Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Differentiation

To do

- Weekly recap
- Real world applications
- Hands on practice
- Questions

Chapter 5 | Differentiation

Derivatives and change

Discrete change

- change between two measures of a concept at two distinct, discrete moments in time

→ first difference between two observations over a discrete interval

Instantaneous change

- change at a specific point in time
- derivative of function $f(x)$ with respect to x tells us the instantaneous rate of change of the function at each point

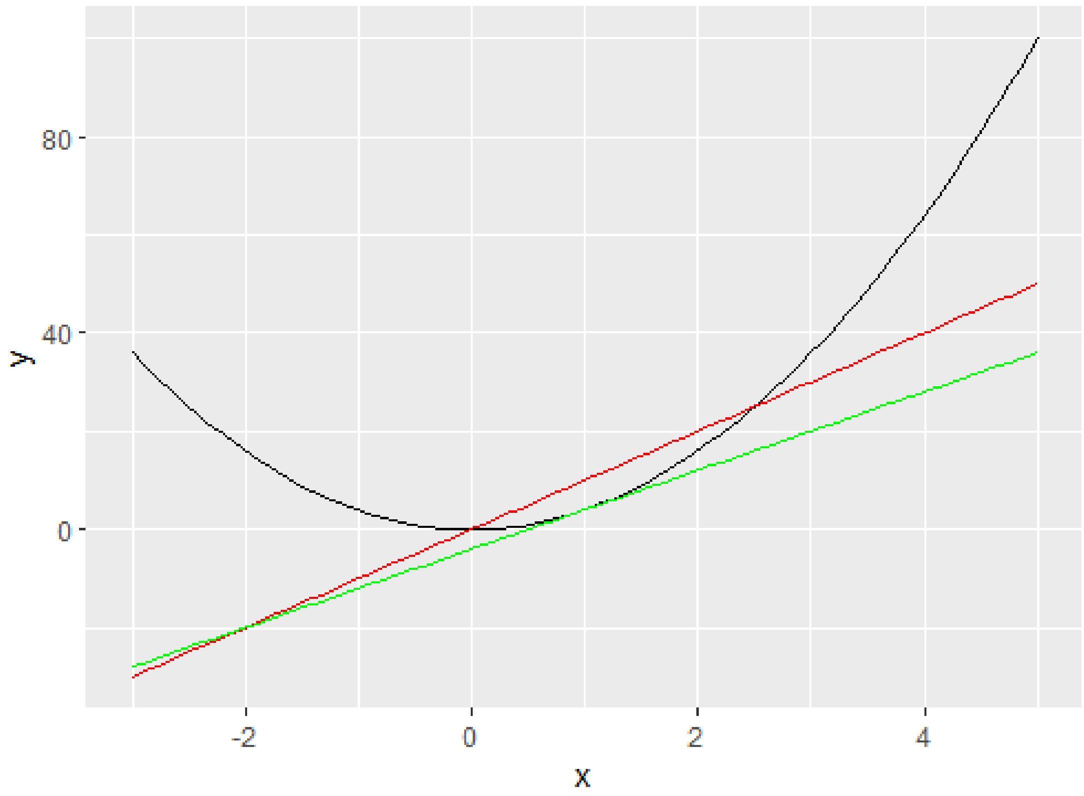
(First) derivative

- describes reactivity to change in function's output based on input argument x

Secants and tangents

Note: ‚secare‘ means to intersect and ‚tangere‘ means to touch in Latin!

- secants graphically represent discrete change
  - $f(x) = mx + b$
  - intercept b as point 1
  - find mx to reach point 2
- tangents graphically represent instantaneous change at $x = a$
  - $t(x) = f'(a) \cdot (x - a) + f(a)$
  - $f(a)$ function f evaluated at a
  - $f'(a)$ first derivative of function f evaluated at a



Hint:

- $t(x) = f'(a) \cdot (x - a) + f(a)$
- $f(a)$ function f evaluated at a
- $f'(a)$ first derivative of function f evaluated at a

Hands on – Secants and tangents

Task: Find the tangent of $f(x) = x^3 + 2x^2 + 5x - 4$ at $x = 5$

Hands on – Secants and tangents

Solution:

$$f(x) = x^3 + 2x^2 + 5x - 4$$
$$t(x) = f'(a) \cdot (x - a) + f(a)$$

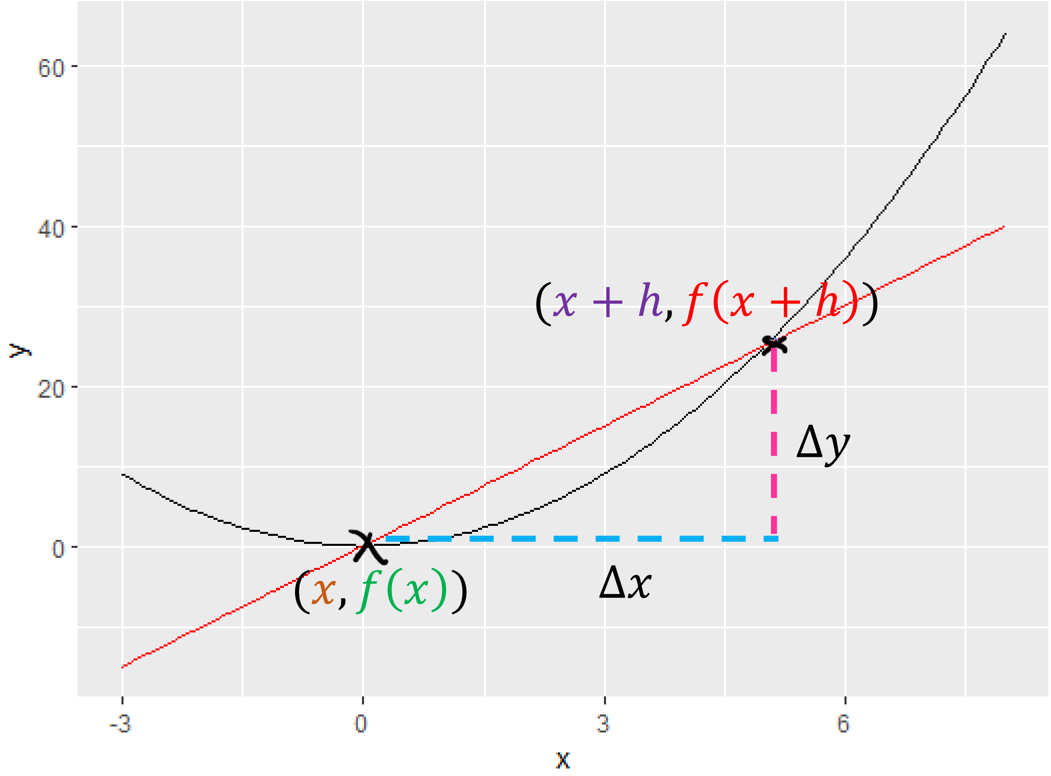
- $f(5) = 5^3 + 2 \cdot 5^2 + 5 \cdot 5 \ - 4 = 125 + 50 + 25 - 4 = 196$
- $f'(x) = 3x^2 + 4x + 5$
- $f'(5) = 3 \cdot 5^2 + 4 \cdot 5 + 5 = 75 + 20 + 5 = 100$

→$t(x) = 100 \cdot (x - 5) + 196 = 100x \ - 500 + 196$

→$t(x) = 100x \ - 304$

The first derivative and secants

- derivative of function $f(x)$ with respect to x tells us the instantaneous rate of change of the function at each point



Slope = $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$

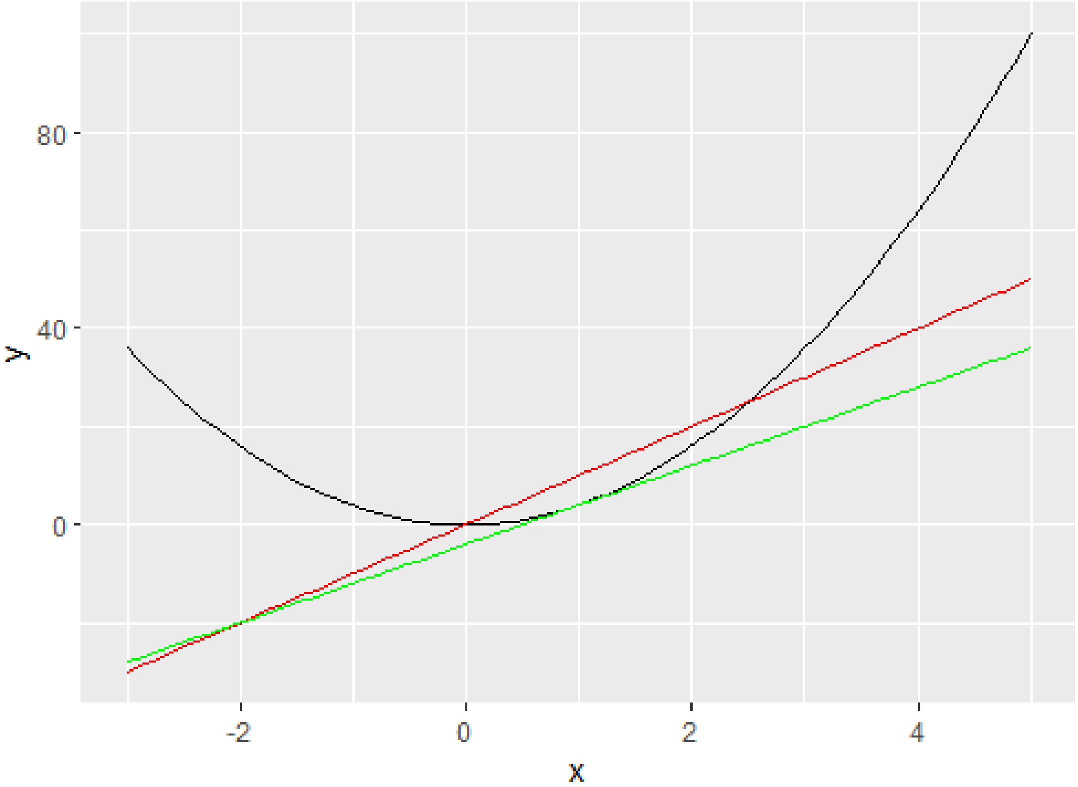
Derivatives, tangents and secants

Slope of secant

$$\frac{f(x+h)-f(x)}{h}$$

Slope of tangent

$$\frac{d}{dx}f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



Real world applications – motion time graphs

position: $s = \Delta\, distance$

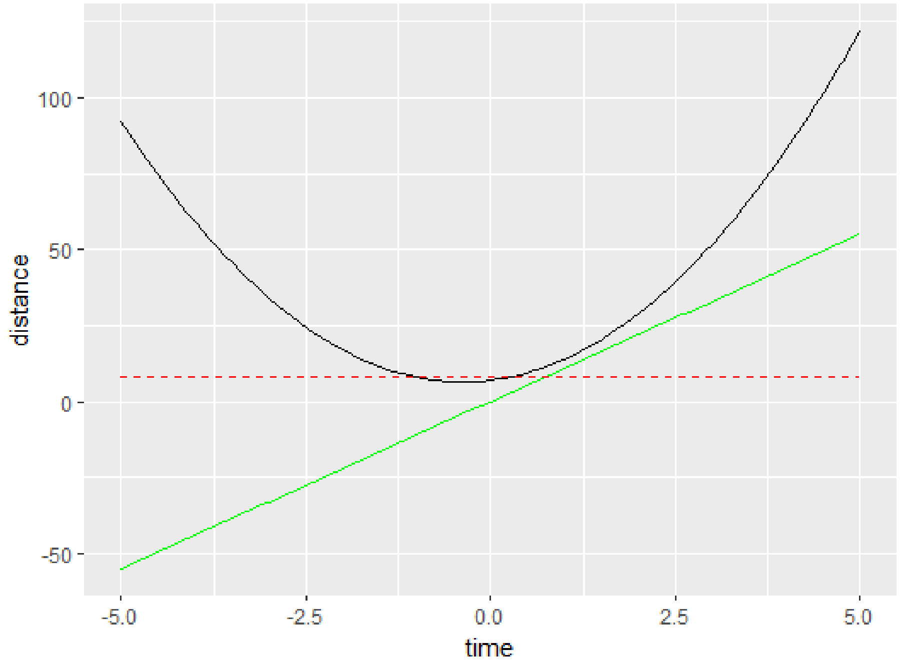
velocity: $\mathrm{v} = \dfrac{\Delta\, distance}{\Delta\, time}$

acceleration: $a = \dfrac{\Delta\, distance}{\Delta\, time^2}$

Let‘s think derivatives:

- let s(t) be the position of an object
- then:
  - v(t) = s’(t)
  - a(t) = v’(t) or s’’(t)

- let $s(t) = 4t^2 + 3t + 7$
- then $v(t) = 8t + 3$
- and $a(t) = 8$



Real world applications – motion time graphs

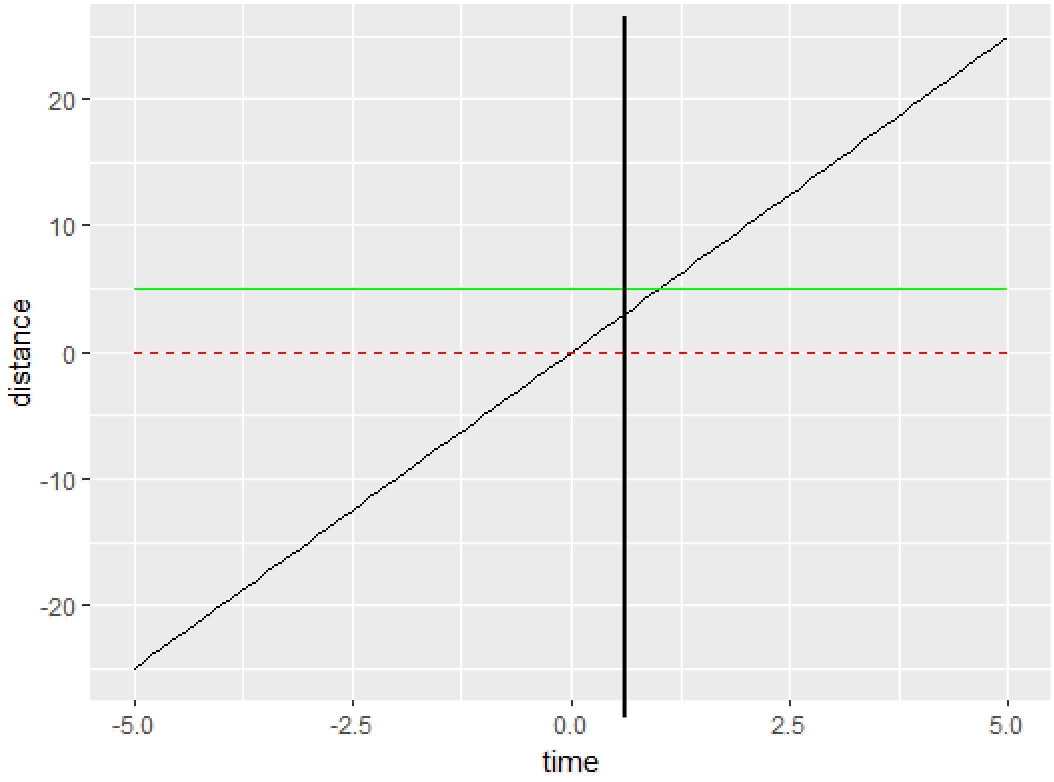
Task: Team up with a partner

1) Discuss why the graph looks the way it does!

2) Find $s(t), \textcolor{green}{v(t)}$ and $\textcolor{red}{a(t)}$!

Solution:

$$s(t) = 5t + 0$$
$$v(t) = 5$$
$$a(t) = 0$$

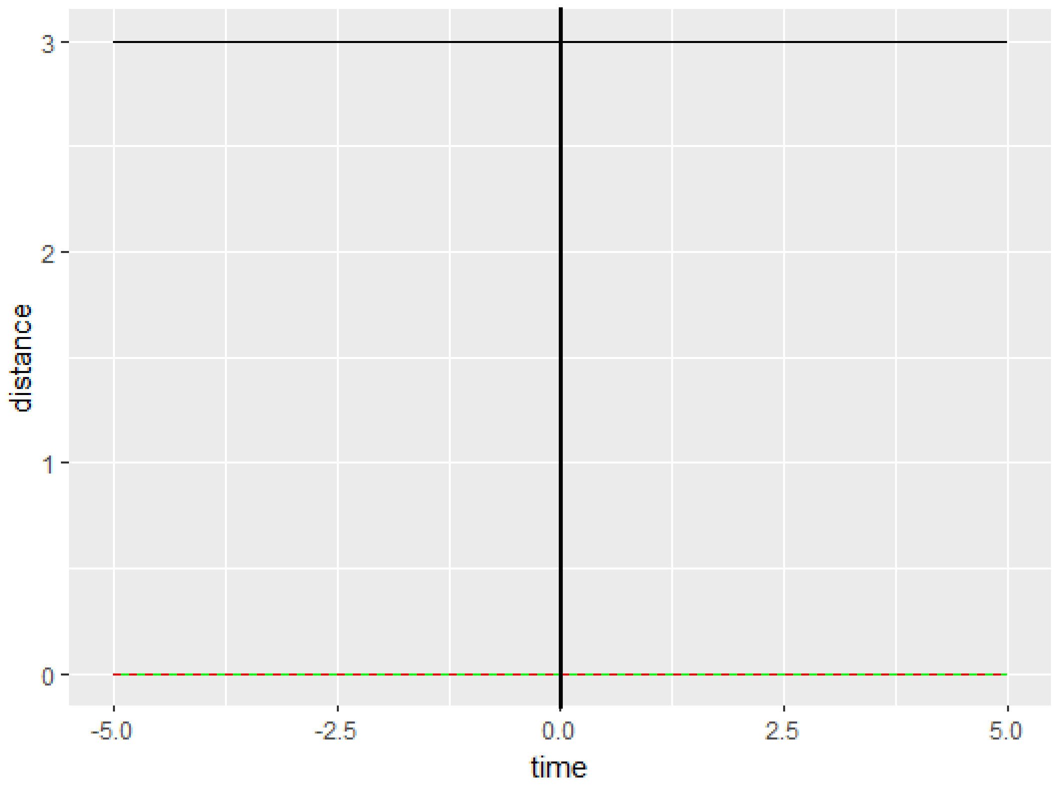


Real world applications – motion time graphs

Task: Team up with a partner

1) Discuss why the graph looks the way it does!

2) Find $s(t), v(t)\ and\ a(t)$!



Solution:

$$s(t) = 3$$
$$v(t) = 0$$
$$a(t) = 0$$

Hands on – definition of derivative

Task: Solve using the definition of derivative

$$\boxed{\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

1) $\frac{d}{dx}3x^2$

2) $\frac{d}{dx}x^3$

3) $\frac{d}{dx}4x^3 - x + 1$

Hands on – definition of derivative

1) $\frac{d}{dx}3x^2$

$= \lim\limits_{\mathrm{h}\to 0}\frac{3(x+h)^2-3x^2}{h} = \lim\limits_{\mathrm{h}\to 0}\frac{3x^2+6xh+3h^2-3x^2}{h}$

$= \lim\limits_{\mathrm{h}\to 0}\frac{6\mathrm{xh}+3\mathrm{h}^2}{h} = \lim\limits_{h\to 0}6x+3h = 6x+3\cdot 0$

$= 6x$

2) $\frac{d}{dx}x^3$

$= \lim\limits_{h\to 0}\frac{(x+h)^3-x^3}{h}$

$= \lim\limits_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim\limits_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim\limits_{h\to 0}3x^2+3xh+h^2$

$= 3x^2+3x\cdot 0+0^2 = 3x^2$

Hands on – definition of derivative

Solution:

3) $\frac{d}{dx}4x^3 - x + 1 = \lim_{h \to 0} \frac{4(x+h)^3 - (x+h) + 1 - (4x^3 - x + 1)}{h}$

$$= \lim_{h \to 0} \frac{4x^3 + 12x^2h + 12xh^2 + 4h^3 - x - h + 1 - 4x^3 + x - 1}{h}$$
$$= \lim_{h \to 0} \frac{12x^2h + 12xh^2 + 4h^3 - h}{h}$$
$$= \lim_{h \to 0} 12x^2 + 12xh + 4h^2 - 1$$
$$= 12x^2 + 12x \cdot 0 + 4.0^2 - 1$$
$$= 12\mathrm{x}^2 - 1$$

Notation

- **Leibniz**
  - $\frac{d}{dx}f(x)$ or $\frac{dy}{dx}$ or $\frac{df}{dx}$
  - $\frac{d^n}{dx^n}f(x)$
- **Newton**
  - $y = f(x) \rightarrow \dot{y}$ and $\ddot{y}$
- **Lagrange**
  - $f'(x)$ and $f''(x)$
- **Euler**
  - $D_x f(x)$

Chapter 6 | Rules of Differentiation

Hands on – rules of differentiation

Task: Come up with two functions to differentiate on your own – swap with a partner and find their derivatives!

Table 6.1: List of Rules of Differentiation

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$
Power rule	$(x^n)' = nx^{n-1}$
Exponential rule 1	$(e^x)' = e^x$
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules	$(\sin(x))' = \cos(x)$
	$(\cos(x))' = -\sin(x)$
	$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules	Treat each piece separately

Moore & Siegel, 2013, p.130

Hands on – Product rule

Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Example:

$$\frac{d}{dx}\left((x^3+2x)(2x^2-x)\right) = (3x^2+2)(2x^2-x) + (x^3+2x)(4x-1)$$
$$= (6x^4 - 3x^3 + 4x^2 - 2x) + (4x^4 - x^3 + 8x^2 - 2x)$$
$$= 10x^4 - 4x^3 + 12x^2 - 4x$$

Hands on – Product rule

Rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Task: Find the first derivatives using the product rule!

1) $\frac{d}{dx}(4x + 1) \cdot (2x^2 - 2x)$

2) $\frac{d}{dx}x^3 \cdot e^x$

Hands on – Product rule

Rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Solution:

1) $\frac{d}{dx} = (4x + 1) \cdot (2x^2 - 2x) = 4(2\mathrm{x}^2 - 2\mathrm{x}) + (4\mathrm{x} + 1)(4\mathrm{x} - 2)$

$= 24\mathrm{x}^2 - 12\mathrm{x} - 2$

1) $\frac{d}{dx} = x^3 \cdot e^x = 3x^2e^x + x^3e^x$

Hands on – Quotient rule

Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

Example:

$$\frac{d}{dx}\left(\frac{x^2}{3x-6}\right) = \frac{(2x)(3x-6) - 3(x^2)}{(3x-6)^2}$$

$$= \frac{6x^2 - 12x - 3x^2}{(3x-6)^2}$$

Hands on – Quotient rule

Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

Task: Find the first derivatives using the quotient rule!

1) $\frac{d}{dx}\left(\frac{x^2+6}{2x-7}\right)$

2) $\frac{d}{dx}\left(\frac{e^x}{x}\right)$

Hands on – Quotient rule

Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

Solution:

1) $\frac{d}{dx}\left(\frac{x^2+6}{2x-7}\right) = \frac{2x\cdot(2x-7)-2(x^2+6)}{(2x-7)^2} = \frac{2(x^2-7x-6)}{(2x-7)^2} = \frac{2x^2-14x-12}{(2x-7)^2}$

2) $\frac{d}{dx}\left(\frac{e^x+x}{x}\right) = \frac{(e^x+1)(x)-1(e^x+x)}{x^2} = \frac{xe^x-e^x}{x^2}$

Hands on – Chain rule

Rule:

$$\frac{d}{dx}\left(g(f(x))\right) = g'\left(f(x)\right)f'(x)$$

Example:

$$\frac{d}{dx}(3x^2 + 2x)^2 = 2(3x^2 + 2x) \cdot (6x + 2)$$

Hands on – Chain rule

Rule:

$$\frac{d}{dx}(g(f(x))) = g'\big(f(x)\big)f'(x)$$

Task: Find the first derivatives using the chain rule

1) $\frac{d}{dx}(4x^2 + x)^3$

2) $\frac{d}{dx}\ \mathrm{e}^{4\mathrm{x}+1}$

Hands on – Chain rule

Rule:

$$\frac{d}{dx}\big(g(f(x))\big) = g'\big(f(x)\big)f'(x)$$

Solution:

1) $\frac{d}{dx}(4x^2 + x)^3 = 3(4x^2 + x)^2 \cdot (8x + 1)$

2) $\frac{d}{dx}\ \mathrm{e}^{4\mathrm{x}+1} = 4e^{4x+1}$

Hands on – rules of differentiation

Task: Apply the rules of differentiation to find the derivatives of…

1) $f(x) = 6$
2) $f(x) = x^8$
3) $f(x) = 27x^3 + 5x^2 - x + 13$
4) $f(x) = \; ax^n - 1$
5) $f(x) = (5x + 1)^3$
6) $f(x) = e^{3x}$
7) $f(x) = \; \frac{x^2+1}{x+1}$
8) $f(x) = \left(\frac{2x^2+3}{x+5}\right)^2$

Hands on – rules of differentiation

Solution:

1) $f'(x) = 0$

2) $f'(x) = 8x^7$

3) $f'(x) = 81x^2 + 10x \quad - 1$

4) $f'(x) = \ anx^{n-1}$

5) $f'(x) = \ 3(5x+1)^2 \cdot 5 = 15(5x+1)^2$

6) $f'(x) = 3e^{3x}$

7) $f'(x) = \frac{2x(x+1) - 1(x^2+1)}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$

8) $f'(x) = 2\left(\frac{2x^2+3}{x+5}\right) \cdot \left(\frac{4x(x+5) - 1(2x^2+3)}{(x+5)^2}\right) = \frac{2(2x^2+3)(2x^2+20x-3)}{(x+5)^3}$

Partial derivatives

- we are interested in the slope in direction of x, while keeping y fixed – and vice versa
- same rules, treat every variable as a constant to whose respect we are **not** differentiating!
- to denote a **partial derivative**, we either use $\frac{\partial}{\partial x}$ or $f_x'(x)$

$$f(x, y, z) = 3y^2z^4 - 5xz^2 + 2x^3$$

$$f_x'(x, y, z) = -5z^2 + 6x^2$$
$$f_y'(x, y, z) = 6yz^4$$
$$f_z'(x, y, z) = 12y^2z^3 - 10xz$$

Hands on – partial derivatives

Task: Find the partial first derivative with respect to z!

1) $\frac{\partial}{\partial z} 9x^2 + 3z^2$

2) $\frac{\partial}{\partial z} 8xyz^2 + 10x^2y^2 + 12x^2y + 14x^2z^2$

Hands on – partial derivatives

Solution:

1) $\frac{\partial}{\partial z} 9x^2 + 3z^2 = 6z$

2) $\frac{\partial}{\partial z} 8xyz^2 + 10x^2y^2 + 12x^2y + 14x^2z^2 = 28x^2z + 16xyz$

Throwback: Limits – Rule of L'Hospital

Have you seen limits like these ones?

1) $\lim_{x \to 2} \frac{x^2+x-6}{x^2-3x+2} = \frac{2^2+2-6}{2^2-3 \cdot 2+2} = \frac{0}{0}$

2) $\lim_{x \to \infty} \frac{1}{x} \cdot \ln(x) = \frac{1}{\infty} \cdot \ln(\infty) = 0 \cdot \infty$

To find the limit, we will apply the rule of L'Hospital!

Finding the limit might look impossible…but that is not always the case!

Throwback: Limits – Rule of L‘Hospital

Rule of L‘Hospital:

- **indeterminate limits** of the form $\frac{\mathbf{0}}{\mathbf{0}}$ and $\frac{\infty}{\infty}$ can at times be solved by differentiation of the expression!

  → instead of evaluating the limit at argument x right away, we differentiate both numerator and denominator separately and **plug in x afterwards**!

- if the limit is still of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, **we may try again**

  → try to find patterns – should we keep differentiating or stop?

  → we may simplify expressions to ‘reach’ $\frac{\mathbf{0}}{\mathbf{0}}$ and $\frac{\infty}{\infty}$ and apply L’Hospital (example 2)

Throwback: Limits – Rule of L'Hospital

Examples:

1) $\lim_{x \to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x \to 2} \frac{\frac{d}{dx}x^2+x-6}{\frac{d}{dx}x^2-3x+2} = \lim_{x \to 2} \frac{2x+1}{2x-3} = \frac{2 \cdot 2+1}{2 \cdot 2-3} = \frac{5}{1} = 5$

2) $\lim_{x \to \infty} \frac{1}{x} \cdot \ln(x) = \lim_{x \to \infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$

Hands on – L‘Hospital

Question: Can we apply the rule of L‘Hospital to find

$$\lim_{x\to 4} = \frac{4x+3}{2x-8} \; ?$$

Answer: Nope, plugging in $x = 4$ yields: $\frac{4\cdot 4+3}{2\cdot 4-8} = \frac{19}{0} \neq \frac{0}{0} \; or \, \frac{\infty}{\infty}$

Time for your questions

- Any questions during the week?
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