Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Integration

To do

- Weekly recap
- Real world applications
- Hands on practice
- Questions

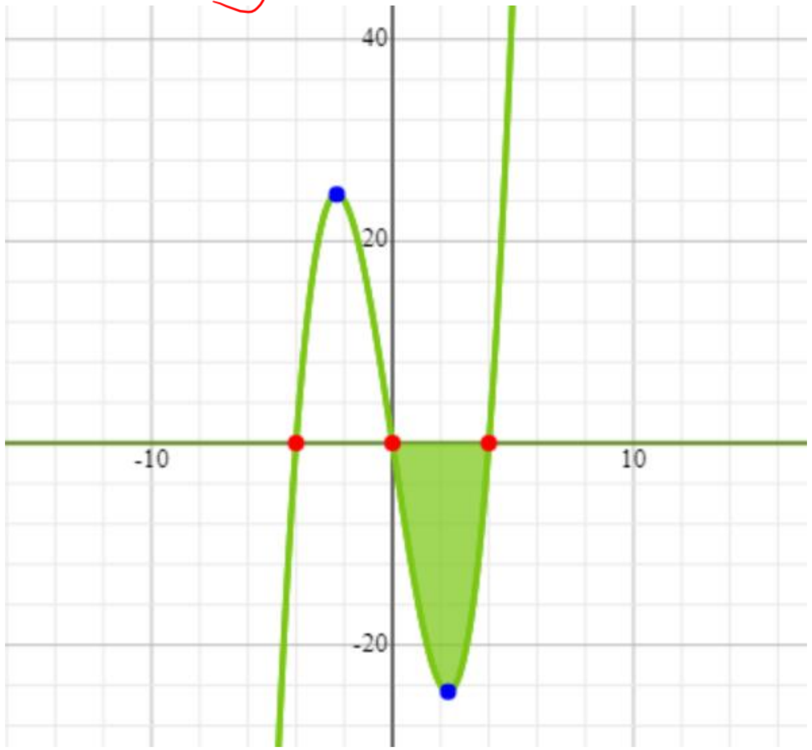
Chapter 7 | Integration

Integration

$$\begin{tikzcd} F(x) & f(x) \arrow[l, bend left, "\int"] \arrow[r, bend left, "\frac{d}{dx}"] & f'(x) \arrow[l, bend left, "\int"] \end{tikzcd}$$

Antiderivative

- the antiderivative is the function we get via integration
- NEVER forget to add a +C to your antiderivative to include constants lost in translation



Integral

- the integral is the 'area under the curve' between the bounds **a** and **b**

Let function *f* be $f(x) = x^3 - 16x$

- $F(x) = \frac{x^4}{4} - 8x^2 + C$
- $\int_0^4 x^3 - 16x \, dx = -64$

Definitive and indefinite integrals

Definite integrals

$$\int_0^4 x^3 - 16x\,dx = -64$$

- evaluate integral of $f(x)$ with **limits a** and **b** → evaluate antiderivative at upper limit $F(b)$ minus lower limit $F(a)$
- will give you a number as a result!

Indefinite integrals

$$\int x^3 - 16x\,dx = F(x) + C$$

$$= \frac{x^4}{4} - 8x^2 + C$$

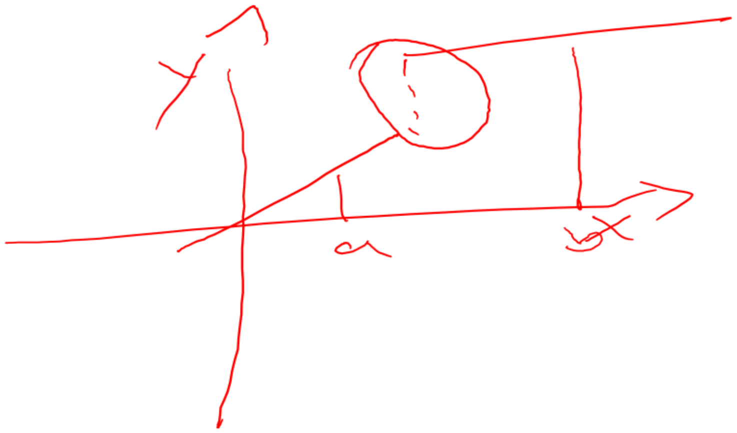
- what function did we differentiate to get $f(x)$? → 'backwards' differentiation
- result in an antiderivative!

  → always add +C to add any constant lost in translation!

Fundamental theorem of calculus

- According to 1st **fundamental theorem of calculus**, we can evaluate a definite integral if integrand $f(x)$ is continuous on [a, b]:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$



1. check if function is continuous
2. find antiderivative $F(x)$
3. evaluate $F(x)$ for upper and lower limits b and a
4. compute integral

Integration 'Algorithm'

$x^4 \quad F(x) \longrightarrow \frac{1}{n+1} x^{n+1}$

$n-1$

$f(x) \quad F'(x) = \frac{x^{n+1}}{n+1}$

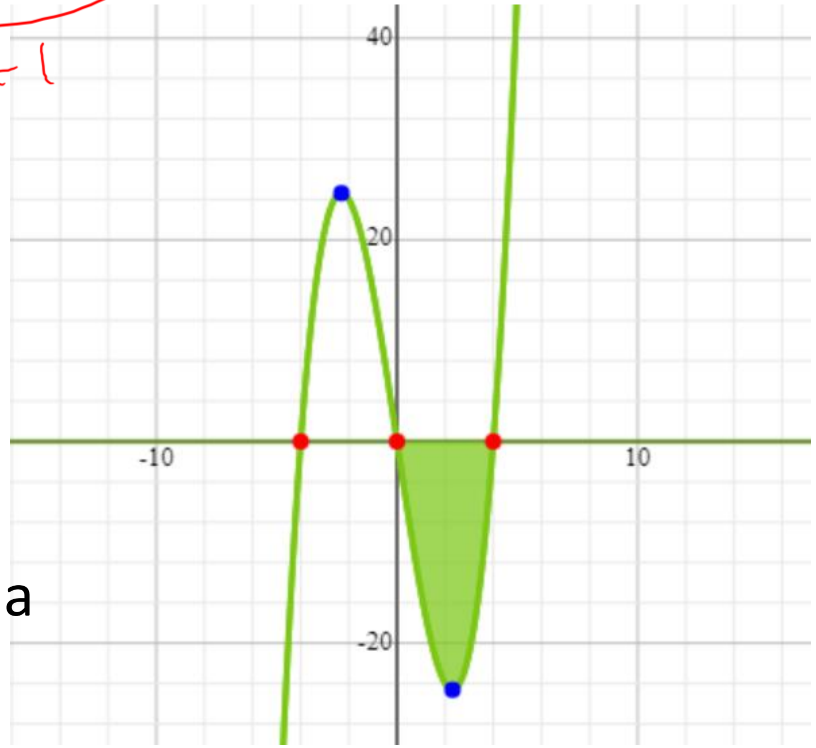
$n+1$

Let's return to $\int_0^4 x^3 - 16x\, dx$

1. is $f(x)$ **continuous** on [0, 4]?
   →'graphic' test: yep, looks good!
   →or check via limits… ☺

2. find **antiderivative**

$$\int x^3 - 16x^1\, dx = F(x) + C = \frac{x^4}{4} - 8x^2 + C$$



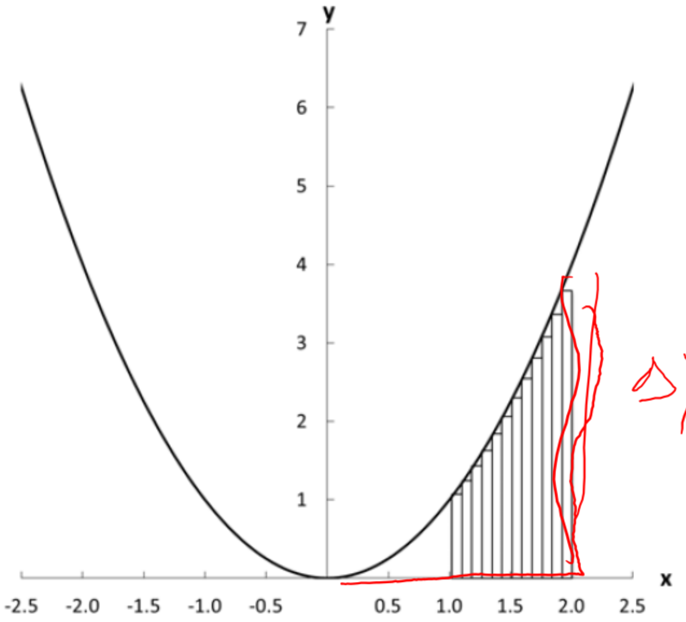
3. **evaluate** $F(x)$ for **upper** and **lower limits** b and a

and 4. compute **integral**

$$F(4) - F(0) = \frac{4^4}{4} - 8 \cdot 4^2 - \left(\frac{0^4}{4} - 8 \cdot 0^2\right) = (64 - 128) - 0 = -64$$

Intuition – Riemann sums and integrals

- please check out some further info on Riemann sums: https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Active_Calculus_(Boelkins_et_al.)/04%3A_The_Definite_Integral/4.02%3A_Riemann_Sums (22.11.2023)
- or check out wikipedia for a more detailed description: https://en.wikipedia.org/wiki/Riemann_sum (22.11.2023)
- check out **Moore and Siegel (2013)** on pp. 135 for a quick discussion of Riemann sums!



Δy

Figure 7.2: Area under $y = x^2$ from $x = 1$ to $x = 2$ with Rectangles

Δx

Hands on – rules of bounds

Task: team up with a partner and discuss the rules of bounds! Apply them to $f(x) = 2x + 1$ for [2, 5] and make a sketch!

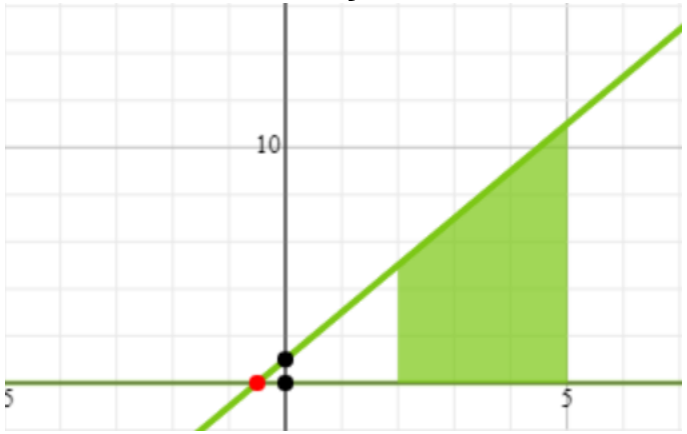
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

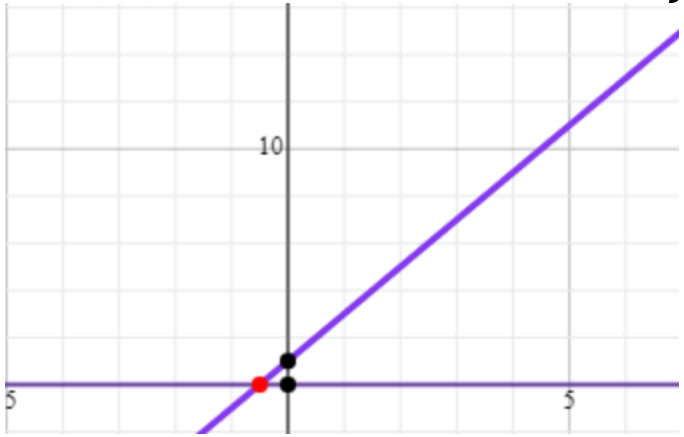
$$\int_a^a f(x)\,dx = 0$$

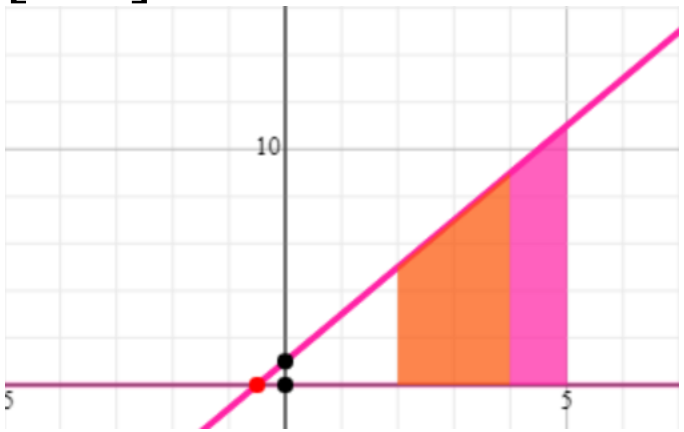
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \;\; for \;\; c \in [a, b]$$

Hands on – rules of bounds

Solution: $f(x) = 2x^{1} + 1\,, F(x) = x^2 + 1x + C,\ for\ [2, 5]$







$$\int_2^5 2x + 1\,dx = -\int_5^2 2x + 1\,dx = 24$$

$$\int_2^2 2x + 1\,dx = 0$$

$$\int_2^5 2x + 1\,dx = \int_2^4 2x + 1\,dx + \int_4^5 2x + 1\,dx = 24$$

Rules of integration … (the most important)

Exponent rule 1 and 2:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad for \;\; n \neq -1$$

$$\int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + C$$

Exponential rule 1 and 2:

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln(a)} + C$$

Log rules 1 and 2:

$$\int \ln(x)\, dx = x \ln(x) - x + C$$

$$\int \log_a(x) dx = \frac{x \ln(x) - x}{\ln(a)} + C$$

Hands on – Antiderivative

Task: Find the antiderivatives by solving the indefinite integrals

$$\int e^4 dx$$

$$\int 3x^5 + 22\, dx$$

$$\int \frac{1}{4x}\, dx$$

$$\int 2a^x\, dx$$

$$\int \ln(5x)\, dx$$

Hands on

Solution:

$$\int e^4 dx = e^4 x + C$$

$$\int 3x^5 + 22\, dx = \frac{x^6}{2} + 22x + C$$

$$\int \frac{1}{4x}\, dx = \frac{1}{4}\int \frac{1}{\mathrm{x}} dx = \frac{1}{4}\ln|x| + C$$

$$\int 2a^x\, dx = \frac{2a^x}{\ln(a)} + C \longrightarrow \int 2a^x dx = 2\int a^x dx$$

$$\int \ln(5x)\, dx = x\ln(5x) - x + C$$

Integration by substitution

- the '**chain rule of integration**'

$$\int_a^b f\big(g(u)\big)g'(u)\,du = \int_{g(a)}^{g(b)} f(x)\,dx$$

- **intuition**: locate the 'chained' function $f\big(g(x)\big)$

  → find the term that is easy to differentiate $g(x) \rightarrow g'(x)$!

1) prepare substitution
   1) find substitute term u
   2) solve for x
   3) differentiate g(u)
   4) replace integration variable
2) substitution
3) integration
4) 'substitute backwards'

Integration by substitution

$\int_a^b e^{3x} dx$

Example: $f(x) = e^{3x}$ $\qquad \int_a^b \overbrace{f}^{f(x):\text{ outer function}}(\underbrace{g(u)}_{\text{inner function}})\overbrace{g'(u)\,du}^{dx} = \int_{g(a)}^{g(b)} f(x)\,dx$

1. **preparation**
   1. find substitute term u (← 'inner function')

      $u = 3x,$
   2. solve for x

      $x = \frac{u}{3} \;\rightarrow\; g(u) = \frac{1}{3}u$
   3. differentiate g(u)

      $g'(u) = \frac{1}{3}$
   4. replace integration variable

      $dx = g'(u)du = \frac{1}{3}du$

Integration by substitution

Example: $f(x) = e^{3x}$

$$\int_a^b \overset{f(x):\text{ outer function}}{f}\Big(\underbrace{g(u)}_{\text{inner function}}\Big)\overbrace{g'(u)\,du}^{dx} = \int_{g(a)}^{g(b)} f(x)\,dx$$

2. substitution

$\int e^{3x}\,dx$ with: $u = 3x$, and $dx = \frac{1}{3}\,du$

$$\int e^u \cdot \frac{1}{3}\,du = \frac{1}{3}\int e^u\,du$$

3. integration

$$F(u) = \frac{1}{3} \cdot e^u + C$$

4. substitute ‘backwards’

$$F(x) = \frac{1}{3} \cdot e^{3x} + C = \frac{1}{3}e^{3x} + C$$

Integration by parts

Intuition: two functions $f(x)$ and $g'(x)$ are multiplied with each other and $g'(x)$ is an 'easy' derivative!

→ 'product rule' of integration

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

1. Choose $f(x)$ and $g'(x)$
2. Find $f(x)$ → $f'(x)$ and $g'(x)$ → $g(x)$
3. plug your functions into the integration by parts formula
4. simplify and solve

Integration by parts

Example: $\int x^3 \ln(x)\, dx$ $\quad\quad$ $\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$

1. **Choose $\boldsymbol{f(x)}$ and $\boldsymbol{g'(x)}$**

   $f(x) = \ln(x) \;\; and \;\; g'(x) = x^3$

2. **Find $\boldsymbol{f(x)} \rightarrow \boldsymbol{f'(x)}$ and $\boldsymbol{g'(x)} \rightarrow \boldsymbol{g(x)}$**

   $f'(x) = \frac{1}{x} \;\; and \;\; g(x) = \frac{x^4}{4}$

3. **Plug your functions into** $\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$

   $\int x^3 \ln(x)\, dx = \ln(x) \cdot \left(\frac{x^4}{4}\right) - \int \frac{1}{x} \cdot \left(\frac{x^4}{4}\right) dx$

Integration by parts

Example: $\int x^3 \ln(x)\, dx \qquad \int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$

4. simplify and solve

$$\int x^3 \ln(x)\, dx = \ln(x) \cdot \left(\frac{x^4}{4}\right) - \int \frac{1}{x} \cdot \left(\frac{x^4}{4}\right) dx$$

$$= \frac{x^4 \ln(x)}{4} - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4 \ln(x)}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + C$$

$$= \frac{x^4 \ln(x)}{4} - \frac{x^4}{16} + C$$

When to use which method?

Integration by substitution

- ‘chain rule’ of integration
- there is an ‘outer function’, containing an ‘inner function’
- the ‘inner function’ can be differentiated easily

→ **rule of thumb:** try substitution first!

Integration by parts

- ‘product rule’ of integration
- two functions are multiplied with each other
- one function is an ‘easy’ derivative
- the other can be differentiated

→ you will get a feeling for when to use which rule with practice

Hands on – Integration by parts & substitution

Task: Find the antiderivatives using substitution and integration by parts as you see fit!

1) $\int \sqrt{x+1} \quad dx$
2) $\int xe^x \ dx$
3) $\int \ln(x^2)\, dx$ $\rightarrow$ $\int \ln(x^2) \cdot 1\, dx$
4) $\int x^2 e^x dx$
5) $\int (x^2+1)^4\, 2x\, dx$
6) $\int xe^{x^2} dx$

2

$$\int f(x) \cdot g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$\int x e^x \,dx$$

1. choose $f(x) \to x$
$g'(x) \to e^x$

2. find $f'(x) = 1$
$g(x) = e^x$

3. and 4. $\int x e^x \,dx = x \cdot e^x - \int 1 \cdot e^x \,dx$

$= x e^x - 1 \cdot \int e^x \,dx$

$= x e^x - e^x + C$

② $\int x e^{x} dx$

and the other way around as one of you suggested: this is definitely possible, but more work and the potential of making a mistake greater!

1. choose $f(x) = e^{x}$

$\wedge$ $g'(x) = x$

2. find $f'(x) = e^{x}$

$\wedge$ $g(x) = \frac{1}{2} x^{2}$

3. and u.

$$\int x e^{x} dx = e^{x} \cdot \frac{1}{2} x^{2} - \int e^{x} \cdot \frac{1}{2} x^{2} dx$$

$$= \frac{1}{2} x^{2} e^{x} - \frac{1}{2} \underbrace{e^{x} \cdot x^{2} dx}_{x^{2} e^{x} - 2(x e^{x} - e^{x}) + C}$$

NOTE: this requires another integration by parts – which is possible, but more complicated! Make your life easier! :-)

$$= \frac{1}{2} x^{2} e^{x} - \frac{1}{2} \left(x^{2} e^{x} - 2(x e^{x} - e^{x}) \right) + C$$

$$= e^{x}(x - 1) + C = x e^{x} - e^{x} + C$$

don't be tricked!

③ $\int \ln(x^2)\,dx = \int 1 \cdot \ln(x^2)\,dx$

<u>1. choose</u> $f(x) = \ln(x^2)$

$g'(x) = 1$ $\quad g'(f(x)) \cdot f'(x)$

chain rule!

<u>2. find</u> $f'(x) = \frac{1}{x^2} \cdot 2x = \frac{2x}{x^2} = \frac{2}{x}$

$g(x) = x$

<u>3. and.</u> $\int 1 \cdot \ln(x^2)\,dx = x \cdot \ln(x^2) - \int \frac{2}{x} \cdot x\,dx$

$= x \cdot \ln(x^2) - 2\int \frac{x}{x}\,dx$

$= \underbrace{x \cdot \ln(x^2) - 2x + C}$

you already know this as part of your integration rules set! this is where it comes from ;-)

④ $\int x^2 e^x \, dx$

1. choose $f(x) = x^2$

$\wedge$ $g'(x) = e^x$

2. find $f'(x) = 2x$

$\wedge$ $g(x) = e^x$

3 and 4. $\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \underbrace{\int x e^x \, dx}_{x e^x - e^x + C}$

$= x^2 e^x - 2(x e^x - e^x) + C$

① $\int \sqrt{x+1}\,dx = \int \sqrt{u} \cdot 1 \cdot du = 1 \cdot \int \sqrt{u}\,du = 1 \cdot \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

2. substitute

move constant

3. integrate

4. substitute backwards

1. Prepare

1. find u : $x+1$
2. solve for x:
   $x = u-1$
3. $g'(u) = 1$
4. replace dx
   $dx = g'(u)du = 1\,du$

Note: Integration of roots

Power rule: $\int x^u\,dx = \frac{x^{u+1}}{u+1} + C$

$\sqrt[2]{x^1} = x^{\frac{1}{2}} \Rightarrow \int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{2}{3} x^{\frac{3}{2}} + C$

② $$\int (x^2+1)^4\, 2x\, dx = 2\int \underbrace{(x^2+1)}_{u}{}^4\, \underbrace{x}_{(u-1)^{\frac{1}{2}}}\, \underbrace{dx}_{\frac{1}{2}(u-1)^{-\frac{1}{2}}du} = 2\int (u)^4 (u-1)^{\frac{1}{2}}\, \frac{1}{2}(u-1)^{-\frac{1}{2}}\, du$$

move constant

1. Prepare

1. find u: x^2+1
2. solve for $x = \sqrt{u-1}$
   ↳ $g(u) = \sqrt{u-1} = (u-1)^{\frac{1}{2}}$
3. $\to g'(u) = \frac{1}{2}(u-1)^{-\frac{1}{2}}$
4. replace dx:
   $dx = \frac{1}{2}(u-1)^{-\frac{1}{2}}\, du$

$$= 2\cdot\frac{1}{2}\int (u)^4 \underbrace{(u-1)^{\frac{1}{2}}(u-1)^{-\frac{1}{2}}}_{(u-1)^{\frac{1}{2}+(-\frac{1}{2})} = (u-1)^0 = 1}\, du$$

$$= 1\cdot\int (u)^4 \cdot 1\, du = 1\cdot\int (u)^4\, du$$

$$= \frac{u^5}{5} + C = \frac{(x^2+1)^5}{5} + C$$

③ $\int x e^{x^2} dx = \int u^{\frac{1}{2}} e^u \frac{1}{2} u^{-\frac{1}{2}} du = \frac{1}{2} \int e^u \underbrace{\cancel{u^{\frac{1}{2}}} \cancel{u^{-\frac{1}{2}}}}_{u^{\frac{1}{2}+(-\frac{1}{2})} = u^0 = 1} du = \frac{1}{2} \int e^u du$

more constant

$= \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$

1. prepare

1. find u: x^2

2. solve for x:

$x = \sqrt{u} = u^{\frac{1}{2}}$

3. $\rightarrow g'(u) = \frac{1}{2} u^{-\frac{1}{2}}$

4. replace dx

$dx = \frac{1}{2} u^{-\frac{1}{2}} du$

Hands on – Integration by parts & substitution

Solution:

1) $\int \sqrt{x+1}\ \ dx = \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

2) $\int xe^x\ dx = xe^x - e^x + C$

3) $\int \ln(x^2)\, dx = x\ln(x^2) - 2x + C = 2x\ln(x) - 2x + C$

4) $\int x^2e^x dx = x^2e^x - 2(e^x x - e^x) + C$

5) $\int (x^2+1)^4\, 2x\, dx = \frac{(x^2+1)^5}{5} + C$

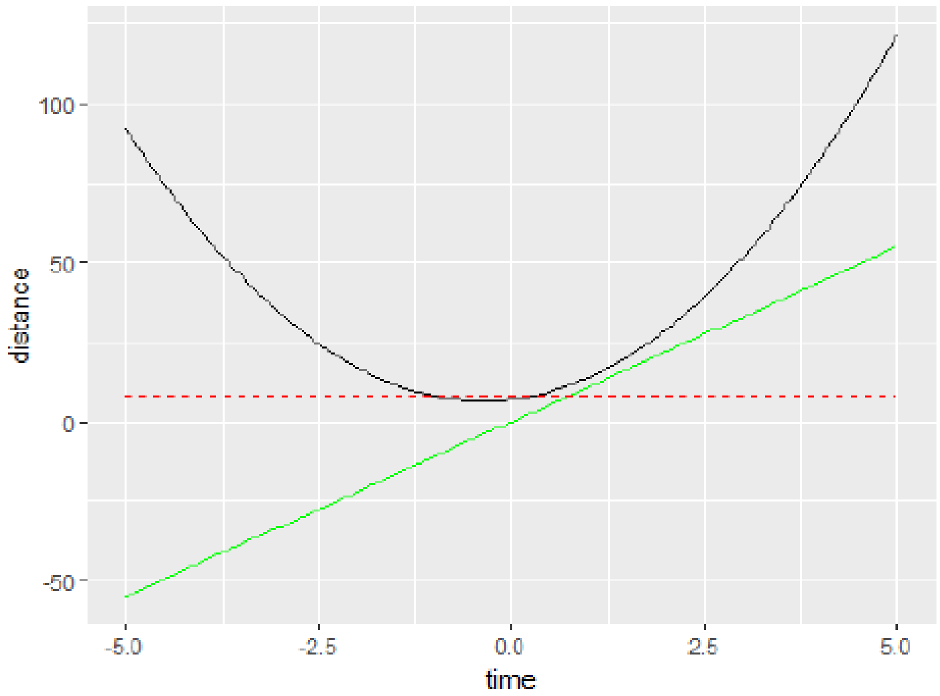
6) $\int xe^{x^2} dx = \frac{e^{x^2}}{2} + C$

Real world applications – Motion time graphs

position: $s = \Delta\, distance$

velocity: $\mathrm{v} = \dfrac{\Delta\, distance}{\Delta\, time}$

acceleration: $a = \dfrac{\Delta\, distance}{\Delta\, time^2}$



Let‘s think derivatives:

- let $a(t)$ be the acceleration of an object
- then:
  - $v(t) = A(t)$
  - $s(t) = V(t)$ or $A''(t)$

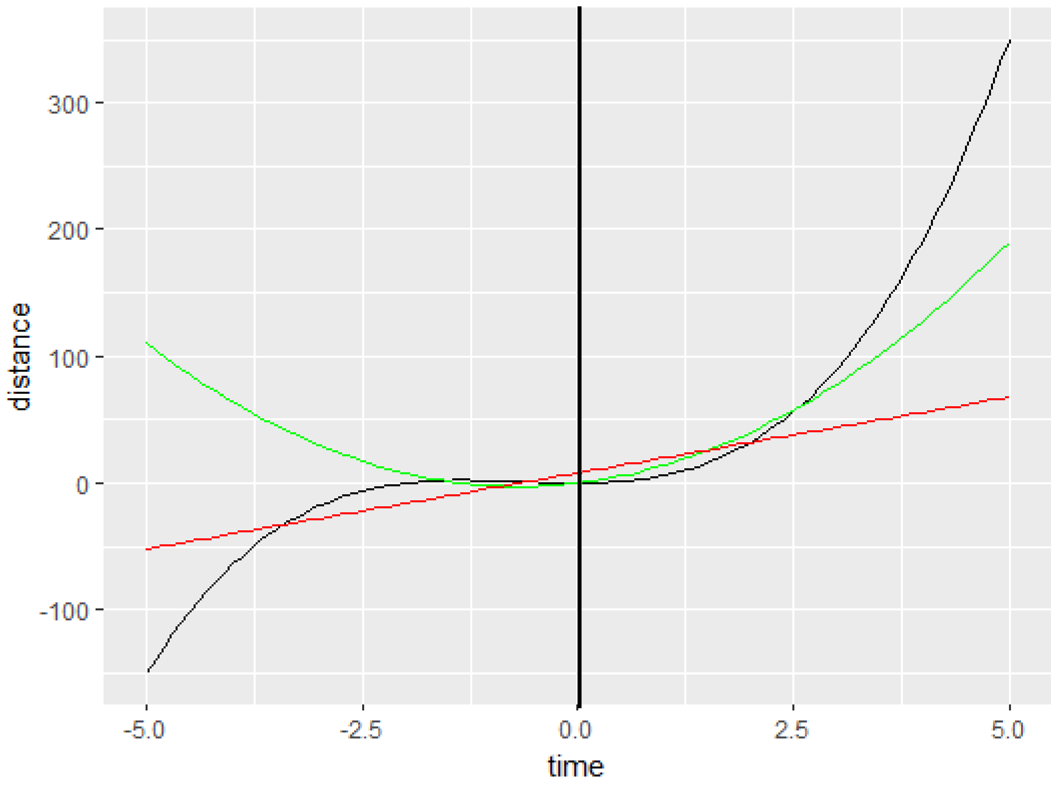
Real world applications – Motion time graphs

Task: Team up with a partner

1) Discuss why the graph looks the way it does!

2) Let $\color{red}{a(t)} = 12t + 8$

→ find $s(t)$ *and* $\color{green}{v(t)}$!



Solution:

$$\color{red}{a(t)} = 12t + 8$$
$$\color{green}{v(t)} = 6t^2 + 8t$$
$$s(t) = 2t^3 + 4t^2$$

Real world applications – Integration

Video game‘s physics

- motion and movement simulation, graphics

Medical field

- analysis of drug efficiency and processes in the body

Credit cards

- companies use differential calculus to calculate the minimum payable amount based e.g., on payment due date

All sorts of fun applications in probability theory

- see you in two weeks!

Time for your questions

- Any questions during the week?
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