Hydromechanics - Principal Stress Output - Hollow Sphere

Analytical Solution

To test the correct output of the principal stresses, a thick walled hollow sphere is simulated with a low pressure on the inside - see Kolditz et al. [2015] for more details. A 2D axial symmetric model is used for this simulation. The principal stresses in this model are equal to the radial and circumferential stresses with the analytical solution:

$$\sigma_{rr} = \frac{p_a R_a^3 - p_i R_i^3}{R_i^3 - R_a^3} - \frac{R_a^3 R_i^3 (p_a - p_i)}{(R_i^3 - R_a^3) r^3} \tag{1}$$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \frac{p_a R_a^3 - p_i R_i^3}{R_i^3 - R_a^3} + \frac{R_a^3 R_i^3 (p_a - p_i)}{2(R_i^3 - R_a^3) r^3} \tag{2}$$

Additionally the displacement result of the simulation is also compared with the analytical solution:

$$u_r = \frac{R_i^3 p_i r}{E(R_i^3 - R_a^3)} \left[\left(\frac{p_a}{p_i} \left(\frac{R_a}{R_i} \right)^3 - 1 \right) (1 - 2\nu) + \left(\frac{p_a}{p_i} - 1 \right) \frac{1+\nu}{2} \left(\frac{R_a}{r} \right)^3 \right] \tag{3}$$

With the parameters

$$p_i = 10^3\,\mathrm{Pa} \qquad p_a = 1.013\,25 \cdot 10^5\,\mathrm{Pa} \qquad E = 128 \cdot 10^9\,\mathrm{Pa}$$
$$R_i = 0.175\,\mathrm{m} \qquad R_a = 0.225\,\mathrm{m} \qquad \nu = 0.3$$

the simulation was executed and showed results in good agreement with the analytical solution.

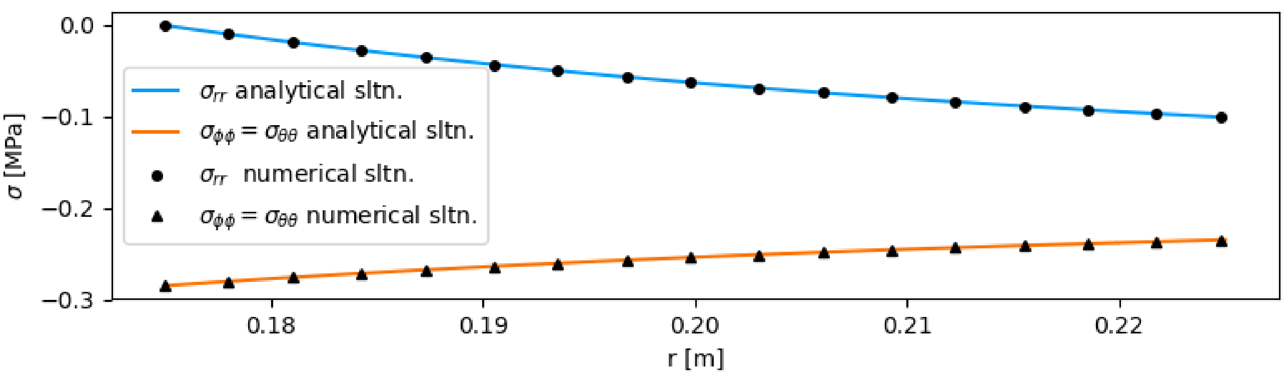

Fig. 1: Radial and circumferential stresses

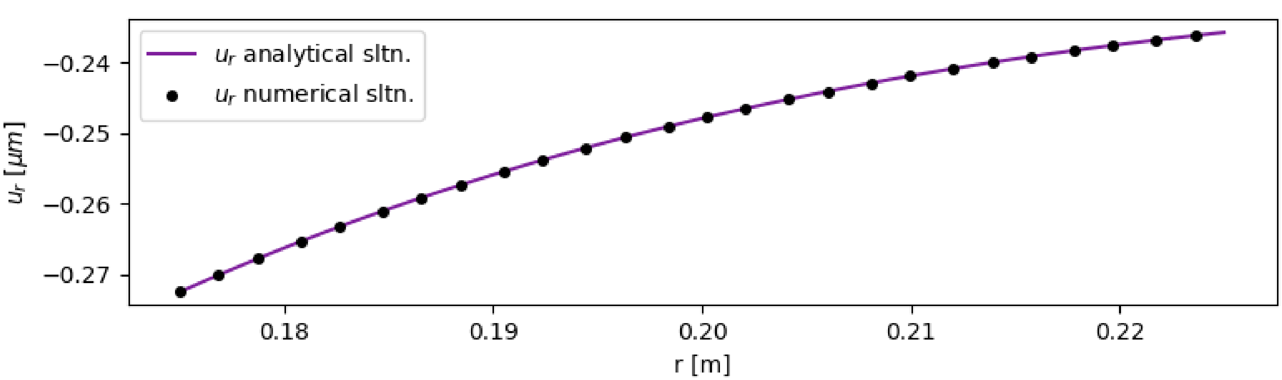

Fig. 2: Radial displacement

Relative Error and Grid Convergence

Since the relative errors of displacement and principal stresses are not sufficiently small with a coarse mesh, the simulation was executed with three different mesh sizes to analyze the grid convergence (4k, 16k and 64k elements).

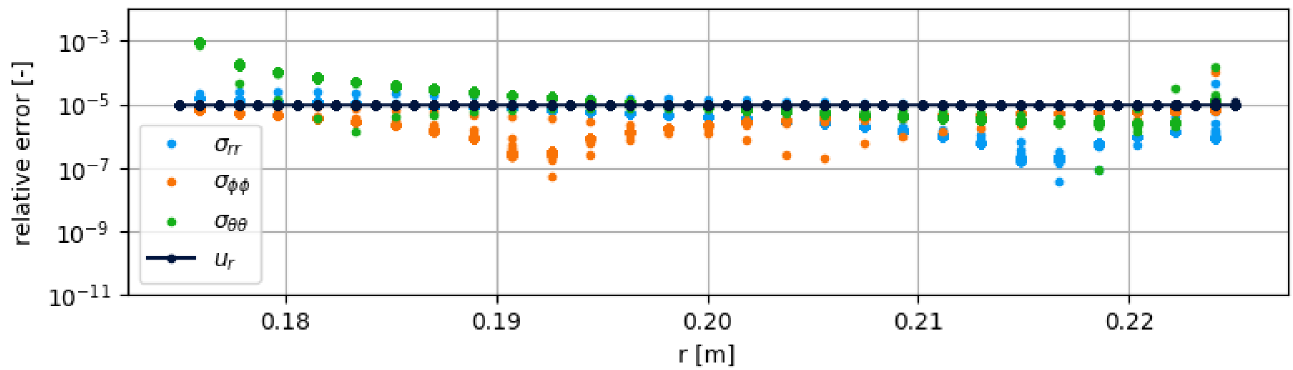

Fig. 3: Relative error for the principal stresses and rad. displacement (4000 elements)

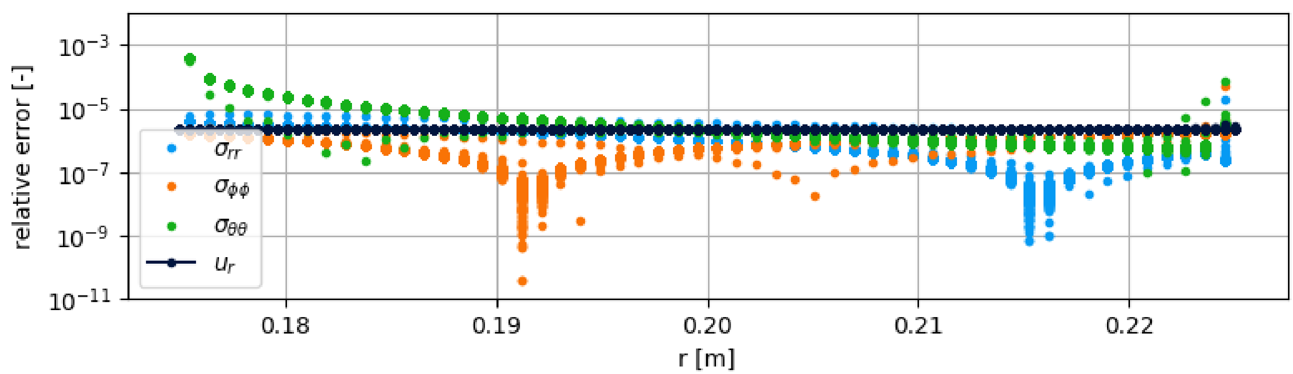

Fig. 4: Relative error for the principal stresses and rad. displacement (16000 elements)

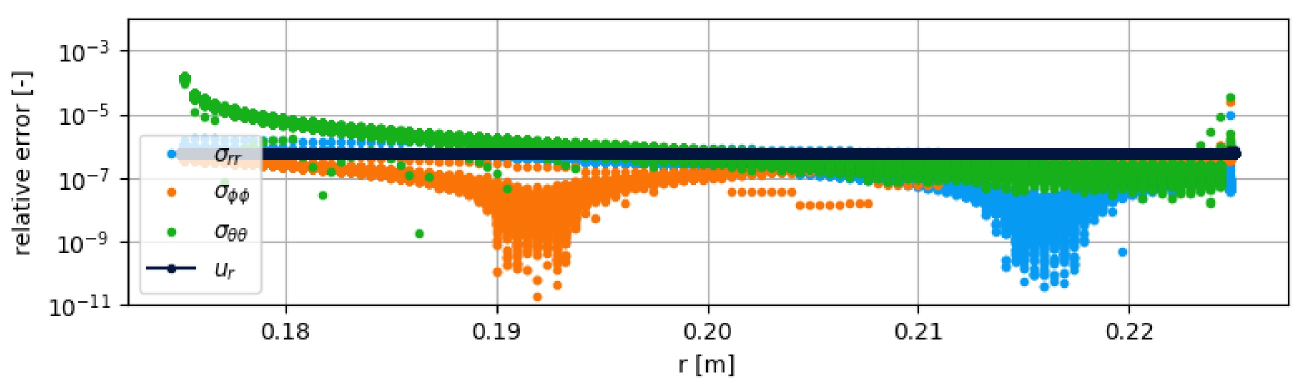

Fig. 5: Relative error for the principal stresses and rad. displacement (64000 elements)

The average relative error of each variable was calculated in all three cases to determine the order of convergence p (see Slater [2008] for more information)

$$p = \ln\left(\frac{f_3 - f_2}{f_2 - f_1}\right) / \ln(r) \tag{4}$$

With f_3, f_2, f_1 being the average relative errors for the meshes with 64k, 16k and 4k elements and r being the the grid refinement ratio which in this case equals two. As a result we get

$$p(u_r) = 1.999$$
$$p(\sigma_{rr}) = 2.021$$
$$p(\sigma_{\phi\phi}) = 2.007$$
$$p(\sigma_{\theta\theta}) = 1.991$$

This means that the principal stresses and the radial displacement converge with a second order and approach the analytical solution even further with a finer mesh. But it is still not clear why the displacement is not orders of magnitude lower than the principal stresses.

Comparison of NormalTraction and Functional Neumann Boundary Condition

To illustrate the versatility of function-expressions in OGS6 project files, note that you can replace NormalTraction BC's with Neumann BC's when you disassemble them into their components. In the example of this test, the following boundary condition:

```
<boundary_condition>
    <mesh>sphere_inside</mesh>
    <type>NormalTraction</type>
    <parameter>pressure_load_inside</parameter>
</boundary_condition>
```

---

can be replaced with:

```
<parameter>
    <name>pressure_load_inside_x</name>
    <mesh>sphere_inside</mesh>
    <type>Function</type>
    <expression>x*pressure_load_inside/inner_radius</expression>
</parameter>
<parameter>
    <name>pressure_load_inside_y</name>
    <mesh>sphere_inside</mesh>
    <type>Function</type>
    <expression>y*pressure_load_inside/inner_radius</expression>
</parameter>
...
<boundary_condition>
    <mesh>sphere_inide</mesh>
    <type>Neumann</type>
    <component>0</component>
    <parameter>pressure_load_inside_x</parameter>
</boundary_condition>
<boundary_condition>
    <mesh>sphere_inside</mesh>
    <type>Neumann</type>
    <component>1</component>
    <parameter>pressure_load_inside_y</parameter>
</boundary_condition>
```

It might be a more complicated variant in this case, but there may also be other cases where it could proof practical. The following figures show the relative difference between the two variants for all the meshes. Unsurprisingly the error decreases with a finer mesh.

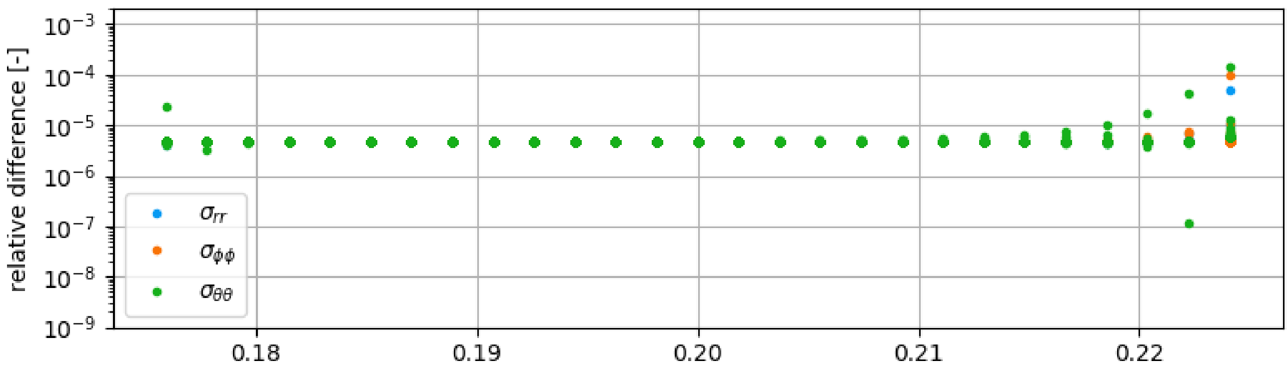

Fig. 6: Relative difference between solutions with NormalTraction-BC and Neumann-BC (4000 elements)

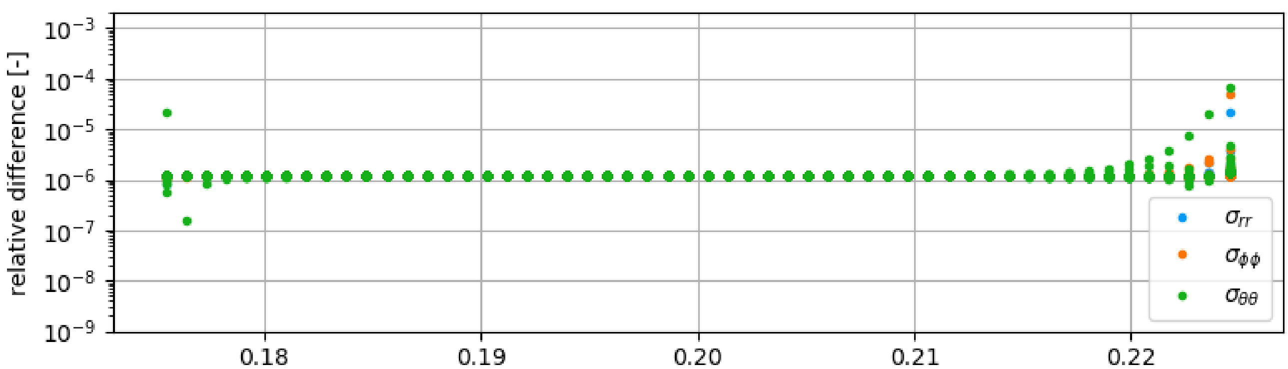

Fig. 7: Relative difference between solutions with NormalTraction-BC and Neumann-BC (16000 elements)

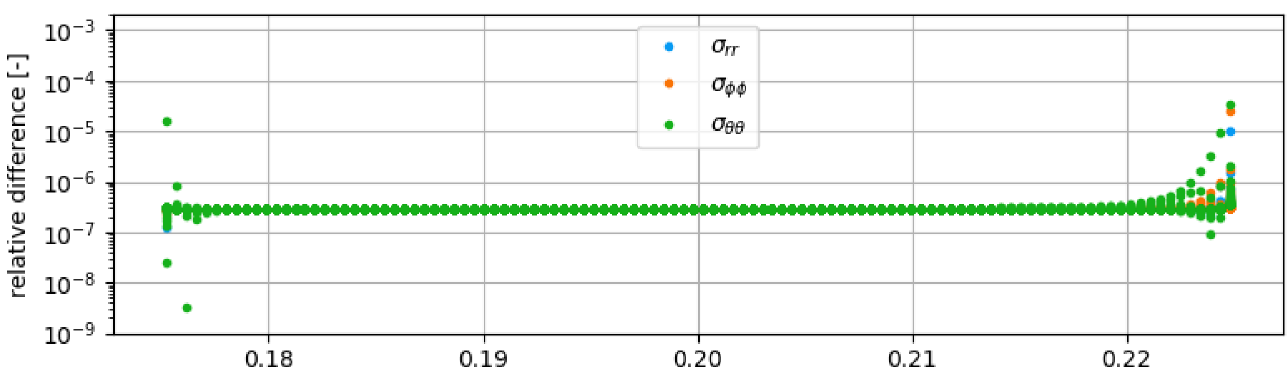

Fig. 8: Relative difference between solutions with NormalTraction-BC and Neumann-BC (64000 elements)

References

Olaf Kolditz, Hua Shao, Wenqing Wang, and Sebastian Bauer. *Thermo-Hydro-Mechanical-Chemical Processes in Fractured Porous Media: Modelling and Benchmarking.* Springer International Publishing, Cham, 2015. ISBN 978-3-319-11893-2. doi: 10.1007/978-3-319-11894-9.

John W. Slater. Examining spatial (grid) convergence. Website, 2008. `https://www.grc.nasa.gov/www/wind/valid/tutorial/spatconv.html`.