Tutorial – Mathematics for Social Scientists

Winter semester 2024/25

Introduction to Probability

To do

- short extrema recap
- weekly recap
- real world applications
- hands on practice
- questions

Chapter 8 | Extrema - Recap

Extrema Recap

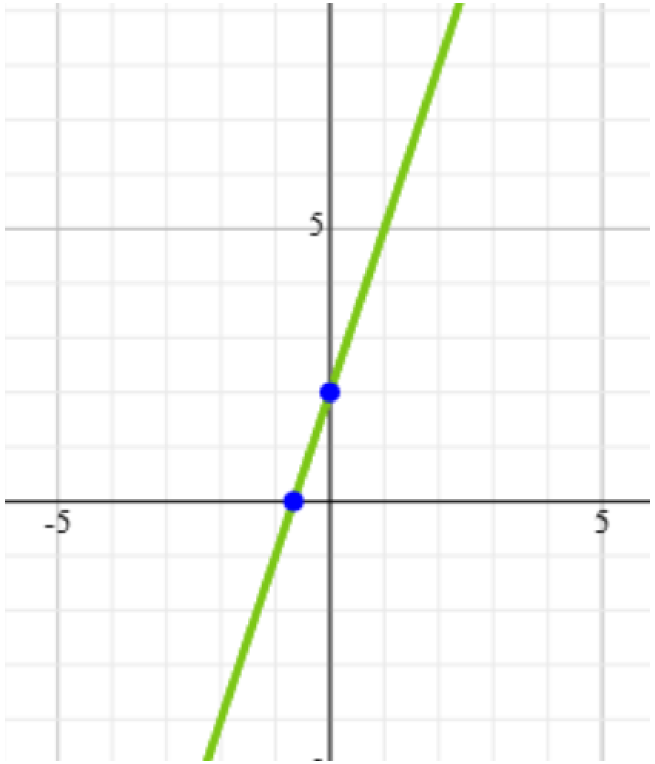
Does $f(x) = 3x + 2$ have extrema?

→No, f(x) increases in a strictly monotonic fashion and if we inspect it in its entirety, it does not have extrema

… what about $f(x) = 3x + 2 \;\; for \; x \in [1, 2]$?

→Yes! Once we zoom into an interval and only inspect parts of f(x), we can find extrema!

→f(x) then has a global min at P(1|5) and a global max at P(2|8)



Endpoint Extrema

Endpoint extrema are valid extrema

→ they occur only(!) at the boundary points of an interval – if we inspect f(x) at such an interval

Endpoints can be **both** local and global extrema

→ an extreme point is considered 'local' when we find no min or max within the direct neighbourhood of our extreme point that is higher/lower in magnitude

→ it is considered 'global', when it is the highest/lowest in magnitude over the defined domain!

The defined **domain** may be the function in its entirety – or just the specified interval!

Chapter 9 | An Introduction to Probability

Probability vocabulary

Probability

- a measure of an event‘s likelihood

Likelihood

- the plausibility of an event to occur

Sample space

- a set of all possible outcomes

Outcome

- a realized event of e.g., an experiment/ the DGP

(Random) Event

- uncertain outcome, follows probability distribution

Probability of an event $\Pr(e) = \frac{\# \, outcomes \; of \; interest}{\# \, outcomes \; in \; sample \; space}$

Hands on – probability definitions

Task: Given one roll of a fair die, what is the probability that the number rolled is less than 3?

- what is an outcome?
- what is the sample space S?
- what is the event e?
- what is the $\Pr(e)$?

Hands on – probability definitions

Solution:

- what is an outcome?
  → one side of the die – a 'number'
- what is the sample space S?
  → $S = \{1, 2, 3, 4, 5, 6\}$
- what is the event e?
  → $e = \{1, 2\}$
- what is $\Pr(e)$?
  → $\Pr(e) = \frac{\#\, outcomes\ of\ interest}{\#\, outcomes\ in\ sample\ space} = \frac{2}{6} = 0.\overline{33}$

Hands On – Venn diagram refresher

Task: Draw the Venn Diagrams for the events A and B in the overall sample space S for:

- the complement of A A^C or $\bar{A}$
- the subset of A and B $A \subset B$
- the Union of A and B $A \cup B$
- the intersection of A and B $A \cap B$

Hint: You have seen all of these already in class…

Hands On – Venn diagram refresher

Solution:

Probability rules – joint probabilities

Addition rule

- applies whenever an event is the union of two other events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Multiplication rule

- applies whenever an event is the intersection of events A and B
→ then events A and B must occur simultaneously

$$P(A \cap B) = P(A) \cdot P(B|A)$$
$$= P(B) \cdot P(A|B)$$

Addition and multiplication rule

Probability rules

Complement rule

- applies whenever we ‘flip’ an event

$$P(A') = 1 - P(A)$$
$$P(A) + P(A') = 1$$

Conditional rule

- applies whenever an event needs another event to occur first!

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Independence, mutually exclusive, collectively exhaustive

Independence

- two events A and B are considered **independent** if the occurrence of one event **does NOT depend** on the occurrence of the other
- $P(A) = P(A|B)$

Mutually exclusive

- **mutually exclusive** events have **NO** outcome in common – they **CANNOT** occur together!

→ also called disjoint events

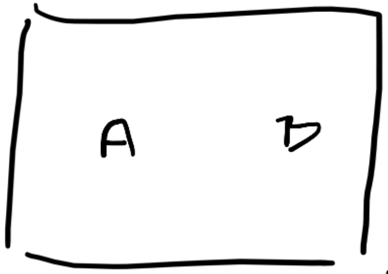
- **Collectively exhaustive**
  - the **union** of all events covers the entire sample space – at least one event in the sample space **MUST** occur

Independence and mutually exclusive events

Which of these depicts independent events?

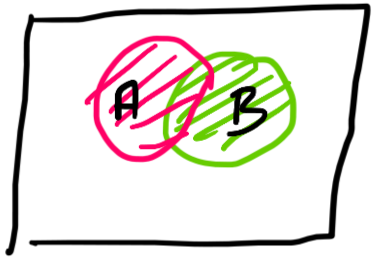
A: Heads

B: Tails



A: Heads

B: Jördis will have lunch



Independence:

$P(A) = P(A|B)$

I can both throw a coin head's up AND I can assure you, I will have lunch!

$P(A \cap B) = P(A) \cdot P(B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A|B) = \frac{P(A) \cdot P(B)}{P(B)}$

mutually exclusive:

$P(A \cap B) = 0$

NO intersection

NO outcomes in common!

Hands on – probability rules

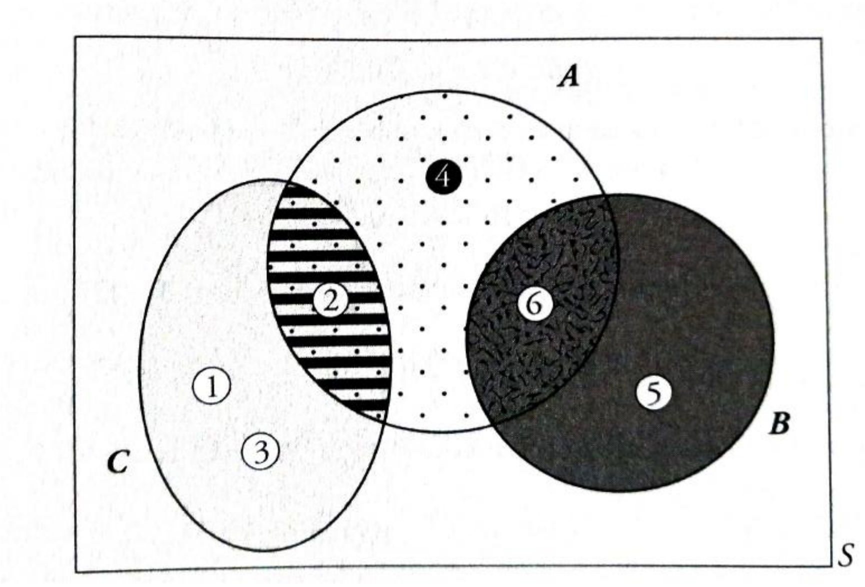
Task: The sample space for throwing a die is $S = \{1, 2, 3, 4, 5, 6\}$. Suppose events A, B and C are defined as follows:

A = Getting an even number {2, 4, 6}

B = Getting at least 5 = {5, 6}

C = Getting at most 3 = {1, 2, 3}

Find the probability of each of these events and its complement. Then, find the union, intersection and conditional probability of each pair of events.



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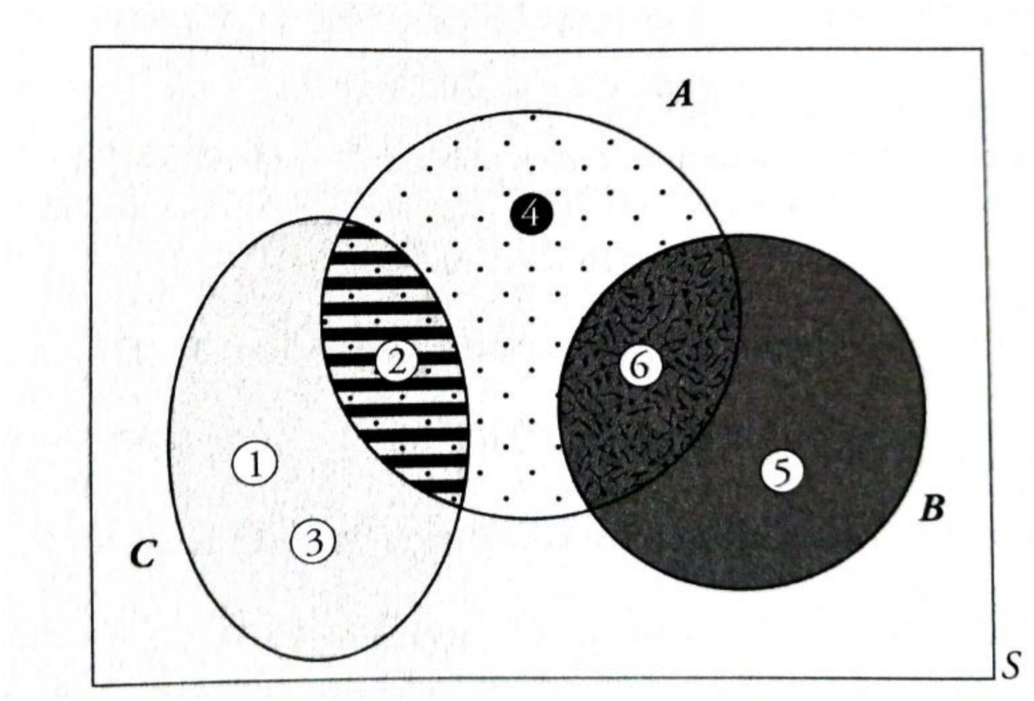
Hands on – probability rules

Solution:

- A = Getting an even number {2, 4, 6}
- B = Getting at least 5 = {5, 6}
- C = Getting at most 3 = {1, 2, 3}



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Probability:

- $P(A) = \frac{3}{6} = 0.5$
- $P(B) = \frac{2}{6} = 0.\overline{3}$
- $P(C) = \frac{3}{6} = 0.5$

Hands on – probability rules

Solution:

- A = Getting an even number {2, 4, 6}
- B = Getting at least 5 = {5, 6}
- C = Getting at most 3 = {1, 2, 3}

Complement:

A’ = Getting an odd number = {1, 3, 5}

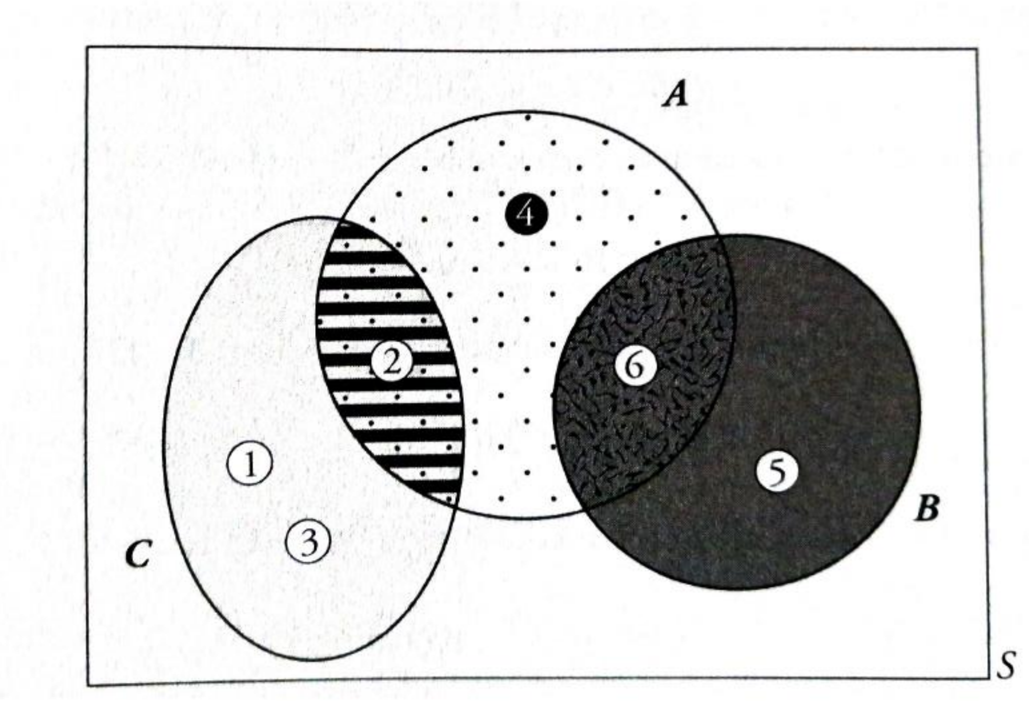
- $P(A') = \frac{3}{6} = 0.5 = 1 - P(A)$

B‘ = Getting a number less than 5 = {1, 2, 3, 4}

- $P(B') = \frac{4}{6} = 0.\overline{6} = 1 - P(B)$

C‘ = Getting a number larger than 3 = {4, 5, 6}

- $P(C') = \frac{3}{6} = 0.5 = 1 - P(C)$



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Hands on – probability rules

Solution:

- A = Getting an even number {2, 4, 6}
- B = Getting at least 5 = {5, 6}
- C = Getting at most 3 = {1, 2, 3}

Union:

$(A \cup B)$ = Getting an even nr. or one $\geq 5 = \{2, 4, 5, 6\}$

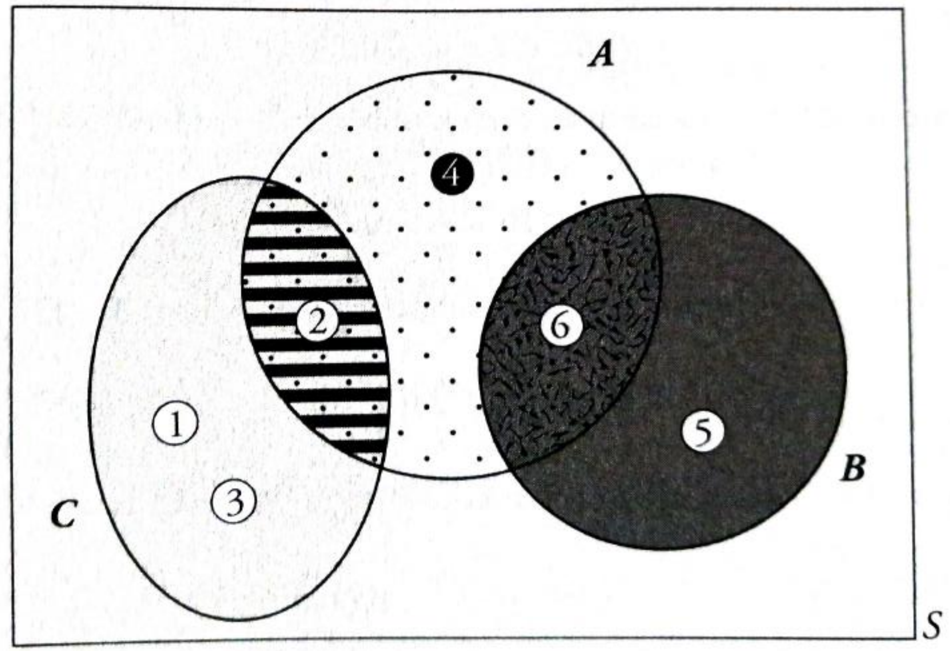
- $P(A \cup B) = \frac{4}{6} = 0.\overline{6}$

$(A \cup C)$= Getting an even nr. or one ≤ 3 = {1, 2, 3, 4, 6}

- $P(A \cup C) = \frac{5}{6} = 0.8\overline{3}$

$(B \cup C)$ = Getting a nr. that is at most 3 or at least 5 or both = {1, 2, 3, 5, 6}

- $P(B \cup C) = \frac{4}{6} = 0.8\overline{3}$



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Hands on – probability rules

Solution:

- A = Getting an even number {2, 4, 6}
- B = Getting at least 5 = {5, 6}
- C = Getting at most 3 = {1, 2, 3}

Intersection:

$(A \cap B)$ = Getting an even nr. that is at least 5 = {6}

- $P(A \cap B) = \frac{1}{6} = 0.1\overline{6}$

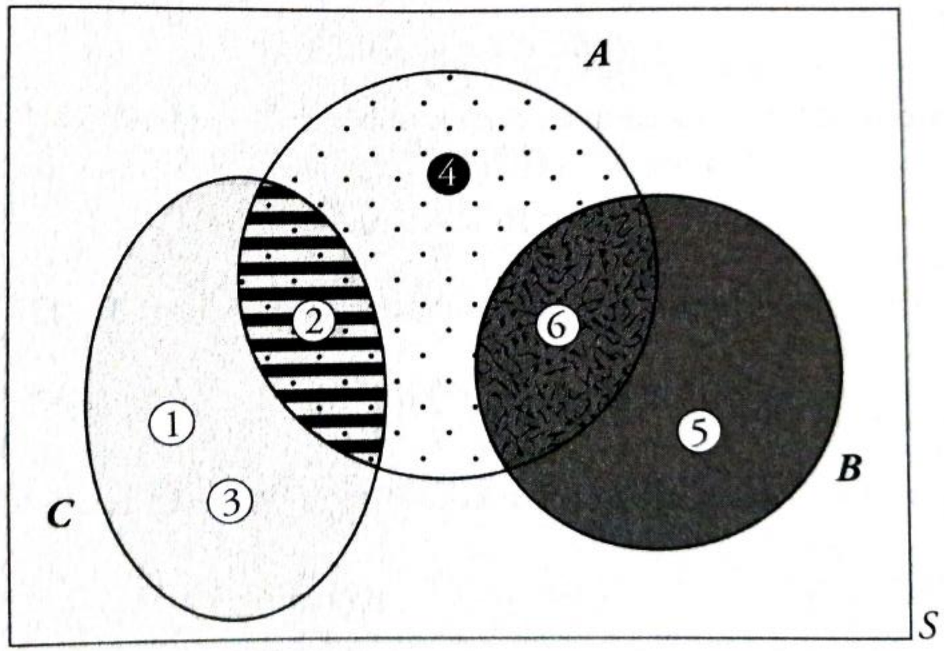
$(A \cap C)$ = Getting an even nr. that is at most 3 = {2}

- $P(A \cap C) = \frac{1}{6} = 0.1\overline{6}$

$(B \cap C)$ = Getting a nr. that is at most 3 and at least 5 = {}

- $P(B \cap C) = \frac{0}{6} = 0.00$

→ B and C are mutually exclusive events!



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Hands on – probability rules

Solution:

- A = Getting an even number {2, 4, 6}
- B = Getting at least 5 = {5, 6}
- C = Getting at most 3 = {1, 2, 3}

Conditional event:

(A|C) = Getting an even nr. that it is at most 3 = {2}

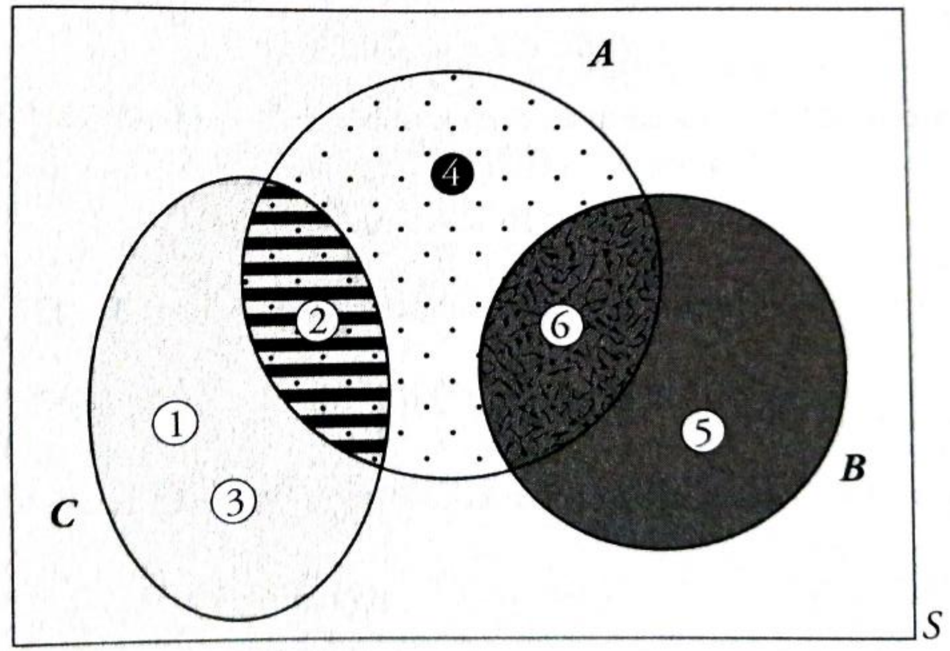
- $P(A|C) = \frac{1}{3} = 0.\overline{3}$

(A|B) = Getting an even nr. given it is $\geq$ 5 = {6}

- $P(A|B) = \frac{1}{2} = 0.5$

(B|C) = Getting at least 5 given that the nr. is at most 3 = ∅

- $P(B|C) = 0$



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Probability rules

Want even more practice?

- come up with your own scenarios and go to https://www.cuemath.com/data/probability-rules/
  to fact check your results! ☺

Combinations and permutations

Combinations

- Order DOES NOT matter!

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = n_{C_r}$$

→ use when you have to 'select' k things out of n

Permutations

- Order DOES matter!

$$P(n,k) = \frac{n!}{(n-k)!} = n_{P_r}$$

→ use when you have to 'arrange' k things out of n

Hands on – combinations and permutations

Task: Let there be a bag with three coloured balls (one red, one yellow, one blue), how many ways are there to draw two balls given:

- order does not matter?
- order does matter?

Hands on – combinations and permutations

Solution: Let there be a bag with three coloured balls (one red, one yellow, one blue), how many ways are there to draw two balls given:

- order does not matter?

  {(red & yellow), (red & blue), (yellow & blue)}

  $$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2} = 3$$

- order does matter?

  {(red & yellow), (yellow & red), (red & blue), (blue & red), (yellow & blue), (blue & yellow)}

  $$P(3,2) = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$$

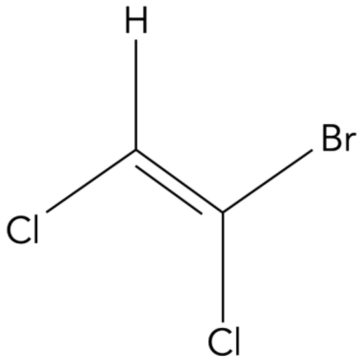
Real world applications – molecules

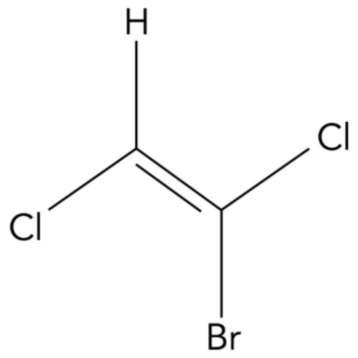
Task: Let a linear hydrocarbon of nine singly bonded carbon atoms be given, at the end of which there is an OH group.

$$CH_3 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - OH$$

a) How many different isomers can be obtained by substituting **two chlorine atoms**, if only one chlorine atom may be placed at each carbon atom?

b) How many different isomers do you get if you use **one chlorine atom** and **one bromine atom** instead of two chlorine atoms?





→**NOTE**: The images do not exactly depict the molecule of the task ☺

→**Isomers**: molecules constituted of the same number of atoms per element – but **put together differently**

Real world applications – molecules

Task: Let a linear hydrocarbon of nine singly bonded carbon atoms be given, at the end of which there is an OH group.

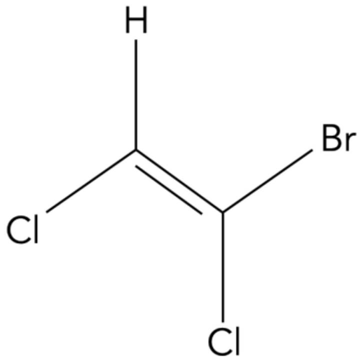
$$CH_3 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - CH_2 - OH$$

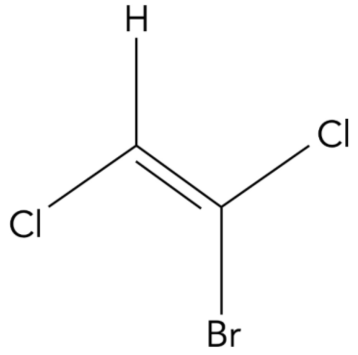
a) How many different isomers can be obtained by substituting **two chlorine atoms**, if only one chlorine atom may be placed at each carbon atom?

$\frac{9!}{7! \cdot 2!} = 36$ different isomers

b) How many different isomers do you get if you use **one chlorine atom** and **one bromine atom** instead of two chlorine atoms?

$\frac{9!}{7! \cdot 1! \cdot 1!} = 72$ different isomers





→**NOTE**: The images do not exactly depict the molecule of the task ☺

Bayes‘ Theorem

Remember the probability rules?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P\ (B|A) \cdot P(A)}{P(B)}$$

Multiplication rule:

- $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Law of total probability:

- events B_i to B_n partition the entire sample space S

$$P(A) = \sum_{i=1}^{N} P(A \cap B_i) = \sum_{i=1}^{N} P(A|B_i) \cdot P(B_i)$$

Bayes’ Theorem

- events B_i to B_n partition the entire sample space S
- $P(A) > 0$
- then for $i = 1, \dots, n$:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}$$

Bayes' Theorem

- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{P(A \cap B)}{P(A)} P(A)}{P(B)} = \frac{P\,(B|A) \cdot P(A)}{P(B)}$

→ we are interested in the probability of two events A and B with P(A) and P(B) > 0 occurring

→ especially the probability of A occurring, given that B occurred via the probability of B occurring, given A

→ joint and conditional probabilities

Bayes‘ Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

- Note that P(B) and P(~B) = 1
  - P(A|B) – the conditional probability of event A given B
  - P(B|A) – the conditional probability of event B given A
  - P(A) – a priori probability of A
  - P(B) – a priori probability of B

Real world applications – Bayes‘ Theorem

How do we interpret Bayes‘ Theorem under the idea of H being a research hypothesis of interest and E some evidence we observed?

$$P(H|E) = \frac{P(E|H)(H)}{P(E)}$$

- $P(H)$ **the ‘priori probability**’ of our hypothesis
- $P(H|E)$ the probability of our hypothesis after we have observed evidence in favour, or against it! ← ‘**posterior probability’**
- $\frac{P(E|H)}{P(E)}$ is called **the likelihood ratio** and $P(E|H)$ the **likelihood** of how well our hypothesis explains the evidence

→ under Bayes, this statement reads as: The probability of our hypothesis after observing evidence is equal to the probability of our hypothesis multiplied with the likelihood ratio

Hands on – Bayes‘ Theorem

Task: 1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get a positive test result. 9,6% of women without breast cancer will receive a false positive on their test results. If a woman in this age group receives a positive test result, what is the probability that she actually has breast cancer?

Tips:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

- Define events A and B!
- Find $P(A),\ P(\sim A),\ P(B|A),\ P(B|\sim A)$!

Hands on – Bayes‘ rule

Solution:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|{\sim}A)P({\sim}A)}$$

A = breast cancer

B = positive test

$$P(A) = 0.01,\ P({\sim}A) = 0.99,\ P(B|A), = 0.80,\ P(B|{\sim}A) = 0.096$$

$$P(A|B) = \frac{0.80 \cdot 0.01}{0.80 \cdot 0.01 + 0.096 \cdot 0.99} = \frac{0.008}{0.10304} = 0.0776 = 7.76\%$$

→ On average, 7.76% of women who receive a positive test result actually have breast cancer.

Real world applications – Bayesian Stats

- **Statistics**
  - Bayesian Stats – available here at uni (usually)
- **Bioinformatics**
  - e.g., used to classify and sort complex & noisy data like DNA samples
- **Your own e-mail accounts ☺**
  - using the 'naïve Bayes' classificatory' email programs are capable to filter incoming mails and sort them into folders – including unwanted spam! This is why before deleting spam that got into your main folders, you should always classify them as spam! You will train your mailing-program this way

Odds and odds ratio

Odds

- the odds of ONE event are given by

$$Pr(y) = \frac{Pr(y)}{Pr(\sim y)}$$

Odds ratio

- the odds ratio of TWO events are given by

$$\frac{\frac{Pr(x_1)}{Pr(\sim x_1)}}{\frac{Pr(x_2)}{Pr(\sim x_2)}}$$

→ expresses relationship between two independent events occurring!

→ important for MLE

Hands on – odds and odds ratio

Task:

1. If $P(y) = 0.60$, what are the odds that y occurs?
2. If the odds of x_1 are 2:1 and the odds of x_2 are 1:2, what is the odds ratio of $x_1 \colon x_2$?

Hands on – odds and odds ratio

Solution:

1. If $P(y) = 0.60$, what are the odds that y occurs?

   $$\mathrm{P}(y) = \frac{0.60}{0.40} = 1.5 \quad [1.5{:}1]$$

2. If the odds of x_1 are 2:1 and the odds of x_2 are 1:2, what is the odds ratio of $x_1 : x_2$?

   $$\frac{\frac{2}{1}}{\frac{1}{2}} = \frac{4}{1} = 4$$

Relative risk – RR

Intuition:

- origin lies in medical field: Do certain groups of people fall ill more easily due to a given risk factor?
- ‘the ratio of two probabilities‘ ranging from 0 to ∞

Interpretation

- $RR < 1$ - risk factor decreases risk of falling ill / prob in first ‘group’ of ratio is larger
- $RR = 1$ - risk factor has no apparent effect/risk factor is equal for two groups
- $RR > 1$ - risk factor increases risk of falling ill / prob in first ‘group’ of ratio is smaller

Example

- P(war|one major power) / P(war| $\sim$ major power) = 1.95
  → the group with at least one major power is more likely to start a war!
- P(war|both autocracies) / P(war|mixed) = 0.67
  → the mixed group is more likely to start a war!

Relative risk – RR

Formula:

$$RR := \frac{P(\#\, of\ cases\ exposed\ to\ risk\ factor)}{P(\#\, of\ cases\ NOT\ exposed\ to\ risk\ factor)}$$

	With risk factor	**Without risk factor**
Positive cases	a	b
Negative cases	c	d

Percentage change:

$$\widehat{RR} = \frac{\frac{a}{(a+c)}}{\frac{b}{(b+d)}}$$

Hands on – more practice

Task:

1) If A, B and C are mutually exclusive and collectively exhaustive events and P(A) = 0.25 and P(B) = 0.35, what is the joint probability of A or C? $P(A \cup C)$?

2) Let $P(A)$ = 0.3 and $P(A \cup B) = 0.5$. Find $P(B)$, assuming both events are independent

3) Solve $\binom{8}{5}$

4) Winning a lottery requires drawing 6 numbers from a possibility of 49 numbers, where the order of numbers being drawn does not matter. What is the probability of winning the lottery?

Hands on – more practice

Solution:

1)

$$P(C) = 1 - P(A) - P(B) = 0$$

$$P(A \cup C) = P(A) + P(C) = 0.25 + 0.4 = 0.65$$

2)

$$P(A \cup B) = P(A) + P(B) \ - P(A)P(B)$$

$$0.5 = 0.3 + P(B) - 0.3 \cdot P(B)$$

$$P(B)(1 - 0.3) = 0.2 \ \leftarrow \textit{factorisation!}$$

$$P(B) = \frac{2}{7}$$

Hands on – more practice

3) Solve $\binom{8}{5}$

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{6 \cdot 7 \cdot 8}{6} = 56$$

4) Winning a lottery requires drawing 6 numbers from a possibility of 49 numbers, where the order of numbers being drawn does not matter. What is the probability of winning the lottery?

$$P(A) = \frac{1}{\binom{49}{6}}$$

$$\binom{49}{6} = \frac{49!}{6! \cdot 43!} = 13{,}983{,}816$$

$$P(A) = \frac{1}{13{,}983{,}816} = 0.0000000715$$

Time for your questions

- Any questions during the week?
  - joerdis.strack@uni-konstanz.de