Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Functions and Relations

To do

- Weekly recap
- Real world applications
- Hands on practice
- Questions

- Upcoming Deadline: 19.11.2024 10:00 AM CET | Problem Set 01 | Algebra

Chapter 3 | Functions and Relations

Functions

Functions $\boldsymbol{f(x): A \rightarrow B}$

- **‘f maps A into B’**
- describe the relationship between two variables as a unique one-to-one mapping where each value of the domain A is mapped to one value of the codomain B

  → if this mapping is NOT unique, we are talking about a **correspondence**
- values reached by $x \in A$ are known as **image**

  → the **image** is a subset of the **codomain B**

Function composition

- We are 'sending' the result of $f(x)$ through $\mathrm{g}(x)$

→ ‚g of f of x'

- **NOTE**: keep domain conditions in mind! some functions might be defined for e.g. $\mathbb{R}^+$

$$(g \circ f)(x) = g\big(f(x)\big)$$

- **Example**:

$$f(x) = 3x - 4 \text{ and } \mathrm{g}(x) = x^2 \text{ for } x = 2$$

$$g\big(f(x)\big) = (3x - 4)^2$$

$$g\big(f(2)\big) = (3 \cdot 2 - 4)^2 = (2)^2 = 4$$

Hands on – Function composition

Task: Solve $g(f(x)) for\ x = 2$!

1) $f(x) = 6x$ *and* $g(x) = x^3$

2) $f(x) = x + \frac{3}{4}$ *and* $g(x) = x + 2$

① $g(f(x)) = (6x)^3$

$= (6 \cdot 2)^3 = 12^3 = 1728$

② $g(f(x)) = (x + \frac{3}{4}) + 2$

$= x + \frac{3}{4} + 2$

$= x + \frac{3 + 8}{4}$

$= x + \frac{11}{4}$

$= 2 + \frac{11}{4}$

$= \frac{8 + 11}{4} = \frac{19}{4} = 4.75$

Hands on – Function composition

Solution:

1) $g\big(f(2)\big) = (6 \cdot 2)^3 = 12^3 = \ 1728$

2) $g\big(f(2)\big) = \left(x + \frac{3}{4}\right) + 2 = \left(2 + \frac{3}{4}\right) + 2 = 2.75 + 2 = 4.75$

Further practice: https://www.mathsisfun.com/sets/functions-composition.html

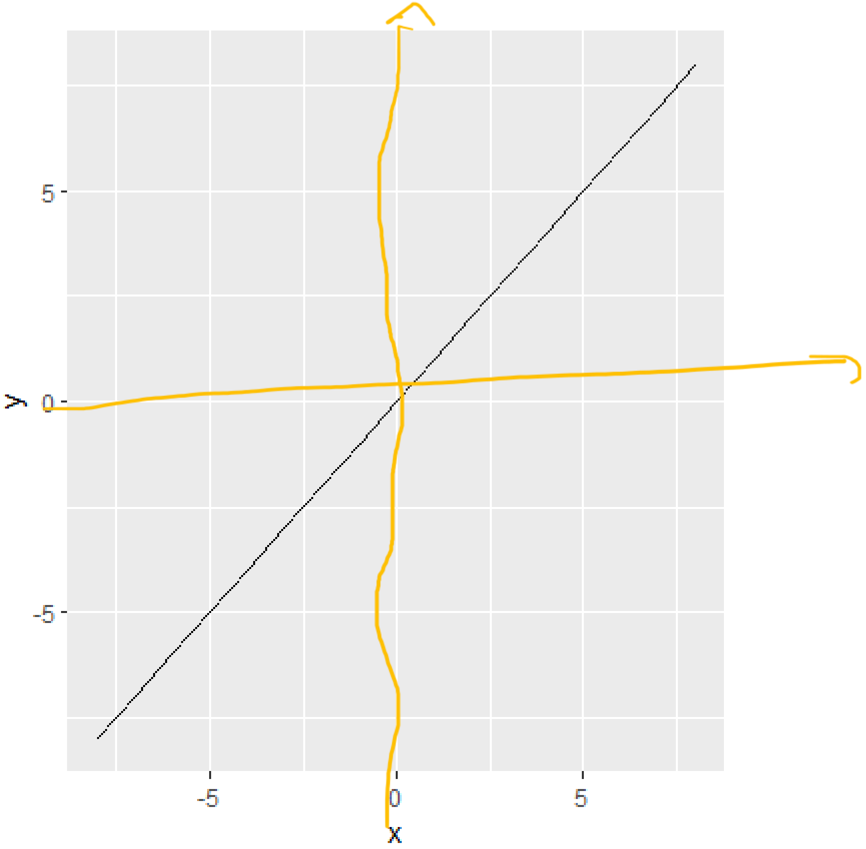
Inverse and Identity functions

Inverse functions – ‘Inverse’

- functions that return identity function when composed with their original functions
- $f^{-1}(x): B \rightarrow A$
- ‘invertible functions’ have an inverse!

Identity function

- returns value of input **argument x:** $f(x) = x$
  - ➢ $f(5) = 5$
  - ➢ $f0 = 0$
  - ➢ $f(-5) = -5$



Inverse functions

Algorithm for $f^{-1}(x)\colon B \to A$

1) replace $f(x)$ with y in original function
2) 'switch' instances of x and y (any variables) in original function
3) solve for y
4) change y to $f^{-1}(x)$

Example: find $f^{-1}(x)$ of $f(x) = 3x - 4$

Inverse functions

Example: Find $f^{-1}(x)$ of $\mathrm{f}(x) = 3x - 4$!

1) replace $f(x)$ with y in original function
   $y = 3x - 4$
2) 'switch' instances of x and y (any variables) in original function
   x$= 3y \ - 4$
3) solve for y
   $x + 4 = 3y \mid \div 3$
   $\mathrm{y} = \frac{x+4}{3}$
4) change y to $f^{-1}(x)$
   $f^{-1}(x) = \frac{x+4}{3}$

Hands on – Inverse functions

Task: Find the respective inverse of the following functions!

1) $f(x) = 2x + 6$

2) $g(x) = x^2 - 1$

3) $h(x) = \frac{1}{3}x + 10$

① $f(x) = 2x + 6$

1. $y = 2x + 6$
2. $x = 2y + 6 \quad | -6$
3. $x - 6 = 2y \quad | :2$

$y = \frac{x-6}{2}$

4. $f^{-1}(x) = \frac{x-6}{2}$

② $g(x) = x^2 - 1$

1. $y = x^2 - 1$
2. $x = y^2 - 1 \quad | +1$
3. $x + 1 = y^2 \quad | \sqrt{}$

$y = \sqrt{x+1}$

4. $f^{-1}(x) = \sqrt{x+1}$

③ $h(x) = \frac{1}{3}x + 10$

1. $y = \frac{1}{3}x + 10$

2. $x = \frac{1}{3}y + 10 \quad | -10$

3. $x - 10 = \frac{1}{3}y \quad | : \frac{1}{3}$

$y = \frac{3}{1} \cdot x - \frac{3}{1} \cdot 10$

$y = 3x - 30$

4. $f^{-1}(x) = 3x - 30$

What happens when we compose $h(x)$ with $h^{-1}(x)$? Will we receive the identity? for $x = 2$

$h(h^{-1}(x)) = \frac{1}{3}(3x - 30) + 10$

$= \frac{1}{3}(3 \cdot 2 - 30) + 10$

$= \frac{1}{3}(6 - 30) + 10$

$= \frac{-24}{3} + 10$

$= -8 + 10$

$= \underline{\underline{2 = x = y}}$

Hands on – Inverse functions

Solution:

1) $f^{-1}(x) = \frac{x-6}{2}$

2) $g^{-1}(x) = \sqrt{x+1}$ ← **Note**: We typically imply both $\sqrt{x+1}$ and $-\sqrt{x+1}$

3) $h^{-1}(x) = 3x - 30$

Further practice: https://www.mathsisfun.com/sets/function-inverse.html

Hands on – Inverse functions & function composition

Task: Are these functions inverses of each other? Show using function composition! Check, if the composed functions produce the identity function!

1) $f(x) = 2x - 4$ and $g(x) = \frac{x+4}{2}$

2) $f(x) = 4x + 3$ and $g(x) = \frac{x-4}{3}$

$f(x) = 2x - 4 \qquad g(x) = \dfrac{x+4}{2}$

① Find inverse

$f^{-1}(x) = ?$

$y = 2x - 4$

$x = 2y - 4 \quad | +4$

$x + 4 = 2y \quad | :2$

$y = \dfrac{x+4}{2}$

$f^{-1}(x) = \dfrac{x+4}{2} = g(x)$

② composition

$x = 2$

$f(g(x)) = ?$

$2\left(\dfrac{x+4}{2}\right) - 4 = 2\left(\dfrac{2+4}{2}\right) - 4$

$= \cancel{2}\left(\dfrac{6}{\cancel{2}}\right) - 4$

$= 6 - 4 = 2$

$x = 2 = y$

$f(x) = 4x + 3 \quad ; \quad g(x) = \frac{x-4}{3}$

1. Find inverse

$$f(x) = 4x + 3$$

$$y = 4x + 3$$

$$x = 4y + 3 \quad | -3$$

$$x - 3 = 4y \quad | :4$$

$$y = \frac{x-3}{4}$$

$$f^{-1}(x) = \frac{x-3}{4} \neq g(x) = \frac{x-4}{3}$$

g(x) is NOT the inverse of f(x)!

2. Function composition

for e.g. $x = 2$

$$f(g(x)) = 4\left(\frac{x-4}{3}\right) + 3$$

$$= 4\left(\frac{2-4}{3}\right) + 3$$

$$= 4\left(\frac{-2}{3}\right) + 3$$

$$= \frac{4\cdot(-2)}{3} + 3$$

$$= -\frac{8}{3} + 3$$

$$= \frac{1}{3} \neq 2$$

composing f(x) with g(x) does NOT yield the identity!

Hands on – Inverse functions & function composition

Solution:

1) Yes!

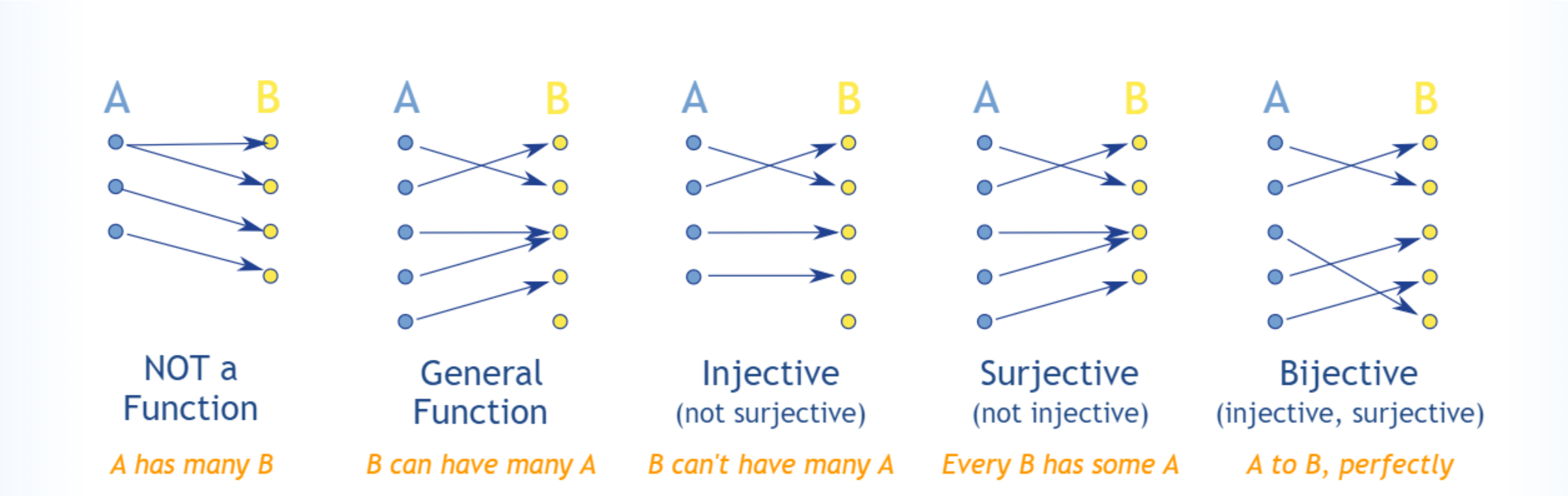
$$\rightarrow f\left(g(x)\right) = 2\left(\frac{x+4}{2}\right) - 4 = x$$

2) No!

$$\rightarrow f\left(g(x)\right) = \frac{4x}{3} - \frac{16}{3} + 3 \neq x$$

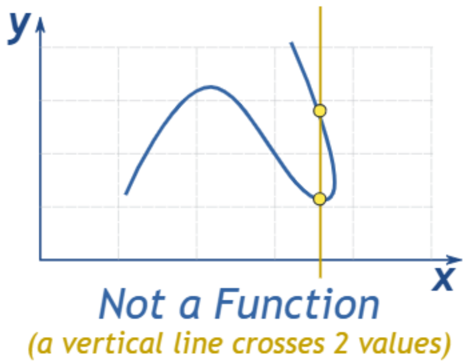
Injective, bijective, surjective functions…

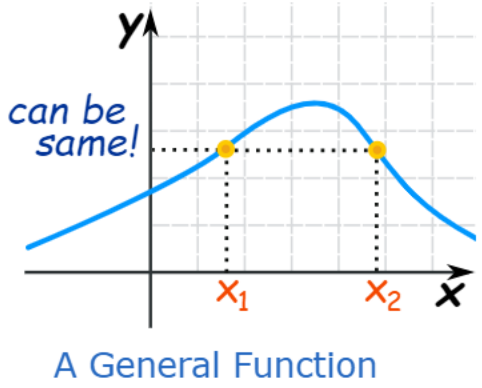
… are classes of functions that describe, how arguments x are mapped to images y

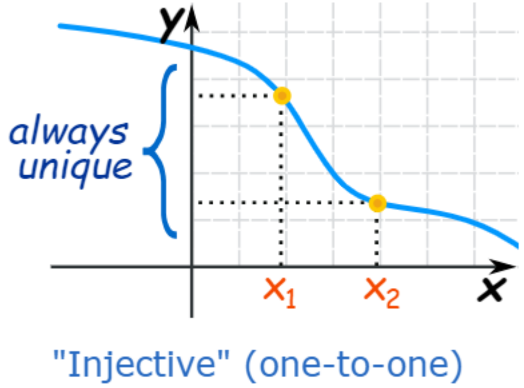


https://www.mathsisfun.com/sets/injective-surjective-bijective.html

Injective, bijective, surjective functions







A function f is…

- **injective** if and only if whenever $f(x) = f(y), x = y$
- **surjective** iff f(A) = B or for every ***y*** in ***B***, there is at least one ***x*** in ***A*** such that $f(x) = y$
- **bijective** (from set ***A*** to ***B***) if, for every ***y*** in ***B***, there is exactly one ***x*** in ***A*** such that $f(x) = y$

https://www.mathsisfun.com/sets/injective-surjective-bijective.html

Monotonicity

Monotonicity is a concept to describe **order:**

- a function f is called **monotonically increasing**, if for every
$$x \leq y, f(x) \leq f(y)$$
so that f preserves order

- a function f is called **monotonically decreasing**, if for every
$$x \geq y, f(x) \geq f(y)$$
so that f preserves order

Monotonicity

Table 3.2: Monotonic Function Terms

Term	Meaning
Increasing	Function increases on subset of domain
Decreasing	Function decreases on subset of domain
Strictly increasing	Function always increases on subset of domain
Strictly decreasing	Function always decreases on subset of domain
Weakly increasing	Function does not decrease on subset of domain
Weakly decreasing	Function does not increase on subset of domain
(Strict) monotonicity	Order preservation; function (strictly) increasing over domain

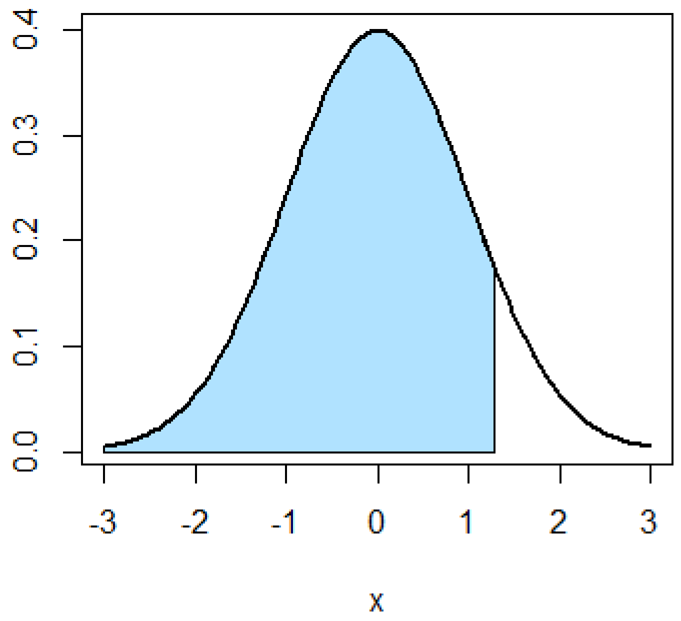
Moore & Siegel, 2013, p.51

NOTE: ALL strictly monotonic functions are invertible due to a strict one-to-one mapping!

Real world applications - Monotonicity

Monotonicity describes **strength** of **relationships** between **variables**!

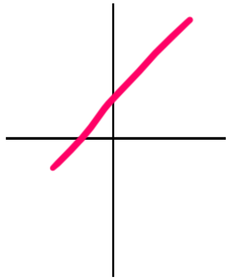
- think about correlation and probability theory!
- if $\boldsymbol{X}$ is a **RV**, its cumulative distribution function is a **monotonically increasing** function!
- $F_X(x) = P(X \leq x)$



Linear functions & equations

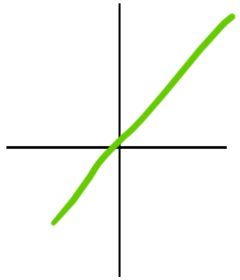
- **linear equations** in slope-intercept form $f(x) = \overset{\text{Slope}}{mx} + \overset{\text{Intercept}}{b}$
  - consist only of terms like x^1 and $x^0 = 1$ multiplied by constants
  - are also called 'affine function'

Note: the intercept $b \neq 0$



- **linear functions** are of the same form but additionally satisfy … because they are fixed at the origin! $f(x) = mx + 0$
  - additivity – superposition $f(x_1 + x_2) = f(x_1) + f(x_2)$
  - scaling – homogeneity $f(ax) = a \cdot f(x)$ for all a

Note: the intercept is at the origin!



- Note: We often call the equation above a 'linear function' – even though it does not satisfy the scaling and additivity properties!

Linear functions & equations - Additivity

$\alpha \neq 0$

$y = f(x) = \alpha + \beta x$

Linear functions $y = f(x) = \beta(x)$

$f(x_1 + x_2) = 0 + \beta(x_1 + x_2) = \beta x_1 + \beta x_2$

$f(x_1) + f(x_2) = \beta x_1 + \beta x_2$

Linear equations/affine functions

$f(x_1 + x_2) = \alpha + \beta(x_1 + x_2) = \alpha + \beta x_1 + \beta x_2$

$f(x_1) + f(x_2) = (\beta x_1 + \alpha) + (\beta x_2 + \alpha)$

$\alpha + \beta x_1 + \beta x_2 \neq 2\alpha + \beta x_1 + \beta x_2$

$\alpha \neq 2\alpha$

Linear functions & equations - Scaling

$\alpha \neq 0$

$y = f(x) = \alpha + \beta x$

Linear functions $y = f(x) = \beta(x)$

$f(ax) = \beta(ax) = a\beta(x)$

$af(x) = a\beta(x)$

//

Linear equations/affine functions

$f(ax) = \alpha + (\beta(ax)) = \alpha + a\beta x$

$af(x) = a\alpha + a\beta x$

$\alpha + a\beta x \neq a\alpha + a\beta x$

$\alpha \neq a\alpha$

Real world applications – linear equation

But don't you worry, there are many applications of linear equations, including your potentially favorite one – random variables!

Distribution of parameters of random variables:

- Let X be a RV with expected value $E(X)$ and variance $Var(X)$
  → generate a new RV using the linear transformation of X:
- $Y = a + bX$ with expected value $\mathrm{E(Y) = a + b} \cdot E(X)$ and $\mathrm{Var(Y) = b^2} \cdot Var(X)$

→ if X is distributed normally, Y will be distributed normally, too!

Exponents, roots, logarithms

Idea: Let's look at b^n

- How do I solve for x in $b^n = x$?
  → **exponents**
- How do I solve for n in $b^n = x$?
  → **logarithms**
- How do I solve for b in $b^n = x$?
  → **radicals/roots**

Exponentials

$x^1 = x$

$x^0 = 1$

$x^{-1} = \frac{1}{x}$

$x^m x^n = x^{m+n}$

$\frac{x^m}{x^n} = x^{m-n}$

$(x^m)^n = x^{mn}$

$(xy)^n = x^n y^n$

$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

$x^{-n} = \frac{1}{x^n}$

$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$

Logarithms

Logarithmic form: $log_b m = x$

Exponential form: $b^x = m$

$$\ln x = log_{e^x}$$

$$\ln e^x = x$$

$$log\ 10^x = x$$

$$log_n\ n^x = x$$

$$log_b(x) = \ log_b(n) \rightarrow x = n$$

$$log_b(m) + log_b(n) = log_b(mn)$$

$$log_b(m) - log_b(n) = log_b\left(\frac{m}{n}\right)$$

$$k \cdot log_b(m) = log_b\left(m^k\right)$$

$$\log_b(m) = \frac{\log m}{\log b}$$

Radicals/Roots

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$$

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

$$\sqrt{a^n} = (\sqrt{a})^n = a^{\frac{n}{2}}$$

→ **even more rules (you probably won‘t need):**
https://www.mathwords.com/s/square_root_rules.htm

Time for your questions

- Any questions during the week?
  - joerdis.strack@uni-konstanz.de