Exercise 28. Prove: If there exists a countable set of utility functions that represents a complete preference relation, then this relation is representable in the ordinary sense.

Exercise 29.$^{\text{H}}$ Let $\succsim$ be a reflexive partial order on $\mathbb{R}$ such that $x \succ y$ iff $x > y + 1$ for any $x, y \in \mathbb{R}$. Is there a $\mathcal{U} \subseteq \mathbb{R}^{\mathbb{R}}$ that represents $\succsim$?

Exercise 30. Define the partial order $\succsim$ on $\mathbb{N} \times \mathbb{R}$ by $(m, x) \succsim (n, y)$ iff $m = n$ and $x \geq y$. Is there a $\mathcal{U} \subseteq \mathbb{R}^{\mathbb{N}\times\mathbb{R}}$ that represents $\succsim$?

Suppose $\succsim$ is represented by $\{u, v\} \subseteq \mathbb{R}^X$. One interpretation we can give to this situation is that the individual with the preference relation $\succsim$ is a person who deems two dimensions relevant for comparing the alternatives in X. (Think of a potential graduate student who compares the graduate programs that she is admitted in according to the amount of financial aid they provide and the reputation of their programs.) Her preferences over the first dimension are represented by the utility function u, and the second by v. She then judges the value of an alternative x on the basis of its performance on both dimensions, that is, by the 2-vector $(u(x), v(x))$, and prefers this alternative to $y \in X$ iff x performs better than y in both of the dimensions, that is, $(u(x), v(x)) \geq (u(y), v(y))$. The utility representation notion we advance above is thus closely related to decision-making with multiple objectives.[19]

[19] Of course, this is an "as if" interpretation. The primitive of the model is $\succsim$, so when $\mathcal{U}$ represents $\succsim$, we may only think "as if" each member of $\mathcal{U}$ measures (completely) how the agent feels about a particular dimension of the alternatives.

Let me elaborate on this a bit. Suppose the agent indeed attributes two dimensions to the alternatives, and ranks the first one with respect to u and the second with respect to v, but she can compare some of the alternatives even in the absence of dominance in both alternatives. More concretely, suppose $\succsim$ is given as: $x \succsim y$ iff $U_\alpha(x) \geq U_\alpha(y)$ for all $\alpha \in [1/3, 2/3]$, where $U_\alpha := \alpha u + (1-\alpha)v$ for all real α. Then, $\succsim$ is represented by $\mathcal{U} := \{U_\alpha : 1/3 \leq \alpha \leq 2/3\}$, and while we may interpret "as if" the agent views every member of $\mathcal{U}$ measuring the value of a dimension of an alternative (so it is "as if" there are uncountably many dimensions for her), we in fact know here that each member of $\mathcal{U}$ corresponds instead to a potential aggregation of the values of the actual dimensions relevant to the problem.

At any rate, this formulation generalizes the usual utility representation notion we studied above. Moreover, it does not cause any information loss; the potential incompleteness of $\succsim$ is fully reflected in the set $\mathcal{U}$ that represents $\succsim$. Finally, it makes working with incomplete preference relations analytically less difficult, for it is often easier to manipulate vector-valued functions than preorders.

So, what sort of preference relations can be represented by means of a *set of* utility functions? It turns out that the answer is not very difficult, especially if one is prepared to adopt our usual order-denseness requirement. In what follows, given any preference relation $\succsim$ on a nonempty set X, we write $x \bowtie y$ when x and y are $\succsim$*-incomparable*, that is, when neither $x \succsim y$ nor $y \succsim x$ holds. Observe that this defines $\bowtie$ as an irreflexive binary relation on X.

Proposition 10.[20] *Let X be a nonempty set, and $\succsim$ a preference relation on X. If X contains a countable $\succsim$-dense subset, then there exists a nonempty set $\mathcal{U} \subseteq \mathbb{R}^X$ which represents $\succsim$.*

Proof. Once again we will prove the claim assuming that $\succsim$ is a partial order; the extension to the case of preorders is straightforward. Assume that X contains a countable $\succsim$-dense subset, and let $\mathcal{U}$ be the collection of all $u \in \mathbb{R}^X$ such that $x \succ y$ implies $u(x) > u(y)$ for any $x, y \in X$. By Lemma 1, $\mathcal{U}$ is nonempty. We wish to show that $\mathcal{U}$ actually represents $\succsim$. Evidently, this means that, for any $x, y \in X$ with $x \bowtie y$, there exist at least two functions u and v in $\mathcal{U}$ such that $u(x) > u(y)$ and $v(y) > v(x)$. (Why?)

Fix any $x^*, y^* \in X$ with $x^* \bowtie y^*$, and pick an arbitrary $w_o \in \mathcal{U}$. Define $w \in [0,1]^X$ by $w(z) := \frac{w_o(z)}{1+|w_o(z)|}$, and note that w is also a Richter-Peleg utility function for $\succsim$. (We have also used the same trick when proving Proposition 9, remember?) Now let

$$Y := \{z \in X : z \succ x^* \;\text{ or }\; z \succ y^*\},$$

and define $u, v \in \mathbb{R}^X$ as

$$u(x) := \begin{cases} w(z)+4, & \text{if } z \in Y \\ 3, & \text{if } z = x^* \\ 2, & \text{if } z = y^* \\ w(z), & \text{otherwise} \end{cases} \quad \text{and} \quad v(x) := \begin{cases} w(z)+4, & \text{if } z \in Y \\ 2, & \text{if } z = x^* \\ 3, & \text{if } z = y^* \\ w(z), & \text{otherwise} \end{cases}.$$

We leave it for you to verify that both u and v are Richter-Peleg utility functions for $\succsim$. Thus $u, v \in \mathcal{U}$ and we have $u(x^*) > u(y^*)$ while $v(y^*) > v(x^*)$. ■

Exercise 31. Complete the proof of Proposition 10.

[20] A special case of this result was obtained in Ok (2002) where you can also find some results that guarantee the *finiteness* of the representing set of utility functions.

Exercise 32.[H] Define the partial order $\succsim$ on $\mathbb{R}_+$ as: $x \succsim y$ iff $x \in \mathbb{Q}_+$ and $y \in \mathbb{R}_+ \backslash \mathbb{Q}$. Show that there is no countable $\succsim$-dense subset of $\mathbb{R}_+$, but there exists an $\mathcal{U} \subseteq \mathbb{R}^{\mathbb{R}_+}$ with $|\mathcal{U}| = 2$ that represents $\succsim$.

All this is nice, but it is clear that we need stronger utility representation results for applications. For instance, at present we have no way of even speaking about representing a preference relation by a *continuous* utility function (or a set of such functions). In fact, a lot can be said about this issue, but this requires us first to go through a number of topics in real analysis. We will come back to this problem in Chapters C and D when we are better prepared to tackle it.