

# ISLR | Chapter 4 Exercises

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*7/3/2018*

## Conceptual

### 1

$$f(\alpha) = \text{Var}(\alpha X + (1 - \alpha)Y)$$

Using the statistical property that  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$ , the above equation can be rewritten as:

$$f(\alpha) = \text{Var}(\alpha X) + \text{Var}((1 - \alpha)Y) + 2\text{Cov}(\alpha X, (1 - \alpha)Y)$$

Then, using the statistical property that  $\text{Var}(cX) = c^2\text{Var}(X)$  and  $\text{Cov}(aX, bY) = ab\text{Cov}(X, Y)$ , the equation can once again be rewritten as:

$$f(\alpha) = \alpha^2\text{Var}(X) + (1 - \alpha)^2\text{Var}(Y) + 2\alpha(1 - \alpha)\text{Cov}(X, Y)$$

Multiplying the  $\alpha(1 - \alpha)$  comes out to:

$$f(\alpha) = \alpha^2\text{Var}(X) + (1 - \alpha)^2\text{Var}(Y) + 2(\alpha - \alpha^2)\text{Cov}(X, Y)$$

By then taking the partial derivative of  $f(\alpha)$  with respect to  $\alpha$ , the slope of the function at a given alpha can be obtained:

$$\frac{\partial f(\alpha)}{\partial \alpha} = 2\alpha\sigma_X^2 + 2(1 - \alpha)(-1)\sigma_Y^2 + 2(1 - 2\alpha)\sigma_{XY}$$

Divide by 2:

$$\frac{\partial f(\alpha)}{\partial \alpha} = \alpha\sigma_X^2 + (-1 + \alpha)\sigma_Y^2 + (1 - 2\alpha)\sigma_{XY}$$

Expand the second and third terms in the equation:

$$\frac{\partial f(\alpha)}{\partial \alpha} = \alpha\sigma_X^2 + -\sigma_Y^2 + \alpha\sigma_Y^2 + \sigma_{XY} - 2\alpha\sigma_{XY}$$

Factor  $\alpha$  out of all possible terms:

$$\frac{\partial f(\alpha)}{\partial \alpha} = \alpha(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) - \sigma_Y^2 + \sigma_{XY}$$

Divide each term by  $(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY})$ :

$$\frac{\partial f(\alpha)}{\partial \alpha} = \alpha - \frac{\sigma_Y^2 + \sigma_{XY}}{(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY})}$$

Since the goal is to minimize the equation, setting the partial derivative to zero will return an equation that is a minimum.

$$0 = \alpha - \frac{\sigma_Y^2 + \sigma_{XY}}{(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY})}$$

Subtract  $\alpha$

$$-\alpha = - \frac{\sigma_Y^2 + \sigma_{XY}}{(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY})}$$

Multiply by -1:

$$\alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}$$

## 2

- **A.** Since a bootstrapped sample contains  $N$  observations of the original sample of the population, each sample being chosen at random with replacement, the probability that the first observation in a bootstrapped sample is *not* the  $j$ th observation is  $\frac{n-1}{n}$ .
- **B.** The probability that the second bootstrap observation is *not* the  $j$ th observation is  $\left(\frac{n-1}{n}\right)^2$ .
- **C.** Since a bootstrapped sample contains  $N$  observations, the probability that the  $j$ th observation ( $x_j$ ) is *not* in the bootstrapped sample ( $S_b$ ) is:

$$P(x_j \text{ not in } S_b) = \left(\frac{n-1}{n}\right)^n$$

Which can be simplified to:

$$P(x_j \text{ not in } S_b) = \left(1 - \frac{1}{n}\right)^n$$

- **D.** Since the probability that the  $j$ th observation is *not* in the bootstrap sample is  $\left(1 - \frac{1}{n}\right)^n$ , the probability that the  $j$ th observation *is* in the bootstrap sample would be the complement,  $1 - \left(1 - \frac{1}{n}\right)^n$ . When  $n = 5$ , this comes out to  $1 - \left(1 - \frac{1}{5}\right)^5 = 0.67232 = 67.23\%$
- **E.**

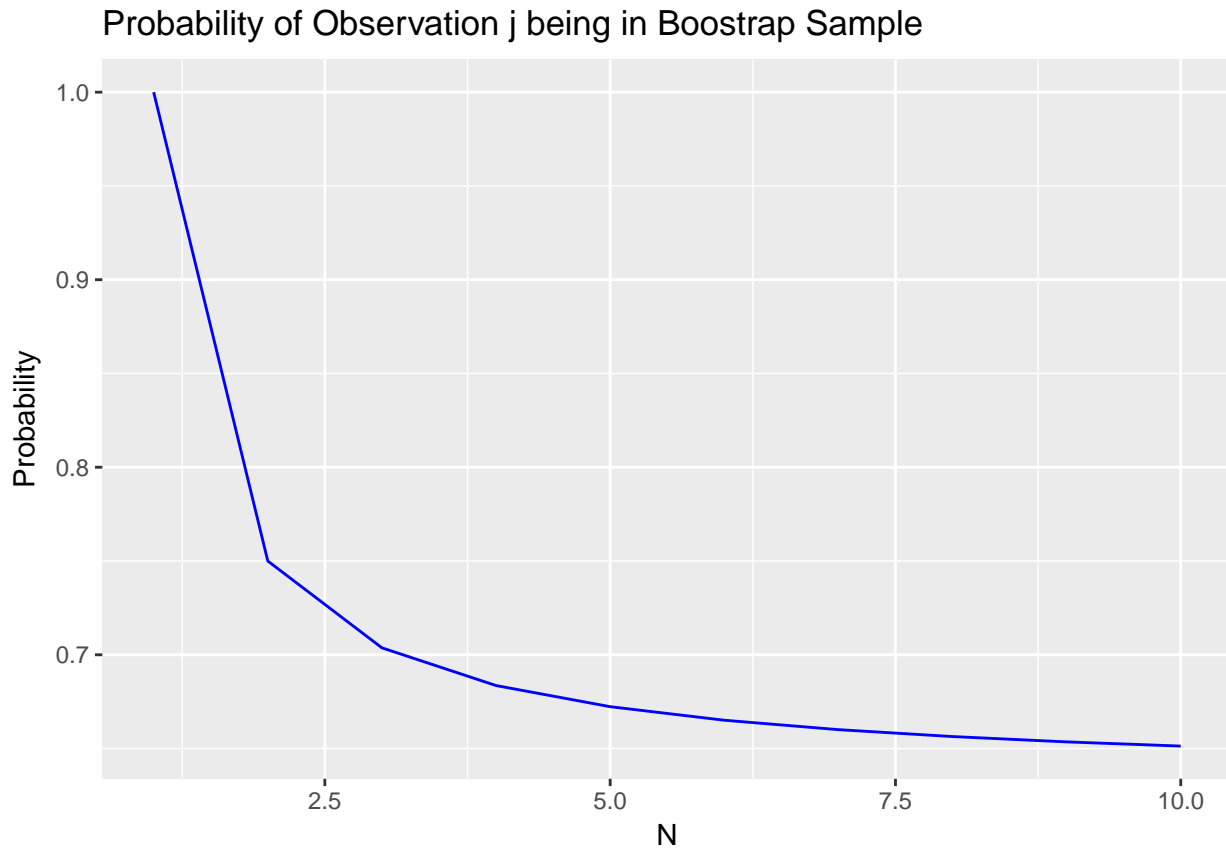
$$1 - \left(1 - \frac{1}{100}\right)^{100} = 0.6339677 = 63.40\%$$

- **F.**

$$1 - \left(1 - \frac{1}{100}\right)^{100} = 0.632139 = 63.21\%$$

- **G.** It is clear that as  $N$  increases the probability that the  $j$ th observation is in the bootstrap sample asymptotically approaches 0.632. The below plot illustrates this phenomenon (only displaying 1 to 10 for illustration purposes)

```
library(ggplot2)
x <- 1:100000
y <- 1 - (1 - (1/x))^x
df <- data.frame(x, y)
display_df <- df[1:10,]
ggplot(display_df, aes(x = x, y = y)) +
  geom_line(color = 'blue') +
  labs(x = "N", y = "Probability",
       title = "Probability of Observation j being in Bootstrap Sample")
```



- **H.** The below code is showing mathematically what the plot above shows; that the limit of the function  $1 - \left(1 - \frac{1}{x}\right)^x$  as  $x$  approaches infinity is 0.632.

```
store <- rep(NA, 10000)
for (i in 1:10000) {
  store[i] <- sum(sample(1:100, replace = TRUE) == 4) > 0
}
mean(store)
```

```
## [1] 0.637
```

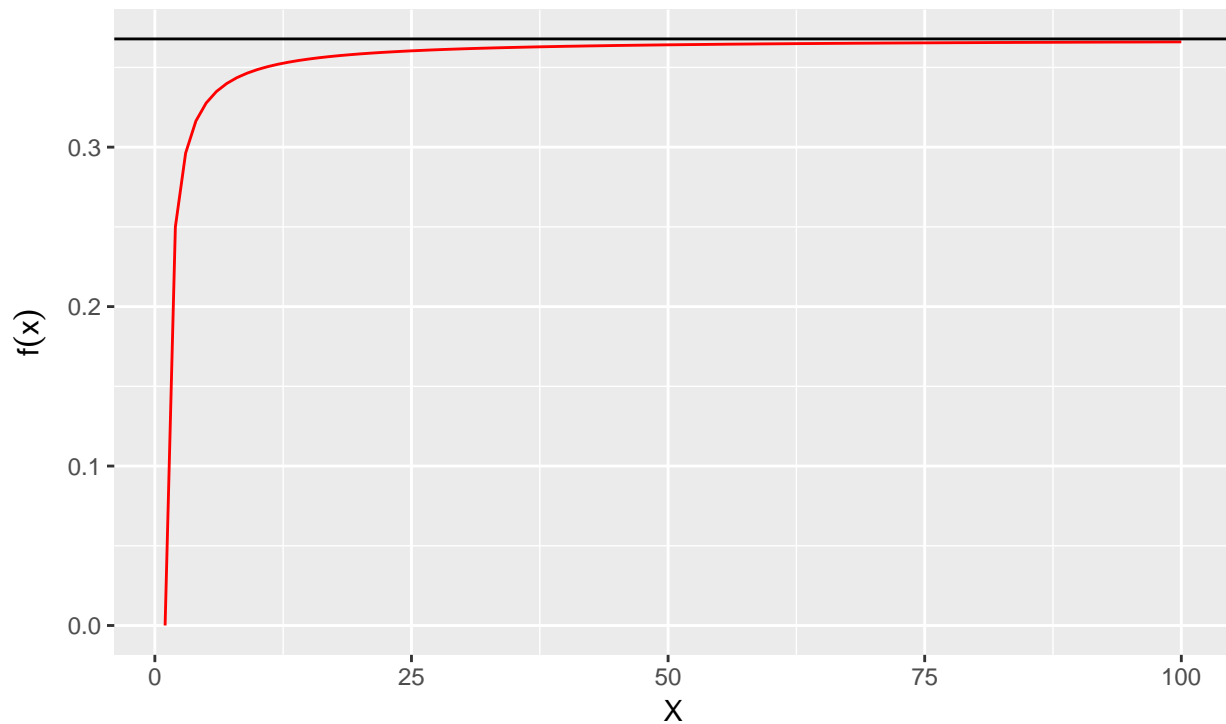
This can be written as:

$$\lim_{x \rightarrow \infty} \left( 1 - \left( 1 - \frac{1}{x} \right)^x \right) = 0.632$$

However, the inner part of that equation,  $\lim_{x \rightarrow \infty} \left( 1 - \frac{1}{x} \right)^x$ , simplifies to  $\frac{1}{e}$ , proven by plot below:

```
x <- 1:100
y <- (1 - (1/x))^x
asymptote <- rep(1/exp(1), 100)
df <- data.frame(x, y, asymptote)
ggplot(df, aes(x = x, y = y)) +
  geom_line(color = 'red') +
  geom_hline(aes(yintercept = asymptote)) +
  labs(x = "X", y = expression(f(x))) +
  ggtitle(expression(lim((1 - over(1, "x"))^"x", x %>% infinity) == frac(1, e)))
```

$$\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x = \frac{1}{e}$$



Therefore:

$$\lim_{x \rightarrow \infty} \left(1 - \left(1 - \frac{1}{x}\right)^x\right) = 0.632 = 1 - \frac{1}{e}$$

### 3

- **A.** K-Fold cross validation is the process of randomly dividing the entire data set into  $K$  separate subsets. A statistical model can then be trained on  $K - 1$  of those subsets, and the final  $K$ th subset is used to test the model on unseen data, returning an estimate of the test error. This is performed  $K$  times, each time using a different subset as to estimate the test error. This results in  $K$  separate estimates of the testing error, which can be averaged to get the cross validated error estimate.
- **B.**
  - i.* There are a couple advantages of K-Fold CV over the validation set approach. First, K-Fold CV will return more than one estimate of the testing error, allowing insight into the variance of

the testing error. In addition to this, since the number of observations in the training data set using the validation set approach is less than the number of observations used in the training data set in K-Fold CV, the validation set approach will typically overestimate the testing error. This is due to the fact that a model has a better chance of modeling the true relationship within the data the more observations too which it has access.

ii. LOOCV also has a couple downsides relative to K-Fold CV. First and foremost, since a total of  $N$  models are fit to the data, there is a large increase in computation time over K-Fold CV when  $K$  is equal to the usual 5 or 10. In addition to this, since there are  $N$  total models and each of the  $N$  models consists of  $N - 1$  observations, *each of the  $N$  models will be trained on nearly identical data*. This leads to the CV error estimates being highly correlated, which corresponds to high variance and low bias.

## 4

Bootstrapping the origin sample would allow an estimate of the standard deviation of a prediction  $Y$  for a given predictor  $X$ . By randomly drawing  $N$  observations from the data set **with replacement**, one can create  $Z$  bootstrapped data sets. The model can be fit to each of these bootstrapped data sets, and then a prediction  $Y$  can be recorded for a constant  $X$  with each of the models. This will result in  $Z$  estimates of the prediction  $Y$ . The standard deviation of these predictions can then be computed using the formula:

$$\sigma_Y = \sqrt{\frac{\sum_i^Z (\hat{y}_i - \bar{y})^2}{n - 1}}$$

## Applied

## 5

- A.

```
suppressPackageStartupMessages(library(ISLR))
suppressPackageStartupMessages(library(caret))
log_mod <- glm(default ~ income + balance, data = Default, family = 'binomial')
```

- B.

```
# i
set.seed(5)
sample_size <- floor(0.5*nrow(Default))
train_idx <- sample(seq_len(nrow(Default)), size = sample_size)
train <- Default[train_idx,]
test <- Default[-train_idx,]

# ii
lm_fit <- glm(default ~ income + balance, data = train, family = 'binomial')

# iii
y_hat <- predict(lm_fit, newdata = test, type = 'response')
preds <- rep('No', nrow(test))
preds[y_hat > 0.5] <- 'Yes'

# iv
```

```
cm <- confusionMatrix(as.factor(preds), as.factor(test$default), 'Yes')
cm$overall[1]
```

```
## Accuracy
## 0.9754
```

- C. Testing three different random seeds leads to the model performing pretty well, with the Accuracy floating from 97% - 98% and therefore a validation error rate of 2% - 3%.

```
results <- c()
for (seed in sample(100, 3)) {
  # train/validation split
  set.seed(seed = seed)
  sample_size <- floor(0.5*nrow(Default))
  train_idx <- sample(seq_len(nrow(Default)), size = sample_size)
  train <- Default[train_idx,]
  test <- Default[-train_idx,]

  # logistic regression modeling
  lm_fit <- glm(default ~ income + balance, data = train, family = 'binomial')

  # prediction
  y_hat <- predict(lm_fit, newdata = test, type = 'response')
  preds <- rep('No', nrow(test))
  preds[y_hat > 0.5] <- 'Yes'

  # model evaluation
  cm <- confusionMatrix(as.factor(preds), as.factor(test$default), 'Yes')
  acc <- cm$overall[1]
  results <- c(results, acc)
}
results
```

```
## Accuracy Accuracy Accuracy
## 0.9738 0.9760 0.9732
```

- D. It doesn't seem as though adding the one hot encoded *Student* variable adds any predictive power to the model. The validation error of 2.68% is within the range of the error rates with *Student* excluded from the model.

```
# train/validation split
set.seed(seed = 200)
sample_size <- floor(0.5*nrow(Default))
train_idx <- sample(seq_len(nrow(Default)), size = sample_size)
train <- Default[train_idx,]
test <- Default[-train_idx,]

# logistic regression modeling
lm_fit <- glm(default ~ ., data = train, family = 'binomial')

# prediction
y_hat <- predict(lm_fit, newdata = test, type = 'response')
preds <- rep('No', nrow(test))
preds[y_hat > 0.5] <- 'Yes'

# model evaluation
```

```
cm <- confusionMatrix(as.factor(preds), as.factor(test$default), 'Yes')
acc <- cm$overall[1]
```

```
# validation error
```

```
1 - acc
```

```
## Accuracy
```

```
## 0.0268
```