Assignment 0 Key

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1   Limits

Problem 1   Compute

$$\lim_{x \to 3} \frac{5x^2 - 8x - 13}{x^2 - 5}$$

Problem 1 Answer

$$\lim_{x \to 3} \frac{5x^2 - 8x - 13}{x^2 - 5} = \frac{5(3)^2 - 8(3) - 13}{(3)^2 - 5}$$
$$= \frac{8}{2} = 2$$

Problem 2   Compute

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$

Problem 2 Answer   Substitution of 3 into the expression will result in the indeterminate form 0/0 so factor out the expressions causing the indeterminate form

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x^2 - 9)(x^2 + 9)}{(x - 3)(2x + 1)}$$
$$= \lim_{x \to 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{(x - 3)(2x + 1)}$$
$$= \lim_{x \to 3} \frac{(x + 3)(x^2 + 9)}{(2x + 1)}$$
$$= \frac{((3) + 3)((3)^2 + 9)}{2(3) + 1}$$
$$= \frac{108}{7}$$

Problem 3   Compute

$$\lim_{x \to 4} \frac{3 - \sqrt{x + 5}}{x - 4}$$

Problem 3 Answer

$$= \lim_{x \to 4} \frac{3 - \sqrt{x+5}}{x - 4} \cdot \frac{3 + \sqrt{x+5}}{3 + \sqrt{x+5}}$$

$$= \lim_{x \to 4} \frac{9 - (x+5)}{(x-4)(3 + \sqrt{x+5})}$$

$$= \lim_{x \to 4} \frac{4 - x}{(x-4)(3 + \sqrt{x+5})}$$

$$= \lim_{x \to 4} \frac{-(x-4)}{(x-4)(3 + \sqrt{x+5})}$$

$$= \lim_{x \to 4} \frac{-1}{3 + \sqrt{x+5}}$$

$$= \frac{-1}{3 + \sqrt{4+5}}$$

$$= -\frac{1}{6}$$

Problem 4   Consider the values of constants a and b so that $\lim_{x \to 2} f(x)$ exists and is equal to $f(2)$ where $f(x)$ is defined as below.

$$f(x) = \begin{cases} a + bx & \text{if } x > 2 \\ 3 & \text{if } x = 2 \\ b - ax^2 & \text{if } x < 2 \end{cases}$$

Problem 4 Answer

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (a + bx) = a + 2b = 3$$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (b - ax^2) = b - 4a = 3$$

The above expressions give us a system of equations

$$a + 2b = 3$$

$$b - 4a = 3$$

Solving the system of 2 equations in 2 unknowns gives us $a = -\frac{1}{3}$ and $b = \frac{5}{3}$.

Problem 5   Compute the following limit.

$$\lim_{x \to \infty} \frac{100}{x^2 + 5}$$

Problem 5 Answer

$$\lim_{x \to \infty} \frac{100}{x^2 + 5} = \frac{100}{\infty} = 0$$

Problem 6   Compute

$$\lim_{x \to \infty} (3x^3 - 1000x^2)$$

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Problem 6 Answer Substitution yields an indeterminate form "$\infty - \infty$" which we circumvent by factoring.

$$= \lim_{x \to 3} x^2 (3x - 1000)$$

$$= \infty \cdot \infty$$

which means that the limit does not exist.

Problem 7 Compute

$$\lim_{x \to \infty} \frac{7x^2 + x - 100}{2x^2 - 5x}$$

Problem 7 Answer

$$= \lim_{x \to \infty} \frac{7x^2 + x - 100}{2x^2 - 5x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{7x^2}{x^2} + \frac{x}{x^2} - \frac{100}{x^2}}{\frac{2x^2}{x^2} - \frac{5x}{x^2}}$$

$$= \lim_{x \to \infty} \frac{7 + \frac{1}{x} - \frac{100}{x^2}}{2 - \frac{5}{x}}$$

$$= \frac{7 + 0 - 0}{2 - 0}$$

$$= \frac{7}{2}$$

Problem 8 Compute

$$\lim_{x \to \infty} \left(3^x + 3^{2x} \right)^{\frac{1}{x}}$$

Problem 8 Answer

$$= \lim_{x \to \infty} \left(3^x + \left[3^2 \right]^x \right)^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \left(3^x + 9^x \right)^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \left(9^x \left[\frac{3^x}{9^x} + \frac{9^x}{9^x} \right] \right)^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \left(9^x \left[\left(\frac{3}{9} \right)^x + 1 \right] \right)^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \left(9^x \left[\left(\frac{1}{3} \right) + 1 \right] \right)^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \left(9^x \right)^{\frac{1}{x}} \cdot \left[\left(\frac{1}{3} \right)^x + 1 \right]^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} 9 \left[\left(\frac{1}{3} \right)^x + 1 \right]^{\frac{1}{x}}$$

$$= (9)[0 + 1]^0$$

$$= 9$$

Problem 9   Compute $\lim_{x \to 0^+} x \cdot \ln x$.

Problem 9 Answer

$$= \lim_{x \to 0^+} \frac{\ln x}{1/x}$$
$$= \frac{\ln 0}{1/0^+}$$

which leads to the indeterminate form $-\infty/\infty$ so use l'Hopital's Rule.

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$
$$= \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{x^2}{-1}$$
$$= \lim_{x \to 0^+} (-x)$$
$$= 0$$

Problem 10   Compute $\lim_{x \to 0^+} x \cdot (\ln x)^2$.

Problem 10 Answer

$$= \lim_{x \to 0^+} \frac{(\ln x)^2}{1/x} = \frac{\infty}{\infty}$$

Use l'Hopital's Rule

$$= \lim_{x \to 0^+} \frac{2 \ln x \cdot (1/x)}{\frac{-1}{x^2}} = \frac{\infty}{\infty}$$

So use l'Hopital's Rule again

$$= \lim_{x \to 0^+} \frac{2 \cdot 1/x}{\frac{1}{x^2}}$$
$$= \lim_{x \to 0^+} 2x$$
$$= 2 \cdot 0$$
$$= 0$$

Problem 11   Compute $\lim_{x \to 0} (1 - x)^{1/x}$.

Problem 11 Answer   Rewrite problem

$$\lim_{x \to 0} (1 - x)^{1/x} = \lim_{x \to 0} e^{\ln(1-x)^{1/x}}$$
$$= \lim_{x \to 0} e^{(1/x) \cdot \ln(1-x)}$$
$$= e^{\lim_{x \to 0} \frac{\ln(1-x)}{x}}$$
$$= e^{(\ln 1)/0} = e^{0/0}$$

Use l'Hopital's rule for the indeterminate form

$$= e^{\lim_{x \to 0} \frac{\frac{1}{1-x} \cdot (-1)}{1}}$$

$$= e^{-1/(1-0)}$$

$$= e^{-1}$$

$$= \frac{1}{e}$$

Problem 12 Consider the following functions for $w \geq 0$ and $0 < \sigma < 1$.

$$u(w) = \frac{w^{1-\sigma} - 1}{1 - \sigma}$$

$$u(w) = \ln(w)$$

$$u(w) = \sqrt{w}$$

For each function above, find the following limits (if they exist) where $u'(\cdot)$ and $u''(\cdot)$ represent the first and second derivatives respectively.

(i) $\lim_{w \to 0} -u''(w)/u'(w)$.

(ii) $\lim_{w \to \infty} -u''(w)/u'(w)$.

(iii) $\lim_{w \to 0} (-u''(w) \cdot w)/u'(w)$.

(iv) $\lim_{w \to \infty} (-u''(w) \cdot w)/u'(w)$.

Problem 13 Differentiate $y = x^x$ (with respect to x).

Problem 13 Answer $x^x(1 + \ln x)$.

Problem 14 Differentiate $y = x^{e^x}$ (with respect to x).

Problem 14 Answer $x^{e^x - 1} e^x (1 + x \ln x)$

Problem 15 Compute the following limits associated with the functions $f(x) = |x|$.

$$\lim_{h \to 0} \frac{f(-2 + h) - f(-2)}{h}$$

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$\lim_{h \to 0} \frac{f(3 + h) + f(3)}{h}$$

Problem 16 Define the intervals (if any) over which the following function is continuous.

$$f(x) = \frac{7x^5 + x - 2}{x^2 - 4}$$

Problem 16 Answer $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

Problem 17   Show that there is a root of the equation $3x^7 - 2x^5 + x - 1 = 0$ between 0 and 1.

Problem 17 Answer   We see that $f(0) = -1 < 0$ and $f(1) = 3 - 2 + 1 - 1 = 1 > 0$. By the Intermediate Value Theorem it follows that there exists a number c in $(0, 1)$ such that $f(c) = 0$ since $f(x)$ is continuous and 0 is between $f(0)$ and $f(1)$.

Problem 18   Compute the following limit associated with $f(x) = x^2 - 8x + 9$.

$$\lim_{x \to a} \frac{(x^2 - 8x + 9) - (a^2 - 8a + 9)}{x - a}$$

Problem 18 Answer $2a - 8$.

Problem 19   If the function $f(x)$ is differentiable in the interval (a, b) and $|f'(x)| \le B < \infty$ for all x in the interval (a, b) then is the maximum change in the function over any sub interval $(c, d) \subseteq (a, b)$ finite or infinite? Prove it.

Problem 20   If the function $f(x)$ is differentiable on (a, b), but not continuously differentiable, then is f continuous everywhere on (a, b)? Prove it.

Problem 21   Can you differentiate the expression $2x = 1$? If so what is the derivative? If not, why not?

Problem 22   Find the derivative of the function

$$f(x) = \frac{\sqrt[4]{x}}{x^{-1}\sqrt{x^5}}$$

Problem 22 Answer

$$f'(x) = -\frac{5}{4x^2 \sqrt[4]{x}}$$

Problem 23   Differentiate the function

$$f(x) = \frac{3x^2 - 5\sqrt{x}}{6x^4}$$

Problem 23 Answer

$$f'(x) = -x^{-3} + \frac{35}{12}x^{-9/2}$$

Problem 24   Find the linearization of the function $f(x) = \sqrt[3]{1 + x}$ at $a = 0$ and use it to approximate the numbers $f(-0.05)$ and $f(0.1)$. Are these approximations overestimates or underestimates?

Problem 24 Answer

$$L(x) = f(0) + f'(0)(x - 0) = 1 + \frac{x}{3}$$
$$L(-0.05) = 0.9833 \approx 0.98305 \quad \text{over estimate}$$
$$L(0.1) = 1.0333 \approx 1.0323 \quad \text{over estimate}$$

Problem 25 Let $f(x) = (3x - 5)/(4 - 2x)$ and find $f^{-1}(x)$. Then compare $f'(x)$ and $\frac{d}{dx} f^{-1}(x)$ and describe the relationship (if any).

Problem 25 Answer The inverse function is

$$f^{-1}(x) = \frac{4x + 5}{3 + 2x}$$

Problem 26 Compute the derivative of the following function, $H(p)$, with respect to p and use it to show the answer to the following questions.

$$H(p) = p \log_2 \left(\frac{1}{p} \right) + (1 - p) \log_2 \left(\frac{1}{1 - p} \right)$$

(i) Is there a global maximum and minimum over the interval $[0, 1]$, if so, what is it?

(ii) If so, over what subset of $[0, 1]$, if any, is the function increasing?

(iii) If so, over what subset of $[0, 1]$, if any, is the function decreasing?

Problem 26 Answer

(i) Yes, there is a global max because it is a continuous function on a compact set. The maximum is $p^* = 0.5$.

(ii) The function increases over $[0, 0.5]$.

(iii) The function decreases over $[0.5, 1]$.

Problem 27 Consider the function $f(x) = x^4 e^x$ with domain all real numbers.

(i) Find the x-value(s) of all roots (x-intercepts) of f.

(ii) Find the x- and y-value(s) of all critical points and identify each as a local max, local min, or neither.

(iii) Find the x- and y-value(s) of all global extrema and identify each as a global max or global min.

(iv) Find the x-value(s) of all inflection points.

Problem 27 Answer

(i) The equation $x^4 e^x = 0$ means $x^4 = 0 \implies x = 0$ so the only root is $x = 0$.

(ii) $f'(x) = 4x^3 e^x + x^4 e^x = 0 \implies x = 0, -4$. Plugging in values shows us that f has a local max at $(-4, 256e^{-4})$ and a local min at $(0, 0)$.

(iii) There is a global minimum at $(0, 0)$, but there is no global maximum since $f(x) \to \infty$ as $x \to \infty$.

(iv) Look for solutions to $f''(x) = 0$ which yields $x = \{0, -2, -6\}$. Checking the value of f and f'' at those points tells us that the inflection points are $x = \{-2, -6\}$ but not $x = 0$.

Problem 28   Use the Intermediate Value Theorem to show that $f(x) = x^3 - 2x - 1$ has a root on $[1, 2]$.

IVT: if f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$. For the function given above, $f(1) = -2$ and $f(2) = 3$. Since 0 is a number between -2 and 3, the IVT says there is a number c between 1 and 2 such that $f(c) = 0$; this c is the desired root.

Problem 29   Does the Extreme Value Theorem say anything about the function $f(x) = x^2$ on each of the following intervals? If so what does it say? In either case, explain why.

(i) $[1, 4]$

(ii) $(1, 4)$

Problem 29 Answer

(i) f has a maximum and a mininum on $[1, 4]$ by EVT.

(ii) Because $(1, 4)$ is not compact, the EVT does not apply and so although there may be, the theorem doesn't say anything to us in this case.

Problem 30   Find the value of the constant c that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on the interval $[0, 3]$.

Problem 30 Answer   MVT: If f is continuous on $[a, b]$, then there is a number c between a and b such that $f'(c) = (f(b) - f(a))/(b - a)$.

For the provided function we have $\frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10$. And $f'(x) = 3x^2 + 1$ so we know $f'(c) = 3c^2 + 1$. Solving $f'(c) = 3c^2 + 1 = 10$ for c yields $c = \sqrt{3}$.

Problem 31   For the equation $x^3 + y^3 = \ln(xy) - 1$ use implicit differentiation to find dy/dx.

Problem 31 Answer

$$\frac{dy}{dx} = \frac{y - 3x^3 y}{3xy^3 - x} = \frac{\frac{1}{x} - 3x^2}{3y^2 - \frac{1}{y}}$$

Problem 32   Consider the function

$$f(x) = \begin{cases} b - x^2 & \text{if } x < 3 \\ ax & \text{if } x \geq 3 \end{cases}$$

(i) What condition(s) must be placed on the constants a and b in order for f to be continuous on $(-\infty, \infty)$?

(ii) For what values of the constants a and b will f be differentiable on $(-\infty, \infty)$?

Problem 32 Answer

(i) The only problem area is $x = 3$ so we want $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$. This gives us limits $b - 9$ and $3a$ so we need $3a = b - 9 \iff a = (1/3)b - 3$.

(ii) Diffentiate piecewise to find $f'(x)$ for $x < 3$ and $x > 3$ giving us $-2x$ and a respectively. Existence will require that $\lim_{x\to 3^-} f'(x) = \lim_{x\to 3^+} f'(x) = f'(3)$. The piecewise limits of the derivatives are -6 and a so $a = -6$ and using the equation found before we know $3(-6) = b - 9$ which implies $-18 + 9 = -9 = b$ so we have $\{a = -6, b = -9\}$.

Problem 33   Consider the function $f(x) = \ln(x^2)$. Find the fourth-order Taylor polynomial for $f(x)$ centered at $x_0 = 1$.

Problem 33 Answer

$$P_3(x) = 2(x - 1) - (x - 1)^2 + \frac{2}{3}(x - 1)^3 - \frac{1}{2}(x - 1)^4$$

Problem 34   Let X be a random variable with a probability density function (PDF) $f(x)$

$$f(x) = \begin{cases} ce^{-x/3} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Remember that a PDF has the property that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

(i) Find the value of the constant c that makes $f(x)$ a valid PDF.

(ii) Find the probability that $X \leq 1/4$.

Problem 34 Answer

(i) $c = -\frac{1}{3}$.

(ii) $1 - e^{-1/12}$.

Problem 35   Find the derivative of the following function

$$G(x) = \int_1^{\sin x} t\,dt$$

Problem 35 Answer

$$G'(x) = \sin x \cdot \cos x$$

Problem 36   Find the derivative of the following function

$$G(x) = \int_{72}^{\ln(x)} t\,dt$$

Problem 36 Answer

$$G'(x) = \frac{\ln x}{x}$$

Problem 37 Suppose that $\sum_{n=1}^{\infty} a_n$ represents a convergent series and that no term of the series equals zero, i.e., $a_n \neq 0$ for all $n = 1, 2, \ldots$. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.

Problem 37 Answer If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $a_n \to 0$ (i.e., the terms go to zero). If $a_n \to 0$, then $|1/a_n| \to \infty$. As the terms of $\sum_{n=1}^{\infty} 1/a_n$ do not go to zero, by the nth term test for divergence, the series diverges.

Problem 38 Determine whether the give sequence is increasing, decreasing, or not monotonic. Is the sequence bounded? On the basis of what you find, does the series converge, diverge or can't be determined?

$$a_n = \frac{1}{5^n}$$

Problem 38 Answer The sequence is **decreasing** and the sequence is **bounded** between 0 and 1 and by the monotonic sequence theorem the sequence is **convergent**.

Problem 39 Differentiate the function $y = 3(x^2 - 1)^3(x^2 + 1)^5$.

Problem 39 Answer

$$y' = 18x(x^2 - 1)^2(x^2 + 1)^5 + 30x(x^2 - 1)^3(x^2 + 1)^4$$