Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Vectors and Matrices

To do

- weekly recap
- real world applications
- hands on practice
- questions

Chapter 12 | Vectors and Matrices

Scalars, vectors, matrices

Scalars $\boldsymbol{x}$

- a quantity that is defined by a numerical value

Vectors $\vec{\boldsymbol{x}}$ $(\ddot{x}, \boldsymbol{x})$

- consist of elements and components and can be used **to express parallel shifts/linear translations** (e.g. movement)

Matrices $\boldsymbol{X}$

- format: $A_{rows\ x\ columns}$

$$A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \qquad A_{3\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Matrices: Determinants

Matrices $\boldsymbol{X}$

- format: $A_{rows\ x\ columns}$

$$A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

A **quadratic** matrix has a **determinant**!

- the determinant tells us, if A is invertible: if |A| ≠ 0
- if a system of equations has a unique solution
- we typically denote the determinant as $\det(A)\ or\ |A|$
- to compute the determinant, apply either the rule of Sarrus ( - method) or apply Laplace expansion for matrices of large size
- more on that later…

Vectors

Vector addition

Component-wise addition:

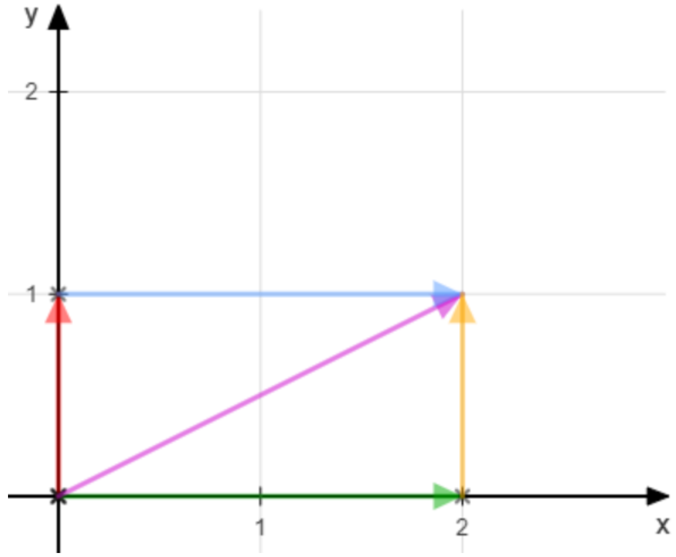
- **Rule**: $\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

  Example: $\vec{x} + \vec{y} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3+2 \\ 5+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Scalar multiplication:

- **Rule**: $\lambda \cdot \vec{x} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$

  Example: $4 \cdot \vec{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \cdot 3 \\ 4 \cdot 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$



Real world applications – back to physics ☺

Let‘s return to physics for a moment!

- Imagine an object e.g., a molecule on which two forces act:
  - gravitational force $\overrightarrow{f_1} = \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix} N,$
  - flow force $\overrightarrow{f_2} = \begin{pmatrix} 1N \\ 1N \\ 0 \end{pmatrix}$

If we add both vectors, we obtain the entire force acting upon the molecule:

$$\vec{f} = \overrightarrow{f_1} + \overrightarrow{f_2} = \begin{pmatrix} 1N \\ 1N \\ -m \cdot g \end{pmatrix} N$$

→we can now use $\vec{f}$ to obtain the molecules acceleration!

$$m \cdot \vec{a} = \vec{f} \rightarrow \vec{a} = \frac{\vec{f}}{m}$$

Euclidian Norm and scalar product

The ‘**length**’ of a vector $\vec{x}$ is computed by squaring all components, summing them up and finally taking the square root of the sum

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Example: $\vec{v} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$

$$\|\vec{v}\| = \sqrt{3^2 + 7^2 + 2^2} = \sqrt{9 + 49 + 4}$$

$$= \sqrt{62} \approx 7.874$$

The ‘scalar product’ results, as the name suggests, in a scalar, where matching components are multiplied, and products summed up

$$< \vec{x}, \vec{y} > = \left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_3 \end{pmatrix} \right\rangle$$

$$:= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

→ Note: two vectors $\vec{x}, \vec{y}$ are called ‘orthogonal’, iff $< \vec{x}, \vec{y} > = 0$

→ 90° degree angle

Scalar product – useful properties

- **Symmetry**

  $\langle \overrightarrow{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

- **Positivity** ☺

  $\langle \vec{x}, \vec{x} \rangle > 0 \; for \; all \; \vec{x} \;\; \neq 0$

- **Zero element**

  $\langle \vec{x}, 0 \rangle = 0$

- **Linearity**

  $\langle \vec{x} + \lambda \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \lambda \langle \vec{z}, \vec{y} \rangle$

- **Note**: $\langle \vec{x}, \vec{x} \rangle = ||\vec{x}||^2$ ← scalar product is equal to squared Euclidian norm!

Hands on – Euclidian Norm and scalar product

Task: Compute the Euclidian lengths of the following vectors and their scalar products!

- $\vec{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
- $\vec{y} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$
- $\vec{z} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Hands on – Euclidian Norm and scalar product

Solution:

$$||\vec{x}|| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$||\vec{y}|| = \sqrt{8^2 + 2^2} = \sqrt{64 + 4} = \sqrt{68} \approx 8.25$$

$$||\vec{z}|| = \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17} \approx 4.12$$

$$< \vec{x}, \vec{y} > = 3 \cdot 8 + 4 \cdot 2 = 24 + 8 = 32$$

$$< \vec{x}, \vec{z} > = 3 \cdot 4 + 4 \cdot 1 = 12 + 4 = 16$$

$$< \vec{y}, \vec{z} > = 8 \cdot 4 + 2 \cdot 1 = 32 + 2 = 34$$

Cauchy-Schwartz inequality

When $\vec{x}\ \ and\ \ \vec{y}$ are elements of a real (or complex) vector room with a defined scalar product, then $|\langle\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}\rangle|^{\mathbf{2}} \leq \langle\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}\rangle \cdot \langle\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}\rangle$ holds

- **In other words**: the positive, squared scalar product of two vectors $|\langle\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}\rangle|^{\mathbf{2}}$ $\vec{x}\ \ and\ \ \vec{y}$ must be smaller than or equal to the product of their inner products $\langle\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}\rangle \cdot \langle\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}\rangle$
- **Note**: Equality holds, when both vectors are linearly dependent!
  →next week ☺
- Substantial to proof the Heisenberg uncertainty principle in quantum mechanics

Matrices

Hands on – Types of matrices

Task: Identify the following matrices as diagonal, identity, square, symmetric, triangular, or none of the above (note all that apply)

1) $A = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

2) $B = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 2 & 5 \\ 1 & 5 & 0 \end{bmatrix}$

3) $C = \begin{bmatrix} -1 & 3 & 1 \\ 6 & -2 & 5 \\ 1 & 5 & -1 \\ 1 & 0 & 7 \end{bmatrix}$

Hints: If you need a refresher, have a look at Moore & Siegel, 2013, p.284

Hands on – Types of matrices

Solution:

1) $A = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \rightarrow diagonal, square, symmetric, triangular$

2) $B = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 2 & 5 \\ 1 & 5 & 0 \end{bmatrix} \rightarrow square, symmetric$

3) $C = \begin{bmatrix} -1 & 3 & 1 \\ 6 & -2 & 5 \\ 1 & 5 & -1 \\ 1 & 0 & 7 \end{bmatrix} \rightarrow none\ of\ the\ above$

Matrix transposition

- when transposing a matrix, we switch its components over its diagonal
  - $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$
  - B= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
  - $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow C^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$
  - $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow D^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Computation with matrices

Addition

- matrices must match in dimensions!

$$A + B = (a_{ij} + b_{ij})$$
$$= \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

- Example:

$$C + D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Subtraction

- same here – otherwise result is undefined!

$$A - B = (a_{ij} - b_{ij})$$
$$= \begin{bmatrix} a_{11} - b_{11} & \cdots & a_{1n} - b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & \cdots & a_{mn} - b_{mn} \end{bmatrix}$$

- Example:

$$C - D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Computation with matrices

Note: the inner dimension must match for result to be valid! Outer dimensions give shape of new matrix!

Scalar multiplication

- $C = rA \quad where \quad c_{ij} = r \cdot a_{ij}$

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix} \; and \; r = 3$$

$$3A = \begin{bmatrix} 3 \cdot 3 & 3 \cdot 1 \\ 3 \cdot 5 & 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 15 & 12 \end{bmatrix}$$

Multiplication

- $C_{n \times p} = A_{n \times m} B_{m \times p} \; \rightarrow c_{i,j} = \sum_{k=1}^{m} a_{ik} b_{jk}$

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix}, \; B = \begin{pmatrix} -2 & 1 \\ 6 & -1 \end{pmatrix}$$

$$AB = \begin{bmatrix} (3 \cdot (-2)) + (1 \cdot 6) & (3 \cdot 1) + (1 \cdot -1) \\ (5 \cdot (-2)) + (4 \cdot 6) & (5 \cdot 1) + (4 \cdot -1) \end{bmatrix}$$

$$AB = \begin{bmatrix} -6 + 6 & 3 - 1 \\ -10 + 24 & 5 - 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 2 \\ 14 & 1 \end{bmatrix}$$

Computation with matrices

Division

$$A \div B = A \cdot B^{-1}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$$

$$A \div B = A \cdot B^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{11}{7} \\ 0 & 1 \end{bmatrix}$$

Hands on – Matrices

Task:

1) A^T
2) $3 \cdot B$
3) BC
4) CB
5) AC
6) $D+A$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 7 & 3 & 4 \\ -1 & 1 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -7 \\ 4 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 9 \\ 1 & 3 \\ 4 & 7 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & -3 & 1 \\ -5 & -2 & -4 \\ -3 & 6 & 2 \end{bmatrix}$$

Hands on - matrices

Solution:

1) $A^T = \begin{bmatrix} 1 & 7 & -1 \\ 0 & 3 & 1 \\ -2 & 4 & 5 \end{bmatrix}$

2) $3 \cdot B = \begin{bmatrix} -4 \cdot 3 & -7 \cdot 3 \\ 4 \cdot 3 & 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} -12 & -21 \\ 12 & 9 \end{bmatrix}$

3) $BC = \emptyset \rightarrow \ dimensions\ do\ NOT\ match$!

4) $CB = \begin{bmatrix} 5 & 9 \\ 1 & 3 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} -4 & -7 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 16 & -8 \\ 8 & 2 \\ 12 & -7 \end{bmatrix}$

Hands on - matrices

Solution:

$$5)\ \ AC = \begin{bmatrix} 1 & 0 & -2 \\ 7 & 3 & 4 \\ -1 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 & 9 \\ 1 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 54 & 100 \\ 16 & 29 \end{bmatrix}$$

$$6)\ \ D + A = \begin{bmatrix} 2 & -3 & 1 \\ -5 & -2 & -4 \\ -3 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -2 \\ 7 & 3 & 4 \\ -1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -3 & -1 \\ 2 & 1 & 0 \\ -4 & 7 & 7 \end{bmatrix}$$

Determinant – Rule of Sarrus

- Rule of Sarrus or 'Butterfly Method' to obtain a matrix' determinant |A| or Det(A)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$|A| = [a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}]$$

Determinant – Rule of Sarrus

- Rule of Sarrus or 'Butterfly Method' to obtain a matrix' determinant |A| or Det(A)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$|A| = [a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}]$$

Determinant – Rule of Sarrus

- Rule of Sarrus or 'Butterfly Method' to obtain a matrix' determinant |A| or Det(A)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$|A| = [a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}]$$

Determinant – Laplace expansion

Laplace expansion via identification of minors

- choose whichever row or column – preferably one containing 0s
- expansion using i^{th} row:
  - $|A| = \sum_{j=1}^{n} a_{ij} \cdot (-1)^{i+j} \cdot D_{ij}$
- expansion using j^{th} column
  - $|A| = \sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+j} \cdot D_{ij}$

Example:

$M_{ij} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

$M_{21} = \quad |M_{ij}| = 1 \cdot 0 - 2 \cdot 0 = 0$

$A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 4 \\ 1 & 2 & 0 \end{bmatrix}$ → let's develop by 2nd row

$|A| = \sum_{j=1}^{3} a_{2j} \cdot (-1)^{2+j} \cdot D_{2j}$

$|A| = a_{21} \cdot (-1)^{2+1} \cdot D_{21} + a_{22} \cdot (-1)^{2+2} \cdot D_{22} + a_{23} \cdot (-1)^{2+3} \cdot D_{23}$

$|A| = 2 \cdot (-1)^{2+1} \cdot 0 + 5 \cdot (-1)^{2+2} \cdot 0 + 4 \cdot (-1)^{2+3} \cdot 5$

$= 4 \cdot (-1)^{5} \cdot 5$

$= -20$

$M_{23} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

Inverse

A matrix $A_{n\times x}$ is invertible, if $Det_A \neq 0$ and there is a matrix $B_{n\times n}$ where AB and $AB = I_{n\times n}$ → $AA^{-1} = I_{n\times n}$

General rule:

$$A^{-1} = \frac{1}{|A|} C^T$$

Rule for $\boldsymbol{A_{2\times 2}}$:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}$$

1) find determinant

$$|A| = (2)(3) - (3)(-1) = 6 + 3 = 9$$

2) switch diagonal elements and signs of a_{12} and a_{21}

$$A^{-1} = \frac{1}{9} \begin{bmatrix} 3 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/9 & 2/9 \end{bmatrix}$$

Inverse – algorithm

Let‘s have an example of the general rule $A^{-1} = \frac{1}{|A|} C^T$

- $A = \begin{bmatrix} 4 & -3 & 2 \\ 2 & 0 & 1 \\ -1 & 6 & 5 \end{bmatrix}$

1) find $Det(A)$

- $|A| = \begin{bmatrix} 4 & -3 & 2 \\ 2 & 0 & 1 \\ -1 & 6 & 5 \end{bmatrix} \begin{matrix} 4 & -3 \\ 2 & 0 \\ -1 & 6 \end{matrix}$
- $= \left[(4 \cdot 0 \cdot 5) + \left((-3) \cdot 1 \ \cdot (-1)\right) + (2 \cdot 2 \cdot 6)\right] - \left[\left((-1) \cdot 0 \cdot 2\right) + (6 \cdot 1 \cdot 4) + \left(5 \cdot 2 \cdot (-3)\right)\right]$
- $= 27 - (-6) = 33$

Inverse

NOTE: Get rid of row i and column j in a_{ij}, so for M_{11} get rid of row 1 and column 1

2) find all minors of $A = \begin{bmatrix} 4 & -3 & 2 \\ 2 & 0 & 1 \\ -1 & 6 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} 4 & -3 & 2 \\ 2 & 0 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

- $M_{11} = (-6)$ $M_{12} = 11$ $M_{13} = 12$
- $M_{21} = (-27)$ $M_{22} = 22$ $M_{23} = 21$
- $M_{31} = (-3)$ $M_{32} = 0$ $M_{33} = 6$

$$M_{11} = [0 \cdot 5] - [6 \cdot 1] = (-6)$$

3) transform into Cofactors

- $C_{11} = (-6)$ $C_{12} = (-11)$ $C_{13} = 12$
- $C_{21} = 27$ $C_{22} = 22$ $C_{23} = (-21)$
- $C_{31} = (-3)$ $C_{32} = 0$ $C_{33} = 6$

$$\begin{bmatrix} b_{11}(+) & b_{12}(-) & b_{13}(+) \\ b_{21}(-) & b_{22}(+) & b_{23}(-) \\ b_{31}(+) & b_{32}(-) & b_{33}(+) \end{bmatrix}$$

Inverse

4) find adjoint matrix (transpose cofactor matrix)

- $C = \begin{bmatrix} -6 & -11 & 12 \\ 27 & 22 & -21 \\ -3 & 0 & 6 \end{bmatrix} \rightarrow \quad C^T = \begin{bmatrix} -6 & 27 & -3 \\ -11 & 22 & 0 \\ 12 & -21 & 6 \end{bmatrix}$

5) solve for inverse $A^{-1} = \frac{1}{|A|} C^T$

- $A^{-1} = \frac{1}{|A|} C^T = \frac{1}{33} \begin{bmatrix} -6 & 27 & -3 \\ -11 & 22 & 0 \\ 12 & -21 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{2}{11} & \frac{9}{11} & -\frac{1}{11} \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{4}{11} & -\frac{7}{11} & \frac{2}{11} \end{bmatrix}$

Hands on – Inverse

Task: Find the inverse of the following matrices, when possible!

$$1)\ A = \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$$

$$2)\ B = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$$

$$3)\ M = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 9 & 1 \\ 5 & 1 & 4 \end{bmatrix}$$

Hands on – Inverse

Solution:

1) $A = \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$

$$|A| = (5 \cdot 1) - \left((-1) \cdot 2\right)$$
$$= 5 - (-2) = 7$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 1 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{bmatrix}$$

2) $B = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$

$$|B| = (1 \cdot 10) - (2 \cdot 5) = 0$$

→ the inverse does not exist!

Hands on – Inverse

Solution:

3) $M = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 9 & 1 \\ 5 & 1 & 4 \end{bmatrix}$

$|M| = 50$

$C = \begin{bmatrix} 35 & 5 & -45 \\ 9 & 7 & -13 \\ -11 & -3 & 27 \end{bmatrix}$

$C^T = \begin{bmatrix} 35 & 9 & -11 \\ 5 & 7 & -3 \\ -45 & -13 & 27 \end{bmatrix}$

$$M^{-1} = \frac{1}{|M|} C^T = \frac{1}{50} \begin{bmatrix} 35 & 9 & -11 \\ 5 & 7 & -3 \\ -45 & -13 & 27 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{35}{50} & \frac{9}{50} & -\frac{11}{50} \\ \frac{5}{50} & \frac{7}{50} & -\frac{3}{50} \\ -\frac{45}{50} & -\frac{13}{50} & \frac{27}{50} \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{7}{10} & \frac{9}{50} & -\frac{11}{50} \\ \frac{1}{10} & \frac{7}{50} & -\frac{3}{50} \\ \frac{.-9}{50} & -\frac{13}{50} & \frac{27}{50} \end{bmatrix}$$

Properties of matrices and vectors

- p.278

Table 12.1: Matrix and Vector Properties

Associative property	$(AB)C = A(BC)$
Additive distributive property	$(A+B)C = AC + BC$
Scalar commutative property	$xAB = (xA)B = A(xB) = ABx$

Table 12.2: Matrix and Vector Transpose Properties

Inverse	$(A^T)^T = A$
Additive property	$(A+B)^T = A^T + B^T$
Multiplicative property	$(AB)^T = B^T A^T$
Scalar multiplication	$(cA)^T = cA^T$
Inverse transpose	$(A^{-1})^T = (A^T)^{-1}$
If A is symmetric	$A^T = A$

Table 12.3: Matrix Determinant Properties

Transpose property	$\det(A) = \det(A^T)$
Identity matrix	$\det(I) = 1$
Multiplicative property	$\det(AB) = \det(A)\det(B)$
Inverse property	$\det(A^{-1}) = \frac{1}{\det(A)}$
Scalar multiplication $(n \times n)$	$\det(cA) = c^n \det(A)$
If A is triangular or diagonal	$\det(A) = \prod_{i=1}^{n} a_{ii}$

Table 12.4: Matrix Inverse Properties

Inverse	$(A^{-1})^{-1} = A$
Multiplicative property	$(AB)^{-1} = B^{-1}A^{-1}$
Scalar multiplication $(n \times n)$	$(cA)^{-1} = c^{-1}A^{-1}$ if $c \neq 0$

Time for your questions

- Any questions during the week?
  - joerdis.strack@uni-konstanz.de