Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Basics and Preliminaries

GitHub: https://github.com/joerdisstrack/tutorial_mathematics_social_science

To do

- introduction: problem sets
- weekly recap
  - basics and preliminaries
  - intro to algebra
- hands on practice
- questions

Introduction

Problem Sets:

- There will be four problem sets throughout the semester, each is worth 12.5% of your final grade
- There will be 1 PS for each main block:
  - Algebra – 05/11/2024
  - Calculus ID – 26/11/2024
  - Probability – 07/01/2025
  - Multivariate Calculus – 21/01/2025

- Note that these dates are preliminary and may change throughout the semester

POL-30410: Mathematics for Social Scientists

WiSe 2024-2025

October 2024						
S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

November 2024						
S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

December 2024						
S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

January 2025						
S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

February 2025						
S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	

March 2025						
S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

DATE	DESCRIPTION	MOORE & SIEGEL	
22/10/24	Introduction and Course Overview Preliminaries	1	
29/10/24	Algebra Review	2	
05/11/24	Functions, Relations, & Utility Limits and Continuity, Sequences & Series, More On Sets	3-4	
12/11/24	Calculus Fundamentals (Differentiation)	5-6	I
19/11/24	The Integral	7	
26/11/24	Extrema in One Dimension	8	
03/12/24	An Introduction to Probability	9	II
10/12/24	Discrete Distributions	10	
17/12/24	Continuous Distributions	11	
07/01/25	Introduction to Linear Algebra	12	III
14/01/25	Vector Spaces and Systems of Equations	13	
21/01/25	Eigenvalues and Markov Chains	14	
28/01/25	Introduction to Multivariate Calculus	15	IV
04/02/25	Multivariate Optimization	16	
20/02/25	Exam		
21/03/25	Resit Exam		

Introduction

Problem Sets:

- You have 1 week to complete each PS and to hand in a **scan** in **pdf format** via Ilias **within** the deadline
- You should name your file → PSX_matrikelnr.pdf e.g. PS1/1234567
- Please write your last name and matrikel nr. /Student ID on the last page of your paper version before you scan it

Introduction

There are a bunch of good helping hands out there to support your learning process:

- Wolfram Alpha
- Symbolab
- R, python, MATLAB
- Chat GPT
- GeoGebra
- etc…

→ Do NOT rely on these too much! You CANNOT use them during the exam

Chapter 1 | Preliminaries

Preliminary vocab

Theory

- a set of statements involving concepts and concern relationships among abstract concepts

Statements

- comprise assumptions, propositions, corollaries, and hypotheses

Assumptions are asserted by us

- **propositions** and corollaries are deduced from these assumptions
    - **hypotheses** are derived from these deductions and then empirically challenged

Preliminary vocab

Concepts

- inventions that human beings create to help them understand the world and may take on different values

Variables

- indicators we develop to measure our concepts
- mathematically they take on different values in given sets

Constants

- concept or a measure that has a single value for a given set

Sets

- describe variables as **discrete** or **continuous**
- **discrete**
  - a variable is discrete if each one of its possible values can be associated with a single integer
- **continuous**:
  - a variable is continuous if its values cannot be assigned a single integer

  → typically assumed to be drawn from subset of real numbers

- sets give the domain – the range of values – a concept may take

Table 1.1: Common Sets

Notation	Meaning
$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real (rational and irrational) numbers
$\mathbb{C}$	Complex numbers
Subscript: $\mathbb{N}_+$	Positive (negative) values of the set
Superscript: $\mathbb{N}^d$	Dimensionality (number of dimensions)

Moore and Siegel, 2013, p. 5

Types of sets

Solution set
- all solutions to a problem

Sample space
- contains all values a variable can take on

Spaces
- sets with some structure – e.g. the difference between elements in $\mathbb{Z}$

Finite sets
- have fixed cardinality – e.g. all integers between 1 and 10

Infinite sets … do not
- all numbers in $\mathbb{Z}$

Uncountable sets
- cannot be classified using cardinality – e.g. all decimal numbers between 1 and 3

Tuple
- an ordered pair

Singleton
- only one element

Empty set
- contains no element

Universal set
- contains ALL elements

Ordered sets
- order of elements must be maintained

Unordered sets
- order does not matter

Operators

The classics:

- addition, subtraction, multiplication, division

Sum operator

$$\sum_{i}^{n} x_i$$

- the sum of x_i over the range from $i = 1$ through $i = 4$

$$\sum_{i=1}^{4} x_{i=1+2+3+4=10}$$

Multiplication operator

$$\prod_{i}^{n} x_i$$

- the product of x_i over the range from $i = 1$ through $i = 4$

$$\prod_{i=1}^{4} x_i = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

Set operators

Union

- $A \cup B$

Intersection

- $A \cap B$

Difference

- $A \setminus B$

Complement

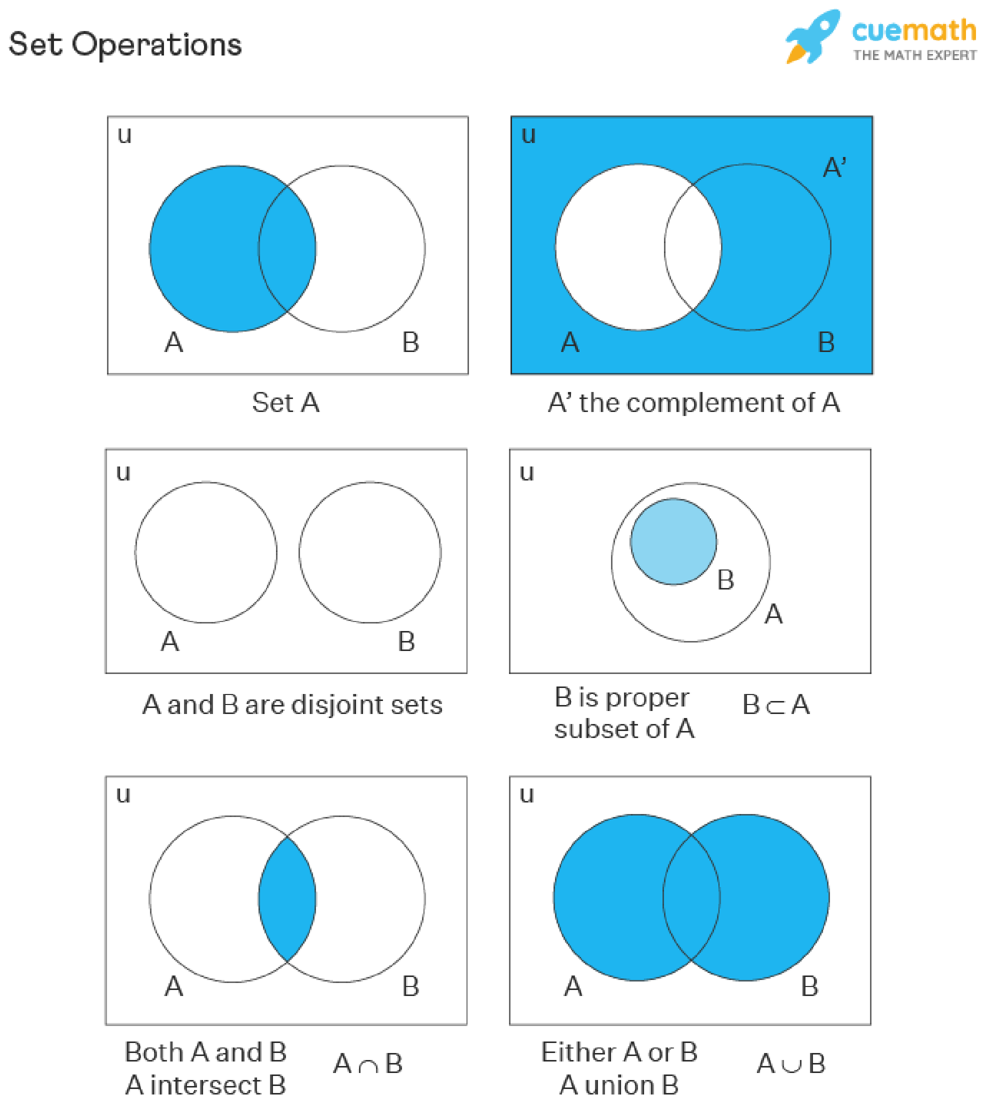
- $\neg B$

Partition P of M

- $P = \{\{blue\}, \{green\}\}$ and $M = \{blue, green\}$

Cartesian Product

- $A \times B$



Hands on – Set operators

Task: Let $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{2, 5, 8, 9\}$ from the universal set U= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Assuming that we do not include the same number as duplicates, find the following:

- $A \cap C$
- $A \cup C$
- $B \cap C$
- $A \setminus C$
- $\neg B$
- $(A \cup C) \setminus \mathrm{B}$
- $(A \cap B) \setminus C$
- $\neg(A \cup C)$

Hands on – Set operators

Solution:

- $A \cap C = \{5, 9\}$
- $A \cup C = \{1, 2, 3, 5, 7, 8, 9\}$
- $B \cap C = \{2, 8\}$
- $A \backslash C = \{1, 3, 7\}$
- $\neg B = A$
- $(A \cup C) \backslash \mathrm{B} = A$
- $(A \cap B) \backslash C = \{ \quad \}$
- $\neg(A \cup C) = \{4, 6, 10\}$

Hands on – Partitions

Task: Find all partitions of $M = \{1, 3, 5\}$

Hands on – Partitions

Solution: M has five partitions:

- $P_1 = \{\{1, 3, 5\}\}$
- $P_2 = \{\{1\}, \{3, 5\}\}$
- $P_3 = \{\{3\}, \{1, 5\}\}$
- $P_4 = \{\{5\}, \{1, 3\}\}$
- $P_5 = \{\{1\},\ \{3\},\ \{5\}\}$

Set operators

Mutually exclusive

- intersection equal to the empty set, i.e., sets with no elements in their intersection

Collectively exhaustive

- a group of sets is collectively exhaustive if together the sets constitute the universal set

Relations

- **used to compare** variables, constants and concepts via >, ≥, ≤, <, =, ≠
- **binary relation**
  - ordered by size (a, b) or a > b

- **functions** are relations, too!
- **consider a function f(x)**
  - **domain**

    → The domain consists of all possible values that x can take on
  - **range**

    → The range consists of all possible values y takes on given x

Level of measurement

Difference of kind

- nominal – distinction by name, type [Greens, SPD, CDU, …]

Difference of degree

- ordinal – distinction by order, size [language ability on your CV]
- interval – same difference between each element [$\mathbb{Z}$ - set of all integers, temperature]
- ratio – ‚meaningful' or true 0 as starting point [length in metres]

	Distinct categories	Meaningful order	Equal spacing	True zero
Nominal	✓			
Ordinal	✓	✓		
Interval	✓	✓	✓	
Ratio	✓	✓	✓	✓

Proofs

Axioms and assumptions

- stated to begin and assumed as true

Proposition

- considered as true based on prior assumptions

Theorem

- a proven proposition

Lemma

- a theorem of ‚little interest' used as a prior step to solve another problem

Corollary

- proposition following from the proof of a 2nd proposition which requires no further proof

Proofs

Direct proofs

- proof by deduction
- proof by exhaustion
- proof by construction
- proof by induction

Indirect proofs

- counterexample
- contradiction

Proof by induction

Initial step

- provide base case for assumption $A(1)$
- necessary to show validity often for n = 1

Inductive hypothesis

- assume that $A(n)$ for $n \in \mathbb{N}$ is true
- this step requires no computation, it can be a sentence you learn by heart ☺

Inductive step

- increment n by one and prove that $A(n+1)$ is true

→ if case is true for both n and $n+1$ we know our case is true for $\mathrm{n} \in \mathbb{N}$

Proof by induction

Let‘s look at Gauss $\sum_{k=0}^{n} k = \frac{n \cdot (n+1)}{2}$ holds for $\forall\, n \in \mathbb{N}$

Initial step for $n = 1$

$$\sum_{k=0}^{1} 0 + 1 = \frac{1 \cdot (1+1)}{2} = 1$$

Inductive hypothesis

→ statement A(n) holds for any $n \in \mathbb{N}$

Inductive step for $n + 1$

$$\sum_{k=0}^{n+1} k = (n+1) + \sum_{k=0}^{n} k = (n+1) + \frac{n \cdot (n+1)}{2}$$

$$= \frac{2\,(n+1)}{2} + \frac{n\,(n+1)}{2} = \frac{2\,(n+1) + n(n+1)}{2} = \frac{(n+2)(n+1)}{2}$$

Proof by induction – an example

Chapter 2 | Algebra

Algebraic properties

Associative properties

- $a + (b + c) = (a + b) + c$ and $a(b \cdot c) = (a \cdot b)c$

Commutative property

- $a + b = b + a$ and $a \cdot c = c \cdot a$

Distributive property

- $a(b + c) = ab + ac$

Identity property

- there exists a zero such that $x + 0 = x$ and $x \cdot 1 = x$

Inverse property

- there exists a $-x$ such that $-x \ \cdot x = 0$ and $x^{-1} \cdot x = 1$

FOIL and PEMDAS

FOIL

→ First, Outer, Inner, Last

$$(3y-4)(5+2y) = 3y\cdot 5 = 15y$$
$$(3y-4)(5+2y) = 3y\cdot 2y = 6y^2$$
$$(3y-4)(5+2y) = (-4)\cdot 5 = (-20)$$
$$(3y-4)(5+2y) = (-4)\cdot 2y = (-8y)$$
$$= 15y + 6y^2 - 20 - 8y$$
$$= 6y^2 + 7y - 20$$

PEMDAS

→ Please Excuse My Dear Aunt Sally

1) Parentheses
2) Exponents
3) Multiplication
4) Division
5) Addition
6) Subtraction

Ratios, proportions and percentages

Ratio of x to y = $\frac{x}{y}$

→ may be negative, range typically between 0 and ∞

Proportion of x and y = $\frac{x}{x+y}$

→ ranges from 0 to 1

Percentage $\frac{x}{x+y} \cdot 100$

→ ranges from 0 to 100

Proportional change $\frac{x_{t+1}-x_t}{x_t}$

→ $\frac{80.3-75.4}{75.4} \cdot 100 \cong 6.5\%$ ← **percentage change** ☺

Fractions

Addition

Same denominator

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

Different denominator

$$\frac{a}{b}+\frac{c}{d}=\frac{a\cdot d}{b\cdot d}+\frac{c\cdot b}{d\cdot b}=\frac{ad+cb}{bd}$$

Subtraction

Same denominator

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

Different denominator

$$\frac{a}{b}-\frac{c}{d}=\frac{a\cdot d}{b\cdot d}-\frac{c\cdot b}{d\cdot b}=\frac{ad-cb}{bd}$$

Multiplication

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

Division

$$\frac{a}{b}:\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Fractions

Expanding fractions

$$\frac{a}{b} = \frac{a}{b} \cdot \frac{e}{e} = \frac{ae}{be}$$

What about double fractions?

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b} \cdot b}{c \cdot b} = \frac{a}{cb}$$

or…

$$\frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a \cdot 1}{c \cdot b} = \frac{a}{cb}$$

→ let‘s build ‚bridges‘

Shortening fractions

$$\frac{a}{b} = \frac{c \cdot e}{d \cdot e} = \frac{c}{d}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \cdot d}{c \cdot b} = \frac{ad}{bc}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

Factoring

Algorithm:

1) Look for **common factors** and ‚factor them out‘
2) Check if a **binomial/identity** applies
3) **Repeat steps 1** and **2** until completion

$$(a+b)(a-b) = (a-b)^2$$
$$(a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a-b)(a-b) = a^2 - 2ab + b^2$$
$$(a+b)(a^2 - ab + b^2) = a^3 + b^3$$
$$(a-b)(a^2 + ab + b^2) = a^3 - b^3$$
$$a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$$
$$a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$$

Factoring

$$4z^2 + 20z$$
$$= \ 4(z^2 + 5z)$$
$$= \ 4z(z + 5)$$

Both of these are correct!

→ we often choose the version without exponent

$$9z^2 - 36$$
$$= \ (9z)^2 - 6^2$$
$$= \ (9z + 6)(9z - 6)$$

It may come in handy to know certain factor identities and (quadratic) binomials

→ $(a + b)(a - b) = \ (a - b)^2$

Quadratic polynomials

Typically of form: $a^2 + bx + c = 0$

→note that a cannot be 0!

Quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{-2a}$$

p/q formula

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Fun with quadratic binomials… ☺

Binomial formulas

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Binomial coefficient

Solving… the square

Algorithm:

1) Divide by quadratic‘s coefficient and move constant to RHS
2) Divide x‘s coefficient by 2, square it and add it to both sides of the equation
3) Factor LHS into $(a \pm b)$ and simplify RHS
4) Take square root of both sides
   → remember: Solution on RHS will be of sign $\pm$
5) Solve for x

Solving… the square

1) Divide by quadratic‘s coefficient and move constant to RHS
2) Divide x‘s coefficient by 2, square it and add it to both sides of the equation
3) Factor LHS into $(a \pm b)$ and simplify RHS
4) Take square root of both sides
   → Remember: Solution on RHS will be of sign $\pm$
5) Solve for x

$$4x^2 + 18x + 8 = 0 \mid \div 4$$
$$x^2 + \frac{18}{4}x + 2 = 0 \mid -2$$
$$x^2 + \frac{18}{4}x = -2 \mid + \left(\frac{\frac{18}{4}}{\frac{2}{1}}\right)^2$$
$$x^2 + \frac{18}{8}x + \left(\frac{18}{8}\right)^2 = -2 + \left(\frac{18}{8}\right)^2$$
$$(x + 2.25)^2 = 3.0625 \mid \sqrt{}$$
$$x + 2.25 = \pm 1.75 \mid -2.25$$
$$x_1 = -0.5$$
$$x_2 = -4$$

Hands on – completing the square & quadratics

Task: Complete the square and apply the quadratic formula!

1) $x^2 - 3x + 2$

2) $x^2 + 10x + 16$

3) $x^2 + x - 12$

Hands on – completing the square & quadratics

- Solve the square for: $x^2 - 3x + 2$

1) Divide by quadratic‘s coefficient and move constant to RHS
2) Divide x‘s coefficient by 2, square it and add it to both sides of the equation
3) Factor LHS into $(a \pm b)$ and simplify RHS
4) Take square root of both sides
   → remember: Solution on RHS will be of sign $\pm$
5) Solve for x

Hands on – completing the square & quadratics

Hands on – completing the square & quadratics

Solution:

1) $x^2 - 3x + 2$
→ $x = 2 \; or \; x = 1$

2) $x^2 + 10x + 16$
→ $x = (-8) \; or \; x = (-2)$

3) $x^2 + x \; - 12$
→ $x = 3 \; or \; x = (-4)$

Hands on – Factoring and fractions

Task:

1) $\frac{9}{x+4} = 7$

2) $\frac{x-5}{x-2} = 1 - \frac{x+1}{x-2}$

3) Factor: $2x + 4$

4) Factor: $5x + 10xy$

Hands on – Factoring and fractions

Solution:

1) $\frac{9}{x+4} = 7$

$$\frac{9}{x+4} = 7 \mid \cdot (x+4)$$

$$9 = 7 \cdot (x+4)$$

$$9 = 7x + 28 \mid -28$$

$$-19 = 7x$$

$$x = -\frac{19}{7}$$

2) $\frac{x-5}{x-2} = 1 - \frac{x+1}{x-2}$

$$\frac{x-5}{x-2} = 1 - \frac{x+1}{x-2} \mid \cdot (x-2)$$

$$x - 5 = x - 2 - (x+1)$$

$$x - 5 = x - 2 - x - 1$$

$$x - 5 = -3 \mid +5$$

$$x = 2 \rightarrow \emptyset$$

Hands on – Factoring and fractions

Solution:

3) Factor: $2x + 4$

→$2(x + 2)$

→$4(0.5x + 1)$

…

4) Factor: $5x + 10xy$

→$5(x + 2xy)$

→$5\mathrm{x}(1 + 2\mathrm{y})$

→$\mathrm{x}(5 + 10\mathrm{y})$

…

Solving…

Equations

1) Focus on variable of interest
2) Combine like terms
3) Check your answer
4) Make use of identities
5) Note: Equivalent transformation!!
   → execute same operation on each side

$$2x - 6 = 4 \mid +6$$
$$2x = 10 \mid \div 2$$
$$x = 5$$

Inequalities

←do the same thing…

But whenever you multiply or divide by a negative number – flip the inequality symbol!

$$2x - 6 > 4 \mid +6$$
$$2x > 10 \mid \div 2$$
$$x > 5$$

$$-2x - 6 > 4 \mid +6$$
$$-2x > 10 \mid \div (-2)$$
$$x < -5$$

Modulo – Division with Remainder

Idea: Sometimes, we are **not interested** in a **fractions/decimal** numbers as a **result of division**

Solution: Inspect the **remainder** after division!

We use modulo to find this remainser after division:

- **4 mod 3 = 4 % 3 = 1** → if we divide 4 by 3, 3 fits into 4 one time, with a **single 1** being left over… this is the **remainder**!

What does congruence mean?

- two numbers a and b are congruent mod n, if they have the same remainder when divided by n!
- $a \equiv b \ (mod\ n) \rightarrow 5 \equiv 9 \ (mod\ 4)$

Real World Applications - Modulo

- **Hash functions** in algorithms, e.g. sorting
  - set array of fixed size 10 and sort elements from ‘bucket’ into it
  - for each element key, calculate index = key % 10
  - → if slot empty, place element key there or handle collisions
- **Shuffling algorithms** to e.g. systematically shift numbers
- **Cryptography**

- **Time series data** and **Cyclical data**
  - use modulo operations to **identify** the **date of the week**
  - 0 = Sunday, 1 = Monday, ..., 6 = Saturday:
  - set **reference date** like October 27th 2024 ← Sunday
  - for **new date** i: day-difference % 7 = |27 – i| % 7 = day of the week
  - considering today: |27 – 31| % 7 = 4 mod 7 = 4 → Thursday!

Boolean Arithmatic

Input: Boolean Values

- True (1) and False (0)

Basic Boolean Operators:

- AND ∧: Returns True only if both inputs are True!
  → 1 AND 1 = 1, any other input combination yields 0
- OR ∨: Returns True if at least one input is True → 1 OR 1 = 1, 1 OR 0 = 1, 0 OR 1 = 1, and only 0 OR 0 = 0
- NOT ¬ : Inverts the value: NOT 1 = 0 and NOT 0 = 1.

- **Combination of Operators:**
  - combine operators to form complex expressions, which are evaluated in a specific order

- **Implication A → B :**
  - meaning "If A, then B“
  - our implication is false only if A is true and B is false; otherwise, it’s true
  - we do not care about anything else happening next to B!
    → but based on the implication, if A happens, B must follow suit!

Boolean Arithmatic

Task: You are a social scientist and want to analyse voting behaviour in a group of lawmakers on a recent policy vote on climate change. Lawmakers' support might depend on two main conditions:

- **A**: The lawmaker aligns with the ruling party's position on climate change
- **B**: The lawmaker's constituents (main group of voters) support the climate change policy

How do these influence, whether the lawmaker votes ‘Yes’ on the policy?

Based on your observations, the following **rule of voting V** holds: A lawmaker votes in favour of the climate change policy if they align with the ruling party or if their constituents support the policy, unless neither condition is met.

Create a truth table and fill in all possible outcomes!

Boolean Arithmatic

Hints:

1. Formulate an implication
2. Setup your truth table
3. Fill it in!

A: Support Party	B: Support Voters	V: Votes Yes
1	1	
1	0	
0	1	
0	0	

Boolean Arithmatic

Hints:

1. Formulate an implication: Our implication is A V B → V
2. Setup your truth table
3. Fill it in!

A: Support Party	B: Support Voters	V: Votes Yes
1	1	1
1	0	1
0	1	1
0	0	0

Time for your questions

- Any questions during the week?
  - → joerdis.strack@uni-konstanz.de