Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Vector spaces and systems of equations

To do

- Weekly recap
- Real world applications
- Hands on practice
- Questions

Chapter 13 | Vector spaces and systems of equations

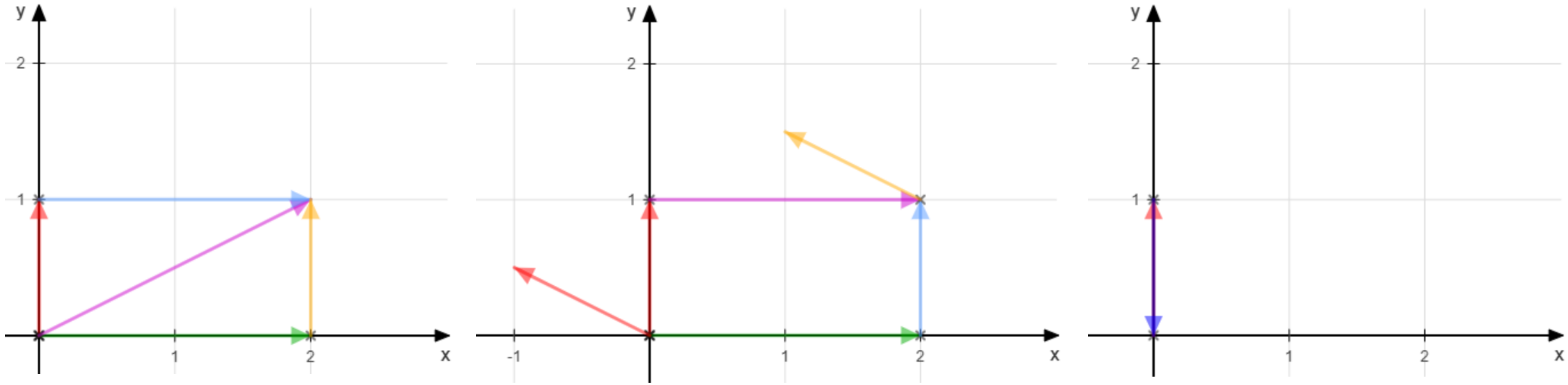
Vector spaces

Let V be a nonempty set of defined objects, e.g., vectors, on which **addition** can be performed for any elements $\vec{x}, \vec{y} \in V$ and denote it by $\vec{x} + \vec{y}$. Let **scalar multiplication** be a second operation defined for a real number $\lambda \in \mathbb{R}$ and any element $\vec{x} \in \mathbb{R}$ as $\lambda\vec{x}$.

Vector spaces

V is a vector space, if the following properties hold for all of its elements:

- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- there exists a vector $\vec{0} \in \mathbb{R}$, such that $\vec{x} + \vec{0} = \vec{x} \;\; \forall \, \vec{x} \in V$
- there exists a unique vector $-\vec{x} \in V \; \forall \, \vec{x} \in V$, such that $-\vec{x} + \vec{x} = \vec{0}$



Linear combinations & independence

- a vector $\vec{v} \in V \in \mathbb{R}$ is a linear combination of vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, \cdots, \overrightarrow{x_n}$, if there exist scalars $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that

$$a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + \cdots a_n\overrightarrow{x_n} = \vec{v}$$

- vectors are said to be linearly independent, if there exists no nontrivial linear combination of them that equals the zero vector:

$$a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + \cdots a_n\overrightarrow{x_n} = \vec{0}$$

- on the other hand, vectors are said to be linearly dependent, if there exist scalars $a_1, a_2, \cdots, a_n \in \mathbb{R}$ – not all zero – such that:

$$a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + \cdots a_n\overrightarrow{x_n} = \vec{0}$$

Spanning vectors & linear hull

The linear span / linear hull of a set of vectors S contains all linear combinations of vectors in S. Imagine them as giving you ‚access‘ to every vector in S via linear combinations

- two linearly independent vectors span a plane
- S is a generator / generating system of V – S spans V – S is a spanning set of V
  - →these are all equivalent

Examples:

- The linear hull (and a basis!) of $\mathbb{R}^3$ is spanned by {(-1, 0, 0), (0, 1, 0), (0, 0, 1)}
- the space of polynomials is spanned by the set on monomials x^n for non-negative integers n

Vector bases

The basis $\boldsymbol{B} \subseteq V$ of a vector space V consists of n linearly independent spanning vectors to 'open up' $\mathbb{R}^n$ and has the following properties:

- every element in V can be depicted as a linear combination of B
- B is the minimal generating system of V → V is a linear hull of B
  - → if we remove an element from B, this property no longer holds!
- B is the maximum subset of V – if an element is added to B, it is not longer linearly independent
  - → B is a linearly independent generating system of V

Maxtrix & vector rank

- The rank of matrices and vectors describes the maximum number of linearly independent rows/columns:
  - $rank(A), rk(A)$ and $rank(f), rk(f)$
  - reduce to row echelon form
  - count rows/columns that are non-zero!
- a quadratic matrix is of full rank (regular & invertible), iff:
  - its rank is equal to its number of rows/columns
  - its determinant is different from 0
  - no eigenvalue is equal to 0

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 7 \\ 0 & -4 & 6 \end{pmatrix} \mid \text{III} + 2\text{II}$$

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 20 \end{pmatrix} \to rank(A) = 3$$

- $|\boldsymbol{A}| = 80 \neq 0$
- **Eigenvalues**:

$$\lambda_1 = 2,$$
$$\lambda_2 = 4 + 2\sqrt{6}i,$$
$$\lambda_3 = 4 - 2\sqrt{6}i$$

→ A is of full rank & regular

Hands on – Rank

Task: Find the rank of the following matrices!

1) $A = \begin{bmatrix} 3 & 7 \\ 6 & 14 \end{bmatrix}$
2) $B = \begin{bmatrix} 1 & 5 & 11 \\ 2 & 3 & 0 \end{bmatrix}$
3) $\mathrm{C} = \begin{bmatrix} 4 & 1 & 3 \\ 13 & 4 & 1 \\ 10 & 2.5 & 7.5 \\ 5 & 2 & 0 \end{bmatrix}$

Hands on – Rank

Solution:

1) $A = \begin{bmatrix} 3 & 7 \\ 6 & 14 \end{bmatrix} \rightarrow II\text{-}2I = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} \rightarrow rank(A) = 1$

2) $B = \begin{bmatrix} 1 & 5 & 11 \\ 2 & 3 & 0 \end{bmatrix} \rightarrow II\text{-}2I = \begin{bmatrix} 1 & 5 & 11 \\ 0 & -7 & -22 \end{bmatrix} \rightarrow rank(B) = 2$

3) $C = \begin{bmatrix} 4 & 1 & 3 \\ 13 & 4 & 1 \\ 10 & 2.5 & 7.5 \\ 5 & 2 & 0 \end{bmatrix} \rightarrow III - 2.5I, II - 3.25I, IV \ - 1.25I, IV \ - II, swap\ II\ and\ IV$

$= \begin{bmatrix} 4 & 1 & 3 \\ 0 & 0.75 & -8.75 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow rank(C) = 3$

Systems of equations – Substitution

Algorithm:

1) reduce one equation to I: $\lambda = ax + by + cz$
2) plug I into II and III

   → obtain reduced forms of II and III
3) repeat steps 1) and 2) on reduced forms

   → obtain first variable

   → use to solve for other variables

Systems of equations – Substitution

Example:

$$\begin{cases} 2x + y + 3z = 4 \\ 4x - 3y + 4z = (-3) \\ 3x + 2y - 5z = 2 \end{cases}$$

1) **solve**

$2x + y + 3z = 4$

→ $x = 2 - \frac{1}{2}y - \frac{3}{2}z$

2) **plug I into II**

$4(2 - \frac{1}{2}y - \frac{3}{2}z) - 3y + 4z = (-3)$

$8 - 2y - 6z - 3y + 4z = (-3)$

$8 - 5y - 2z = (-3) \mid -8$

$-5y - 2z = (-11)$

- **plug I into II**

$3(2 - \frac{1}{2}y - \frac{3}{2}z) + 2y - 5z = 2$

$6 - \frac{3}{2}y - \frac{9}{2}z + 2y - 5z = 2$

$\frac{1}{2}y - \frac{19}{20}z = (-4)$

Systems of equations – Substitution

Example:

$$\begin{cases} x = 2 - \frac{1}{2}y - \frac{3}{2}z \\ -5y - 2z = (-11) \\ \frac{1}{2}y - \frac{19}{20}z = -4 \end{cases}$$

- **repeat steps 1) and 2)**

$-5y - 2z = (-11)$

→ $-2z = (-11) + 5y$

→ $z = 5.5 - 2.5y$

- **plug II into III**

$\frac{1}{2}y - 9.5(5.5 - 2.5y) = (-4)$

$\frac{1}{2}y - 52.25 + 23.75y = (-4)$

$24.25y = 48.25 | \div 24.25$

$y \approx 1.99$

- **solve for z**

$-5 \cdot 1.99 - 2z = (-11) \mid +9.95$

$-2z = -1.05 \mid \div (-2)$

$z = 0.525$

Systems of equations – Substitution

Example:

$$\begin{cases} x = 2 - \frac{1}{2}y - \frac{3}{2}z \\ -5y - 2z = (-11) \\ \frac{1}{2}y - \frac{19}{20}z = -4 \end{cases}$$

→ **solve for x**

$$x = 2 - \frac{1}{2} \cdot 1.99 - \frac{3}{2} \cdot 0.525$$

$$x = 0.2165$$

→ $x = 0.2175 \approx 0.22,\ y \approx 1.99,\ z = 0.525 \approx 0.53$

→ Note that I rounded in between!

Systems of equations – Elimination/Gauss

Algorithm:

1) switch rows/equations to approach step-form
2) use multiples of rows to obtain step-form
3) solve for variables $x, y, \cdots, n$

→ **Goal:** $\begin{matrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{matrix}$

Systems of equations – Elimination/Gauss

Example:

$$\begin{cases} 2x - 2y + 4z = 0 \\ -4x + 2y - 12z = 0 \\ 2x - 4z = 3 \end{cases}$$

1) **switch rows/equations**
- we could switch row II and III
- … but we don‘t have to, so we won‘t ;)

2) **use multiples of rows to obtain step-form**

$2x{-}2y{+}4z = 0$	
$-4x + 2y - 12z = 0$	
$2x + 0\text{y} - 4z = 3$	III - I
$2x{-}2y{+}4z = 0$	
$-4x + 2y - 12z = 0$	II + 2I
$0\text{x} + 2\text{y} - 8z = 3$	
$2x{-}2y{+}4z = 0$	
$0\text{x} - 2y - 4z = 0$	
$0\text{x} + 2\text{y} - 8z = 3$	III + II

Systems of equations – Elimination/Gauss

- **Example**:

$2x{-}2y{+}4z = 0$	
$0 - 2y - 4z = 0$	
$0 + 2y - 8z = 3$	III + II
$2x{-}2y{+}4z = 0$	
$0 - 2y - 4z = 0$	
$0 + 0 - 12z = 3$	

3) solve for z, y, x!

$$z = -\frac{3}{12} = -0.25$$

$$-2y = 4(-0.25)$$

$$y = 0.5$$

$$2x - 2 \cdot 0.5 + 4 \cdot (-0.25)$$

$$x = \frac{2}{2} = 1$$

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ -4x + 2y - 12z &= 0 \\ 2x + 0y - 4z &= 3 \end{aligned} \quad \Bigg| \; \text{III} - \text{I}$$

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ -4x + 2y - 12z &= 0 \\ (2x - 2x) + (0y - (-2y)) + (-4z - 4z) &= 3 - 0 \end{aligned}$$

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ -4x + 2y - 12z &= 0 \quad \text{II} + 2\text{I} \\ 0x + 2y - 8z &= 3 \end{aligned}$$

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ (-4x + 2 \cdot 2x) + (2y + 2 \cdot (-2y)) + (-12z + 2 \cdot 4z) &= 0 \cdot 2 \cdot 0 \\ 0x + 2y - 8z &= 3 \end{aligned}$$

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ 0x - 2y - 4z &= 0 \\ 0x + 2y - 8z &= 3 \quad \text{III} + \text{II} \end{aligned}$$

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ 0x - 2y - 4z &= 0 \\ (0x + 0x) + (2y + (-2y)) + (-8z + (-4z)) &= 3 + 0 \end{aligned}$$

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ 0x - 2y - 4z &= 0 \\ 0x + 0y - 12z &= 3 \end{aligned}$$

$$\Rightarrow z = \frac{3}{-12} = \underline{\underline{-0.25}}$$

$$\Rightarrow y = \frac{4 \cdot (-0.25)}{-2} = \frac{-1}{-2} = \underline{\underline{0.5}}$$

$$\Rightarrow x = \frac{(-4 \cdot (-0.25) + 2 \cdot 0.5)}{2} = \frac{2}{2} = \underline{\underline{1}}$$

Systems of equations – Matrix inversion

Algorithm:

1) convert SoE into Matrix A and vector $\vec{b}$
2) find $|A|$
3) compute A^{-1}
4) apply $\vec{x} \ = \ A^{-1} \cdot \vec{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \ and \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Systems of equations – Matrix inversion

Example:

$$\begin{cases} 2x + 6y - 3z = 8 \\ 3x - 2y + 2z = 4 \\ \;\; x + 3y + z = 1 \end{cases}$$

1) convert into A and $\vec{b}$

$$A = \begin{bmatrix} 2 & 6 & -3 \\ 3 & -2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$$

2) find $|A|$

$$|A| = [(2 \cdot (-2) \cdot 1) + (6 \cdot 2 \cdot 1) + ((-3) \cdot 3 \cdot 3)] \\ -\left[\left(1 \cdot (-2) \cdot (-3)\right) + (3 \cdot 2 \cdot 2) + (1 \cdot 3 \cdot 6)\right]$$

$$|A| = -55 \neq 0$$

→ A has an inverse and is of full rank!

3) find $A^{-1} = \frac{1}{|A|} C^T$

M11 = -8	M12 = 1	M13 = 11
M21 = 15	M22 = 5	M23 = 0
M31 = 6	M32 = 13	M33 = -22

Systems of equations – Matrix inversion

Example:

3) find $\boldsymbol{A^{-1} = \frac{1}{|A|} C^T}$

C11 = (-8)	C12 = (-1)	C13 = 11
C21 = (-15)	C22 = 5	C23 = 0
C31 = 6	C32 = (-13)	C33 = -(22)

C= $\begin{bmatrix} -8 & -1 & 11 \\ -15 & 5 & 0 \\ 6 & -13 & -22 \end{bmatrix}$ and $\mathrm{C}^{\mathrm{T}} = \begin{bmatrix} -8 & -15 & 6 \\ -1 & 5 & -13 \\ 11 & 0 & -22 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{-55} \begin{bmatrix} -8 & -15 & 6 \\ -1 & 5 & -13 \\ 11 & 0 & -22 \end{bmatrix} = \begin{bmatrix} \frac{8}{55} & \frac{3}{11} & -\frac{6}{55} \\ \frac{1}{55} & -\frac{1}{11} & \frac{13}{55} \\ -\frac{1}{5} & 0 & \frac{2}{5} \end{bmatrix}$$

Systems of equations – Matrix inversion

Example:

4) apply $\boldsymbol{\vec{x} = A^{-1} \cdot \vec{b}}$

$$\begin{bmatrix} \frac{8}{55} & \frac{3}{11} & -\frac{6}{55} \\ \frac{1}{55} & -\frac{1}{11} & \frac{13}{55} \\ -\frac{1}{5} & 0 & \frac{2}{5} \end{bmatrix} \cdot \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{64}{55} + \frac{12}{11} - \frac{6}{55} \\ \frac{8}{55} - \frac{4}{11} + \frac{13}{55} \\ -\frac{8}{5} + 0 + \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{118}{55} \\ \frac{1}{55} \\ -\frac{6}{5} \end{pmatrix}$$

→ $x \approx 2.15 \;, y \approx 0.02 \;, z = (-1.2)$

Systems of equations – Cramer‘s rule

Algorithm:

1) convert SoE into Matrix A and vector $\vec{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \; and \; \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

2) find $|A|$
3) replace each column j with vector $\vec{b}$ to obtain B_j
4) compute determinants of B_j
5) apply Cramer’s rule $x_i = \frac{|B_j|}{|A|}$

Systems of equations – Cramer‘s rule

Example:

$$\begin{cases} 4x - 3y + z = 2 \\ 2y + 2z = 6 \\ x - 5y + 4z = (-7) \end{cases}$$

1) convert into A and $\vec{b}$

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 0 & 2 & 2 \\ 1 & -5 & 4 \end{bmatrix} \text{ and } \vec{b} = \begin{pmatrix} 2 \\ 6 \\ -7 \end{pmatrix}$$

2) find $|A|$

$$|A| = \left[(4 \cdot 2 \cdot 4) + \left((-3) \cdot 2 \cdot 1\right) + (1 \cdot 0 \cdot (-5))\right]$$
$$-\left[(1 \cdot 2 \cdot 1) + \left((-5) \cdot 2 \cdot 4\right) + (4 \cdot 0 \cdot (-3))\right]$$
$$|A| = 64 \neq 0$$

→ A has a non-negative determinant and is of full rank!

Systems of equations – Cramer‘s rule

Example:

3) replace each column j with vector $\vec{b}$ to obtain B_j

$$B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 6 & 2 & 2 \\ -7 & -5 & 4 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 6 & 2 \\ 1 & -7 & 4 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 4 & -3 & 2 \\ 0 & 2 & 6 \\ 1 & -5 & -7 \end{bmatrix}$$

4) compute determinants of B_j

$$|B_1| = 134$$

$$|B_2| = 150$$

$$|B_3| = 42$$

5) apply Cramer’s rule $x_i = \frac{|B_j|}{|A|}$

$$x = \frac{134}{64} \approx 2.09$$

$$y = \frac{150}{64} \approx 2.34$$

$$z = \frac{42}{64} \approx 0.66$$

… When do I choose which method?

- your task states, which method to choose
- get a feeling, which ones work best for you
- my experiences (considering the exam)
    - → matrix inversion requires a lot of time and has a higher potential for miscalculation
    - →same goes for substitution, however, this might feel the most intuitive to some of you ☺
    - →if you spot many zeros, you might want to choose Gauss or substitution
    - →Cramer’s rule can be a (very!) quick fix – if you feel comfortable with calculating determinants

Hands on – Substitution & Gauss elimination

Task: Solve the following SoE – using substitution and Gauss elimination

1) $\begin{cases} x - 7y = 4 \\ 2x + y = (-2) \end{cases}$

2) $\begin{cases} 4x - 3y = 6 \\ 20x - 15y = (-30) \end{cases}$

3) $\begin{cases} x - y + 2z = 6 \\ 2x + 3y + 2z = 4 \\ 3x + y - z = 8 \end{cases}$

Hands on – SoE

Solution:

$$1)\ \begin{cases} x = -\frac{2}{3} \\ y = -\frac{2}{3} \end{cases}$$

2) No solution!

$$3)\ \begin{cases} x = 1 \\ y = 1 \\ z = 3 \end{cases}$$

Hands on – Cramer‘s rule & matrix inversion

Task: Solve the following SoE – using Cramer‘s rule and matrix inversion

1) $$\begin{cases} 3x - y = 2 \\ 2x - y = (-3) \end{cases}$$

2) $$\begin{cases} x - y + 2z = 6 \\ 2x + 3y + 2z = 11 \\ 3x + 2y + z = 8 \end{cases}$$

Hands on – SoE

Solution:

$$1)\ \begin{cases} x = 5 \\ y = 13 \end{cases}$$

$$2)\ \begin{cases} x = 1 \\ y = 1 \\ z = 3 \end{cases}$$

Time for your questions

- Any questions during the week?
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