Trustworthiness and Expertise: Social Choice and Logic-based Perspectives

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Abstract

This thesis studies problems involving unreliable information. We look at how to aggregate conflicting reports from multiple unreliable sources, how to assess the trustworthiness and expertise of sources, and investigate the extent to which the truth can be found with imperfect information. We take a formal approach, developing mathematical frameworks in which these problems can be formulated precisely and their properties studied. The results are of a conceptual and technical nature, which aim to elucidate interesting properties of the problem at the core abstract level.

In the first half we adopt the axiomatic approach of *social choice theory*. We formulate *truth discovery* – the problem of aggregating reports to estimate true information and reliability of the sources – as a social choice problem. We apply the axiomatic method to investigate desirable properties of such aggregation methods, and analyse a specific truth discovery method from the literature. We go on to study ranking methods for *bipartite tournaments*. This setting can be applied to rank sources according to their accuracy on a number of topics, and is also of independent interest.

In the second half we take a logic-based perspective. We use modal logic to formalise the notion of expertise, and explore connections with knowledge and truthfulness of information. We use this as the foundation for a belief change problem, in which reports must be aggregated to form beliefs about the true state of the world and the expertise of the sources. We again take an axiomatic approach – this time in the tradition of belief revision – where several postulates are proposed as rationality criteria. Finally, we address *truth-tracking*: the problem of finding the truth given non-expert reports. Adapting recent work combining logic with formal learning theory, we investigate the extent to which truth-tracking is possible, and how truth-tracking interacts with rationality.

Contents

Ackowledgements

I would like to thank Bear for being a dog. Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

List of Publications

The content of this thesis is derived from the following publications. [**TODO:** Add descriptions and chapter referencesbeneath each citation?]

- Joseph Singleton and Richard Booth. "An Axiomatic Approach to Truth Discovery". In: *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*. AAMAS 20. Auckland, New Zealand: International Foundation for Autonomous Agents and Multiagent Systems, 2020, pp. 2011–2013. ISBN: 9781450375184
- Joseph Singleton and Richard Booth. "Rankings for Bipartite Tournaments via Chain Editing". In: *Proceedings of the 20th International Conference on Autonomous Agents and MultiAgent Systems*. AAMAS '21. Virtual Event, United Kingdom: International Foundation for Autonomous Agents and Multiagent Systems, 2021, pp. 1236–1244. ISBN: 9781450383073
- Joseph Singleton. "A Logic of Expertise". In: *ESSLLI 2021 Student Session* (2021). URL: https://arxiv.org/abs/2107.10832
- Joseph Singleton and Richard Booth. *Who's the Expert? On Multi-source Belief Change*. 2022. DOI: 10.48550/ARXIV.2205.00077. URL: https://arxiv.org/abs/2205.00077

1 Belief Change with Non-Expert Sources

1.1 Introduction

Consider the following belief change scenario in a hospital. We observe the results of a blood test of patient 1, confirming condition X. Assuming the test is reliable, the AGM paradigm [1] tells us how to revise our beliefs in light of the new information. Dr. A then claims that patient 2 suffers from the same condition, but Dr. B disagrees. Given that doctors specialise in different areas and may make mistakes, who should we trust? Since the **Success** postulate ($\alpha \in K * \alpha$) assumes information is reliable, we are outside the realm of AGM revision, and must instead apply some form of *non-prioritised* revision [58].

Suppose it now emerges that Dr. A had earlier claimed patient 1 did *not* suffer from condition X, contrary to the test results. We now have reason to suspect Dr. A may *lack expertise* on diagnosing X, and may subsequently revise beliefs about Dr. A's domain of expertise and the status of patient 2 (e.g. by opting to trust Dr. B instead).

While simple, this example illustrates the key features of the belief change problem we study: we consider multiple sources, whose expertise is *a priori* unknown, providing reports on various instances of a problem domain. On the basis of these reports we form beliefs both about the expertise of the sources and the state of the world in each instance.

By including a distinguished *completely reliable* source (the test results in the example) we extend AGM revision. In some respects we also extend approaches to non-prioritised revision (e.g. selective revision [45], credibility-limited revision [59], and trust-based revision [16]), which assume information about the reliability of sources is known up front. The problem is also related to *belief merging* [72] which deals with combining belief bases from multiple sources; a detailed comparison will be given in Section 1.7.

Our work is also connected to trust and belief revision, if one interprets trust as *belief in expertise*. As Yasser and Ismail [118] note in recent work, trust and belief are inexorably linked: we should accept reports from sources we believe are trustworthy, and we should trust sources whose reports turn out to be reliable. Trust and belief should also be revised in tandem, so that we may increase or decrease trust in a source as more reports are received, and revoke or reinstate previous reports from a source as its perceived trustworthiness changes.[1]

To unify the trust and belief aspects, we enrich a propositional language with *expertise statements* $\mathsf{E}_i\varphi$, read as "source i has expertise on φ". The output of our belief change problem is then a collection of belief and knowledge sets in the extended language, describing what we *know* and *believe* about the expertise of the sources and the state of the world in each instance. For example, we should *know* reports from the reliable source are true, whereas reports from ordinary sources may only be believed.

Following recent work on logical approaches to expertise [95, 16], we formally model expertise using a partition of propositional valuations for each source. Equivalently, each source has an *indistinguishability equivalence relation* over valuations. A source is an expert on a proposition φ exactly when they can distinguish every φ valuation from every $\neg\varphi$ valuation.[2] As in Singleton [95], we also use *soundness formulas* $\mathsf{S}_i\varphi$, which intuitively say that φ is true *up the expertise of* i. For example, if i has expertise on p but not q, then the conjunction $p \wedge q$ is sound for i whenever p holds, since we can effectively ignore q. Formally, φ is sound for i if the "actual" state of the world is indistinguishable from a φ valuation. Note that expertise does not depend on the "actual" state, whereas soundness does. This provides a crucial link between expertise and truthfulness of information.

We then make the assumption that *sources only report sound propositions*. That is, reports are only false due to sources overstepping the bounds of their expertise. In particular, we assume sources are honest in their reports, and that experts are always right.

Note that in our introductory example, the fact that we had a report from Dr. A on patient 1 (together with reliable information on patient 1) was essential for determining the expertise of Dr. A, and subsequently the status of patient 2. While the patients are independent, reports on one can cause beliefs about the other to change, as we update our beliefs about the expertise of the sources.

In general we consider an arbitrary number of *cases*, which are seen as labels for instances of the domain. For example, a crowdsourcing worker may label multiple images, or a weather forecaster may give predictions for different locations. Each report in the input to the problem then refers to a specific case. Via these

[1] This mutual dependence between trust and belief is also the core idea in *truth discovery* [77].

[2] The relationship between this notion of expertise and *S5 epistemic logic* is explored in a modal logic setting in Singleton [95], and we revisit this connection in Section 1.7.

cases and the presence of the completely reliable source, we are able to model scenarios where some "ground truth" is available, listing how often sources have been correct/incorrect on a proposition (e.g. the *report histories* of Hunter [63]). We can also generalise this scenario, e.g. by having only partial information about "previous" cases.

Throughout the chapter we make the assumption that *expertise is fixed across cases*: the expertise of a source does not depend on the particular instance of the domain we look at. For instance, the expertise of Dr. A is the same for patient 1 as for patient 2. This is a simplifying assumption, and may rule out certain interpretations of the cases (e.g. if cases represent different points in time, it would be natural to let expertise evolve over time).

Contribution. Our contributions are threefold. First, we develop a logical framework for reasoning about the expertise of multiple sources and the state of the world in multiple cases. Second, we formulate a belief change problem within this framework, which allows us to explore how trust and belief should interact and evolve as reports are received from the various sources. Finally, we put forward several postulates and two concrete classes of operators – with a representation result for one class – and analyse these operators with respect to the postulates.

Chapter Outline. In Section 1.2 we develop the formal framework. Section 1.3 introduces the problem and lists some core postulates. We give two constructions and specific example operators in Section 1.4. Section 1.5 introduces some further postulates concerning belief change on the basis of one new report. An analogue of selective revision [45] is presented Section 1.6. Section 1.7 discusses related work, and we conclude in Section 1.8. [**TODO:** mention expertise and selectivity?]

1.2 The Framework

Let $\mathcal{S}$ be a finite set of information sources. For convenience, we assume there is a *completely reliable* source in $\mathcal{S}$, which we denote by $*$. For example, we can treat our first-hand observations as if they are reported by $*$. Other sources besides $*$ will be termed *ordinary sources*. Let $\mathcal{C}$ be a finite set of *cases*, which we interpret as labels for different instances of the problem domain.

Syntax. There are two levels to our formal language. To describe properties of the world in each case $c \in \mathcal{C}$, we assume a fixed finite set $\mathcal{P}$ of propositional variables, and let $\mathcal{L}_0$ denote the set of propositional formulas generated from $\mathcal{P}$ using the usual propositional connectives. We use lower case Greek letters (φ, ψ etc) for formulas in $\mathcal{L}_0$. The classical logical consequence operator will be denoted by Cn_0, and $\equiv$ denotes equivalence of propositional formulas.

The extended *language of expertise* $\mathcal{L}$ additionally describes the expertise of the sources, and is defined by the following grammar:

$$\Phi ::= \varphi \mid \Phi \wedge \Phi \mid \neg\Phi \mid \mathsf{E}_i\varphi \mid \mathsf{S}_i\varphi$$

where $i \in \mathcal{S}$ and $\varphi \in \mathcal{L}_0$. We introduce Boolean connectives $\vee$, $\rightarrow$, $\leftrightarrow$ and $\bot$ as abbreviations. We use upper case Greek letters (Φ, Ψ etc) for formulas in $\mathcal{L}$. For $\Gamma \subseteq \mathcal{L}$, we write $[\Gamma] = \Gamma \cap \mathcal{L}_0$ for the propositional formulas in Γ.

The intuitive reading of $\mathsf{E}_i\varphi$ is *source i has expertise on φ*, i.e., i is able to correctly identify the truth value of φ in any possible state. The intuitive reading of $\mathsf{S}_i\varphi$ is that φ *sound* for i to report: that φ is true up to the expertise of i. That is, the parts of φ on which i has expertise are true. Note that both operators are restricted to propositional formulas, so we will not consider iterated formulas such as $\mathsf{E}_i\mathsf{S}_j\varphi$.

Semantics. Let $\mathcal{V}$ denote the set of propositional valuations over $\mathcal{P}$. For each $\varphi \in \mathcal{L}_0$, the set of valuations making φ true is denoted by $\|\varphi\|$. A *world* $W = \langle \{v_c\}_{c\in\mathcal{C}}, \{\Pi_i\}_{i\in\mathcal{S}} \rangle$ is a possible complete specification of the environment we find ourselves in:

- $v_c \in \mathcal{V}$ is the "true" valuation at case $c \in \mathcal{C}$;
- Π_i is a partition of $\mathcal{V}$ for each $i \in \mathcal{S}$, representing the "true" expertise of source i; and
- Π_* is the unit partition $\{\{v\} \mid v \in \mathcal{V}\}$.

Let $\mathcal{W}$ denote the set of all worlds. Note that the partition corresponding to the distinguished source $*$ is fixed in all worlds as the finest possible partition, reflecting the fact that $*$ is completely reliable.

For any partition Π and valuation v, write $\Pi[v]$ for the unique cell in Π containing v. For a set of valuations U, write $\Pi[U] = \bigcup_{v\in U} \Pi[v]$. For brevity, we write $\Pi[\varphi]$ for $\Pi[\|\varphi\|]$. Then $\Pi[\varphi]$ is the set of valuations indistinguishable from a φ valuation.

For our belief change problem we will be interested in maintaining a collection of several belief sets, describing beliefs about each case $c \in \mathcal{C}$. Towards determining when a world W models such a collection, we define semantics for $\mathcal{L}$ formulas with respect to a world and a case:

$$\begin{aligned} W, c \models \varphi \quad &\iff \quad v_c \in \|\varphi\| \\ W, c \models \mathsf{E}_i\varphi \quad &\iff \quad \Pi_i[\varphi] = \|\varphi\| \\ W, c \models \mathsf{S}_i\varphi \quad &\iff \quad v_c \in \Pi_i[\varphi] \end{aligned}$$

where $i \in \mathcal{S}$, $\varphi \in \mathcal{L}_0$, and the clauses for conjunction and negation are the expected ones. Since $\|\varphi\| \subseteq \Pi_i[\varphi]$ always holds, we have that $\mathsf{E}_i\varphi$ holds iff there is no $\neg\varphi$ valuation which is indistinguishable from a φ valuation (c.f. Booth and Hunter [16]). Note that since each source i has only a single partition Π_i used to interpret the expertise formulas, the truth value of $\mathsf{E}_i\varphi$ does not depend on the case c. On the other hand, $\mathsf{S}_i\varphi$ holds in case c iff the c-valuation of W is indistinguishable from some model of φ. That is, it is consistent with i's expertise that φ is true.

Note that the mapping $2^{\mathcal{V}} \to 2^{\mathcal{V}}$ given by $U \mapsto \Pi[U]$ satisfies the *Kuratowski closure axioms*,[3] so can be considered a closure operator of the set of propositional valuations. Then $W, c \models \mathsf{E}_i\varphi$ iff $\|\varphi\|$ is closed in V, and $W, c \models \mathsf{S}_i\varphi$ iff v_c lies in the closure of $\|\varphi\|$, i.e. φ is true after closing $\|\varphi\|$ along the lines of the expertise of source i. Also note that $\Pi[U] = U$ iff U can be expressed as a union of the partition cells in Π, so that $W, c \models \mathsf{E}_i\varphi$ can alternatively be interpreted as saying φ is a disjunction of stronger formulas on which i also has expertise.

Also note that if φ is a propositional tautology, $\mathsf{E}_i\varphi$ holds for every source i. Thus, all sources are experts on *something*, even if just the tautologies.

Example 1.2.1. *Let us extend the hospital example from the introduction. Let $\mathcal{S} = \{*, a, b\}$ denote the reliable source, Dr. A and Dr. B, and let $\mathcal{C} = \{c_1, c_2\}$ denote patients 1 and 2. Consider propositional variables $\mathcal{P} = \{x, y\}$, standing for condition X and Y respectively. Suppose that Dr. A has expertise on diagnosing condition Y only, whereas Dr. B only has expertise on X. For the sake of the example, suppose that patient 1 suffers from both conditions, and patient 2 suffers only from condition Y. This situation is modelled by the following world $W = \langle \{v_c\}_{c \in \{c_1, c_2\}}, \{\Pi_i\}_{i \in \{*, a, b\}} \rangle$:*

$$v_{c_1} = xy; \qquad v_{c_2} = \bar{x}y;$$
$$\Pi_a = xy, \bar{x}y \mid x\bar{y}, \bar{x}\bar{y}; \quad \Pi_b = xy, x\bar{y} \mid \bar{x}y, \bar{x}\bar{y}.$$

We have $W, c \models \mathsf{E}_a y \wedge \mathsf{E}_b x$ for each $c \in \{c_1, c_2\}$. Also note that $W, c_1 \models x$ (patient 1 suffers from X), $W, c_1 \models \mathsf{S}_a\neg x$ (it is sound for Dr. A to report otherwise; this holds since $\Pi_a[\neg x] = \{xy, \bar{x}y\} \cup \{x\bar{y}, \bar{x}\bar{y}\} \ni xy = v_{c_1}$), but $W, c_1 \models \neg\mathsf{S}_b\neg x$ (the same formula is not *sound for Dr. B; we have $\Pi_b[\neg x] = \{\bar{x}y, \bar{x}\bar{y}\} = \|\neg x\| \not\ni xy = v_{c_1}$).*

Say Φ is *valid* if $W, c \models \Phi$ for all $W \in \mathcal{W}$ and $c \in \mathcal{C}$. For future reference we collect a list of validities.

Proposition 1.2.1. *For any $i \in \mathcal{S}$, $c \in \mathcal{C}$ and $\varphi, \psi \in \mathcal{L}_0$, the following formulas are valid*

1. $\mathsf{S}_i\varphi \leftrightarrow \mathsf{S}_i\psi$ *and* $\mathsf{E}_i\varphi \leftrightarrow \mathsf{E}_i\psi$*, whenever* $\varphi \equiv \psi$
2. $\mathsf{E}_i\varphi \leftrightarrow \mathsf{E}_i\neg\varphi$ *and* $\mathsf{E}_i\varphi \wedge \mathsf{E}_i\psi \to \mathsf{E}_i(\varphi \wedge \psi)$

[3] That is, (i) $\Pi[\emptyset] = \emptyset$, (ii) $U \subseteq \Pi[U]$, (iii) $\Pi[\Pi[U]] = \Pi[U]$, and (iv) $\Pi[U_1 \cup U_2] = \Pi[U_1] \cup \Pi[U_2]$.

3. $\mathsf{E}_i p_1 \wedge \cdots \wedge \mathsf{E}_i p_k \rightarrow \mathsf{E}_i\varphi$, *where* $p_1, \ldots, p_k$ *are the propositional variables appearing in* φ

4. $\mathsf{E}_i\varphi \wedge \mathsf{S}_i\varphi \rightarrow \varphi$, *and* $\mathsf{S}_i\varphi \wedge \neg\varphi \rightarrow \neg\mathsf{E}_i\varphi$

5. $\mathsf{S}_i\varphi \wedge \mathsf{S}_i\neg\varphi \rightarrow \neg\mathsf{E}_i\varphi$

6. $\mathsf{S}_*\varphi \leftrightarrow \varphi$ *and* $\mathsf{E}_*\varphi$

We comment on each property before giving the proof. (1) states syntax-irrelevance properties. (2) says that expertise is symmetric with respect to negation, and closed under conjunctions. Intuitively, symmetry means that i is an expert on φ if they know *whether or not* φ holds. (3) says that expertise on each propositional variable in φ is sufficient for expertise on φ itself. (4) says that, in the presence of expertise, soundness of φ is sufficient for φ to in fact be true. (5) says that if both φ and $\neg\varphi$ are true up to the expertise of i, then i cannot have expertise on φ. Finally, (6) says that the reliable source $*$ has expertise on *all* formulas, and thus φ is sound for $*$ iff it is true.

Proof.

1. If $\varphi \equiv \psi$ then $\|\varphi\| = \|\psi\|$; since the semantics for $\mathsf{S}_i\varphi$ and $\mathsf{E}_i\varphi$ only refer to $\|\varphi\|$ (and likewise for ψ), we have that $\mathsf{S}_i\varphi \leftrightarrow \mathsf{S}_i\psi$ and $\mathsf{E}_i\varphi \leftrightarrow \mathsf{E}_i\psi$ are valid.

2. For the first validity, suppose $W, c \models \mathsf{E}_i\varphi$. Then $\|\varphi\| = \Pi_i[\varphi]$. We show $W, c \models \mathsf{E}_i\neg\varphi$. Indeed, take $v \in \Pi_i[\neg\varphi]$. Then there is $v' \in \|\neg\varphi\|$ such that $v \in \Pi_i[v']$. Thus $v' \in \Pi_i[v]$ also. Supposing for contradiction that $v \in \|\varphi\|$, we get
$$v' \in \Pi_i[v] \subseteq \Pi_i[\varphi] = \|\varphi\|.$$
But then $v' \in \|\neg\varphi\| \cap \|\varphi\| = \emptyset$; contradiction. Hence $v \notin \|\varphi\|$, i.e. $v \in \|\neg\varphi\|$. This shows that $\Pi_i[\neg\varphi] \subseteq \|\neg\varphi\|$, so $W, c \models \mathsf{E}_i\neg\varphi$.

   We have shown that $\mathsf{E}_i\varphi \rightarrow \mathsf{E}_i\neg\varphi$ is valid. For the converse note that, by symmetry, $\mathsf{E}_i\neg\varphi \rightarrow \mathsf{E}_i\neg\neg\varphi$ is valid; since $\mathsf{E}_i\neg\neg\varphi$ is equivalent to $\mathsf{E}_i\varphi$ by (1) we get $\mathsf{E}_i\varphi \leftrightarrow \mathsf{E}_i\neg\varphi$.

   For the second validity, suppose $W, c \models \mathsf{E}_i\varphi \wedge \mathsf{E}_i\psi$. Note that
$$\Pi_i[\varphi \wedge \psi] \subseteq \Pi_i[\varphi] = \|\varphi\|$$
and, similarly, $\Pi_i[\varphi \wedge \psi] \subseteq \|\psi\|$. Hence
$$\Pi_i[\varphi \wedge \psi] \subseteq \|\varphi\| \cap \|\psi\| = \|\varphi \wedge \psi\|,$$
which shows $W, c \models \mathsf{E}_i(\varphi \wedge \psi)$.

3. Let φ be a propositional formula, and let $p_1, \ldots, p_k$ be the variables appearing in φ. Let $\widehat{\mathcal{L}_0} \subseteq \mathcal{L}_0$ be the propositional formulas over $p_1, \ldots p_k$ generated only using conjunction and negation. Then there is some $\psi \in \widehat{\mathcal{L}_0}$ with $\varphi \equiv \psi$.

   Suppose $W, c \models \mathsf{E}_i p_1 \wedge \cdots \wedge \mathsf{E}_i p_k$. By this assumption and the properties in (2), one can show by induction that $W, c \models \mathsf{E}_i \theta$ for all $\theta \in \widehat{\mathcal{L}_0}$. In particular, $W, c \models \mathsf{E}_i \psi$. Since $\varphi \equiv \psi$, we get $W, c \models \mathsf{E}_i \varphi$.

4. Suppose $W, c \models \mathsf{E}_i \varphi \wedge \mathsf{S}_i \varphi$. Then $v_c \in \Pi_i[\varphi] = \|\varphi\|$, so $W, c \models \varphi$. Hence $\mathsf{E}_i \varphi \wedge \mathsf{S}_i \varphi \to \varphi$ is valid. Similarly, $\mathsf{S}_i \varphi \wedge \neg\varphi \to \neg\mathsf{E}_i \varphi$ is valid.

5. Suppose $W, c \models \mathsf{S}_i \varphi \wedge \mathsf{S}_i \neg\varphi$, and, for contradiction, $W, c \models \mathsf{E}_i \varphi$. On the one hand we have $W, c \models \mathsf{E}_i \varphi \wedge \mathsf{S}_i \varphi$, so (4) gives $W, c \models \varphi$. On the other hand, $W, c \models \mathsf{E}_i \varphi$ gives $W, c \models \mathsf{E}_i \neg\varphi$ by (2), so $W, c \models \mathsf{E}_i \neg\varphi \wedge \mathsf{S}_i \neg\varphi$; by (4) again we have $W, c \models \neg\varphi$. But then $W, c \models \varphi \wedge \neg\varphi$ – contradiction.

6. Since the distinguished source $*$ has the unit partition Π_* in any world W, we have $\Pi_*[\varphi] = \|\varphi\|$, so $W, c \models \mathsf{E}_* \varphi$. Similarly, $W, c \models \mathsf{S}_i \varphi$ iff $v_c \in \Pi_*[\varphi] = \|\varphi\|$ iff $W, c \models \varphi$.

□

Case-Indexed Collections. In the remainder of this chapter we will be interested in forming beliefs about each case $c \in \mathcal{C}$. To do so we use collections of belief sets $G = \{\Gamma_c\}_{c \in \mathcal{C}}$, with $\Gamma_c \subseteq \mathcal{L}$, indexed by cases. Say a world W is a *model* of G iff

$$W, c \models \Phi \text{ for all } c \in \mathcal{C} \text{ and } \Phi \in \Gamma_c,$$

i.e. iff W satisfies all formulas in G in the relevant case. Let $\mathrm{mod}(G)$ denote the models of G, and say that G is *consistent* if $\mathrm{mod}(G) \neq \emptyset$. For $c \in \mathcal{C}$, define the *c-consequences*

$$\mathrm{Cn}_c(G) = \{\Phi \in \mathcal{L} \mid \forall W \in \mathrm{mod}(G), W, c \models \Phi\}.$$

We write $\mathrm{Cn}(G)$ for the collection $\{\mathrm{Cn}_c(G)\}_{c \in \mathcal{C}}$.

Example 1.2.2. *Suppose $\mathcal{C} = \{c_1, c_2, c_3\}$, and define G by $\Gamma_{c_1} = \{\mathsf{S}_i(p \wedge q)\}$, $\Gamma_{c_2} = \{\mathsf{E}_i p\}$ and $\Gamma_{c_3} = \{\mathsf{E}_i q\}$. Then, since expertise holds independently of case, any model W of G has $W, c_1 \models \mathsf{E}_i p \wedge \mathsf{E}_i q$. By Proposition 1.2.1 part (3), $W, c_1 \models \mathsf{E}_i(p \wedge q)$. Since W satisfies Γ_{c_1} in case c_1, Proposition 1.2.1 part (4) gives $W, c_1 \models p \wedge q$. Since W was an arbitrary model of G, we have $p \wedge q \in \mathrm{Cn}_{c_1}(G)$, i.e. $p \wedge q$ is a c_1-consequence of G. This illustrates how information about distinct cases can be brought together to have consequences for other cases.*

For two collections $G = \{\Gamma_c\}_{c \in \mathcal{C}}$, $D = \{\Delta_c\}_{c \in \mathcal{C}}$, write $G \sqsubseteq D$ iff $\Gamma_c \subseteq \Delta_c$ for all c, and let $G \sqcup D$ denote the collection $\{\Gamma_c \cup \Delta_c\}_{c \in \mathcal{C}}$. With this notation, the case-indexed consequence operator satisfies analogues of the Tarskian consequence properties.[4]

Say a collection G is *closed* if $\mathrm{Cn}(G) = G$. Closed collections provide an idealised representation of beliefs, which will become useful later on. For instance, when G is closed we have $\mathsf{E}_i\varphi \in \Gamma_c$ iff $\mathsf{E}_i\varphi \in \Gamma_d$ for all $c, d \in \mathcal{C}$; i.e. expertise statements are either present for all cases or for none. We also have $\mathrm{Cn}_0\,[\Gamma_c] = [\Gamma_c]$, i.e. the propositional parts of G are (classically) closed.

In propositional logic, $\|\cdot\|$ is a 1-to-1 correspondence between closed sets of formulas and sets of valuations. This is not so in our setting, since some subsets of $\mathcal{W}$ do not arise as the models of any collection. Instead, we have a 1-to-1 correspondence into a restricted collection of sets of worlds. Borrowing the terminology of Delgrande, Peppas, and Woltran [30], say a set of worlds $S \subseteq \mathcal{W}$ is *elementary* if $S = \mathrm{mod}(G)$ for some collection $G = \{\Gamma_c\}_{c\in\mathcal{C}}$.[5]

Elementariness is characterised by a certain closure condition. Say that two worlds W, W' are *partition-equivalent* if $\Pi_i^W = \Pi_i^{W'}$ for all sources i, and say W is a *valuation combination* from a set $S \subseteq \mathcal{W}$ if for all cases c there is $W_c \in S$ such that $v_c^W = v_c^{W_c}$. Then a set is elementary iff it is closed under valuation combinations of partition-equivalent worlds.

Proposition 1.2.2. *$S \subseteq \mathcal{W}$ is elementary if and only if the following condition holds: for all $W \in \mathcal{W}$ and $W_1, W_2 \in S$, if W is partition-equivalent to both W_1, W_2 and W is a valuation combination from $\{W_1, W_2\}$, then $W \in S$.*

Proof. "if": Suppose the stated condition holds for $S \subseteq \mathcal{W}$. Form a collection $G = \{\Gamma_c\}_{c\in\mathcal{C}}$ by setting $\Gamma_c = \{\Phi \in \mathcal{L} \mid S \subseteq \mathrm{mod}_c(\Phi)\}$. Clearly $S \subseteq \mathrm{mod}(G)$. For the reverse inclusion, suppose $W \in \mathrm{mod}(G)$. For any set of valuations $U \subseteq \mathcal{V}$, let φ_U be any propositional sentence with $\|\varphi_U\| = U$. For each $c \in \mathcal{C}$, consider the sentence

$$\Phi_c = \bigvee_{W'\in S} \left(\varphi_{\{v_c^{W'}\}} \wedge \bigwedge_{i\in\mathcal{S}} \bigwedge_{U\subseteq\mathcal{V}} R_{W',i,U} \right)$$

where

$$R_{W',i,U} = \begin{cases} \mathsf{E}_i\varphi_U, & W', c_0 \models \mathsf{E}_i\varphi_U \\ \neg\mathsf{E}_i\varphi_U, & \text{otherwise} \end{cases}$$

for some fixed case $c_0 \in \mathcal{C}$. It is straightforward to see that each $W' \in S$ satisfies its corresponding disjunct at case c, so $\Phi_c \in \Gamma_c$. Hence $W \in \mathrm{mod}(G)$ implies $W, c \models \Phi_c$ for each c. Consequently, for each c there is some $W_c \in S$ such that (i)

[4] That is, (i) $G \sqsubseteq \mathrm{Cn}(G)$, (ii) $G \sqsubseteq D$ implies $\mathrm{Cn}(G) \sqsubseteq \mathrm{Cn}(D)$, and (iii) $\mathrm{Cn}(\mathrm{Cn}(G)) = \mathrm{Cn}(G)$.

[5] Non-elementary sets can also exist for weaker logics (such as Horn logic [30]) which lack the syntactic expressivity to identify all sets of models. In our framework, $\mathcal{C}$-indexed collections are not expressive enough to specify *combinations of valuations*, since each Γ_c only says something about the valuation for c.

$v_c^W = v_c^{W_c}$; and (ii) for each $i \in \mathcal{S}$ and $U \subseteq V$, $W, c \models \mathsf{E}_i\varphi_U$ iff $W_c, c \models \mathsf{E}_i\varphi_U$. From (i), W is a valuation combination from $\{W_c\}_{c\in\mathcal{C}}$. From (ii) it can be shown that in fact $\Pi_i^W = \Pi_i^{W_c}$ for each c and i; that is, W is partition-equivalent to each W_c. In particular, all the W_c are partition-equivalent to each other. Repeatedly applying the closure condition assumed to hold for S, we see that $W \in S$ as required.

"only if": Suppose S is elementary, i.e. $S = \mathrm{mod}(G)$ for some collection $G = \{\Gamma_c\}_{c\in\mathcal{C}}$, and let W, W_1, W_2 be as in the statement of the proposition. Take $c \in \mathcal{C}$ and $\Phi \in \Gamma_c$. We will show $W, c \models \Phi$. By assumption, there is $n \in \{1, 2\}$ such that $v_c^W = v_c^{W_n}$. It can be shown by induction on $\mathcal{L}$ formulas that, since W and W_n are partition-equivalent and have the same c valuation, $W, c \models \Phi$ iff $W_n, c \models \Phi$. But $W_n \in S = \mathrm{mod}(G)$ implies $W_n, c \models \Phi$, so $W, c \models \Phi$ too. Since $\Phi \in \Gamma_c$ was arbitrary, we have $W \in \mathrm{mod}(G) = S$ as required.

□

1.3 The Problem

With the framework set out, we can formally define the problem. We seek an operator with the following behaviour:

- **Input:** A sequence of reports σ, where each report is a triple $\langle i, c, \varphi\rangle \in \mathcal{S} \times \mathcal{C} \times \mathcal{L}_0$ and $\varphi \not\equiv \bot$. Such a report represents that *source i reports φ to hold in case c*. Note that we only allow sources to make *propositional* reports.
- **Output:** A pair $\langle B^\sigma, K^\sigma\rangle$, where $B^\sigma = \{B_c^\sigma\}_{c\in\mathcal{C}}$ is a collection of *belief sets* $B_c^\sigma \subseteq \mathcal{L}$ and $K^\sigma = \{K_c^\sigma\}_{c\in\mathcal{C}}$ is a collection of *knowledge sets* $K_c^\sigma \subseteq \mathcal{L}$.

1.3.1 Basic Postulates

We immediately narrow the scope of operators under consideration by introducing some basic postulates which are expected to hold. In what follows, say a sequence σ is $*$-*consistent* if for each $c \in \mathcal{C}$ the set $\{\varphi \mid \langle *, c, \varphi\rangle \in \sigma\} \subseteq \mathcal{L}_0$ is classically consistent. Write G_{snd}^σ for the collection with $(G_{\mathsf{snd}}^\sigma)_c = \{\mathsf{S}_i\varphi \mid \langle i, c, \varphi\rangle \in \sigma\}$, i.e. the collection of soundness statements corresponding to the reports in σ.

◇ **Closure** $B^\sigma = \mathrm{Cn}(B^\sigma)$ *and* $K^\sigma = \mathrm{Cn}(K^\sigma)$

◇ **Containment** $K^\sigma \sqsubseteq B^\sigma$

◇ **Consistency** *If σ is $*$-consistent, B^σ and K^σ are consistent*

◇ **Soundness** *If $\langle i, c, \varphi\rangle \in \sigma$, then $\mathsf{S}_i\varphi \in K_c^\sigma$*

◇ **K-bound** $K^\sigma \sqsubseteq \mathrm{Cn}(G_{\mathsf{snd}}^\sigma \sqcup K^\emptyset)$

- ⋄ **Prior-extension** $K^{\emptyset} \sqsubseteq K^{\sigma}$
- ⋄ **Rearrangement** *If σ is a permutation of ρ, then $B^{\sigma} = B^{\rho}$ and $K^{\sigma} = K^{\rho}$*
- ⋄ **Equivalence** *If $\varphi \equiv \psi$ then $B^{\sigma \cdot \langle i,c,\varphi \rangle} = B^{\sigma \cdot \langle i,c,\psi \rangle}$ and $K^{\sigma \cdot \langle i,c,\varphi \rangle} = K^{\sigma \cdot \langle i,c,\psi \rangle}$*

Closure says that the belief and knowledge collections are closed under logical consequence. In light of earlier remarks, this implies that the propositional belief sets $[B_c^{\sigma}]$ are closed under (propositional) consequence, and that $\mathsf{E}_i\varphi \in B_c^{\sigma}$ iff $\mathsf{E}_i\varphi \in B_d^{\sigma}$. **Containment** says that everything which is known is also believed. **Consistency** ensures the output is always consistent, provided we are not in the degenerate case where $*$ gives inconsistent reports. **Soundness** says we *know* that all reports are sound in their respective cases. This formalises our assumption that sources are *honest*, i.e. that false reports only arise due to lack of expertise. By Proposition 1.2.1 part (4) it also implies *experts are always right*: if a source has expertise on their report then it must be true. While **Soundness** places a lower bound on knowledge, **K-bound** places an upper bound: knowledge cannot go beyond the soundness statements corresponding to the reports in σ together with the prior knowledge $K^{\emptyset}$. That is, from the point view of knowledge, a new report of $\langle i, c, \varphi \rangle$ only allows us to learn $\mathsf{S}_i\varphi$ in case c (and to combine this with other reports and prior knowledge). Note that the analogous property for belief is *not* desirable: we want to be more liberal when it comes to beliefs, and allow for *defeasible inferences* going beyond the mere fact that reports are sound. **Prior-extension** says that knowledge after a sequence σ extends the prior knowledge on the empty sequence $\emptyset$. **Rearrangement** says that the order in which reports are received is irrelevant. This can be justified on the basis that we are reasoning about *static worlds* for each case c, so that there is no reason to see more "recent" reports as any more or less important or truthful than earlier ones.[6] Consequently, we can essentially view the input as a *multi-set* of belief sets – one for each source – bringing us close to the setting of belief merging. This postulate also appears as the commutativity postulate **(Com)** in the work of Schwind and Konieczny [94]. Finally, **Equivalence** says that the syntactic form of reports is irrelevant.

Taking all the basic postulates together, the knowledge component K^{σ} is fully determined once $K^{\emptyset}$ is chosen.

Proposition 1.3.1. *Suppose an operator satisfies the basic postulates. Then*

1. $K^{\sigma} = \mathrm{Cn}(G^{\sigma}_{\mathsf{snd}} \sqcup K^{\emptyset})$
2. $K^{\emptyset} = \mathrm{Cn}(\emptyset)$ *iff* $K^{\sigma} = \mathrm{Cn}(G^{\sigma}_{\mathsf{snd}})$ *for all* σ.

Proof.

[6] This argument is from [29].

1. The "$\sqsubseteq$" inclusion is just **K-bound**. For the "$\sqsupseteq$" inclusion, note that $G^{\sigma}_{\mathsf{snd}} \sqsubseteq K^{\sigma}$ by **Soundness**, and $K^{\emptyset} \sqsubseteq K^{\sigma}$ by **Prior-extension**. Hence

$$G^{\sigma}_{\mathsf{snd}} \sqcup K^{\emptyset} \sqsubseteq K^{\sigma}.$$

   By monotonicity of Cn,

$$\mathrm{Cn}(G^{\sigma}_{\mathsf{snd}} \sqcup K^{\emptyset}) \sqsubseteq \mathrm{Cn}(K^{\sigma}) = K^{\sigma}$$

   where we use **Closure** in the final step.

2. "if": Suppose $K^{\sigma} = \mathrm{Cn}(G^{\sigma}_{\mathsf{snd}})$ for all σ. Taking $\sigma = \emptyset$ we obtain

$$K^{\sigma} = \mathrm{Cn}(G^{\emptyset}_{\mathsf{snd}}) = \mathrm{Cn}(\emptyset).$$

   "only if": Suppose $K^{\emptyset} = \mathrm{Cn}(\emptyset)$. Take any sequence σ. By **K-bound**,

$$K^{\sigma} \sqsubseteq \mathrm{Cn}(G^{\sigma}_{\mathsf{snd}} \sqcup \mathrm{Cn}(\emptyset)) = \mathrm{Cn}(G^{\sigma}_{\mathsf{snd}})$$

   On the other hand, **Soundness** and **Closure** give $\mathrm{Cn}(G^{\sigma}_{\mathsf{snd}}) \sqsubseteq K^{\sigma}$. Hence $K^{\sigma} = \mathrm{Cn}(G^{\sigma}_{\mathsf{snd}})$.

□

The choice of $K^{\emptyset}$ depends on the scenario one wishes to model. While $\mathrm{Cn}(\emptyset)$ is a sensible choice if the sequence σ is all we have to go on, we allow $K^{\emptyset} \neq \mathrm{Cn}(\emptyset)$ in case *prior knowledge* is available (for example, the expertise of particular sources may be known ahead of time).

Another important property of knowledge, which follows from the basic postulates, says that *knowledge is monotonic*: knowledge after receiving σ and ρ together is just the case-wise union of K^{σ} and K^{ρ}.

◇ **K-conjunction** $K^{\sigma \cdot \rho} = \mathrm{Cn}(K^{\sigma} \sqcup K^{\rho})$

K-conjunction reflects the idea that one should be cautious when it comes to knowledge: a formula should only be accepted as known if it won't be given up in light of new information.

Proposition 1.3.2. *Any operator satisfying the basic postulates satisfies* ***K-conjunction****.*

Proof. Suppose an operator satisfies the basic postulates, and take sequences σ and ρ. By Proposition 1.3.1,

$$K^{\sigma \cdot \rho} = \mathrm{Cn}(G^{\sigma \cdot \rho}_{\mathsf{snd}} \sqcup K^{\emptyset})$$

Note that $G^{\sigma \cdot \rho}_{\mathsf{snd}} = G^{\sigma}_{\mathsf{snd}} \sqcup G^{\rho}_{\mathsf{snd}}$. Hence we may write

$$\begin{aligned} K^{\sigma \cdot \rho} &= \mathrm{Cn}(G^{\sigma}_{\mathsf{snd}} \sqcup G^{\rho}_{\mathsf{snd}} \sqcup K^{\emptyset}) \\ &= \mathrm{Cn}((G^{\sigma}_{\mathsf{snd}} \sqcup K^{\emptyset}) \sqcup (G^{\rho}_{\mathsf{snd}} \sqcup K^{\emptyset})) \end{aligned}$$

By Proposition 1.3.1 again, we have $K^\sigma = \mathrm{Cn}(G^\sigma_{\mathsf{snd}} \sqcup K^\emptyset)$ and $K^\rho = \mathrm{Cn}(G^\rho_{\mathsf{snd}} \sqcup K^\emptyset)$. It is easily verified that for any collections G, D, we have

$$\mathrm{Cn}(G \sqcup D) = \mathrm{Cn}(\mathrm{Cn}(G) \sqcup \mathrm{Cn}(D)).$$

Consequently,

$$\begin{aligned} K^{\sigma\cdot\rho} &= \mathrm{Cn}(\mathrm{Cn}(G^\sigma_{\mathsf{snd}} \sqcup K^\emptyset) \sqcup \mathrm{Cn}(G^\rho_{\mathsf{snd}} \sqcup K^\emptyset)) \\ &= \mathrm{Cn}(K^\sigma \sqcup K^\rho) \end{aligned}$$

as required for **K-conjunction**. □

The postulates also imply some useful properties linking *trust* (seen as belief in expertise) and *belief/knowledge*.

Proposition 1.3.3. *Suppose an operator satisfies the basic postulates. Then*

1. *If* $\varphi \in K^\sigma_c$ *and* $\neg\psi \in \mathrm{Cn}_0(\varphi)$ *then* $\neg\mathsf{E}_i\psi \in K^{\sigma\cdot\langle i,c,\psi\rangle}_c$.
2. *If* $\langle i, c, \varphi\rangle \in \sigma$ *and* $\mathsf{E}_i\varphi \in B^\sigma_c$ *then* $\varphi \in B^\sigma_c$.

Proof.

1. Suppose $\varphi \in K^\sigma_c$ and $\neg\psi \in \mathrm{Cn}_0(\varphi)$. Write $\rho = \sigma\cdot\langle i, c, \psi\rangle$. By **Soundness**, $\mathsf{S}_i\psi \in K^\rho_c$. By **K-conjunction**, $\varphi \in K^\sigma_c \subseteq (K^\sigma \sqcup K^{\langle i,c,\psi\rangle})_c \subseteq \mathrm{Cn}_c(K^\sigma \sqcup K^{\langle i,c,\psi\rangle}) = K^\rho_c$. Since $\neg\psi \in \mathrm{Cn}_0(\varphi)$ and $\varphi \in K^\rho_c$, **Closure** gives $\neg\psi \in K^\rho_c$. Recalling from Proposition 1.2.1 part (4) that $\mathsf{S}_i\psi \wedge \neg\psi \to \neg\mathsf{E}_i\psi$, **Closure** gives $\neg\mathsf{E}_i\psi \in K^\rho_c$, as desired.

2. Suppose $\langle i, c, \varphi\rangle \in \sigma$ and $\mathsf{E}_i\varphi \in B^\sigma_c$. By **Soundness** and **Containment**, $\mathsf{S}_i\varphi \in B^\sigma_c$. From Proposition 1.2.1 part (4) again we have $\mathsf{E}_i\varphi \wedge \mathsf{S}_i\varphi \to \varphi$. By **Closure**, $\varphi \in B^\sigma_c$.

□

(1) expresses how knowledge can negatively affect trust: we should distrust sources who make reports we know to be false. (2) expresses how trust affects belief: we should believe reports from trusted sources. It can also be seen as a form of *success* for ordinary sources, and implies AGM success when $i = *$ (by Proposition 1.2.1 part (6) and **Closure**). We illustrate the basic postulates by formalising the introductory hospital example.

Example 1.3.1. *Set* $\mathcal{S}, \mathcal{C}$ *and* $\mathcal{P}$ *as in Example 1.2.1, and consider the sequence*

$$\sigma = (\langle *, c_1, x\rangle, \langle a, c_2, x\rangle, \langle b, c_2, \neg x\rangle, \langle a, c_1, \neg x\rangle).$$

*What do we know on the basis of this sequence, assuming the basic postulates? First note that by **Soundness**, Proposition 1.2.1 part (6) and **Closure**, the report from $*$ gives $x \in K^{\sigma}_{c_1}$, i.e. reliable reports are known. **Soundness** also gives $\mathsf{S}_a x \wedge \mathsf{S}_b \neg x \in K^{\sigma}_{c_2}$. Combined with Proposition 1.2.1 parts (2), (4) and **Closure**, this yields $\neg(\mathsf{E}_a x \wedge \mathsf{E}_b x) \in K^{\sigma}_c$ for all c, formalising the intuitive idea that Drs. A and B cannot both be experts on X, since they give conflicting reports. Considering the final report from a, we get $x \wedge \mathsf{S}_a \neg x \in K^{\sigma}_{c_1}$, and thus $\neg\mathsf{E}_a x \in K^{\sigma}_c$ by **Closure**. So in fact Dr. A is known to be a non-expert on X.*

*What about beliefs? The basic postulates do not require beliefs to go beyond knowledge, so we cannot say much in general. An "optimistic" operator, however, may opt to believe that sources are experts unless we know otherwise, and thus maximise the information that can be (defeasibly) inferred from the sequence (in the next section we will introduce concrete operators obeying this principle). In this case we may believe that at least one source has expertise on x (i.e. $\mathsf{E}_a x \vee \mathsf{E}_b x \in B^{\sigma}_c$). Combined with $\neg\mathsf{E}_a x \in K^{\sigma}_c$, **Closure** and **Containment**, we get $\mathsf{E}_b x \in B^{\sigma}_{c_2}$. Symmetry of expertise together with Proposition 1.3.3 part (2) then gives $\neg x \in B^{\sigma}_{c_2}$, i.e. we trust Dr. B in the example and believe patient 2 does not suffer from condition X.*

1.3.2 Model-Based Operators

While an operator is a purely syntactic object, it will be convenient to specify K^{σ} and B^{σ} in semantic terms by selecting a set of *possible* and *most plausible* worlds for each sequence σ. We call such operators *model-based*.

Definition 1.3.1. *An operator is* model-based *if for every σ there are sets $\mathcal{X}_\sigma, \mathcal{Y}_\sigma \subseteq \mathcal{W}$ such that (i) $\mathcal{X}_\sigma \supseteq \mathcal{Y}_\sigma$; (ii) $\Phi \in K^{\sigma}_c$ iff $W, c \models \Phi$ for all $W \in \mathcal{X}_\sigma$; and (iii) $\Phi \in B^{\sigma}_c$ iff $W, c \models \Phi$ for all $W \in \mathcal{Y}_\sigma$.*

In other words, K^{σ}_c (resp., B^{σ}_c) contains the formulas which hold at case c in *all worlds* in $\mathcal{X}_\sigma$ (resp., $\mathcal{Y}_\sigma$). It follows from the relevant definitions that $\mathcal{X}_\sigma \subseteq \mathrm{mod}(K^{\sigma})$, and equality holds if and only if $\mathcal{X}_\sigma$ is elementary (similarly for $\mathcal{Y}^{\sigma}$ and B^{σ}). Model-based operators are characterised by our first two basic postulates.

Theorem 1.3.1. *An operator satisfies **Closure** and **Containment** if and only if it is model-based.*

Proof. For ease of notation in what follows, write $\mathrm{mod}_c(\Phi) = \{W \in \mathcal{W} \mid W, c \models \Phi\}$.

"if": Suppose an operator $\sigma \mapsto \langle B^{\sigma}, K^{\sigma} \rangle$ is model-based. For **Closure**, we need to show that $B^{\sigma}_c \supseteq \mathrm{Cn}_c(B^{\sigma})$ and $K^{\sigma}_c \supseteq \mathrm{Cn}_c(K^{\sigma})$, for each c. Take any $\Phi \in \mathrm{Cn}_c(B^{\sigma})$. Then $\mathrm{mod}(B^{\sigma}) \subseteq \mathrm{mod}_c(\Phi)$. From the relevant definitions, one can easily check that $\mathcal{Y}_\sigma \subseteq \mathrm{mod}(B^{\sigma})$, so we have $\mathcal{Y}_\sigma \subseteq \mathrm{mod}_c(\Phi)$. That is, $W, c \models \Phi$ for all $W \in \mathcal{Y}_\sigma$. By definition of model-based operators, $\Phi \in B^{\sigma}_c$. The fact that $K^{\sigma}_c \supseteq \mathrm{Cn}_c(K^{\sigma})$ follows by an identical argument upon noticing that $\mathcal{X}_\sigma \subseteq \mathrm{mod}(K^{\sigma})$.

Containment follows from $\mathcal{X}_\sigma \supseteq \mathcal{Y}_\sigma$: if $\Phi \in K_c^\sigma$ then $W, c \models \Phi$ for all $W \in \mathcal{X}_\sigma$, and in particular this holds for all $W \in \mathcal{Y}_\sigma$. Hence $\Phi \in B_c^\sigma$, so $K^\sigma \sqsubseteq B^\sigma$.

"only if": Suppose an operator satisfies **Closure** and **Containment**. For any σ, set

$$\mathcal{X}_\sigma = \mathrm{mod}(K^\sigma)$$
$$\mathcal{Y}_\sigma = \mathrm{mod}(B^\sigma)$$

We show the three properties required in Definition 1.3.1. $\mathcal{X}_\sigma \supseteq \mathcal{Y}_\sigma$ follows from **Containment** and the definition of a model of a collection. For the second property, note that $\Phi \in K_c^\sigma$ iff $\Phi \in \mathrm{Cn}_c(K^\sigma)$ by **Closure**, i.e. iff $\mathrm{mod}(K^\sigma) \subseteq \mathrm{mod}_c(\Phi)$. By choice of $\mathcal{X}_\sigma$, this holds exactly when $W, c \models \Phi$ for all $W \in \mathcal{X}_\sigma$, as required. The third property is proved using an identical argument. □

Since we take **Closure** and **Containment** to be fundamental properties, all operators we consider from now on will be model-based. We introduce our first concrete operator.

Definition 1.3.2. *Define the model-based operator* weak-mb *by*

$$\mathcal{X}_\sigma = \mathcal{Y}_\sigma = \{W \mid W, c \models \mathsf{S}_i\varphi \textit{ for all } \langle i, c, \varphi\rangle \in \sigma\}.$$

That is, the possible worlds $\mathcal{X}_\sigma$ are exactly those satisfying the soundness constraint for each report in σ, i.e. false reports are only due to lack of expertise of the corresponding source. Syntactically, $K^\sigma = B^\sigma = \mathrm{Cn}(G^\sigma_{\mathsf{snd}})$.

Clearly weak-mb satisfies **Soundness**, and one can show that it satisfies all of the basic postulates of Section 1.3.1.[7] In fact, it is the *weakest* operator satisfying **Closure**, **Containment** and **Soundness**, in that for any other operator $\sigma \mapsto \langle \hat{B}^\sigma, \hat{K}^\sigma\rangle$ with these properties we have $B^\sigma \sqsubseteq \hat{B}^\sigma$ and $K^\sigma \sqsubseteq \hat{K}^\sigma$ for any σ.

Example 1.3.2. *Consider* weak-mb *applied to the sequence $\sigma = (\langle *, c, p\rangle, \langle i, c, \neg p \wedge q\rangle)$. By **Soundness**, **Closure** and the validities from Proposition 1.2.1, we have $p \in K_c^\sigma$ and $\neg\mathsf{E}_i p \in K_c^\sigma$. In fact, by **Closure**, we have $\neg\mathsf{E}_i p \in K_d^\sigma$ for all cases d. However, we cannot say much about q: neither q, $\neg q$, $\mathsf{E}_i q$ nor $\neg\mathsf{E}_i q$ are in $B_c^\sigma = K_c^\sigma$.*

1.4 Constructions

For model-based operators in Definition 1.3.1, the sets $\mathcal{X}_\sigma$ and $\mathcal{Y}_\sigma$ – which determine knowledge and belief – can depend on σ in a completely arbitrary manner. This lack of structure leads to very wide class of operators, and one cannot say much about them in general beyond the satisfaction of **Closure** and **Containment**. In this section we specialise model-based operators by providing two constructions.

[7] For **Consistency**, note that for any $*$-consistent sequence σ one can form a world W such that v_c is a model of all reports from $*$ at case c, and $\Pi_i = \{\mathcal{V}\}$ for all $i \neq *$. This satisfies all the soundness constraints, so $W \in \mathcal{X}_\sigma = \mathcal{Y}_\sigma$.

1.4.1 Conditioning Operators

Intuitively, $\mathcal{Y}_\sigma$ is supposed to represent the *most plausible* worlds among the possible worlds in $\mathcal{X}_\sigma$. This suggests the presence of a *plausibility ordering* on $\mathcal{X}_\sigma$, which is used to select $\mathcal{Y}_\sigma$. For our first construction we take this approach: we condition a fixed plausibility total preorder[8] on the knowledge $\mathcal{X}_\sigma$, and obtain $\mathcal{Y}_\sigma$ by selecting the minimal (i.e. most plausible) worlds.

Definition 1.4.1. *An operator is a* conditioning operator *if there is a total preorder $\leq$ on $\mathcal{W}$ and a mapping $\sigma \mapsto \langle \mathcal{X}_\sigma, \mathcal{Y}_\sigma \rangle$ as in Definition 1.3.1 such that $\mathcal{Y}_\sigma = \min_\leq \mathcal{X}_\sigma$ for all σ.*

Note that $\leq$ is independent of σ: it is fixed before receiving any reports. All conditioning operators are model-based by definition. Clearly $\mathcal{Y}_\sigma$ is determined by $\mathcal{X}_\sigma$ and the plausibility order, so that to define a conditioning operator it is enough to specify $\leq$ and the mapping $\sigma \mapsto \mathcal{X}_\sigma$. Write $W \simeq W'$ iff both $W \leq W'$ and $W' \leq W$. We now present examples of how such an ordering can be defined.

Definition 1.4.2. *Define the conditioning operator* var-based-cond *by setting $\mathcal{X}_\sigma$ in the same way as* weak-mb *in Definition 1.3.2, and $W \leq W'$ iff $r(W) \leq r(W')$, where*

$$r(W) = -\sum_{i \in \mathcal{S}} \left| \{ p \in \mathcal{P} \mid \Pi_i^W[p] = \|p\| \} \right| .$$

var-based-cond aims to trust each source on *as many propositional variables* as possible. One can check that var-based-cond satisfies the basic postulates.

Example 1.4.1. *Revisiting the sequence $\sigma = (\langle *, c, p \rangle, \langle i, c, \neg p \wedge q \rangle)$ from Example 1.3.2 with* var-based-cond*, the knowledge set K_c^σ is the same as before, but we now have $q \wedge \mathsf{E}_i q \in B_c^\sigma$. This reflects the "credulous" behaviour of the ranking $\leq$: while it is not possible to believe i is an expert on p, we should believe they* are *an expert on q so long as this does not conflict with soundness. For the propositional beliefs generally, we have $[B_c^\sigma] = \mathrm{Cn}_0(p \wedge q)$. That is,* var-based-cond *takes the q part of the report from i (on which i is credulously trusted) while ignoring the $\neg p$ part (which is false due to report from $*$).*

Definition 1.4.3. *Define a conditioning operator* part-based-cond *with $\mathcal{X}_\sigma$ as for* var-based-cond*, and $\leq$ defined by the ranking function*

$$r(W) = -\sum_{i \in \mathcal{S}} |\Pi_i^W|.$$

part-based-cond aims to maximise the *number of cells* in the sources' partitions, and thereby maximise the number of propositions on which they have expertise. Unlike var-based-cond, the propositional variables play no special role. As expected, part-based-cond satisfies the basic postulates.

[8] A total preorder is a reflexive, transitive and total relation.

Example 1.4.2. *Applying* part-based-cond *to σ from Examples 1.3.2 and 1.4.1, we no longer extract q from the report of i: $q \notin B_c^\sigma$ and $\mathsf{E}_i q \notin B_c^\sigma$. Instead, we have $[B_c^\sigma] = \mathrm{Cn}_0(p)$, and $\mathsf{E}_i(p \vee q) \in B_c^\sigma$.*

Note that both var-based-cond and part-based-cond are based on the general principle of maximising the expertise of sources, subject to the constraint that all reports are sound. This intuition is formalised by the following postulate for conditioning operators. In what follows, write $W \preceq W'$ iff Π_i^W refines $\Pi_i^{W'}$ for all $i \in \mathcal{S}$, i.e. if all sources have broadly more expertise in W than in W'.[9]

◇ **Refinement** *If $W \preceq W'$ then $W \leq W'$*

Since $\preceq$ is only a partial order on $\mathcal{W}$ there are many possible total extensions; var-based-cond and part-based-cond provide two specific examples.

We now turn to an axiomatic characterisation of conditioning operators. Taken with the basic postulates from Section 1.3.1, conditioning operators can be characterised using an approach similar to that of Delgrande, Peppas, and Woltran [30] in their account of *generalised AGM belief revision*.[10] This involves a technical property Delgrande, Peppas, and Woltran call **Acyc**, which finds its roots in the *Loop* property of Kraus, Lehmann, and Magidor [74].

◇ **Duplicate-removal** *If $\rho_1 = \sigma \cdot \langle i, c, \varphi \rangle$ and $\rho_2 = \rho_1 \cdot \langle i, c, \varphi \rangle$ then $B^{\rho_1} = B^{\rho_2}$ and $K^{\rho_1} = K^{\rho_2}$*

◇ **Conditional-consistency** *If K^σ is consistent then so is B^σ*

◇ **Inclusion-vacuity** $B^{\sigma \cdot \rho} \sqsubseteq \mathrm{Cn}(B^\sigma \sqcup K^\rho)$*, with equality if $B^\sigma \sqcup K^\rho$ is consistent*

◇ **Acyc** *If $\sigma_0, \ldots, \sigma_n$ are such that $K^{\sigma_j} \sqcup B^{\sigma_{j+1}}$ is consistent for all $0 \leq j < n$ and $K^{\sigma_n} \sqcup B^{\sigma_0}$ is consistent, then $K^{\sigma_0} \sqcup B^{\sigma_n}$ is consistent*

Inclusion-vacuity is so-named since it is analogous to the combination of *Inclusion* and *Vacuity* from AGM revision, if one informally views $B^{\sigma \cdot \rho}$ as the revision of B^σ by K^ρ. **Conditional-consistency** is another consistency postulate, which follows from **Consistency**, **Closure** and **Soundness**. **Acyc** is the analogue of the postulate of Delgrande, Peppas, and Woltran, which rules out cycles in the plausibility order constructed in the representation result.

As with the result of Delgrande, Peppas, and Woltran, a technical condition beyond Definition 1.4.1 is required to obtain the characterisation: say that a conditioning operator is *elementary* if for each σ the sets of worlds $\mathcal{X}_\sigma$ and $\mathcal{Y}_\sigma = \min_\leq \mathcal{X}_\sigma$ are elementary.[11]

[9] Π refines Π' if $\forall A \in \Pi$, $\exists B \in \Pi'$ such that $A \subseteq B$.

[10] Note that while the result is similar, our framework is not an instance of theirs.

[11] Equivalently, there is a total preorder $\leq$ such that $\mathrm{mod}(B^\sigma) = \min_\leq \mathrm{mod}(K^\sigma)$ for all σ.

Theorem 1.4.1. *Suppose an operator satisfies the basic postulates of Section 1.3.1.*[12] *Then it is an elementary conditioning operator if and only if it satisfies* ***Duplicate-removal****,* ***Conditional-consistency****,* ***Inclusion-vacuity*** *and* ***Acyc****.*

The proof is roughly follows the lines of Theorem 4.9 in [30], although some differences arise due to the form of our input as finite sequences of reports. First, we need a preliminary result.

Lemma 1.4.1. *For any model-based operator and sequence σ, $\mathcal{X}_\sigma = \mathrm{mod}(K^\sigma)$ iff $\mathcal{X}_\sigma$ is elementary, and $\mathcal{Y}_\sigma = \mathrm{mod}(B^\sigma)$ iff $\mathcal{Y}_\sigma$ is elementary.*

Proof. We prove the result for $\mathcal{X}_\sigma$ and K^σ only. The "only if" direction is clear from the definition of an elementary set. For the "if" direction, suppose $\mathcal{X}_\sigma$ is elementary, i.e. $\mathcal{X}_\sigma = \mathrm{mod}(G)$ for some collection G. Since $\Phi \in K^\sigma_c$ iff $\mathcal{X}_\sigma \subseteq \mathrm{mod}_c(\Phi)$, we have $K^\sigma_c = \mathrm{Cn}_c(G)$, i.e. $K^\sigma = \mathrm{Cn}(G)$. Consequently $\mathrm{mod}(K^\sigma) = \mathrm{mod}(\mathrm{Cn}(G)) = \mathrm{mod}(G) = \mathcal{X}_\sigma$. □

We will prove the following result – slightly more general than Theorem 1.4.1 – from which Theorem 1.4.1 immediately follows.

Proposition 1.4.1. *Suppose an operator satisfies* ***Closure****,* ***Containment****,* ***K-conjunction*** *and* ***Equivalence****. Then it is an elementary conditioning operator if and only if it satisfies* ***Rearrangement****,* ***Duplicate-removal****,* ***Conditional-consistency****,* ***Inclusion-vacuity*** *and* ***Acyc****.*

Proof. Take some operator $\sigma \mapsto \langle B^\sigma, K^\sigma \rangle$ satisfying **Closure**, **Containment**, **K-conjunction** and **Equivalence**.

"if": Suppose the operator in question additionally satisfies **Rearrangement**, **Duplicate-removal**, **Conditional-consistency**, **Inclusion-vacuity** and **Acyc**. For any σ, set

$$\mathcal{X}_\sigma = \mathrm{mod}(K^\sigma)$$

$$\mathcal{Y}_\sigma = \mathrm{mod}(B^\sigma)$$

Then – by **Closure** and **Containment** as shown in the proof of Theorem 1.3.1 – our operator is model based corresponding to this choice of $\mathcal{X}_\sigma$ and $\mathcal{Y}_\sigma$. Clearly both are elementary. We will construct a total preorder $\leq$ over $\mathcal{W}$ such that $\mathcal{Y}_\sigma = \min_\leq \mathcal{X}_\sigma$; this will show the operator is an elementary conditioning operator.

First, fix a function $c : \mathcal{L}_0/{\equiv} \to \mathcal{L}_0$ which chooses a fixed representative of each equivalence class of logically equivalent propositional formulas, i.e. any mapping such that $c([\varphi]_\equiv) \equiv \varphi$. To simplify notation, write $\widehat{\varphi}$ for $c([\varphi]_\equiv)$. Then $\varphi \equiv \widehat{\varphi}$. Write $\widehat{\mathcal{L}_0} = \{\widehat{\varphi} \mid \varphi \in \mathcal{L}_0\}$. Note that $\widehat{\mathcal{L}_0}$ is finite (since we work with only finitely

[12] Strictly speaking, we only need **Closure**, **Containment**, **K-conjunction**, **Equivalence** and **Rearrangement**.

many propositional variables) and every formula in $\mathcal{L}_0$ is equivalent to exactly one formula in $\widehat{\mathcal{L}_0}$. For a sequence σ, let $\widehat{\sigma}$ be the result of replacing each report $\langle i, c, \varphi \rangle$ with $\langle i, c, \widehat{\varphi} \rangle$. Note that by **Rearrangement** and **Equivalence**, $\mathcal{X}_{\widehat{\sigma}} = \mathcal{X}_\sigma$ and $\mathcal{Y}_{\widehat{\sigma}} = \mathcal{Y}_\sigma$.

Now, for any world W, set

$$\mathcal{R}(W) = \{\langle i, c, \varphi \rangle \in \mathcal{S} \times \mathcal{C} \times \widehat{\mathcal{L}_0} \mid W \in \mathcal{X}_{\langle i,c,\varphi \rangle}\}$$

Note that $\mathcal{R}(W)$ is finite. For any pair of worlds W_1, W_2, let $\rho(W_1, W_2)$ be some enumeration of $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$. We establish some useful properties of $\rho(W_1, W_2)$.

Claim 1.4.1. *If $\rho(W_1, W_2) \neq \emptyset$, $W_1, W_2 \in \mathcal{X}_{\rho(W_1,W_2)}$.*

Proof. By **K-conjunction**, for any sequences σ, ρ we have $K^{\sigma\cdot\rho} = \mathrm{Cn}(K^\sigma \sqcup K^\rho)$. Taking the models of both sides, we have $\mathcal{X}_{\sigma\cdot\rho} = \mathcal{X}_\sigma \cap \mathcal{X}_\rho$. It follows that for $\rho(W_1, W_2) \neq \emptyset$,

$$\mathcal{X}_{\rho(W_1,W_2)} = \bigcap_{\langle i,c,\varphi \rangle \in \rho(W_1,W_2)} \mathcal{X}_{\langle i,c,\varphi \rangle}$$

If $\langle i, c, \varphi \rangle \in \rho(W_1, W_2)$ then $W_1, W_2 \in \mathcal{X}_{\langle i,c,\varphi \rangle}$ by definition. Hence $W_1, W_2 \in \mathcal{X}_{\rho(W_1,W_2)}$. □

Claim 1.4.2. *If a sequence σ contains no equivalent reports (i.e. no distinct tuples $\langle i, c, \varphi \rangle$, $\langle i, c, \psi \rangle$ with $\varphi \equiv \psi$) and $W_1, W_2 \in \mathcal{X}_\sigma$, there is a sequence δ such that $W_1, W_2 \in \mathcal{X}_\delta$ and $\rho(W_1, W_2)$ is a permutation of $\widehat{\sigma} \cdot \delta$.*

Proof. If $\sigma = \emptyset$ then we can simply take $\delta = \rho(W_1, W_2)$. So suppose $\sigma \neq \emptyset$. By the same argument as in the proof of Claim 1.4.1, we have

$$\mathcal{X}_\sigma = \bigcap_{\langle i,c,\varphi \rangle \in \sigma} \mathcal{X}_{\langle i,c,\varphi \rangle}$$

Take any $\langle i, c, \varphi \rangle \in \widehat{\sigma}$. Then $\varphi \in \widehat{\mathcal{L}_0}$, and there is $\psi \equiv \varphi$ such that $\langle i, c, \psi \rangle \in \sigma$. By **Equivalence**, we have

$$W_1, W_2 \in \mathcal{X}_\sigma \subseteq \mathcal{X}_{\langle i,c,\psi \rangle} = \mathcal{X}_{\langle i,c,\varphi \rangle}$$

i.e. $\langle i, c, \varphi \rangle \in \mathcal{R}(W_1) \cap \mathcal{R}(W_2)$. Hence $\langle i, c, \varphi \rangle$ appears in $\rho(W_1, W_2)$. By the assumption that σ contains no equivalent reports, $\widehat{\sigma}$ contains no duplicates. It follows that $\rho(W_1, W_2)$ can be permuted so that $\widehat{\sigma}$ appears as a prefix. Taking δ to be the sequence that remains after $\widehat{\sigma}$ in this permutation, we clearly have that $\rho(W_1, W_2)$ is a permutation of $\widehat{\sigma} \cdot \delta$. Since $\sigma \neq \emptyset$ implies $\widehat{\sigma} \neq \emptyset$ and thus $\rho(W_1, W_2) \neq \emptyset$, by **Rearrangement**, **K-conjunction** and Claim 1.4.1 we get

$$W_1, W_2 \in \mathcal{X}_{\rho(W_1,W_2)} = \mathcal{X}_{\widehat{\sigma}\cdot\delta} = \mathcal{X}_{\widehat{\sigma}} \cap \mathcal{X}_\delta \subseteq \mathcal{X}_\delta$$

and we are done. □

Now define a relation R on $\mathcal{W}$ by

$$WRW' \iff W = W' \text{ or } W \in \mathcal{Y}_{\rho(W,W')}$$

We have that any world in $\mathcal{Y}_\sigma$ R-precedes all worlds $\mathcal{X}_\sigma$.

Claim 1.4.3. *If $W \in \mathcal{Y}_\sigma$, then for all $W' \in \mathcal{X}_\sigma$ we have WRW'*

Proof. By **Rearrangement**, **Equivalence** and **Duplicate-removal**, we may assume without loss of generality that σ contains no distinct equivalent reports.

Let $W \in \mathcal{Y}_\sigma$ and $W' \in \mathcal{X}_\sigma$. Then $W \in \mathcal{X}_\sigma$ too. By Claim 1.4.2 and **Rearrangement**, there is some sequence δ such that $\mathcal{Y}_{\rho(W,W')} = \mathcal{Y}_{\widehat{\sigma}\cdot\delta}$ and $W, W' \in \mathcal{X}_\delta$. Consequently $W \in \mathcal{Y}_\sigma \cap \mathcal{X}_\delta = \mathcal{Y}_{\widehat{\sigma}} \cap \mathcal{X}_\delta$. Thus $B^{\widehat{\sigma}} \sqcup K^\delta$ is consistent. From **Inclusion-vacuity** we get

$$\mathcal{Y}_{\widehat{\sigma}\cdot\delta} = \mathcal{Y}_{\widehat{\sigma}} \cap \mathcal{X}_\delta$$

Thus

$$W \in \mathcal{Y}_{\widehat{\sigma}} \cap \mathcal{X}_\delta = \mathcal{Y}_{\widehat{\sigma}\cdot\delta} = \mathcal{Y}_{\rho(W,W')}$$

so WRW' as required. □

Now let $\leq_0$ be the transitive closure of R. Then $\leq_0$ is a (partial) preorder. By Claim 1.4.3, every world in $\mathcal{Y}_\sigma$ is $\leq_0$-minimal in $\mathcal{X}_\sigma$. In fact, the converse is also true.

Claim 1.4.4. *If $W \in \mathcal{X}_\sigma$ and there is no $W' \in \mathcal{X}_\sigma$ with $W' <_0 W$, then $W \in \mathcal{Y}_\sigma$.*

Proof. As before, assume without loss of generality that σ contains no distinct equivalent reports.

Take W as in the statement of the claim. Then $\mathcal{X}_\sigma \neq \emptyset$, so $\mathcal{Y}_\sigma \neq \emptyset$ by **Conditional-consistency**. Let $W' \in \mathcal{Y}_\sigma$. By Claim 1.4.3, $W'RW$ and thus $W' \leq_0 W$. But by assumption, $W' \not<_0 W$. So we must have $W \leq_0 W'$. By definition of $\leq_0$ as the transitive closure of R, there are $W_0, \ldots W_n$ such that $W_0 = W$, $W_n = W'$ and

$$W_j R W_{j+1} \qquad (0 \leq j < n)$$

Without loss of generality, $n > 0$ and each of the W_j are distinct. From the definition of R, we therefore have that

$$W_j \in \rho(W_j, W_{j+1}) \qquad (0 \leq j < n)$$

Now set

$$\rho_j = \rho(W_j, W_{j+1}) \qquad (0 \leq j < n)$$
$$\rho_n = \rho(W_0, W_n)$$

Since $W'RW$, i.e. W_nRW_0, we in fact have $W_j \in \mathcal{Y}_{\rho_j}$ for all j (including $j = n$). For $j < n$, we also have $W_{j+1} \in \mathcal{X}_{\rho_j}$.[13] Consequently, for $j < n$ we have

$$W_{j+1} \in \mathcal{X}_{\rho_j} \cap \mathcal{Y}_{\rho_{j+1}}$$

i.e. $K^{\rho_j} \sqcup B^{\rho_{j+1}}$ is consistent. Moreover, $W_0 \in \mathcal{X}_{\rho_n} \cap \mathcal{Y}_{\rho_0}$, so $K^{\rho_n} \sqcup B^{\rho_0}$ is consistent. We can now apply **Acyc**: we get that $K^{\rho_0} \sqcup B^{\rho_n}$ is also consistent. On the one hand, **Inclusion-vacuity** and consistency of $K^{\rho_n} \sqcup B^{\rho_0}$ gives

$$B^{\rho_0 \cdot \rho_n} = \mathrm{Cn}(B^{\rho_0} \sqcup K^{\rho_n})$$

On the other, consistency of $B^{\rho_n} \sqcup K^{\rho_0}$ and **Rearrangement** gives

$$B^{\rho_0 \cdot \rho_n} = B^{\rho_n \cdot \rho_0} = \mathrm{Cn}(B^{\rho_n} \sqcup K^{\rho_0})$$

Combining these and taking models, we find

$$\mathcal{Y}_{\rho_0} \cap \mathcal{X}_{\rho_n} = \mathcal{Y}_{\rho_n} \cap \mathcal{X}_{\rho_0}$$

In particular, since W_0 lies in the set on the left-hand side, we have $W_0 \in \mathcal{Y}_{\rho_n}$.

Now, since $W_0, W_n \in \mathcal{X}_\sigma$ and $\rho_n = \rho(W_0, W_n)$, Claim 1.4.2 gives that there is δ with $W_0, W_n \in \mathcal{X}_\delta$ such that ρ_n is a permutation of $\widehat{\sigma} \cdot \delta$. Recalling that $W_n = W' \in \mathcal{Y}_\sigma = \mathcal{Y}_{\widehat{\sigma}}$ by assumption, we have $W_n \in \mathcal{Y}_{\widehat{\sigma}} \cap \mathcal{X}_\delta$, i.e. $B^{\widehat{\sigma}} \sqcup K^\delta$ is consistent. Applying **Inclusion-vacuity** once more, we get

$$B^{\rho_n} = B^{\widehat{\sigma} \cdot \delta} = \mathrm{Cn}(B^{\widehat{\sigma}} \sqcup K^\delta) = \mathrm{Cn}(B^\sigma \sqcup K^\delta)$$

Taking models of both sides,

$$\mathcal{Y}_{\rho_n} = \mathcal{Y}_\sigma \cap \mathcal{X}_\delta \subseteq \mathcal{Y}_\sigma$$

But we already saw that $W_0 \in \mathcal{Y}_{\rho_n}$. Hence $W_0 \in \mathcal{Y}_\sigma$. Since $W_0 = W$, we are done. $\square$

To complete the proof we extend $\leq_0$ to a *total* preorder and show that this does not affect the minimal elements of each $\mathcal{X}_\sigma$. Indeed, let $\leq$ be any total preorder extending $\leq_0$ and preserving strict inequalities, i.e. $\leq$ such that (i) $W \leq_0 W'$ implies $W \leq W'$; and (ii) $W <_0 W'$ implies $W < W'$.[14]

Claim 1.4.5. *For any sequence σ, $\mathcal{Y}_\sigma = \min_\leq \mathcal{X}_\sigma$*

[13] If $\rho_j \neq \emptyset$ this follows from Claim 1.4.1. Otherwise, $W_{j+1} \in \mathcal{Y}_{\rho_{j+1}} \subseteq \mathcal{X}_{\rho_{j+1}} = \mathcal{X}_{\rho_{j+1} \cdot \emptyset} = \mathcal{X}_{\rho_{j+1}} \cap \mathcal{X}_\emptyset \subseteq \mathcal{X}_\emptyset = \mathcal{X}_{\rho_j}$ by **K-conjunction**.

[14] Such $\leq$ always exists. Indeed, note that $\leq_0$ induces a partial order on the equivalence classes of $\mathcal{W}$ with respect to the symmetric part of $\leq_0$ given by $W \simeq_0 W'$ iff $W \leq_0 W'$ and $W' \leq_0 W$. This partial order can be extended to a linear order $\leq^*$ on the equivalence classes. Taking $W \leq W'$ iff $[W] \leq^* [W']$, we obtain a total preorder on $\mathcal{W}$ with the desired properties.

Proof. Take any σ. For the left-to-right inclusion, take $W \in \mathcal{Y}_\sigma$. Then $W \in \mathcal{X}_\sigma$. Let $W' \in \mathcal{X}_\sigma$. By Claim 1.4.3, WRW', so $W \leq_0 W'$ and $W \leq W'$. Hence W is $\leq$-minimal in $\mathcal{X}_\sigma$.

For the right-to-left inclusion, take $W \in \min_\leq \mathcal{X}_\sigma$. Then for any $W' \in \mathcal{X}_\sigma$ we have $W \leq W'$. In particular, $W' \not< W$. By property (ii) of $\leq$, we have $W' \not<_0 W$. Since W' was an arbitrary member of $\mathcal{X}_\sigma$ and $W \in \mathcal{X}_\sigma$, the conditions of Claim 1.4.4 are satisfied, and we get $W \in \mathcal{Y}_\sigma$. □

This shows that our operator is an elementary conditioning operator as required.

"only if": Now suppose the operator is an elementary conditioning operator. i.e. there is a total preorder $\leq$ on $\mathcal{W}$ and a mapping $\sigma \mapsto \langle \mathcal{X}_\sigma, \mathcal{Y}_\sigma \rangle$ such that for each σ, $\mathcal{Y}_\sigma = \min_\leq X_\sigma$, $\mathcal{X}_\sigma$ and $\mathcal{Y}_\sigma$ are elementary, and K^σ, B^σ are determined by $\mathcal{X}_\sigma$, $\mathcal{Y}_\sigma$ respectively according to Definition 1.3.1. By elementariness and Lemma 1.4.1, $\mathcal{X}_\sigma = \mathrm{mod}(K^\sigma)$ and $\mathcal{Y}_\sigma = \mathrm{mod}(B^\sigma)$.

The following claim will be useful at various points.

Claim 1.4.6. *Suppose σ and ρ are such that $\mathcal{X}_\sigma = \mathcal{X}_\rho$. Then $K^\sigma = K^\rho$ and $B^\sigma = B^\rho$.*

Proof. Since the total preorder $\leq$ is fixed, we have

$$Y_\sigma = \min_\leq \mathcal{X}_\sigma = \min_\leq \mathcal{X}_\rho = \mathcal{Y}_\rho$$

Now, $\mathcal{X}_\sigma = \mathcal{X}_\rho$ means $\mathrm{mod}(K^\sigma) = \mathrm{mod}(K^\rho)$, so $\mathrm{Cn}(K^\sigma) = \mathrm{Cn}(K^\rho)$. By **Closure**, $K^\sigma = K^\rho$. Similarly, $\mathcal{Y}_\sigma = \mathcal{Y}_\rho$ gives $B^\sigma = B^\rho$. □

We take the postulates to be shown in turn.

- **Rearrangement**: Suppose σ is a permutation of ρ. Without loss of generality, $\sigma, \rho \neq \emptyset$. Repeated application of **K-conjunction** gives

$$\mathcal{X}_\sigma = \bigcap_{\langle i,c,\varphi \rangle \in \sigma} \mathcal{X}_{\langle i,c,\varphi \rangle}$$

  Since σ and ρ contain exactly the same reports – just in a different order – commutativity and associativity of intersection of sets gives $\mathcal{X}_\sigma = \mathcal{X}_\rho$. **Rearrangement** follows from Claim 1.4.6.

- **Duplicate-removal**: Let σ, ρ_1 and ρ_2 be as in the statement of **Duplicate-removal**. Then by **K-conjunction**,

$$\begin{aligned}
\mathcal{X}_{\rho_2} &= \mathcal{X}_{\rho_1 \cdot \langle i,c,\varphi \rangle} \\
&= \mathcal{X}_{\rho_1} \cap \mathcal{X}_{\langle i,c,\varphi \rangle} \\
&= \mathcal{X}_{\sigma \cdot \langle i,c,\varphi \rangle} \cap \mathcal{X}_{\langle i,c,\varphi \rangle} \\
&= \mathcal{X}_\sigma \cap \mathcal{X}_{\langle i,c,\varphi \rangle} \cap \mathcal{X}_{\langle i,c,\varphi \rangle} \\
&= \mathcal{X}_\sigma \cap \mathcal{X}_{\langle i,c,\varphi \rangle} \\
&= \mathcal{X}_{\rho_1}
\end{aligned}$$

and we may conclude by Claim 1.4.6.

- **Conditional-consistency**: Suppose K^σ is consistent, i.e. $\mathcal{X}_\sigma \neq \emptyset$. Since $\mathcal{W}$ is finite, $\mathcal{X}_\sigma$ is finite and thus some $\leq$-minimal world must exist in $\mathcal{X}_\sigma$. Hence $\mathcal{Y}_\sigma \neq \emptyset$, so B^σ is consistent.

- **Inclusion-vacuity**: Take any sequences σ, ρ. First we show $B^{\sigma\cdot\rho} \sqsubseteq \mathrm{Cn}(B^\sigma \sqcup K^\rho)$, or equivalently, $\mathcal{Y}_{\sigma\cdot\rho} \supseteq \mathcal{Y}_\sigma \cap \mathcal{X}_\rho$. Suppose $W \in \mathcal{Y}_\sigma \cap \mathcal{X}_\rho$. Since $\mathcal{Y}_\sigma \subseteq \mathcal{X}_\sigma$, we have $W \in \mathcal{X}_\sigma \cap \mathcal{X}_\rho = \mathcal{X}_{\sigma\cdot\rho}$ by **K-conjunction**. We need to show W is minimal. Take any $W' \in \mathcal{X}_{\sigma\cdot\rho}$. Then $W' \in \mathcal{X}_\sigma$, so $W \in \mathcal{Y}_\sigma = \min_\leq \mathcal{X}_\sigma$ gives $W \leq W'$. Hence $W \in \min_\leq X_{\sigma\cdot\rho} = \mathcal{Y}_{\sigma\cdot\rho}$.

  Now suppose $B^\sigma \sqcup K^\rho$ is consistent, i.e. $\mathcal{Y}_\sigma \cap \mathcal{X}_\rho \neq \emptyset$. Take some $\widehat{W} \in \mathcal{Y}_\sigma \cap \mathcal{X}_\rho$. We need to show $B^{\sigma\cdot\rho} \sqsupseteq \mathrm{Cn}(B^\sigma \sqcup K^\rho)$, i.e. $\mathcal{Y}_{\sigma\cdot\rho} \subseteq \mathcal{Y}_\sigma \cap \mathcal{X}_\rho$. To that end, let $W \in \mathcal{Y}_{\sigma\cdot\rho}$. Then $W \in \mathcal{X}_{\sigma\cdot\rho} = \mathcal{X}_\sigma \cap \mathcal{X}_\rho \subseteq \mathcal{X}_\rho$, so we only need to show $W \in \mathcal{Y}_\sigma$. Take any $W' \in \mathcal{X}_\sigma$. Then $\widehat{W} \in \mathcal{Y}_\sigma$ gives $\widehat{W} \leq W'$. But $\widehat{W} \in \mathcal{X}_\sigma \cap \mathcal{X}_\rho = \mathcal{X}_{\sigma\cdot\rho}$ and $W \in \mathcal{Y}_{\sigma\cdot\rho}$ gives $W \leq \widehat{W}$. By transitivity of $\leq$, we have $W \leq W'$. Hence $W \in \min_\leq \mathcal{X}_\sigma = \mathcal{Y}_\sigma$.

- **Acyc**: Let $\sigma_0, \ldots, \sigma_n$ be as in the statement of **Acyc**. Without loss of generality, $n > 0$. Then there are $W_0, \ldots, W_n$ such that

  $$\begin{aligned} &W_j \in \mathcal{X}_{\sigma_j} \cap \mathcal{Y}_{\sigma_{j+1}} \qquad (0 \leq j < n) \\ &W_n \in \mathcal{X}_{\sigma_n} \cap \mathcal{Y}_{\sigma_0} \end{aligned}$$

  Note that $W_j \in \mathcal{X}_{\sigma_j}$ for all j. For $j < n$, we also have $W_j \in \mathcal{Y}_{\sigma_{j+1}} = \min_\leq X_{\sigma_{j+1}}$. It follows that $W_j \leq W_{j+1}$ for such j, so

  $$W_0 \leq \cdots \leq W_n$$

  But we also have $W_n \in \mathcal{Y}_{\sigma_0} = \min_\leq \mathcal{X}_{\sigma_0}$ and $W_0 \in \mathcal{X}_{\sigma_0}$, so $W_n \leq W_0$. By transitivity of $\leq$, the chain flattens: we have

  $$W_0 \simeq \cdots \simeq W_n$$

  Now note that since $W_{n-1} \in \mathcal{Y}_{\sigma_n}$, W_{n-1} is minimal in $\mathcal{X}_{\sigma_n}$. But $W_n \in \mathcal{X}_{\sigma_n}$ and $W_{n-1} \simeq W_n$ by the above, so in fact $W_n \in \mathcal{Y}_{\sigma_n}$ too. Hence

  $$\begin{aligned} W_n &\in \mathcal{Y}_{\sigma_0} \cap \mathcal{Y}_{\sigma_n} \\ &\subseteq \mathcal{X}_{\sigma_0} \cap \mathcal{Y}_{\sigma_n} \\ &= \mathrm{mod}(K^{\sigma_0} \sqcup B^{\sigma_n}) \end{aligned}$$

  i.e. $K^{\sigma_0} \sqcup B^{\sigma_n}$ is consistent, as required for **Acyc**.

$\square$

Note that while the requirement in Theorem 1.4.1 that $\mathcal{X}_\sigma$ and $\mathcal{Y}_\sigma$ are elementary is a technical condition,[15] the characterisation in Proposition 1.2.2 implies a simple sufficient condition for elementariness.

Proposition 1.4.2. *Suppose $\leq$ is such that $W \simeq W'$ whenever W and W' are partition-equivalent. Then $\min_{\leq} S$ is elementary for any elementary set $S \subseteq \mathcal{W}$.*

Proof. We use the characterisation of elementary sets from Proposition 1.2.2. Take $S \subseteq \mathcal{W}$ elementary. Suppose $W \in \mathcal{W}$, $W_1, W_2 \in \min_{\leq} S$ are such that W is partition-equivalent to both W_1, W_2 and W is a valuation combination from $\{W_1, W_2\}$. By hypothesis we have $W \simeq W_1 \simeq W_2$.

Now since $\min_{\leq} S \subseteq S$, we have $W_1, W_2 \in S$. Since S is elementary, $W \in S$. But now $W \simeq W_1$ and $W_1 \in \min_{\leq} S$ gives $W \in \min_{\leq} S$. This shows the required closure property for $\min_{\leq} S$, and we are done. $\square$

Proposition 1.4.2 implies that var-based-cond and part-based-cond are elementary. Indeed, for both operators $\mathcal{X}_\sigma = \mathrm{mod}(G^\sigma_{\mathsf{snd}})$ so is elementary by definition. Since the ranking $\leq$ for each operator only depends on the partitions of worlds, $\mathcal{Y}_\sigma = \min_{\leq} \mathcal{X}_\sigma$ is elementary also.

1.4.2 Score-Based Operators

The fact that the plausibility order $\leq$ of a conditioning operator is fixed may be too limiting. For example, consider

$$\sigma = (\langle i, c, p\rangle, \langle j, c, \neg p\rangle, \langle i, d, p\rangle).$$

If one sets $\mathcal{X}_\sigma$ to satisfy the soundness constraints (i.e. as in weak-mb), there is a possible world $W_1 \in \mathcal{X}_\sigma$ with $W_1, d \models \neg \mathsf{E}_i p \wedge \mathsf{E}_j p \wedge \neg p$ (i.e. W_1 sides with source j and p is false at d) and another world $W_2 \in \mathcal{X}_\sigma$ with $W_2, d \models \mathsf{E}_i p \wedge \neg \mathsf{E}_j p \wedge p$ (i.e. W_2 sides with source i). Appealing to symmetry, one may argue that neither world is *a priori* more plausible than the other, so any fixed plausibility order should have $W_1 \simeq W_2$. If these worlds are maximally plausible (e.g. if taking the "optimistic" view outlined in Example 1.3.1), conditioning gives $p \notin B^\sigma_d$ and $\neg p \notin B^\sigma_d$. However, there is an argument that W_2 should be considered more plausible than W_1 *given the sequence σ*, since W_2 validates the final report $\langle i, d, p\rangle$ whereas W_1 does not. Consequently, there is an argument that we should in fact have $p \in B^\sigma_d$.[16] This shows that we need the plausibility order to be responsive to the input sequence for adequate belief change.[17]

[15] **Inclusion-vacuity** may fail for non-elementary conditioning.

[16] At the very least, the case $p \in B^\sigma_d$ should not be *excluded*.

[17] In Section 1.5 we make this argument more precise by providing an impossibility result which shows conditioning operators with some basic properties cannot accept p in sequences such as this.

As a result of this discussion, we look for operators whose plausibility ordering can depend on σ. One approach to achieve this in a controlled way is to have a ranking for each *report* $\langle i, c, \varphi \rangle$, and combine these to construct a ranking for each sequence σ. We represent these rankings by *scoring functions*, and call the resulting operators *score-based*.

Definition 1.4.4. *An operator is* score-based *if there is a mapping $\sigma \mapsto \langle \mathcal{X}_\sigma, \mathcal{Y}_\sigma \rangle$ as in Definition 1.3.1 and functions $r_0 : \mathcal{W} \to \mathbb{N} \cup \{\infty\}$, $d : \mathcal{W} \times (\mathcal{S} \times \mathcal{C} \times \mathcal{L}_0) \to \mathbb{N} \cup \{\infty\}$ such that $\mathcal{X}_\sigma = \{W \mid r_\sigma(W) < \infty\}$ and $\mathcal{Y}_\sigma = \operatorname{argmin}_{W \in \mathcal{X}_\sigma} r_\sigma(W)$, where*

$$r_\sigma(W) = r_0(W) + \sum\nolimits_{\langle i,c,\varphi \rangle \in \sigma} d(W, \langle i, c, \varphi \rangle).$$

Here $r_0(W)$ is the *prior implausibility score* of W, and $d(W, \langle i, c, \varphi \rangle)$ is the *disagreement score* for world W and $\langle i, c, \varphi \rangle$. The set of most plausible worlds $\mathcal{Y}_\sigma$ consists of those W which minimise the sum of the prior implausibility and the total disagreement with σ. Note that by summing the scores of each report $\langle i, c, \varphi \rangle$ with equal weight, we treat each report independently. Score-based operators generalise elementary conditioning operators with **K-conjunction**.

Proposition 1.4.3. *Any elementary conditioning operator satisfying **K-conjunction** is score-based.*

Proof. Take any elementary conditioning operator corresponding to some mapping $\sigma \mapsto \langle \mathcal{X}_\sigma, \mathcal{Y}_\sigma \rangle$ and total preorder $\leq$, and suppose **K-conjunction** holds. Write

$$k(W) = |\{W' \in \mathcal{W} \mid W' \leq W\}|$$

Then we have $W \leq W'$ iff $k(W) \leq k(W')$. Set

$$r_0(W) = \begin{cases} \infty, & W \notin \mathcal{X}_\emptyset \\ k(W), & W \in \mathcal{X}_\emptyset \end{cases},$$

$$d(W, \langle i, c, \varphi \rangle) = \begin{cases} \infty, & W \notin \mathcal{X}_{\langle i,c,\varphi \rangle} \\ 0, & W \in \mathcal{X}_{\langle i,c,\varphi \rangle} \end{cases}.$$

For any sequence σ, repeated applications of **K-conjunction** (and the fact that $\mathcal{X}_\sigma$ is elementary) give $r_\sigma(W) < \infty$ iff $W \in \mathcal{X}_\sigma$. Similarly, the choice of r_0 gives $\operatorname{argmin}_{W \in \mathcal{X}_\sigma} r_\sigma(W) = \min_{\leq} \mathcal{X}_\sigma = \mathcal{Y}_\sigma$. Hence the operator is score-based.

□

We now give a concrete example.

Definition 1.4.5. *Define a score-based operator* excess-min *by setting $r_0(W) = 0$ and*

$$d(W, \langle i, c, \varphi \rangle) = \begin{cases} |\Pi_i^W[\varphi] \setminus \|\varphi\||, & W, c \models \mathsf{S}_i\varphi \\ \infty, & \textit{otherwise.} \end{cases}$$

The set of possible worlds $\mathcal{X}_\sigma$ is the same as for the earlier operators. All worlds are *a priori* equiplausible according to r_0. The disagreement score d is defined as the number of propositional valuations in the "excess" of $\Pi_i^W[\varphi]$ which are not models of φ, i.e. the number of $\neg\varphi$ valuations which are indistinguishable from some φ valuation. The intuition here is that *sources tend to only report formulas on which they have expertise*. The minimum score 0 is attained exactly when i has expertise on φ; other worlds are ordered by how much they deviate from this ideal.

One can verify that excess-min satisfies the basic postulates of Section 1.3.1. It can also be seen that $\mathcal{X}_\sigma$ and $\mathcal{Y}_\sigma$ are elementary, and excess-min fails **Duplicate-removal** and **Inclusion-vacuity**. It follows from Theorem 1.4.1 that excess-min is *not* a conditioning operator.[18]

Example 1.4.3. *To illustrate the differences between* excess-min *and conditioning, consider a more elaborate version of the example given at the start of this section:*

$$\sigma = (\langle i, c, p \to q\rangle, \langle j, c, p \to \neg q\rangle, \langle *, c, p\rangle, \langle i, d, p\rangle, \langle i, d, q\rangle).$$

Here the reports of i and j in case c are consistent, but inconsistent when taken with the reliable information p from $*$. Should we believe q or $\neg q$? Both our conditioning operators var-based-cond *and* part-based-cond *decline to decide, and have $[B_c^\sigma] = \mathrm{Cn}_0(p)$. However, since* excess-min *takes into account each report in the sequence, the fact that i reports both p and q in case d leads to $\mathsf{E}_i p \wedge \mathsf{E}_i q \in B_c^\sigma$. This gives $\mathsf{E}_i(p \to q) \in B_c^\sigma$ by Proposition 1.2.1 part (3), so we can make use of the report from i in case c: we have $[B_c^\sigma] = \mathrm{Cn}_0(p \wedge q)$. This example shows that score-based operators can be* more credulous *than conditioning operators (e.g. we can believe $\mathsf{E}_i p$ when i reports p), and can consequently hold stronger propositional beliefs.*

1.5 One-Step Revision

The postulates of Section 1.3.1 only set out very basic requirements for an operator. In this section we introduce some more demanding postulates which address how beliefs should change when a sequence σ is extended by a new report $\langle i, c, \varphi\rangle$. In view of **Rearrangement**, we do not view this process as *revision* of B^σ by $\langle i, c, \varphi\rangle$, but rather as *reinterpretation* of σ in light of a new report $\langle i, c, \varphi\rangle$. The postulates we introduce can therefore be seen as *coherency* requirements, which place some constraints on this reinterpretation.

First, we address how propositional beliefs should be affected by reliable information.

[18] We will later give an alternative proof of this fact, via an impossibility result for conditioning operators (Proposition 1.5.3).

◇ **AGM-$*$** *For any σ and $c \in \mathcal{C}$ there is an AGM operator $\star$ for $[B_c^\sigma]$ such that* $\left[B_c^{\sigma\cdot\langle *,c,\varphi\rangle}\right] = [B_c^\sigma] \star \varphi$ *whenever* $\neg\varphi \notin K_c^\sigma$

AGM-$*$ says that receiving information from the reliable source $*$ acts in accordance with the well-known AGM postulates [1] for propositional belief revision (provided we are not in the degenerate case where the new report φ was already *known* to be false). Since AGM revision operators are characterised by total preorders over valuations [56, 66], it is no surprise that our order-based constructions are consistent with **AGM-$*$**.

[TODO: flow in what follows.]

Proposition 1.5.1. var-based-cond, part-based-cond *and* excess-min *satisfy* ***AGM-$*$***.

We require some preliminary results. For a case $c \in \mathcal{C}$ and valuation $v \in \mathcal{V}$, write $\mathcal{W}_{c\,:\,v} = \{W \in \mathcal{W} \mid v_c^W = v\}$ for the set of worlds whose c valuation is v.

Lemma 1.5.1. *For any model-based operator, sequence σ, case c, and valuation v in$\mathcal{V}$,*

$$v \in \|[B_c^\sigma]\| \iff \mathcal{Y}_\sigma \cap \mathcal{W}_{c\,:\,v} \neq \emptyset$$

Proof. " $\implies$ ": We show the contrapositive. Suppose $\mathcal{Y}_\sigma \cap \mathcal{W}_{c\,:\,v} = \emptyset$. Let ψ be any propositional formula such that $\|\psi\| = \mathcal{V} \setminus \{v\}$. Now for any $W \in \mathcal{Y}_\sigma$, we have $W \notin \mathcal{W}_{c\,:\,v}$, i.e. $v_c^W \neq v$. Hence $v_c^W \in \|\psi\|$, so $W, c \models \psi$. By definition of the belief set of a model-based operator, we have $\psi \in B_c^\sigma$. But ψ is a propositional formula, so $\psi \in [B_c^\sigma]$. Since $v \notin \|\psi\|$, we have $v \notin \|[B_c^\sigma]\|$.

" $\impliedby$ ": Suppose there is some $W \in \mathcal{Y}_\sigma \cap \mathcal{W}_{c\,:\,v}$. Let $\varphi \in [B_c^\sigma]$. Then, in particular, $\varphi \in B_c^\sigma$, so $W, c \models \varphi$ by $W \in \mathcal{Y}_\sigma$ and the definition of the model-based belief set. That is, $v = v_c^W \in \|\varphi\|$. Since $\varphi \in [B_c^\sigma]$ was arbitrary, we have $v \in \|[B_c^\sigma]\|$. □

We have a sufficient condition for **AGM-$*$** for score-based operators.

Lemma 1.5.2. *Suppose a score-based operator is such that for each $c \in \mathcal{C}$ and $\varphi \in \mathcal{L}_0$ there is a constant $M \in \mathbb{N}$ with*

$$d(W, \langle *, c, \varphi\rangle) = \begin{cases} M, & W, c \models \varphi \\ \infty, & W, c \models \neg\varphi \end{cases}$$

for all W. Then ***AGM-$*$*** *holds.*

Proof. Take a score-based operator with the stated property. Let σ be a sequence and take $c \in \mathcal{C}$. Without loss of generality, there is some $\varphi \in \mathcal{L}_0$ such that $\neg\varphi \notin K_c^\sigma$ (otherwise **AGM-$*$** trivially holds). Since any score-based operator is model-based and therefore satisfies **Closure**, we have that K^σ is inconsistent iff $K_c^\sigma = \mathcal{L}$. But since K_c^σ does not contain $\neg\varphi$, it must be the case that K^σ is consistent.

Now, set

$$k(v) = \min\{r_\sigma(W) \mid W \in \mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,v}\}$$

where $\min \emptyset = \infty$. Note that $k(v) = \infty$ if and only if $\mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,v} = \emptyset$. Then k defines a total preorder $\preceq$ on valuations, where $v \preceq v'$ iff $k(v) \leq k(v')$. Define a propositional revision operator $\star$ for $[B_c^\sigma]$ by

$$[B_c^\sigma] \star \varphi = \{\psi \in \mathcal{L}_0 \mid \min_\preceq \|\varphi\| \subseteq \|\psi\|\}$$

To show that $\star$ satisfies the AGM postulates (for $[B_c^\sigma]$) it is sufficient to show that the models of $[B_c^\sigma]$ are exactly the $\preceq$-minimal valuations.

Claim 1.5.1. $\|[B_c^\sigma]\| = \min_\preceq \mathcal{V}$.

Proof. "$\subseteq$": let $v \in \|[B_c^\sigma]\|$. By Lemma 1.5.1, there is some $W \in \mathcal{Y}_\sigma \cap \mathcal{W}_{c\,:\,v}$. Since $W \in \mathcal{X}_\sigma$ too, by definition of k we have $k(v) \leq r_\sigma(W) < \infty$. Now let $v' \in \mathcal{V}$. Without loss of generality assume $k(v') < \infty$. Then there is some $W' \in \mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,v'}$ such that $k(v') = r_\sigma(W')$. But $W' \in \mathcal{X}_\sigma$ and $W \in \mathcal{Y}_\sigma$ gives $r_\sigma(W) \leq r_\sigma(W')$, so

$$k(v) \leq r_\sigma(W) \leq r_\sigma(W') = k(v')$$

i.e. $v \preceq v'$. Hence v is $\preceq$-minimal.

"$\supseteq$": let $v \in \min_\preceq \mathcal{V}$. Since K^σ is consistent, there is some $\hat{W} \in \mathcal{X}_\sigma$. Writing $\hat{v} = v_c^{\hat{W}}$, we have $\hat{W} \in \mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,\hat{v}}$, so $v \preceq \hat{v}$ implies

$$k(v) \leq k(\hat{v}) \leq r_\sigma(\hat{W}) < \infty$$

Hence there must be some $W \in \mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,v}$ such that $k(v) = r_\sigma(W)$. We claim that, in fact, $W \in \mathcal{Y}_\sigma$. Indeed, for any $W' \in \mathcal{X}_\sigma$ we have $v \preceq v_c^{W'}$, so

$$r_\sigma(W) = k(v) \leq k(v_c^{W'}) \leq r_\sigma(W')$$

That is, $W \in \mathcal{Y}_\sigma \cap \mathcal{W}_{c\,:\,v}$. By Lemma 1.5.1, $v \in \|[B_c^\sigma]\|$.

□

So, $\star$ is indeed an AGM operator for $[B_c^\sigma]$. Now take $\varphi \in \mathcal{L}_0$ such that $\neg\varphi \notin K_c^\sigma$. Write $\rho = \sigma \cdot \langle *, c, \varphi\rangle$. We claim the following.

Claim 1.5.2. $\|[B_c^\rho]\| = \min_\preceq \|\varphi\|$.

Proof. "$\subseteq$": let $v \in \|[B_c^\rho]\|$. By Lemma 1.5.1 again, there is some $W \in \mathcal{Y}_\rho \cap \mathcal{W}_{c\,:\,v}$. Since $\langle *, c, \varphi\rangle \in \rho$ and $d(W, \langle *, c, \varphi\rangle) \leq r_\rho(W) < \infty$, we must have $W, c \models \varphi$ by the assumed property of the score function d. Hence $v = v_c^W \in \|\varphi\|$.

Now since $\mathcal{Y}_\rho \subseteq \mathcal{X}_\rho$, we have $W \in \mathcal{Y}_\rho \subseteq \mathcal{X}_\rho \subseteq \mathcal{X}_\sigma$, so $W \in \mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,v}$. By definition of k, we have $k(v) \leq r_\sigma(W)$. Take any $v' \in \|\varphi\|$. Without loss of generality, assume $k(v') < \infty$, so that there is some $W' \in \mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,v'}$ with $k(v') = r_\sigma(W')$.

Since $v_c^{W'} = v' \in \|\varphi\|$, we have $W', c \models \varphi$. Consequently, by the property of d again, $d(W', \langle *, c, \varphi \rangle) = M$. Since $W' \in \mathcal{X}_\sigma$ gives $r_\sigma(W') < \infty$, it follows that

$$r_\rho(W') = r_\sigma(W') + M < \infty$$

so $W' \in \mathcal{X}_\rho$.

Recall that $W, c \models \varphi$ too, so $d(W, \langle *, c, \varphi \rangle) = M$ also. From $W \in \mathcal{Y}_\rho$ and $W' \in \mathcal{X}_\rho$ we get

$$\begin{aligned} r_\sigma(W) &= r_\rho(W) - M \\ &\leq r_\rho(W') - M \\ &= r_\rho(W') - d(W', \langle *, c, \varphi \rangle) \\ &= r_\sigma(W') \end{aligned}$$

This yields

$$k(v) \leq r_\sigma(W) \leq r_\sigma(W') = k(v')$$

and $v \preceq v'$ as required.

"$\supseteq$": let $v \in \min_\preceq \|\varphi\|$. Since $\neg\varphi \notin K_c^\sigma$, there is some $\hat{W} \in \mathcal{X}_\sigma$ such that $\hat{W}, c \models \varphi$. Writing $\hat{v} = v_c^{\hat{W}}$, we have $\hat{v} \in \|\varphi\|$. Hence $v \preceq \hat{v}$. This implies

$$k(v) \leq k(\hat{v}) \leq r_\sigma(\hat{W}) < \infty$$

so there must be some $W \in \mathcal{X}_\sigma \cap \mathcal{W}_{c\,:\,v}$ with $k(v) = r_\sigma(W)$. Since $v_c^W = v \in \|\varphi\|$, we have $W, c \models \varphi$. By the assumed property of d, we get $d(W, \langle *, c, \varphi \rangle) = M$. Hence

$$r_\rho(W) = r_\sigma(W) + d(W, \langle *, c, \varphi \rangle) = r_\sigma(W) + M < \infty$$

so $W \in \mathcal{X}_\rho$ too. We will show that $W \in \mathcal{Y}_\rho$. Let $W' \in \mathcal{X}_\rho$. Then we must have $d(W', \langle *, c, \varphi \rangle) = M$ and $W', c \models \varphi$. That is, $v_c^{W'} \in \|\varphi\|$. By minimality of v, we have $v \preceq v_c^{W'}$. Noting that $W' \in \mathcal{X}_\rho \subseteq \mathcal{X}_\sigma$, we get

$$r_\sigma(W) = k(v) \leq k(v_c^{W'}) \leq r_\sigma(W')$$

Consequently,

$$r_\rho(W) = r_\sigma(W) + M \leq r_\sigma(W') + M = r_\rho(W')$$

This shows $W \in \mathcal{Y}_\rho$, i.e. $\mathcal{Y}_\rho \cap \mathcal{W}_{c\,:\,v} \neq \emptyset$. By Lemma 1.5.1, we are done.

□

Noting that $\|[B_c^\sigma]\| \star \varphi = \min_\preceq \|\varphi\|$, it follows from Claim 1.5.2 that $\mathrm{Cn}_0([B_c^\rho]) = \mathrm{Cn}_0([B_c^\sigma] \star \varphi)$. But $[B_c^\rho]$ is deductively closed by **Closure**, and $[B_c^\sigma] \star \varphi$ is deductively closed by construction. Hence $[B_c^\rho] = [B_c^\sigma] \star \varphi$, as required for **AGM-$*$**.

□

As a consequence of Proposition 1.4.3 (and the construction of d in its proof), one can apply Lemma 1.5.2 with $M = 0$ for conditioning operators with **K-conjunction** and a certain natural property.

Corollary 1.5.1. *Suppose an elementary conditioning operator satisfying* ***K-conjunction*** *has the property that*

$$W \in \mathcal{X}_{\langle *,c,\varphi\rangle} \iff W, c \models \varphi$$

Then ***AGM-$*$*** *holds.*

We can now prove Proposition 1.5.1.

Proof of Proposition 1.5.1. For the conditioning operators var-based-cond and part-based-cond, it is easily verified that the condition in Corollary 1.5.1 holds, and thus **AGM-$*$** does also.

For the score-based operator excess-min, we may use Lemma 1.5.2 with $M = 0$. □

Thus, we do indeed extend AGM revision in the case of reliable information. What about non-reliable information? First note that the analogue of **AGM-$*$** for ordinary sources $i \neq *$ is *not* desirable. In particular, we should not have the **Success** postulate:

$$\varphi \in B_c^{\sigma\cdot\langle i,c,\varphi\rangle}.$$

Indeed, the sequence in Example 1.3.2 with $\varphi = \neg p \land q$ already shows that **Success** would conflict with the basic postulates. However, there are weaker modifications of **Success** which may be more appropriate. We consider two such postulates.

- ◇ **Cond-success** *If* $\mathsf{E}_i\varphi \in B_c^\sigma$ *and* $\neg\varphi \notin B_c^\sigma$*, then* $\varphi \in B_c^{\sigma\cdot\langle i,c,\varphi\rangle}$
- ◇ **Strong-cond-success** *If* $\neg(\mathsf{E}_i\varphi \land \varphi) \notin B_c^\sigma$*, then* $\varphi \in B_c^{\sigma\cdot\langle i,c,\varphi\rangle}$

Cond-success says that if i is deemed an expert on φ, which is consistent with current beliefs, then φ is accepted after i reports it. That is, the acceptance of φ is *conditional* on prior beliefs about the expertise of i (on φ). **Strong-cond-success** weakens the antecedent by only requiring that $\mathsf{E}_i\varphi$ and φ are jointly consistent with current beliefs (i.e. i need not be considered an expert on φ). In other words, we should believe reports if there is no reason not to. It is easily shown that **Closure** and **Strong-cond-success** implies **Cond-success**. We once again revisit our examples. [**TODO:** flow, again.]

Proposition 1.5.2. var-based-cond, part-based-cond *and* excess-min *satisfy* ***Cond-success****, and* excess-min *additionally satisfies* ***Strong-cond-success****.*

As a first step in the proof, we present sufficient conditions for conditioning operators to satisfy **Cond-success**. In fact, we do not need to impose any condition on the total preorder $\leq$: a natural constraint on the mapping $\sigma \mapsto \mathcal{X}_\sigma$ (together with some basic postulates) is enough.

Lemma 1.5.3. *Suppose an elementary conditioning operator satisfies **K-conjunction**, **Soundness** and*

$$W, c \models \varphi \implies W \in \mathcal{X}_{\langle i,c,\varphi \rangle}$$

*Then **Cond-success** holds.*

Proof. Suppose an elementary conditioning operator corresponding to the mapping $\sigma \mapsto \langle \mathcal{X}_\sigma, \mathcal{Y}_\sigma \rangle$ and total preorder $\leq$ satisfies **K-conjunction**, **Soundness** and has the stated property.

Let σ be a sequence and $c \in \mathcal{C}$. Suppose $\mathsf{E}_i\varphi \in B_c^\sigma$ and $\neg\varphi \notin B_c^\sigma$. Write $\rho = \sigma \cdot \langle i, c, \varphi \rangle$. We need to show $\varphi \in B_c^\rho$.

By $\neg\varphi \notin B_c^\sigma$, there is some $W \in \mathcal{Y}_\sigma$ such that $W, c \models \varphi$. Hence $W \in \mathcal{X}_{\langle i,c,\varphi \rangle}$. By elementariness and **K-conjunction**, we have $\mathcal{X}_\rho = \mathcal{X}_\sigma \cap \mathcal{X}_{\langle i,c,\varphi \rangle}$. Since $W \in \mathcal{Y}_\sigma \subseteq \mathcal{X}_\sigma$, we get $W \in \mathcal{X}_\rho$.

Now take any $W' \in \mathcal{Y}_\rho$. Then W' is $\leq$-minimal in $\mathcal{X}_\rho$, so $W' \leq W$. But W is $\leq$-minimal in $\mathcal{X}_\sigma$, so $W' \in \mathcal{Y}_\rho \subseteq \mathcal{X}_\rho \subseteq \mathcal{X}_\sigma$ gives $W' \in \mathcal{Y}_\sigma$ also. Consequently, $\mathsf{E}_i\varphi \in B_c^\sigma$ means $W', c \models \mathsf{E}_i\varphi$. On the other hand, **Soundness** together with $\langle i, c, \varphi \rangle \in \rho$ and $W' \in \mathcal{X}_\rho$ means $W', c \models \mathsf{S}_i\varphi$. Hence $W', c \models \mathsf{E}_i\varphi \wedge \mathsf{S}_i\varphi$. From Proposition 1.2.1 part (4), we get $W', c \models \varphi$.

We have shown that φ holds in case c at an arbitrary world in $\mathcal{Y}_\rho$. Hence $\varphi \in B_c^\rho$, as required.

□

Similarly, we have sufficient conditioning for score-based operators to satisfy **Strong-cond-success**: the postulate follows if worlds in which i makes a expert, truthful report are strictly more plausible than worlds in which i makes a false report.

Lemma 1.5.4. *Suppose a score-based operator is such that for any $i \in \mathcal{S}$, $c \in \mathcal{C}$, $\varphi \in \mathcal{L}_0$ and $W, W' \in \mathcal{W}$,*

$$W, c \models \mathsf{E}_i\varphi \wedge \varphi \text{ and } W', c \models \neg\varphi \implies d(W, \langle i, c, \varphi \rangle) < d(W', \langle i, c, \varphi \rangle)$$

*Then **Strong-cond-success** holds.*

Proof. Suppose a score-based operator has the stated property. Take σ such that $\neg(\mathsf{E}_i\varphi \wedge \varphi) \notin B_c^\sigma$. Write $\rho = \sigma \cdot \langle i, c, \varphi \rangle$. We need to show that $\varphi \in B_c^\rho$.

First note that by $\neg(\mathsf{E}_i\varphi \wedge \varphi) \notin B_c^\sigma$ and the definition of B^σ for score-based operators, there is $W \in \mathcal{Y}_\sigma$ such that $W, c \models \mathsf{E}_i\varphi \wedge \varphi$.

Take any $W' \in \mathcal{Y}_\rho$. Suppose, for the sake of contradiction, that $W', c \not\models \varphi$. Then by the hypothesised property of the score function d, we have

$$d(W, \langle i, c, \varphi\rangle) < d(W', \langle i, c, \varphi\rangle)$$

Now, $W \in \mathcal{Y}_\sigma$ and $W' \in \mathcal{Y}_\rho \subseteq \mathcal{X}_\rho \subseteq \mathcal{X}_\sigma$ gives $r_\sigma(W) \leq r_\sigma(W')$. Thus

$$\begin{aligned} r_\rho(W) &= r_\sigma(W) + d(W, \langle i, c, \varphi\rangle) \\ &\leq r_\sigma(W') + d(W, \langle i, c, \varphi\rangle) \\ &< r_\sigma(W') + d(W', \langle i, c, \varphi\rangle) \\ &= r_\rho(W') < \infty \end{aligned}$$

i.e. $r_\rho(W) < r_\rho(W') < \infty$. But this means $W \in \mathcal{X}_\rho$ and W' is not minimal in $\mathcal{X}_\rho$ under r_ρ, contradicting $W' \in \mathcal{Y}_\rho$. Hence $W', c \models \varphi$.

Since W' was an arbitrary member of $\mathcal{Y}_\rho$, we have shown $\varphi \in B_c^\rho$, and thus **Strong-cond-success** is shown. □

The main result now follows.

Proof of Proposition 1.5.2. For the conditioning operators var-based-cond and part-based-cond, **Cond-success** follows from Lemma 1.5.3 since $W, c \models \varphi$ implies $W, c \models \mathsf{S}_i\varphi$. For the score-based operator excess-min, one can easily check that the condition in Lemma 1.5.4 holds, and thus **Strong-cond-success** and **Cond-success** follow. □

By omission, the reader may suppose that the conditioning operators fail **Strong-cond-success**. This is correct, and we can in fact say even more: *no* conditioning operator with a few basic properties – all of which are satisfied by var-based-cond and part-based-cond – can satisfy **Strong-cond-success**. In what follows, for a permutation $\pi : \mathcal{S} \to \mathcal{S}$ with $\pi(*) = *$, write $\pi(W)$ for the world with $v_c^{\pi(W)} = v_c^W$ and $\Pi_i^{\pi(W)} = \Pi_{\pi(i)}^W$. We have an impossibility result.

Proposition 1.5.3. *No elementary conditioning operator satisfying the basic postulates can simultaneously satisfy the following properties:*

1. $K^\emptyset = \mathrm{Cn}(\emptyset)$
2. *If π is a permutation of $\mathcal{S}$ with $\pi(*) = *$, $W \simeq \pi(W)$*
3. ***Refinement***
4. ***Strong-cond-success***

However, any proper subset of (1) - (4) is satisfiable.

(1) says that before any reports are received, we only know tautologies. As remarked earlier, this is not an *essential* property, but is reasonable when no prior knowledge is available. (2) is an anonymity postulate: it says that permuting the "names" of sources does not affect the plausibility of a world, and is a desirable property in light of (1). **Refinement**, introduced in Section 1.4.1, says that worlds in which all sources have more expertise are preferred.

Proof. Take distinct sources $i_1, i_2 \in \mathcal{S} \setminus \{*\}$, distinct cases $c, d \in \mathcal{C}$, and distinct valuations $v_1, v_2 \in \mathcal{V}$. Let $\varphi_1, \varphi_2 \in \mathcal{L}_0$ be propositional formulas with $\|\varphi_k\| = v_k$ ($k \in \{1, 2\}$). Suppose for contradiction that some elementary conditioning operator – satisfying the basic postulates – has the stated properties.

Define a sequence

$$\sigma = (\langle *, c, \varphi_1 \vee \varphi_2 \rangle, \langle i_1, c, \varphi_1 \rangle, \langle i_2, c, \varphi_2 \rangle).$$

Let $\Pi_\perp$ denote the unit partition $\{\{u\} \mid u \in \mathcal{V}\}$, and let $\widehat{\Pi}$ denote the partition

$$\{\{v_1, v_2\}\} \cup \{\{u\} \mid u \in \mathcal{V} \setminus \{v_1, v_2\}\},$$

i.e. the partition obtained from $\Pi_\perp$ by merging the cells of v_1 and v_2.

Consider worlds W_1, W_2 given by

$$v_{c'}^{W_k} = v_k \qquad (c' \in \mathcal{C})$$

$$\Pi_i^{W_k} = \begin{cases} \widehat{\Pi}, & (k = 1 \text{ and } i = i_2) \text{ or } (k = 2 \text{ and } i = i_1) \\ \Pi_\perp, & \text{otherwise} \end{cases}$$

That is, W_1 has v_1 as its valuation for all cases, i_2 has partition $\widehat{\Pi}$, and all other sources have the finest partition $\Pi_\perp$; similarly W_2 has v_2 for its valuations and all sources except i_1 have $\Pi_\perp$.

Let $\leq$ denote the total preorder associated with the conditioning operator.

Claim 1.5.3. *$W_1 \simeq W_2$.*

Proof. Let π be the permutation of $\mathcal{S}$ which swaps i_1 and i_2. It is easily observed that $\pi(W_1)$ is partition-equivalent to W_2. By reflexivity of partition refinement, $\pi(W_1) \preceq W_2$ and $W_2 \preceq \pi(W_1)$. By **Refinement**, we get $\pi(W_1) \simeq W_2$. By property (2), $W_1 \simeq \pi(W_1)$. By transitivity of $\simeq$ we get $W_1 \simeq W_2$ as desired. ☐

Now, from the basic postulates, property (1) and Proposition 1.3.1 we have $K^\sigma = \mathrm{Cn}(G^\sigma_{\mathsf{snd}})$. By elementariness and Lemma 1.4.1, we get $\mathcal{X}_\sigma = \mathrm{mod}(K^\sigma) = \mathrm{mod}(G^\sigma_{\mathsf{snd}})$. It is easily checked that both W_1 and W_2 satisfy the soundness statements corresponding to σ, and thus $W_1, W_2 \in \mathrm{mod}(G^\sigma_{\mathsf{snd}}) = \mathcal{X}_\sigma$.

Claim 1.5.4. *$W_1, W_2 \in \mathcal{Y}_\sigma$.*

Proof. We show W_1 and W_2 are $\leq$-minimal in $\mathcal{X}_\sigma$. Take any $W \in \mathcal{X}_\sigma$. Then $W \in \mathrm{mod}(G^\sigma_{\mathsf{snd}})$, so $W, c \models \mathsf{S}_*(\varphi_1 \vee \varphi_2)$, i.e. $V^W_c \in \{v_1, v_2\}$. We consider two cases.

- **Case 1** ($v^W_c = v_1$). By $W \in \mathrm{mod}(G^\sigma_{\mathsf{snd}})$ again we have $W, c \models \mathsf{S}_{i_2}\varphi_2$, i.e

$$v_1 = v^W_c \in \Pi^W_{i_2}[\varphi_2] = \Pi^W_{i_2}[v_2].$$

  It follows that $\{v_1, v_2\} \subseteq \Pi^W_{i_2}[v_2]$, and that $\widehat{\Pi}$ refines $\Pi^W_{i_2}$. Since $\widehat{\Pi}$ is the partition of i_2 in W_1, and all other sources have the finest partition $\Pi_\perp$, we get $W_1 \preceq W$. By **Refinement**, $W_1 \leq W$. Since $W_1 \simeq W_2$ we have $W_2 \leq W$ also.

- **Case 2** ($v^W_c = v_2$). Applying a near-identical argument to that used in case 1 with soundness of the report $\langle i_1, c, \varphi_1\rangle$, we get $W_1, W_2 \leq W$.

In either case, both $W_1 \leq W$ and $W_2 \leq W$, so $W_1, W_2 \in \mathcal{Y}_\sigma$. □

Now we consider case d. Since

$$W_1, d \models \mathsf{E}_{i_1}\varphi_1 \wedge \varphi_1$$

and $W_1 \in \mathcal{Y}_\sigma$, $\neg(\mathsf{E}_{i_1}\varphi_1 \wedge \varphi_1) \notin B^\sigma_d$. Writing $\rho = \sigma \cdot \langle i_1, d, \varphi_1\rangle$, we get from **Strong-cond-success** that $\varphi_1 \in B^\rho_d$.

Note that $W_2, d \models \mathsf{S}_{i_1}\varphi_1$, so $W_2 \in \mathrm{mod}(G^\rho_{\mathsf{snd}}) = \mathrm{mod}(K^\rho) = \mathcal{X}_\rho$. Since W_2 is $\leq$-minimal in $\mathcal{X}_\sigma$ and

$$X_\rho = \mathrm{mod}(G^\rho_{\mathsf{snd}}) \subseteq \mathrm{mod}(G^\sigma_{\mathsf{snd}}) = \mathcal{X}_\sigma,$$

W_2 is also $\leq$-minimal in $\mathcal{X}_\rho$, i.e. $W_2 \in \mathcal{Y}_\rho$. Now $\varphi_1 \in B^\rho_d$ gives $W_2, d \models \varphi_1$. Since $v^{W_2}_d = v_2$ and $\|\varphi_1\| = \{v_1\}$, this means $v_1 = v_2$. But v_1 and v_2 were assumed to be distinct: contradiction.

[**TODO:** Show that any strict subset is satisfiable] □

Proposition 1.5.3 highlights an important difference between conditioning and score-based operators, and hints that a fixed plausibility order may be too restrictive: we need to allow the order to be responsive to new reports in order to satisfy properties such as **Strong-cond-success**.

1.6 Selective Change

In the previous section we saw how a single formula φ may be accepted when it is received as an additional report. But what can we say about propositional beliefs when taking into account the *whole sequence* σ? To investigate this we introduce an analogue of *selective revision* [45], in which propositional beliefs are formed by "selecting" part of each input report (intuitively, some part consistent with the source's expertise). In what follows, write $\sigma \upharpoonright c = \{\langle i, \varphi\rangle \mid \langle i, c, \varphi\rangle \in \sigma\}$ for the c-reports in σ.

Definition 1.6.1. *A* selection scheme *is a mapping f assigning to each $*$-consistent sequence σ a function $f_\sigma : \mathcal{S} \times \mathcal{C} \times \mathcal{L}_0 \to \mathcal{L}_0$ such that $f_\sigma(i, c, \varphi) \in \mathrm{Cn}_0(\varphi)$. An operator is* selective *if there is a selection scheme f such that for all $*$-consistent σ and $c \in \mathcal{C}$,*

$$[B_c^\sigma] = \mathrm{Cn}_0(\{f_\sigma(i, c, \varphi) \mid \langle i, \varphi\rangle \in \sigma \restriction c\}).$$

Thus, an operator is selective if its propositional beliefs in case c are formed by weakening each c-report and taking their consequences. Note that for $\sigma = \emptyset$ we get $[B_c^\sigma] = \mathrm{Cn}_0(\emptyset)$, so selectivity already rules out non-tautological prior propositional beliefs. Also note that in the presence of **Closure**, **Containment** and **Soundness**, selectivity implies that $[B_c^\sigma] = [B_c^\rho]$, where ρ is obtained by replacing each report $\langle i, c, \varphi\rangle$ with $\langle *, c, f_\sigma(i, c, \varphi)\rangle$.

Selectivity can be characterised by a natural postulate placing an upper bound on the propositional part of B_c^σ. In what follows, let $\Gamma_c^\sigma = \{\varphi \in \mathcal{L}_0 \mid \exists i \in \mathcal{S} : \langle i, \varphi\rangle \in \sigma \restriction c\}$.

$\diamond$ **Boundedness** *If σ is $*$-consistent, $[B_c^\sigma] \subseteq \mathrm{Cn}_0(\Gamma_c^\sigma)$*

Boundedness says that the propositional beliefs in case c should not go beyond the consequences of the formulas reported in case c. In some sense this can be seen as an iterated version of **Inclusion** from AGM revision, in the case where $[B_c^\sigma] = \mathrm{Cn}_0(\emptyset)$. We have the following characterisation.

Theorem 1.6.1. *A model-based operator is selective if and only if it satisfies **Boundedness**.*

Proof. "if": Suppose a model-based operator satisfies **Boundedness**. Take any $*$-consistent σ. For $c \in \mathcal{C}$, set

$$M_c = \|[B_c^\sigma]\|.$$

By **Boundedness**, we have $M_c \supseteq \|\Gamma_c^\sigma\|$. Now set

$$F_\sigma(i, c, \varphi) = \|\varphi\| \cup M_c.$$

Define a selection function f_σ by letting $f_\sigma(i, c, \varphi)$ be any formula with $\|f_\sigma(i, c, \varphi\|) = F_\sigma(i, c, \varphi)$. Since $F_\sigma(i, c, \varphi)$ contains the models of φ, clearly $f_\sigma(i, c, \varphi) \in \mathrm{Cn}_0(\varphi)$. Therefore f is indeed a selection function.

We claim that, for any $c \in \mathcal{C}$,

$$M_c = \bigcap_{\langle i, \varphi\rangle \in \sigma \restriction c} F_\sigma(i, c, \varphi).$$

The "$\subseteq$" inclusion is clear since, by definition, $F_\sigma(i, c, \varphi) \supseteq M_c$. For the "$\supseteq$" inclusion, suppose for contradiction that there is some $v \in \bigcap_{\langle i, \varphi\rangle \in \sigma \restriction c} F_\sigma(i, c, \varphi)$ with $v \notin M_c$.

Take any $\varphi \in \Gamma_c^\sigma$. Then there is $i \in \mathcal{S}$ such that $\langle i, \varphi \rangle \in \sigma \upharpoonright c$, and hence $v \in F_\sigma(i, c, \varphi)$. But $v \notin M_c$ by assumption, so $v \in \|\varphi\|$. This shows $v \in \|\Gamma_c^\sigma\|$. But $\|\Gamma_c^\sigma\| \subseteq M_c$ by **Boundedness**, so $v \in M_c$; contradiction.

From this we get

$$\begin{aligned} \|[B_c^\sigma]\| &= M_c \\ &= \bigcap_{\langle i,\varphi\rangle \in \sigma\upharpoonright c} F_\sigma(i, c, \varphi) \\ &= \bigcap_{\langle i,\varphi\rangle \in \sigma\upharpoonright c} \|f_\sigma(i, c, \varphi\|) \\ &= \|\{f_\sigma(i, c, \varphi) \mid \langle i, \varphi \rangle \in \sigma \upharpoonright c\}\| \end{aligned}$$

Since $[B_c^\sigma]$ is deductively closed (by **Closure**, which holds for all model-based operators), we get

$$[B_c^\sigma] = \mathrm{Cn}_0\left(\{f_\sigma(i, c, \varphi) \mid \langle i, \varphi \rangle \in \sigma \upharpoonright c\}\right)$$

as required for selectivity.

"only if": Suppose a model-based operator is selective according to some selection scheme f. Take any $*$-consistent σ and $c \in \mathcal{C}$. Write

$$\Delta = \{f_\sigma(i, c, \varphi) \mid \langle i, \varphi \rangle \in \sigma \upharpoonright c\}.$$

so that $[B_c^\sigma] = \mathrm{Cn}_0(\Delta)$. For $\langle i, \varphi \rangle \in \sigma \upharpoonright c$ we have $f_\sigma(i, c, \varphi) \in \mathrm{Cn}_0(\varphi) \subseteq \mathrm{Cn}_0(\Gamma_c^\sigma)$ from the definition of a selection scheme and the fact that $\varphi \in \Gamma_c^\sigma$. Hence $\Delta \subseteq \mathrm{Cn}_0(\Gamma_c^\sigma)$, so

$$[B_c^\sigma] = \mathrm{Cn}_0(\Delta) \subseteq \mathrm{Cn}_0(\mathrm{Cn}_0(\Gamma_c^\sigma)) = \mathrm{Cn}_0(\Gamma_c^\sigma)$$

as required for **Boundedness**. □

[TODO: flow.]

This characterisation in Theorem 1.6.1 allows us to easily analyse when conditioning and score-based operators are selective. In the case of conditioning operators with $K^\emptyset = \mathrm{Cn}(\emptyset)$, we in fact have a precise characterisation. First, some terminology: say that a world W *refines* W' *at* c if for all $i \in \mathcal{S}$ we have $\Pi_i^W[v_c^W] \subseteq \Pi_i^{W'}[v_c^{W'}]$. Intuitively, this means each source is more knowledgable in case c in world W than they are in W'. Write $\mathcal{W}_{c\,:\,v} = \{W \in \mathcal{W} \mid v_c^W = v\}$ for the set of worlds whose c valuation is v. We have the following.

Proposition 1.6.1. *Suppose an elementary conditioning operator satisfies the basic postulates and has $K^\emptyset = \mathrm{Cn}(\emptyset)$. Then it is selective if and only if for all W, c, v there is $W' \in \mathcal{W}_{c\,:\,v}$ such that $W' \leq W$ and W refines W' at all cases $d \neq c$.*

While the condition on $\leq$ in Proposition 1.6.1 is somewhat technical, it is implied by the very natural *partition-equivalence* property from Section 1.2. Consequently, var-based-cond and part-based-cond are selective. For the score-based operator excess-min, one can show **Boundedness** holds directly using a property of the disagreement scoring function d similar to the property of $\leq$ above. Consequently, excess-min is also selective.

We first state some preliminary results.

Lemma 1.6.1. *Suppose W refines W' at c. Then for any $i \in \mathcal{S}$ and $\varphi \in \mathcal{L}_0$,*

$$W, c \models \mathsf{S}_i\varphi \implies W', c \models \mathsf{S}_i\varphi$$

Proof. Suppose $W, c \models \mathsf{S}_i\varphi$. Then $v_c^W \in \Pi_i^W[\varphi]$, i.e. $\|\varphi\| \cap \Pi_i^W[v_c^W] \neq \emptyset$. By refinement, $\Pi_i^W[v_c^W] \subseteq \Pi_i^{W'}[v_c^{W'}]$. Hence $\|\varphi\| \cap \Pi_i^{W'}[v_c^{W'}] \neq \emptyset$, so $v_c^{W'} \in \Pi_i^{W'}[\varphi]$. That is, $W', c \models \mathsf{S}_i\varphi$. $\square$

Lemma 1.6.2. *For any $W \in \mathcal{W}$ and $c \in \mathcal{C}$, there is a $*$-consistent sequence σ – containing only reports for case c – such that for all $W' \in \mathcal{W}$,*

$$W' \in \mathrm{mod}(G^\sigma_{\mathsf{snd}}) \iff W \textit{ refines } W' \textit{ at } c.$$

Proof. For a valuation $v \in \mathcal{V}$, let $\varphi(v)$ be a propositional formula such that $\|\varphi(v\|) = \{v\}$. Take σ to be any enumeration of reports of the form

$$\langle i, c, \varphi(v) \rangle$$

where $i \in \mathcal{S}$ and $v \in \Pi_i^W[v_c^W]$. Note that such a sequence exists since there are only finitely many sources and valuations. Clearly σ contains only c-reports. Since Π_*^W is the unit partition, the only report from $*$ is $\langle *, c, \varphi(v_c^W) \rangle$. Hence σ is $*$-consistent. We show the desired equivalence.

$\implies$: Suppose $W' \in \mathrm{mod}(G^\sigma_{\mathsf{snd}})$. Take any $i \in \mathcal{S}$. We need to show $\Pi_i^W[v_c^W] \subseteq \Pi_i^{W'}[v_c^{W'}]$. Take $v \in \Pi_i^W[v_c^W]$. By construction of σ, $\langle i, c, \varphi(v) \rangle \in \sigma$. Hence $W', c \models \mathsf{S}_i\varphi(v)$, i.e. $v_c^{W'} \in \Pi_i^{W'}[\varphi(v)] = \Pi_i^{W'}[v]$. This shows $v \in \Pi_i^{W'}[v_c^{W'}]$ as required.

$\impliedby$: Suppose W refines W' at c. Take any $\langle i, c, \varphi(v) \rangle \in \sigma$. Then $v \in \Pi_i^W[v_c^W]$, so $v_c^W \in \Pi_i^W[v] = \Pi_i^W[\varphi(v)]$. This shows $W, c \models \mathsf{S}_i\varphi(v)$, and Lemma 1.6.1 gives $W', c \models \mathsf{S}_i\varphi(v)$. Hence $W' \in \mathrm{mod}(G^\sigma_{\mathsf{snd}})$.

$\square$

Proof of Proposition 1.6.1. Take an elementary conditioning operator with the basic postulates and $K^\emptyset = \mathrm{Cn}(\emptyset)$.

"if": Suppose the stated property holds. Since all conditioning operators are model-based, by Theorem 1.6.1 it suffices to show **Boundedness**. To that end, let σ be $*$-consistent and take $c \in \mathcal{C}$. We need $[B_c^\sigma] \subseteq \mathrm{Cn}_0(\Gamma_c^\sigma)$; or equivalently, by **Closure**, $\|[B_c^\sigma]\| \supseteq \|\Gamma_c^\sigma\|$.

Take any $v \in \|\Gamma^\sigma_c\|$. Since σ is $*$-consistent, B^σ is consistent by **Consistency**. Hence $\mathcal{Y}_\sigma \neq \emptyset$. Take any $W \in \mathcal{Y}_\sigma$. By the property in the statement of the result, there is $W' \in \mathcal{W}_{c\,:\,v}$ such that $W' \leq W$ and W refines W' at all cases $d \neq c$.

We claim $W' \in \mathcal{X}_\sigma$. By Proposition 1.3.1, elementariness and Lemma 1.4.1, we have $\mathcal{X}_\sigma = \mathrm{mod}(K^\sigma) = \mathrm{mod}(G^\sigma_{\mathsf{snd}})$. Take any $\langle i, d, \varphi \rangle \in \sigma$. We consider cases.

- **Case 1** ($d = c$). Here $\langle i, \varphi \rangle \in \sigma \restriction c$, so $\varphi \in \Gamma^\sigma_c$. Hence $v \in \|\Gamma^\sigma_c\| \subseteq \|\varphi\|$. Since $W' \in \mathcal{W}_{c\,:\,v}$, v is the c-valuation of W'. Hence $W', c \models \varphi$, and $W', c \models \mathsf{S}_i\varphi$ follows.

- **Case 2** ($d \neq c$). By assumption, W refines W' at d. Since $W \in \mathcal{Y}_\sigma \subseteq \mathcal{X}_\sigma$, we have $W, d \models \mathsf{S}_i\varphi$. By Lemma 1.6.1, $W', d \models \mathsf{S}_i\varphi$ also.

We have shown $W' \in \mathrm{mod}(G^\sigma_{\mathsf{snd}}) = \mathcal{X}_\sigma$. Now recall that $W \in \mathcal{Y}_\sigma$ – so W is $\leq$-minimal in $\mathcal{X}_\sigma$ – and $W' \leq W$. Thus W' is also $\leq$-minimal in $\mathcal{X}_\sigma$, i.e. $W' \in \mathcal{Y}_\sigma$. Since $W' \in \mathcal{W}_{c\,:\,v}$ also, we have by Lemma 1.5.1 that $v \in \|[B^\sigma_c]\|$, as required.

"only if": Suppose our operator is selective, i.e. satisfies **Boundedness**. To show the desired property holds, take any W, c and v. Enumerate $\mathcal{C} \setminus \{c\}$ as $\{d_1, \ldots, d_N\}$. By Lemma 1.6.2, for each $1 \leq n \leq N$ there is a $*$-consistent sequence σ_n such that

$$\mathrm{mod}(G^{\sigma_n}_{\mathsf{snd}}) = \{W' \in \mathcal{W} \mid W \text{ refines } W' \text{ at } d_n\}.$$

Now, let φ and ψ be formulas with $\|\varphi\| = \{v\}$ and $\|\psi\| = \{v^W_c\}$. Let ρ be the concatenation

$$\rho = \sigma_1 \cdots \sigma_n \cdot \langle *, c, \varphi \vee \psi \rangle.$$

Note that ρ is $*$-consistent, since each σ_n is (and only refers to case d_n). We may therefore apply **Boundedness** for case c. Taking models of both sides yields

$$\|[B^\rho_c]\| \supseteq \|\Gamma^\rho_c\| = \|\varphi \vee \psi\| = \{v, v^W_c\}.$$

In particular, $v \in \|[B^\rho_c]\|$. By Lemma 1.5.1, there is some $W' \in \mathcal{Y}_\rho \cap \mathcal{W}_{c\,:\,v}$.

We show W' has the required properties. First note that since W refines itself at each d_n, we have $W \in \mathrm{mod}(G^{\sigma_n}_{\mathsf{snd}})$. Clearly $W, c \models \psi$, so $W, c \models \mathsf{S}_*(\varphi \vee \psi)$ too. Thus $W \in \mathrm{mod}(G^\rho_{\mathsf{snd}}) = \mathcal{X}_\rho$ (using $K^\emptyset = \mathrm{Cn}(\emptyset)$). Since $W' \in \mathcal{Y}_\rho = \min_{\leq} \mathcal{X}_\rho$, we get $W' \leq W$ as required.

Next, take any case $d \neq c$. Then there is some n such that $d = d_n$. Since $W' \in \mathcal{Y}_\rho \subseteq \mathcal{X}_\rho = \mathrm{mod}(G^\rho_{\mathsf{snd}}) \subseteq \mathrm{mod}(G^{\sigma_n}_{\mathsf{snd}})$, we get that W refines W' at d. This completes the proof.

□

1.6.1 Case Independence

In the definition of a selection scheme, we allow $f_\sigma(i, c, \varphi)$ to depend on the case c. If one views $f_\sigma(i, c, \varphi)$ as a weakening of φ which accounts for the lack of expertise of i, this is somewhat at odds with other aspects of the framework, where expertise is independent of case. For this reason it is natural to consider *case independent* selective schemes.

Definition 1.6.2. *A selection scheme f is* case independent *if $f_\sigma(i, c, \varphi) \equiv f_\sigma(i, d, \varphi)$ for all $*$-consistent σ and $i \in \mathcal{S}$, $c, d \in \mathcal{C}$ and $\varphi \in \mathcal{L}_0$.*

Say an operator is *case-independent-selective* if it is selective according to some case independent scheme. This stronger notion of selectivity can again be characterised by a postulate which bounds propositional beliefs. For any set of cases $H \subseteq \mathcal{C}$, sequence σ and $c \in \mathcal{C}$, write

$$\begin{aligned}\Gamma_c^{\sigma,H} = \{\varphi \in \mathcal{L}_0 \mid \exists i \in \mathcal{S} : &\langle i, \varphi\rangle \in \sigma \restriction c \\ &\text{and } \forall d \in H : \langle i, \varphi\rangle \notin \sigma \restriction d\}.\end{aligned}$$

$\diamond$ **H-Boundedness** *For any $*$-consistent σ, $H \subseteq \mathcal{C}$ and $c \in \mathcal{C}$,*

$$[B_c^\sigma] \subseteq \mathrm{Cn}_0\left(\Gamma_c^{\sigma,H} \cup \bigcup_{d \in H} [B_d^\sigma]\right)$$

Note that **Boundedness** is obtained as the special case where $H = \emptyset$. We illustrate with an example.

Example 1.6.1. *Consider case c in the following sequence:*

$$\sigma = (\langle i, c, p\rangle, \langle j, c, q\rangle, \langle j, d, q\rangle, \langle k, d, r\rangle)$$

***Boundedness** requires that $[B_c^\sigma] \subseteq \mathrm{Cn}_0(\{p, q\})$. However, the instance of **H-Boundedness** with $H = \{d\}$ makes use of the fact that j reports q in both cases c and d, and requires $[B_c^\sigma] \subseteq \mathrm{Cn}_0(\{p\} \cup [B_d^\sigma])$. This also has an interesting implication for case d: if $\varphi \in [B_c^\sigma]$, then $p \to \varphi \in [B_d^\sigma]$. This follows since $\beta \in \mathrm{Cn}_0(\{\alpha\} \cup \Gamma)$ iff $\alpha \to \beta \in \mathrm{Cn}_0(\Gamma)$ for $\alpha, \beta \in \mathcal{L}_0$. Intuitively, this says that if p (from i) and q (from j) is enough to accept φ in case c, then φ is accepted in case d* if p is, *given that the report of q from j is repeated for d.*

The characterisation is as follows.

Theorem 1.6.2. *A model-based operator is case-independent-selective if and only if it satisfies **H-Boundedness**.*

Proof. "only if": Suppose a model-based operator is selective according to some case-independent scheme f. Take any $*$-consistent σ, $H \subseteq \mathcal{C}$ and $c \in \mathcal{C}$. For any case d, write $M_d = \|[B_d^\sigma]\|$. Note that with c_0 an arbitrary fixed case, and writing $F_\sigma(i,\varphi) = \|f_\sigma(i, c_0, \varphi\|)$, we have by case-independent-selectivity that

$$M_d = \bigcap_{\langle i,\varphi\rangle \in \sigma\restriction d} F_\sigma(i,\varphi).$$

By closure, it is sufficient for **H-Boundedness** to show that

$$M_c \supseteq \|\Gamma_c^{\sigma,H}\| \cap \bigcap_{d\in H} M_d. \tag{1.1}$$

Take any v in the set on the right-hand side. To show $v \in M_c$, take any $\langle i,\varphi\rangle \in \sigma \restriction c$. If $\varphi \in \Gamma_c^{\sigma,H}$, then clearly

$$\begin{aligned} v &\in \|\Gamma_c^{\sigma,H}\| \\ &\subseteq \|\varphi\| \\ &\subseteq \|f_\sigma(i,c,\varphi\|) \\ &= F_\sigma(i,\varphi) \end{aligned}$$

(where we use $f_\sigma(i,c,\varphi) \in \mathrm{Cn}_0(\varphi)$). Otherwise, $\varphi \notin \Gamma_c^{\sigma,H}$. Since $\langle i,\varphi\rangle \in \sigma \restriction c$, this means there is $d \in H$ such that $\langle i,\varphi\rangle \in \sigma \restriction d$. Hence $v \in M_d$ gives $v \in F_\sigma(i,\varphi)$. This shows the inclusion in (1.1), and we are done.

"if": Suppose a model-based operator satisfies **H-Boundedness**. Let σ be a $*$-consistent sequence. As before, write M_c for $\|[B_c^\sigma]\|$. For $i \in \mathcal{S}$ and $c \in \mathcal{C}$, write

$$\mathcal{C}(i,\varphi) = \{c \in \mathcal{C} \mid \langle i,\varphi\rangle \in \sigma \restriction c\},$$

and set

$$F_\sigma(i,\varphi) = \|\varphi\| \cup \bigcup_{c\in\mathcal{C}(i,\varphi)} M_c.$$

Define f by letting $f_\sigma(i,c,\varphi)$ be any propositional formula with $\|f_\sigma(i,c,\varphi\|) = F_\sigma(i,\varphi)$. Then f is a case-independent selection scheme. We show our operator is selective according to f; by closure of $[B_c^\sigma]$ for each c, it suffices to show

$$M_c = \bigcap_{\langle i,\varphi\rangle \in \sigma\restriction c} F_\sigma(i,\varphi).$$

Fix c. For the left-to-right inclusion, suppose $v \in M_c$. Take any $\langle i,\varphi\rangle \in \sigma \restriction c$. Then $c \in \mathcal{C}(i,\varphi)$, so $F_\sigma(i,\varphi) \supseteq M_c$ and thus $v \in F_\sigma(i,\varphi)$ as required.

For the right-to-left inclusion, suppose v lies in the intersection. Set

$$H = \{d \in \mathcal{C} \mid v \in M_d\}.$$

Apply **H-Boundedness** and taking the models of both sides, we obtain

$$M_c \supseteq \|\Gamma_c^{\sigma,H}\| \cap \bigcap_{d\in H} M_d. \tag{1.2}$$

Clearly $v \in \bigcap_{d \in H} M_d$ by definition of H. Let $\varphi \in \Gamma_c^{\sigma,H}$. Then there is $i \in \mathcal{S}$ such that $\langle i, \varphi \rangle \in \sigma \restriction c$, and consequently $v \in F_\sigma(i, \varphi)$. We claim $v \in \|\varphi\|$. If not, by definition of $F_\sigma(i, \varphi)$ we must have $v \in \bigcup_{d \in \mathcal{C}(i,\varphi)} M_d$, i.e. there is $d \in \mathcal{C}$ such that $\langle i, \varphi \rangle \in \sigma \restriction d$ and $v \in M_d$. On the one hand, $\varphi \in \Gamma_c^{\sigma,H}$ implies $d \notin H$. On the other, $v \in M_d$ gives $d \in H$ directly by the definition of H: contradiction. This shows $v \in \|\varphi\|$. Since φ was arbitrary, we have $v \in \|\Gamma_c^{\sigma,H}\|$. By (1.2) we get $v \in M_c$, and the proof is complete.

□

The question of whether our concrete operators satisfy **H-Boundedness** (equivalently, whether they are case-independent-selective) is still open.

1.6.2 Expertise and Selectivity

In the existing literature on selective belief change (e.g. [45, 16]), the selection function typically acts as a means to separate out the part of new information on which the reporting sources is *credible*, or *trusted*. In our framework, this property of the selection scheme f can be captured as follows.

Definition 1.6.3. *A selection scheme f is* expertise-compatible *(EC) with an operator $\sigma \mapsto \langle B^\sigma, K^\sigma \rangle$ if for all $*$-consistent σ and $\langle i, c, \varphi \rangle \in \sigma$,*

$$\mathsf{E}_i f_\sigma(i, c, \varphi) \in B_c^\sigma$$

Say an operator is *EC-selective* if it is selective according to some expertise-compatible scheme. While EC-selectivity may appear natural on first glance, we argue that it can be overly restrictive when expertise is derived from the input sequence itself. For example, consider the sequence

$$\sigma = (\langle i, c, p \rangle, \langle j, c, p \rangle, \langle i, d, p \rangle, \langle i, d, \neg p \rangle)$$

By **Soundness** and **Closure**, we cannot have both $\mathsf{E}_i p$ and $\mathsf{E}_j p$ in B_c^σ. Ideas of symmetry (**[TODO:** check for typos: there is no symmetry in the sequence as written]) suggest that neither can we pick one of i or j over the other, so that in fact it is reasonable to have neither $\mathsf{E}_i p$ nor $\mathsf{E}_j p$ in B_c^σ. Consequently – assuming p is the only propositional variable – the only formulas weaker than p on which i and j are believed to have expertise are tautologies. Any EC scheme f must therefore have $f_\sigma(i, c, p) \equiv f_\sigma(j, c, p) \equiv \top$. Consequently, EC-selectivity would imply $[B_c^\sigma] = \mathrm{Cn}_0(\top)$. This is a very conservative stance: while there is total consensus for p in case c, p cannot be believed due to disagreement elsewhere. This also conflicts with the "optimistic" attitude described in Example 1.3.1. According to that view we should have $\mathsf{E}_i p \vee \mathsf{E}_j p \in B_c^\sigma$, but this implies $p \in B_c^\sigma$ by **Soundness**, **Containment** and **Closure**.

The core issue here is that for $\mathsf{E}_i\varphi$ to be believed, i needs to be trusted on φ in *every* maximally plausible world. If these worlds look very different – e.g. W_1 trusts i but not j, and vice versa in W_2 – then EC-selectivity requires reports to be significantly weakened before expertise is believed in all worlds.

This discussion also hints that propositional beliefs for EC-selective operators are fully determined by the expertise part of B^σ, together with the reports in σ. If the knowledge K^σ is formed according to the soundness constraints of σ (as in our examples so far), we in fact have a characterisation result to this effect. The following lemma, which may also be of independent interest, is required. For a set of worlds $S \subseteq \mathcal{W}$, write $\Pi_i^S = \bigvee_{W \in S} \Pi_i^W$ for the join of the i-partitions of worlds in S.

Lemma 1.6.3. *If a model-based operator is EC-selective and satisfies **Soundness**, then*

$$\|[B_c^\sigma]\| = \bigcap_{\langle i,\varphi\rangle \in \sigma \upharpoonright c} \Pi_i^{\mathcal{Y}_\sigma}[\varphi]$$

for all $*$-consistent σ and $c \in \mathcal{C}$.

[TODO: Write up from online notes.]

Now write G_σ^{snd} for the collection with $(G_\sigma^{\mathsf{snd}})_c = \{\mathsf{S}_i\varphi \mid \langle i,\varphi\rangle \in \sigma \upharpoonright c\}$. For any collection G, write $E(G)$ for the sub-collection of formulas of the form $\mathsf{E}_i\varphi$.

Theorem 1.6.3. *Suppose a model-based operator satisfies **Consistency** and $K^\sigma = \mathrm{Cn}(G_\sigma^{\mathsf{snd}})$ for all σ. Then it is EC-selective if and only if*

$$[B_c^\sigma] = [\mathrm{Cn}_c(K^\sigma \sqcup E(B^\sigma))]$$

for all $*$-consistent σ and $c \in \mathcal{C}$.

[TODO: Write up from online notes.]

Finally, note that Lemma 1.6.3 immediately implies selectivity with respect to any scheme f such that $\|f_\sigma(i,c,\varphi)\| = \Pi_i^{\mathcal{Y}_\sigma}[\varphi]$. Since the right-hand side does not depend on the case c, we get the following corollary.

Corollary 1.6.1. *If a model-based operator is EC-selective and satisfies **Soundness**, then it is case-independent-selective.*

1.7 Related Work

In this section we discuss related work.

Belief Merging. In the framework of Konieczny and Pérez [72], a merging operator Δ maps a multiset of propositional formulas $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ and an integrity constraint μ to a formula $\Delta_\mu(\Phi)$. Here φ_i represents the input from source i, and $\Delta_\mu(\Phi)$ represents the merged result. Various operators and postulates have been proposed in the literature; see [71] for a review.

This can be seen as the special case of our framework with a single case c: for Φ, μ we consider the sequence $\sigma_{\Phi,\mu}$ where $*$ reports μ and each source i reports φ_i. Any operator then gives rise to a merging operator $\Delta_\mu(\Phi) = \bigwedge \left[B_c^{\sigma_{\Phi,\mu}}\right]$. Note that our basic postulates imply $\Delta_\mu(\Phi) \vdash \mu$ – the **IC0** postulate of Konieczny and Pérez [72]. We leave it to future work to determine which other merging postulates hold.

We go beyond this setting by considering multiple cases and explicitly modelling expertise (and trust, via beliefs about expertise). While it may be possible to model expertise *implicitly* in belief merging (for example, say i is not trusted on ψ if $\Delta_\mu(\Phi) \not\vdash \psi$ when $\varphi_i \vdash \psi$), bringing expertise to the object level allows us to express more complex beliefs about expertise, such as $\mathsf{E}_a x \vee \mathsf{E}_b x$ in Example 1.3.1. It also facilitates postulates which refer directly to expertise, such as the weakenings of **Success** in Section 1.5.

However, our problem is more specialised than merging, since we focus specifically on conflicting information due to lack of expertise. Belief merging may be applied more broadly to other types of *information fusion*, e.g. subjective beliefs or goals [54], where notions of objective expertise do not apply. While our framework *could* be applied in these settings, our postulates may no longer be desirable.

Epistemic Logic. Our notions of expertise and soundness are related to *S5 knowledge* from epistemic logic [107]. In such logics, an agent *knows* φ at a state x if φ holds at all states y "accessible" from x. Knowledge is thus determined by an *epistemic accessibility relation*, which describes the distinctions between states the agent can make. The logic of S5 arises when this relation is an equivalence relation (or equivalently, a partition).

Our previous work [95] – in which expertise and soundness were introduced in a modal logic framework – showed that "expertise models" are in 1-to-1 correspondence with S5 models, such that $\mathsf{E}\varphi$ holds iff $\mathsf{A}(\varphi \to \mathsf{K}\varphi)$ holds in the S5 model, where A is the universal modality. By symmetry of expertise, we can also replace φ with its negation. Thus, expertise has a precise epistemic interpretation: it is the ability to *know whether* φ holds in *any possible state*. Similarly, $\mathsf{S}\varphi$ translates to $\neg\mathsf{K}\neg\varphi$. That is, φ is sound exactly when the source does not *know* φ is false.

In the present framework, if we set $W, c \models \mathsf{K}_i\varphi$ iff $\Pi_i[v_c] \subseteq \|\varphi\|$ and $W, c \models \mathsf{A}\Phi$ iff $\forall v : \; W_{c=v}, c \models \Phi$, where $W_{c=v}$ is the world obtained from W by setting $v'_c = v$, then we have $\mathsf{E}_i\varphi \equiv \mathsf{A}(\varphi \to \mathsf{K}_i\varphi)$ and $\mathsf{S}_i\varphi \equiv \neg\mathsf{K}_i(\neg\varphi)$. While K_i is not quite an S5 modality (the **5** axiom requires iterating K_i, which is not possible in our framework), this shows the fundamental link between expertise, soundness and knowledge.

1.8 Conclusion

Summary. In this chapter we studied a belief change problem – extending the classical AGM framework – in which beliefs about the state of the world in multiple cases, as well as expertise of multiple sources, must be inferred from a sequence of reports. This allowed us to take a fresh look at the interaction between trust (seen as *belief in expertise*) and belief. By inferring the expertise of the sources from the reports, we have generalised some earlier approaches to non-prioritised revision which assume expertise (or reliability, credibility, priority etc) is known up-front (e.g. [45, 59, 16, 29]). We went on to propose some concrete belief change operators, and explored their properties through examples and postulates.

We saw that conditioning operators satisfy some desirable properties, and our concrete instances make useful inferences that go beyond weak-mb. However, we have examples in which intuitively plausible inferences are blocked, and conditioning is largely incompatible with **Strong-cond-success**. Score-based operators, and in particular excess-min, offer a way around these limitations, but may come at the expense of some other seemingly reasonable postulates, such as **Duplicate-removal**.

Future Work. There are many possibilities for future work. Firstly, we have a representation result only for conditioning operators. A characterisation of score-based operators – either the class in general or the specific operator excess-min – remains to be found. This would help to further clarify the differences between conditioning and score-based operators. We have also not considered any computational issues. Determining the complexity of calculating the results of our example operators, and the complexity for conditioning and score-based operators more broadly, is left to future work. Secondly, there is scope for deeper postulate-based analysis. For example, there should be postulates governing how beliefs change in case c in response to reports in case d. We could also consider more postulates relating trust and belief, and compare these postulates with those of Yasser and Ismail [118]. Moreover, there are many weaker version of **Success** which have been considered in the literature (e.g. in [45, 59, 16]); we should compare these against our **Cond-success** and **Strong-cond-success** in future work.

Finally, our framework only deals with three levels of trust on a proposition: we can believe $\mathsf{E}_i\varphi$, believe $\neg\mathsf{E}_i\varphi$, or neither. Future work could investigate how to extend our semantics to talk about *graded expertise*, and thereby permit more fine-grained *degrees of trust* [63, 118, 29].

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