Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Eigenvalues and Eigenvectors

To do

- Weekly recap
- Real world applications
- Hands on practice
- Questions

Chapter 14 | Eigenvectors & Eigenvalues

Eigenpairs in the social sciences

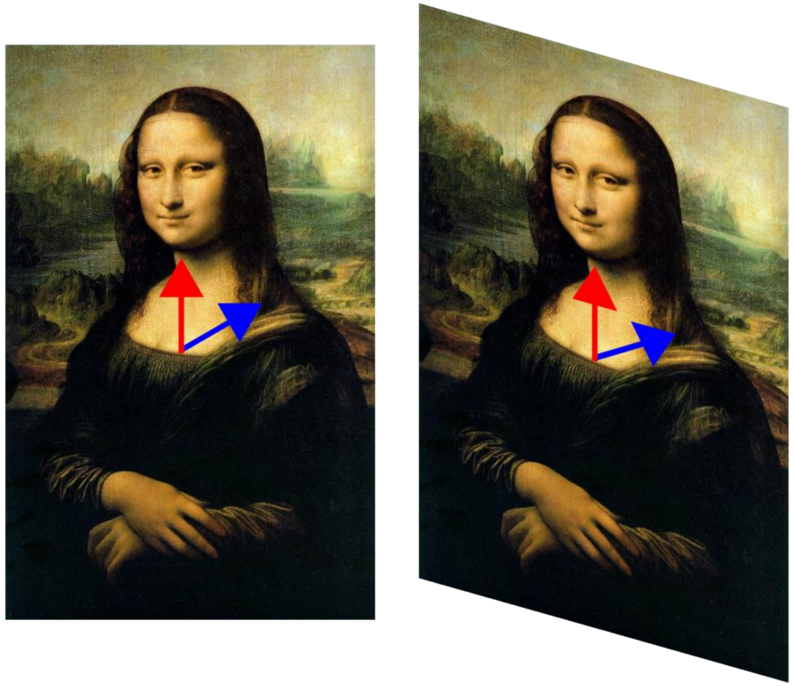
For a square matrix A, **eigenvalues** $\boldsymbol{\lambda}$ and **eigenvectors** $\vec{\boldsymbol{v}}$ make the following equation true:

$$A\vec{v} = \lambda\vec{v}$$

… but intuitively speaking, what are they?

- **eigenvalues** are characteristic for matrix A
- **eigenvectors** are fixed in direction when A is applied to them

→ this is why scalar multiplication must be defined for vector spaces! A 2D graphic is defined on a plane using x- and y-ccoordinates



Source picture: https://de.wikipedia.org/wiki/Eigenwerte_und_Eigenvektoren#/media/Datei:Mona_Lisa_with_eigenvector.png, 16.20.2023

Eigenpairs in the social sciences

Let‘s recall our **calculus** and **analysis** sessions

- why do we study minima and maxima of functions?
- why do we try to solve for roots of polynomials?

→they offer **information** about our **system**! This is what we want to do to gain **inference** ☺

Eigenvectors (and –**values**) are important, because they **describe** and **limit** the **shape** of our **system**!

- they bound a line (or plane) into shape by restraining its linear transformation!
- we are interested in eigenpairs because they describe the shape and behaviour of our system based on changes in data (our matrix A) → linear transformation

Eigenvalues

Eigenvalues λ are specific scalars to a quadratic matrix A and define, how certain eigenvectors $\vec{v}$ change under linear transformation:

$$A\vec{v} = \lambda\vec{v}$$

- 'eigen' is German for 'own' or in this context 'characteristic
- eigenvalues can be understood as 'characteristic values'
- or to stick with our previous example:

  → characteristic 'roots' that are found when we solve A's characteristic polynomial

$$\lambda^2 + \lambda + \beta = 0 \leftarrow \text{for } A_{2\times 2}$$

Eigenvalues – Algorithm

Algorithm:

1) multiply RHS by identity matrix

   $A\vec{v} = \lambda\vec{v} \rightarrow A\vec{v} = \lambda I\vec{v}$

2) factor by $\vec{v}$ and set equal to zero

   $\vec{v}(A - \lambda I) = 0$

3) compute determinant

   $|A - \lambda I| = [(a_{11} - \lambda) \cdot (a_{22} - \lambda)] - [(a_{12} \cdot a_{21})]$

4) solve for λ

   $\lambda^2 + \lambda + \beta = 0$

Computing Eigenvalues

Example: $A = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$

1) $A\vec{v} = \lambda\vec{v} \rightarrow A\vec{v} = \lambda I \vec{v}$

$$\begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix} \vec{v} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

2) factor by $\vec{v}$ and set equal to zero: $\vec{v}\ (A - \lambda I) = 0$

$$\vec{v} \left(\begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

$$\vec{v} \left(\begin{bmatrix} 6 - \lambda & 1 \\ -2 & 3 - \lambda \end{bmatrix} \right) = 0$$

Computing Eigenvalues

Example:

3) compute determinant $|A - \lambda I|$

$$|A - \lambda I| = [(6 - \lambda) \cdot (3 - \lambda)] - [1 \cdot (-2)] = \lambda^2 - 9\lambda + 20$$

4) solve for λ

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \cdot 1 \cdot 20}}{2} = \frac{9 \pm \sqrt{1}}{2}$$

$$\lambda_1 = 5$$

$$\lambda_2 = 4$$

Hands on – Eigenvalues

Task: Compute the eigenvalues for the respective matrices below!

1) $A = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix}$

2) $B = \begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix}$

Hands on – Eigenvalues

Solution:

$$1)\ A = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix} \vec{v} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

$$\vec{v}\left(\begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\vec{v}\left(\begin{bmatrix} 4-\lambda & 3 \\ 6 & 1-\lambda \end{bmatrix}\right) = 0$$

$$|A - \lambda I| = [(4-\lambda)(1-\lambda)] - [3 \cdot 6]$$

$$= \lambda^2 - 5\lambda - 14$$

$$\lambda_{1,2} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-14)}}{2}$$

$$\lambda_1 = 7$$

$$\lambda_2 = (-2)$$

Hands on – Eigenvalues

Solution:

$2)\ B = \begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix}$

$$\begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix} \vec{v} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

$$\vec{v} \left(\begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

$$\vec{v} \left(\begin{bmatrix} (-8) - \lambda & -4 \\ 6 & 2 - \lambda \end{bmatrix} \right) = 0$$

$$|A - \lambda I| = [((-8) - \lambda)(2 - \lambda)] - [6 \cdot (-4)]$$

$$= \lambda^2 + 6\lambda + 8$$

$$\lambda_{1,2} = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 8}}{2}$$

$$\lambda_1 = (-2)$$

$$\lambda_2 = (-4)$$

Eigenvectors

An **eigenvector** $\vec{v}$ to a **quadratic matrix** $\boldsymbol{A}$ over an **object** $\boldsymbol{K}$ characterizes its linear transformation:

- any eigenvector $\vec{v}$ must be different from the Nullvector $\vec{0}$
- an eigenvector $\vec{v}$ is NOT 'definitely' defined as it can be scaled by A's eigenvalues λ
- iff $\vec{v}$ is an eigenvector, every $\boldsymbol{\alpha\vec{v}} \in \boldsymbol{K}$ will be an eigenvector, too
- such a scaling / linear transformation does **not** change the direction of $\vec{v}$

→an **eigenvalue** $\boldsymbol{\lambda}$ has an **infinite** number of **eigenvectors** $\vec{\boldsymbol{v}}$

→ an **eigenvector** $\vec{\boldsymbol{v}}$ only belongs to **ONE eigenvalue** $\boldsymbol{\lambda}$

- **Note**: Only **quadratic matrices** have eigenvectors/values

Eigenvectors – Algorithm

Algorithm:

1) start with $A\vec{v} = \lambda\vec{v}$ and plug in the eigenvalues

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}$$

2) multiply to get system of equations

$$\begin{matrix} a_{11}x + a_{12}y = \\ a_{21}x + a_{22}y = \end{matrix} \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}$$

3) plug in $x = 1$…

4) … and solve for variables $y, \cdots, n$ to find eigenvector $\begin{pmatrix} x \\ y \\ \vdots \\ n \end{pmatrix}$

Computing Eigenvectors

Example: $A = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$ with eigenvalues $\lambda_1 = 5, \lambda_2 = 4$

1) and 2) start with $A\vec{v} = \lambda\vec{v}$, plug in eigenvalues, multiply & simplify

$$6x + y = 5x$$
$$-2x + 3y = 5y$$

3) and 4) plug in $x = 1$ and solve for y

$\mathbf{starting\ with\ \lambda_1 = 5}$

$$6 \cdot 1 + y = 5 \cdot 1 \rightarrow y = (-1)$$
$$(-2) \cdot 1 + 3y = 5y \rightarrow (-2) = 2y \rightarrow y = (-1)$$

Computing Eigenvectors

Example: $A = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$ with eigenvalues $\lambda_1 = 5, \lambda_2 = 4$

3) and 4) plug in $x = 1$ and solve for y

starting with $\boldsymbol{\lambda_2 = 4}$
$6 \cdot 1 + y = 4 \cdot 1 \rightarrow y = (-2)$
$(-2) \cdot 1 + 3y = 4y \rightarrow y = (-2)$

→ our eigenvectors for $x = 1$ are $\overrightarrow{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Hands on – Eigenvectors

Task: Compute the eigenvectors for the following matrices!

1) $A = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix}$

2) $B = \begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix}$

Hands on – Eigenvectors

Solution:

1) $A = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix}$

- $\boldsymbol{A\vec{v} = \lambda\vec{v}}$ **and set up SoE**

$$\begin{cases} 4x + 3y = 7x \\ 6x + y = 7y \end{cases}$$

- **plug in $\boldsymbol{x = 1}$ and $\boldsymbol{\lambda = 7}$ solve for $\boldsymbol{y}$**

$$\begin{cases} 4 \cdot 1 + 3y = 7 \cdot 1 \\ 6 \cdot 1 + y = 7y \end{cases}$$

$$\begin{cases} 3y = 3 \ \rightarrow y = 1 \\ 6y \ = 6 \ \rightarrow y = 1 \end{cases} \rightarrow \overrightarrow{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- **plug in $\boldsymbol{x = 1}$ and $\boldsymbol{\lambda = (-2)}$ solve for $\boldsymbol{y}$**

$$\begin{cases} 4 \cdot 1 + 3y = (-2) \cdot 1 \\ 6 \cdot 1 + y = (-2)y \end{cases}$$

$$\begin{cases} 3y = (-6) \ \rightarrow y = (-2) \\ (-3)y \ = 6 \ \rightarrow y = (-2) \end{cases} \rightarrow \overrightarrow{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Hands on – Eigenvectors

Solution:

2) $B = \begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix}$

- $\boldsymbol{A\vec{v} = \lambda\vec{v}}$ **and set up SoE**

$$\begin{cases} (-8)x + (-4)y = (-2)x \\ 6x + 2y = (-2)y \end{cases}$$

- **plug in $\boldsymbol{x = 1}$ and $\boldsymbol{\lambda = (-2)}$ solve for $\boldsymbol{y}$**

$$\begin{cases} (-8)\cdot 1 + (-4)y = (-2)\cdot 1 \\ 6\cdot 1 + 2y = (-2)y \end{cases}$$

$$\begin{cases} (-4)y = 6 \ \rightarrow y = -1.5 \\ (-4)y = 6 \ \rightarrow y = -1.5 \end{cases} \rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}$$

- **plug in $\boldsymbol{x = 1}$ and $\boldsymbol{\lambda = (-4)}$ solve for $\boldsymbol{y}$**

$$\begin{cases} (-8)\cdot 1 + (-4)y = (-4)\cdot 1 \\ 6\cdot 1 + 2y = (-4)y \end{cases}$$

$$\begin{cases} (-4)y = (-4) \ \rightarrow y = (-1) \\ (-6)y = 6 \ \rightarrow y = (-1) \end{cases} \rightarrow \vec{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Real world applications – Eigenvalues & -vectors

Quantum mechanics and Wave Theory

- incredibly important in quantum mechanics, physics, maths and chemistry… basically all the nature sciences
- anything that can be described as an oscillating system (musical instrument, particles, molecules, bridges…) relies on eigenpairs

  → these systems have 'favourite' frequencies – so called resonance frequencies – which are described using eigenfunctions, -values, -vectors

  → we WANT to encourage oscillation in musical instruments

  → we DO NOT want to encourage oscillation in bridges… (see below!)

- https://www.youtube.com/watch?v=j-zczJXSxnw

Matrix diagonalization

A quadratic matrix A is called diagonalizable, if:

- there is an inverse Q with which the result of $\mathrm{Q}^{-1}AQ$ is a diagonal matrix
- geometric and algebraic multiplicity of eigenvalues (and eigenvectors) are equal
- $A's$ characteristic polynomial can be expressed entirely using linear factors
- $A's$ characteristic polynomial has as many roots as A has dimensions

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 6 & 5 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow D_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Matrix diagonalization – Algorithm

Algorithm:

1) Find A's eigenvalues
2) Check, if A is diagonalizable:

   → Find A's eigenvectors
   → Find A's characteristic polynomial and its roots

3) Find diagonal matrix $D_A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

← if you're tasked with finding D_A, you can most likely expect A to be diagonalizable!

Matrix diagonalization – Inverse matrix

How to find the inverse of a diagonal matrix?

→ if you multiply a diagonal matrix with its inverse, the result will be the identity matrix!

Note: the inverse of D_A is composed of D_A's reciprocal values!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hands on – Matrix diagonalization

Task: Diagonalize the following matrices!

→ **Hint**: Solution will be of form $D_A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

1) $A = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix}$

2) $B = \begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix}$

Hands on – Matrix diagonalization

Solution:

1) $A = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix} \rightarrow \mathrm{D_A} = \begin{bmatrix} 7 & 0 \\ 0 & (-2) \end{bmatrix}$

2) $B = \begin{bmatrix} -8 & -4 \\ 6 & 2 \end{bmatrix} \rightarrow \mathrm{D_B} = \begin{bmatrix} (-2) & 0 \\ 0 & (-4) \end{bmatrix}$

Real world applications – Matrix diagonalization

Why do we diagonalize matrices?

→ because it makes our lives **easier**!

- matrix multiplication
- scalar multiplication
- addition and subtraction
- inverses…
- transpose matrices…
- you name it

→ diagonal matrices are of a ‘very-easy-to-work-with’ form ;)

Matrices and OLS

Linear model → OLS

you should know this: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \varepsilon$

but do you know this: $\vec{y} = X\vec{b} + \vec{e}$

$$\vec{e} = \vec{y} - X\vec{b}$$

minimized sum of squared residuals $\vec{e}^T\vec{e}$

$$\vec{e}^T\vec{e} = (\vec{y} - X\vec{b})^T(\vec{y} - X\vec{b})$$

$$X^T X\vec{b} = X^T\vec{y}$$

after minimization:
if $X^T X$ is non-singular

$$\hat{\beta} = (X^T X)^{-1} X^T\vec{y}$$

Matrices and OLS – Epidemiology

Source: Mathematics II for Chemistry, Life Science and Nano Science |Dr. Stefan Frei

t_k	0	1	2	3
i_k	58 716	56 998	54 984	53 450
$j_k := \ln(i_k)$	10.98	10.95	10.91	10.89

Source: RKI

let's estimate the 'Reproduction figure R_0' for Germany back in spring 2020...

Assumption: Infektionszahl follows the relationship: $i(t) = i_0 \exp\left(\frac{(R_0-1)t}{t_{int}}\right)$

→ let's scale the exp. growth!

$$j(t) := \ln(i(t)) = \underbrace{\ln(i_0)}_{j_0} + \frac{(R_0-1)t}{t_{int}}$$

Matrices and OLS – Epidemiology

we want to model: $j(t_k) = j_k \Rightarrow j_0 + \frac{t_k}{t_{inf}} R_0 = \frac{t_k}{t_{inf}} j_k$ for $k = 1, \dots, 4$
(over time)

set t_{inf} to 6 days $\Rightarrow t_{inf} = 6$

$$\begin{pmatrix} 1 & \frac{t_1}{6} \\ 1 & \frac{t_2}{6} \\ 1 & \frac{t_3}{6} \\ 1 & \frac{t_4}{6} \end{pmatrix} \underset{\text{parameters}}{\begin{pmatrix} j_0 \\ R_0 \end{pmatrix}} = \begin{pmatrix} \frac{t_1}{6} + j_1 \\ \frac{t_2}{6} + j_2 \\ \frac{t_3}{6} + j_3 \\ \frac{t_4}{6} + j_4 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{6} \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} j_0 \\ R_0 \end{pmatrix}}_{=\vec{x}} = \underbrace{\begin{pmatrix} 10.98 \\ 11.12 \\ 11.24 \\ 11.39 \end{pmatrix}}_{=\vec{b}}$$

Matrices and OLS – Epidemiology

$$\Rightarrow \quad A^T A \vec{x} = A^T \vec{b} \; : \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{matrix} 44.73 \\ 11.29 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{6} \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} 10.98 \\ 11.12 \\ 11.24 \\ 11.39 \end{pmatrix}$$

$A^T A =$

$$\begin{pmatrix} (1\cdot 1)+(1\cdot 1)+(1\cdot 1)+(1\cdot 1) & (0\cdot 1)+(\frac{1}{6}\cdot 1)+(\frac{1}{3}\cdot 1)+(\frac{1}{2}\cdot 1) \\ (0\cdot 1)+(\frac{1}{6}\cdot 1)+(\frac{1}{3}\cdot 1)+(\frac{1}{2}\cdot 1) & (0\cdot 0)+(\frac{1}{6}\cdot\frac{1}{6})+(\frac{1}{3}\cdot\frac{1}{3})+(\frac{1}{2}\cdot\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} = A^T A$$

Matrices and OLS – Epidemiology

$$\Rightarrow \quad A^T A \vec{x} = A^T \vec{b} \; : \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{matrix} 44.73 \\ 11.29 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{6} \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} 10.98 \\ 11.12 \\ 11.24 \\ 11.39 \end{pmatrix}$$

$$A^T A =$$

$$\begin{pmatrix} (1\cdot 1)+(1\cdot 1)+(1\cdot 1)+(1\cdot 1) & (0\cdot 1)+(\frac{1}{6}\cdot 1)+(\frac{1}{3}\cdot 1)+(\frac{1}{2}\cdot 1) \\ (0\cdot 1)+(\frac{1}{6}\cdot 1)+(\frac{1}{3}\cdot 1)+(\frac{1}{2}\cdot 1) & (0\cdot 0)+(\frac{1}{6}\cdot\frac{1}{6})+(\frac{1}{3}\cdot\frac{1}{3})+(\frac{1}{2}\cdot\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} = A^T A$$

Matrices and OLS – Epidemiology

$$\Rightarrow \quad A^T A \vec{x} = A^T \vec{b} \; : \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{matrix} 44.73 \\ 11.29 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{6} \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} 10.98 \\ 11.12 \\ 11.24 \\ 11.39 \end{pmatrix}$$

$$A^T A =$$

$$\begin{pmatrix} (1\cdot1)+(1\cdot1)+(1\cdot1)+(1\cdot1) & (0\cdot1)+(\frac{1}{6}\cdot1)+(\frac{1}{3}\cdot1)+(\frac{1}{2}\cdot1) \\ (0\cdot1)+(\frac{1}{6}\cdot1)+(\frac{1}{3}\cdot1)+(\frac{1}{2}\cdot1) & (0\cdot0)+(\frac{1}{6}\cdot\frac{1}{6})+(\frac{1}{3}\cdot\frac{1}{3})+(\frac{1}{2}\cdot\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} = A^T A$$

Matrices and OLS – Epidemiology

$$\Rightarrow \quad A^T A \vec{x} = A^T \vec{b} \quad : \quad \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} 44.73 \\ 11.29 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{6} \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} j_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 10.98 \\ 11.12 \\ 11.24 \\ 11.35 \end{pmatrix}$$

$$A^T A =$$

$$\begin{pmatrix} (1\cdot 1)+(1\cdot 1)+(1\cdot 1)+(1\cdot 1) & (0\cdot 1)+(\frac{1}{6}\cdot 1)+(\frac{1}{3}\cdot 1)+(\frac{1}{2}\cdot 1) \\ (0\cdot 1)+(\frac{1}{6}\cdot 1)+(\frac{1}{3}\cdot 1)+(\frac{1}{2}\cdot 1) & (0\cdot 0)+(\frac{1}{6}\cdot\frac{1}{6})+(\frac{1}{3}\cdot\frac{1}{3})+(\frac{1}{2}\cdot\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & \frac{7}{18} \end{pmatrix} = A^T A$$

Matrices and OLS – Epidemiology

→ apply Gauß-Elimination: $\quad \left| \; II + \left(-\frac{1}{4} I\right) \right.$

$$\left(\begin{array}{cc|c} 4 & 1 & 44.73 \\ 1 & 7/18 & 11.29 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 4 & 1 & 44.73 \\ 0 & \frac{5}{36} & 0.1075 \end{array}\right)$$

→ substitute backwards:

$$x_2 = R_0 = 0.1075 \cdot \frac{36}{5} \approx 0.774$$

$$x_1 = j_0 \approx \frac{1}{4}(44.73 - 0.774) = 10.989$$

$$\Rightarrow i_0 = \exp(j_0) \approx 59219 \qquad \Rightarrow \boxed{i(t) = 59219 \exp\left(\frac{0.774 - 1}{t_{inf}} t\right)}$$

Time for your questions

- Any questions during the week?
  - joerdis.strack@uni-konstanz.de