Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Extrema in One Dimension

To do

- Weekly recap
- Real world applications
- Hands on practice
- Questions

- Upcoming Deadline: 10.12.2024 – Assignment 02

Chapter 8 | Extrema in one dimension

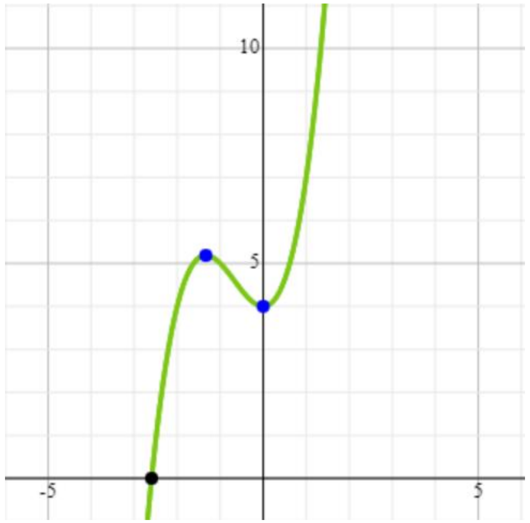
Extrema

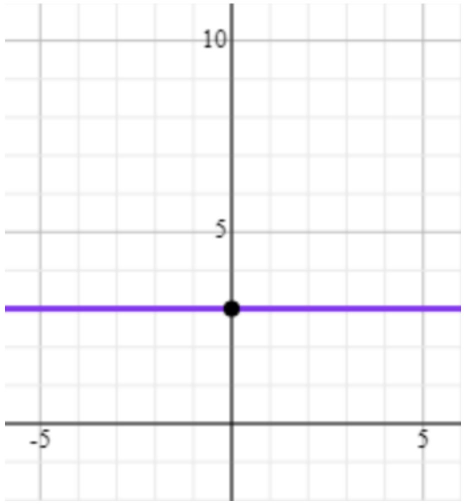
- Let‘s get to know our functions!
  - **Maximum** - $f(x_0) \geq f(x)\ \forall\, x\ \in [a, b]$
  - **Minimum** - $f(x_0) \leq f(x)\ \forall\, x\ \in [a, b]$

- function f must be differentiable (must be defined for interval [a, b] and the slope must be non-zero)
  - $f'(x) = 0$
  - $f''(\mathrm{x}) \neq 0$ for minimum or maximum

- Why use differentiation?

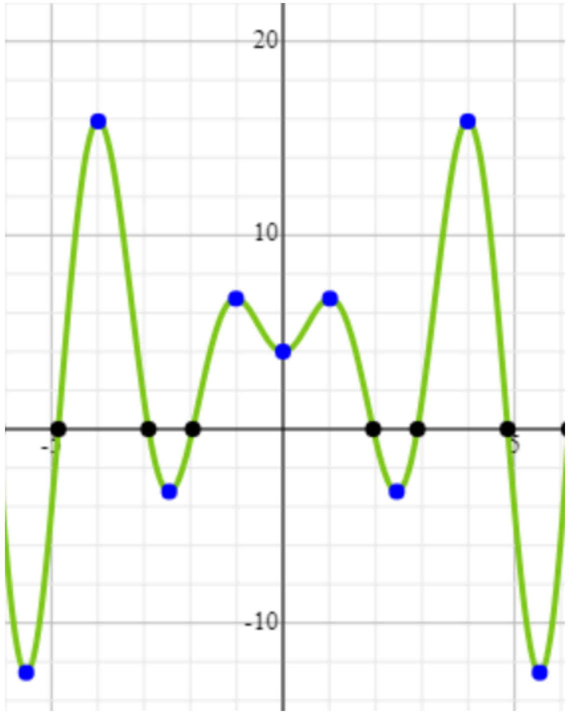
→ we are interested in points where f'(x)'s rate of change takes on 0 momentarily, before in- or decreasing!





Local and global extrema

$f(x) = 3x \cdot \sin(2x) + 4$



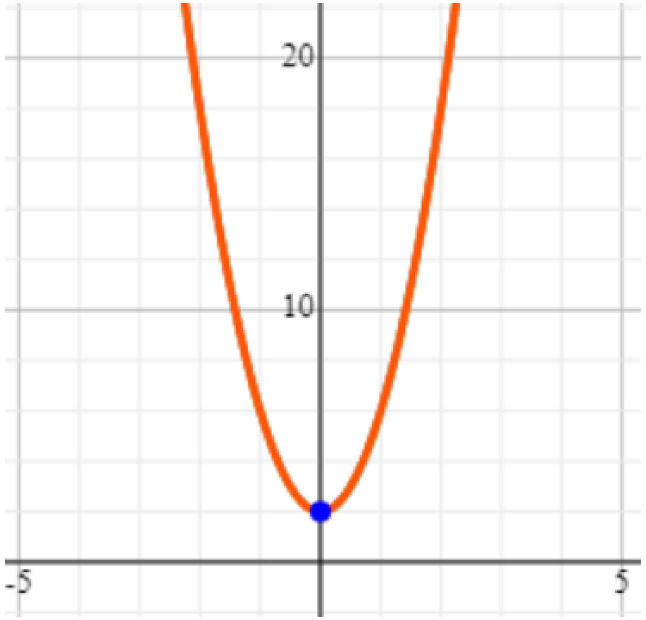
Global

- the ‚highest‘ value in range f can take on
- the ‘lowest’ value in range f can take on

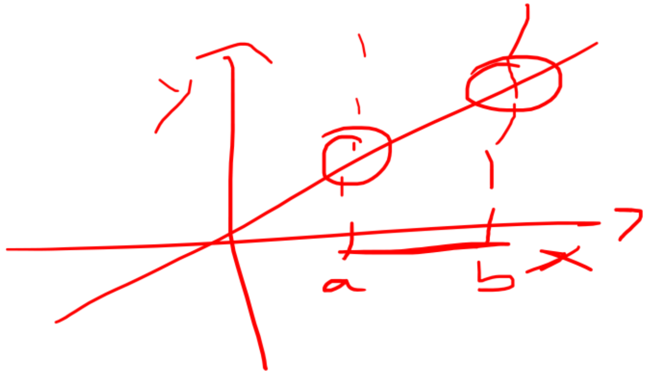
Local

- still extremum… but not the most extreme in I [a, b]

$f(x) = 4x^2 + 2$



Finding extrema



General

1) find $\boldsymbol{f'(x)}$
2) set $\boldsymbol{f'(x_0) = 0}$ and **solve** for all $\boldsymbol{x_0}$

   → stationary points

3) find $\boldsymbol{f''(x)}$
4) for each stationary point, plug in $\boldsymbol{x_0}$ **into** $\boldsymbol{f''(x)}$ and obtain all extrema, inflection & saddle points
5) plug each point into $f(x)$ to **find** the according **value for** $\boldsymbol{y}$

WITH given interval [a, b]

6) evaluate function‘s values at **lower limit** $\boldsymbol{f(a)}$ and **upper limit** $\boldsymbol{f(b)}$
7) compare all extrema and **find global minimum** and **maximum** for interval [a, b]

→ VERY detailed algorithm in Moore & Siegel, 2013, p.168 ☺

Minimum, maximum, inflection or saddle point?

Moore & Siegel, 2013, p.168

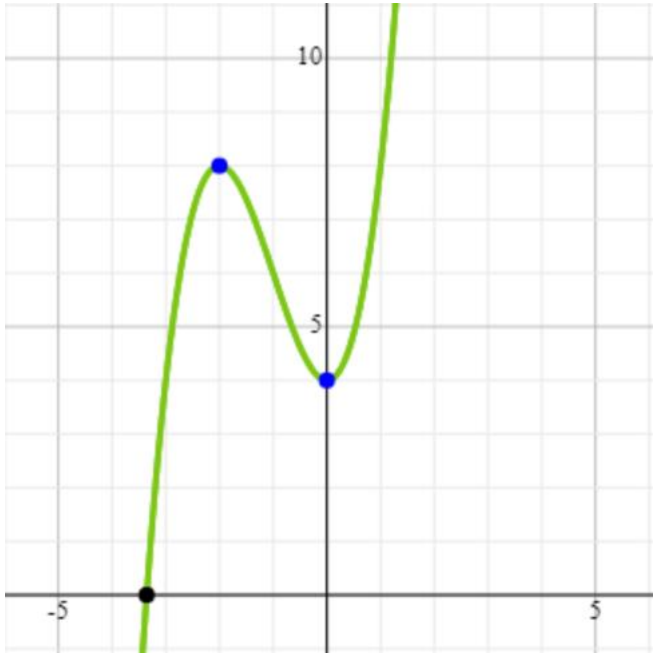
- If $f''(x^*) < 0$, $f(x)$ has a local maximum at x^*.
- If $f''(x^*) > 0$, $f(x)$ has a local minimum at x^*.
- If $f''(x^*) = 0$, x^* may be an inflection point. To check this:
  a) Calculate higher-order derivatives ($f'''(x)$, $f^{(4)}(x)$, etc.) until you find the first one that is non-zero at x^*. Call the order of this derivative n.
  b) If n is odd, then this x^* is an inflection point and not an extremum. Do not include it in further steps.
  c) If n is even and $f^{(n)}(x^*) < 0$, $f(x)$ has a local maximum at x^*.
  d) If n is even and $f^{(n)}(x^*) > 0$, $f(x)$ has a local minimum at x^*.

Extreme value theorem:

- a real-valued function that is continuous and differentiable on a closed and bounded interval [a, b] **must attain both global min and max at least once**!

Hands on – finding extrema

- **Task**: find all extrema of function $f(x) = x^3 + 3x^2 + 4$ for $x \in$ [-3, 2]
- **Hints**:
  - find $f'(x)$
  - find $f''(x)$
  - find all stationary points and obtain x_0
  - find all values for y for all x_0



$f(x) = x^3 + 3x^2 + 4, \; x \in [-3, 2]$

1. $f'(x) = 3x^2 + 6x$

2. set $f'(x) = 0$

$$f'(x) = 0$$

$$3x^2 + 6x = 0$$

$$3x(x+2) = 0$$

$$x_1 = 3\cdot(-2)\cdot(0) \Rightarrow (-2)$$

$$x_2 = 0$$

$$x_3 = (-3)$$

$$x_4 = 2$$

3. find $f''(x) \Rightarrow f''(x) = 6x + 6$

4. $x_1 \to f''(x)$

$$f''(-2) = 6\cdot(-2) + 6 = (-6)$$

$\to$ max

$x_2 \to f''(x)$

$$f''(0) = 6\cdot 0 + 6 = 6$$

$\to$ min

5. find y-values

plug x_0 into $f(x) = x^3 + 3x^2 + 4$

$f(x_1) = (-2)^3 + 3 \cdot (-2)^2 + 4 = 8 \quad P(-2|8)$

$f(x_2) = 0^3 + 3 \cdot 0^2 + 4 = 4 \quad P(0|4)$

6. evaluate bounds

$f(x_3) = (-3)^3 + 3 \cdot (-3)^2 + 4 = 4 \quad P(-3|4)$

$f(x_4) = 2^3 + 3 \cdot 2^2 + 4 = 24 \quad P(2|24)$

7. compare!

$x_1 \Rightarrow (-2) \Rightarrow P(-2|8) \rightarrow$ local max

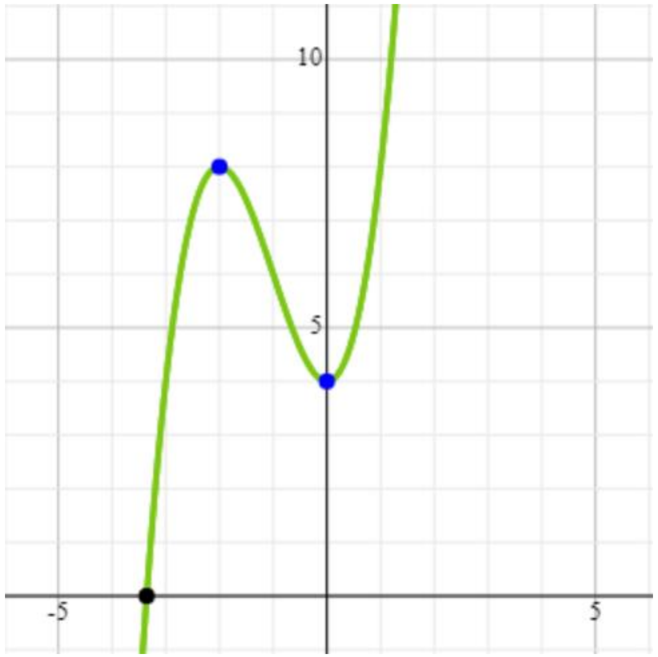
$x_2 \Rightarrow 0 \Rightarrow P(0|4) \rightarrow$ global min

$x_3 = (-3) \Rightarrow P(-3|4) \rightarrow$ endpoint min

$x_4 = 2 \Rightarrow P(2|24) \Rightarrow$ ~~endpoint~~ max
global

Hands on – finding extrema

- **Task**: find all extrema of function $f(x) = x^3 + 3x^2 + 4$ for [-3, 2]
- **Hints**:
  - find $f'(x) = 3x^2 + 6x$
  - find $f''(x) = 6x + 6$
  - find all stationary points and obtain x_0
    - $x = -3, x = -2, x = 0, x = 2$
  - find all values for y for all x_0
    - global minimum $(-3, 4)$
    - global minimum $(0, 4)$
    - local maximum $(-2, 8)$
    - global maximum $(2, 24)$



Inflection and saddle points

Inflection points

- does not have to be stationary point, but if it is, then **not** a local extremum but a saddle point!
- sign of curvature of function changes
- **‘car steering’ test**: Imagine yourself driving a car – do you have to make an S-curve to follow the curvature of the graph?

Saddle points

- stationary point that is **not** a local extremum
- slope is equal to zero **for ALL directions of graph** – **tangent** is horizontal → there is **no** sign change before and after saddle point!
- **conditions:**

$$f'(x_0) = 0$$
$$f''(x_0) = 0$$
$$f'''(x_0) \neq 0$$

Inflection and saddle points – Algorithms

- **conditions:**

$$f'(x_0) = 0$$
$$f''(x_0) = 0$$
$$f'''(x_0) \neq 0$$

Inflection points

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points
(or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0 and changes sign

→plug in $x_0 = a$ into $f'(x)$ and
check if $f'(x_0) \neq 0$

Saddle points

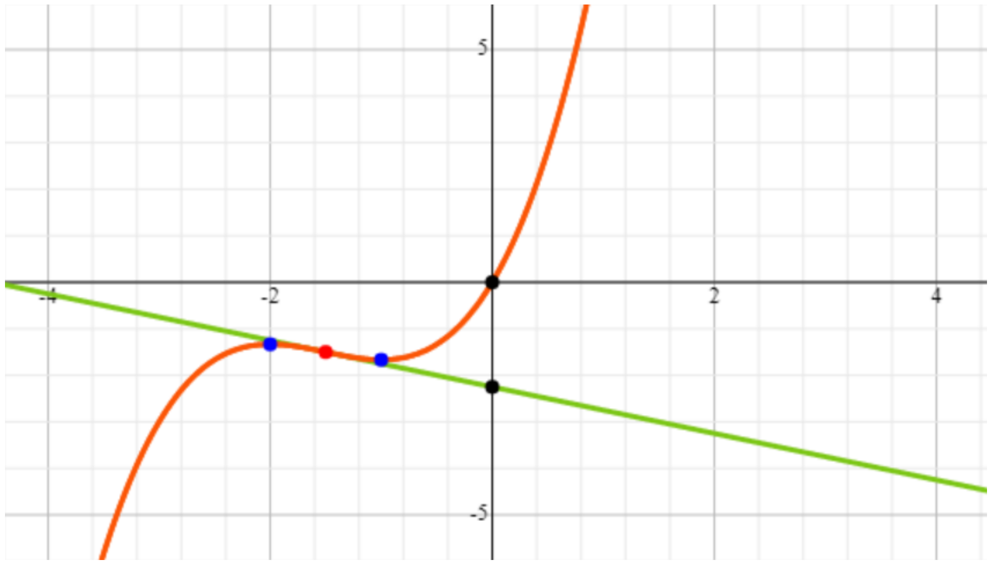
- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points
(or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0 and changes sign

→plug in $x_0 = a$ into $f'(x)$ and
check if $f'(x_0) = 0$

Inflection and saddle points

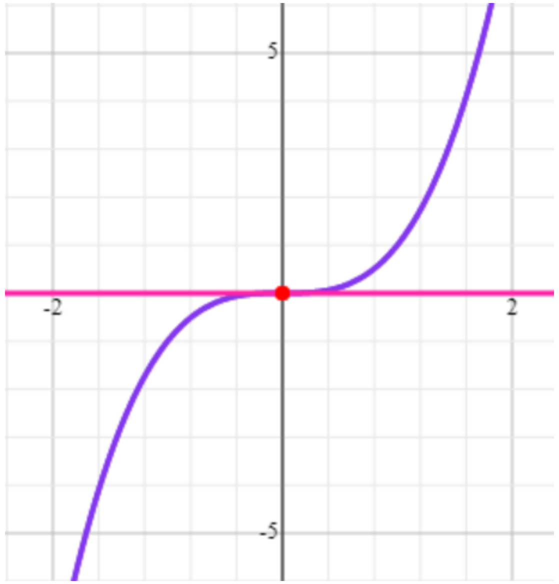
Inflection points

- inflection point at (-1,5, -1,5) for $f(x) = \frac{2}{3}x^3 + 3x^2 + 4x$



Saddle points

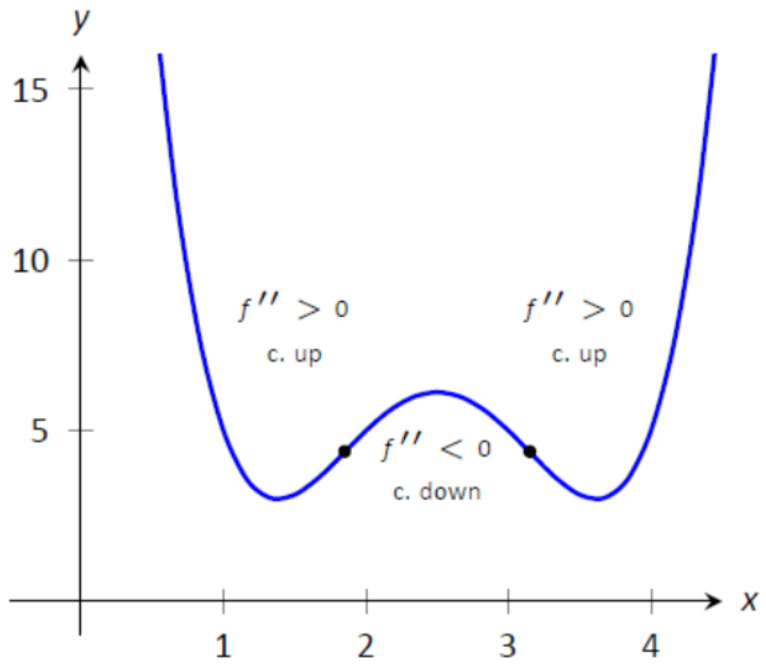
- both an inflection and saddle point at (0, 0) for $f(x) = x^3$



Concavity, convexity and inflection points ☺

Intuition: inflection point tells us, where the sign of function f changes

- to describe **concavity** & **convexity**, f must be differentiable at least twice on interval I [a, b]
  - graph of f is convex if $f'' > 0$ → gradient is increasing
  - graph of f is concave if $f'' < 0$ → gradient is decreasing

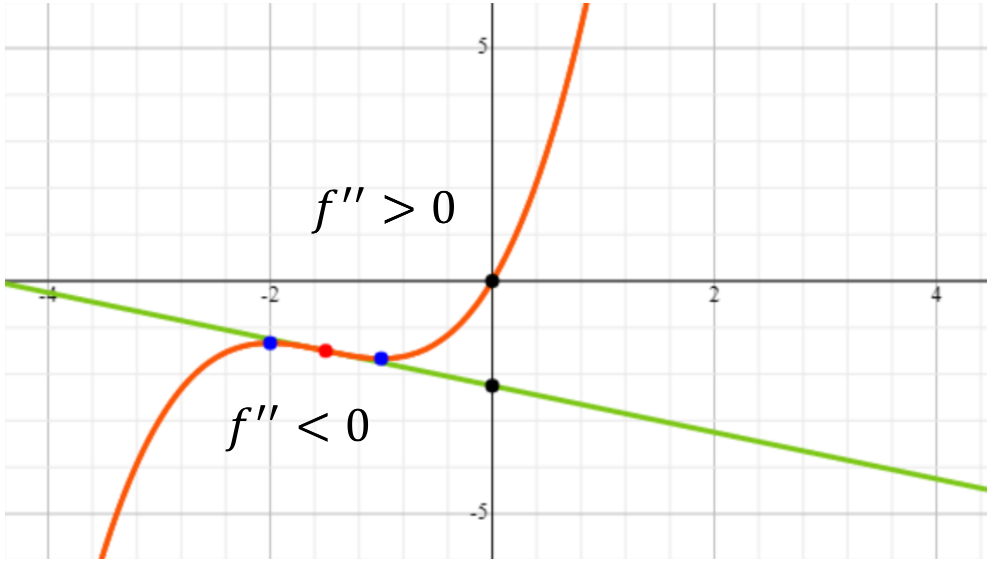


https://math.libretexts.org/Bookshelves/Calculus/Calculus_3e_(Apex)/03%3A_The_Graphical_Behavior_of_Functions/3.04%3A_Concavity_and_the_Second_Derivative

Concavity and convexity

Graphically: graph of f is **convex** if it lies above its tangent line at inflection point $(x_0$, y) and **concave** if it lies below its tangent line

- **inflection point** at (-1,5, -1,5) for $f(x) = \frac{2}{3}x^3 + 3x^2 + 4x$
- **concave** – below the 'mountains'
- **convex** above the 'valleys' ☺



Hands on – saddle points

Let function $\boldsymbol{f}$ be $f(x) = -\frac{2}{3}x^3 + 2x^2 - 2x + 4$

Task: find the inflection/saddle points of f and discuss where f is concave or convex

Hints:

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points (or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0
- describe sign changes around inflection point and classify concavity/convexity using $f'(x)$

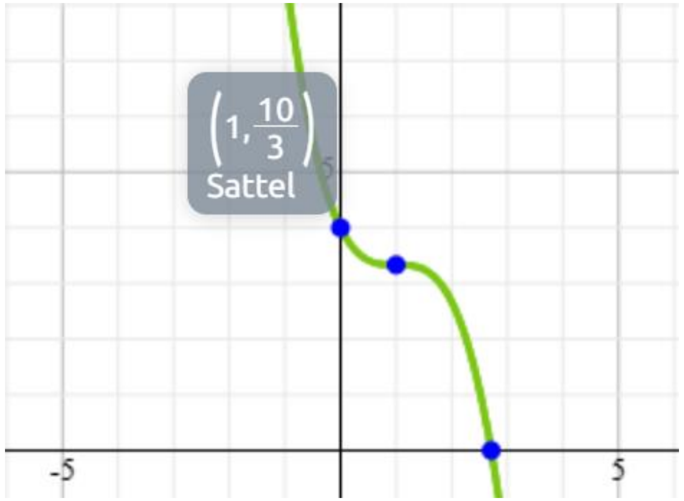
$f(x) = -\frac{2}{3}x^3 + 2x^2 - 2x + 4$ $\quad -\frac{3 \cdot 2}{3} = -\frac{6}{3} = -2$

1. find $f'(x) = 3 \cdot \left(-\frac{2}{3}\right)x^2 + 4x - 2 = -2x^2 + 4x - 2$

2. find $f''(x) = -4x + 4$

3. set $f''(x) = 0$ to find stationary points

spoiler....



$$-4x + 4 = 0$$

$$-4(x - 1) = 0$$

$$x_0 = 1$$

$$x_1 = 0$$

4. second derivative test!

$f''(1) = -4 \cdot 1 + 4 = 0$ ← obv. x_0 is NOT an extremum!

$f''(0) = -4 \cdot 0 + 4 = 4 \neq 0$ → NOT an inflection point!

5. find $f'''(x) = -4 \neq 0 \Rightarrow$ ALL conditions are met: $f'(x_0) = 0$

$f''(x_0) = 0$

$f'''(x_0) \neq 0$

6. inflection or saddle?

plug x_0 into $f'(x)$ & check if $f'(x_0) = 0$

$$f'(1) = (-2)\cdot 1^2 + 4\cdot 1 - 2 = (-2) + 4 - 2 = 0$$

$\Rightarrow$ that means, we found a saddle!

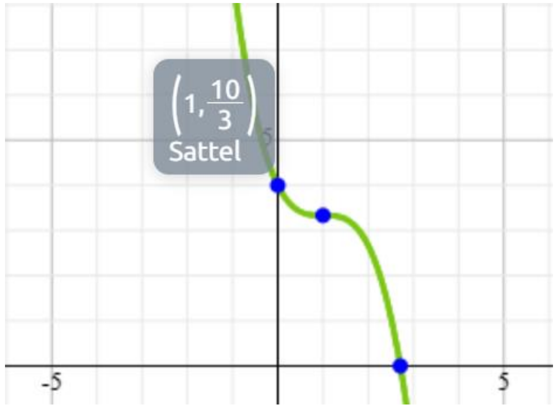
7. find corresponding y-value:

$$f(1) = -\frac{2}{3}\cdot 1^3 + 2\cdot 1^2 - 2\cdot 1 + 4$$

$$= -\frac{2}{3} + 2 - 2 + 4 = -\frac{2}{3} + 4 = -\frac{2}{3} + \frac{12}{3} = \frac{10}{3}$$

$f(x)$ has a saddle point at $(1 \mid \frac{10}{3}) \rightarrow$

$\rightarrow$ observe that there is NO imminent sign change around the saddle



Hands on – inflection points, concavity, convexity

Let function $\boldsymbol{f}$ be $f(x) = -3x^3 - 3x + 1$

Task: find the inflection/saddle points of f and discuss where f is concave or convex

Algorithm:

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points (or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0
- describe sign changes around inflection point and classify concavity/convexity using $f'(x)$

$f(x) = 3x^3 - 3x + 1$

find $f'(x) = 9x^2 - 3$

find $f''(x) = 18x$

set $f''(x) = 0$ and find inflection points:

$18x = 0$

$x = 0$ ← potential inflection point at $x_0 = 0$

find $f'''(x) = 18 \neq 0$

inflection or saddle point?

plug in $x_0 = 0$ into $f'(x)$ and check if $f'(x_0) = 0$

$f'(0) = 9 \cdot 0^2 - 3 = -3 \neq 0$ ← $f'(0) \neq 0$ → inflection point!

obtain corresponding y-value:

$f(0) = 3 \cdot 0^3 - 3 \cdot 0 + 1 = 1$

→ $f(x)$ has an inflection point at $(0,1)$

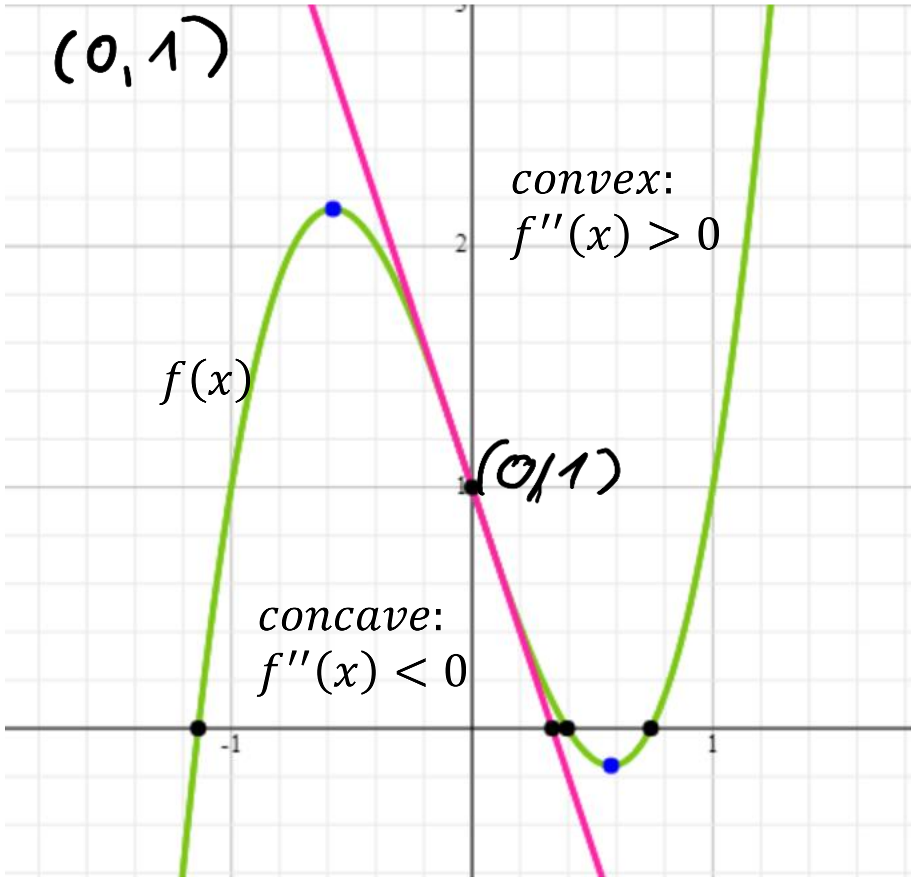
DISCUSSION

→ $f(x)$ has an inflection point at $(0,1)$

→ if we plot the tangent at $x=0$, we can clearly see where $f(x)$ is concave and convex.

→ at inflection point $(0/1)$ the sign of the 2^{nd} derivative changes!

→ 'car steering wheel' test



Taylor series

Intuition:

- first derivative states if function f increases or decreases
- second derivative describes f‘s curvature
- … can we ‚build‘ a function with all relevant info from its derivatives?

Answer:

- Yes, via the Taylor series!

$$
\begin{aligned}
f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.
\end{aligned}
$$

Moore & Siegel, 2013, p.161

Taylor series

Let's look at the function g(x) = e^x. Noting the fact that the kth order derivative of g(x) is also g(x), the expansion of g(x) about x=a, is given by:

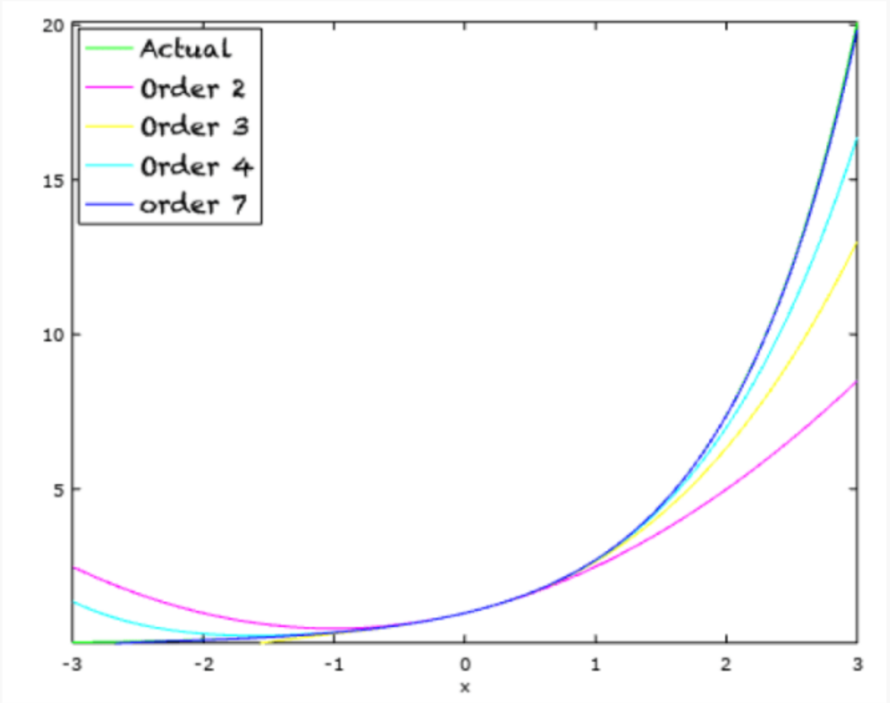
$$e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \ldots + \frac{e^a}{k!}(x-a)^k + \ldots$$

Hence, around x=0, the series expansion of g(x) is given by (obtained by setting a=0):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

The polynomial of order k generated for the function e^x around the point x=0 is given by:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!}$$



Polynomials of varying degrees that approximate e^x

https://machinelearningmastery.com/a-gentle-introduction-to-taylor-series/, 28.11.2023

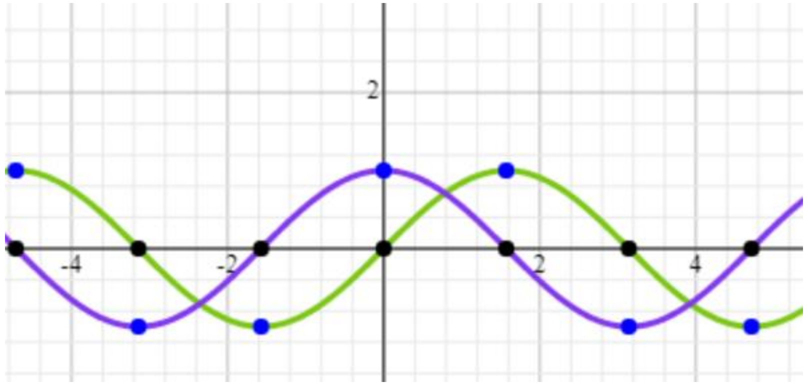
Taylor series - example

Task: derive the Taylor series for $f(x) = \sin(x)$ with $a = 0$

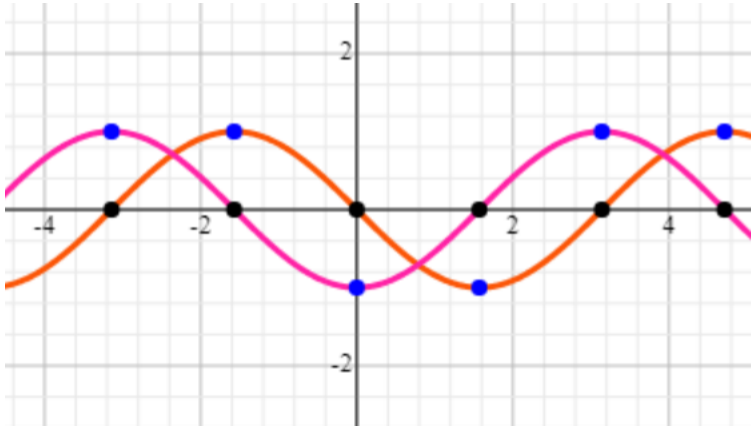
Algorithm:

- find n^{th} derivative of f
- plug in starting point $a = 0$
- plug result for each derivative into Taylor series

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$



$f(x) = \sin(x)$ *and* $h(x) = \cos(x)$



$g(x) = -\sin(x)$ *and* $j(x) = -\cos(x)$

Taylor series - example

Solution: derive the Taylor series for $f(x) = \sin(x)$ with $a = 0$

Find $\boldsymbol{n^{th}}$ derivative of $\boldsymbol{f}$

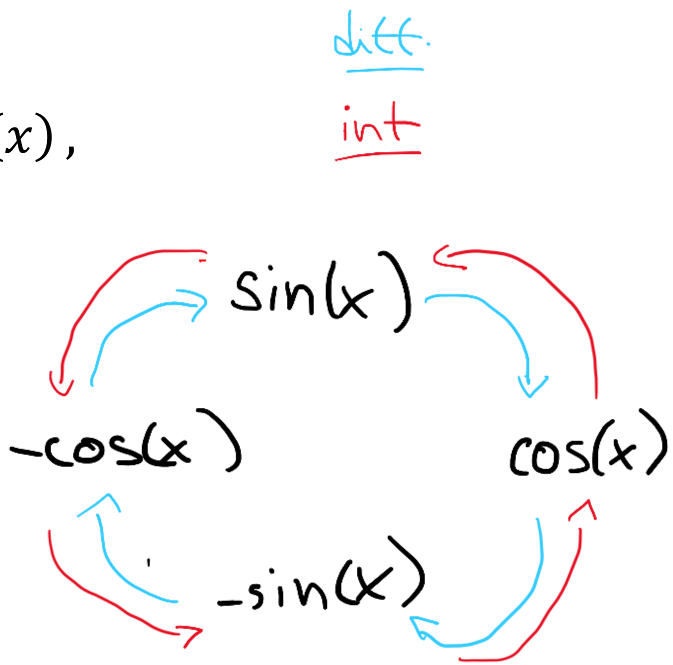
$f(x) = sin(x), f'(x) = \cos(x)\,, f''(x) = -\sin(x)\,, f'''(x) = -\cos(x)\,,$

$f^{(4)} = \sin(x)\,, \dots.$ *mind differentiation rules*

Plug in starting point $\boldsymbol{a = 0}$

$f(0) = 0, f'(0) = 1, f''(0) = 0,\ f'''(0) = -1,$

$f^{(4)}(0) = 0 \dots.$ *mind differentiation rules*



Taylor series - example

Solution: derive the Taylor series for $f(x) = \sin(x)$ with $a = 0$

Plug in starting point $\boldsymbol{a = 0}$

$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(x) \quad = -1,$

$f^{(4)}(0) = 0$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Plug result for each derivative into Taylor series

$\sin(x) = 0 + \frac{1}{1!}(x) + 0 + \frac{-1}{3!}(x^3) + 0 + \frac{1}{5!}(x^5) + 0 + \frac{-1}{7!}(x^7) + \cdots$

$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Note: every ‚even' term cancels out as it becomes equal to 0

with a bit of practice and the more derivatives you compute, you will find a pattern!

Real world applications – Taylor series

- **Nature science**:
  - The Taylor series is applied to approach complex functions for e.g., molecular movement or energy exchange in systems
- **Machine learning:**
  - Taylors a lot in various model training algorithms

- If you would like to learn a bit more about Tayloring, give this video a go!
  - https://www.3blue1brown.com/lessons/taylor-series

Time for your questions

- Any questions during the week?
    - joerdis.strack@uni-konstanz.de