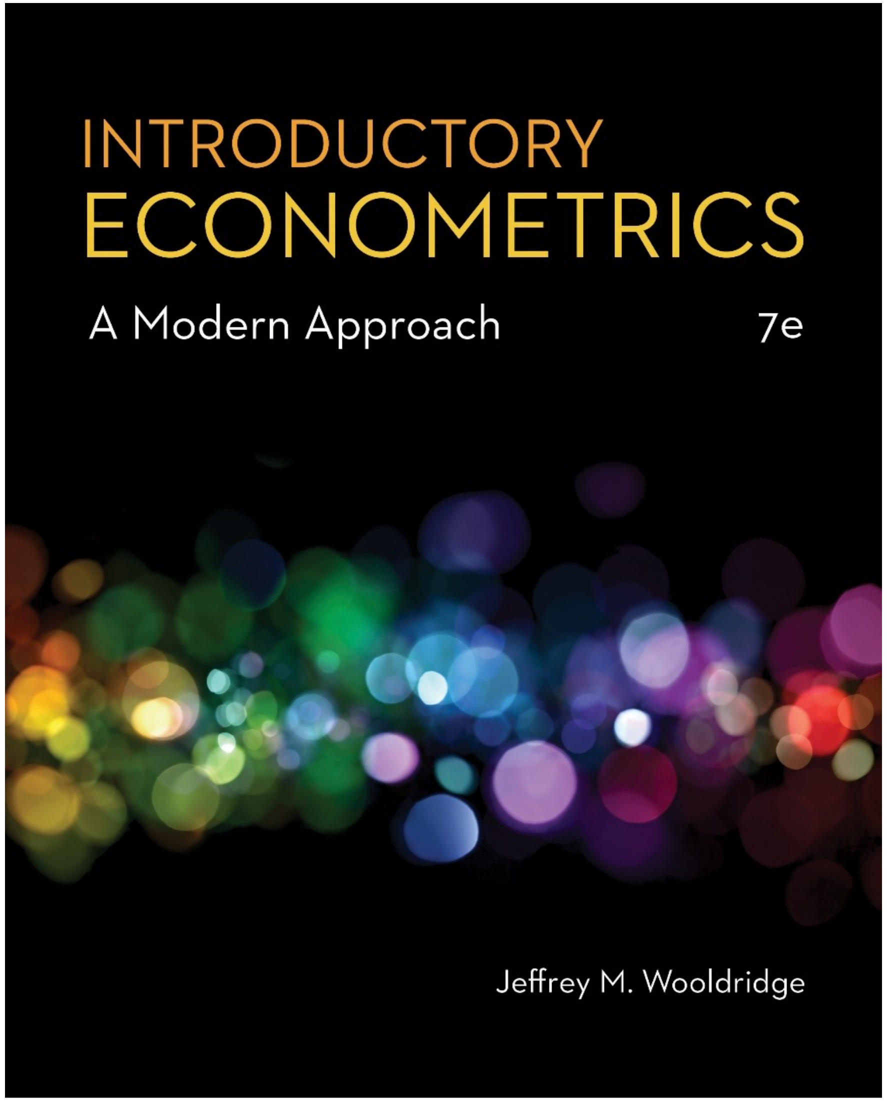
Chapter 12

Serial Correlation and Heteroskedasticity in Time Series Regressions



Serial Correlation and Heteroskedasticity in Time Series Regressions (1 of 18)

- **Properties of OLS with serially correlated errors**
- OLS still unbiased and consistent if errors are serially correlated.
- Correctness of R-squared also does not depend on serial correlation.
- OLS standard errors and tests will be invalid if there is serial correlation.
- OLS will not be efficient anymore if there is serial correlation.

Serial Correlation and Heteroskedasticity in Time Series Regressions (2 of 18)

- **Serial correlation and the presence of lagged dependent variables**

- Is OLS inconsistent if there are serial correlation and lagged dependent variables?

- No: Including enough lags so that TS.3‘ holds guarantees consistency.

- Including too few lags will cause an omitted variable problem and serial correlation because some lagged dependent variables end up in the error term.

Serial Correlation and Heteroskedasticity in Time Series Regressions (3 of 18)

- **Correcting for serial correlation with strictly exogenous regressors**
  - Under the assumption of AR(1) errors, one can transform the model so that it satisfies all GM-assumptions. For this model, OLS is BLUE.

$$y_t = \beta_0 + \beta_1 x_t + u_t \leftarrow$$ Simple case of regression with only one explanatory variable. The general case works analogusly.

$$\rho y_{t-1} = \rho\beta_0 + \rho\beta_1 x_{t-1} + \rho u_{t-1} \leftarrow$$ Lag and multiply by ρ

$$\Rightarrow \quad y_t - \rho y_{t-1} = \beta_0(1-\rho) + \beta_1(x_t - \rho x_{t-1}) + u_t - \rho u_{t-1}$$

$$u_t = \rho u_{t-1} + e_t \Leftrightarrow u_t - \rho u_{t-1} = e_t \leftarrow$$ The transformed error satisfies the GM-assumptions.

- Problem: The AR(1)-coefficient is not known and has to be estimated

Serial Correlation and Heteroskedasticity in Time Series Regressions (4 of 18)

- **Correcting for serial correlation (contd)**
- Replacing the unknown AR(1) coefficient ρ with an estimated value leads to an FGLS estimator.
- There are two variants:
  - Cochrane-Orcutt estimation omits the first observation
  - Prais-Winsten estimation adds a transformed first observation
- In smaller samples, Prais-Winsten estimation should be more efficient

- Comparing OLS and FGLS with autocorrelation
  - For consistency of FGLS more than TS.3‘ is needed (e.g. TS.3) because the transformed regressors include variables from different periods.
  - If OLS and FGLS differ dramatically this might indicate violation of TS.3.

Serial Correlation and Heteroskedasticity in Time Series Regressions (5 of 18)

- **FGLS Estimation of the AR(1) Model**
- Under the assumption of strict exogeneity, we can correct for serial correlation using FGLS.

$$y_t = \beta_0 + \beta_1 x_{1t} + \cdots + \beta_k x_{kt} + u_t$$
$$u_t = \rho u_{t-1} + e_t$$

Regress y_t on $x_{1t}, \ldots, x_{kt}$ to obtain $\hat{u}_t$
Regress $\hat{u}_t$ on $\hat{u}_{t-1}$ to obtain $\hat{\rho}$

With an estimate of the AR(1) parameter, we can transform the original model:

$$\tilde{y}_t = \beta_0 \tilde{x}_{0t} + \beta_1 \tilde{x}_{1t} + \cdots + \beta_k \tilde{x}_{kt} + error$$

For $t = 1$: $\tilde{y}_t = (1-\hat{\rho}^2)^{.5} y_t$, $\tilde{x}_{0t} = (1-\hat{\rho}^2)^{.5}$, $\tilde{x}_{jt} = (1-\hat{\rho}^2)^{.5} x_{jt}$

For $t \geq 2$: $\tilde{y}_t = y_t - \hat{\rho} y_{t-1}$, $\tilde{x}_{0t} = 1 - \hat{\rho}$, $\tilde{x}_{jt} = x_{jt} - \hat{\rho} x_{jt-1}$

Serial Correlation and Heteroskedasticity in Time Series Regressions (6 of 18)

- **Serial correlation-robust inference after OLS**
  - In the presence of serial correlation, OLS standard errors overstate statistical significance because there is less independent variation.
  - One can compute serial correlation-robust standard errors after OLS.
  - This is useful because FGLS requires strict exogeneity and assumes a very specific form of serial correlation (AR(1) or, generally, AR(q)).
  - Serial correlation-robust standard errors:

$$se(\hat{\beta}_j) = \left[\text{"}se(\hat{\beta}_j)\text{"} / \hat{\sigma} \right]^2 \sqrt{\hat{v}}$$

The usual OLS standard errors are normalized and then "inflated" by a correction factor.

- Serial correlation-robust F- and t-tests are also available.

Serial Correlation and Heteroskedasticity in Time Series Regressions (7 of 18)

- **How to compute Newey-West Standard Errors**

In general, $se(\hat{\beta}_1) = \left[\frac{"se(\hat{\beta}_1)"}{\hat{\sigma}}\right]^2 \sqrt{\hat{v}}$, where "$se(\hat{\beta})$" is the usual OLS standard error.

For a chosen integer g: $\hat{v} = \sum_{t=1}^{n} \hat{a}_t^2 + 2\sum_{h=1}^{g}\left[1 - \frac{h}{g+1}\right]\sum_{t=h+1}^{n} \hat{a}_t \hat{a}_{t-h}$

$\hat{a}_t = \hat{r}_t \hat{u}_t$

$\hat{r}_t$ is the residual from a regression of x_{1t} on $x_{2t}, \ldots, x_{kt}$

- The choice of the integer g is open to debate.

1. A preliminary estimate is to use the integer part of $4(n/100)^{2/9}$
2. Stock and Watson suggest the integer part of $(3/4)n^{1/3}$
3. Others have suggested using the integer part of $n^{1/4}$

The resulting value of g does not vary too much between these methods

Serial Correlation and Heteroskedasticity in Time Series Regressions (8 of 18)

- **Discussion of serial correlation-robust standard errors**
- The formulas are also robust to heteroskedasticity; they are therefore called “heteroskedasticity and autocorrelation consistent” (=HAC).
- HAC SEs lagged in use behind heteroskedasticity robust errors for several reasons.
  - We generally have more observations with cross-sections than for time series.
  - Newey-West SEs can be poorly behaved if there is substantial serial correlation and the sample size is small.
  - The bandwidth g must be chosen by the researcher and the SEs can be sensitive to the choice of g.
- That said, HAC SE's are now widespread in use.

Serial Correlation and Heteroskedasticity in Time Series Regressions (9 of 18)

- **Testing for serial correlation**
- Testing for AR(1) serial correlation with strictly exogenous regressors

$$y_t = \beta_0 + \beta_1 x_{t1} + \ldots + \beta_k x_{tk} + u_t$$

$$u_t = \rho u_{t-1} + e_t$$ ← AR(1) model for serial correlation (with an i.i.d. series e_t)

Replace true unobserved errors by estimated residuals

Test $H_0 : \rho = 0$ in $\hat{u}_t = \rho \hat{u}_{t-1} + error$

- Example: Static Phillips curve (see above)

$$\hat{\rho} = .573, t = 4.93, p-value = .000$$ ← Reject null hypothesis of no serial correlation

Serial Correlation and Heteroskedasticity in Time Series Regressions (10 of 18)

- **The Durbin-Watson test under classical assumptions**
  - Under assumptions TS.1 – TS.6, the Durbin-Watson test is an exact test (whereas the previous t-test is only valid asymptotically).

$$DW = \sum_{t=2}^{n} (\hat{u}_t - \hat{u}_{t-1})^2 / \sum_{t=2}^{n} \hat{u}_t^2 \approx 2(1-\hat{\rho})$$

$$H_0 : \rho = 0 \quad \text{vs.} \quad H_1 : \rho > 0$$

Reject if $DW < d_L$, fail to reject if $DW > d_U$

Unfortunately, the Durbin-Watson test works with a lower and an upper bound for the critical value. In the area between the bounds the test result is inconclusive.

- Example: Static Phillips curve (see above)

$$DW = .80 < d_L = 1.32 \longleftarrow$$ Reject null hypothesis of no serial correlation

Serial Correlation and Heteroskedasticity in Time Series Regressions (11 of 18)

- **Testing for AR(1) serial correlation with general regressors**
  - The t-test for autocorrelation can be easily generalized to allow for the possibility that the explanatory variables are not strictly exogenous:

$$\hat{u}_t = \alpha_0 + \alpha_1 x_{t1} + \ldots + \alpha_k x_{tk} + \rho \hat{u}_{t-1} + error$$

The test now allows for the possibility that the strict exogeneity assumption is violated.

Test for $H_0 : \rho = 0$

- General Breusch-Godfrey test for AR(q) serial correlation

$$\hat{u}_t = \alpha_0 + \alpha_1 x_{t1} + \ldots + \alpha_k x_{tk} + \rho_1 \hat{u}_{t-1} + \ldots + \rho_q \hat{u}_{t-q} + \ldots$$

Test $H_0 : \rho_1 = \cdots = \rho_q = 0$

Serial Correlation and Heteroskedasticity in Time Series Regressions (12 of 18)

- **Heteroskedasticity in time series regressions**

- Heteroskedasticity usually receives less attention than serial correlation.

- Heteroskedasticity-robust standard errors also work for time series.

- Heteroskedasticity is automatically corrected for if one uses the serial correlation-robust formulas for standard errors and test statistics.

Serial Correlation and Heteroskedasticity in Time Series Regressions (13 of 18)

- **Testing for heteroskedasticity**

- The usual heteroskedasticity tests assume absence of serial correlation.

- Before testing for heteroskedasticity one should therefore test for serial correlation first, using a heteroskedasticity-robust test if necessary.

- After serial correlation has been corrected for, test for heteroskedasticity.

Serial Correlation and Heteroskedasticity in Time Series Regressions (14 of 18)

- **Example: Serial correlation and homoskedasticity in the EMH**

$$return_t = \beta_0 + \beta_1 return_{t-1} + u_t$$

Test equation for the EMH

$$\underset{}{\hat{u}_t} = \underset{(.147)}{.122} - \underset{(.647)}{.645}\, return_{t-1} + \underset{(.648)}{.646}\, \hat{u}_{t-1}$$

$$n = 689, R^2 = .0015, \bar{R}^2 = -.0015$$

Test for serial correlation:
No evidence for serial correlation

$$\hat{u}_t^2 = \underset{(0.43)}{4.66} - \underset{(0.201)}{1.104}\, return_{t-1} + residual_t$$

$$n = 689, R^2 = .042$$

Test for heteroskedasticity:
Strong evidence for heteroskedasticity

Serial Correlation and Heteroskedasticity in Time Series Regressions (15 of 18)

- **Autoregressive Conditional Heteroskedasticity (ARCH)**
  - Even if there is no heteroskedasticity in the usual sense (the error variance depends on the explanatory variables), there may be heteroskedasticity in the sense that the variance depends on how volatile the time series was in previous periods.

$$Var(u_t|X, u_{t-1}, u_{t-2}, \ldots) = \alpha_0 + \alpha_1 u_{t-1}^2 \longleftarrow \text{ARCH(1) model}$$

- Consequences of ARCH in static and distributed lag models
  - If there are no lagged dependent variables among the regressors, i.e. in static or distributed lag models, OLS remains BLUE under TS.1-TS.5.
  - Also, OLS is consistent etc. for this case under assumptions TS.1'-TS.5'
  - As explained, in this case, assumption TS.4 still holds under ARCH

Serial Correlation and Heteroskedasticity in Time Series Regressions (16 of 18)

- **Consequences of ARCH in dynamic models**
  - In dynamic models, i.e. models including lagged dependent variables, the homoskedasticity assumption TS.4 will necessarily be violated:

$$Var(u_t|\mathbf{X}) = Var(u_t|\mathbf{x}_1, y_0, \ldots \mathbf{x}_n, y_{n-1}) = \alpha_0 + \alpha_1 u_{t-1}^2$$

because $u_t = y_t - E(y_t|\mathbf{x}_t, y_{t-1}, \mathbf{x}_{t-1}, y_{t-2}, \ldots)$

  - This means the error variance indirectly depends on explanatory variables.
  - In this case, heteroskedasticity-robust standard error and test statistics should be computed, or a FGLS/WLS-procedure should be applied.
  - Using a FGLS/WLS-procedure will also increase efficiency.

Serial Correlation and Heteroskedasticity in Time Series Regressions (17 of 18)

- **Example: Testing for ARCH-effects in stock returns**

$$return_t = \beta_0 + \beta_1 return_{t-1} + u_t$$ ← Are there ARCH-effects in these errors?

$$Var(u_t|u_{t-1}) = E(u_t^2|u_{t-1}) = \alpha_0 + \alpha_1 u_{t-1}^2$$

$$\Rightarrow \quad u_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + v_t$$ ← Estimating equation for ARCH(1) model

$$\underset{}{\hat{u}_t^2} = \underset{(0.44)}{2.95} + \underset{(.036)}{.337}\ \hat{u}_{t-1}^2 + residual_t$$

$$n = 688, R^2 = .1136$$

There are statistically significant ARCH effects: If returns were particularly high or low (squared returns were high) in the previous period, they tend to be particularly high or low again, i.e. high volatility is followed by high volatility.

Serial Correlation and Heteroskedasticity in Time Series Regressions (18 of 18)

- **An FGLS procedure for serial correlation and heteroskedasticity**
- We can model heteroskedasticity and serial correlation and correct for both with a combined weighted least squares AR(1) procedure.

$y_t = \beta_0 + \beta_1 x_{1t} + \cdots + \beta_k x_{kt} + u_t$

→ Regress y_t on $x_{1t}, \ldots, x_{kt}$ to obtain $\hat{u}_t$

Regress $\log(\hat{u}_t^2)$ on $x_{1t}, \ldots, x_{kt}$ and obtain the fitted values $\hat{g}_t$

Define $\hat{h}_t = \exp(\hat{g}_t)$

Estimate the transformed equation below with Cochrane-Orcutt or Prais-Winsten methods:

$$\frac{y_t}{\sqrt{\hat{h}_t}} = \beta_0 \frac{1}{\sqrt{\hat{h}_t}} + \beta_1 \frac{x_{1t}}{\sqrt{\hat{h}_t}} + \cdots + \beta_k \frac{x_{kt}}{\sqrt{\hat{h}_t}} + error_t$$