

# Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Integration

# To do

- Weekly recap
- Real world applications
- Hands on practice
- Questions

# Chapter 7 | Integration

# Integration

## Antiderivative

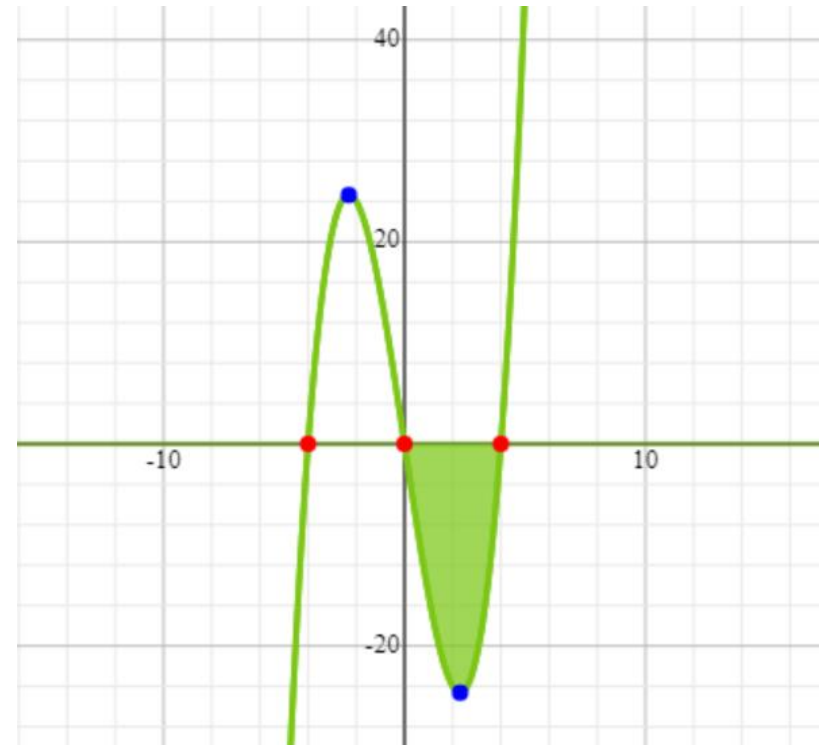
- the antiderivative is the function we get via integration
- **NEVER** forget to add a  $+C$  to your antiderivative to include constants lost in translation

## Integral

- the integral is the 'area under the curve' between the bounds  $a$  and  $b$

Let function  $f$  be  $f(x) = x^3 - 16x$

- $F(x) = \frac{x^4}{4} - 8x^2 + C$
- $\int_0^4 x^3 - 16x \, dx = -64$



# Definitive and indefinite integrals

## Definite integrals

$$\int_0^4 x^3 - 16x \, dx = -64$$

- **evaluate integral** of  $f(x)$  with **limits a** and **b**  $\rightarrow$  evaluate antiderivative at upper limit  $F(b)$  minus lower limit  $F(a)$
- will give you a **number** as a result!

## Indefinite integrals

$$\begin{aligned} \int x^3 - 16x \, dx &= F(x) + C \\ &= \frac{x^4}{4} - 8x^2 + C \end{aligned}$$

- what function did we differentiate to get  $f(x)$ ?  $\rightarrow$  **'backwards'** differentiation
- result in an **antiderivative!**  
 $\rightarrow$  always add **+C** to add any constant lost in translation!

# Fundamental theorem of calculus

- According to 1<sup>st</sup> **fundamental theorem of calculus**, we can evaluate a **definite** integral if **integrand**  $f(x)$  is **continuous** on  $[a, b]$ :

$$\int_a^b f(x) dx = F(b) - F(a)$$

1. check if function is continuous
2. find antiderivative  $F(x)$
3. evaluate  $F(x)$  for upper and lower limits  $b$  and  $a$
4. compute integral

# Integration 'Algorithm'

Let's return to  $\int_0^4 x^3 - 16x \, dx$

1. is  $f(x)$  **continuous** on  $[0, 4]$ ?

→ 'graphic' test: yep, looks good!

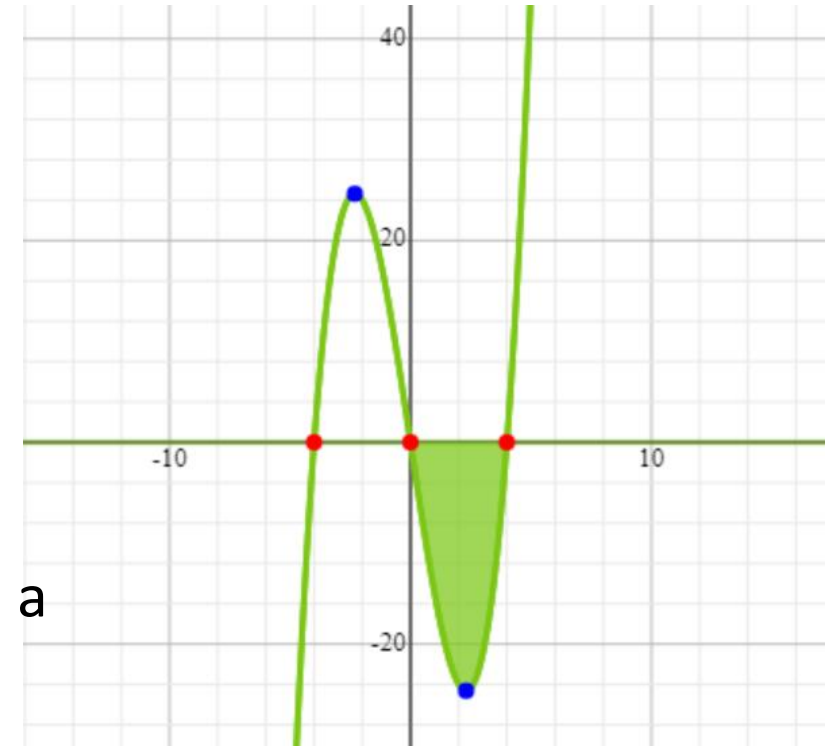
→ or check via limits... 😊

2. find **antiderivative**

$$\int x^3 - 16x \, dx = F(x) + C = \frac{x^4}{4} - 8x^2 + C$$

3. **evaluate**  $F(x)$  for **upper** and **lower limits** b and a  
and 4. compute **integral**

$$F(4) - F(0) = \frac{4^4}{4} - 8 \cdot 4^2 - \left( \frac{0^4}{4} - 8 \cdot 0^2 \right) = (64 - 128) - 0 = -64$$



# Intuition – Riemann sums and integrals

- please check out some further info on **Riemann sums**:  
[https://math.libretexts.org/Bookshelves/Calculus/Book%3A\\_Active\\_Calculus\\_\(Boelkins\\_et\\_al.\)/04%3A\\_The\\_Definite\\_Integral/4.02%3A\\_Riemann\\_Sums](https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Active_Calculus_(Boelkins_et_al.)/04%3A_The_Definite_Integral/4.02%3A_Riemann_Sums) (22.11.2023)
- or check out wikipedia for a more detailed description:  
[https://en.wikipedia.org/wiki/Riemann\\_sum](https://en.wikipedia.org/wiki/Riemann_sum) (22.11.2023)
- check out **Moore and Siegel (2013)** on pp. 135 for a quick discussion of Riemann sums!

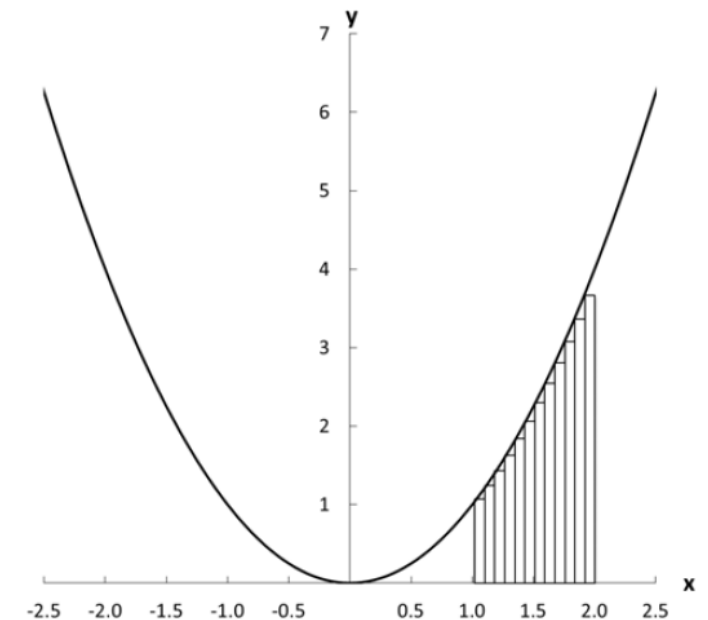


Figure 7.2: Area under  $y = x^2$  from  $x = 1$  to  $x = 2$  with Rectangles



# Hands on – rules of bounds

**Task:** team up with a partner and discuss the rules of bounds! Apply them to  $f(x) = 2x + 1$  for  $[2, 5]$  and make a sketch!

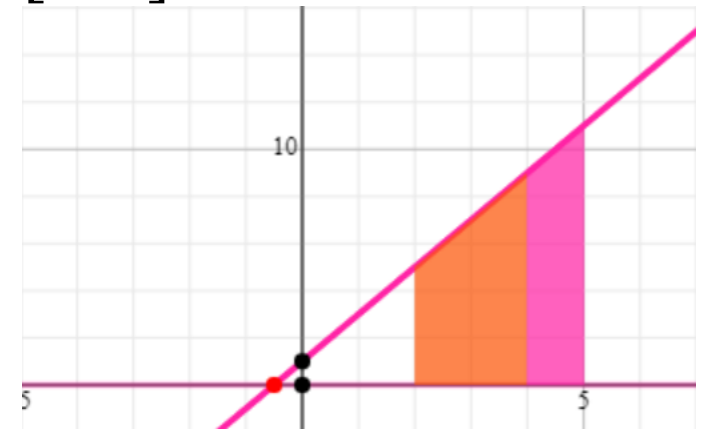
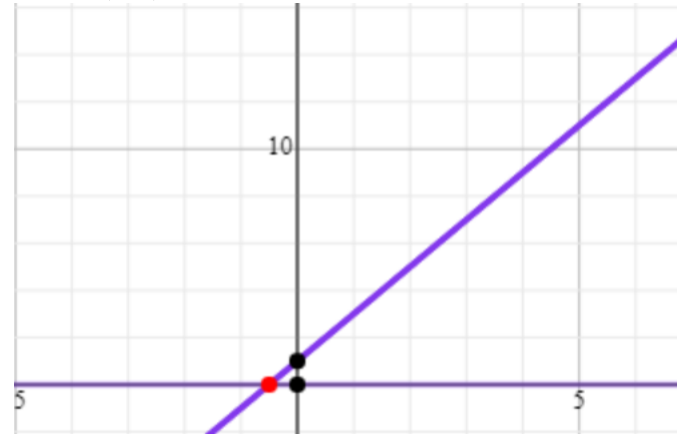
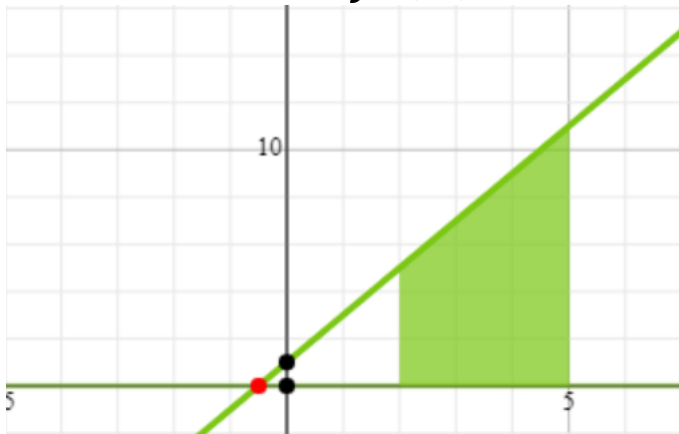
$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \text{ for } c \in [a, b]$$

# Hands on – rules of bounds

**Solution:**  $f(x) = 2x + 1$ ,  $F(x) = x^2 + 1x + C$ , for  $[2, 5]$



$$\int_2^5 2x + 1 dx = - \int_5^2 2x + 1 dx = 24$$

$$\int_2^2 2x + 1 dx = 0$$

$$\int_2^5 2x + 1 dx = \int_2^4 2x + 1 dx + \int_4^5 2x + 1 dx = 24$$

# Rules of integration ... (the most important)

## Exponent rule 1 and 2:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1 \qquad \int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + C$$

## Exponential rule 1 and 2:

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln(a)} + C$$

## Log rules 1 and 2:

$$\int \ln(x) dx = x \ln(x) - x + C \qquad \int \log_a(x) dx = \frac{x \ln(x) - x}{\ln(a)} + C$$

# Hands on – Antiderivative

**Task:** Find the antiderivatives by solving the indefinite integrals

$$\int e^4 dx$$

$$\int 3x^5 + 22 dx$$

$$\int \frac{1}{4x} dx$$

$$\int 2a^x dx$$

$$\int \ln(5x) dx$$

# Hands on

## Solution:

$$\int e^4 dx = e^4 x + C$$

$$\int 3x^5 + 22 dx = \frac{x^6}{6} + 22x + C$$

$$\int \frac{1}{4x} dx = \frac{1}{4} \int \frac{1}{x} dx = \frac{1}{4} \ln|x| + C$$

$$\int 2a^x dx = \frac{2a^x}{\ln(a)} + C$$

$$\int \ln(5x) dx = x \ln(5x) - x + C$$

# Integration by substitution

- the ‘**chain rule of integration**’

$$\int_a^b f(g(u))g'(u) du = \int_{g(a)}^{g(b)} f(x) dx$$

- **intuition:** locate the ‘**chained**’  
**function**  $f(g(x))$

→ find the **term** that is **easy to differentiate**  $g(x) \rightarrow g'(x)$ !

- 1) prepare substitution
  - 1) find substitute term **u**
  - 2) solve for **x**
  - 3) differentiate **g(u)**
  - 4) replace **integration variable**
- 2) substitution
- 3) integration
- 4) ‘substitute backwards’

# Integration by substitution

**Example:**  $f(x) = e^{3x}$

$$\int_a^b \underbrace{f(g(u))}_{\text{inner function}} \underbrace{g'(u) du}_{\text{outer function}} = \int_{g(a)}^{g(b)} f(x) dx$$

## 1. preparation

1. find substitute term  $u$  ( $\leftarrow$  'inner function')

$$u = 3x,$$

2. solve for  $x$

$$x = \frac{u}{3} \rightarrow g(u) = \frac{1}{3}u$$

3. differentiate  $g(u)$

$$g'(u) = \frac{1}{3}$$

4. replace **integration variable**

$$dx = g'(u)du = \frac{1}{3}du$$

# Integration by substitution

Example:  $f(x) = e^{3x}$

$$\int_a^b \underbrace{f(g(u))}_{\text{inner function}} \underbrace{g'(u) du}_{\text{outer function } dx} = \int_{g(a)}^{g(b)} f(x) dx$$

## 2. substitution

$$\int e^{3x} dx \quad \text{with: } u = 3x, \text{ and } dx = \frac{1}{3} du$$

$$\int e^u \cdot \frac{1}{3} du = \frac{1}{3} \int e^u du$$

## 3. integration

$$F(u) = \frac{1}{3} \cdot e^u + C$$

## 4. substitute 'backwards'

$$F(x) = \frac{1}{3} \cdot e^{3x} + C = \frac{1}{3} e^{3x} + C$$



# Integration by parts

**Intuition:** two functions  $f(x)$  and  $g'(x)$  are multiplied with each other and  $g'(x)$  is an 'easy' derivative!

→ 'product rule' of integration

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

1. Choose  $f(x)$  and  $g'(x)$
2. Find  $f(x) \rightarrow f'(x)$  and  $g'(x) \rightarrow g(x)$
3. plug your functions into the integration by parts formula
4. simplify and solve

# Integration by parts

**Example:**  $\int x^3 \ln(x) dx$        $\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$

**1. Choose  $f(x)$  and  $g'(x)$**

$$f(x) = \ln(x) \text{ and } g'(x) = x^3$$

**2. Find  $f(x) \rightarrow f'(x)$  and  $g'(x) \rightarrow g(x)$**

$$f'(x) = \frac{1}{x} \text{ and } g(x) = \frac{x^4}{4}$$

**3. Plug your functions into  $\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$**

$$\int x^3 \ln(x) dx = \ln(x) \cdot \left(\frac{x^4}{4}\right) - \int \frac{1}{x} \cdot \left(\frac{x^4}{4}\right) dx$$

# Integration by parts

**Example:**  $\int x^3 \ln(x) dx$        $\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$

## 4. simplify and solve

$$\begin{aligned}\int x^3 \ln(x) dx &= \ln(x) \cdot \left(\frac{x^4}{4}\right) - \int \frac{1}{x} \cdot \left(\frac{x^4}{4}\right) dx \\&= \frac{x^4 \ln(x)}{4} - \frac{1}{4} \int x^3 dx \\&= \frac{x^4 \ln(x)}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + C \\&= \frac{x^4 \ln(x)}{4} - \frac{x^4}{16} + C\end{aligned}$$

# When to use which method?

## Integration by substitution

- ‘chain rule’ of integration
- there is an ‘outer function’, containing an ‘inner function’
- the ‘inner function’ can be differentiated easily

→ **rule of thumb:** try substitution first!

## Integration by parts

- ‘product rule’ of integration
- two functions are multiplied with each other
- one function is an ‘easy’ derivative
- the other can be differentiated

→ you will get a feeling for when to use which rule with practice

# Hands on – Integration by parts & substitution

**Task:** Find the antiderivatives using substitution and integration by parts as you see fit!

$$1) \int \sqrt{x+1} \, dx$$

$$2) \int x e^x \, dx$$

$$3) \int \ln(x^2) \, dx$$

$$4) \int x^2 e^x \, dx$$

$$5) \int (x^2 + 1)^4 2x \, dx$$

$$6) \int x e^{x^2} \, dx$$

# Hands on – Integration by parts & substitution

## Solution:

$$1) \int \sqrt{x+1} \, dx = \frac{2}{3} (x+1)^{\frac{3}{2}} + C$$

$$2) \int x e^x \, dx = x e^x - e^x + C$$

$$3) \int \ln(x^2) \, dx = x \ln(x^2) - 2x + C = 2x \ln(x) - 2x + C$$

$$4) \int x^2 e^x \, dx = x^2 e^x - 2(e^x x - e^x) + C$$

$$5) \int (x^2 + 1)^4 2x \, dx = \frac{(x^2+1)^5}{5} + C$$

$$6) \int x e^{x^2} \, dx = \frac{e^{x^2}}{2} + C$$

# Real world applications – Motion time graphs

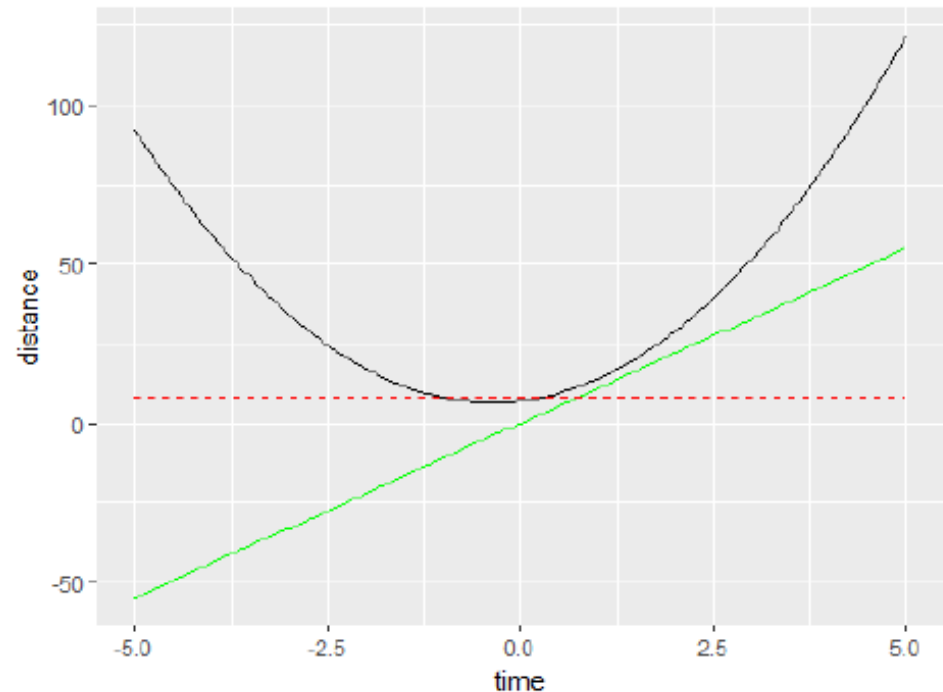
position:  $s = \Delta \text{distance}$

velocity:  $v = \frac{\Delta \text{distance}}{\Delta \text{time}}$

acceleration:  $a = \frac{\Delta \text{distance}}{\Delta \text{time}^2}$

## Let's think derivatives:

- let  $a(t)$  be the acceleration of an object
- then:
  - $v(t) = A(t)$
  - $s(t) = V(t)$  or  $A''(t)$



# Real world applications – Motion time graphs

**Task:** Team up with a partner

1) Discuss why the graph looks the way it does!

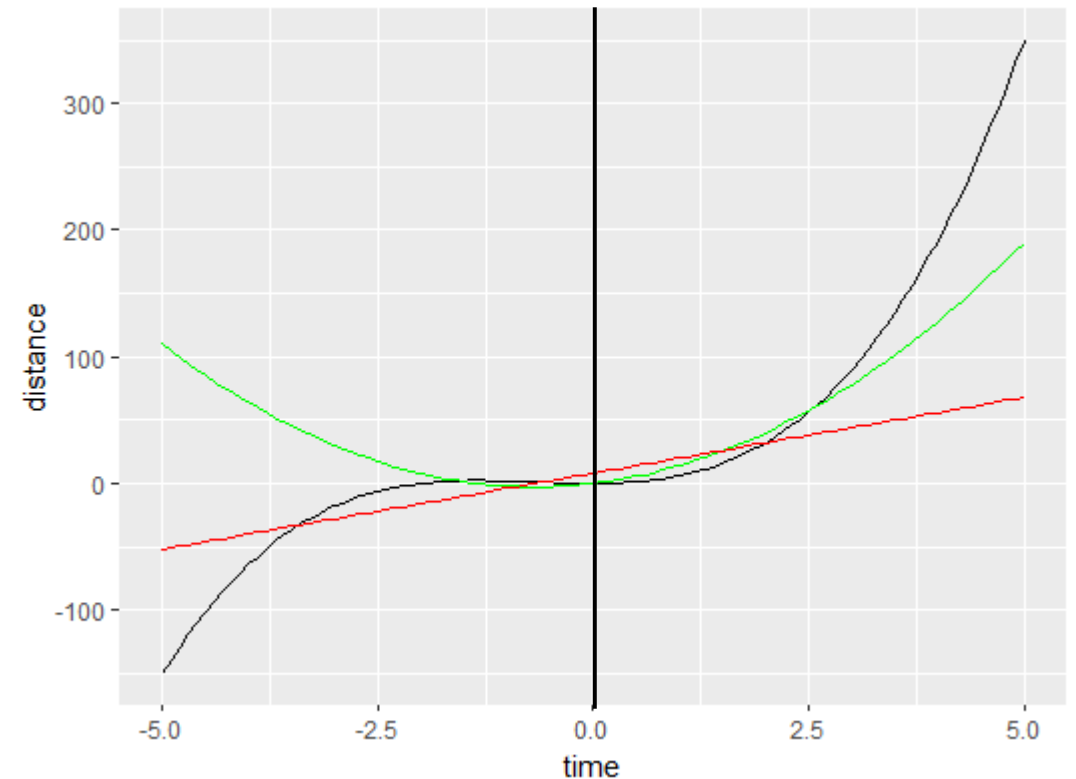
2) Let  $a(t) = 12t + 8$   
→ find  $s(t)$  and  $v(t)$  !

**Solution:**

$$a(t) = 12t + 8$$

$$v(t) = 6t^2 + 8t$$

$$s(t) = 2t^3 + 4t^2$$





# Real world applications – Integration

## **Video game's physics**

- motion and movement simulation, graphics

## **Medical field**

- analysis of drug efficiency and processes in the body

## **Credit cards**

- companies use differential calculus to calculate the minimum payable amount based e.g., on payment due date

## **All sorts of fun applications in probability theory**

- see you in two weeks!

# Time for your questions

- Any questions during the week?
  - [joerdis.strack@uni-konstanz.de](mailto:joerdis.strack@uni-konstanz.de)

