

Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Extrema in One Dimension

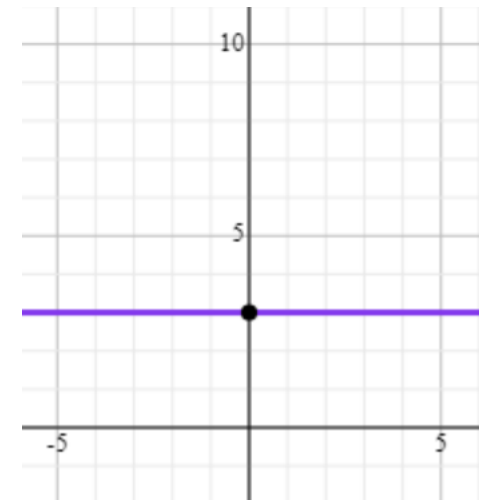
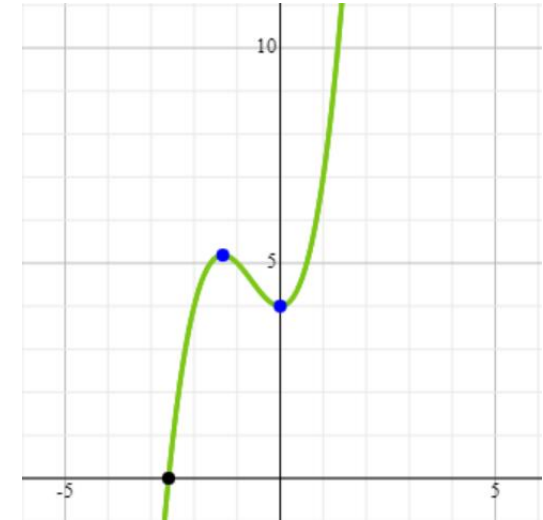
To do

- Weekly recap
 - Real world applications
 - Hands on practice
 - Questions
-
- Upcoming Deadline: 10.12.2023 – Assignment 02

Chapter 8 | Extrema in one dimension

Extrema

- Let's get to know our functions!
 - **Maximum** - $f(x_0) \geq f(x) \forall x \in [a, b]$
 - **Minimum** - $f(x_0) \leq f(x) \forall x \in [a, b]$
- function f must be **differentiable** (must be **defined** for **interval** $[a, b]$ and the **slope** must be **non-zero**)
 - $f'(x) = 0$
 - $f''(x) \neq 0$ for minimum or maximum
- Why use **differentiation**?
 - we are interested in points where $f'(x)$'s **rate of change** takes on **0 momentarily**, before in- or decreasing!



Local and global extrema

$$f(x) = 3x \cdot \sin(2x) + 4$$



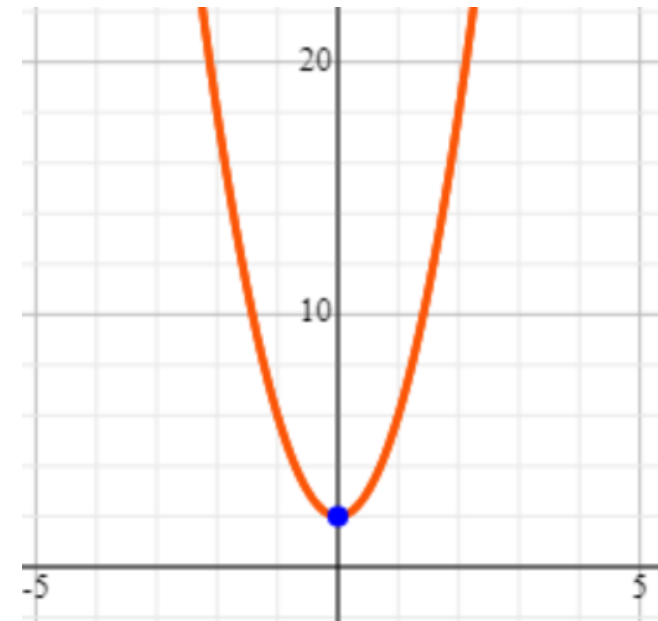
Global

- the ‚highest‘ value in range f can take on
- the ‘lowest’ value in range f can take on

Local

- still extremum... but not the most extreme in $I [a, b]$

$$f(x) = 4x^2 + 2$$



Finding extrema

General

- 1) find $f'(x)$
- 2) set $f'(x_0) = 0$ and **solve** for all x_0
→ stationary points
- 3) find $f''(x)$
- 4) for each stationary point, plug in x_0 **into** $f''(x)$ and obtain all extrema, inflection & saddle points
- 5) plug each point into $f(x)$ to **find** the according **value for y**

WITH given interval $[a, b]$

- 6) evaluate function's values at **lower limit** $f(a)$ and **upper limit** $f(b)$
- 7) compare all extrema and **find global minimum** and **maximum** for interval $[a, b]$

→ VERY detailed algorithm in Moore & Siegel, 2013, p.168 😊

Minimum, maximum, inflection or saddle point?

Moore & Siegel, 2013, p.168

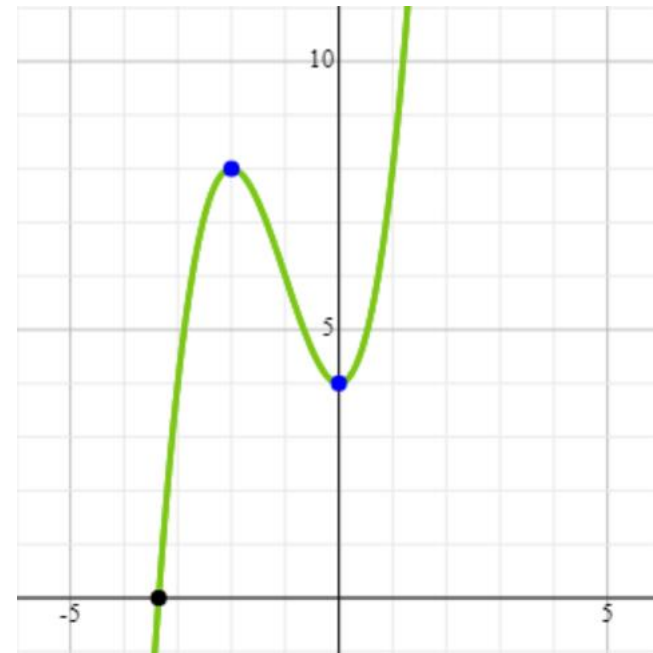
- If $f''(x^*) < 0$, $f(x)$ has a local maximum at x^* .
- If $f''(x^*) > 0$, $f(x)$ has a local minimum at x^* .
- If $f''(x^*) = 0$, x^* may be an inflection point. To check this:
 - a) Calculate higher-order derivatives ($f'''(x)$, $f^{(4)}(x)$, etc.) until you find the first one that is non-zero at x^* . Call the order of this derivative n .
 - b) If n is odd, then this x^* is an inflection point and not an extremum. Do not include it in further steps.
 - c) If n is even and $f^{(n)}(x^*) < 0$, $f(x)$ has a local maximum at x^* .
 - d) If n is even and $f^{(n)}(x^*) > 0$, $f(x)$ has a local minimum at x^* .

Extreme value theorem:

- a **real-valued** function that is **continuous** and **differentiable** on a **closed and bounded interval** $[a, b]$ **must attain both global min and max at least once!**

Hands on – finding extrema

- **Task:** find all extrema of function $f(x) = x^3 + 3x^2 + 4$ for $x \in [-3, 2]$
- **Hints:**
 - find $f'(x)$
 - find $f''(x)$
 - find all stationary points and obtain x_0
 - find all values for y for all x_0

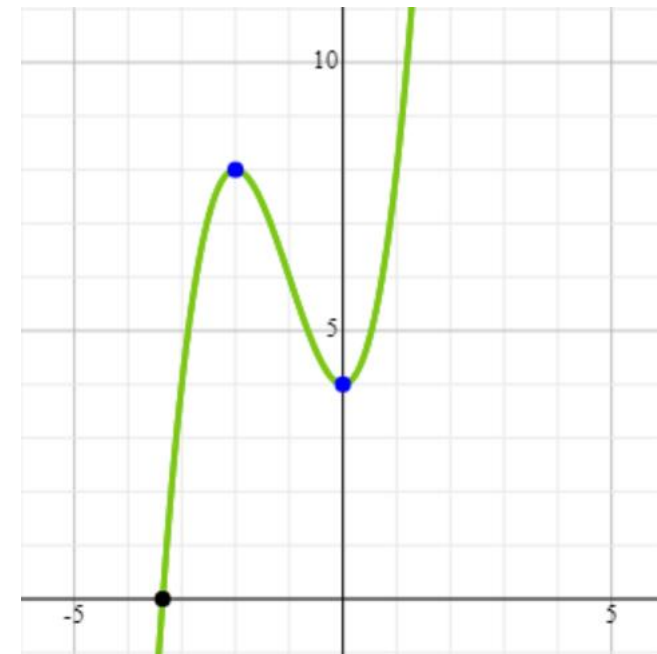


Hands on – finding extrema

- **Task:** find all extrema of function $f(x) = x^3 + 3x^2 + 4$ for $[-3, 2]$

- **Hints:**

- find $f'(x) = 3x^2 + 6x$
- find $f''(x) = 6x + 6$
- find all stationary points and obtain x_0
 - $x = -3, x = -2, x = 0, x = 2$
- find all values for y for all x_0
 - global minimum $(-3, 4)$
 - global minimum $(0, 4)$
 - local maximum $(-2, 8)$
 - global maximum $(2, 24)$



Inflection and saddle points

Inflection points

- does not have to be stationary point, but if it is, then **not** a **local extremum but a saddle point!**
- **sign** of curvature of function **changes**
- **‘car steering’ test:** Imagine yourself driving a car – do you have to make an S-curve to follow the curvature of the graph?

Saddle points

- stationary point that is **not** a **local extremum**
- **slope** is equal to **zero** for **ALL directions of graph** – **tangent** is **horizontal** → there is **no sign change** before and after saddle point!
- **conditions:**
$$f'(x_0) = 0$$
$$f''(x_0) = 0$$
$$f'''(x_0) \neq 0$$

Inflection and saddle points – Algorithms

Inflection points

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points
(or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0 and changes sign
→ plug in $x_0 = a$ into $f'(x)$ and
check if $f'(x_0) \neq 0$

• conditions:

$$\begin{aligned}f'(x_0) &= 0 \\f''(x_0) &= 0 \\f'''(x_0) &\neq 0\end{aligned}$$

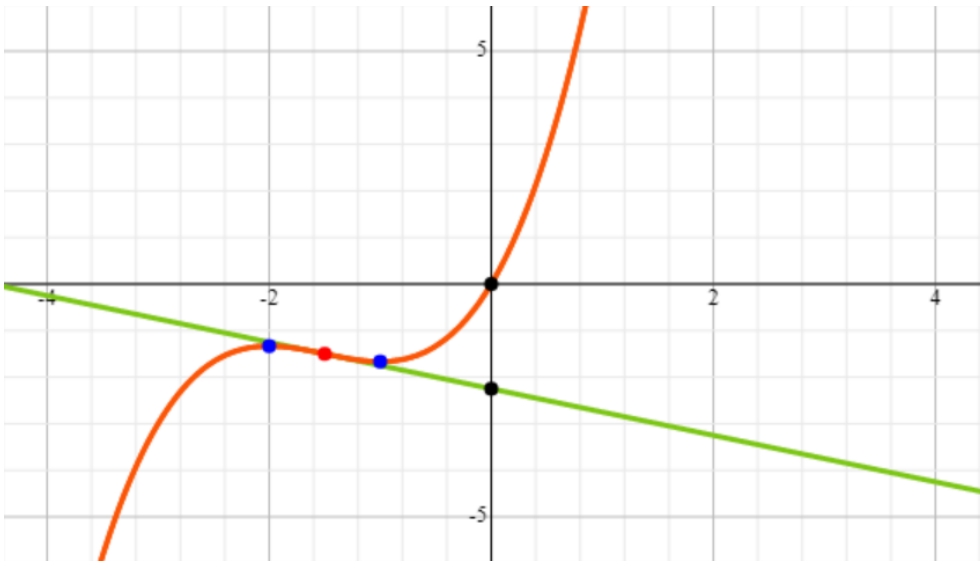
Saddle points

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points
(or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0 and changes sign
→ plug in $x_0 = a$ into $f'(x)$ and
check if $f'(x_0) = 0$

Inflection and saddle points

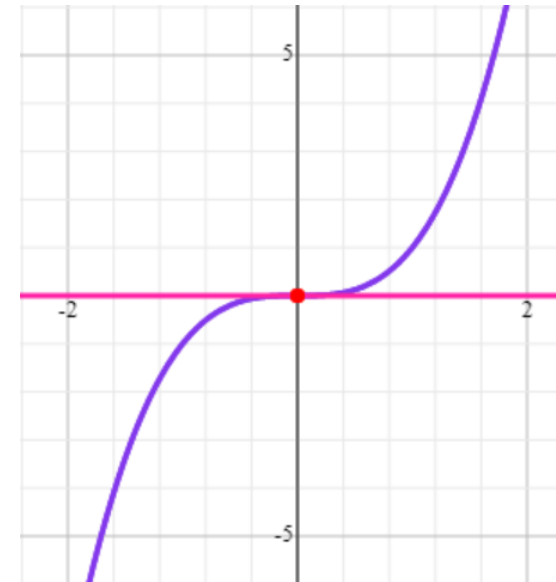
Inflection points

- inflection point at $(-1,5, -1,5)$ for $f(x) = \frac{2}{3}x^3 + 3x^2 + 4x$



Saddle points

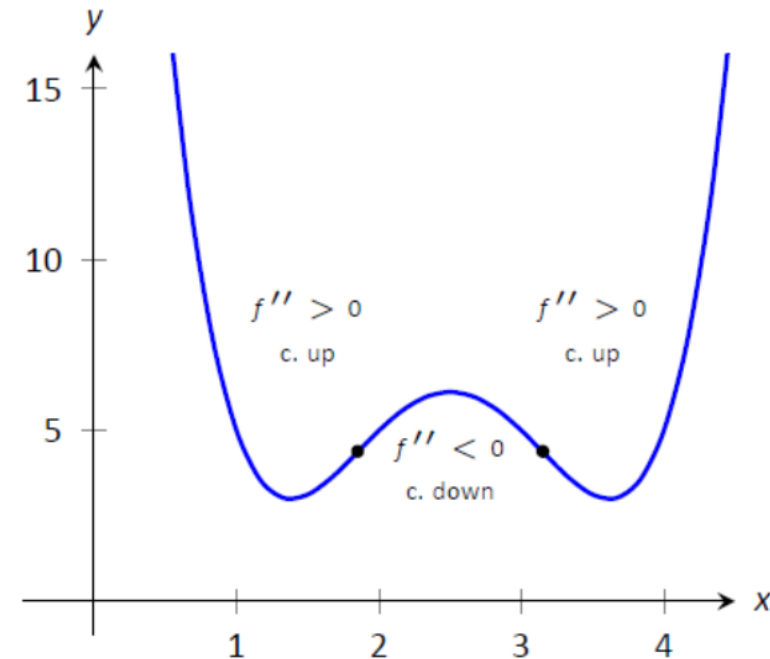
- both an inflection and saddle point at $(0, 0)$ for $f(x) = x^3$



Concavity, convexity and inflection points 😊

Intuition: **inflection point** tells us, where the sign of function f changes

- to describe **concavity & convexity**, f must be **differentiable** at least **twice** on interval I $[a, b]$
 - graph of f is convex if $f'' > 0$
→ gradient is increasing
 - graph of f is concave if $f'' < 0$
→ gradient is decreasing

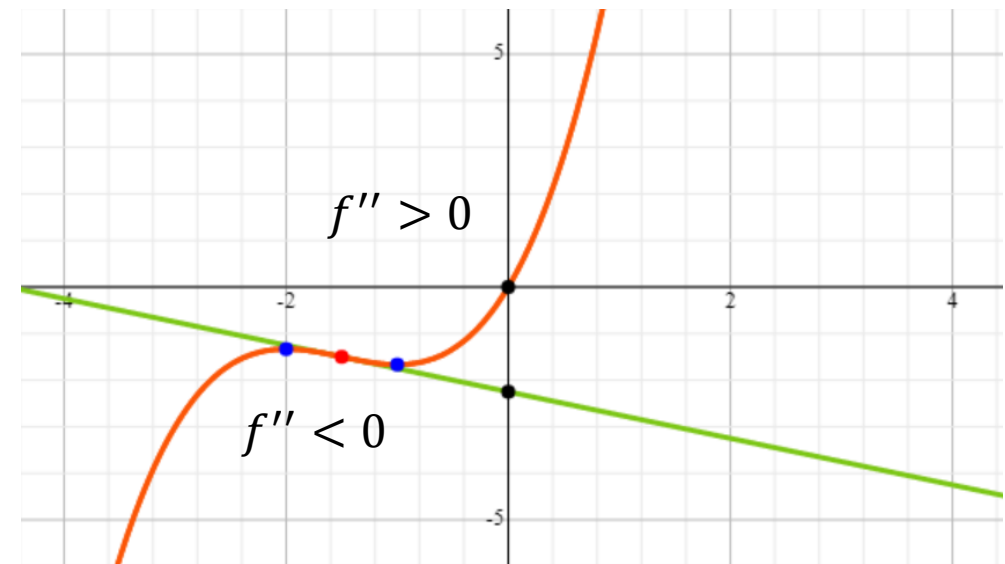


[https://math.libretexts.org/Bookshelves/Calculus/Calculus_3e_\(Apex\)/03%3A_The_Graphical_Behavior_of_Functions/3.04%3A_Concavity_and_the_Second_Derivative](https://math.libretexts.org/Bookshelves/Calculus/Calculus_3e_(Apex)/03%3A_The_Graphical_Behavior_of_Functions/3.04%3A_Concavity_and_the_Second_Derivative)

Concavity and convexity

Graphically: graph of f is **convex** if it lies **above** its **tangent line** at inflection point (x_0, y) and **concave** if it lies **below** its **tangent line**

- **inflection point** at $(-1,5, -1,5)$
for $f(x) = \frac{2}{3}x^3 + 3x^2 + 4x$
- **concave** – below the ‘mountains’
- **convex** above the ‘valleys’ 😊



Hands on – saddle points

Let function f be $f(x) = -\frac{2}{3}x^3 + 2x^2 - 2x + 4$

Task: find the inflection/saddle points of f and discuss where f is concave or convex

Hints:

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points (or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0
- describe sign changes around inflection point and classify concavity/convexity using $f'(x)$

Hands on – inflexion points, concavity, convexity

Let function f be $f(x) = -3x^3 - 3x + 1$

Task: find the inflection/saddle points of f and discuss where f is concave or convex

Algorithm:

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points (or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0
- describe sign changes around inflection point and classify concavity/convexity using $f'(x)$

$$f(x) = 3x^3 - 3x + 1$$

$$\text{find } f'(x) = 9x^2 - 3$$

$$\text{find } f''(x) = 18x$$

set $f''(x) = 0$ and find inflection points:

$$18x = 0$$

$$x = 0$$

← potential inflection point at $x_0 = 0$

$$\text{find } f'''(x) = 18 \neq 0$$

inflection or saddle point?

plug in $x_0 = 0$ into $f'(x)$ and check if $f'(x_0) = 0$

$$f'(0) = 9 \cdot 0^2 - 3 = -3 \neq 0 \quad \leftarrow f'(0) \neq 0 \rightarrow \text{inflection point!}$$

obtain corresponding y-value:

$$f(0) = 3 \cdot 0^3 - 3 \cdot 0 + 1 = \underline{\underline{1}}$$

→ $f(x)$ has an inflection point at $(0,1)$

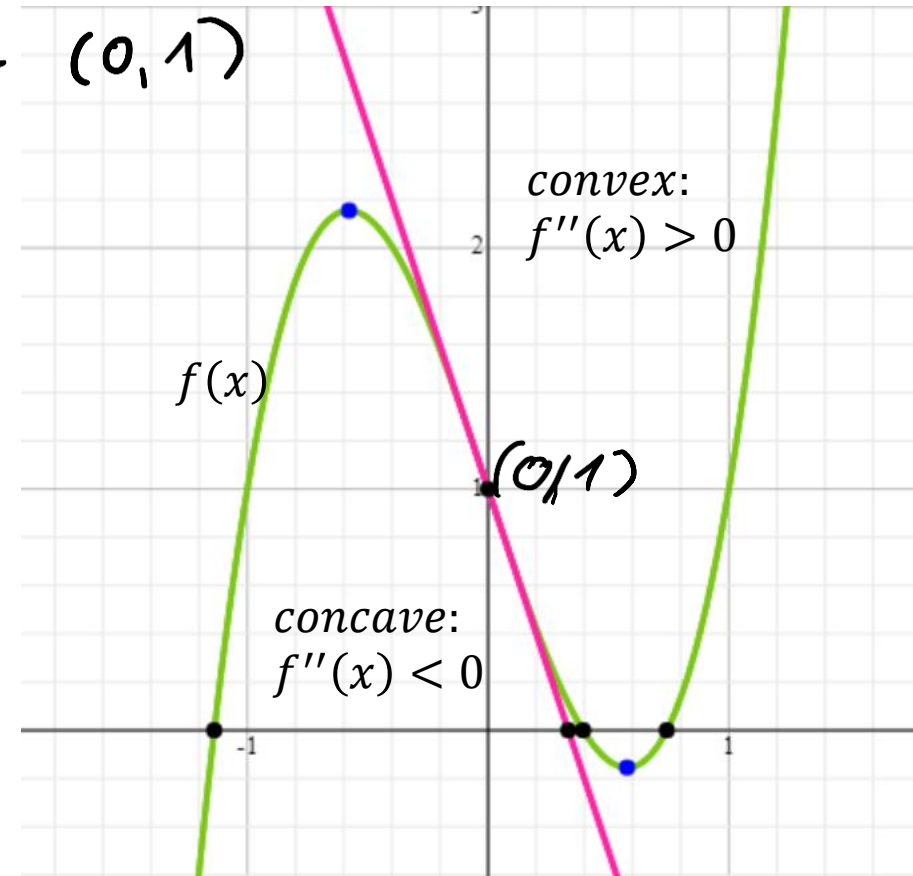
DISCUSSION

→ $f(x)$ has an inflection point at $(0,1)$

→ if we plot the tangent at $x=0$,
we can clearly see where
 $f(x)$ is concave and convex.

→ at inflection point $(0/1)$
the sign of the 2nd derivative
changes!

→ 'car steering wheel' test



Taylor series

Intuition:

- first derivative states if function f increases or decreases
- second derivative describes f 's curvature
- ... can we ,build' a function with all relevant info from its derivatives?

Answer:

- Yes, via the Taylor series!

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

Moore & Siegel, 2013, p.161

Taylor series

Let's look at the function $g(x) = e^x$. Noting the fact that the k th order derivative of $g(x)$ is also $g(x)$, the expansion of $g(x)$ about $x=a$, is given by:

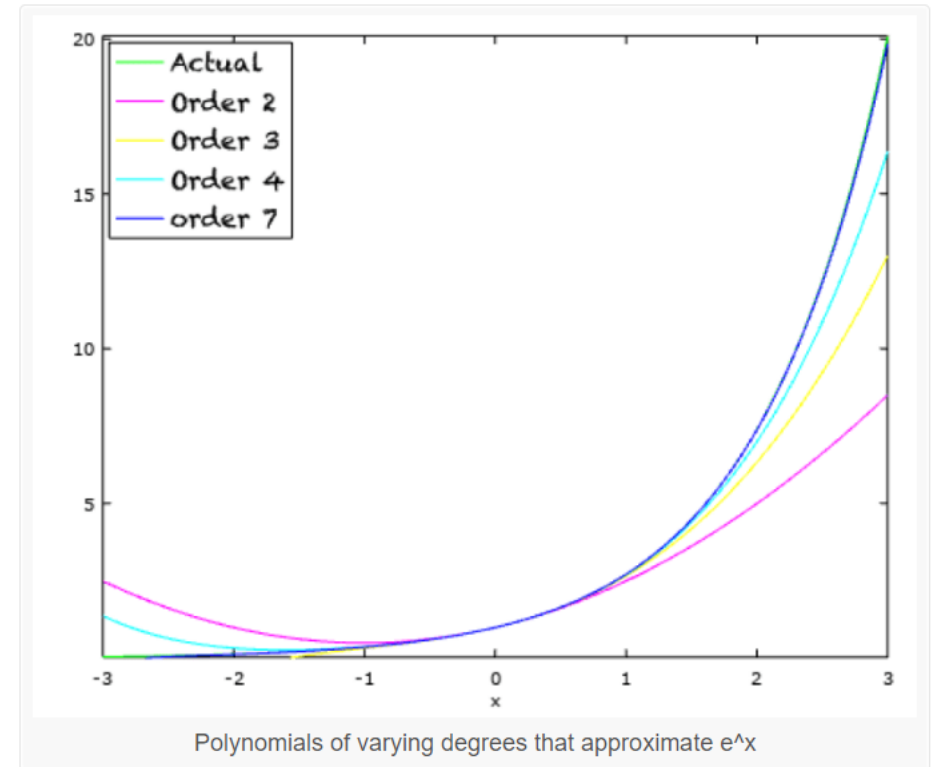
$$e^a + e^a(x - a) + \frac{e^a}{2!}(x - a)^2 + \dots + \frac{e^a}{k!}(x - a)^k + \dots$$

Hence, around $x=0$, the series expansion of $g(x)$ is given by (obtained by setting $a=0$):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

The polynomial of order k generated for the function e^x around the point $x=0$ is given by:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!}$$



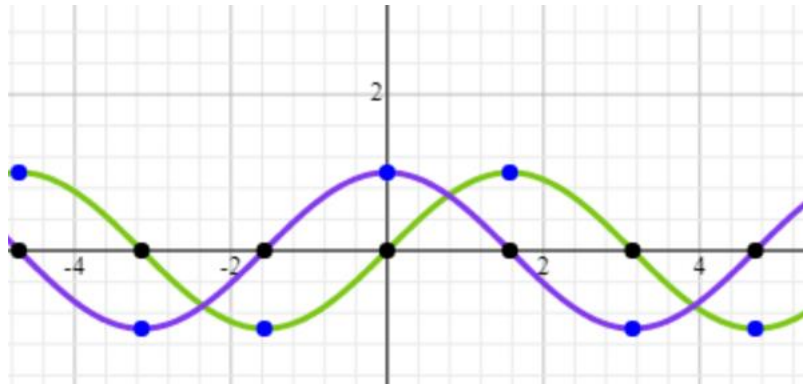
<https://machinelearningmastery.com/a-gentle-introduction-to-taylor-series/>, 28.11.2023

Taylor series - example

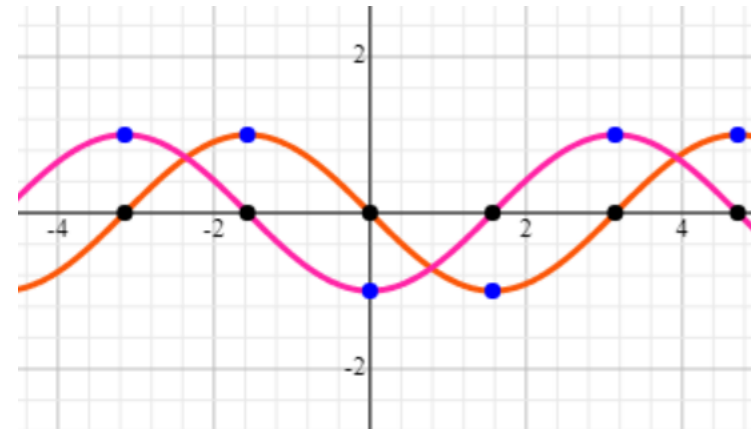
Task: derive the Taylor series for $f(x) = \sin(x)$ with $a = 0$

Algorithm:

- find n^{th} derivative of f
- plug in starting point $a = 0$
- plug result for each derivative into Taylor series



$$f(x) = \sin(x) \text{ and } h(x) = \cos(x)$$



$$g(x) = -\sin(x) \text{ and } j(x) = -\cos(x)$$

Taylor series - example

Solution: derive the Taylor series for $f(x) = \sin(x)$ with $a = 0$

Find n^{th} derivative of f

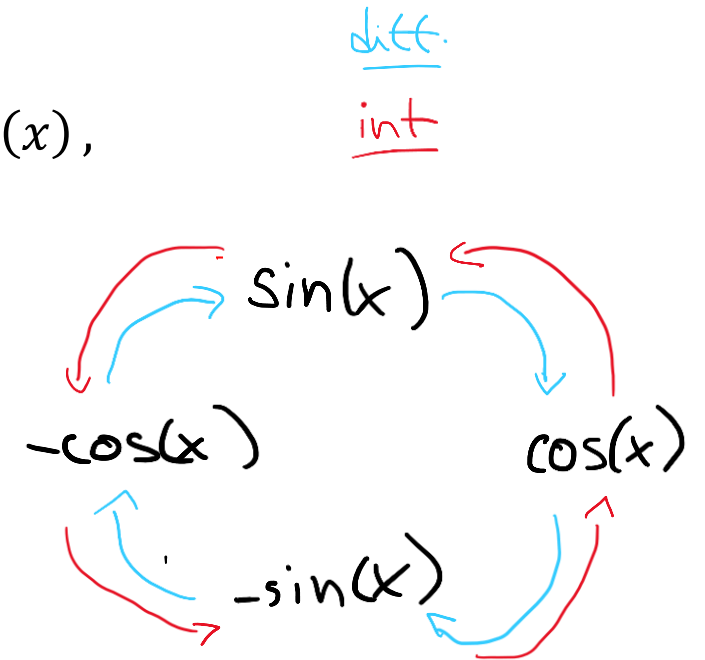
$$f(x) = \sin(x), f'(x) = \cos(x), f''(x) = -\sin(x), f'''(x) = -\cos(x),$$

$$f^{(4)}(x) = \sin(x), \dots \text{mind differentiation rules}$$

Plug in starting point $a = 0$

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1,$$

$$f^{(4)}(0) = 0 \dots \text{mind differentiation rules}$$



Taylor series - example

Solution: derive the Taylor series for $f(x) = \sin(x)$ with $a = 0$

Plug in starting point $a = 0$

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(x) = -1, \\ f^{(4)}(0) = 0$$

Plug result for each derivative into Taylor series

$$\sin(x) = 0 + \frac{1}{1!}(x) + 0 + \frac{-1}{3}(x^3) + 0 + \frac{1}{5!}(x^5) + 0 + \frac{-1}{7!}(x^7) + \dots$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Real world applications – Taylor series

- **Nature science:**

- The Taylor series is applied to approach complex functions for e.g., molecular movement or energy exchange in systems

- **Machine learning:**

- Taylors a lot in various model training algorithms

- If you would like to learn a bit more about Tayloring, give this video a go!

- <https://www.3blue1brown.com/lessons/taylor-series>

Time for your questions

- Any questions during the week?
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