Handling Unreliable Information

A thesis submitted in partial fulfilment of the requirement for the degree of Doctor of Philosophy

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Abstract

This is a short summary of the thesis. What a great thesis it's going to be.

Contents

Ackowledgements

I would like to thank Bear for being a dog. Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

List of Publications

The content of this thesis is derived from the following publications.

- Joseph Singleton and Richard Booth. *Who's the Expert? On Multi-source Belief Change*. 2022. DOI: 10.48550/ARXIV.2205.00077. URL: https://arxiv.org/abs/2205.00077

  This work appears in some chapter later on.

- And another

Part I

Introduction and Motivation

1 Introduction

- Overall theme: how should we deal with unreliable information?
- We want to:
  - Aggregate conflicting reports (crowdsourcing, news)
  - Assess the trustworthiness of information sources
  - Understand what reliability, trustworthiness and expertise *mean*
  - Find the truth with imperfect information
- This thesis offers two main perspectives on these general themes
  - **Social choice theory.**
    * By posing the aggregation problem as one of social choice, we can apply the axiomatic method to investigate desirable properties of aggregation methods. We can then analyse and evaluate such methods in a formal and principled way.
    * Related ranking problems can be addressed through the lens of social choice.
  - **Logic and knowledge representation.**
    * We develop a logical system to formalise notions of expertise, and explore connections with knowledge and information.
    * We use these formal notions to express the aggregation problem in logical terms, taking an alternative look at the problems of the first part of the thesis. We use what is essentially still an axiomatic approach, but now in the tradition of knowledge representation and rational belief change.
    * This logical model is well-suited to investigate *truth-tracking*: the question of when we can find the truth given that not all sources are experts.
- Note that while there are many links between the two major parts, they are not tightly connected and may be read independently.

2 Thesis Outline

Part II

Social Choice Perspectives

3 Introduction

- Describe what we mean by social choice?
- Overview of how our stuff will relate to the COMSOC literature?

4 Truth Discovery

4.1 Introduction

There is an increasing amount of data available in today's world, particularly from the web, social media platforms and crowdsourcing systems. The openness of such platforms makes it simple for a wide range of users to share information quickly and easily, potentially reaching a wide international audience. It is inevitable that amongst this abundance of data there are *conflicts*, where data sources disagree on the truth regarding a particular object or entity. For example, low-quality sources may mistakenly provide erroneous data for topics on which they lack expertise.

Resolving such conflicts and determining the true facts is therefore an important task. Truth discovery has emerged as a set of techniques to achieve this by considering the *trustworthiness* of sources [42, 33, 8]. The general principle is that true facts are those claimed by trustworthy sources, and trustworthy sources are those that claim believable facts. Application areas include real-time traffic navigation [23], drug side-effect discovery [45], crowdsourcing and social sensing [70, 60, 44].

For a simple example of a situation where trust can play an important role in conflict resolution, consider the following example.

Example 1. *Let o and p represent two images for which crowdsourcing workers are asked to provide labels (in the truth discovery terminology, o and p are called* objects*). Consider workers (the data sources) s, t, u and v who put forward potential labels f, g for o, and h, i for p, as shown in Fig. 4.1; such potential answers are termed* facts*. In the graphical representation, sources, facts and objects are shown from left to right, and the edges indicate claims made by sources and the objects to which facts relate.*

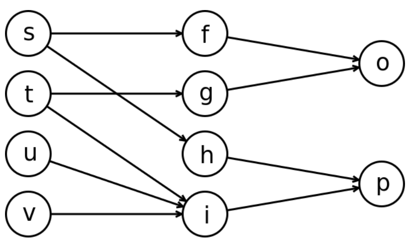

Figure 4.1: Illustrative example of a dataset to which truth discovery can be applied with data sources $\{s, t, u, v\}$, facts $\{f, g, h, i\}$ and objects $\{o, p\}$

Without considering trust information, the label for o appears a tie, with both options f and g receiving one vote from sources s and t respectively.

Taking a trust-aware approach, however, we can look beyond object o to consider the trustworthiness *of s and t. Indeed, when it comes to object p, t agrees with two extra sources u and v, whereas s disagrees with everyone. In principle there could be* hundreds *of extra sources here instead of just two, in which case the effect would be even more striking. We may conclude that s is* less trustworthy *than t. Returning to o, we see that g is supported by a more trustworthy source, and conclude that it should be accepted over f.*

Many truth discovery algorithms have been proposed in the literature with a wide range of techniques used, e.g. iterative heuristic-based methods [52, 28], probabilistic models [68], maximum likelihood estimation and optimisation-based methods [43], and neural network models [39, 46, 61]. It is common for such algorithms to be evaluated empirically by running them against real-world or synthetic datasets for which the true facts are already known; this allows *accuracy* and other metrics to be calculated, and permits comparison between algorithms (see [59] for a systematic empirical evaluation of this kind). This may be accompanied by some theoretical analysis, such as calculating run-time complexity [33], proving convergence of an iterative algorithm [69], or proving convergence to the 'true' facts under certain assumptions on the distribution of source trustworthiness [64, 63, 31].

A limitation of this kind of analysis is that the results only apply narrowly to particular algorithms, due to the assumptions made (for instance, that claims from sources follow a particular probability distribution). Such assumptions can be problematic in domains where the desired truth is somewhat 'fuzzy'; for example, image classification problems and determining the copyright status of books.[1]

In this work we take first steps towards a more general approach, in which we aim to study truth discovery without reference to any specific methodology or probabilistic framework. To do so we note the similarities between truth discovery and problems such as judgment aggregation [25], voting theory [73] ranking and recommendation systems [1, 2, 4, 58] in which the *axiomatic approach* of social choice has been successfully applied. In taking the axiomatic approach one aims to formulate *axioms* that encode intuitively desirable properties that an algorithm may possess. The interaction between these axioms can then be studied; typical results include *impossibility results*, where it is shown that a set of axioms cannot hold simultaneously, and *characterisation results*, where it is shown that a set of axioms are uniquely satisfied by a particular algorithm.

Such analysis brings a new *normative* perspective to the truth discovery literature. This complements empirical evaluation: in addition to seeing how well an algorithm performs in practise on test datasets, one can check how well it does against theoretical properties that any 'reasonable' algorithm should satisfy. The satisfaction (or failure) of such properties then shines new light on the *intuitive behaviour* of an algorithm, and may guide development of new ones.

With this in mind, we develop a simplified framework for truth discovery in which axioms can be formulated, and go on to give both an impossibility result and an axiomatic characterisation of a baseline voting algorithm. We also analyse the class of *recursive* truth discovery algorithms, which includes many existing examples

[1] https://www.nytimes.com/2019/08/19/technology/amazon-orwell-1984.html

from the literature. In particular, we analyse the well-known algorithm *Sums* [52] with respect to the axioms.

However, as a first step towards a social choice perspective of truth discovery, our framework involves a number of simplifying assumptions not commonly made in the truth discovery literature.

- **Manipulation and collusion.** Some of our axioms assume sources are not *manipulative*: they provide claims in good faith, and do not aim to misinform or artificially improve their standing with respect to the truth discovery algorithm. We also assume sources act independently, i.e. they do not *collude* with or *copy* one another.
- **Object correlations.** We do not model correlations between the objects of interest in the truth discovery problem. For example, in a crowdsourcing setting it may be known in advance that two objects o and p are similar, so that the true labels for o and p are correlated; this cannot be expressed in our framework.
- **Ordinal outputs.** For the most part, the outputs of our truth discovery methods consist of *rankings* of the sources and facts. Thus, we describe when a source is considered *more trustworthy* than another, but do not assign precise numerical values representing trustworthiness. This breaks with tradition in the truth discovery literature, but is a common point of view in social choice theory.

At first glance these are strong assumptions, and rule out potential applications of our version of truth discovery. However, we argue that the problem is non-trivial even in this simplified setting, and that interesting axioms can still be put forth. The framework as set out here lays the groundwork for these assumptions to be lifted in future work.

The paper is organised as follows. Our framework is introduced and formally defined in the next section. Section 4.3 provides examples of truth discovery algorithms from the literature expressed in the framework. In Section 4.4 we formally introduce the axioms and present an impossibility result showing a subset of these cannot all be satisfied simultaneously. The examples of Section 4.3 are then revisited in Section 4.5, where we analyse them with respect to the axioms and propose modifications to resolve some axiom failures. In Section 4.6 we extend the framework to allow variable domains of sources, facts and objects, and give an impossibility result similar to that of Section 4.4. We discuss the axioms in Section 4.7 and related work in Section 4.8. We conclude in Section 4.9. Missing proofs are given in Appendix A.

4.2 An idealised framework for truth discovery

In this section we define our formal framework, which provides the key definitions required for theoretical discussion and analysis of truth discovery methods.

For most of the paper, we consider a fixed domain of finite and mutually disjoint sets $\mathcal{S}$, $\mathcal{F}$ and $\mathcal{O}$ throughout, called the *sources*, *facts* and *objects* respectively. All definitions and axioms will be stated with respect to these sets.[2]

[2]We generalise to variable domains in Section 4.6.

4.2.1 Truth discovery networks

A core definition of the framework is that of a *truth discovery network*, which represents the input to a truth discovery problem. We model this as a tripartite graph with certain constraints on its structure, in keeping with approaches taken throughout the truth discovery literature [68, 33].

Definition 1. *A* truth discovery network *(hereafter a* TD network*) is a directed graph $N = (V, E)$ where $V = \mathcal{S} \cup \mathcal{F} \cup \mathcal{O}$, and $E \subseteq (\mathcal{S} \times \mathcal{F}) \cup (\mathcal{F} \times \mathcal{O})$ has the following properties:*

1. *For each $f \in \mathcal{F}$ there is a unique $o \in \mathcal{O}$ with $(f, o) \in E$, denoted $\mathsf{obj}_N(f)$. That is, each fact is associated with exactly one object.*

2. *For $s \in \mathcal{S}$ and $o \in \mathcal{O}$, there is at most one directed path from s to o. That is, sources cannot claim multiple facts for a single object.*

3. *$(\mathcal{S} \times \mathcal{F}) \cap E$ is non-empty. That is, at least one claim is made.*

We will say that s claims *f when $(s, f) \in E$. Let $\mathcal{N}$ denote the set of all TD networks.*

Figure 4.1 (page 6) provides an example of a TD network. Note that there is no requirement that a source makes a claim for *every* object, or even that a source makes any claims at all. This reflects the fact that truth discovery datasets are in practise extremely sparse, i.e. each individual source makes few claims. Conversely, we allow for facts that receive no claims from any sources.

Also note that the object associated with a fact f is not fixed across all networks. This is because we view facts as *labels* for information that sources may claim, not the pieces of information themselves. Similarly, we consider objects simply as labels for real-world entities. Whilst a particular piece of information has a fixed entity to which it pertains, the labels do not.[3]

A special case of our framework is the binary case in which every object has exactly two associated facts. This brings us close to the setting studied in *judgment aggregation* [25] and, specifically (since sources do not necessarily claim a fact associated to every object) to the setting of *binary aggregation with abstentions* [14, 19]. An important difference, however, is that for simplicity we do not assume any *constraints* on the possible configurations of true facts across *different* objects. That is, any combination of facts is feasible. In judgment aggregation such an assumption has the effect of neutralising the impossibility results that arise in that domain (see, e.g., [14]). We shall see that that is not the case in our setting.

To simplify the notation in what follows, for a network $N = (V, E)$ we write $\mathsf{facts}_N(s) = \{f \in \mathcal{F} : (s, f) \in E\}$ for the set of facts claimed by a source s, and $\mathsf{src}_N(f) = \{s \in \mathcal{S} : (s, f) \in E\}$ for the set of sources claiming a fact f.

[3] For example, when implementing truth discovery algorithms in practise it is common to assign integer IDs to the 'facts' and 'objects'; the algorithm then operates using only the integer IDs. In this case there is no reason to require that fact 17 is always associated with object 4, for example, and the same principle applies in our framework.

4.2.2 Truth discovery operators

Having defined the input to a truth discovery problem, the output must be defined. Contrary to many approaches in the truth discovery literature which output numeric *trust scores* for sources and *belief scores* for facts [68, 52, 28, 72, 70, 71], we consider the primary output to be *rankings* of the sources and facts. To the extent that we do consider numeric scores, it is only to induce a ranking. This is because we are chiefly interested in *ordinal properties* rather than quantitative values. Indeed, for the theoretical analysis we wish to perform it is only important that a source is *more trustworthy* than another; the particular numeric scores produced by an algorithm are irrelevant.

Moreover, the scores produced by existing algorithms may have no semantic meaning [52], and so referring to numeric values is not meaningful when comparing across algorithms. In this case it is only the rankings of sources and facts that can be compared, which is further motivation for our choice. This point of view is also common across the social choice literature.

However, numerical scores do provide valuable information for comparing sources and facts given a *fixed* algorithm. For example, the magnitude of the difference in trust scores for sources s and t tells us something about *confidence*: a small difference indicates low confidence in distinguishing s and t – even if one is ranked above the other – whereas a large difference indicates high confidence. In this sense our decision to primarily deal with ordinal outputs (and ordinal axioms) is another simplifying assumption compared to typical truth discovery settings.

For a set X, let $\mathcal{L}(X)$ denote the set of all total preorders on X, i.e. the set of transitive, reflexive and complete binary relations on X. We define a *truth discovery operator* as a function which maps networks to rankings of sources and facts.

Definition 2. *An* ordinal truth discovery operator *T (hereafter* TD operator*) is a mapping $T : \mathcal{N} \to \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{F})$. We shall write $T(N) = (\sqsubseteq_N^T, \preceq_N^T)$, i.e. $\sqsubseteq_N^T$ is a total preorder on $\mathcal{S}$ and $\preceq_N^T$ is a total preorder on $\mathcal{F}$.*

Intuitively, the relation $\sqsubseteq_N^T$ is a measure of *source trustworthiness* in the network N according to T, and $\preceq_N^T$ is a measure of *fact believability*; $s_1 \sqsubseteq_N^T s_2$ means that source s_2 is *at least as trustworthy* as source s_1, and $f_1 \preceq_N^T f_2$ means fact f_2 is *at least as believable* as fact f_1. The notation $\sqsubset_N^T$ and $\simeq_N^T$ will be used to denote the strict and symmetric orders induced by $\sqsubseteq_N^T$ respectively. For fact rankings, $\prec_N^T$ and $\approx_N^T$ are defined similarly. Note that for simplicity the fact ranking $\preceq_N^T$ compares *all* facts, even those associated with different objects in N.

To capture existing truth discovery methods we introduce *numerical operators*, which assign each source a numeric *trust score* and each fact a *belief score*.

Definition 3. *A* numerical TD operator *is a mapping $T : \mathcal{N} \to \mathbb{R}^{\mathcal{S} \cup \mathcal{F}}$, i.e. T assigns to each TD network N a function $T(N) = T_N : \mathcal{S} \cup \mathcal{F} \to \mathbb{R}$. For $s \in \mathcal{S}$, $T_N(s)$ is the* trust score *for s in the network N according to T; for $f \in \mathcal{F}$, $T_N(f)$ is the* belief score *for f. The set of all numerical TD operators will be denoted by $\mathcal{T}_{Num}$.*

Note that any numerical operator T naturally induces an ordinal operator $\hat{T}$, where $s_1 \sqsubseteq_N^{\hat{T}} s_2$ iff $T_N(s_1) \leq T_N(s_2)$, and $f_1 \preceq_N^{\hat{T}} f_2$ iff $T_N(f_1) \leq T_N(f_2)$. Henceforth we shall write $\sqsubseteq_N^T$, $\preceq_N^T$ without explicitly defining the induced ordinal operator $\hat{T}$.

It is worth noting that yet other truth discovery algorithms output neither rankings nor numeric scores for facts, but only a single 'true' fact for each object [43, 18, 66]. This is also the approach taken in judgment aggregation, where an *aggregation rule* selects which formulas are to be taken as true. In the case of finitely many possible facts, such algorithms can be modelled in our framework as numerical operators where $T_N(f) = 1$ for each identified 'true' fact f, and $T_N(g) = 0$ for other facts g. To go in the reverse direction and obtain the 'true' facts according to an operator, one may simply select the set of facts for each object that rank maximally.

4.3 Examples of truth discovery operators

Our framework can capture some operators that have been proposed in the truth discovery literature. In this section we provide two concrete examples: *Voting*, which is a simple approach commonly used as a baseline method, and *Sums* [52]. We go on to outline the class of *recursive operators* – of which *Sums* is an instance – which contains many more examples from the literature.

4.3.1 Voting

In *Voting*, we consider each source to cast 'votes' for the facts they claim, and facts are ranked according to the number of votes received. Clearly this method disregards the source trustworthiness aspect of truth discovery, as a vote from one source carries as much weight as a vote from any other. As such, *Voting* cannot be considered a serious contender for truth discovery. It is nonetheless useful as a simple baseline method against which to compare more sophisticated methods.

Definition 4. Voting *is the numerical operator defined as follows: for any network $N \in \mathcal{N}$, $s \in \mathcal{S}$ and $f \in \mathcal{F}$, $T_N(s) = 1$ and $T_N(f) = |\mathsf{src}_N(f)|$.*

Consider the network N shown in Fig. 4.1. Facts f, g and h each receive one vote, whereas i receives 3. The fact ranking induced by *Voting* is therefore $f \approx g \approx h \prec i$. On the other hand, all sources receive a trust score of 1 and therefore rank equally.

4.3.2 Sums

Sums [52] is a simple and well-known operator adapted from the *Hubs and Authorities* [36] algorithm for ranking web pages. The algorithm operates iteratively and recursively, assigning each source and fact a sequences of scores, with the final scores taken as the limit of the sequence.

Initially, scores are fixed at a constant value of $1/2$. The trust score for each source is then updated by summing the belief score of its associated facts. Similarly, belief scores are updated by summing the trust scores of the associated sources. To prevent these scores from growing without bound as the algorithm iterates, they are normalised at each iteration by dividing each trust score by the maximum across all sources (belief scores are normalised similarly).

Expressed in our framework, we have that if T is the (numerical) operator giving the scores at iteration n, then the pre-normalisation scores at iteration $n+1$ are given by T', where

$$T'_N(s) = \sum_{f \in \mathsf{facts}_N(s)} T_N(f); \quad T'_N(f) = \sum_{s \in \mathsf{src}_N(f)} T'_N(s) \tag{4.1}$$

Consider again the network N shown in Fig. 4.1. It can be shown that, with T denoting the limiting scores from *Sums* with normalisation, we have $T_N(s) = 0$, $T_N(t) = 1$, and $T_N(u) = T_N(v) = \sqrt{2}/2$. The induced ranking of sources is therefore $s \sqsubset u \simeq v \sqsubset t$.

For fact scores, we have $T_N(f) = 0$, $T_N(g) = \sqrt{2} - 1$, $T_N(h) = 0$ and $T_N(i) = 1$, so the ranking is $f \approx h \prec g \prec i$. Note that fact g fares better under *Sums* than *Voting*, due to its association with the highly-trusted source t.

4.3.3 Recursive truth discovery operators

The iterative and recursive aspect of *Sums* is hoped to result in the desired mutual dependence between trust and belief scores: namely that sources claiming high-belief facts are seen as trustworthy, and vice versa. In fact, this recursive approach is near universal across the truth discovery literature (see for instance [65, 23, 71, 43, 28, 72]). As such it is appropriate to identify the class of *recursive operators* as an important subset of $\mathcal{T}_{Num}$. To make a formal definition we first define an *iterative operator*.

Definition 5. *An* iterative operator *is a sequence* $(T^n)_{n \in \mathbb{N}}$ *of numerical operators. An iterative operator is said to* converge *to a numerical operator* T^* *if* $\lim_{n \to \infty} T^n_N(z) = T^*_N(z)$ *for all networks* N *and* $z \in \mathcal{S} \cup \mathcal{F}$. *In such case the iterative operator can be identified with the ordinal operator induced by its limit* T^*.

Note that it is possible that an iterative operator $(T^n)_{n \in \mathbb{N}}$ converges for only a subset of networks. In such case we can consider $(T^n)_{n \in \mathbb{N}}$ to converge to a 'partial operator' and identify it with the induced partial ordinal operator; that is, a partial function $\mathcal{N} \to \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{F})$. Recursive operators can now be defined as those iterative operators where T^{n+1} can be obtained from T^n.

Definition 6. *An iterative operator* $(T^n)_{n \in \mathbb{N}}$ *is said to be* recursive *if there is a function* $U : \mathcal{T}_{Num} \to \mathcal{T}_{Num}$ *such that* $T^{n+1} = U(T^n)$ *for all* $n \in \mathbb{N}$.

In this context the mapping $U : \mathcal{T}_{Num} \to \mathcal{T}_{Num}$ is called the *update function*, and the initial operator T^1 is called the *prior operator*. For a prior operator T and update function U, we write $\mathsf{rec}(T, U)$ for the associated recursive operator; that is, $T^1 = T$ and $T^{n+1} = U(T^n)$.

Returning to *Sums*, we see that Eq. (4.1) defines a mapping $\mathcal{T}_{Num} \to \mathcal{T}_{Num}$ and consequently an update function U^{Sums}. The normalisation step can be considered a separate update function norm which maps any numerical operator T to T', where[4]

$$T'_N(s) = \frac{T_N(s)}{\max_{x \in \mathcal{S}} |T_N(x)|}, \quad T'_N(f) = \frac{T_N(f)}{\max_{y \in \mathcal{F}} |T_N(y)|}$$

It can then be seen that *Sums* is the recursive operator $\mathsf{rec}(T^{\text{fixed}}, \mathsf{norm} \circ U^{\text{Sums}})$, where $T^{\text{fixed}}_N \equiv 1/2$.

Many other existing algorithms proposed in the literature can also be realised as recursive operators in the framework, such as *Investment*, *PooledInvestment* [52], *TruthFinder* [68], LDT [71] and others.

4.4 Axioms for truth discovery

Having laid out the formal framework, we now introduce axioms for truth discovery. To start with, we consider axioms which encode a desirable theoretical property that we believe any 'reasonable' operator T should satisfy. Several properties of this nature can be obtained by adapting existing axioms from the social choice literature (e.g. from voting [12], ranking systems [58, 1] and judgement aggregation [25]), to our framework.

However, the correspondence between truth discovery and classical social choice problems – such as voting – has its limits. To show this, we translate the famous Independence of Irrelevant Alternatives (IIA) axiom [5] to our setting, and argue that it is actually an *undesirable* property. Indeed, it will be seen that this translated axiom, in combination with two basic desirable axioms, leads to *Voting*-like behaviour in every network, which is undesirable for the reasons given in Section 4.3.1. Furthermore, a slight strengthening of the IIA axiom completely characterises the fact ranking component of *Voting*. These results formalise the intuition that truth discovery's consideration of source-trustworthiness leads to fundamental differences from classical social choice problems.

Afterwards, we will revisit the specific operators of the previous section to check which axioms are satisfied.

4.4.1 Coherence

As mentioned previously, a guiding principle of truth discovery is that sources claiming highly believed facts should be seen as trustworthy, and that facts backed by highly trusted sources should be seen as believable.

Whilst this intuition is difficult to formalise in general, it is possible to do so in particular cases where there are obvious means by which to compare the set of facts for two sources (and vice versa). This situation is considered in the axiomatic analysis of ranking and reputation systems under the name *Transitivity* [58, 1], and we adapt it to truth discovery in this section. A preliminary definition is required.

Definition 7. *Let T be a TD operator, N be a TD network and $Y, Y' \subseteq \mathcal{F}$. We say Y is* less believable *than Y' with respect to N and T if there is a bijection $\varphi : Y \to Y'$ such that $f \preceq^T_N \varphi(f)$ for each $f \in Y$, and $\hat{f} \prec^T_N \varphi(\hat{f})$ for some $\hat{f} \in Y$.*

For $X, X' \subseteq \mathcal{S}$ we define X less trustworthy *than X' with respect to N and T in a similar way.*

In plain English, Y less believable than Y' means that the facts in each set can be paired up in such a way that each fact in Y' is at least as believable as

[4] If $\max_{x\in\mathcal{S}} |T_N(x)| = 0$ then the above is ill-defined; we set $T'_N(s) = 0$ for all s in this case. Fact belief scores are defined similarly if the maximum is 0.

its counterpart in Y, and at least one fact in Y' is strictly more believable. Now, consider a situation where $\mathsf{facts}_N(s_1)$ is less believable than $\mathsf{facts}_N(s_2)$. In this case the intuition outlined above tells us that s_2 provides 'better' facts, and should thus be seen as more trustworthy than s_1. A similar idea holds if $\mathsf{src}_N(f_1)$ is less trustworthy than $\mathsf{src}_N(f_2)$ for some facts f_1, f_2. We state this formally as our first axiom.

Axiom 1 (Coherence). *For any network N, $\mathsf{facts}_N(s_1)$ less believable than $\mathsf{facts}_N(s_2)$ implies $s_1 \sqsubset_N^T s_2$, and $\mathsf{src}_N(f_1)$ less trustworthy than $\mathsf{src}_N(f_2)$ implies $f_1 \prec_N^T f_2$.*

Coherence can be broken down into two sub-axioms: *Source-Coherence*, where the first implication regarding source rankings is satisfied; and *Fact-Coherence*, where the second implication is satisfied. We take Coherence to be a fundamental desirable axiom for TD operators.

4.4.2 Symmetry

Our next axiom requires that rankings of sources and facts should not depend on their 'names', but only on the structure of the network. To state it formally, we need a notion of when two networks are essentially the same but use different names.

Definition 8. *Two TD networks N and N' are* equivalent *if there is a graph isomorphism π between them that preserves sources, facts and objects, i.e., $\pi(s) \in \mathcal{S}$, $\pi(f) \in \mathcal{F}$ and $\pi(o) \in \mathcal{O}$ for all $s \in \mathcal{S}$, $f \in \mathcal{F}$ and $o \in \mathcal{O}$. In such case we write $\pi(N)$ for N'.*

Axiom 2 (Symmetry). *Let N and $N' = \pi(N)$ be equivalent networks. Then for all $s_1, s_2 \in \mathcal{S}$, $f_1, f_2 \in \mathcal{F}$, we have $s_1 \sqsubseteq_N^T s_2$ iff $\pi(s_1) \sqsubseteq_{N'}^T \pi(s_2)$ and $f_1 \preceq_N^T f_2$ iff $\pi(f_1) \preceq_{N'}^T \pi(f_2)$.*

In the theory of voting in social choice, Symmetry as above is expressed as two axioms: *Anonymity*, where output is insensitive to the names of voters, and *Neutrality*, where output is insensitive to the names of alternatives [73]. Analogous axioms are also used in judgment aggregation.

Symmetry can also be broken down into sub-axioms where the above need only hold for a subset of permutations π satisfying some condition: *Source-Symmetry* (where π must leave facts and objects fixed) and *Fact-Symmetry* (where π leaves sources and objects fixed). For truth discovery we have the additional notion of objects, and thus *Object-Symmetry* can defined be similarly.

4.4.3 Fact ranking axioms

Next, we introduce axioms that dictate the ranking of particular facts in cases where there is an 'obvious' ordering. *Unanimity* and *Groundedness* express the idea that if all sources are in agreement about the status of a fact, then an operator should respect this in its verdict. Two obvious ways in which sources can be in agreement are when *all* sources believe a fact is true, and when *none* believe a fact is true.

Axiom 3 (Unanimity). *Suppose $N \in \mathcal{N}$, $f \in \mathcal{F}$, and $\mathsf{src}_N(f) = \mathcal{S}$. Then for any other $g \in \mathcal{F}$, $g \preceq_N^T f$.*

Axiom 4 (Groundedness). *Suppose $N \in \mathcal{N}$, $f \in \mathcal{F}$, and $\mathsf{src}_N(f) = \emptyset$. Then for any other $g \in \mathcal{F}$, $f \preceq_N^T g$.*

That is, f cannot do better than to be claimed by all sources when T satisfies Unanimity, and cannot do worse than to be claimed by none when T satisfies Groundedness. Unanimity here is a truth discovery rendition of the same axiom in judgment aggregation, and can also be compared to the *weak Paretian* property in voting [12]. Groundedness is a version of the same axiom studied in the analysis of collective annotation [40].

The next axiom is a monotonicity property, which states that if f receives extra support from a new source s, then its ranking should receive a strictly positive boost.[5] Note that we do not make any judgement on the new ranking of s.

Axiom 5 (Monotonicity). *Suppose $N \in \mathcal{N}$, $s \in \mathcal{S}$, $f \in \mathcal{F} \setminus \mathsf{facts}_N(s)$. Write E for the set of edges in N, and let N' be the network in which s claims f; i.e. the network with edge set*

$$E' = \{(s, f)\} \cup E \setminus \{(s, g) : g \neq f, \mathsf{obj}_N(g) = \mathsf{obj}_N(f)\}$$

Then for all $g \neq f$, $g \preceq_N^T f$ implies $g \prec_{N'}^T f$.

Note that the axioms in this section assume sources do not have 'negative' trust levels. That is, we assume that support from even the most untrustworthy source still has a *positive* effect on the believability of a fact. Consequently, these axioms are not suitable in the presence of knowledgable but malicious sources who always claim false facts. Indeed, otherwise a fact claimed only by a 'negative' source should rank strictly *worse* than a fact with no sources, but this goes against Groundedness. Similarly, receiving extra support from a negative source should worsen a fact's ranking, contrary to Monotonicity. Moreover, Monotonicity implicitly assumes sources act independently, i.e. they do not *collude* with one another.[6]

While these assumptions may appear somewhat strong, we argue that this 'simple' case – with no 'negative' sources or collusion – is already non-trivial and permits interesting axiomatic analysis. We therefore view Unanimity, Groundedness and Monotonicity as desirable properties for TD operators.

4.4.4 Independence axioms

We now come to exploring the differences between truth discovery and other social choice problems via *independence* axioms. In voting, this takes the form of Independence of Irrelevant Alternatives (IIA), which requires that the ranking of two alternatives A and B depends only on the individual assessments of A and B, not on some 'irrelevant' alternative C.

An analogous truth discovery axiom states that the ranking of facts f_1 and f_2 for some object o depends only on the claims relating to o. Intuitively, this is *not* a desirable property. Indeed, we have already seen in Example 1 that the claims for object p in the network from Fig. 4.1 can play an important role in determining the ranking of f and g for object o, but the adapted IIA axiom precludes this.

This undesirability can be made precise. First, we must state the axiom formally.

[5] One could also consider the weak version, in which we only require $g \preceq_{N'}^T f$ in the consequent; we discuss this in Section 4.7.

[6] Note that collusion has been studied in the truth discovery literature (e.g. [20, 7, 21]).

Axiom 6 (Per-object Independence (POI))**.** *Let $o \in \mathcal{O}$. Suppose N_1, N_2 are networks such that $F_o = \mathsf{obj}_{N_1}^{-1}(o) = \mathsf{obj}_{N_2}^{-1}(o)$ and $\mathsf{src}_{N_1}(f) = \mathsf{src}_{N_2}(f)$ for each $f \in F_o$. Then the restrictions of $\preceq_{N_1}^T$ and $\preceq_{N_2}^T$ to F_o are equal; that is, $f_1 \preceq_{N_1}^T f_2$ iff $f_1 \preceq_{N_2}^T f_2$ for all $f_1, f_2 \in F_o$.*

Considering Fig. 4.1 again, POI implies that the ranking of f and g remains the same if the claims for h and i are removed. But in this case, Symmetry implies $f \approx g$. Similarly, the ranking of h and i remains the same if the claims for f and g are removed. In this case, Symmetry together with Monotonicity implies $h \prec i$, since $|\mathsf{src}_N(h)| < |\mathsf{src}_N(i)|$.

This observation forms the basis of the following result, which formalises the undesirability of POI: in the presence of our less controversial requirements of Symmetry and Monotonicity, it forces *Voting*-like behaviour within $\mathsf{obj}_N^{-1}(o)$ for each $o \in \mathcal{O}$. We note that, for the special case of binary networks, similar results have been shown in the literature on binary aggregation with abstentions [14].

Theorem 1. *Let T be any operator satisfying Symmetry, Monotonicity and POI. Then for any $N \in \mathcal{N}$, $o \in \mathcal{O}$ and $f_1, f_2 \in \mathsf{obj}_N^{-1}(o)$ we have $f_1 \preceq_N^T f_2$ iff $|\mathsf{src}_N(f_1)| \leq |\mathsf{src}_N(f_2)|$.*

Proof (sketch). We will sketch the main ideas of the proof here with some technical details omitted; see Appendix A for the full proof. Let N be a network, o be an object and $f_1, f_2 \in \mathsf{obj}_N^{-1}(o)$. Consider N' obtained by removing from N all claims for objects other than o. By POI, we have $f_1 \preceq_N^T f_2$ iff $f_1 \preceq_{N'}^T f_2$. Since $|\mathsf{src}_N(f_j)| = |\mathsf{src}_{N'}(f_j)|$ also ($j \in \{1, 2\}$), it is sufficient for the proof to show that $f_1 \preceq_{N'}^T f_2$ iff $|\mathsf{src}_{N'}(f_1)| \leq |\mathsf{src}_{N'}(f_2)|$.

For the 'if' direction, first suppose $|\mathsf{src}_{N'}(f_1)| = |\mathsf{src}_{N'}(f_2)|$. Let π be the permutation which swaps f_1 with f_2 and swaps each source in $\mathsf{src}_{N'}(f_1)$ with one in $\mathsf{src}_{N'}(f_2)$; then we have $\pi(N') = N'$, and Symmetry of T gives $f_1 \approx_{N'}^T f_2$. In particular $f_1 \preceq_{N'}^T f_2$ as required.

Otherwise, $|\mathsf{src}_{N'}(f_2)| - |\mathsf{src}_{N'}(f_1)| = k > 0$. Consider N'' where k sources from $\mathsf{src}_{N'}(f_2)$ are removed, and all other claims remain. By Symmetry as above, $f_1 \approx_{N''}^T f_2$. Applying Monotonicity k times we can produce N' from N'' and get $f_1 \prec_{N'}^T f_2$ as desired.

For the 'only if' statement, suppose $f_1 \preceq_{N'}^T f_2$ but, for contradiction, $|\mathsf{src}_{N'}(f_1)| > |\mathsf{src}_{N'}(f_2)|$. Applying Monotonicity again as above we get $f_1 \succ_{N'}^T f_2$ and the required contradiction. $\square$

Recall that Coherence formalises the idea that source-trustworthiness should inform the fact ranking, and vice versa. Clearly *Voting* does not conform to this idea, and in fact even the object-wise voting patterns in Theorem 1 are incompatible with Coherence. This can easily be seen in the network in Fig. 4.1 where, regarding object p, we have $|\mathsf{src}_N(h)| < |\mathsf{src}_N(i)|$ (hence $h \prec_N^T i$) and, regarding object o, we have $|\mathsf{src}_N(f)| = |\mathsf{src}_N(g)|$ (hence $f \approx_N^T g$). Hence $\mathsf{facts}_N(s)$ is less believable than $\mathsf{facts}_N(t)$. If Coherence held this would give $s \sqsubset_N^T t$, but then $\mathsf{src}_N(f)$ is less trustworthy than $\mathsf{src}_N(g)$, giving $f \prec_N^T g$ – a contradiction. From this discussion and Theorem 1 we obtain as a corollary the following first impossibility result for truth discovery.

Theorem 2. *There is no TD operator satisfying Coherence, Symmetry, Monotonicity and POI.*

Given that Theorem 1 characterises the fact ranking of *Voting* for facts relating to a single object, it is natural to ask if there is a stronger form of independence that guarantees this behaviour across *all* facts. As our next result shows, the answer is *yes*, and the necessary axiom is obtained by ignoring the role of objects altogether for fact ranking.

Axiom 7 (Strong Independence)**.** *For any networks N_1, N_2 and facts f_1, f_2, if $\mathsf{src}_{N_1}(f_j) = \mathsf{src}_{N_2}(f_j)$ for each $j \in \{1, 2\}$ then $f_1 \preceq^T_{N_1} f_2$ iff $f_1 \preceq^T_{N_2} f_2$.*

That is, the ranking of two facts f_1 and f_2 is determined solely by the sources claiming f_1 and f_2. In particular, the fact-object affiliations and claims for facts other than f_1, f_2 are irrelevant when deciding on f_1 versus f_2. Note that Strong Independence implies POI. We have the following result.

Theorem 3. *Suppose $|\mathcal{O}| \geq 3$. Then an operator T satisfies Strong Independence, Monotonicity and Symmetry if and only if for any network N and $f_1, f_2 \in \mathcal{F}$ we have*

$$f_1 \preceq^T_N f_2 \iff |\mathsf{src}_N(f_1)| \leq |\mathsf{src}_N(f_2)|$$

Theorem 3 can be seen as a characterisation of the class of TD operators that rank facts in the same way as *Voting*. The proof is similar to that of Theorem 1, but uses a different transformation to obtain a modified network N' in the first step.

We have established that neither POI nor Strong Independence are satisfactory axioms for truth discovery, and a weaker independence property is required. Figure 4.1 can help us once again in this regard. Whereas POI and Strong Independence would say that facts h and i are irrelevant to f, the argument with Coherence for Theorem 2 suggests otherwise due the indirect links via the sources. We therefore propose that only when there is no (undirected) path between two nodes can we consider them to be truly irrelevant to each other. That is, nodes are relevant to each other iff they lie in the same *connected component* of the network.

Our final rendering of independence states that the ordering of two facts in the same connected component does not depend on any claims outside of the component, and similarly for sources.

Axiom 8 (Per-component Independence (PCI))**.** *For any TD networks N_1, N_2 with a common connected component G, the restrictions of $\sqsubseteq^T_{N_1}$ and $\sqsubseteq^T_{N_2}$ to $G \cap \mathcal{S}$ are equal, and the restrictions of $\preceq^T_{N_1}$ and $\preceq^T_{N_2}$ to $G \cap \mathcal{F}$ are equal; that is, $s_1 \sqsubseteq^T_{N_1} s_2$ iff $s_1 \sqsubseteq^T_{N_2} s_2$ and $f_1 \preceq^T_{N_1} f_2$ iff $f_1 \preceq^T_{N_2} f_2$ for $s_1, s_2 \in G \cap \mathcal{S}$ and $f_1, f_2 \in G \cap \mathcal{F}$.*

In analogy with Source/Fact Coherence and Source/Fact Symmetry, it is possible to split the two requirements of PCI into sub-axioms Source-PCI (in which only the constraint on source ranking is imposed) and Fact-PCI (in which only the fact ranking is constrained).

Note that while our framework can be easily adapted to require *by definition* that a network is itself connected (and therefore has only one connected component), we have found that datasets with multiple connected components do indeed occur in practise.[7] This means that failure of PCI is a real issue, and consequently we consider PCI to be another core axiom that all reasonable operators should satisfy.

[7] For example, the *Book* and *Restaurant* datasets found at the following web page each contain two connected components: http://lunadong.com/fusionDataSets.htm

4.5 Satisfaction of the axioms

With the axioms formally defined, we can now consider whether they are satisfied by the example operators of Section 4.3. *Voting* can be analysed outright; for *Sums* we require some preliminary results giving sufficient conditions for iterative and recursive operators to satisfy various axioms. It will be seen that neither *Voting* nor *Sums* satisfy all our desirable axioms, but it is possible to modify each operator to gain some improvement with respect to the axioms.

4.5.1 Voting

As the simplest operator, we consider *Voting* first. The following theorem shows that all axioms except Coherence are satisfied. Since Coherence is a fundamental principle of truth discovery, and we actually consider POI and Strong Independence to be *undesirable*, this formally rules out *Voting* as a viable operator.

Theorem 4. Voting *satisfies Symmetry, Unanimity, Groundedness, Monotonicity, POI, Strong Independence and PCI.* Voting *does not satisfy Coherence.*

The proof is straightforward, and is deferred to Appendix A. Note that once Symmetry, Monotonicity and POI are shown, the fact that *Voting* fails Coherence follows from our impossibility result (Theorem 2), and Fig. 4.1 serves as an explicit counterexample.

4.5.2 Iterative and recursive operators

In this section we give sufficient conditions for iterative and recursive operators to satisfy various axioms. These results will be useful in what follows when analysing *Sums*, although they may also be applied more generally to other operators.

Coherence. To analyse whether the limit of a recursive operator satisfies Coherence, we consider how the update function U behaves when the difference in belief scores between the facts of s_1 and s_2 is 'small' (and similarly for the sources of f_1, f_2). To that end, we introduce a numerical variant of a set of facts Y being 'less believable' than Y'.

Definition 9. *Let T be a numerical TD operator, N a network, $Y, Y' \subseteq \mathcal{F}$ and $\varepsilon, \rho > 0$. We say Y is (ε, ρ)-*less believable *than Y' with respect to N and T if there is a bijection $\varphi : Y \to Y'$ such that $T_N(f) - T_N(\varphi(f)) \leq \varepsilon$ for all $f \in Y$, and $T_N(\hat{f}) - T_N(\varphi(\hat{f})) \leq \varepsilon - \rho$ for some $\hat{f} \in Y$.*

*For $X, X' \subseteq \mathcal{S}$, we define X (ε, ρ)-*less trustworthy *than X' similarly.*

This generalises Definition 7 by relaxing the requirement that $f \preceq_N^T \varphi(f)$, and instead requiring that f can only be more believable than $\varphi(f)$ by some threshold $\varepsilon > 0$. Definition 7 is recovered in the limiting case $\varepsilon \to 0$. We obtain a sufficient condition on the update function U for a recursive operator to satisfy Source-Coherence.

Lemma 1. *Let $U : \mathcal{T}_{Num} \to \mathcal{T}_{Num}$. For any prior operator T^{prior}, $\mathsf{rec}(T^{prior}, U)$ satisfies Source-Coherence if the following condition is satisfied: there exist $C, D > 0$ such that for* *all networks N and numerical operators T it holds that if* $\mathsf{facts}_N(s_1)$ *is* (ε, ρ)*-less believable than* $\mathsf{facts}_N(s_2)$ *with respect to* N *and* T*, then* $T'_N(s_1) - T'_N(s_2) \leq C\varepsilon - D\rho$*, where* $T' = U(T)$*.*

The proof of Lemma 1 uses the following result, the proof of which is a straightforward application of the definition of the limit.

Lemma 2. *Let N be a truth discovery network and $(T^n)_{n\in\mathbb{N}}$ be a convergent iterative operator with limit T^*. Then for $f_1, f_2 \in \mathcal{F}$, $f_1 \preceq_N^{T^*} f_2$ if and only if*

$$\forall \varepsilon > 0\ \exists K \in \mathbb{N} : \forall n \geq K : T_N^n(f_1) - T_N^n(f_2) \leq \varepsilon$$

Also, $f_1 \prec_N^{T^*} f_2$ if and only if

$$\exists \rho > 0 : \forall \varepsilon > 0\ \exists K \in \mathbb{N} : \forall n \geq K : T_N^n(f_1) - T_N^n(f_2) \leq \varepsilon - \rho$$

Analogous statements for source rankings also hold.

Lemma 1. Let N be a network. Suppose U has the stated property and that $\mathsf{rec}(T^{\text{prior}}, U) = (T^n)_{n\in\mathbb{N}}$ converges to T^*. Suppose $\mathsf{facts}_N(s_1)$ is less trustworthy than $\mathsf{facts}_N(s_2)$ with respect to N and T^* under a bijection φ. We must show that $s_1 \sqsubset_N^{T^*} s_2$.

Now, there is some $\hat{f} \in \mathsf{facts}_N(s_1)$ with $\hat{f} \prec_N^{T^*} \varphi(\hat{f})$. The second part of Lemma 2 therefore applies; let ρ be as given there. Now let $\varepsilon > 0$. Since $f \preceq_N^{T^*} \varphi(f)$ for each $f \in \mathsf{facts}_N(s_1)$, we may apply Lemma 2 with $f, \varphi(f)$ and $\bar{\varepsilon} = \varepsilon / C$ to get that there is $K \in \mathbb{N}$ such that

$$T_N^n(f) - T_N^n(\varphi(f)) \leq \bar{\varepsilon}$$

and

$$T_N^n(\hat{f}) - T_N^n(\varphi(\hat{f})) \leq \bar{\varepsilon} - \rho$$

for all $n \geq K$. In other words, $\mathsf{facts}_N(s_1)$ is $(\bar{\varepsilon}, \rho)$-less believable than $\mathsf{facts}_N(s_2)$ with respect to N and T^n for all $n \geq K$.

Now, recall that $T^{n+1} = U(T^n)$. For $m \geq K' = K + 1$ we therefore have, applying our condition on U,

$$T_N^m(s_1) - T_N^m(s_2) \leq C\bar{\varepsilon} - D\rho = \varepsilon - D\rho$$

Since $D\rho$ is positive and does not depend on ε, we get $s_1 \sqsubset_N^{T^*} s_2$ by Lemma 2. This shows that T^* satisfies Source-Coherence. □

A similar result gives conditions under which Fact-Coherence is satisfied.

Lemma 3. $\mathsf{rec}(T^{prior}, U)$ *satisfies Fact-Coherence if there exist $E, F > 0$ such that for all networks N and numerical operators T it holds that if* $\mathsf{src}_N(f_1)$ *is* (ε, ρ)*-less trustworthy than* $\mathsf{src}_N(f_2)$ *with respect to N and T', then $T'_N(f_1) - T'_N(f_2) \leq E\varepsilon - F\rho$, where $T' = U(T)$.*

Proof. The proof proceeds in an identical way to Lemma 1; the only difference is that we may simply take $K' = K$ in the final step. □

Note that there is asymmetry between Lemma 1 and Lemma 3 – in the condition on U in Lemma 1 we have $\mathsf{facts}_N(s_1)$ (ε, ρ)-less trustworthy than $\mathsf{facts}_N(s_2)$ with respect to T, whereas in Lemma 3 the corresponding condition is with respect to $T' = U(T)$. This reflects the manner in which *Sums* and other TD operators are typically defined: source trust scores are updated based on the fact scores of the previous iteration, whereas fact belief scores are updated based on the (new) trust scores in the *current* iteration.

Also note that the above results still hold if U has the stated property only for 'small' ε; that is, if there is a constant $0 < \lambda < 1$ such that the property holds for all ρ and for all $\varepsilon < \lambda\rho$.

Symmetry and PCI. When considering either Symmetry or PCI for an iterative operator $(T^n)_{n\in\mathbb{N}}$, it is not enough to know that each T^n satisfies the relevant axiom. The following example illustrates this fact for Symmetry.

Example 2. *Fix some $\hat{f} \in \mathcal{F}$, and define an iterative operator by*

$$T^n_N(s) = 1$$

$$T^n_N(f) = \begin{cases} |\mathsf{src}_N(f)| + (1 - \frac{1}{n+1}) & \text{if } |\mathsf{src}_N(f)| = |\mathsf{src}_N(\hat{f})| \\ |\mathsf{src}_N(f)| & \text{otherwise} \end{cases}$$

That is, each T^n is a modification of Voting *in which we boost the score of all facts tied with $\hat{f}$ under* Voting *by $1 - \frac{1}{n+1}$. Since this additional weight is (strictly) less than 1 for each n, the ordinal operator induced by T^n is simply* Voting, *and therefore satisfies Symmetry. However, it is easy to see that the limit operator T^* has $T^*_N(\hat{f}) = |\mathsf{src}_N(\hat{f})| + 1$; this means T^* uses extra information beyond the structure of the network N in its ranking (namely, the identity of a selected fact $\hat{f}$) which violates Symmetry.*

Using a similar tactic, one can construct a sequence of numerical operators $(T^n)_{n\in\mathbb{N}}$ such that each T^n satisfies PCI, but the limit operator T^* does not.

Fortunately, there is a natural strengthening of both Symmetry and PCI for numerical operators which *is* preserved in the limit. Let us say that a numerical operator T satisfies *numerical Symmetry* if for any equivalent networks $N, \pi(N)$ we have $T_N(z) = T_{\pi(N)}(\pi(z))$ for all $z \in \mathcal{S} \cup \mathcal{F}$. Similarly, T satisfies *numerical PCI* if for any networks N_1 and N_2 with a common connected component G, we have $T_{N_1}(z) = T_{N_2}(z)$ for all $z \in G \cap (\mathcal{S} \cup \mathcal{F})$. Clearly numerical Symmetry implies Symmetry, and numerical PCI implies PCI. The following result is immediate.

Lemma 4. *Suppose $(T^n)_{n\in\mathbb{N}}$ converges to T^*. Then*

- *If T^n satisfies numerical Symmetry for each $n \in \mathbb{N}$, then T^* satisfies Symmetry.*
- *If T^n satisfies numerical PCI for each $n \in \mathbb{N}$, then T^* satisfies PCI.*

As a consequence of Lemma 4, any recursive operator $\mathsf{rec}(T^{\text{prior}}, U)$ satisfies Symmetry whenever T^{prior} satisfies numerical Symmetry and U *preserves* numerical Symmetry, in the sense that $U(T)$ satisfies numerical Symmetry whenever T does (and similarly for PCI).

Unanimity, Groundedness and Monotonicity. In contrast to Symmetry and PCI, both Unanimity and Groundedness *are* preserved when taking the limit of an iterative operator.

Lemma 5. *Suppose* $(T^n)_{n\in\mathbb{N}}$ *converges to* T^**. Then*

- *If* T^n *satisfies Unanimity for each* $n \in \mathbb{N}$*, then* T^* *satisfies Unanimity.*
- *If* T^n *satisfies Groundedness for each* $n \in \mathbb{N}$*, then* T^* *satisfies Groundedness.*

For Monotonicity, we require the following (stronger) property to hold for each T^n.

Definition 10. *A numerical operator* T *satisfies* Improvement *if for each* N, N' *and* f *as in the statement of Monotonicity, we have* $\delta(f) > \delta(g)$ *for all* $g \neq f$*, where*

$$\delta(g) = T_{N'}(g) - T_N(g)$$

In this case we write $\rho_{N,N'} = \min_{g\neq f} (\delta(f) - \delta(g)) > 0$*.*

Here $\delta(g)$ is the amount by which the belief score for g increases when going from the network N to N'. Improvement simply says that when adding a new source to a fact f, it is f that sees the largest increase.

Proposition 1. *Suppose* $(T^n)_{n\in\mathbb{N}}$ *converges to* T^**, and* T^n *satisfies Improvement for each* $n \in \mathbb{N}$*. Suppose also that* $\inf_{n\in\mathbb{N}} \rho^n_{N,N'} > 0$ *for each* N, N' *arising in the statement of Monotonicity. Then* T^* *satisfies Monotonicity.*

Proof. Let N, N' and f be as in the statement of Monotonicity, and suppose $g \preceq^{T^*}_N f$ for some $g \neq f$. We will show $g \prec^{T^*}_{N'} f$ using Lemma 2.

Write $\rho^* = \inf_{n\in\mathbb{N}} \rho^n_{N,N'} > 0$ and let $\varepsilon > 0$. Since $g \preceq^{T^*}_N f$, there is $K \in \mathbb{N}$ such that $T^n_N(g) - T^n_N(f) \le \varepsilon$ for all $n \ge K$. For such n, we have

$$\begin{aligned} T^n_{N'}(g) - T^n_{N'}(f) &= (T^n_N(g) + \delta^n(g)) - (T^n_N(f) + \delta^n(f)) \\ &= \underbrace{T^n_N(g) - T^n_N(f)}_{\le\varepsilon} - \underbrace{(\delta^n(f) - \delta^n(g))}_{\ge \rho^n_{N,N'}} \\ &\le \varepsilon - \rho^n_{N,N'} \\ &\le \varepsilon - \rho^* \end{aligned}$$

By Lemma 2, we have $g \prec^{T^*}_{N'} f$ as required. □

The requirement that $\inf_{n\in\mathbb{N}} \rho^n_{N,N'} > 0$ is a technical condition which ensures the *strict* inequality $g \prec^{T^*}_{N'} f$ holds in the limit, as required for Monotonicity. If this condition fails T^* still satisfies a natural 'weak Monotonicity' axiom, in which the strict inequality $g \prec^{T^*}_{N'} f$ is replaced with $g \preceq^{T^*}_{N'} f$.

4.5.3 Sums

We come to the axiomatic analysis of *Sums*. Coherence and the simpler axioms are satisfied here, and the undesirable independence axioms (POI and Strong Independence) are not. However, Monotonicity and PCI do *not* hold. Since PCI is one of our most important axioms that we expect any reasonable operator to satisfy, this potentially limits the usefulness of *Sums* in practise.

Theorem 5. Sums *satisfies Coherence, Symmetry, Unanimity and Groundedness.* Sums *does not satisfy POI, Strong Independence, PCI or Monotonicity.*

Proof (sketch). Symmetry, Unanimity and Groundedness can be easily shown from Lemma 4 and Lemma 5; the details can be found in the appendix. In the remainder of the proof, $(T^n)_{n\in\mathbb{N}}$ will denote the iterative operator *Sums*, T^* will denote the limit operator, and $U = \mathsf{norm} \circ U^{\mathsf{Sums}}$ will denote the update function for *Sums*.

Coherence. We will show Source-Coherence using Lemma 1. The argument for Fact-Coherence is similar (using Lemma 3) and can be found in the appendix.

Suppose $N \in \mathcal{N}$, $T \in \mathcal{T}_{Num}$, $\varepsilon, \rho > 0$, and $\mathsf{facts}_N(s_1)$ is (ε, ρ)-less believable than $\mathsf{facts}_N(s_2)$ with respect to N and T under a bijection $\varphi : \mathsf{facts}_N(s_1) \to \mathsf{facts}_N(s_2)$. By definition there is $\hat{f} \in \mathsf{facts}_N(s_1)$ such that $T_N(\hat{f}) - T_N(\varphi(\hat{f})) \leq \varepsilon - \rho$. By the remark after the proof of Lemma 1, we may assume without loss of generality that $\varepsilon < \frac{1}{|\mathcal{F}|}\rho$.

Recall that the update function for *Sums* is $U = \mathsf{norm} \circ U^{\mathsf{Sums}}$. Write $T' = U^{\mathsf{Sums}}(T)$ and $\tilde{T} = U(T) = \mathsf{norm}(U^{\mathsf{Sums}}(T))$ so that $\tilde{T} = \mathsf{norm}(T')$. We must show that $\tilde{T}_N(s_1) - \tilde{T}_N(s_2) \leq C\varepsilon - D\rho$ for some constants $C, D > 0$.

Note at this stage that it is possible to further weaken the hypotheses of Lemma 1: the result follows if U has the stated property not for *all* operators T, but only for those such that $T = T^n$ for some $n \in \mathbb{N}$. Next, note that if $T'_N(x) = 0$ for all $x \in \mathcal{S}$ then trust and belief scores are 0 in all subsequent iterations, and thus all sources rank equally in the limit T^*. But this means the hypothesis for Source-Coherence cannot be satisfied (there are no strict inequalities). We may therefore assume without loss of generality that $T'_N(x) \neq 0$ for at least one $x \in \mathcal{S}$. Therefore, by definition of norm,

$$\tilde{T}_N(s) = \alpha T'_N(s)$$

where

$$\alpha = \frac{1}{\max_{x\in\mathcal{S}} |T'_N(x)|}$$

Applying the definition of U^{Sums} and using the pairing of $\mathsf{facts}_N(s_1)$ and $\mathsf{facts}_N(s_2)$

via φ, we have

$$\begin{aligned}
\tilde{T}_N(s_1) - \tilde{T}_n(s_2) &= \alpha[T'_N(s_1) - T'_N(s_2)] \\
&= \alpha \left[\sum_{f \in \mathsf{facts}_N(s_1)} T_N(f) - \sum_{f \in \mathsf{facts}_N(s_1)} T_N(\varphi(f)) \right] \\
&= \alpha \sum_{f \in \mathsf{facts}_N(s_1)} \Big(T_N(f) - T_N(\varphi(f)) \Big) \\
&= \underbrace{\alpha}_{>0} \left[\underbrace{\Big(T_N(\hat{f}) - T_N(\varphi(\hat{f})) \Big)}_{\leq \varepsilon - \rho} + \sum_{f \in \mathsf{facts}_N(s_1) \setminus \{\hat{f}\}} \underbrace{\Big(T_N(f) - T_N(\varphi(f)) \Big)}_{\leq \varepsilon} \right] \\
&\leq \alpha \left[\varepsilon - \rho + \sum_{f \in \mathsf{facts}_N(s_1) \setminus \{\hat{f}\}} \varepsilon \right] \\
&\leq \alpha \cdot \underbrace{\Big(|\mathcal{F}|\varepsilon - \rho \Big)}_{<0}
\end{aligned}$$

To complete the proof, we need to find a lower bound for α that is independent of T and N (note that a *lower* bound on α is required since $|\mathcal{F}|\varepsilon - \rho$ is negative). It is here that we use the assumption that $T = T^n$ for some $n \in \mathbb{N}$. Since $T^n_N(x) \in [0, 1]$ for any $n \in \mathbb{N}$ and $x \in S$, we have

$$|T'_N(x)| = T'_N(x) = \sum_{f \in \mathsf{facts}_N(x)} \underbrace{T_N(f)}_{\leq 1} \leq |\mathsf{facts}_N(x)| \leq |\mathcal{F}|$$

and so

$$\alpha = \frac{1}{\max\limits_{x \in \mathcal{S}} |T'_N(x)|} \geq \frac{1}{|\mathcal{F}|}$$

Combining this with the above bound for $\tilde{T}_N(s_1) - \tilde{T}_n(s_2)$, we get

$$\tilde{T}_N(s_1) - \tilde{T}_n(s_2) \leq \frac{1}{|\mathcal{F}|} \Big(|\mathcal{F}|\varepsilon - \rho \Big) = \varepsilon - \frac{1}{|\mathcal{F}|}\rho$$

Taking $C = 1$ and $D = \frac{1}{|\mathcal{F}|}$, the hypotheses of Lemma 1 are satisfied; thus *Sums* satisfies Source-Coherence.

POI, Strong Independence, PCI and Monotonicity. The remaining axioms are handled by counterexamples derived from the network shown in Fig. 4.2. It can be shown that, if N denotes this network, we have $T^*_N(f) = T^*_N(g) = 0$, so $f \approx^{T^*}_N g$.

Let N' denote the network whose claims are just those of the top connected component. Then it can be shown that $T^*_{N'}(f) = 1$ and $T^*_{N'}(g) = 0$, i.e. $g \prec^{T^*}_{N'} f$. However it is easily verified that our three independence axioms, if satisfied, would each imply $f \preceq^{T^*}_N g$ iff $f \preceq^{T^*}_{N'} g$. Therefore none of POI, Strong Independence and PCI can hold for *Sums*.

For Monotonicity, consider the network N'' obtained from N by removing the edge (u, g). Then we still have $T^*_{N''}(f) = T^*_{N''}(g) = 0$, and in particular $f \preceq^{T^*}_{N''} g$.

Returning to N amounts to adding extra support for the fact g. Monotonicity would give $f \prec_N^{T^*} g$ here, but this is clearly false. Hence Monotonicity is not satisfied by *Sums*.

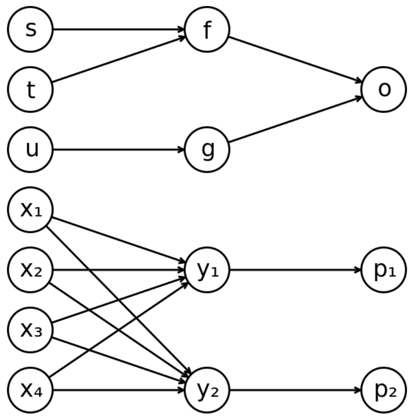

Figure 4.2: Network which yields counterexamples for POI, Strong Independence, PCI and Monotonicity for *Sums*

□

The key to the counterexamples derived from Fig. 4.2 in the above proof lies in the lower connected component, which – restricted to $\mathcal{S} \cup \mathcal{F}$ – is a *connected* bipartite graph. That is, each source x_i claims all facts in the component, and each fact y_j is claimed by all sources in the component. Moreover, sources elsewhere in the network claim fewer facts than the x_i, and facts elsewhere are claimed by fewer sources than the y_j.

Since *Sums* assigns scores by a simple sum, this results in the scores for the x_i and y_j dominating those of the other sources and facts. The normalisation step then divides these scores by the (comparatively large) maximum. As the next result shows, under certain conditions this causes scores to decrease *exponentially* and become 0 in the limit. In particular, we can generate pathological examples such as Fig. 4.2 where a whole connected component receives scores of 0, which leads to failure of Monotonicity and the independence axioms.

Proposition 2. *Let N be a network. Suppose there is $X \subseteq \mathcal{S}$, $Y \subseteq \mathcal{F}$ such that*

1. $\mathsf{facts}_N(x) = Y$ *for each* $x \in X$
2. $\mathsf{src}_N(y) = X$ *for each* $y \in Y$
3. $\mathsf{facts}_N(s) \cap Y = \emptyset$ *and* $|\mathsf{facts}_N(s)| \leq \frac{|Y|}{2}$ *for each* $s \in \mathcal{S} \setminus X$
4. $\mathsf{src}_N(f) \cap X = \emptyset$ *and* $|\mathsf{src}_N(f)| \leq \frac{|X|}{2}$ *for each* $f \in \mathcal{F} \setminus Y$

Then, with $(T^n)_{n\in\mathbb{N}}$ denoting Sums, *for all $n > 1$ we have*

$$T_N^n(s) \leq \frac{1}{2^{n-1}} \quad (s \in \mathcal{S} \setminus X)$$

$$T_N^n(f) \leq \frac{1}{2^{n-1}} \quad (f \in \mathcal{F} \setminus Y)$$

$$T_N^n(x) = 1 \quad (x \in X)$$
$$T_N^n(y) = 1 \quad (y \in Y)$$

In particular, if T^* denotes the limit of Sums *then $T_N^*(s) = T_N^*(f) = 0$ for all $s \in \mathcal{S} \setminus X$ and $f \in \mathcal{F} \setminus Y$.*

Proof. We proceed by induction. The result is easy to show in the base case $n = 2$ since $|\mathsf{facts}_N(s)| \leq \frac{1}{2}|\mathsf{facts}_N(x)|$ for any $x \in X$ and $s \notin X$ (and similarly for facts). Assume the result holds for some $n > 1$. Write $T' = U^{\mathsf{Sums}}(T^n)$, so that $T^{n+1} = \mathsf{norm}(T')$. If $s \notin X$ then $\mathsf{facts}_N(s) \subseteq \mathcal{F} \setminus Y$, so

$$T'_N(s) = \sum_{f \in \mathsf{facts}_N(s)} \underbrace{T_N^n(f)}_{\leq \frac{1}{2^{n-1}}} \leq \frac{|\mathsf{facts}_N(s)|}{2^{n-1}} \leq \frac{\frac{1}{2}|Y|}{2^{n-1}} = \frac{|Y|}{2^{(n+1)-1}}$$

Similarly, if $f \notin Y$ then $\mathsf{src}_N(f) \subseteq \mathcal{S} \setminus X$, so

$$T'_N(f) = \sum_{s \in \mathsf{src}_N(f)} \underbrace{T'_N(s)}_{\leq \frac{|Y|}{2^{(n+1)-1}}} \leq \frac{|\mathsf{src}_N(f)| \cdot |Y|}{2^{(n+1)-1}} \leq \frac{\frac{1}{2}|X| \cdot |Y|}{2^{(n+1)-1}} = \frac{|X| \cdot |Y|}{2^{(n+2)-1}}$$

On the other hand, the fact that $T_N^n(x) = T_N^n(y) = 1$ for $x \in X$ and $y \in Y$ gives

$$T'_N(x) = \sum_{y \in Y} T_N^n(y) = |Y|$$

$$T'_N(y) = \sum_{x \in X} T'_N(x) = |X| \cdot |Y|$$

Clearly the $x \in X$ and $y \in Y$ are the sources and facts with maximal trust and belief scores, respectively. This means that after normalisation via norm, $T_N^{n+1}(x) = T_N^{n+1}(y) = 1$ and for $s \notin X$ and $f \notin Y$,

$$T_N^{n+1}(s) = \frac{T'_N(s)}{|Y|} \leq \frac{1}{2^{(n+1)-1}}$$

$$T_N^{n+1}(f) = \frac{T'_N(f)}{|X| \cdot |Y|} \leq \frac{1}{2^{(n+2)-1}} \leq \frac{1}{2^{(n+1)-1}}$$

This shows that the claim holds for $n + 1$; by induction, the proof is complete. □

4.5.4 Modifying *Voting* and *Sums*

So far we have seen that neither of the basic operators *Voting* or *Sums* are completely satisfactory with respect to the axioms of Section 4.4. Armed with the knowledge of how and why certain axioms fail, one may wonder whether it is possible to modify the operators accordingly so that the axioms *are* satisfied. Presently we shall show that this is partially possible both in the case of *Voting* and *Sums*.

4.5.4.1 Voting

A core problem with *Voting* is that it fails Coherence. Indeed, all sources are ranked equally regardless of the 'votes' for facts, so in some sense it is obvious that the source ranking does not cohere with the fact ranking.[8] An easy improvement is to explicitly construct the source ranking to guarantee Source-Coherence.

Definition 11. *For a network N, define a binary relation $\lhd_N$ on $\mathcal{S}$ by $s_1 \lhd_N s_2$ iff* $\mathsf{facts}_N(s_1)$ *is less believable than* $\mathsf{facts}_N(s_2)$ *with respect to* Voting. *The numerical operator* SC-Voting *(Source-Coherence Voting) is defined by*

$$T_N^{SCV}(s) = |\{t \in \mathcal{S} : t \lhd_N s\}|, \quad T_N^{SCV}(f) = |\mathsf{src}_N(f)|$$

It can be seen that *SC-Voting* satisfies Source-Coherence, which is a significant improvement over regular *Voting*. Since $\lhd_N$ relies on 'global' properties on N, however, this comes at the expense of Source-PCI. Satisfaction of the other axioms is inherited from *Voting*.

Theorem 6. SC-Voting *satisfies Source-Coherence, Symmetry, Unanimity, Groundedness, Monotonicity, Fact-PCI, POI and Strong Independence. It does not satisfy Fact-Coherence or Source-PCI.*

The following properties of $\lhd_N$ are useful for showing Source-Coherence.

Lemma 6. *$\lhd_N$ is transitive and irreflexive.*

Proof. For transitivity, suppose $s \lhd_N t$ and $t \lhd_N u$. Then $\mathsf{facts}_N(s)$ is less believable than $\mathsf{facts}_N(t)$ (with respect to *Voting*) via some bijection $\varphi : \mathsf{facts}_N(s) \to \mathsf{facts}_N(t)$, and $\mathsf{facts}_N(t)$ is less believable than $\mathsf{facts}_N(u)$ via some bijection $\psi : \mathsf{facts}_N(t) \to \mathsf{facts}_N(u)$. It is easily seen that $\mathsf{facts}_N(s)$ is less believable than $\mathsf{facts}_N(u)$ via the composition $\theta = \psi \circ \varphi$, so $s \lhd_N u$.

For irreflexivity, suppose for contradiction that $s \lhd_N s$ for some $s \in \mathcal{S}$, i.e. $F = \mathsf{facts}_N(s)$ is less believable than itself under some bijection $\varphi : F \to F$. Then $f \preceq_N^T \varphi(f)$ for each $f \in F$, and there is $\hat{f}$ such that $\hat{f} \prec_N^T \varphi(\hat{f})$. Iterating applications of φ, we get

$$\hat{f} \prec_N^T \varphi(\hat{f}) \preceq_N^T \varphi(\varphi(\hat{f}) \preceq_N^T \cdots \preceq_N^T \varphi^n(\hat{f}) \tag{4.2}$$

for each $n \geq 1$, where φ^n is the n-th iterate of φ and T denotes *Voting*.

Since F is finite, the sequence $\varphi(\hat{f}), \varphi(\varphi(\hat{f})), \ldots$ must repeat at some point, i.e. there is $i < j$ such that $\varphi^i(\hat{f}) = \varphi^j(\hat{f})$. But then injectivity of φ implies that $\hat{f} = \varphi^{j-i}(\hat{f})$. Taking $n = j - i$ in Eq. (4.2) we get $\hat{f} \prec_N^T \hat{f}$ – a contradiction. □

Theorem 6 (sketch). Note that *SC-Voting* inherits Unanimity, Groundedness, Monotonicity, Fact-PCI, POI and Strong Independence from *Voting*, since these axioms only refer to the rankings of facts (which is the same for *SC-Voting* as for *Voting*).

We take the remaining axioms in turn. To simplify notation, write $W_N(s) = \{t \in \mathcal{S} : t \lhd_N s\}$ in what follows.

[8] Fact-Coherence is vacuously satisfied, however: since all sources rank equally we can never have $\mathsf{src}_N(f_1)$ less trustworthy than $\mathsf{src}_N(f_2)$.

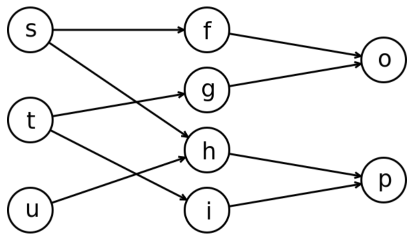

Figure 4.3: Fact-Coherence counterexample for *SC-Voting*

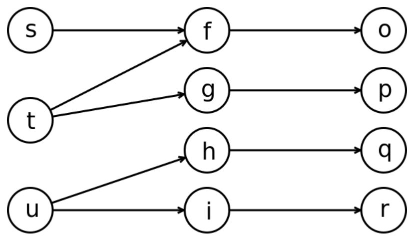

Figure 4.4: Source-PCI counterexample for *SC-Voting*

Source-Coherence. Suppose facts$_N(s_1)$ is less believable than facts$_N(s_2)$ with respect to T^{SCV}. We need to show $s_1 \sqsubset_N^{T^{SCV}} s_2$.

Note that since the fact ranking for T^{SCV} coincides with *Voting*, we have $s_1 \lhd_N s_2$. Transitivity of $\lhd_N$ gives $W_N(s_1) \subseteq W_N(s_2)$. Moreover, $s_1 \in W_N(s_2)$ but by irreflexivity, $s_1 \notin W_N(s_1)$. Therefore $W_N(s_1) \subset W_N(s_2)$, which means $T_N^{SCV}(s_1) = |W_N(s_1)| < |W_N(s_2)| = T_N^{SCV}(s_2)$, i.e. $s_1 \sqsubset_N^{T^{SCV}} s_2$ as required.

Symmetry. Since the fact ranking of T^{SCV} is the same as *Voting*, which satisfies Symmetry, we only need to show that $s_1 \sqsubseteq_N^{T^{SCV}} s_2$ iff $\pi(s_1) \sqsubseteq_{\pi(N)}^{T^{SCV}} \pi(s_2)$ for all equivalent networks $N, \pi(N)$ and $s_1, s_2 \in \mathcal{S}$.

In can be shown, and we do so in the appendix, that the Symmetry of *Voting* implies a symmetry property for $\lhd_N$ and $\lhd_{\pi(N)}$: we have $s_1 \lhd_N s_2$ iff $\pi(s_1) \lhd_{\pi(N)} \pi(s_2)$. Consequently, $t \in W_N(s_i)$ iff $\pi(t) \in W_{\pi(N)}(\pi(s_i))$; in particular, $|W_N(s_i)| = |W_{\pi(N)}(\pi(s_i))|$. This means

$$\begin{aligned} s_1 \sqsubseteq_N^{T^{SCV}} s_2 &\iff |W_N(s_1)| \leq |W_N(s_2)| \\ &\iff |W_{\pi(N)}(\pi(s_1))| \leq |W_{\pi(N)}(\pi(s_2))| \\ &\iff \pi(s_1) \sqsubseteq_{\pi(N)}^{T^{SCV}} \pi(s_2) \end{aligned}$$

as required for Symmetry.

Fact-Coherence. Consider the network shown in Fig. 4.3. We have $f \approx g \approx i \prec h$. Source-Coherence between s and t gives $t \sqsubset s$. If Fact-Coherence held we would then get $g \prec f$, but this is not the case.

Source-PCI. Let N_1 denote the top connected component of the network shown in Fig. 4.4, and let N_2 denote the network as a whole. The fact ranking is the same in both networks: $g \approx h \approx i \prec f$. In N_1 sources s and t claim a different number of facts, so neither $s \lhd_{N_1} t$ nor $t \lhd_{N_1} s$. Consequently $W_{N_1}(s) = W_{N_1}(t) = \emptyset$ and $s \simeq_{N_1}^{T^{SCV}} t$.

In N_2 sources t and u can be compared for Source-Coherence, and we see that $u \lhd_{N_2} t$ since $i \preceq_{N_2}^{T^{SCV}} g$ and $h \prec_{N_2}^{T^{SCV}} f$. Hence $W_{N_2}(t) = \{u\}$ and $W_{N_2}(s) = \emptyset$, which means $s \sqsubset_{N_2}^{T^{SCV}} t$. This contradicts Source-PCI, which requires the ranking of s and t to be the same in both networks. □

Note that the idea behind *SC-Voting* can be generalised beyond *Voting*: it is possible to define $\lhd_N$ in terms of *any* operator T, and to construct a new operator T' whose source ranking is given according to $\lhd_N$ as above, and whose fact ranking coincides with that of T. This ensures T' satisfies Source-Coherence whilst keeping the existing fact ranking from T.

Moreover we can go in the other direction and ensure *Fact*-Coherence whilst retaining the source ranking of T by defining a relation $\blacktriangleleft_N$ on $\mathcal{F}$ in a analogous manner to $\lhd_N$, and proceeding similarly.

4.5.4.2 Sums

Our main concern with *Sums* is the failure of PCI and Monotonicity. We have seen that this is in some sense caused by the normalisation step: in Fig. 4.2 the scores of f, g etc go to 0 in the limit after dividing the 'global' maximum score across the network, which happens to come from a different connected component.

A natural fix for PCI is to therefore divide by the maximum score *within each component*. In this case the score for a source s depends only on the structure of the connected component in which it lies, which is exactly what is required for PCI.

However, this approach does not negate the undesirable effects of Proposition 2. Indeed, suppose the network in Fig. 4.2 was modified so that fact y_1 is associated with object o instead of p_1. Clearly Proposition 2 still applies after this change, and all sources and facts shown now belong to the same connected component. Therefore the 'per-component *Sums*' operator gives the same result as *Sums* itself, and in particular our Monotonicity counterexample still applies. Perhaps even worse, one can show that Coherence fails for this operator. We consider the loss of Coherence too high a price to pay for PCI.

Instead, let us take a step back and consider if normalisation is truly necessary. On the one hand, without normalisation the trust and belief scores are unbounded and therefore do not converge. On the other, we are not interested in the numeric scores for their own sake, but rather for the *rankings* that they induce. It may be possible that whilst the scores diverge without normalisation, the induced rankings *do* converge to a fixed one, which we may take as the 'ordinal limit'. This is in fact the case. We call this new operator *UnboundedSums*.

Definition 12. UnboundedSums *is the recursive operator* $\mathsf{rec}(T^{prior}, U^{Sums})$ *where* $T_N^{prior}(s) = 1$, $T_N^{prior}(f) = |\mathsf{src}_N(f)|$ *and* U^{Sums} *is defined as in Section 4.3.2.*[9]

[9] Note that to simplify proof of ordinal convergence we use a different prior operator to *Sums*, but this does not effect the operator in any significant way.

Theorem 7. UnboundedSums *is ordinally convergent in the following sense: there is an ordinal operator T^* such that for each network N there exists $J_N \in \mathbb{N}$ such that $T^n_N(s_1) \leq T^n_N(s_2)$ iff $s_1 \sqsubseteq^{T^*}_N s_2$ for all $n \geq J_N$ and $s_1, s_2 \in \mathcal{S}$ (and similarly for facts).*

That is, the rankings induced by T^n remain constant after J_N iterations, and are identical to those of T^*.

Proof. The proof will use some results from linear algebra, so we will work with a matrix and vector representation of *UnboundedSums*. Fix an enumeration $\mathcal{S} = \{s_1, \ldots, s_k\}$ of $\mathcal{S}$ and $\mathcal{F} = \{f_1, \ldots, f_l\}$ of $\mathcal{F}$. Write M for the $k \times l$ matrix given by

$$[M]_{ij} = \begin{cases} 1 & \text{if } s_i \in \mathsf{src}_N(f_j) \\ 0 & \text{otherwise} \end{cases} \qquad (1 \leq i \leq k, 1 \leq j \leq l)$$

We also write v_n and w_n for the vectors of trust and belief scores of *UnboundedSums* at iteration n; that is

$$\begin{aligned} v_n &= [T^n_N(s_1), \ldots, T^n_N(s_k)]^\top \in \mathbb{R}^k \\ w_n &= [T^n_N(f_1), \ldots, T^n_N(f_l)]^\top \in \mathbb{R}^l \end{aligned}$$

where $(T^n)_{n\in\mathbb{N}}$ denotes *UnboundedSums*.

Multiplication by M encodes the update step of *UnboundedSums*: it is easily shown that $v_{n+1} = Mw_n$ and $w_{n+1} = M^\top v_{n+1}$. Writing $A = MM^\top \in \mathbb{R}^{k\times k}$, we have $v_{n+1} = Av_n$, and therefore $v_{n+1} = A^n v_1$.

To show that the rankings of *UnboundedSums* remain constant after finitely many iterations, we will show that for each $s_p, s_q \in \mathcal{S}$ there is $J_{pq} \in \mathbb{N}$ such that $\operatorname{sign}([v_n]_p - [v_n]_q)$ is constant for all $n \geq J_{pq}$. Since $[v_n]_p$ and $[v_n]_q$ are the trust scores of s_p and s_q respectively in the n-th iteration, this will show that the ranking of s_p and s_q remains the same after J_{pq} iterations. Since there are only finitely many pairs of sources, we may then take J_N as the maximum value of J_{pq} over all pairs (p, q), and the entire source ranking $\sqsubseteq^{T^n}_N$ of *UnboundedSums* remains constant for $n \geq J_N$. An almost identical argument can be carried out for the fact ranking, and these together will prove the result.

So, fix $s_p, s_q \in \mathcal{S}$. Write $\delta_n = [v_n]_p - [v_n]_q$. First note that $A = MM^\top$ is symmetric, so the *spectral theorem* gives the existence of k orthogonal eigenvectors $x_1, \ldots, x_k$ for A [6, Theorem 7.29]. Let $\lambda_1, \ldots, \lambda_k$ be the corresponding eigenvalues. Form a $(k \times k)$-matrix P whose i-th column is x_i, and let $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$. Then A can be diagonalised as $A = PDP^{-1}$. It follows that for any $n \in \mathbb{N}$, $A^n = PD^nP^{-1}$.

Now, since $x_1, \ldots, x_k$ are orthogonal, P is an orthogonal matrix, i.e. $P^\top = P^{-1}$. Hence $A^n = PD^nP^\top$. Note that

$$PD^n = \begin{bmatrix} x_1 \mid \ldots \mid x_k \end{bmatrix} \begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{bmatrix} = \begin{bmatrix} \lambda_1^n x_1 \mid \ldots \mid \lambda_k^n x_k \end{bmatrix}$$

and

$$P^\top v_1 = \begin{bmatrix} x_1 \\ - \\ \vdots \\ - \\ x_k \end{bmatrix} v_1 = \begin{bmatrix} x_1 \cdot v_1 \\ \vdots \\ x_k \cdot v_1 \end{bmatrix}$$

which means

$$v_{n+1} = A^n v_1 = PD^n P^\top v_1 = \left[\lambda_1^n x_1 \mid \ldots \mid \lambda_k^n x_k\right] \begin{bmatrix} x_1 \cdot v_1 \\ \vdots \\ x_k \cdot v_1 \end{bmatrix} = \sum_{i=1}^{k} (x_i \cdot v_1) \lambda_i^n x_i$$

We obtain an explicit formula for δ_{n+1}:

$$\delta_{n+1} = [v_n]_p - [v_n]_q = \sum_{i=1}^{k} (x_i \cdot v_1) \lambda_i^n \left([x_i]_p - [x_i]_q\right) = \sum_{i=1}^{k} r_i \lambda_i^n \tag{4.3}$$

where $r_i = (x_i \cdot v_1)\left([x_i]_p - [x_i]_q\right)$. Note that r_i does not depend on n.

Now, it is easy to see that $A = MM^\top$ is *positive semi-definite*, which means its eigenvalues $\lambda_1, \ldots, \lambda_k$ are all non-negative. We re-index the sum in Eq. (4.3) by grouping together the λ_i which are equal, to get

$$\delta_{n+1} = \sum_{t=1}^{K} R_t \mu_t^n$$

where $K \leq k$, each R_t is a sum of the r_i (whose corresponding λ_i are equal), and the μ_t are distinct and non-negative. Assume without loss of generality that $\mu_1 > \mu_2 > \cdots > \mu_K \geq 0$. If $R_t = 0$ for all t, then clearly $\operatorname{sign}(\delta_{n+1}) = \operatorname{sign}(0) = 0$ which is constant, so we are done. Otherwise, let T be the minimal t such that $R_t \neq 0$. We may also assume $\mu_T > 0$ (otherwise we necessarily have $\mu_T = 0, T = K$ and $\operatorname{sign}(\delta_{n+1}) = \operatorname{sign}(R_T \cdot 0^n)$ which is again constant 0). Observe that

$$\delta_{n+1} = R_T \mu_T^n + \sum_{t=T+1}^{K} R_t \mu_t^n = \mu_T^n \left[R_T + \sum_{t=T+1}^{K} R_t \left(\frac{\mu_t}{\mu_T}\right)^n\right]$$

By our assumption on the ordering of the μ_t, we have $\mu_t < \mu_T$ in the sum. Consequently $|\mu_t/\mu_T| < 1$, and $(\mu_t/\mu_T)^n \to 0$ as $n \to \infty$. This means

$$\lim_{n\to\infty} \left[R_T + \sum_{t=T+1}^{K} R_t \underbrace{\left(\frac{\mu_t}{\mu_T}\right)^n}_{\to 0}\right] = R_T \neq 0$$

Since this limit is non-zero, there is $J_{pq} \in \mathbb{N}$ such that the sign of term in square brackets is equal to $S = \operatorname{sign} R_T \in \{1, -1\}$ for all $n \geq J_{pq}$. Finally, for such n we have

$$\operatorname{sign} \delta_{n+1} = \operatorname{sign}\left(\underbrace{\mu_T^n}_{>0} \left[R_T + \sum_{t=T+1}^{K} R_t \left(\frac{\mu_t}{\mu_T}\right)^n\right]\right) = \operatorname{sign}\left(R_T + \sum_{t=T+1}^{K} R_t \left(\frac{\mu_t}{\mu_T}\right)^n\right) = S$$

i.e. $\operatorname{sign} \delta_n$ is constant for $n \geq J_{pq} + 1$. This completes the proof.[10] □

[10] The argument which shows that the difference between fact belief scores is also eventually constant in sign is almost identical. Write $B = M^\top M$, and observe that $w_{n+1} = B^n w_1$. Since B is also symmetric and positive semi-definite, the proof goes through as above.

In light of Theorem 7, we may consider *UnboundedSums* itself as an ordinal operator T^*, where $s \sqsubseteq_N^{T^*} t$ iff $s \sqsubseteq_N^{T^{J_n}} t$ for each network N (and similarly for the fact ranking). Moreover, with the normalisation problems aside, *UnboundedSums* provides an example of a principled operator satisfying our two key axioms – Coherence and PCI. In particular, we escape the undesirable behaviour of *Sums* in Fig. 4.2; whereas *Sums* trivialises the ranking of sources and facts in the upper connected component, *UnboundedSums* allows meaningful ranking (e.g. we have $g \prec f$). In particular, the counterexample for Monotonicity for *Sums* is no longer a counterexample for *UnboundedSums*. We conjecture that *UnboundedSums* also satisfies Monotonicity, but this remains to be proven. For example, we have experimentally verified that *UnboundedSums* satisfies all the specific instances of Monotonicity with the starting network N as in Fig. 4.1.

Theorem 8. UnboundedSums *satisfies Coherence, Symmetry, Unanimity, Groundedness and PCI.* UnboundedSums *does not satisfy POI and Strong Independence.*

Proof (sketch). The proof that *UnboundedSums* satisfies Symmetry, PCI, Unanimity and Groundedness is routine, and we leave the details to the appendix. In what follows, let $(T^n)_{n\in\mathbb{N}}$ denote *UnboundedSums*, T^* denote the ordinal limit of *UnboundedSums*, and for a network N let J_N be as in Theorem 7. Then the rankings in N induced by T^n for $n \geq J_N$ are the same as T^*.

Coherence. First we show Source-Coherence. Let N be a network and suppose $\mathsf{facts}_N(s_1)$ is less believable than $\mathsf{facts}_N(s_2)$ with respect to N and T^*. Let φ and $\hat{f}$ be as in the definition of less believable.

Let $n \geq J_N$. Then $f \preceq_N^{T^*} \varphi(f)$ and $\hat{f} \prec_N^{T^*} \varphi(\hat{f})$ for each $f \in \mathsf{facts}_N(s_1)$ means $T_N^n(f) \leq T_N^n(\varphi(f))$ and $T_N^n(\hat{f}) < T_N^n(\varphi(\hat{f}))$. Hence

$$\begin{aligned}
T_N^{n+1}(s) &= \sum_{f\in\mathsf{facts}_N(s_1)} T_N^n(f) \\
&= T_N^n(\hat{f}) + \sum_{f\in\mathsf{facts}_N(s_1)\setminus\{\hat{f}\}} T_N^n(f) \\
&< T_N^n(\varphi(\hat{f})) + \sum_{f\in\mathsf{facts}_N(s_1)\setminus\{\hat{f}\}} T_N^n(\varphi(f)) \\
&= \sum_{f\in\mathsf{facts}_N(s_1)} T_N^n(\varphi(f)) \\
&= \sum_{g\in\mathsf{facts}_N(s_2)} T_N^n(g) \\
&= T_N^{n+1}(s_2)
\end{aligned}$$

i.e. $T_N^{n+1}(s_1) < T_N^{n+1}(s_2)$. But T_N^{n+1} gives the same ranking as T_N^n and therefore the same ranking as T^*, so we get $s_1 \sqsubset_N^{T^*} s_2$ as required.

For Fact-Coherence, suppose $\mathsf{src}_N(f_1)$ is less trustworthy than $\mathsf{src}_N(f_2)$ with respect to N and T^*. Again, let $n \geq J_N$ and φ, $\hat{s}$ be as in the definition of less trustworthy. Recall that belief scores for facts in T_N^n are obtained from trust scores in T_N^n; applying the same argument as above we get $T_N^n(f_1) < T_N^n(f_2)$ and consequently $f_1 \preceq_N^{T^*} f_2$ as required. Hence T^* satisfies Coherence.

Table 4.1: Satisfaction of the axioms for the various operators. Recall that POI and Strong Independence are undesirable properties.

	Voting	SC-Voting	Sums	U-Sums
Source-Coherence	X	✓	✓	✓
Fact-Coherence	✓	X	✓	✓
Symmetry	✓	✓	✓	✓
Unanimity	✓	✓	✓	✓
Ground.	✓	✓	✓	✓
Mon.	✓	✓	X	?
Source-PCI	✓	X	X	✓
Fact-PCI	✓	✓	X	✓
POI	✓	✓	X	X
Str. Indep.	✓	✓	X	X

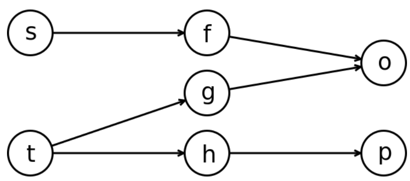

Figure 4.5: Counterexample for POI and Strong Independence for *UnboundedSums*

POI and Strong Independence. To show POI and Strong Independence are not satisfied, consider the network N shown in Fig. 4.5. It can be seen (e.g. by induction) that

$$T^n_N(f) = 1, \quad T^n_N(g) = 2^{n-1}$$

for all $n \in \mathbb{N}$. Consequently $f \prec^{T^*}_N g$.[11]

Now let N' be the network in which the claim (t, h) is removed. Since $\mathsf{src}_N(f) = \mathsf{src}_{N'}(f) = \{s\}$ and $\mathsf{src}_N(g) = \mathsf{src}_{N'}(g) = \{t\}$, both POI and Strong Independence imply $f \preceq^{T^*}_N g$ iff $f \preceq^{T^*}_{N'} g$. Therefore assuming either of POI or Strong Independence we get $f \prec^{T^*}_{N'} g$. However is is also clear that

$$T^n_{N'}(f) = T^n_{N'}(g) = 1$$

for all $n \in \mathbb{N}$, so $f \approx^{T^*}_{N'} g$. This is a contradiction, so neither POI nor Strong Independence are satisfied. □

To summarise this section, Table 4.1 shows which axioms are satisfied by each of the operators.

4.6 Variable domain truth discovery

So far, we have considered an arbitrary but fixed (finite) domain of sources, facts and objects $(\mathcal{S}, \mathcal{F}, \mathcal{O})$. Our operators and axioms were defined with respect to this

[11] Note that g ranks higher than f in this network simply because t makes more claims than s, and each fact is claimed only by a single source. This questionable property of *UnboundedSums* is inherited from *Sums*.

domain. However, the operators do not *depend* on the domain: they can be defined for *any* choice of $\mathcal{S}$, $\mathcal{F}$ and $\mathcal{O}$. In this section we generalise the framework so that these sets are no longer fixed. This allows new situations to be modelled, such as new sources entering the network. Adapting the definition of a TD operator to this case, we can then see how the ranking of facts changes as new sources are added. Such variable domain operators are then analogues of *variable electorate voting rules* in social theory.

Formally, let $\mathbb{S}$, $\mathbb{F}$ and $\mathbb{O}$ be countably infinite sets of sources, facts and objects respectively. A *domain* is a triple $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$, where $\mathcal{S} \subseteq \mathbb{S}$, $\mathcal{F} \subseteq \mathbb{F}$ and $\mathcal{O} \subseteq \mathbb{O}$ are finite, non-empty sets. We think of $\mathbb{S}$, $\mathbb{F}$ and $\mathbb{O}$ as being the 'universe' of possible sources, facts and objects, and a domain as the (finite) sets of entities under consideration in a particular TD problem. Given a domain $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$, we define $\mathcal{D}$-networks and $\mathcal{D}$-operators as in Definitions 1 and 2.

Definition 13. *A* variable domain operator T *is a mapping which maps each domain* $\mathcal{D}$ *to a* $\mathcal{D}$*-operator* $T_{\mathcal{D}}$.

Note that for a domain $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$ and a $\mathcal{D}$-network N, $\sqsubseteq_N^{T_{\mathcal{D}}}$ and $\preceq_N^{T_{\mathcal{D}}}$ are rankings only over the set of sources $\mathcal{S}$ and $\mathcal{F}$ in the domain $\mathcal{D}$, *not* all of $\mathbb{S}$ and $\mathbb{F}$. If $\mathcal{D}$ is clear from context, we write $\sqsubseteq_N^T$ and $\preceq_N^T$ without explicit reference to the domain.

Since all the axioms so far were stated with respect to a fixed but arbitrary domain, they can be easily lifted to the variable domain case. For instance, we say a variable domain operator T satisfies Coherence if $T_{\mathcal{D}}$ satisfies the instance of Coherence for domain $\mathcal{D}$, for all $\mathcal{D}$, and similar for the other axioms.

But we can now go further, and introduce axioms which make use of *several* domains. First, we generalise Symmetry to act across domains. Say networks N, N' in domains $\mathcal{D}, \mathcal{D}'$ respectively are *equivalent* if there is a graph isomorphism π between them such that $\pi(s) \in \mathcal{S}'$, $\pi(f) \in \mathcal{F}'$ and $\pi(o) \in \mathcal{O}'$ for all $s \in \mathcal{S}$, $f \in \mathcal{F}$ and $o \in \mathcal{O}$.

Axiom 9 (Isomorphism)**.** *Let* N *and* $N' = \pi(N)$ *be equivalent networks. Then for all* $s_1, s_2 \in \mathcal{S}$, $f_1, f_2 \in \mathcal{F}$, *we have* $s_1 \sqsubseteq_N^T s_2$ *iff* $\pi(s_1) \sqsubseteq_{N'}^T \pi(s_2)$ *and* $f_1 \preceq_N^T f_2$ *iff* $\pi(f_1) \preceq_{N'}^T \pi(f_2)$.

Like Symmetry, Isomorphism simply says that operators only care about the *structure* of the network, not the particular 'names' chosen for sources, facts and objects. Symmetry is obtained as the special case where N and N' are equivalent when seen as networks in a common domain $\mathcal{D}$. All the operators of Sections 4.3 and 4.5.4 satisfy Isomorphism.

Next we introduce a new monotonicity property. In what follows, for a network $N = (V, E)$ in domain $(\mathcal{S}, \mathcal{F}, \mathcal{O})$, $f \in \mathcal{F}$ and $\mathcal{S}' \subseteq \mathbb{S}$ finite and disjoint from $\mathcal{S}$, write $N + (\mathcal{S}', f)$ for the network in domain $(\mathcal{S} \cup \mathcal{S}', \mathcal{F}, \mathcal{O})$ with edge set $E \cup \{(s, f) \mid s \in \mathcal{S}'\}$, i.e. the extension of N where a collection of 'fresh' sources $\mathcal{S}'$ each claim f. For example, Fig. 4.6 shows $N + (\mathcal{S}', h)$ for the network N from Fig. 4.1 and new sources $\mathcal{S}' = \{x_1, \ldots, x_4\}$.

Axiom 10 (Eventual Monotonicity)**.** *Let* $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$ *be a domain and* N *a* $\mathcal{D}$*-network. Then for all* $f, g \in \mathcal{F}$, $f \neq g$, *there is a finite, non-empty set* $\mathcal{S}' \subseteq \mathbb{S}$ *with* $\mathcal{S} \cap \mathcal{S}' = \emptyset$ *and* $g \prec_{N+(\mathcal{S}', f)}^T f$.

This axiom says that, given any pair of distinct facts f, g, it is possible to add enough new claims for f to make f strictly more believable than g. Intuitively, this is less demanding that Monotonicity, which requires that f can become strictly more believable than g with the addition of just *one* additional claim. Note that Eventual Monotonicity is not possible to state in the fixed domain case (e.g. consider N already containing claims from all the available sources in $\mathcal{S}$).

When paired with Isomorphism, Eventual Monotonicity takes on a form similar to postulates for *Improvement* and *Majority* operators in belief merging [37, 38]: there is a threshold $n \in \mathbb{N}$ such that f becomes strictly more believable than g after n new claims are added for f. That is, the identities of the new sources $\mathcal{S}'$ are irrelevant; it is just the *size* of $\mathcal{S}'$ that matters. We require a preliminary lemma.

Lemma 7. *Suppose a variable domain operator T satisfies Isomorphism. Let $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$ be a domain, N a network in $\mathcal{D}$ and $f \in \mathcal{F}$. Then for all non-empty, finite sets $\mathcal{S}'_1, \mathcal{S}'_2 \subseteq \mathbb{S}$ disjoint from $\mathcal{S}$ with $|\mathcal{S}'_1| = |\mathcal{S}'_2|$,*

$$\preceq^T_{N+(\mathcal{S}'_1, f)} = \preceq^T_{N+(\mathcal{S}'_2, f)}$$

Proof. Write $\mathcal{D}_1 = (\mathcal{S} \cup \mathcal{S}'_1, \mathcal{F}, \mathcal{O})$ and $\mathcal{D}_2 = (\mathcal{S} \cup \mathcal{S}'_2, \mathcal{F}, \mathcal{O})$. Then $N + (\mathcal{S}'_i, f)$ is a network in domain $\mathcal{D}_i$ (for $i \in \{1, 2\}$). Since $|\mathcal{S}'_1| = |\mathcal{S}'_2|$ by assumption, there is a bijection $\varphi : \mathcal{S}'_1 \to \mathcal{S}'_2$. Define a mapping π from $\mathcal{D}_1$ to $\mathcal{D}_2$ by

$$\pi(s) = \begin{cases} s, & s \in \mathcal{S} \\ \varphi(s), & s \in \mathcal{S}'_1 \end{cases} \quad (s \in \mathcal{S} \cup \mathcal{S}'_1)$$

and $\pi(g) = g$, $\pi(o) = o$ for $g \in \mathcal{F}$ and $o \in \mathcal{O}$. Then it is easily verified that π is an isomorphism from $N + (\mathcal{S}'_1, f)$ to $N + (\mathcal{S}'_2, f)$. For $g_1, g_2 \in \mathcal{F}$, we have $g_1 \preceq^T_{N+(\mathcal{S}'_1, f)} g_2$ iff $\pi(g_1) \preceq^T_{N+(\mathcal{S}'_2, f)} \pi(g_2)$ by Isomorphism. Since $\pi(g_1) = g_1$ and $\pi(g_2) = g_2$, this shows $\preceq^T_{N+(\mathcal{S}'_1, f)} = \preceq^T_{N+(\mathcal{S}'_2, f)}$. □

Note that since $\mathbb{S}$ is infinite and domains are finite, for any $n \in \mathbb{N}$ and any domain $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$ there is always some $\mathcal{S}' \subseteq \mathbb{S}$, disjoint from $\mathcal{S}$, with $|\mathcal{S}'| = n$.

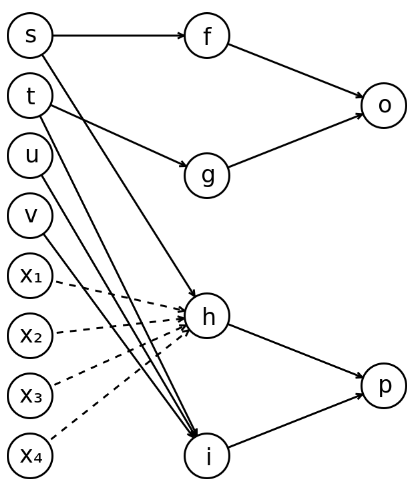

Figure 4.6: $N + (\mathcal{S}', h)$, where N is the network from Fig. 4.1 and $\mathcal{S}' = \{x_1, \dots, x_4\}$.

For operators T satisfying Isomorphism, write $\preceq^T_{N+(n\times f)}$ for $\preceq^T_{N+(\mathcal{S}',f)}$; Lemma 7 guarantees this is well-defined (i.e. does not depend on the particular choice of $\mathcal{S}'$). That is, $\preceq^T_{N+(n\times f)}$ is the fact ranking resulting from adding n new claims for f from fresh sources. We obtain an equivalent characterisation of Eventual Monotonicity, whose proof is almost immediate given Lemma 7.

Proposition 3. *Suppose T satisfies Isomorphism. Then T satisfies Eventual Monotonicity if and only if for all domains $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$, all networks N in $\mathcal{D}$ and distinct $f, g \in \mathcal{F}$, there is $n \in \mathbb{N}$ such that $g \prec^T_{N+(n\times f)} f$.*

Proof (sketch). 'if': To show Eventual Monotonicity, take any $\mathcal{S}' \subseteq \mathbb{S} \setminus \mathcal{S}$ of size n.

'Only if': Given that Eventual Monotonicity holds, simply take $n = |\mathcal{S}'|$. □

We can now show that all operators studied so far – when lifted to the variable domain case – satisfy Eventual Monotonicity.

Theorem 9. Voting, Sums, SC-Voting *and* UnboundedSums *satisfy Eventual Monotonicity.*

Proof (sketch). Let $\mathcal{D} = (\mathcal{S}, \mathcal{F}, \mathcal{O})$ be a domain, N a network in $\mathcal{D}$ and $f, g \in \mathcal{F}$. Given that Isomorphism holds for each operator, we sketch the proof via Proposition 3.

For *Voting* and *SC-Voting*, we may simply take $n = 1 + |\mathsf{src}_N(g)|$. For *Sums* and *UnboundedSums*, take $n = 2|\mathcal{S}| \cdot |\mathcal{F}|$. Write $N' = N + (\mathcal{S}', f)$ for some $\mathcal{S}' \subseteq \mathbb{S} \setminus \mathcal{S}$ with $|\mathcal{S}'| = n$

If $(T^k)_{k\in\mathbb{N}}$ denotes *Sums*, one can show by induction that $T^k_{N'}(f) = 1$ and $T^k_{N'}(h) \leq \frac{1}{2}$ for any $h \neq f$ and $k > 1$, and thus $g \prec^{T^{\mathit{Sums}}}_{N'} f$.

Similarly, letting $(T^k)_{k\in\mathbb{N}}$ denote *UnboundedSums*, one can show by induction that $T^k_{N'}(f) > T^k_{N'}(h)$ for $h \neq f$, and thus $g \prec^{T^{\mathit{UnboundedSums}}}_{N'} g$.

□

To conclude this section, we show that the impossibility result of Theorem 2 holds in the variable domain case if one replaces Monotonicity with Eventual Monotonicity and Symmetry with Isomorphism.

Theorem 10. *There is no variable domain operator satisfying Coherence, Isomorphism, Eventual Monotonicity and POI.*

Proof. For contradiction, suppose T is an operator satisfying the stated axioms. Let N be the network from Fig. 4.1, viewed as a network in domain $(\{s,t,u,v\}, \{f,g,h,i\}, \{o,p\})$. Applying Eventual Monotonicity with i and h, we have that there is N' with $i \prec^T_{N'} h$, where $N' = N + (\mathcal{S}', h)$ for some $\mathcal{S}' \subseteq \mathbb{S} \setminus \{s,t,u,v\}$. Since N' only adds claims for p-facts, POI applied to object o and Isomorphism give $f \approx^T_{N'} g$ (e.g. consider π which simply swaps s with t and f with g). From Source-Coherence we get $t \sqsubset^T_{N'} s$. But $\mathsf{src}_{N'}(f) = \{s\}$ and $\mathsf{src}_{N'}(g) = \{t\}$, so Fact-Coherence gives $g \prec^T_{N'} f$: contradiction! □

4.7 Discussion

In this section we discuss the axioms and their limitations. First, the version of Monotonicity we consider is a strict one: a new claim for f gives f a *strictly* positive boost in the fact believability ranking. This is also the case for Eventual Monotonicity in the variable domain case, where we require that some number of new claims make f strictly more believable than any other fact g. As noted in Section 4.4.3, this assumes there is no *collusion* between sources. Indeed, suppose sources s_1, s_2 are in collusion. For example, s_2 may agree to unconditionally back up all claims made by s_1. In this case a claim of f from s_1 alone should carry just as much weight as the claim from both s_1 and s_2. However, Monotonicity requires that f becomes strictly more believable when moving to the latter case.

A natural solution is to simply relax the strictness requirement. We obtain the following weak version of Monotonicity.

Axiom 11 (Weak Monotonicity)**.** *Let N, s, f, N' be as in the statement of Monotonicity. Then for all $g \neq f$, $g \preceq_N^T$ implies $g \preceq_{N'}^T f$.*

Weak Monotonicity only says says that extra support for a fact f does not make f *less* believable. Clearly Monotonicity implies Weak Monotonicity, but not vice versa. In the collusion example, an operator may select to leave the fact ranking unchanged when a new report of f from s_2 arrives; this is inconsistent with Monotonicity but consistent with Weak Monotonicity. The weak analogue of Eventual Monotonicity can be defined in the same way.

In the same spirit, one could consider versions of Coherence only using weak comparisons. Say $\mathsf{facts}_N(s_1)$ is *weakly less believable* than $\mathsf{facts}_N(s_2)$ iff the condition in Definition 7 holds, but without the requirement that some $\hat{f} \in \mathsf{facts}_N(s_1)$ is strictly less believable than its counterpart $\varphi(\hat{f})$ in $\mathsf{facts}_N(s_2)$, and define $\mathsf{src}_N(f_1)$ weakly less trustworthy than $\mathsf{src}_N(f_2)$ in a similar way. The weak analogue of Coherence is as follows.

Axiom 12 (Weak Coherence)**.** *For any network N, $\mathsf{facts}_N(s_1)$ weakly less believable than $\mathsf{facts}_N(s_2)$ implies $s_1 \sqsubseteq_N^T s_2$, and $\mathsf{src}_N(f_1)$ weakly less trustworthy than $\mathsf{src}_N(f_2)$ implies $f_1 \preceq_N^T f_2$.*

Note that Coherence does *not* imply Weak Coherence. This is because Weak Coherence relaxes both the consequent *and the antecedent* in the implications in the statement of the axiom. Whereas Coherence imposes no constraint when $\mathsf{facts}_N(s_1)$ is only weakly less believable than $\mathsf{facts}_N(s_2)$, Weak Coherence requires $s_1 \sqsubseteq_N^T s_2$. Consequently, the 'weakness' of Weak Coherence refers to its use of weak comparisons between sources and facts, not its logical strength in relation to Coherence.

A natural question now arises. Does the impossibility result of Theorem 2 still hold with these new axioms? We have an answer in the negative: the 'flat' operator, which sets all sources and facts equally ranked in all networks, satisfies all the axioms of the would-be impossibility.

Proposition 4. *Define an operator T by $s_1 \simeq_N^T s_2$ and $f \approx_N^T f_2$ for all networks N, sources s_1, s_2 and facts f_1, f_2. Then T satisfies Coherence, Weak Coherence, Symmetry, Weak Monotonicity and POI.*

Proof. Coherence holds vacuously since we can never have $\mathsf{facts}_N(s_1)$ less believable than $\mathsf{facts}_N(s_2)$ or $\mathsf{src}_N(f_1)$ less believable than $\mathsf{src}_N(f_2)$. Since *any* weak ranking holds for T, the other axioms are straightforward to see. □

This shows that (strict) Monotonicity is required for the impossibility result, since the result is no longer true when relaxing to Weak Monotonicity.

We now consider the new axioms in relation to the operators. First, Weak Coherence.

Proposition 5. Voting, Sums *and* UnboundedSums *satisfy Weak Coherence*

Proof (sketch).

Voting. Since $s_1 \sqsubseteq_N^{T^{Voting}} s_2$ always holds, Weak Source-Coherence clearly holds. For Weak Fact-Coherence, suppose $\mathsf{src}_N(f_1)$ is weakly less trustworthy than $\mathsf{src}_N(f_2)$. Then there is a bijection $\varphi : \mathsf{src}_N(f_1) \to \mathsf{src}_N(f_2)$, so $|\mathsf{src}_N(f_1)| = |\mathsf{src}_N(f_2)|$. By definition of *Voting*, $f_1 \approx_N^{T^{Voting}} f_2$. In particular, $f_1 \prec_N^{T^{Voting}} f_2$.

Sums. First, one may adapt Definition 9 to a numerical variant of a set of facts Y being *weakly* less believable than Y', by dropping all references to ρ. We then have an analogue of Lemma 1, and Weak Coherence for *Sums* follows by an argument similar to the one used to show Coherence using Lemma 1.

UnboundedSums. The proof that *UnboundedSums* satisfies Coherence can be adapted in a straightforward way to show Weak Coherence. □

Proposition 5 indicates that Weak Coherence may in fact be too weak to capture the original intuition behind Coherence – namely, that there should be a mutual dependence between trustworthy sources and believable facts – since it does not even rule out *Voting*. Instead, Weak Coherence can be seen as a simple requirement which only rules out undesirable behaviour, and complements (strict) Coherence.

Since Monotonicity implies Weak Monotonicity, it is clear that *Voting* satisfies Weak Monotonicity. We conjecture that Weak Monotonicity also holds for *Sums* and *UnboundedSums*, but this remains to be proven.[12]

4.8 Related work

In this section we discuss related work.

Ranking systems. Altman and Tennenholtz [1] initiated axiomatic study of ranking systems. First we discuss their framework in relation to ours, and then turn to their axioms. In their framework, a ranking system F maps any (finite) directed graph $G = (V, E)$ to a total preorder $\leq_G^F$ on the vertex set V. In their view this is a variation of the classical social choice setting, in which the set of voters and alternatives coincide. Nodes $v \in V$ "vote" on their peers in V by a form of approval

[12] Indeed, we conjectured in Section 4.5 that the stronger axiom (strict) Monotonicity holds for *UnboundedSums*. As with Monotonicity, experimental evidence from various starting networks N suggests that Weak Monotonicity is likely to hold.

voting [41]: an edge $v \to u$ is interpret as a vote for u from v. A ranking system then outputs a ranking of V, following the general intuition that the more "votes" v receives (i.e. the more incoming edges), the higher v should rank. As with the ranking of facts in truth discovery, this does not necessarily mean ranking nodes simply by the *number* of votes received, since the *quality* of the voters should also be taken in account. For example, a ranking system may prioritise nodes which receive few votes from highly ranked nodes over those with many votes from lower ranked nodes.

Note that our truth discovery networks N are themselves directed graphs on the vertex set $\mathcal{S} \cup \mathcal{F} \cup \mathcal{O}$. However, naively applying a ranking system to N directly makes little sense: sources never receive any "votes", and the resulting ranking includes objects, which do not need to be ranked in our setting. Perhaps a more sensible approach is to consider the bipartite graph $G_N = (V_N, E_N)$ associated with a network N, where

$$V_N = \mathcal{S} \cup \mathcal{F}, \qquad E_N = \bigcup_{(s,f) \in N} \{(s,f),(f,s)\}.$$

That is, we take the edges from sources to facts together with the reversal of such edges. The edges in G_N have some intuitive interpretation: a source votes for the facts which it claims are true, and a fact votes for the sources who vouch for it. Any ranking system F thus gives rise to a truth discovery operator, where $s_1 \sqsubseteq_N^T s_2$ iff $s_1 \preceq_{G_N}^F s_2$, and similar for facts.

However, some characteristic aspects of the truth discovery problem are lost in this translation to ranking systems. Notably, the objects play no role at all in G_N. Sources and facts are also treated symmetrically, where they perhaps should not be. For example, a fact f receiving more claims than g is beneficial for f, all else being equal (see Monotonicity), but a source s claiming more facts than t does not tell us anything about the relative trustworthiness of s and t.

While other choices of G_N may be possible to alleviate some of these problems, we believe the truth discovery is sufficiently specialised beyond general graph ranking so that a bespoke modelling is required to capture its nuances appropriately. Our framework provides this novel contribution.

In [1], Altman and Tennenholtz also introduce axioms for ranking systems. Their first set of axioms deal with the transitive effects of voting when the alternatives are the voters themselves. As mentioned in Section 4.4, these axioms provided the inspiration for Coherence. The core idea is that if the predecessors of a node v are weaker than those of u, then v should be ranked below u. If v additionally has *more* predecessors, v should rank *strictly* below. Coherence applies this idea both in the direction of sources-to-facts (Fact-Coherence) and from facts-to-sources (Source-Coherence). A notable difference is that we only consider the case where the number of sources for two facts (or the number of facts, for two sources) is the same. For example, a source claiming more facts does not give it the strict boost Transitivity would dictate. Under the mapping $N \mapsto G_N$ described above, any ranking system satisfying Transitivity induces a truth discovery operator which satisfies Coherence.

The other axiom in [1] is an independence axiom RIIA (ranked independence of irrelevant alternatives), which adapts the classical IIA axiom from social choice theory to the ranking system setting, although in a different manner to our independence axioms of Section 4.4.4. We describe the axiom in rough terms, deferring to the paper for the technical details. Suppose the relative ranking of u_1's predecessors compared to u_2's predecessors is the same as that of v_1's compared to v_2's. Then RIIA requires $u_1 \leq u_2$ iff $v_1 \leq v_2$. Informally, "the relative ranking of two agents must only depend on the pairwise comparison of the ranks of their predecessors" [1]. While we do not have an analogous axiom, the idea can be adapted to truth discovery networks. Intuitively, such an axiom would state that the ranking of two facts depends only on comparisons between their corresponding sources (and similar for the ranking of sources).

However, the main result of Altman and Tennenholtz is an impossibility: Transitivity is incompatible with RIIA. Moreover, the result remains true even when restricting to bipartite graphs, such as G_N described above. Accordingly, we can expect a similar impossibility result to hold in the truth discovery setting between Coherence and any analogue of RIIA.

PageRank. PageRank [51] is a well-known algorithm for ranking web pages based on the hyperlink structure of the web, viewed as a directed graph. It has also been studied through the lens of social choice and characterised axiomatically [2, 62].[13] In [2] the authors propose several *invariance axioms*, each of which requires that the ranking of pages is not affected by a certain transformation of the web graph. For example, the axiom *Self Edge* says that adding a self loop from a page a to itself does not change the relative ranking of other pages, and results in a strictly positive boost for a (c.f. Monotonicity). However, if we identify a truth discovery network N with the graph G_N as described above, most of the transformations involved do not respect the bipartite, symmetric structure of G_N. That is, the transformed graph does not correspond to any $G_{N'}$, for a network N'. Consequently, the PageRank axioms have no truth discovery counterpart in our setting. The only exception is *Isomorphism*, where the transformation in question is graph isomorphism; this axiom is analogous to our Symmetry and Isomorphism axioms. However, since PageRank is similar to the *Hubs and Authorities* [36] algorithm on which Sums is based – which also uses the link structure of the web to rank pages – we expect there may be additional axioms which can be expressed both for general graphs *and* truth discovery networks, satisfied by PageRank and Sums. We leave the task of finding such axioms to future work.

4.9 Summary

In this paper we formalised a mathematical framework for truth discovery. While a number of simplifying assumptions were made compared to the mainstream truth discovery literature, we are able to express several algorithms in the framework. This provided the setting for the axiomatic method of social choice to be applied. To our knowledge, this is the first such axiomatic treatment in this context.

[13] WÄĚs and Skibski [62] axiomatise the *numerical scores* of PageRank, whereas Altman and Tennenholtz [2] axiomatise the resulting ranking. Moreover, WÄĚs and Skibski point out that Altman and Tennenholtz in fact only consider a simplified version of PageRank called *Katz prestige*, defined only for strongly connected graphs.

It was possible to adapt many axioms from social choice theory and related areas. In particular, the *Transitivity* axiom studied in the context of ranking systems [58, 1] took on new life in the form of Coherence, which we consider a core axiom for TD operators. We proceeded to establish the differences between our setting and classical social choice by considering independence axioms. This led to an impossibility result and an axiomatic characterisation of the baseline *Voting* method.

On the practical side, we analysed the existing TD algorithm *Sums* and found that, surprisingly, it fails PCI. This is a serious issue for *Sums* which has not been discussed in the literature to date, and its discovery here highlights the benefits of the axiomatic method. To resolve this, we suggested a modification to *Sums* – which we call *UnboundedSums* – for which PCI *is* satisfied. However, while *UnboundedSums* resolves axiomatic problems with *Sums*, it may introduce computational difficulties (since the numeric scores involved grow without bound). We leave further investigation of such issues to future work.

A restriction of our analysis is that only one 'real-world' algorithm was considered. Further axiomatic analysis of algorithms provides a deeper understanding of how algorithms operate on an intuitive level, but is made difficult by the complexity of the state-of-the-art truth discovery methods. New techniques for establishing the satisfaction (or otherwise) of axioms would be helpful in this regard.

There is also scope for extensions to our model of truth discovery in the framework itself. For example, even in the variable domain setting of Section 4.6, we make the somewhat simplistic assumption that there are only finitely many possible facts for sources to claim. This effectively means we can only consider *categorical values*; modelling an object whose domain is the set of real numbers, for example, is not straightforward in our framework.

Next, our model does not account for any associations or constraints between objects, whereas in reality the belief in a fact for one object may strengthen or weaken our belief in other facts for related objects. These types of constraints or correlations have been studied both on the theoretical side (e.g. in judgment aggregation) and practical side in truth discovery [65].

The axioms can also be further refined to relax some of the simplifying assumptions we make regarding source attitudes; e.g. that they do not collude or attempt to manipulate. Most notably, Monotonicity should be weakened to account for such sources.

Finally, it may be argued that truth discovery as formulated in this paper risks simply to find *consensus* among sources, rather than the *truth*. To remedy this, the framework could be extended to model the possible states of the world and thus the *ground truth* (c.f. [47]). Upon doing so one could investigate how well, and under what conditions, an operator is able to recover the truth from a TD network. Such truth-tracking methods have also been studied in judgment aggregation and belief fusion [27, 34].

5 Bipartite Tournaments

5.1 Introduction

A tournament consists of a finite set of players equipped with a *beating relation* describing pairwise comparisons between each pair of players. Determining a ranking of the players in a tournament has applications in voting in social choice [13] (where players represent alternatives and x beats y if a majority of voters prefer x over y), paired comparisons analysis [32] (where players may represent products and the beating relation the preferences of a user), search engines [57], sports tournaments [10] and other domains.

In this paper we introduce *bipartite tournaments*, which consist of two disjoint sets of players A and B such that comparisons only take place between players from opposite sets. We consider ranking methods which produce two rankings for each tournament – one for each side of the bipartition. Such tournaments model situations in which two different kinds of entity compete *indirectly* via matches against entities of the opposite kind. The notion of competition may be abstract, which allows the model to be applied in a variety of settings. An important example is education [35], where A represents students, B exam questions, and student a 'beats' question b by answering it correctly. Here the ranking of students reflects their performance in the exam, and the ranking of questions reflects their *difficulty*. The simultaneous ranking of both sides allows one ranking to influence the other; e.g. so that students are rewarded for correctly answering difficult questions. This may prove particularly useful in the context of crowdsourced questions provided by students themselves, which may vary in their difficulty (see for example the PeerWise system [17]).

A related example is *truth discovery* [42, 54]: the task of finding true information on a number of topics when faced with conflicting reports from sources of varying (but unknown) reliability. Many truth discovery algorithms operate iteratively, alternately estimating the reliability of sources based on current estimates of the true information, and obtaining new estimates of the truth based on source reliability levels. The former is an instance of a bipartite tournament; similar to the education example, A represents data sources, B topics of interest, and a defeats b by providing true information on topic b (according to the current estimates of the truth). Applying a bipartite tournament ranking method at this step may therefore facilitate development of *difficulty-aware* truth discovery algorithms, which reward sources for providing accurate information on difficult topics [29]. Other application domains include the evaluation of generative models in machine learning [50] (where A represents generators and B discriminators) and solo sports contests (e.g. where A represents golfers and B golf courses).

In principle, bipartite tournaments are a special case of *generalised* tournaments [32, 56, 16], which allow intensities of victories and losses beyond a binary win or loss (thus permitting draws or multiple comparisons), and drop the requirement that every player is compared to all others. However, many existing ranking methods in the literature do not apply to bipartite tournaments due to the violation of an *irreducibility* requirement, which requires that the tournament graph be strongly connected. In any case, bipartite tournament ranking presents a unique problem – since we aim to rank players with only indirect information available – which we believe is worthy of study in its own right.

In this work we focus particularly on ranking via *chain graphs* and *chain editing*. A chain graph is a bipartite graph in which the neighbourhoods of vertices on one side form a chain with respect to set inclusion. A (bipartite) tournament of this form represents an 'ideal' situation in which the capabilities of the players are perfectly nested: weaker players defeat a subset of the opponents that stronger players defeat. In this case a natural ranking can be formed according to the set of opponents defeated by each player. These rankings respect the tournament results in an intuitive sense: if a player a defeats b and b' ranks worse than b, then a must defeat b' also. Unfortunately, this perfect nesting may not hold in reality: a weak player may win a difficult match by coincidence, and a strong player may lose a match by accident. With this in mind, Jiao, Ravi, and Gatterbauer [35] suggested an appealing ranking method for bipartite tournaments: apply *chain editing* to the input tournament – i.e. find the minimum number of edge changes required to form a chain graph – and output the corresponding rankings. Whilst their work focused on algorithms for chain editing and its variants, we look to study the properties of the ranking method itself through the lens of computational social choice.

Contribution. Our primary contribution is the introduction of a class of ranking mechanisms for bipartite tournaments defined by chain editing. We also provide a new probabilistic characterisation of chain editing via maximum likelihood estimation. To our knowledge this is the first in-depth study of chain editing as a ranking mechanism. Secondly, we introduce a new class of 'chain-definable' mechanisms by relaxing the minimisation constraint of chain editing in order to obtain tractable algorithms and to resolve the failure of an important anonymity axiom.

Paper outline. In Section 5.2 we define the framework for bipartite tournaments and introduce chain graphs. Section 5.3 outlines how one may use chain editing to rank a tournament, and characterises the resulting mechanisms in a probabilistic setting. Axiomatic properties are considered in Section 5.4. Section 5.5 defines a concrete scheme for producing chain-editing-based rankings. Section 5.6 introduces new ranking methods by relaxing the chain editing requirement. Related work is discussed in Section 5.7, and we conclude in Section 5.8. Note that some proofs are omitted and can be found in Appendix B.

5.2 Preliminaries

In this section we define our framework for bipartite tournaments, introduce chain graphs and discuss the link between them.

5.2.1 Bipartite Tournaments

Following the literature on generalised tournaments [32, 56, 16], we represent a tournament as a matrix, whose entries represent the results of matches between participants. In what follows, $[n]$ denotes the set $\{1, \dots, n\}$ whenever $n \in \mathbb{N}$.

Definition 14. *A* bipartite tournament *– hereafter simply a* tournament *– is a triple (A, B, K), where $A = [m]$ and $B = [n]$ for some $m, n \in \mathbb{N}$, and K is an $m \times n$ matrix with $K_{ab} \in \{0, 1\}$ for all $(a, b) \in A \times B$. The set of all tournaments will be denoted by $\mathcal{K}$.*

Here A and B represent the two sets of players in the tournament.[1] An entry K_{ab} gives the result of the match between $a \in A$ and $b \in B$: it is 1 if a defeats b and 0 otherwise. Note that we do not allow for the possibility of draws, and every $a \in A$ faces every $b \in B$. When there is no ambiguity we denote a tournament simply by K, with the understanding that $A = [\text{rows}(K)]$ and $B = [\text{columns}(K)]$.

The *neighbourhood* of a player $a \in A$ in K is the set $K(a) = \{b \in B \mid K_{ab} = 1\} \subseteq B$, i.e. the set of players which a defeats. The neighbourhood of $b \in B$ is the set $K^{-1}(b) = \{a \in A \mid K_{ab} = 1\} \subseteq A$, i.e. the set of players defeating b.

Given a tournament K, our goal is to place a ranking on each of A and B. We define a ranking *operator* for this purpose.

Definition 15. *An* operator *φ assigns each tournament K a pair $\varphi(K) = (\preceq_K^\varphi, \sqsubseteq_K^\varphi)$ of total preorders on A and B respectively.*[2]

For $a, a' \in A$, we interpret $a \preceq_K^\varphi a'$ to mean that a' is ranked *at least as strong* as a in the tournament K, according to the operator φ (similarly, $b \sqsubseteq_K^\varphi b'$ means b' is ranked at least as strong as b). The strict and symmetric parts of $\preceq_K^\varphi$ are denoted by $\prec_K^\varphi$ and $\approx_K^\varphi$.

As a simple example, consider φ_{count}, where $a \preceq_K^{\varphi_{\mathsf{count}}} a'$ iff $|K(a)| \leq |K(a')|$ and $b \sqsubseteq_K^{\varphi_{\mathsf{count}}} b'$ iff $|K^{-1}(b)| \geq |K^{-1}(b')|$. This operator simply ranks players by number of victories. It is a bipartite version of the *points system* introduced by Rubinstein [53], and generalises *Copeland's rule* [13].

5.2.2 Chain Graphs

Each bipartite tournament K naturally corresponds to a bipartite graph G_K, with vertices $A \sqcup B$ and an edge between a and b whenever $K_{ab} = 1$.[3] The task of ranking a tournament admits a particularly simple solution if this graph happens to be a *chain graph*.

[1] Note that A and B are not disjoint as sets: 1 is always contained in both A and B, for instance. This poses no real problem, however, since we view the number 1 merely a *label* for a player. It will always be clear from context whether a given integer should be taken as a label for a player on the A side or the B side.

[2] A total preorder is a transitive and complete binary relation.

[3] $A \sqcup B$ is the *disjoint union* of A and B, which we define as $\{(a, \mathcal{A}) \mid a \in A\} \cup \{(b, \mathcal{B}) \mid b \in B\}$, where $\mathcal{A}$ and $\mathcal{B}$ are constant symbols.

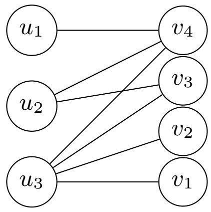

Figure 5.1: An example of a chain graph

Definition 16 ([67])**.** *A bipartite graph $G = (U, V, E)$ is a* chain graph *if there is an ordering $U = \{u_1, \ldots, u_k\}$ of U such that $N(u_1) \subseteq \cdots \subseteq N(u_k)$, where $N(u_i) = \{v \in V \mid (u_i, v) \in E\}$ is the neighbourhood of u_i in G.*

In other words, a chain graph is a bipartite graph where the neighbourhoods of the vertices on one side can be ordered so as to form a chain with respect to set inclusion. It is easily seen that this nesting property holds for U if and only if it holds for V. Figure 5.1 shows an example of a chain graph.

Now, as our terminology might suggest, the neighbourhood $K(a)$ of some player $a \in A$ in a tournament K coincides with the neighbourhood of the corresponding vertex in G_K. If G_K is a chain graph we can therefore enumerate A as $\{a_1, \ldots, a_m\}$ such that $K(a_i) \subseteq K(a_{i+1})$ for each $1 \leq i < m$. This indicates that each a_{i+1} has performed *at least as well* as a_i in a strong sense: every opponent which a_i defeated was also defeated by a_{i+1}, and a_{i+1} may have additionally defeated opponents which a_i did not.[4] It seems only natural in this case that one should rank a_i (weakly) below a_{i+1}. Appealing to transitivity and the fact that each $a \in A$ appears as *some* a_i, we see that any tournament K where G_K is a chain graph comes pre-equipped with a natural total preorder on A, where a' ranks higher than than a if and only if $K(a) \subseteq K(a')$. The duality of the neighbourhood-nesting property for chain graphs implies that B can also be totally preordered, with b' ranked higher than b if and only if $K^{-1}(b) \supseteq K^{-1}(b')$.[5] Moreover, these total preorders relate to the tournament results in an important sense: if a defeats b and b' ranks worse than b, then a must defeat b' also. That is, the neighbourhood of each $a \in A$ is *downwards closed* w.r.t the ranking of B, and the neighbourhood of each $b \in B$ is *upwards closed* in A.

Tournaments corresponding to chain graphs will be said to satisfy the *chain property*, and will accordingly be called *chain tournaments*. We give a simpler (but equivalent) definition which does not refer to the underlying graph G_K. First, define relations $\leqslant_K^{\mathcal{A}}, \leqslant_K^{\mathcal{B}}$ on A and B respectively by $a \leqslant_K^{\mathcal{A}} a'$ iff $K(a) \subseteq K(a')$ and $b \leqslant_K^{\mathcal{B}} b'$ iff $K^{-1}(b) \supseteq K^{-1}(b')$, for any tournament K.

Definition 17. *A tournament K has the* chain property *if $\leqslant_K^{\mathcal{A}}$ is a total preorder.*

According to the duality principle mentioned already, the chain property implies that $\leqslant_K^{\mathcal{B}}$ is also a total preorder. Note that the relations $\leqslant_K^{\mathcal{A}}$ and $\leqslant_K^{\mathcal{B}}$ are analogues of the *covering relation* for non-bipartite tournaments [13].

[4] Note that this is a more robust notion of performance than comparing the neighbourhoods of a_i and a_{i+1} by *cardinality*, which may fail to account for differences in the strength of opponents when counting wins and losses.

[5] Note that the ordering of the Bs is reversed compared to the As, since the larger $K^{-1}(b)$ the *worse* b has performed.

Example 3. *Consider* $K = \left[\begin{smallmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 1&1&1&1 \end{smallmatrix}\right]$. *Then* $K(1) \subset K(2) \subset K(3)$, *so* K *has the chain property. In fact,* K *is the tournament corresponding to the chain graph* G *from Figure 5.1.*

5.3 Ranking via Chain Editing

We have seen that chain tournaments come equipped with natural rankings of A and B. Such tournaments represent an 'ideal' situation, wherein the abilities of the players on both sides of the tournament are perfectly nested. Of course this may not be so in reality: the nesting may be broken by some $a \in A$ winning a match it ought not to by chance, or by losing a match by accident.

One idea for recovering a ranking in this case, originally suggested by Jiao, Ravi, and Gatterbauer [35], is to apply *chain editing*: find the minimum number of edge changes required to convert the graph G_K into a chain graph. This process can be seen as correcting the 'noise' in an observed tournament K to obtain an ideal ranking. In this section we introduce the class of operators producing rankings in this way.

5.3.1 Chain-minimal Operators

To define chain-editing in our framework we once again present an equivalent definition which does not refer to the underlying graph G_K: the number of edge changes between graphs can be replaced by the *Hamming distance* between tournament matrices.

Definition 18. *For* $m, n \in \mathbb{N}$, *let* $\mathcal{C}_{m,n}$ *denote the set of all* $m \times n$ *chain tournaments. For an* $m \times n$ *tournament* K, *write* $\mathcal{M}(K) = \arg\min_{K' \in \mathcal{C}_{m,n}} d(K, K') \subseteq \mathcal{K}$ *for the set of chain tournaments closest to* K *w.r.t the Hamming distance* $d(K, K') = |\{(a, b) \in A \times B \mid K_{ab} \neq K'_{ab}\}|$. *Let* $m(K)$ *denote this minimum distance.*

Note that chain editing, which is NP-hard in general [35], amounts to finding a single element of $\mathcal{M}(K)$.[6] We comment further on the computational complexity of chain editing in Section 5.7. The following property characterises chain editing-based operators φ.

Axiom 13 (chain-min)**.** *For every tournament* K *there is* $K' \in \mathcal{M}(K)$ *such that* $\varphi(K) = (\leqslant^{\mathcal{A}}_{K'}, \leqslant^{\mathcal{B}}_{K'})$.

That is, the ranking of K is obtained by choosing the neighbourhood-subset rankings for some closest chain tournament K'. Operators satisfying **chain-min** will be called *chain-minimal*.

Example 4. *Consider* $K = \left[\begin{smallmatrix} 1&0&1&0 \\ 1&1&0&0 \\ 0&1&1&1 \end{smallmatrix}\right]$. K *does not have the chain property, since neither* $K(1) \subseteq K(2)$ *nor* $K(2) \subseteq K(1)$. *The set* $\mathcal{M}(K)$ *consists of four tournaments a distance of 2 from* K*:*

$$\mathcal{M}(K) = \left\{ \begin{bmatrix} 1&\mathbf{1}&1&0 \\ 1&1&0&0 \\ \mathbf{1}&1&1&1 \end{bmatrix}, \begin{bmatrix} 1&0&\mathbf{0}&0 \\ 1&1&0&0 \\ \mathbf{1}&1&1&1 \end{bmatrix}, \begin{bmatrix} 1&0&1&0 \\ 1&\mathbf{0}&0&0 \\ \mathbf{1}&1&1&1 \end{bmatrix}, \begin{bmatrix} 1&0&1&0 \\ 1&1&\mathbf{1}&0 \\ \mathbf{1}&1&1&1 \end{bmatrix} \right\}$$

[6] The decision problem associated with chain editing – which in tournament terms is the question of whether $m(K) \leq k$ for a given integer k – is NP-complete [22].

The corresponding rankings are $(213, \{12\}34)$*,* $(123, 12\{34\})$*,* $(213, 13\{24\})$ *and* $(123, \{13\}24)$*.*[7]

Example 4 shows that there is no unique chain-minimal operator, since for a given tournament K there may be several closest chain tournaments to choose from. In Section 5.5 we introduce a principled way to single out a *unique* chain tournament and thereby construct a well-defined chain-minimal operator.

5.3.2 A Maximum Likelihood Interpretation

So far we have motivated **chain-min** as a way to fix errors in a tournament and recover the ideal or *true* ranking. In this section we make this notion precise by defining a probabilistic model in which chain-minimal rankings arise as maximum likelihood estimates. The maximum likelihood approach has been applied for (non-bipartite) tournaments (e.g. the Bradley-Terry model [11, 32]), voting in social choice theory [24], truth discovery [60], belief merging [26] and other related problems.

In this approach we take an epistemic view of tournament ranking: it is assumed there exists a true 'state of the world' which determines the tournament results along with objective rankings of A and B. A given tournament K is then seen as a *noisy observation* derived from the true state, and a *maximum likelihood estimate* is a state for which the probability of observing K is maximal.

More specifically, a state of the world is represented as a vector of *skill levels* for the players in A and B.[8]

Definition 19. *For a fixed size $m \times n$, a* state of the world *is a tuple $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, where $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^n$ satisfies the following properties:*

$$\forall a, a' \in A \quad (x_a < x_{a'} \implies \exists b \in B : x_a < y_b \leq x_{a'}) \tag{5.1}$$

$$\forall b, b' \in B \quad (y_b < y_{b'} \implies \exists a \in A : y_b \leq x_a < y_{b'}) \tag{5.2}$$

where $A = [m]$, $B = [n]$. Write $\Theta_{m,n}$ for the set of all $m \times n$ states.

For $a \in A$, x_a is the *skill level* of a in state θ (and similarly for y_b). These skill levels represent the true capabilities of the players in A and B in state θ: a is capable of defeating b if and only if $x_a \geq y_b$. Note that (5.1) suggests a simple form of *explainability*: a' can only be strictly more skilful than a if there is some $b \in B$ which *explains* this fact, i.e. some b which a' can defeat but a cannot ((5.2) is analogous for the Bs). These conditions are intuitive if we assume that skill levels are relative to the sets A and B currently under consideration (i.e. they do not reflect the abilities of players in future matches against new contenders outside of A or B). Finally note that our states of the world are *richer* than the output of an operator, in contrast to other work in the literature [11, 32, 24]. Specifically, a state θ contains extra information in the form of comparisons between A and B.

[7] Here $a_1a_2a_3$ is shorthand for the ranking $a_1 \prec a_2 \prec a_3$ of A, and similar for B. Elements in brackets are ranked equally.

[8] For simplicity we use numerical skill levels here, although it would suffice to have a partial preorder on $A \sqcup B$ such that each $a \in A$ is comparable with every $b \in B$.

Noise is introduced in the observed tournament K via *false positives* (where $a \in A$ defeats a more skilled $b \in B$ by accident) and *false negatives* (where $a \in A$ is defeated by an inferior $b \in B$ by mistake).[9] The noise model is therefore parametrised by the false positive and false negative rates $\boldsymbol{\alpha} = \langle \alpha_+, \alpha_- \rangle \in [0,1]^2$, which we assume are the same for all $a \in A$.[10] We also assume that noise occurs independently across all matches.

Definition 20. *Let $\boldsymbol{\alpha} = \langle \alpha_+, \alpha_- \rangle \in [0,1]^2$. For each $m, n \in \mathbb{N}$ and $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \Theta_{m,n}$, consider independent binary random variables X_{ab} representing the outcome of a match between $a \in [m]$ and $b \in [n]$, where*

$$P_{\boldsymbol{\alpha}}(X_{ab} = 1 \mid \theta) = \begin{cases} \alpha_+, & x_a < y_b \\ 1 - \alpha_-, & x_a \geq y_b \end{cases} \tag{5.3}$$

$$P_{\boldsymbol{\alpha}}(X_{ab} = 0 \mid \theta) = \begin{cases} 1 - \alpha_+, & x_a < y_b \\ \alpha_-, & x_a \geq y_b \end{cases} \tag{5.4}$$

This defines a probability distribution $P_{\boldsymbol{\alpha}}(\cdot \mid \theta)$ over $m \times n$ tournaments by

$$P_{\boldsymbol{\alpha}}(K \mid \theta) = \prod_{(a,b) \in [m] \times [n]} P_{\boldsymbol{\alpha}}(X_{ab} = K_{ab} \mid \theta)$$

Here $P_{\boldsymbol{\alpha}}(K \mid \theta)$ is the probability of observing the tournament results K when the false positive and negative rates are given by $\boldsymbol{\alpha}$ and the true state of the world is θ. Note that the four cases in (5.3) and (5.4) correspond to a false positive, true positive, true negative and false negative respectively. We can now define a maximum likelihood operator.

Definition 21. *Let $\boldsymbol{\alpha} \in [0,1]^2$ and $m, n \in \mathbb{N}$. Then $\theta \in \Theta_{m,n}$ is a* maximum likelihood estimate *(MLE) for an $m \times n$ tournament K w.r.t $\boldsymbol{\alpha}$ if $\theta \in \arg\max_{\theta' \in \Theta_{m,n}} P_{\boldsymbol{\alpha}}(K \mid \theta')$. An operator φ is a* maximum likelihood operator *w.r.t $\boldsymbol{\alpha}$ if for any $m, n \in \mathbb{N}$ and any $m \times n$ tournament K there is an MLE $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \Theta_{m,n}$ for K such that $a \preceq^{\varphi}_{K} a'$ iff $x_a \leq x_{a'}$ and $b \sqsubseteq^{\varphi}_{K} b'$ iff $y_b \leq y_{b'}$.*

Now, consider the tournament K_θ associated with each state $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, given by $[K_\theta]_{ab} = 1$ if $x_a \geq y_b$ and $[K_\theta]_{ab} = 0$ otherwise. Note that K_θ is the unique tournament with non-zero probability when there are no false positive or false negatives. Expressed in terms of K_θ, the MLEs take a particularly simple form if $\alpha_+ = \alpha_-$, i.e. if false positives and false negatives occur at the same rate.

Lemma 8. *Let $\boldsymbol{\alpha} = \langle \beta, \beta \rangle$ for some $\beta < \frac{1}{2}$. Then θ is an MLE for K if and only if $\theta \in \arg\min_{\theta' \in \Theta_{m,n}} d(K, K_{\theta'})$.*

Proof (sketch). Let K be an $m \times n$ tournament. It can be shown (and we do so in the appendix) that for any $\theta \in \Theta_{m,n}$

$$P_{\boldsymbol{\alpha}}(K \mid \theta) = \Big(\prod_{a \in A} \alpha_+^{|K(a) \setminus K_\theta(a)|} (1 - \alpha_-)^{|K(a) \cap K_\theta(a)|}$$
$$(1 - \alpha_+)^{|B \setminus (K(a) \cup K_\theta(a))|} \alpha_-^{|K_\theta(a) \setminus K(a)|} \Big)$$

[9]Note that a false positive for a is a false negative for b and vice versa.

[10] This is a strong assumption, and it may be more realistic to model the false positive/negative rates as a function of x_a. We leave this to future work.

Plugging in $\alpha_+ = \alpha_- = \beta$ and simplifying, one can obtain

$$P_{\boldsymbol{\alpha}}(K \mid \theta) = c \prod_{a \in A} \left(\frac{\beta}{1-\beta} \right)^{|K(a) \triangle K_\theta(a)|}$$

where $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$ is the symmetric difference of two sets X and Y, and $c = (1-\beta)^{|A| \cdot |B|}$ is a positive constant that does not depend on θ. Now, $P_{\boldsymbol{\alpha}}(K \mid \theta)$ is positive, and is maximal when its logarithm is. We have

$$\begin{aligned}
\log P_{\boldsymbol{\alpha}}(K \mid \theta) &= \log c + \log \left(\frac{\beta}{1-\beta} \right) \sum_{a \in A} |K(a) \triangle K_\theta(a)| \\
&= \log c + \log \left(\frac{\beta}{1-\beta} \right) d(K, K_\theta)
\end{aligned}$$

Since $\log c$ is constant and $\beta < 1/2$ implies $\log \left(\frac{\beta}{1-\beta} \right) < 0$, it follows that $\log P_{\boldsymbol{\alpha}}(K \mid \theta)$ is maximised exactly when $d(K, K_\theta)$ is minimised, which proves the result. □

This result characterises the MLE states for K as those for which K_θ is the closest to K. As it turns out, the tournaments K_θ that arise in this way are exactly those with the chain property.

Lemma 9. *An $m \times n$ tournament K has the chain property if and only if $K = K_\theta$ for some $\theta \in \Theta_{m,n}$.*

The proof of Lemma 9 relies crucially on (5.1) and (5.2) in the definition of a state. Combining all the results so far we obtain our first main result: the maximum likelihood operators for $\boldsymbol{\alpha} = \langle \beta, \beta \rangle$ are exactly the chain-minimal operators.

Theorem 11. *Let $\boldsymbol{\alpha} = \langle \beta, \beta \rangle$ for some $\beta < \frac{1}{2}$. Then φ is a maximum likelihood operator w.r.t $\boldsymbol{\alpha}$ if and only if φ satisfies **chain-min**.*

Proof (sketch). First note that by Lemma 8, a state θ is an MLE for an $m \times n$ tournament K iff K_θ is closest to K amongst all other tournaments $\{K_{\theta'} \mid \theta' \in \Theta_{m,n}\}$. But by Lemma 9, this set is exactly the $m \times n$ tournaments with the chain property. It follows from the definition of $\mathcal{M}(K)$ that θ is an MLE if and only if $K_\theta \in \mathcal{M}(K)$. Consequently, $K' \in \mathcal{M}(K)$ if and only if $K' = K_\theta$ for some MLE θ for K. We see that **chain-min** can be equivalently stated as follows: for all K there exists an MLE θ such that $\varphi(K) = (\leqslant^{\mathcal{A}}_{K_\theta}, \leqslant^{\mathcal{B}}_{K_\theta})$. Using properties (5.1) and (5.2) in Definition 19 for θ it is straightforward to show that $a \leqslant^{\mathcal{A}}_{K_\theta} a'$ iff $x_a \leq x_{a'}$ and $b \leqslant^{\mathcal{B}}_{K_\theta} b'$ iff $y_b \leq y_{b'}$ for all $a, a' \in A$, $b, b' \in B$ (where $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$). This means that the above reformulation of **chain-min** coincides with the definition of a maximum likelihood operator, and we are done. □

Similar results can be obtained for other limiting values of $\boldsymbol{\alpha}$. If $\alpha_+ = 0$ and $\alpha_- \in (0, 1)$ then the MLE operators correspond to *chain completion*: finding the minimum number of edge *additions* required to make G_K a chain graph. This models situations where false positives never occur, although false negatives may (e.g. numerical entry questions in the case where A represents students and B exam questions [35]). Similarly, the case $\alpha_- = 0$ and $\alpha_+ \in (0, 1)$ corresponds to *chain deletion*, where edge additions are not allowed.

5.4 Axiomatic analysis

Chain-minimal operators have theoretical backing in a probabilistic sense due to the results of Section 5.3.2, but are they appropriate ranking methods in practise? To address this question we consider the *normative* properties of chain-minimal operators via the axiomatic method of social choice theory. We formulate several axioms for bipartite tournament ranking and assess whether they are compatible with **chain-min**. It will be seen that an important *anonymity* axiom fails for all chain-minimal operators; later in Section 5.5 we describe a scenario in which this is acceptable and define a class of concrete operators for this case, and in Section 5.6 we relax the **chain-min** requirement in order to gain anonymity.

5.4.1 The Axioms

We will consider five axioms – mainly adaptations of standard social choice properties to the bipartite tournament setting.

Symmetry Properties. We consider two symmetry properties. The first is a classic *anonymity* axiom, which says that an operator φ should not be sensitive to the 'labels' used to identify participants in a tournament. Axioms of this form are standard in social choice theory; a tournament version goes at least as far back as [53].

We need some notation: for a tournament K and permutations $\sigma : A \to A$, $\pi : B \to B$, let $\sigma(K)$ and $\pi(K)$ denote the tournament obtained by permuting the rows and columns of K by σ and π respectively, i.e. $[\sigma(K)]_{ab} = K_{\sigma^{-1}(a),b}$ and $[\pi(K)]_{ab} = K_{a,\pi^{-1}(b)}$. Note that in the statement of the axioms we omit universal quantification over K, $a, a' \in A$ and $b, b' \in B$ for brevity.

Axiom 14 (anon)**.** *Let $\sigma : A \to A$ and $\pi : B \to B$ be permutations. Then $a \preceq_K^\varphi a'$ iff $\sigma(a) \preceq_{\pi(\sigma(K))}^\varphi \sigma(a')$.*

Our second axiom is specific to bipartite tournaments, and expresses a *duality* between the two sides A and B: given the two sets of conceptually disjoint entities participating in a bipartite tournament, it should not matter which one we label A and which one we label B. We need the notion of a *dual tournament*.

Definition 22. *The* dual tournament *of K is $\overline{K} = \mathbf{1} - K^\top$, where $\mathbf{1}$ denotes the matrix consisting entirely of 1s.*

$\overline{K}$ is essentially the same tournament as K, but with the roles of A and B swapped. In particular, $A_K = B_{\overline{K}}$, $B_K = A_{\overline{K}}$ and $K_{ab} = 1$ iff $\overline{K}_{ba} = 0$. Also note that $\overline{\overline{K}} = K$. The duality axiom states that the ranking of the Bs in K is the same as the As in $\overline{K}$.

Axiom 15 (dual)**.** *$b \sqsubseteq_K^\varphi b'$ iff $b \preceq_{\overline{K}}^\varphi b'$.*

Whilst **dual** is not necessarily a universally desirable property – one can imagine situations where A and B are not fully abstract and should not be treated symmetrically – it is important to consider in any study of bipartite tournaments. Note that **dual** implies $a \preceq_K^\varphi a'$ iff $a \sqsubseteq_{\overline{K}}^\varphi a'$, so that a **dual**-operator can be defined by giving the ranking for one of A or B only, and defining the other by duality. This explains our choice to define **anon** (and subsequent axioms) solely in terms of the A ranking: the analogous anonymity constraint for the B ranking follows from **anon** together with **dual**.

An Independence Property. *Independence axioms* play a crucial role in social choice. We present a bipartite adaptation of a classic axiom introduced in [53], which has subsequently been called *Independence of Irrelevant Matches* [32].

Axiom 16 (IIM). *If K_1, K_2 are tournaments of the same size with identical a-th and a'-th rows, then $a \preceq^{\varphi}_{K_1} a'$ iff $a \preceq^{\varphi}_{K_2} a'$.*

IIM is a strong property, which says the relative ranking of a and a' does not depend on the results of any match not involving a or a'. This axiom has been questioned for generalised tournaments [32], and a similar argument can be made against it here: although each player in A faces the same opponents, we may wish to take the *strength* of opponents into account, e.g. by rewarding victories against highly-ranked players in B. Consequently we do not view **IIM** as an essential requirement, but rather introduce it to facilitate comparison with our work and the existing tournament literature.

Monotonicity Properties. Our final axioms are monotonicity properties, which express the idea that *more victories are better*. The first axiom follows our original intuition for constructing the natural ranking associated with a chain graph; namely that $K(a) \subseteq K(a')$ indicates a' has performed at least as well as a.

Axiom 17 (mon). *If $K(a) \subseteq K(a')$ then $a \preceq^{\varphi}_{K} a'$.*

Note that **mon** simply says $\preceq^{\varphi}_{K}$ extends the (in general, partial) preorder $\leqslant^{A}_{K}$. Yet another standard axiom is *positive responsiveness*.

Axiom 18 (pos-resp). *If $a \preceq^{\varphi}_{K} a'$ and $K_{a',b} = 0$ for some $b \in B$, then $a \prec^{\varphi}_{K+\mathbf{1}_{a',b}} a'$, where $\mathbf{1}_{a',b}$ is the matrix with 1 in position (a', b) and zeros elsewhere.*

That is, adding an extra victory for a should only improve its ranking, with ties now broken in its favour. This version of positive responsiveness was again introduced in [53], where together with **anon** and **IIM** it characterises the *points system* ranking method for round-robin tournaments, which simply ranks players according to the number of victories. The analogous operator in our framework is φ_{count}, and it can be shown that φ_{count} is uniquely characterised by **anon**, **IIM**, **pos-resp** and **dual**. Finally, note that **pos-resp** also acts as a kind of *strategyproofness*: a cannot improve its ranking by deliberately losing a match. Specifically, if $K_{ab} = 1$ and $a \preceq^{\varphi}_{K} a'$, then **pos-resp** implies $a \prec^{\varphi}_{K-\mathbf{1}_{ab}} a'$.

5.4.2 Axiom Compatibility with chain-min

We come to analysing the compatibility of **chain-min** with the axioms. First, the negative results.

Theorem 12. *There is no operator satisfying **chain-min** and any of **anon**, **IIM** or **pos-resp**.*

The counterexample for **anon** is particularly simple: take $K = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$. Swapping the rows and columns brings us back to K, so **anon** implies $1, 2 \in A$ rank equally. However, it is easily seen for every $K' \in \mathcal{M}(K)$, either $K(1) \subset K(2)$ or $K(2) \subset K(1)$, i.e no chain-minimal operator can rank 1 and 2 equally.

The MLE results of Section 5.3.2 provides informal explanation for this result. For K above to arise in the noise model of Definition 20 there must have been two 'mistakes' (false positives or false negatives). This is less likely than a single mistake from just one of $1, 2 \in A$, but the likelihood maximisation forces us to choose one or the other. A similar argument explains the **pos-resp** failure.

It is also worth noting that **anon** only fails at the last step of chain editing, where a single element of $\mathcal{M}(K)$ is chosen. Indeed, the set $\mathcal{M}(K)$ itself *does* exhibit the kind of symmetry one might expect: we have $\mathcal{M}(\pi(\sigma(K))) = \{\pi(\sigma(K')) \mid K' \in \mathcal{M}(K)\}$. This means that an operator which aggregates the rankings from *all* $K' \in \mathcal{M}(K)$ – e.g. any anonymous social welfare function – would satisfy **anon**. The other axioms are compatible with **chain-min**.

Theorem 13. *For each of **dual** and **mon**, there exists an operator satisfying **chain-min** and the stated property.*

Despite the simplicity of **mon**, Theorem 13 is deceptively difficult to prove. We describe operators satisfying **chain-def** and **dual** or **mon** non-constructively by first taking an *arbitrary* chain-minimal operator φ, and using properties of the set $\mathcal{M}(K)$ to produce φ' satisfying **dual** or **mon**. Note also that we have not yet constructed an operator satisfying **dual**, **mon** and **chain-min** simultaneously, although we conjecture that such operators do exist.

5.5 Match-preference operators

The counterexample for **chain-min** and **anon** suggests that chain-minimal operators require some form of tie-breaking mechanism when the tournaments in $\mathcal{M}(K)$ cannot be distinguished while respecting anonymity. While this limits the use of chain-minimal operators as general purpose ranking methods, it is not such a problem if additional information is available to guide the tie-breaking. In this section we introduce a new class of operators for this case.

The core idea is to single out a unique chain tournament close to K by paying attention to not only the *number* of entries in K that need to be changed to produce a chain tournament, *which* entries. Specifically, we assume the availability of a total order on the set of matrix indices $\mathbb{N} \times \mathbb{N}$ (the *matches*) which indicates our willingness to change an entry in K: the higher up (a, b) is in the ranking, the more acceptable it is to change K_{ab} during chain editing.

This total order – called the *match-preference relation* – is fixed for all tournaments K; this means we are dealing with extra information about how tournaments are *constructed in matrix form*, not extra information about any specific tournament K.

One possible motivation for such a ranking comes from cases where matches occur at distinct points in time. In this case the matches occurring more recently are (presumably) more representative of the players' *current* abilities, and we should therefore prefer to modify the outcome of old matches where possible.

For the formal definition we need notation for the *vectorisation* of a tournament K: for a total order $\trianglelefteq$ on $\mathbb{N} \times \mathbb{N}$ and an $m \times n$ tournament K, we write $\mathrm{vec}_{\trianglelefteq}(K)$ for the vector in $\{0,1\}^{mn}$ obtained by collecting the entries of K in the order given by $\trianglelefteq \restriction (A \times B)$,[11] starting with the minimal entry. That is, $\mathrm{vec}_{\trianglelefteq}(K) = (K_{a_1,b_1}, \ldots, K_{a_{mn},b_{mn}})$, where $(a_1, b_1), \ldots, (a_{mn}, b_{mn})$ is the unique enumeration of $A \times B$ such that $(a_i, b_i) \trianglelefteq (a_{i+1}, b_{i+1})$ for each i.

The operator corresponding to $\trianglelefteq$ is defined using the notion of a *choice function*: a function α which maps any tournament K to an element of $\mathcal{M}(K)$. Any such function defines a chain-minimal operator φ by setting $\varphi(K) = (\leqslant^{\mathcal{A}}_{\alpha(K)}, \leqslant^{\mathcal{B}}_{\alpha(K)})$.

Definition 23. *Let $\trianglelefteq$ be a total order on $\mathbb{N} \times \mathbb{N}$. Define an operator $\varphi_{\trianglelefteq}$ according to the choice function*

$$\alpha_{\trianglelefteq}(K) = \underset{K' \in \mathcal{M}(K)}{\arg\min}\ \mathrm{vec}_{\trianglelefteq}(K \oplus K') \tag{5.5}$$

where $[K \oplus K']_{ab} = |K_{ab} - K'_{ab}|$, and the minimum is taken w.r.t the lexicographic ordering on $\{0,1\}^{|A| \cdot |B|}$.[12] *Operators generated in this way will be called* match-preference operators.

Example 5. *Let $\trianglelefteq$ be the lexicographic order*[13] *on $\mathbb{N} \times \mathbb{N}$ so that $\mathrm{vec}_{\trianglelefteq}(K \oplus K')$ is obtained by collecting the entries of $K \oplus K'$ row-by-row, from top to bottom and left to right. Take K from Example 4. Writing $K_1, \ldots, K_4$ for the elements of $\mathcal{M}(K)$ in the order that they appear in Example 4 and setting $v_i = \mathrm{vec}_{\trianglelefteq}(K \oplus K_i)$, we have*

$$v_1 = (0\mathbf{1}00\ 0000\ \mathbf{1}0000); \qquad v_2 = (00\mathbf{1}0\ 0000\ \mathbf{1}0000)$$
$$v_3 = (0000\ 0\mathbf{1}00\ \mathbf{1}0000); \qquad v_4 = (0000\ 00\mathbf{1}0\ \mathbf{1}0000)$$

The lexicographic minimum is the one with the 1 entries as far right as possible, which in this case is v_4. Consequently $\varphi_{\trianglelefteq}$ ranks K according to K_4, i.e. $1 \prec^{\varphi_{\trianglelefteq}}_K 2 \prec^{\varphi_{\trianglelefteq}}_K 3$ and $1 \approx^{\varphi_{\trianglelefteq}}_K 3 \sqsubset^{\varphi_{\trianglelefteq}}_K 2 \sqsubset^{\varphi_{\trianglelefteq}}_K 4$.

To conclude the discussion of match-preference operators, we note that one can compute $\alpha_{\trianglelefteq}(K)$ as the unique closest chain tournament to K w.r.t a *weighted* Hamming distance, and thereby avoid the need to enumerate $\mathcal{M}(K)$ in full as per Eq. (5.5).

Theorem 14. *Let $\trianglelefteq$ be a total order on $\mathbb{N} \times \mathbb{N}$. Then for any $m, n \in \mathbb{N}$ there exists a function $w : [m] \times [n] \to \mathbb{R}_{\geq 0}$ such that for all $m \times n$ tournaments K:*

$$\underset{K' \in \mathcal{C}_{m,n}}{\arg\min}\, d_w(K, K') = \{\alpha_{\trianglelefteq}(K)\} \tag{5.6}$$

where $d_w(K, K') = \sum_{(a,b) \in [m] \times [n]} w(a,b) \cdot |K_{ab} - K'_{ab}|$.

For example, the weights corresponding to $\trianglelefteq$ from Example 5 and $m = 2, n = 3$ are $w = \left[\begin{smallmatrix} 1.5 & 1.25 & 1.125 \\ 1.0625 & 1.03125 & 1.015625 \end{smallmatrix}\right]$.

[11] This denotes the restriction of $\trianglelefteq$ to $A \times B$, i.e. $\trianglelefteq \cap ((A \times B) \times (A \times B))$.

[12] Note that $K \oplus K'$ is 1 in exactly the entries where K and K' differ.

[13] That is, $(a, b) \trianglelefteq (a', b')$ iff $a < a'$ or ($a = a'$ and $b \leq b'$).

5.6 Relaxing chain-min

Having studied chain-minimal operators in some detail, we turn to two remaining problems: **chain-min** is incompatible with **anon**, and computing a chain-minimal operator is NP-hard. In this section we obtain both anonymity and tractability by relaxing the **chain-min** requirement to a property we call *chain-definability*. We go on to characterise the class of operators with this weaker property via a greedy approximation algorithm, single out a particularly intuitive instance, and revisit the axioms of Section 5.4.

5.6.1 Chain-definability

The source of the difficulties with **chain-min** lies in the minimisation aspect of chain editing. A natural way to retain the spirit of **chain-min** without the complications is to require that $\varphi(K)$ corresponds to *some* chain tournament, not necessarily one closest to K. We call this property *chain-definability*.

Axiom 19 (chain-def)**.** *For every $m \times n$ tournament K there is $K' \in \mathcal{C}_{m,n}$ such that $\varphi(K) = (\leqslant^{\mathcal{A}}_{K'}, \leqslant^{\mathcal{B}}_{K'})$.*

Clearly **chain-min** implies **chain-def**. 'Chain-definable' operators can also be cast in the MLE framework of Section 5.3.2 as those whose rankings correspond to *some* (not necessarily MLE) state θ.

At first glance it may seem difficult to determine whether a given pair of rankings correspond to a chain tournament, since the number of such tournaments grows rapidly with m and n. Fortunately, **chain-def** can be characterised without reference to chain tournaments by considering the number of *ranks* of $\preceq^{\varphi}_{K}$ and $\sqsubseteq^{\varphi}_{K}$. In what follows $\mathsf{ranks}(\preceq)$ denotes the number of ranks of a total preorder $\preceq$, i.e. the number of equivalence classes of its symmetric part.

Theorem 15. *φ satisfies **chain-def** if and only if $|\mathsf{ranks}(\preceq^{\varphi}_{K}) - \mathsf{ranks}(\sqsubseteq^{\varphi}_{K})| \leq 1$ for every tournament K.*

5.6.2 Interleaving Operators

According to Theorem 15, to construct a chain-definable operator it is enough to ensure that the number of ranks of $\preceq^{\varphi}_{K}$ and $\sqsubseteq^{\varphi}_{K}$ differ by at most one. A simple way to achieve this is to iteratively select and remove the top-ranked players of A and B simultaneously, until one of A or B is exhausted. We call such operators *interleaving operators*. Closely related ranking methods have been previously introduced for non-bipartite tournaments by Bouyssou [9].

Formally, our procedure is defined by two functions f and g which select the next top ranks given a tournament K and subsets $A' \subseteq A$, $B' \subseteq B$ of the remaining players.

Definition 24. *An* $\mathcal{A}$-selection function *is a mapping $f : \mathcal{K} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for any tournament K, $A' \subseteq A$ and $B' \subseteq B$: (i) $f(K, A', B') \subseteq A'$; (ii) If $A' \neq \emptyset$ then $f(K, A', B') \neq \emptyset$; (iii) $f(K, A', \emptyset) = A'$.*

Similarly, a $\mathcal{B}$-selection function *is a mapping $g : \mathcal{K} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that (i) $g(K, A', B') \subseteq B'$; (ii) If $B' \neq \emptyset$ then $g(K, A', B') \neq \emptyset$; (iii) $g(K, \emptyset, B') = B'$.*

The corresponding interleaving operator ranks players according to how soon they are selected in this way; the earlier the better.

Definition 25. *Let f and g be selection functions and K a tournament. Write $A_0 = A$, $B_0 = B$, and for $i \geq 0$:*

$$A_{i+1} = A_i \setminus f(K, A_i, B_i); \quad B_{i+1} = B_i \setminus g(K, A_i, B_i)$$

For $a \in A$ and $b \in B$, write $r(a) = \max\{i \mid a \in A_i\}$ and $s(b) = \max\{i \mid b \in B_i\}$.[14] *We define the corresponding* interleaving operator *$\varphi = \varphi^{\mathsf{int}}_{f,g}$ by $a \preceq^{\varphi}_{K} a'$ iff $r(a) \geq r(a')$ and $b \sqsubseteq^{\varphi}_{K} b'$ iff $s(b) \geq s(b')$.*

Note that A_i and B_i are the players left remaining after i applications of f and g, i.e. after removing the top i ranks from both sides. Before giving a concrete example, we note that interleaving is not just *one* way to satisfying **chain-def**, it is the *only* way.

Theorem 16. *An operator φ satisfies **chain-def** if and only if $\varphi = \varphi^{\mathsf{int}}_{f,g}$ for some selection functions (f, g).*

Theorem 16 justifies our study of interleaving operators, and provides a different perspective on chain-definability via the selection functions f and g. We come to an important example.

Example 6. *Define the* cardinality-based interleaving operator *$\varphi_{\mathsf{CI}} = \varphi^{\mathsf{int}}_{f,g}$ where $f(K, A', B') = \arg\max_{a \in A'} |K(a) \cap B'|$ and $g(K, A', B') = \arg\min_{b \in B'} |K^{-1}(b) \cap A'|$, so that the 'winners' at each iteration are the As with the most wins, and the Bs with the least losses, when restricting to A' and B' only. We take the $\arg\min/\arg\max$ to be the emptyset whenever A' or B' is empty.*

Table 5.1 shows the iteration of the algorithm for a 4×5 tournament K. In each row i we show K with the rows and columns of $A \setminus A_i$ and $B \setminus B_i$ greyed out, so as to make it more clear how the f and g values are calculated.[15] *For brevity we also write f and g in place of $f(K, A_i, B_i)$ and $g(K, A_i, B_i)$ respectively.*

The r and s values can be read off as 0, 2, 1, 3 for A and 0, 3, 1, 1, 2 for B, giving the ranking on A as $4 \prec 2 \prec 3 \prec 1$, and the ranking on B as $2 \sqsubset 5 \sqsubset 3 \approx 4 \sqsubset 1$. Note also that each $f(K, A_i, B_i)$ is a rank of $\preceq^{\varphi}_{K}$ (and similar for $g(K, A_i, B_i)$), so the rankings can in fact be read off by looking at the f and g columns of Table 5.1.

The interleaving algorithm can also be seen as a greedy algorithm for converting K into a chain graph directly. Indeed, by setting the neighbourhood of each $a \in f(K, A_i, B_i)$ to B_i, and removing each $b \in g(K, A_i, B_i)$ from the neighbourhoods of all $a \in A_{i+1}$, we eventually obtain a chain graph. We show this process in the K'_i column of Table 5.1, where only three entries need to be changed.[16] The selection functions f and g can therefore be seen as *heuristics* with the goal of finding a chain graph 'close' to K.

[14] We show in the appendix that the recursive procedure eventually terminates with A_i and B_i becoming empty (and remaining so) after finitely many iterations, so r and s are well-defined.

[15] Note that while f and g for φ_{CI} are independent of the greyed out entries, we do not require this property for selection functions in general.

[16] In this example $\mathcal{M}(K)$ contains a single tournament a distance of 2 from K, so φ_{CI} makes one more change than necessary.

Table 5.1: Iteration of the interleaving algorithm for φ_{CI}

i	K	A_i	B_i	f	g	K'_i
0	$\begin{bmatrix} 1&1&1&1&0 \\ 0&1&0&0&1 \\ 0&1&0&1&1 \\ 0&1&1&0&0 \end{bmatrix}$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{1\}$	$\{1\}$	$\begin{bmatrix} 1&1&1&1&\mathbf{1} \\ 0&1&0&0&1 \\ 0&1&0&1&1 \\ 0&1&1&0&0 \end{bmatrix}$
1	$\begin{bmatrix} 1&1&1&1&0 \\ 0&1&0&0&1 \\ 0&1&0&1&1 \\ 0&1&1&0&0 \end{bmatrix}$	$\{2,3,4\}$	$\{2,3,4,5\}$	$\{3\}$	$\{3,4\}$	$\begin{bmatrix} 1&1&1&1&\mathbf{1} \\ 0&1&0&0&1 \\ 0&1&\mathbf{1}&1&1 \\ 0&1&\mathbf{0}&0&0 \end{bmatrix}$
2	$\begin{bmatrix} 1&1&1&1&0 \\ 0&1&0&0&1 \\ 0&1&0&1&1 \\ 0&1&1&0&0 \end{bmatrix}$	$\{2,4\}$	$\{2,5\}$	$\{2\}$	$\{5\}$	-
3	$\begin{bmatrix} 1&1&1&1&0 \\ 0&1&0&0&1 \\ 0&1&0&1&1 \\ 0&1&1&0&0 \end{bmatrix}$	$\{4\}$	$\{2\}$	$\{4\}$	$\{2\}$	-
4	-	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	-

The operator φ_{CI} from Example 6 uses simple cardinality-based heuristics, and can be seen as a chain-definable version of φ_{count} (which is not chain-definable). It is also the bipartite counterpart to repeated applications of Copeland's rule [9]. Note that $f(K, A_i, B_i)$ and $g(K, A_i, B_i)$ can be computed in $O(N^2)$ time at each iteration i, where $N = |A| + |B|$. Since there cannot be more than N iterations, it follows that the rankings of φ_{CI} can be computed in $O(N^3)$ time.

5.6.3 Axiom Compatibility

We now revisit the axioms of Section 5.4 in relation to chain-definable operators in general and φ_{CI} specifically. Firstly, the weakening of **chain-min** pays off: **chain-def** is compatible with all our axioms.

Theorem 17. *For each of **anon**, **dual**, **IIM**, **mon** and **pos-resp**, there exists an operator satisfying **chain-def** and the stated property.*

Unfortunately, these cannot all hold at the same time. Indeed, taking $K = \left[\begin{smallmatrix} 0&0&1&1 \\ 0&1&0&1 \end{smallmatrix}\right]^\top$ and assuming **anon** and **pos-resp**, the ranking on A is fully determined as $1 \prec 2 \approx 3 \prec 4$, and $\mathsf{ranks}(\preceq_K^\varphi) = 3$. However, **anon** with **dual** implies the ranking of B is flat, i.e. $\mathsf{ranks}(\sqsubseteq_K^\varphi) = 1$. This contradicts **chain-def** by Theorem 15, yielding the following impossibility result.

Theorem 18. *There is no operator satisfying **chain-def**, **anon**, **dual** and **pos-resp**.*

For the specific operator φ_{CI} we have the following.

Theorem 19. *φ_{CI} satisfies **chain-def**, **anon**, **dual** and **mon**, and does not satisfy **IIM** or **pos-resp**.*

Note that **anon** *is* satisfied. This makes φ_{CI} an important example of a well-motivated, tractable, chain-definable and anonymous operator, meeting the criteria outlined at the start of this section.

5.7 Related Work

On chain graphs. Chain graphs were originally introduced by Yannakakis [67], who proved that *chain completion* – finding the minimum number of edges that when added to a bipartite graph form a chain graph – is NP-complete. Hardness results have subsequently been obtained for chain *deletion* [49] (where only edge deletions are allowed) and chain *editing* [22] (where both additions and deletions are allowed). We refer the reader to the work of Jiao, Ravi, and Gatterbauer [35] and Drange et al. [22] for a more detailed account of this literature. Outside of complexity theory, chain graphs have been studied for their spectral properties in [3, 30], and the more general notion of a *nested colouring* was introduced in [15].

On tournaments in social choice. Tournaments have important applications in the design of voting rules, where an alternative x beats y in a pairwise comparison if a majority of voters prefer x to y. Various *tournament solutions* have been proposed, which select a set of 'winners' from a given tournament.[17] Of particular relevance to our work are the *Slater set* and *Kemeney's rule* [13], which find minimal sets of edges to invert in the tournament graph such that the beating relation becomes a total order.[18] These methods are intuitively similar to chain editing: both involve making minimal changes to the tournament until some property is satisfied. A rough analogue to the Slater set in our framework is the union of the top-ranked players from each $K' \in \mathcal{M}(K)$. Solutions based on the covering relation – such as the *uncovered* and *Banks* set [13] – also bear similarity to chain editing.

Finally, note that directed versions of chain graphs (obtained by orienting edges from A to B and adding missing edges from B to A) correspond to *acyclic tournaments*, and a topological sort of A becomes a linearisation of the chain ranking $\preccurlyeq_K^A$. This suggests a connection between chain deletion and the standard *feedback arc set* problem for removing cycles and obtaining a ranking.

On generalised tournaments. A *generalised* tournament [32] is a pair (X, T), where $X = [t]$ for some $t \in \mathbb{N}$ and $T \in \mathbb{R}_{\geq 0}^{t \times t}$ is a non-negative $t \times t$ matrix with $T_{ii} = 0$ for all $i \in X$. In this formalism each encounter between a pair of players i and j is represented by *two* numbers: T_{ij} and T_{ji}. This allows one to model both intensities of victories and losses (including draws) via the difference $T_{ij} - T_{ji}$, and the case where a comparison is not available (where $T_{ij} = T_{ji} = 0$).

Any $m \times n$ bipartite tournament K has a natural generalised tournament representation via the $(m+n) \times (m+n)$ *anti-diagonal block matrix* $T = \left[\begin{smallmatrix} 0 & K \\ K & 0 \end{smallmatrix}\right]$, where the top-left and bottom-right blocks are the $m \times m$ and $n \times n$ zero matrices respectively. However, such anti-diagonal block matrices are often excluded in the generalised tournament literature due to an assumption of *irreducibility*, which requires that the directed graph corresponding to T is strongly connected. This is not the case in general for T constructed as above, which means not all existing tournament operators (and tournament axioms) are well-defined for bipartite inputs.[19] Consequently, bipartite tournaments are a special case of generalised tournaments *in principle*, but not in practise.

[17] Note that a ranking, such as we consider in this paper, induces a set of winners by taking the maximally ranked players.

[18] Note that like chain editing, Kemeny's rule also admits a maximum likelihood characterisation [24].

[19] We note that Slutzki and Volij [56] side-step the reducibility issue by decomposing T into irreducible components and ranking each separately, although their methods may give only *partial* orders.

5.8 Conclusion

Summary. In this paper we studied chain editing, an interesting problem from computational complexity theory, as a ranking mechanism for bipartite tournaments. We analysed such mechanisms from a probabilistic viewpoint via the MLE characterisation, and in axiomatic terms. To resolve both the failure of an important anonymity axiom and NP-hardness, we weakened the chain editing requirement to one of *chain definability*, and characterised the resulting class of operators by the intuitive interleaving algorithm.

Limitations and future work. The hardness of chain editing remains a limitation of our approach. A possible remedy is to look to one of the numerous variant problems that are polynomial-time solvable [35]; determining their applicability to ranking is an interesting topic for future work. One could develop approximation algorithms for chain editing, possibly based on existing approximations of chain completion [48]. The interleaving operators of Section 5.6.2 go in this direction, but we did not yet obtain any theoretical or experimental bounds on the approximation ratio.

A second limitation of our work lies in the assumptions of the probabilistic model; namely that the true state of the world can be reduced to vectors of numerical skill levels which totally describe the tournament participants. This assumption may be violated when the competitive element of a tournament is *multi-faceted*, since a single number cannot represent multiple orthogonal components of a player's capabilities. Nevertheless, if skill levels are taken as *aggregations* of these components, chain editing may prove to be a useful, albeit simplified, model.

Finally, there is room for more detailed axiomatic investigation. In this paper we have stuck with fairly standard social choice axioms and performed preliminary analysis. However, the indirect nature of the comparisons in a bipartite tournament presents unique challenges; new axioms may need to be formulated to properly evaluate bipartite ranking methods in a normative sense.

Part III

Logic-based Perspectives

6 Introduction

7 Expertise and Knowledge

7.1 Summary

8 Belief Change with Non-Expert Sources

8.1 Summary

9 Truth-Tracking

9.1 Summary

Part IV

Conclusions

10 Conclusion

10.1 Summary

10.2 Future Work

Bibliography

[1] Alon Altman and Moshe Tennenholtz. "Axiomatic Foundations for Ranking Systems". In: *J. Artif. Int. Res.* 31.1 (Mar. 2008), pp. 473–495. ISSN: 1076-9757. URL: http://dl.acm.org/citation.cfm?id=1622655.1622669.

[2] Alon Altman and Moshe Tennenholtz. "Ranking systems: the PageRank axioms". In: *Proceedings of the 6th ACM conference on Electronic commerce*. ACM. 2005, pp. 1–8.

[3] Milica Anđelić et al. "Some new considerations about double nested graphs". In: *Linear Algebra and its Applications* 483 (2015), pp. 323–341.

[4] Reid Andersen et al. "Trust-based Recommendation Systems: An Axiomatic Approach". In: *Proceedings of the 17th International Conference on World Wide Web*. WWW '08. New York, NY, USA: ACM, 2008, pp. 199–208. ISBN: 978-1-60558-085-2. DOI: 10.1145/1367497.1367525. URL: http://doi.acm.org/10.1145/1367497.1367525.

[5] Kenneth J. Arrow. "Social Choice and Individual Values". In: *Ethics* 62.3 (1952), pp. 220–222.

[6] Sheldon Axler. *Linear Algebra Done Right*. Springer International Publishing, Nov. 2014. DOI: 10.1007/978-3-319-11080-6. URL: https://doi.org/10.1007/978-3-319-11080-6.

[7] Raju Balakrishnan and Subbarao Kambhampati. "Sourcerank: Relevance and trust assessment for deep web sources based on inter-source agreement". In: *Proceedings of the 20th international conference on World wide web*. 2011, pp. 227–236.

[8] Laure Berti-Equille and Javier Borge-Holthoefer. "Veracity of data: From truth discovery computation algorithms to models of misinformation dynamics". In: *Synthesis Lectures on Data Management* 7.3 (2015), pp. 1–155.

[9] Denis Bouyssou. "Monotonicity of 'ranking by choosing': A progress report". In: *Social Choice and Welfare* 23.2 (2004), pp. 249–273.

[10] Sándor Bozóki, László Csató, and József Temesi. "An application of incomplete pairwise comparison matrices for ranking top tennis players". In: *European Journal of Operational Research* 248.1 (2016), pp. 211–218.

[11] Ralph Allan Bradley and Milton E. Terry. "Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons". In: *Biometrika* 39.3/4 (1952), pp. 324–345. ISSN: 00063444.

[12] Felix Brandt et al. “Introduction to Computational Social Choice”. In: *Handbook of Computational Social Choice*. Ed. by Felix Brandt et al. 1st. New York, NY, USA: Cambridge University Press, 2016. Chap. 1.

[13] Felix Brandt et al. “Tournament Solutions”. In: *Handbook of Computational Social Choice*. Ed. by Felix Brandt et al. Cambridge University Press, 2016, pp. 57–84.

[14] Zoé Christoff and Davide Grossi. “Binary Voting with Delegable Proxy: An Analysis of Liquid Democracy”. In: *Proc. TARK 2017*. 2017.

[15] David Cook II. “Nested colorings of graphs”. In: *Australasian Journal of Combinatorics* 62.1 (2015), pp. 100–127.

[16] László Csató. “An impossibility theorem for paired comparisons”. In: *Central European Journal of Operations Research* 27.2 (2019), pp. 497–514.

[17] Paul Denny et al. “PeerWise: Students Sharing Their Multiple Choice Questions”. In: *Proceedings of the Fourth International Workshop on Computing Education Research*. ICER ’08. Sydney, Australia: ACM, 2008, pp. 51–58.

[18] Hu Ding, Jing Gao, and Jinhui Xu. “Finding global optimum for truth discovery: Entropy based geometric variance”. In: *Proc. 32nd International Symposium on Computational Geometry (SoCG 2016)*. 2016.

[19] Elad Dokow and Ron Holzman. “Aggregation of binary evaluations with abstentions”. In: *Journal of Economic Theory* 145 (2010), pp. 544–561.

[20] Xin Luna Dong, Laure Berti-Equille, and Divesh Srivastava. “Integrating conflicting data: the role of source dependence”. In: *Proceedings of the VLDB Endowment* 2.1 (2009), pp. 550–561.

[21] Xin Luna Dong, Laure Berti-Equille, and Divesh Srivastava. “Truth Discovery and Copying Detection in a Dynamic World”. In: *Proc. VLDB Endow.* 2.1 (Aug. 2009), pp. 562–573. ISSN: 2150-8097. DOI: 10.14778/1687627.1687691. URL: https://doi.org/10.14778/1687627.1687691.

[22] Pål Grønås Drange et al. “On the threshold of intractability”. In: *Algorithms-ESA 2015*. Springer, 2015, pp. 411–423.

[23] Yang Du et al. “Bayesian Co-Clustering Truth Discovery for Mobile Crowd Sensing Systems”. In: *IEEE Transactions on Industrial Informatics* (2019), pp. 1–1. ISSN: 1551-3203. DOI: 10.1109/TII.2019.2896287.

[24] Edith Elkind and Arkadii Slinko. “Rationalizations of voting rules”. In: *Handbook of Computational Social Choice*. Ed. by Felix Brandt et al. Cambridge University Press, 2016, pp. 169–196.

[25] Ulle Endriss. “Judgment Aggregation”. In: *Handbook of Computational Social Choice*. Ed. by Felix Brandt et al. 1st. New York, NY, USA: Cambridge University Press, 2016. Chap. 17.

[26] Patricia Everaere, Sebastien Konieczny, and Pierre Marquis. “Belief Merging Operators as Maximum Likelihood Estimators”. In: *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI-20*. Ed. by Christian Bessiere. International Joint Conferences on Artificial Intelligence Organization, July 2020, pp. 1763–1769.

[27] Patricia Everaere, Sébastien Konieczny, Pierre Marquis, et al. "The Epistemic View of Belief Merging: Can We Track the Truth?." In: *ECAI*. 2010, pp. 621–626.

[28] Alban Galland et al. "Corroborating Information from Disagreeing Views". In: *Proceedings of the Third ACM International Conference on Web Search and Data Mining*. WSDM '10. New York, New York, USA: ACM, 2010, pp. 131–140. ISBN: 978-1-60558-889-6. DOI: 10.1145/1718487.1718504. URL: http://doi.acm.org/10.1145/1718487.1718504.

[29] Alban Galland et al. "Corroborating Information from Disagreeing Views". In: *Proceedings of the Third ACM International Conference on Web Search and Data Mining*. WSDM '10. New York: ACM, 2010, pp. 131–140. ISBN: 978-1-60558-889-6.

[30] Ebrahim Ghorbani. *Some spectral properties of chain graphs*. 2017. arXiv: 1703.03581 [math.CO].

[31] Arpita Ghosh, Satyen Kale, and Preston McAfee. "Who moderates the moderators?" In: *Proceedings of the 12th ACM conference on Electronic commerce - EC 11*. ACM Press, 2011. DOI: 10.1145/1993574.1993599. URL: https://doi.org/10.1145%2F1993574.1993599.

[32] Julio González-Díaz, Ruud Hendrickx, and Edwin Lohmann. "Paired comparisons analysis: an axiomatic approach to ranking methods". In: *Social Choice and Welfare* 42.1 (2014), pp. 139–169.

[33] Manish Gupta and Jiawei Han. "Heterogeneous Network-based Trust Analysis: A Survey". In: *SIGKDD Explor. Newsl.* 13.1 (Aug. 2011), pp. 54–71. ISSN: 1931-0145. DOI: 10.1145/2031331.2031341. URL: http://doi.acm.org/10.1145/2031331.2031341.

[34] Stephan Hartmann and Jan Sprenger. "Judgment aggregation and the problem of tracking the truth". In: *Synthese* (2012), pp. 209–221.

[35] Yang Jiao, R Ravi, and Wolfgang Gatterbauer. "Algorithms for automatic ranking of participants and tasks in an anonymized contest". In: *International Workshop on Algorithms and Computation*. Springer. 2017, pp. 335–346.

[36] Jon M. Kleinberg. "Authoritative Sources in a Hyperlinked Environment". In: *J. ACM* 46.5 (Sept. 1999), pp. 604–632. ISSN: 0004-5411. DOI: 10.1145/324133.324140. URL: http://doi.acm.org/10.1145/324133.324140.

[37] Sébastien Konieczny and Ramón Pino Pérez. "Improvement Operators". In: *Principles of Knowledge Representation and Reasoning: Proceedings of the Eleventh International Conference, KR 2008, Sydney, Australia, September 16-19, 2008*. Ed. by Gerhard Brewka and Jérôme Lang. AAAI Press, 2008, pp. 177–187. URL: http://www.aaai.org/Library/KR/2008/kr08-018.php.

[38] Sébastien Konieczny and Ramón Pino Pérez. "Merging information under constraints: a logical framework". In: *Journal of Logic and computation* 12.5 (2002), pp. 773–808.

[39] Neema Kotonya and Francesca Toni. "Explainable Automated Fact-Checking for Public Health Claims". In: *Proceedings of the 2020 Conference on Empirical Methods in Natural Language Processing (EMNLP)*. 2020, pp. 7740–7754.

[40] Justin Kruger et al. "Axiomatic Analysis of Aggregation Methods for Collective Annotation". In: *Proceedings of the 2014 International Conference on Autonomous Agents and Multi-agent Systems*. AAMAS '14. Paris, France: International Foundation for Autonomous Agents and Multiagent Systems, 2014, pp. 1185–1192. ISBN: 978-1-4503-2738-1. URL: http://dl.acm.org/citation.cfm?id=2617388.2617437.

[41] Jean-François Laslier and M Remzi Sanver. *Handbook on approval voting*. Springer Science & Business Media, 2010.

[42] Yaliang Li et al. "A Survey on Truth Discovery". In: *SIGKDD Explor. Newsl.* 17.2 (2016), pp. 1–16. ISSN: 1931-0145. DOI: 10.1145/2897350.2897352. URL: http://doi.acm.org/10.1145/2897350.2897352.

[43] Yaliang Li et al. "Conflicts to Harmony: A Framework for Resolving Conflicts in Heterogeneous Data by Truth Discovery". In: *IEEE Transactions on Knowledge and Data Engineering* 28.8 (Aug. 2016), pp. 1986–1999. ISSN: 1041-4347. DOI: 10.1109/TKDE.2016.2559481.

[44] Fenglong Ma et al. "FaitCrowd: Fine Grained Truth Discovery for Crowdsourced Data Aggregation". In: *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. KDD '15. eventplace: Sydney, NSW, Australia. New York, NY, USA: ACM, 2015, pp. 745–754. ISBN: 978-1-4503-3664-2. DOI: 10.1145/2783258.2783314. URL: http://doi.acm.org/10.1145/2783258.2783314.

[45] Fenglong Ma et al. "Unsupervised Discovery of Drug Side-Effects from Heterogeneous Data Sources". In: *Proceedings of the 23rd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. KDD '17. Halifax, NS, Canada: ACM, 2017, pp. 967–976. ISBN: 978-1-4503-4887-4. DOI: 10.1145/3097983.3098129. URL: http://doi.acm.org/10.1145/3097983.3098129.

[46] Jermaine Marshall, Arturo Argueta, and Dong Wang. "A neural network approach for truth discovery in social sensing". In: *2017 IEEE 14th international conference on mobile Ad Hoc and sensor systems (MASS)*. IEEE. 2017, pp. 343–347.

[47] Reshef Meir et al. "Truth Discovery via Proxy Voting". In: *CoRR* abs/1905.00629 (2019). arXiv: 1905.00629. URL: http://arxiv.org/abs/1905.00629.

[48] Assaf Natanzon, Ron Shamir, and Roded Sharan. "A polynomial approximation algorithm for the minimum fill-in problem". In: *SIAM Journal on Computing* 30.4 (2000), pp. 1067–1079.

[49] Assaf Natanzon, Ron Shamir, and Roded Sharan. "Complexity classification of some edge modification problems". In: *Discrete Applied Mathematics* 113.1 (2001), pp. 109–128.

[50] Catherine Olsson et al. *Skill Rating for Generative Models*. 2018. arXiv: 1808.04888 [stat.ML].

[51] Lawrence Page et al. *The PageRank Citation Ranking: Bringing Order to the Web.* Technical Report 1999-66. Stanford InfoLab, Nov. 1999. URL: http://ilpubs.stanford.edu:8090/422/.

[52] Jeff Pasternack and Dan Roth. "Knowing What to Believe (when You Already Know Something)". In: *Proceedings of the 23rd International Conference on Computational Linguistics*. COLING '10. Beijing, China: Association for Computational Linguistics, 2010, pp. 877–885. URL: http://dl.acm.org/citation.cfm?id=1873781.1873880.

[53] Ariel Rubinstein. "Ranking the participants in a tournament". In: *SIAM Journal on Applied Mathematics* 38.1 (1980), pp. 108–111.

[54] Joseph Singleton and Richard Booth. "An Axiomatic Approach to Truth Discovery". In: *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*. AAMAS '20. Auckland, New Zealand: International Foundation for Autonomous Agents and Multiagent Systems, 2020, pp. 2011–2013. ISBN: 9781450375184.

[55] Joseph Singleton and Richard Booth. *Who's the Expert? On Multi-source Belief Change*. 2022. DOI: 10.48550/ARXIV.2205.00077. URL: https://arxiv.org/abs/2205.00077.

[56] Giora Slutzki and Oscar Volij. "Ranking participants in generalized tournaments". In: *International Journal of Game Theory* 33.2 (2005), pp. 255–270.

[57] Giora Slutzki and Oscar Volij. "Scoring of web pages and tournaments—axiomatizations". In: *Social Choice and Welfare* 26.1 (2006), pp. 75–92.

[58] Moshe Tennenholtz. "Reputation Systems: An Axiomatic Approach". In: *Proceedings of the 20th Conference on Uncertainty in Artificial Intelligence*. UAI '04. Banff, Canada: AUAI Press, 2004, pp. 544–551. ISBN: 0-9749039-0-6. URL: http://dl.acm.org/citation.cfm?id=1036843.1036909.

[59] Dalia Attia Waguih and Laure Berti-Equille. "Truth discovery algorithms: An experimental evaluation". In: *arXiv preprint arXiv:1409.6428* (2014).

[60] Dong Wang et al. "On Truth Discovery in Social Sensing: A Maximum Likelihood Estimation Approach". In: *Proceedings of the 11th International Conference on Information Processing in Sensor Networks*. IPSN '12. event-place: Beijing, China. New York, NY, USA: ACM, 2012, pp. 233–244. ISBN: 978-1-4503-1227-1. DOI: 10.1145/2185677.2185737. URL: http://doi.acm.org/10.1145/2185677.2185737.

[61] Yaqing Wang et al. "Eann: Event adversarial neural networks for multi-modal fake news detection". In: *Proceedings of the 24th acm sigkdd international conference on knowledge discovery & data mining*. 2018, pp. 849–857.

[62] Tomasz Wąs and Oskar Skibski. *Axiomatic Characterization of PageRank*. 2020. DOI: 10.48550/ARXIV.2010.08487. URL: https://arxiv.org/abs/2010.08487.

[63] Houping Xiao. "Multi-sourced Information Trustworthiness Analysis: Applications and Theory". PhD thesis. University at Buffalo, State University of New York, 2018.

[64] Houping Xiao et al. "A Truth Discovery Approach with Theoretical Guarantee". In: *Proceedings of the 22Nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. KDD '16. San Francisco, California, USA: ACM, 2016, pp. 1925–1934. ISBN: 978-1-4503-4232-2. DOI: 10.1145/2939672.2939816. URL: http://doi.acm.org/10.1145/2939672.2939816.

[65] Yi Yang, Quan Bai, and Qing Liu. "A probabilistic model for truth discovery with object correlations". In: *Knowledge-Based Systems* 165 (2019), pp. 360–373. ISSN: 0950-7051. DOI: https://doi.org/10.1016/j.knosys.2018.12.004. URL: http://www.sciencedirect.com/science/article/pii/S0950705118305914.

[66] Yi Yang, Quan Bai, and Qing Liu. "On the Discovery of Continuous Truth: A Semi-supervised Approach with Partial Ground Truths". In: *Web Information Systems Engineering – WISE 2018*. Springer International Publishing, 2018, pp. 424–438. DOI: 10.1007/978-3-030-02922-7_29. URL: https://doi.org/10.1007%2F978-3-030-02922-7_29.

[67] Mihalis Yannakakis. "Computing the minimum fill-in is NP-complete". In: *SIAM Journal on Algebraic Discrete Methods* 2.1 (1981), pp. 77–79.

[68] Xiaoxin Yin, Jiawei Han, and Philip S. Yu. "Truth Discovery with Multiple Conflicting Information Providers on the Web". In: *IEEE Transactions on Knowledge and Data Engineering* 20.6 (June 2008), pp. 796–808. ISSN: 1041-4347. DOI: 10.1109/TKDE.2007.190745.

[69] Xiaoxin Yin and Wenzhao Tan. "Semi-supervised Truth Discovery". In: *Proceedings of the 20th International Conference on World Wide Web*. WWW '11. event-place: Hyderabad, India. New York, NY, USA: ACM, 2011, pp. 217–226. ISBN: 978-1-4503-0632-4. DOI: 10.1145/1963405.1963439. URL: http://doi.acm.org/10.1145/1963405.1963439.

[70] Daniel Yue Zhang et al. "On robust truth discovery in sparse social media sensing". In: *2016 IEEE International Conference on Big Data (Big Data)*. 2016-12, pp. 1076–1081. DOI: 10.1109/BigData.2016.7840710.

[71] Liyan Zhang et al. "Latent Dirichlet Truth Discovery: Separating Trustworthy and Untrustworthy Components in Data Sources". In: *IEEE Access* 6 (2018), pp. 1741–1752. ISSN: 2169-3536. DOI: 10.1109/ACCESS.2017.2780182.

[72] Shi Zhi et al. "Modeling Truth Existence in Truth Discovery". In: *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. KDD '15. Sydney, NSW, Australia: ACM, 2015, pp. 1543–1552. ISBN: 978-1-4503-3664-2. DOI: 10.1145/2783258.2783339. URL: http://doi.acm.org/10.1145/2783258.2783339.

[73] William S. Zwicker. "Introduction to the Theory of Voting". In: *Handbook of Computational Social Choice*. Ed. by Felix Brandt et al. 1st. New York, NY, USA: Cambridge University Press, 2016. Chap. 2.

A Proofs for Chapter 4

A.1 Proof of Theorem 1

The following lemma is required before the proof.

Lemma 10. *Suppose a network $N = (V, E)$ contains claims only for a single object $o \in \mathcal{O}$; that is, there exists $o \in \mathcal{O}$ such that $(s, f) \in E$ implies $obj_N(f) = o$ for all $s \in \mathcal{S}, f \in \mathcal{F}$. Then for any Symmetric operator T and $f_1, f_2 \in \mathcal{F}$, $|\mathsf{src}_N(f_1)| = |\mathsf{src}_N(f_2)| > 0$ implies $f_1 \approx_N^T f_2$.*

Proof. Suppose N has the stated property, T satisfies symmetry, and $|\mathsf{src}_N(f_1)| = |\mathsf{src}_N(f_2)| > 0$. Then there is a bijection $\varphi : \mathsf{src}_N(f_1) \to \mathsf{src}_N(f_2)$. Note that since f_1 and f_2 are for the same object no source can claim both facts, i.e. $\mathsf{src}_N(f_1) \cap \mathsf{src}_N(f_2) = \emptyset$.

Define a permutation π by

$$\pi(s) = \begin{cases} \varphi(s) & \text{if } s \in \mathsf{src}_N(f_1) \\ \varphi^{-1}(s) & \text{if } s \in \mathsf{src}_N(f_2) \\ s & \text{otherwise} \end{cases}$$

$$\pi(f) = \begin{cases} f_2 & \text{if } f = f_1 \\ f_2 & \text{if } f = f_2 \\ f & \text{otherwise} \end{cases}$$

and $\pi(o) = o$ for all $o \in \mathcal{O}$. That is, π swaps facts f_1 and f_2, and swaps the sources of f_1 with their counterparts in f_2. Note that $\pi = \pi^{-1}$.

Write $N' = \pi(N)$. We claim that $N' = N$. Write E, E' for the edges in N and N' respectively. First we will show $E \subseteq E'$. Suppose $(s, f) \in E$. There are three cases.

Case 1: $f = f_1$. Here we have $(s, f_1) \in E$, so $s \in \mathsf{src}_N(f_1)$. Consequently $\pi(s) = \varphi(s) \in \mathsf{src}_N(f_2)$, i.e. $(\pi(s), f_2) \in E$. By the definition of a graph isomorphism we get $(\pi(\pi(s)), \pi(f_2)) \in E'$. Noting that $\pi(f_2) = f_1 = f$ and $\pi(\pi(s)) = s$ (since $\pi = \pi^{-1}$), we have $(s, f) \in E'$ as desired.

Case 2: $f = f_2$. Similar to the above case, here we have $s \in \mathsf{src}_N(f_2)$ and so $\pi(s) = \varphi^{-1}(s) \in \mathsf{src}_N(f_1)$, i.e. $(\pi(s), f_1) \in E$. As before, applying the definition of a graph isomorphism and using $\pi = \pi^{-1}$, we get $(s, f) \in E'$.

Case 3: $f \notin \{f_1, f_2\}$. By hypothesis f relates to the same object as f_1 and f_2. This means $s \notin \mathsf{src}_N(f_1)$ and $s \notin \mathsf{src}_N(f_2)$, since otherwise s would make claims for multiple facts for a single object. Hence we have $\pi(s) = s$ and $\pi(f) = f$. This means $(s, f) = (\pi(s), \pi(f)) \in E'$ as required.

To complete the claim $E \subseteq E'$, suppose $(f, o) \in E$. There are again three cases: $f = f_1$, $f = f_2$, or $f \notin \{f_1, f_2\}$. In each case the definition of π and $\pi(N)$ easily yield $(f, o) \in E'$. Hence $E \subseteq E'$.

Now for the reverse direction: we must show $E' \subseteq E$. Let $(x, y) \in E'$. By definition of a graph isomorphism, we have $(\pi^{-1}(x), \pi^{-1}(y)) \in E$. Using $\pi^{-1} = \pi$ and the first part we get $(\pi(x), \pi(y)) = (\pi^{-1}(x), \pi^{-1}(y)) \in E \subseteq E'$. The definition of a graph isomorphism then yields $(x, y) \in E$ and so $E' \subseteq E$. Hence $E = E'$ and $N = N'$.

To conclude the proof, we apply Symmetry of T to get

$$\begin{aligned} f_1 \preceq^T_N f_2 &\iff \pi(f_1) \preceq^T_{N'} \pi(f_2) \\ &\iff f_2 \preceq^T_{N'} f_1 \\ &\iff f_2 \preceq^T_N f_1 \end{aligned}$$

and so $f_1 \approx^T_N f_2$ as required. □

Proof of Theorem 1. Suppose T is an operator satisfying Symmetry, Monotonicity and POI. Let $N \in \mathcal{N}$, $o \in \mathcal{O}$ and $f_1, f_2 \in \mathsf{obj}_N^{-1}(o)$. We need to show that $f_1 \preceq^T_N f_2$ iff $|\mathsf{src}_N(f_1)| \le |\mathsf{src}_N(f_2)|$.

Let N' be the network obtained from N by removing all claims for facts other than those for object o; that is, $N' = (V, E')$ where E is the set of edges in N and

$$E' = (E \cap (\mathcal{S} \times \mathsf{obj}_N^{-1}(o))) \cup (E \cap (\mathcal{F} \times \mathcal{O}))$$

Note that the fact-object affiliations are the same in N' as in N, and the set of sources for each fact in $\mathsf{obj}_N^{-1}(o)$ is the same. Therefore POI applies, and it is sufficient to show that $f_1 \preceq^T_{N'} f_2$ iff $|\mathsf{src}_{N'}(f_1)| \le |\mathsf{src}_{N'}(f_2)|$.

First suppose $|\mathsf{src}_{N'}(f_1)| \le |\mathsf{src}_{N'}(f_2)|$. If $|\mathsf{src}_{N'}(f_1)| = |\mathsf{src}_{N'}(f_2)|$, then we have $f_1 \approx^T_{N'} f_2$ by Symmetry and Lemma 10; in particular $f_1 \preceq^T_{N'} f_2$. Otherwise $|\mathsf{src}_{N'}(f_2)| - |\mathsf{src}_{N'}(f_1)| = k > 0$. Removing k sources from f_2 to obtain a new network N'', we can apply the lemma to get $f_1 \approx^T_{N''} f_2$. We may then add these sources *back* to obtain N' again; k applications of Monotonicity then give $f_1 \prec^T_{N'} f_2$ as required.

To complete the proof we show that $f_1 \preceq^T_{N'} f_2$ implies $|\mathsf{src}_{N'}(f_1)| \le |\mathsf{src}_{N'}(f_2)|$. Indeed, suppose $f_1 \preceq^T_{N'} f_2$ but $|\mathsf{src}_{N'}(f_1) > |\mathsf{src}_{N'}(f_2)|$. Then the argument above gives $f_1 \succ^T_{N'} f_2$, which is clearly a contradiction. Hence the proof is complete. □

A.2 Proof of Theorem 3

The proof of this theorem is similar in spirit to that of Theorem 1. Like in that case, a preliminary result is required first.

Lemma 11. *Let N be a network and $f_1, f_2 \in \mathcal{F}$. Write $o_1 = \mathsf{obj}_N(f_1)$, $o_2 = \mathsf{obj}_N(f_2)$. Suppose N has the following properties:*

1. *There is $o^* \in \mathcal{O} \setminus \{o_1, o_2\}$ such that $f \in \mathcal{F} \setminus \{f_1, f_2\} \implies \mathsf{obj}_N(f) = o^*$; and*

2. $\mathsf{src}_N(f) = \emptyset$ *for all* $f \in \mathcal{F} \setminus \{f_1, f_2\}$.

Then for any operator T satisfying Symmetry, $|\mathsf{src}_N(f_1)| = |\mathsf{src}_N(f_2)|$ *implies* $f_1 \approx_N^T f_2$.

Proof. The proof is similar to that of Lemma 10. Suppose $|src_N(f_1)| = |\mathsf{src}_N(f_2)|$. Write

$$Q_1 = \mathsf{src}_N(f_1) \setminus \mathsf{src}_N(f_2)$$
$$Q_2 = \mathsf{src}_N(f_2) \setminus \mathsf{src}_N(f_1)$$

Then $|Q_1| = |Q_2|$, so there exists a bijection $\varphi : Q_1 \to Q_2$. Define a permutation π as follows:

$$\pi(s) = \begin{cases} \varphi(s) & \text{if } s \in Q_1 \\ \varphi^{-1}(s) & \text{if } s \in Q_2 \\ s & \text{otherwise} \end{cases}$$

$$\pi(f) = \begin{cases} f_2 & \text{if } f = f_1 \\ f_1 & \text{if } f = f_2 \\ f & \text{otherwise} \end{cases}$$

$$\pi(o) = \begin{cases} o_2 & \text{if } o = o_1 \\ o_1 & \text{if } o = o_2 \\ o & \text{otherwise} \end{cases}$$

That is, π swaps f_1 and f_2, swaps o_1 and o_2, and swaps sources in Q_1 with their counterparts in Q_2. Note that $\pi = \pi^{-1}$. Write $N' = \pi(N)$. We claim that $N' = N$. Write E, E' for the edges in N and N' respectively. First we show that $E \subseteq E'$. For this, first suppose $(s, f) \in E$ for some $s \in \mathcal{S}$, $f \in \mathcal{F}$. By definition of E, either $f = f_1$ or $f = f_2$.

Case 1: $f = f_1$. Here $\pi(f) = f_2$, and we have either $s \in Q_1$ or $s \in \mathsf{src}_N(f_1) \cap \mathsf{src}_N(f_2)$. In the first case, $\pi(s) = \varphi(s) \in Q_2 \subseteq \mathsf{src}_N(f_2) = \mathsf{src}_N(\pi(f))$. In the second case $\pi(s) = s \in \mathsf{src}_N(f_2) = \mathsf{src}_N(\pi(f))$. In either case, $(\pi(s), \pi(f)) \in E$.

Applying the definition of a graph isomorphism we get $(\pi(\pi(s)), \pi(\pi(f))) \in E'$. But $\pi = \pi^{-1}$, so this means $(s, f) \in E'$ as desired.

Case 2: $f = f_2$. This case is similar. Here $\pi(f) = f_1$. If $s \in Q_2$, then $\pi(s) = \varphi^{-1}(s) \in Q_1 \subseteq \mathsf{src}_N(f_1) = \mathsf{src}_N(\pi(f))$. Otherwise $s \in \mathsf{src}_N(f_1) \cap \mathsf{src}_N(f_2)$ and $\pi(s) = s \in \mathsf{src}_N(f_1) = \mathsf{src}_N(\pi(f))$. Again, we have $(\pi(s), \pi(f)) \in E$ in either case, so $(s, f) \in E'$.

Note that these two cases cover all possibilities since by hypothesis $\mathsf{src}_N(f) = \emptyset$ if $f \notin \{f_1, f_2\}$.

Next, suppose $(f, o) \in E$. If $f = f_1$ then $o = o_1$, so $(\pi(f), \pi(o)) = (f_2, o_2) \in E$. Similarly if $f = f_2$ then $o = o_2$ and $(\pi(f), \pi(o)) = (f_1, o_1) \in E$. If $f \notin \{f_1, f_2\}$ then $\pi(f) = f$ and $o = o^*$, so $\pi(o) = o$. We see that in all cases, $(\pi(f), \pi(f)) \in E$. Applying the same argument as in case 1 above, we see that $(f, o) \in E'$. This shows $E \subseteq E'$.

To complete the claim that $N = N'$ we must show $E' \subseteq E$. This can be shown using the same argument used in Lemma 10. Indeed, suppose $(x, y) \in E'$. Then by definition of a graph isomorphism, $(\pi^{-1}(x), \pi^{-1}(y)) \in E$. Using the fact that $\pi = \pi^{-1}$ and $E \subseteq E'$ we get $(\pi(x), \pi(y)) \in E'$, which then yields $(x, y) \in E$ as required. Hence $E = E'$ and $N = N'$.

Finally, using Symmetry of T we have

$$\begin{aligned} f_1 \preceq_N^T f_2 &\iff \pi(f_1) \preceq_{\pi(N)}^T \pi(f_2) \\ &\iff f_2 \preceq_{N'}^T f_1 \\ &\iff f_2 \preceq_N^T f_1 \end{aligned}$$

Consequently $f_1 \approx_N^T f_2$. □

Proof of Theorem 3. The 'if' direction is clear since *Voting* satisfies Strong Independence, Monotonicity and Symmetry (see Theorem 4). For the other direction, suppose T satisfies the stated axioms. Let N be a network and $f_1, f_2 \in \mathcal{F}$. We will construct a network N' where all claims for facts other than f_1, f_2 are removed, and these facts are grouped under a single object. To do so, let $o_1 = \mathsf{obj}_N(f_1)$, $o_2 = \mathsf{obj}_N(f_2)$ and take $o^* \in \mathcal{O} \setminus \{o_1, o_2\}$. Define an edge set E' by

$$(s, f) \in E' \iff f \in \{f_1, f_2\} \text{ and } s \in \mathsf{src}_N(f)$$

$$(f, o) \in E' \iff (f \in \{f_1, f_2\} \text{ and } o = \mathsf{obj}_N(f)) \text{ or } (f \notin \{f_1, f_2\} \text{ and } o = o^*)$$

Then let N' be the network with edge set E'. Note that $\mathsf{src}_{N'}(f_j) = \mathsf{src}_N(f_j)$. By Strong Independence it is therefore sufficient to show that $f_1 \preceq_{N'}^T f_2$ iff $|src_{N'}(f_1)| \leq |\mathsf{src}_{N'}(f_2)|$. Note also that N' satisfies the hypothesis of Lemma 11.

Now, suppose $|\mathsf{src}_{N'}(f_1)| \leq |\mathsf{src}_{N'}(f_2)|$. If $|\mathsf{src}_{N'}(f_1)| = |\mathsf{src}_{N'}(f_2)|$ then by Lemma 11 $f_1 \approx_{N'}^T f_2$, and in particular $f_1 \preceq_{N'}^T f_2$.

Otherwise, $|\mathsf{src}_{N'}(f_2)| - |\mathsf{src}_{N'}(f_1)| = k > 0$. Consider N'' where k sources from $\mathsf{src}_{N'}(f_2)$ are removed, and all other claims remain. By the lemma, $f_1 \approx_{N''}^T f_2$. Applying Monotonicity k times we can produce N' from N'' and get $f_1 \prec_{N'}^T f_2$ as desired.

For the other implication, suppose $f_1 \preceq_{N'}^T f_2$ and, for contradiction, $|\mathsf{src}_{N'}(f_1)| > |\mathsf{src}_{N'}(f_2)|$. Applying Monotonicity again as above gives $f_1 \succ_{N'}^T f_2$ and the required contradiction. □

A.3 Proof of Theorem 4

Proof. We will show that *Voting* satisfies Symmetry, Unanimity, Groundedness, Monotonicity, POI, Strong Independence and PCI, and that Coherence is *not* satisfied. For Symmetry and PCI we use the (stronger) numerical variants *numerical Symmetry* and *numerical PCI*, introduced in Section 4.5.2. T will denote the (numerical) *Voting* operator in what follows.

Symmetry. Suppose N and $\pi(N)$ are equivalent networks. Let $f \in \mathcal{F}$. By definition of equivalent networks we have $s \in \mathsf{src}_N(f)$ iff $\pi(s) \in \mathsf{src}_{\pi(N)}(\pi(f))$ for all $s \in \mathcal{S}$. Consequently π restricted to $\mathsf{src}_N(f)$ is a bijection into $\mathsf{src}_{\pi(N)}(\pi(f))$, and hence

$$T_N(f) = |\mathsf{src}_N(f)| = |\mathsf{src}_{\pi(N)}(\pi(f))| = T_{\pi(N)}(\pi(f))$$

Now let $s \in \mathcal{S}$. Clearly we have $T_N(s) = 1 = T_{\pi(N)}(\pi(s))$. Hence T satisfies numerical Symmetry and therefore Symmetry.

Unanimity and Groundedness. Suppose $N \in \mathcal{N}$ and $f \in \mathcal{F}$. If $\mathsf{src}_N(f) = \mathcal{S}$ then for any $g \in \mathcal{F}$,

$$T_N(g) = |\mathsf{src}_N(g)| \leq |\mathcal{S}| = |\mathsf{src}_N(f)| = T_N(f)$$

so $g \preceq_N^T f$ and Unanimity is satisfied. If instead $\mathsf{src}_N(f) = \emptyset$, we have

$$T_N(g) = |\mathsf{src}_N(g)| \geq 0 = |\mathsf{src}_N(f)| = T_N(f)$$

so $f \preceq_N^T g$ and Groundedness is satisfied.

Monotonicity. Let N, N', s and f be as given in the statement of Monotonicity. It is clear that $|\mathsf{src}_{N'}(f)| = |\mathsf{src}_N(f)| + 1$. Also, for any $g \in \mathcal{F}$, $g \neq f$, the set of sources in N' is the same as in N but with s possibly removed. Hence $|\mathsf{src}_{N'}(g)| \leq |\mathsf{src}_N(g)$. Therefore $g \preceq_N^T f$ implies

$$|\mathsf{src}_{N'}(g)| \leq |\mathsf{src}_N(g)| \leq |\mathsf{src}_N(f)| < |\mathsf{src}_{N'}(f)|$$

and so $g \prec_{N'}^T f$ as required.

Independence axioms. Next we show Strong Independence, which implies POI. Suppose $N_1, N_2 \in \mathcal{N}$, $f_1, f_2 \in \mathcal{F}$ and $\mathsf{src}_{N_1}(f_j) = \mathsf{src}_{N_2}(f_j)$ for each $j \in \{1, 2\}$. Clearly we have

$$T_{N_1}(f_j) = |\mathsf{src}_{N_1}(f_j)| = |\mathsf{src}_{N_2}(f_j)| = T_{N_2}(f_j)$$

Consequently

$$\begin{aligned} f_1 \preceq_{N_1}^T f_2 &\iff T_{N_1}(f_1) \leq T_{N_1}(f_2) \\ &\iff T_{N_2}(f_1) \leq T_{N_2}(f_2) \\ &\iff f_1 \preceq_{N_2}^T f_2 \end{aligned}$$

as required for Strong Independence.

For PCI we proceed as with Symmetry by showing numerical PCI. Let N_1, N_2 have a common connected component G. Let $f \in G \cap \mathcal{F}$. By definition of a connected component, $s \in \mathsf{src}_{N_1}(f)$ iff $s \in \mathsf{src}_{N_2}(f)$, so $\mathsf{src}_{N_1}(f) = \mathsf{src}_{N_2}(f)$. Hence

$$T_{N_1}(f) = |\mathsf{src}_{N_1}(f)| = |\mathsf{src}_{N_2}(f)| = T_{N_2}(f)$$

For $s \in G \cap \mathcal{S}$, we trivially have $T_{N_1}(s) = 1 = T_{N_1}(s)$. Hence numerical PCI is satisfied.

Coherence. The violation of Coherence follows from Theorem 2, since we have already shown that Symmetry, Monotonicity and POI are satisfied. □

A.4 Proof of Lemma 2

Proof. The first statement follows easily from the definition of the limit. We shall prove only the second one.

First we prove the 'if' direction. Write $D = T_N^*(f_1) - T_n^*(f_2)$. We need to show that $D < 0$. Write $d_n = T_N^n(f_1) - T_N^n(f_2)$ so that $D = \lim_{n \to \infty} d_n$. Take $\varepsilon = \rho/2 > 0$.

Then for sufficiently large n we have $d_n \leq -\rho/2 < 0$. Taking $n \to \infty$, we have $D = \lim_{n\to\infty} d_n \leq -\rho/2 < 0$ as required.

For the 'only if' direction, suppose $D < 0$. Let $\rho = -D$. Then for any $\varepsilon > 0$, by the definition of the limit there is $K \in \mathbb{N}$ such that $|d_n - D| < \varepsilon$ for $n \geq K$; in particular, $d_n < \varepsilon + D = \varepsilon - \rho$ as required. □

A.5 Proof of Theorem 5

The following results will be helpful to simplify the proof of Theorem 5.

Lemma 12. norm *has the following properties.*

1. norm *preserves numerical Symmetry, in the sense that* $\mathsf{norm}(T)$ *satisfies numerical Symmetry whenever* T *does.*
2. norm *leaves rankings unchanged, in the following sense. For* $T \in \mathcal{T}_{Num}$, $N \in \mathcal{N}$, $s_1, s_2 \in \mathcal{S}$, $f_1, f_2 \in \mathcal{F}$,

$$s_1 \sqsubseteq_N^T s_2 \iff s_1 \sqsubseteq_N^{\mathsf{norm}(T)} s_2$$
$$f_1 \preceq_N^T f_2 \iff f_1 \preceq_N^{\mathsf{norm}(T)} f_2$$

Proof. For part (i), suppose T satisfies numerical Symmetry, and write $T' = U(T)$. Let N and $\pi(N)$ be equivalent networks. First note that

$$\max_{x\in\mathcal{S}} |T_N(x)| = \max_{x\in\mathcal{S}} |T_{\pi(N)}(\pi(x))| = \max_{x\in\mathcal{S}} |T_{\pi(N)}(x)|$$

where the second equality follows since π restricted to $\mathcal{S}$ is a surjection into $\mathcal{S}$ by the definition of equivalent networks. If this maximum is 0, then $T'_N(s) = 0 = T'_{\pi(N)}(s)$ for all $s \in \mathcal{S}$. Otherwise,

$$T'_N(s) = \frac{T_N(s)}{\max\limits_{x\in\mathcal{S}} |T_N(x)|} = \frac{T_{\pi(N)}(\pi(s))}{\max\limits_{x\in\mathcal{S}} |T_{\pi(N)}(x)|} = T'_{\pi(N)}(\pi(s))$$

One can show that $T'_N(f) = T'_{\pi(N)}(\pi(f))$ by an identical argument. Hence $T' = U(T)$ satisfies numerical Symmetry also.

Now we prove part (ii). First suppose $s_1 \sqsubseteq_N^T s_2$. Write $T' = \mathsf{norm}(T)$. We have $T'_N(x) = \alpha T_N(x)$ for some $\alpha \geq 0$ and all $x \in \mathcal{S}$ (either $\alpha = 1/\max_{x\in\mathcal{S}} |T_N(x)|$ or $\alpha = 0$). We therefore have

$$\begin{aligned} s_1 \sqsubseteq_N^T s_2 &\implies T_N(s_1) \leq T_N(s_2) \\ &\implies \alpha T_N(s_1) \leq \alpha T_N(s_2) \\ &\implies T'_N(s_1) \leq T'_N(s_2) \\ &\implies s_1 \sqsubseteq_N^{T'} s_2 \end{aligned}$$

as desired.

Now suppose $s_1 \sqsubseteq_N^{T'} s_2$, i.e. $\alpha T_N(s_1) \leq \alpha T_N(s_2)$. If $\alpha > 0$ then dividing by α readily gives $s_1 \sqsubseteq_N^T s_2$. Otherwise, $\alpha = 0$. This means $\max_{x\in\mathcal{S}} |T_N(x)| = 0$, and thus $T_N(x) = 0$ for all $x \in \mathcal{S}$. In particular $T_N(s_1) = 0 \leq 0 = T_N(s_2)$ so $s_1 \sqsubseteq_N^T s_2$.

The second statement regarding fact ranking may be shown using an identical argument. □

Corollary 1. norm *preserves Coherence, Unanimity, Groundedness and PCI.*

Proof of Theorem 5. Throughout this proof, $(T^n)_{n\in\mathbb{N}}$ will denote the iterative operator *Sums*, T^* will denote the limit operator, and $U = \mathsf{norm}\circ U^{\mathsf{Sums}}$ will denote the update function for *Sums*.

Coherence. Source-Coherence was shown in the body of the paper. The proof that Fact-Coherence is satisfied is similar, and uses Lemma 3. Suppose $N \in \mathcal{N}$, $T = T^n$ for some $n \in \mathbb{N}$, $\varepsilon, \rho > 0$, and $\mathsf{src}_N(f_1)$ is (ε,ρ)-less trustworthy than $\mathsf{src}_N(f_2)$ with respect to N and $\tilde{T}$ under a bijection φ, where $\tilde{T} = U(T)$. Let $\hat{s} \in \mathsf{src}_N(f_1)$ be such that $\tilde{T}_N(s) - \tilde{T}_N(\varphi(s)) \leq \varepsilon - \rho$.

Write $T' = U^{\mathsf{Sums}}(T)$ so that $\tilde{T} = \mathsf{norm}(T')$, and set

$$\alpha = \frac{1}{\max\limits_{x\in\mathcal{S}} |T'_N(x)|}$$

We may assume without loss of generality that $\varepsilon < \frac{1}{|\mathcal{S}|}\rho$. Note that for $s \in \mathcal{S}$, $\tilde{T}_N(s) = \alpha T'_N(s)$ and therefore $T'_N(s) = \frac{1}{\alpha}\tilde{T}_N(s)$. Writing

$$\beta = \frac{1}{\max\limits_{y\in\mathcal{F}} |T'_N(y)|}$$

and applying a similar argument as for showing Source-Coherence, we find

$$\begin{aligned}
\tilde{T}_N(f_1) - \tilde{T}_N(f_2) &= \beta \sum_{s\in\mathsf{src}_N(f_1)} \Big(T'_N(s) - T'_N(\varphi(s))\Big) \\
&= \frac{\beta}{\alpha} \sum_{s\in\mathsf{src}_N(f_1)} \Big(\tilde{T}_N(s) - \tilde{T}_N(\varphi(s))\Big) \\
&= \frac{\beta}{\alpha}\left[\underbrace{\Big(\tilde{T}_N(\hat{s}) - \tilde{T}_N(\varphi(\hat{s}))\Big)}_{\leq\varepsilon-\rho} + \sum_{s\in\mathsf{src}_N(f_1)\setminus\{\hat{s}\}} \underbrace{\Big(\tilde{T}_N(s) - \tilde{T}_N(\varphi(s))\Big)}_{\leq\varepsilon}\right] \\
&\leq \frac{\beta}{\alpha}\cdot\underbrace{\Big(|\mathcal{S}|\varepsilon - \rho\Big)}_{<0}
\end{aligned}$$

Now we need to bound β/α from below. Since we assume $T = T^n$ for some $n \in \mathbb{N}$, for any $y \in \mathcal{F}$ we have

$$|T'_N(y)| = \sum_{s\in\mathsf{src}_N(y)} \underbrace{T'_N(s)}_{\leq|\mathcal{F}|} \leq |\mathsf{src}_N(y)|\cdot|\mathcal{F}| \leq |\mathcal{S}|\cdot|\mathcal{F}|$$

Therefore

$$\beta \geq \frac{1}{|\mathcal{S}|\cdot|\mathcal{F}|}$$

Next, we claim there is some fact $\bar{f} \in \mathcal{F}$ with $T_N(\bar{f}) \geq 1/2$ and $\mathsf{src}_N(\bar{f}) \neq \emptyset$. Indeed, if $T = T^1 = T^{\text{fixed}}$ then take any fact with at least one associated source.[1]

Otherwise, since we assume not all scores are 0 in the limit, there is some $\bar{f}$ with $T_N(\bar{f}) = 1$ due to the application of norm. Clearly $\mathsf{src}_N(\bar{f}) \neq \emptyset$, since we would have $T_N(\bar{f}) = 0$ otherwise.

Let $\bar{x} \in \mathsf{src}_N(\bar{f})$. Then

$$|T'_N(\bar{x})| = T'_N(\bar{x}) = \underbrace{T_N(\bar{f})}_{\geq 1/2} + \underbrace{\sum_{f \in \mathsf{facts}_N(\bar{x}) \setminus \{\bar{f}\}} T_N(f)}_{\geq 0} \geq \frac{1}{2}$$

This means

$$\frac{1}{\alpha} = \max_{x \in \mathcal{S}} |T'_N(x)| \geq |T'_N(\bar{x})| \geq \frac{1}{2}$$

and so, finally,

$$\frac{\beta}{\alpha} \geq \frac{1}{|\mathcal{S}| \cdot |\mathcal{F}|} \cdot \frac{1}{2}$$

Combined with what was shown before, this means

$$\tilde{T}_N(f_1) - \tilde{T}_N(f_2) \leq \frac{1}{2 \cdot |\mathcal{S}| \cdot |\mathcal{F}|} \Big(|\mathcal{S}|\varepsilon - \rho \Big)$$

and Fact-Coherence follows from Lemma 3.

Symmetry. As a consequence of Lemma 4, to show Symmetry it is sufficient to show that T^{fixed} satisfies numerical Symmetry, and that $U = \mathsf{norm} \circ U^{\text{Sums}}$ preserves numerical Symmetry. Since T^{fixed} is constant with value $1/2$, it is clear that numerical Symmetry is satisfied. Moreover, Lemma 12 part (i) already shows that norm preserves numerical Symmetry, so we only need to show that U^{Sums} does.

To that end, suppose $T \in \mathcal{T}_{Num}$ satisfies numerical symmetry, and write $T' = U^{\text{Sums}}(T)$. Let N and $\pi(N)$ be equivalent networks and $s \in \mathcal{S}$. Then

$$T'_{\pi(N)}(\pi(s)) = \sum_{y \in \mathsf{facts}_{\pi(N)}(\pi(s))} T_{\pi(N)}(y)$$

Note that $f \in \mathsf{facts}_N(s)$ iff $\pi(f) \in \mathsf{facts}_{\pi(N)}(\pi(s))$. Rephrased slightly, we have $y \in \mathsf{facts}_{\pi(N)}(\pi(s))$ iff $\pi^{-1}(y) \in \mathsf{facts}_N(s)$. Hence we may make a 'substitution' $f = \pi^{-1}(y)$ and sum over $\mathsf{facts}_N(s)$, i.e.

$$T'_{\pi(N)}(\pi(s)) = \sum_{f \in \mathsf{facts}_N(s)} T_{\pi(N)}(\pi(f))$$

Applying numerical symmetry for T, we get

$$\begin{aligned} T'_{\pi(N)}(\pi(s)) &= \sum_{f \in \mathsf{facts}_N(s)} T_N(f) \\ &= T'_N(s) \end{aligned}$$

Following the same tactic, one may also show that $T'_{\pi(N)}(\pi(f)) = T'_N(f)$ for all $f \in \mathcal{F}$. Hence U^{Sums} preserves numerical Symmetry, and we are done.

[1] Note that this is always possible since a truth discovery network contains at least one claim by definition.

Unanimity and Groundedness. Unanimity and Groundedness can be proved together using Lemma 5 and Corollary 1. By these results it is sufficient that T^{fixed} satisfies Unanimity and Groundedness – this is trivial – and that U^{Sums} preserves them.

Suppose T satisfies Unanimity and Groundedness and write $T' = U^{\text{Sums}}(T)$. Assume without loss of generality that $T = T^n$ for some $n \in \mathbb{N}$ so that $T'_N \geq 0$. Suppose $N \in \mathcal{N}$, $f \in \mathcal{F}$ and that $\mathsf{src}_N(f) = \mathcal{S}$. Let $g \in \mathcal{F}$. We must show that $g \preceq_N^{T'} f$. We have

$$T'_N(g) = \sum_{s \in \mathsf{src}_N(g)} T'_N(s) \leq \sum_{s \in \mathcal{S}} T'_N(s) = T'_N(f)$$

i.e. $g \preceq_N^{T'} f$ as required for Unanimity. For Groundedness, suppose $\mathsf{src}_N(f) = \emptyset$. We must show $f \preceq_N^{T'} g$. Indeed, the sum in the expression for $T'_N(f)$ is taken over the empty set, which by convention is 0. Since $T'_N(g) \geq 0$, we are done. □

A.6 Proof of Theorem 6

Proof. Here we give only the technical details for the argument showing *SC-Voting* satisfies Symmetry, since the results for the other axioms were given in the main text.

Symmetry. Since *Voting* satisfies Symmetry, it is clear that $f_1 \preceq_N^{T^{SCV}} f_2$ iff $\pi(f_1) \preceq_{\pi(N)}^{T^{SCV}} \pi(f_2)$ for any equivalent networks N and $\pi(N)$. We need to show that $s_1 \sqsubseteq_N^{T^{SCV}} s_2$ iff $\pi(s_1) \sqsubseteq_{\pi(N)}^{T^{SCV}} \pi(s_2)$.

First we will show that $\lhd_N$ and $\lhd_{\pi(N)}$ have a similar symmetry property: $s_1 \lhd_N s_2$ iff $\pi(s_1) \lhd_{\pi(N)} \pi(s_2)$. Indeed, suppose $s_1 \lhd_N s_2$. Then there is a bijection $\varphi : \mathsf{facts}_N(s_1) \to \mathsf{facts}_N(s_2)$ with $f \preceq_N^{T^{SCV}} \varphi(f)$, and there is some $\hat{f}$ with $\hat{f} \prec_N^{T^{SCV}} \varphi(\hat{f})$.

It can be seen that π restricted to $\mathsf{facts}_N(s_i)$ is a bijection into $\mathsf{facts}_{\pi(N)}(\pi(s_i))$. Let π_1 and π_2 denote these restrictions for $i = 1, 2$ respectively. Set $\theta = \pi_2 \circ \varphi \circ \pi_1^{-1}$, so that θ maps $\mathsf{facts}_{\pi(N)}(\pi(s_1))$ into $\mathsf{facts}_{\pi(N)}(\pi(s_2))$. As a composition of bijections, θ is itself bijective.

Let $g \in \mathsf{facts}_{\pi(N)}(\pi(s_1))$. Write $f = \pi_1^{-1}(g) \in \mathsf{facts}_N(s_1)$. By the property of φ, we have

$$f \preceq_N^{T^{SCV}} \varphi(f)$$

By the symmetry property of the fact-ranking (which follows from symmetry of *Voting*), we can apply π to the above to get

$$\pi(f) \preceq_{\pi(N)}^{T^{SCV}} \pi(\varphi(f))$$

Since $f \in \mathsf{facts}_N(s_1)$ and $\varphi(f) \in \mathsf{facts}_N(s_2)$, we have $\pi(f) = \pi_1(f)$ and $\pi(\varphi(f)) = \pi_2(\varphi(f))$. Using this fact in the above inequality and recalling $f = \pi^{-1}(g)$ we get

$$g = \pi_1(f) = \pi(f) \preceq_{\pi(N)}^{T^{SCV}} \pi(\varphi(f)) = \pi_2(\varphi(f)) = \pi_2(\varphi(\pi_1^{-1}(g))) = \theta(g)$$

i.e. $g \preceq_{\pi(N)}^{T^{SCV}} \theta(g)$. Applying the same argument with $\hat{g} = \pi_1^{-1}(\hat{f})$ we get $\hat{g} \prec_{\pi(N)}^{T^{SCV}} \theta(\hat{g})$.

This shows that $\text{facts}_{\pi(N)}(\pi(s_1))$ is less believable than $\text{facts}_{\pi(N)}(\pi(s_2))$ with respect to *SC-Voting* (whose fact-ranking coincides with *Voting*) in $\pi(N)$ under θ. Hence $\pi(s_1) \lhd_{\pi(N)} \pi(s_2)$.

We have shown $s_1 \lhd_N s_2 \implies \pi(s_1) \lhd_{\pi(N)} \pi(s_2)$. For the converse implication, apply the same argument starting from $\pi(s_1) \lhd_{\pi(N)} \pi(s_2)$ with the π^{-1}.

Next, we note that for $i = 1, 2$ and any $t \in \mathcal{S}$,

$$\begin{aligned} t \in W_N(s_i) &\iff t \lhd_N s_i \\ &\iff \pi(t) \lhd_{\pi(N)} \pi(s_i) \\ &\iff \pi(t) \in W_{\pi(N)}(\pi(s_i)) \end{aligned}$$

Consequently π restricted to $W_N(s_i)$ is a bijection into $W_{\pi(N)}(\pi(s_i))$, which means $|W_N(s_i)| = |W_{\pi(N)}(\pi(s_i))|$. Finally, this means

$$\begin{aligned} s_1 \sqsubseteq_N^{T^{SCV}} s_2 &\iff |W_N(s_1)| \leq |W_N(s_2)| \\ &\iff |W_{\pi(N)}(\pi(s_1))| \leq |W_{\pi(N)}(\pi(s_2))| \\ &\iff \pi(s_1) \sqsubseteq_{\pi(N)}^{T^{SCV}} \pi(s_2) \end{aligned}$$

as required for Symmetry. □

A.7 Proof of Theorem 8

Proof. Here we show that *UnboundedSums* satisfies Symmetry, PCI, Unanimity and Groundedness, since the other axioms were dealt with in the main body of the paper.

Throughout the proof, let $(T^n)_{n\in\mathbb{N}}$ denote *UnboundedSums*, T^* denote the ordinal limit of *UnboundedSums*, and for a network N let J_N be as in Theorem 7. Then the rankings in N induced by T^n for $n \geq J_N$ are the same as T^*.

Symmetry. In the proof of Theorem 5, we saw that the update function U^{Sums} preserves numerical Symmetry, in the sense that if T satisfies numerical Symmetry then $U^{\text{Sums}}(T)$ does also. Since it is clear that the prior operator for *UnboundedSums* satisfies numerical Symmetry, T^n satisfies numerical Symmetry and consequently Symmetry for all $n \in \mathbb{N}$.

Now, let N and $\pi(N)$ be equivalent networks. Let $J, J' \in \mathbb{N}$ be such that $T^*(N)$ and $T^*(\pi(N))$ are given by T_N^J and $T_{\pi(N)}^{J'}$ respectively and take $n \geq \max\{J, J'\}$. For $s_1, s_2 \in \mathcal{S}$ we have by Symmetry of T^n,

$$\begin{aligned} s_1 \sqsubseteq_N^{T^*} s_2 &\iff s_1 \sqsubseteq_N^{T^n} s_2 \\ &\iff \pi(s_1) \sqsubseteq_{\pi(N)}^{T^n} \pi(s_2) \\ &\iff \pi(s_1) \sqsubseteq_{\pi(N)}^{T^*} \pi(s_2) \end{aligned}$$

as required for Symmetry. Using an identical argument, one can show that $f_1 \preceq_N^{T^*} f_2$ iff $\pi(f) \preceq_{\pi(N)}^{T^*} \pi(f_2)$. Hence T^* satisfies Symmetry.

PCI. As with Symmetry, we will show that T^n satisfies numerical PCI, and consequently PCI, for all $n \in \mathbb{N}$. Let N_1, N_2 be networks with a common connected component G. Let $s \in G \cap \mathcal{S}$ and $f \in G \cap \mathcal{F}$. Note that $\mathsf{facts}_{N_1}(s) = \mathsf{facts}_{N_2}(s)$ and $\mathsf{src}_{N_1}(f) = \mathsf{src}_{N_2}(f)$ since by definition a source is connected to its facts and vice versa. For $n = 1$ we have

$$T^1_{N_1}(s) = 1 = T^1_{N_2}(s)$$

$$T^1_{N_1}(f) = |\mathsf{src}_{N_1}(f)| = |\mathsf{src}_{N_2}(f)| = T^1_{N_2}(f)$$

so T^1 has numerical PCI. Supposing T^n has numerical PCI for some $n \in \mathbb{N}$, we have

$$T^{n+1}_{N_1}(s) = \sum_{g \in \mathsf{facts}_{N_1}(s)} \underbrace{T^n_{N_1}(g)}_{=T^n_{N_2}(g)} = \sum_{g \in \mathsf{facts}_{N_2}(s)} T^n_{N_2}(g) = T^{n+1}_{N_2}(s)$$

and similarly

$$T^{n+1}_{N+1}(f) = T^{n+1}_{N_2}(f)$$

Hence, by induction, T^n has numerical PCI for all $n \in \mathbb{N}$, and we are done.

Unanimity and Groundedness. For Unanimity, suppose $\mathsf{src}_N(f) = \mathcal{S}$. For any $g \in \mathcal{F}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
T^n_N(g) &= \sum_{s \in \mathsf{src}_N(g)} T^n_N(s) \\
&\leq \sum_{s \in \mathsf{src}_N(g)} T^n_N(s) + \sum_{s \in \mathcal{S} \setminus \mathsf{src}_N(g)} T^n_N(s) \\
&= \sum_{s \in \mathcal{S}} T^n_N(s) \\
&= \sum_{s \in \mathsf{src}_N(f)} T^n_N(s) \\
&= T^n_N(f)
\end{aligned}$$

so $g \preceq^{T^n}_N f$ for all $n \in \mathbb{N}$. Since the ranking of T^* corresponds to T^n for large n, we have $g \preceq^{T^*}_N f$ as required

For Groundedness, suppose $\mathsf{src}_N(f) = \emptyset$. Then $T^n_N(f) = 0$ for all $n \in \mathbb{N}$. For any $g \in \mathcal{F}$, we have $T^n_N(g) \geq 0 = T^n_N(f)$. Consequently $f \preceq^{T^n}_N g$ for all $n \in \mathbb{N}$. As above, this gives $f \preceq^{T^*}_N g$ as required. $\square$

B Proofs for Chapter 5

B.1 Proof of Lemma 8

The proof of Lemma 8 requires a lemma of its own.

Lemma 13. *Let K be an $m \times n$ tournament, $\boldsymbol{\alpha} \in [0,1]^2$ and $\theta \in \Theta_{m,n}$. Then*

$$P_{\boldsymbol{\alpha}}(K \mid \theta) = \prod_{a \in A} \alpha_+^{|K(a) \setminus K_\theta(a)|}(1-\alpha_-)^{|K(a) \cap K_\theta(a)|}$$
$$(1-\alpha_+)^{|B \setminus (K(a) \cup K_\theta(a))|}\alpha_-^{|K_\theta(a) \setminus K(a)|}$$

Proof. Write $p_{ab,K}$ for $P_{\boldsymbol{\alpha}}(X_{ab} = K_{ab} \mid \theta)$. Expanding the product in Definition 20, we have

$$P_{\boldsymbol{\alpha}}(K \mid \theta) = \prod_{a \in A} \prod_{b \in B} p_{ab,K}$$

Let $a \in A$. Note that B can be written as the disjoint union $B = B_1 \cup B_2 \cup B_3 \cup B_4$, where

$$\begin{aligned} B_1 &= K(a) \setminus K_\theta(a) \\ B_2 &= K(a) \cap K_\theta(a) \\ B_3 &= B \setminus (K(a) \cup K_\theta(a)) \\ B_4 &= K_\theta(a) \setminus K(a) \end{aligned}$$

Recall that $b \in K_\theta(a)$ iff $x_a \geq y_b$ (where $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$). It follows that

- $b \in B_1$ iff $K_{ab} = 1$ and $x_a < y_b$
- $b \in B_2$ iff $K_{ab} = 1$ and $x_a \geq y_b$
- $b \in B_3$ iff $K_{ab} = 0$ and $x_a < y_b$
- $b \in B_4$ iff $K_{ab} = 0$ and $x_a \geq y_b$

Note that this correspond exactly to the four cases in (5.3) and (5.4) which define $p_{ab,K}$; we have

$$p_{ab,K} = \begin{cases} \alpha_+, & b \in B_1 \\ 1-\alpha_-, & b \in B_2 \\ 1-\alpha_+, & b \in B_3 \\ \alpha_-, & b \in B_4 \end{cases}$$

Consequently

$$\begin{aligned}\prod_{b\in B} p_{ab,K} &= \left(\prod_{b\in B_1}\alpha_+\right)\left(\prod_{b\in B_2}(1-\alpha_-)\right)\left(\prod_{b\in B_3}(1-\alpha_+)\right)\left(\prod_{b\in B_4}\alpha_-\right)\\ &= \alpha_+^{|B_1|}(1-\alpha_-)^{|B_2|}(1-\alpha_+)^{|B_3|}\alpha_-^{|B_4|}\\ &= \alpha_+^{|K(a)\setminus K_\theta(a)|}(1-\alpha_-)^{|K(a)\cap K_\theta(a)|}\\ &\quad (1-\alpha_+)^{|B\setminus(K(a)\cup K_\theta(a))|}\alpha_-^{|K_\theta(a)\setminus K(a)|}\end{aligned}$$

Taking the product over all $a \in A$ we reach the desired expression for $P_{\boldsymbol{\alpha}}(K \mid \theta)$. $\square$

Proof of Lemma 8. Let $\theta \in \Theta_{m,n}$. From Lemma 13 we get

$$\begin{aligned}P_{\boldsymbol{\alpha}}(K \mid \theta) = \prod_{a\in A} &\beta^{|K(a)\setminus K_\theta(a)|+|K_\theta(a)\setminus K(a)|}\\ &(1-\beta)^{|K(a)\cap K_\theta(a)|+|B\setminus(K(a)\cup K_\theta(a))|}\end{aligned}$$

Note that

$$\begin{aligned}|K(a)\setminus K_\theta(a)|+|K_\theta(a)\setminus K(a)| &= |K(a)\bigtriangleup K_\theta(a)|\\ |K(a)\cap K_\theta(a)|+|B\setminus(K(a)\cup K_\theta(a))| &= |B|-|K(a)\bigtriangleup K_\theta(a)|\end{aligned}$$

and so

$$\begin{aligned}P_{\boldsymbol{\alpha}}(K \mid \theta) &= \prod_{a\in A}\beta^{|K(a)\triangle K_\theta(a)|}(1-\beta)^{|B|-|K(a)\triangle K_\theta(a)|}\\ &= \prod_{a\in A}\left(\frac{\beta}{1-\beta}\right)^{|K(a)\triangle K_\theta(a)|}(1-\beta)^{|B|}\\ &= \underbrace{(1-\beta)^{|A|\cdot|B|}}_{=c}\prod_{a\in A}\left(\frac{\beta}{1-\beta}\right)^{|K(a)\triangle K_\theta(a)|}\\ &= c\prod_{a\in A}\left(\frac{\beta}{1-\beta}\right)^{|K(a)\triangle K_\theta(a)|}\end{aligned}$$

where c is a positive constant that does not depend on θ. Now, $P_{\boldsymbol{\alpha}}(K \mid \theta)$ is positive, and is maximal when its logarithm is. We have

$$\begin{aligned}\log P_{\boldsymbol{\alpha}}(K \mid \theta) &= \log c + \sum_{a\in A}|K(a)\bigtriangleup K_\theta(a)|\log\left(\frac{\beta}{1-\beta}\right)\\ &= \log c + \log\left(\frac{\beta}{1-\beta}\right)\sum_{a\in A}|K(a)\bigtriangleup K_\theta(a)|\\ &= \log c + \log\left(\frac{\beta}{1-\beta}\right)d(K, K_\theta)\end{aligned}$$

Noting that $\beta < 1/2$ implies $\log\left(\frac{\beta}{1-\beta}\right) < 0$, it follows that for any $\theta, \theta' \in \Theta_{m,n}$:

$$\begin{aligned}P_{\boldsymbol{\alpha}}(K \mid \theta) \geq P_{\boldsymbol{\alpha}}(K \mid \theta') &\iff \log P_{\boldsymbol{\alpha}}(K \mid \theta) - \log P_{\boldsymbol{\alpha}}(K \mid \theta') \geq 0\\ &\iff \underbrace{\log\left(\frac{\beta}{1-\beta}\right)}_{<0}[d(K, K_\theta) - d(K, K_{\theta'})] \geq 0\\ &\iff d(K, K_\theta) \leq d(K, K_{\theta'})\end{aligned}$$

which proves the result. □

B.2 Proof of Lemma 9

We need a preliminary result.

Lemma 14. *Let $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \Theta_{m,n}$. Then for all $a, a' \in A$ and $b, b' \in B$:*

1. $K_\theta(a) \subseteq K_\theta(a')$ *iff* $x_a \leq x_{a'}$
2. $K_\theta^{-1}(b) \supseteq K_\theta^{-1}(b')$ *iff* $y_b \leq y_{b'}$.

Proof. We prove (1); (2) is shown similarly. Let $a, a' \in A$. First suppose $x_a \leq x_{a'}$. Let $b \in K_\theta(a)$. Then $y_b \leq x_a \leq x_{a'}$, so $b \in K_\theta(a')$ also. This shows $K_\theta(a) \subseteq K_\theta(a')$.

Now suppose $K_\theta(a) \subseteq K_\theta(a')$. For the sake of contradiction, suppose $x_a > x_{a'}$. By (5.1) in the definition of a state (Definition 19), there is $b \in B$ such that $x_{a'} < y_b \leq x_a$. But this means $b \in K_\theta(a) \setminus K_\theta(a')$, which contradicts $K_\theta(a) \subseteq K_\theta(a')$. Thus (1) is proved. □

Proof of Lemma 9. The "if" direction follows from Lemma 14 part (1): if $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ and $a, a' \in A$ then either $x_a \leq x_{a'}$ – in which case $K_\theta(a) \subseteq K_\theta(a')$ – or $x_{a'} < x_a$ – in which case $K_\theta(a') \subseteq K_\theta(a)$. Therefore K_θ has the chain property.

For the "only if" direction, suppose K has the chain property. Define $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ by

$$x_a = |\{a' \in A \mid K(a') \subseteq K(a)\}|$$
$$y_b = \begin{cases} \min\{x_a \mid a \in K^{-1}(b)\}, & K^{-1}(b) \neq \emptyset \\ 1 + |A|, & K^{-1}(b) = \emptyset \end{cases}$$

It is easily that since the neighbourhood-subset relation $\leqslant_K^A$ is a total preorder, we have $K(a) \subseteq K(a')$ if and only if $x_a \leq x_{a'}$. First we show that $K_\theta = K$ by showing that $K_{ab} = 1$ if and only if $[K_\theta]_{ab} = 1$. Suppose $K_{ab} = 1$. Then $a \in K^{-1}(b)$, so $y_b = \min\{x_{a'} \mid a' \in K^{-1}(b)\} \leq x_a$ and consequently $[K_\theta]_{ab} = 1$.

Now suppose $[K_\theta]_{ab} = 1$. Then $x_a \geq y_b$. We must have $K^{-1}(b) \neq \emptyset$; otherwise $y_b = 1 + |A| > |A| \geq x_a$. We can therefore take $\hat{a} \in \arg\min_{a' \in K^{-1}(b)} x_{a'}$. By definition of y_b, $x_{\hat{a}} = y_b \leq x_a$. But $x_{\hat{a}} \leq x_a$ implies $K(\hat{a}) \subseteq K(a)$; since $\hat{a} \in K^{-1}(b)$ this gives $b \in K(\hat{a})$ and $b \in K(a)$, i.e. $K_{ab} = 1$. This completes the claim that $K = K_\theta$.

It only remains to show that θ satisfies conditions (5.1) and (5.2) of Definition 19. For (5.1), suppose $x_a < x_{a'}$. Then $K(a) \subset K(a')$, i.e there is $b \in K(a') \setminus K(a) = K_\theta(a') \setminus K_\theta(a)$. But $b \in K_\theta(a')$ gives $y_b \leq x_{a'}$, and $b \notin K_\theta(a)$ gives $x_a < y_b$; this shows that (5.1) holds.

For (5.2), suppose $y_b < y_{b'}$. Clearly $K^{-1}(b) \neq \emptyset$ (otherwise $y_b = 1 + |A|$ is maximal). Thus there is $a \in K^{-1}(b)$ such that $y_b = x_a$. This of course means $x_a < y_{b'}$; in particular we have $y_b \leq x_a < y_{b'}$ as required for (5.2).

We have shown that $K = K_\theta$ and that $\theta \in \Theta_{m,n}$, and the proof is complete. □

B.3 Proof of Theorem 11

Proof. First we show that for any $m, n \in \mathbb{N}$ and any $m \times n$ tournament K it holds that θ is an MLE state for K if and only if $K_\theta \in \mathcal{M}(K)$.

Indeed, fix some m, n and K. Write $\mathcal{K}_{\Theta_{m,n}} = \{K_\theta \mid \theta \in \Theta_{m,n}\}$. By Lemma 8, θ is an MLE if and only if $d(K, K_\theta) \leq d(K, K_{\theta'})$ for all $\theta' \in \Theta_{m,n}$, i.e. $K_\theta \in \arg\min_{K' \in \mathcal{K}_{\Theta_{m,n}}} d(K, K')$. But by Lemma 9, $\mathcal{K}_{\Theta_{m,n}}$ is just $\mathcal{C}_{m,n}$, the set of all $m \times n$ tournaments with the chain property. We see that $\arg\min_{K' \in \mathcal{K}_{\Theta_{m,n}}} d(K, K') = \arg\min_{K' \in \mathcal{C}_{m,n}} d(K, K') = \mathcal{M}(K)$ by definition of $\mathcal{M}(K)$. This shows that θ is an MLE iff $K_\theta \in \mathcal{M}(K)$.

Now, by definition, φ satisfies **chain-min** iff for every tournament K there is $K' \in \mathcal{M}(K)$ such that $\varphi(K) = (\leqslant^{\mathcal{A}}_{K'}, \leqslant^{\mathcal{B}}_{K'})$. Using Lemma 9 and the above result, $K' \in \mathcal{M}(K)$ if and only if $K' = K_\theta$ for some MLE θ for K. We see that **chain-min** can be equivalently stated as follows: for all tournament K there exists an MLE θ such that $\varphi(K) = (\leqslant^{\mathcal{A}}_{K_\theta}, \leqslant^{\mathcal{B}}_{K_\theta})$. But by Lemma 14 we have $a \leqslant^{\mathcal{A}}_{K_\theta} a'$ iff $x_a \leq x_{a'}$ and $b \leqslant^{\mathcal{B}}_{K_\theta} b'$ iff $y_b \leq y_{b'}$ (where $\theta = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$). The above reformulation of **chain-min** now coincides with the definition of a maximum likelihood operator, and we are done. □

B.4 Proof of Theorem 12

Proof. We take each axiom in turn. Let φ be any operator satisfying **chain-min**.

anon: Consider $K = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, and define permutations $\sigma = \pi = (1\ 2)$, i.e. the permutations which simply swap 1 and 2. It is easily seen that $\pi(\sigma(K)) = K$. Supposing φ satisfied **anon**, we would get $1 \preceq^{\varphi}_K 2$ iff $\sigma(1) \preceq^{\varphi}_{\pi(\sigma(K))} \sigma(2)$ iff $2 \preceq^{\varphi}_K 1$, which implies $1 \approx^{\varphi}_K 2$. On the other hand, we have

$$\mathcal{M}(K) = \{\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right]\}$$

Since φ satisfies **chain-min** and $1, 2 \in A$ rank equally in $\preceq^{\varphi}_K$, there must be $K' \in \mathcal{M}(K)$ such that 1 and 2 rank equally in $\leqslant^{\mathcal{A}}_{K'}$, i.e. $K'(1) = K'(2)$. But clearly there is no such K'; all tournaments in $\mathcal{M}(K)$ have distinct first and second rows. Hence φ cannot satisfy **anon**.

IIM: Suppose φ satisfies **chain-min** and **IIM**. Write

$$K_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}, \quad K_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$$

Note that the first and second rows of K_1 and K_2 are identical, so by **IIM** we have $1 \preceq^{\varphi}_{K_1} 2$ iff $1 \preceq^{\varphi}_{K_2} 2$. Both tournaments have a unique closest chain tournament requiring changes to only a single entry:

$$\mathcal{M}(K_1) = \left\{\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\right\}, \quad \mathcal{M}(K_2) = \left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}\right\}$$

Write ${K_1}'$ and ${K_2}'$ for these nearest chain tournaments respectively. By **chain-min**, we must have $\varphi(K_i) = (\leqslant^{\mathcal{A}}_{{K_i}'}, \leqslant^{\mathcal{B}}_{{K_i}'})$. In particular, $1 \prec^{\varphi}_{K_1} 2$ and $2 \prec^{\varphi}_{K_2} 1$. But this contradicts **IIM**, and we are done.

pos-resp: Suppose φ satisfies **chain-min** and **pos-resp**, and consider

$$K = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}$$

K has a unique closest chain tournament K':

$$\mathcal{M}(K) = \{K'\} = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

chain-min therefore implies $\varphi(K) = (\leqslant^{\mathcal{A}}_{K'}, \leqslant^{\mathcal{B}}_{K'})$. Note that $K'(1) = K'(2)$, so we have $1 \approx^{\varphi}_{K} 2$. In particular, $1 \preceq^{\varphi}_{K} 2$. Since $K_{23} = 0$, we may apply **pos-resp** to get $1 \prec^{\varphi}_{K+\mathbf{1}_{23}} 2$. But $K + \mathbf{1}_{23}$ is just K'. Since the chain property already holds for K', we have $\mathcal{M}(K') = \{K'\}$ and consequently

$$\varphi(K + \mathbf{1}_{23}) = \varphi(K') = (\leqslant^{\mathcal{A}}_{K'}, \leqslant^{\mathcal{B}}_{K'}) = \varphi(K)$$

so in fact $1 \approx^{\varphi}_{K+\mathbf{1}_{23}} 2$, contradicting **pos-resp**. $\square$

B.5 Proof of Theorem 13

For ease of presentation we establish the compatibility of **chain-min** with **dual** and **mon** separately.

Proposition 6. *There exists an operator φ satisfying **chain-min** and **dual**.*

Proposition 7. *There exists an operator φ satisfying **chain-min** and **mon**.*

It is clear that these two propositions will together prove Theorem 13. For Proposition 6 we use the following result.

Lemma 15. *Let K be a tournament. Then*

1. $\leqslant^{\mathcal{B}}_{K} = \leqslant^{\mathcal{A}}_{\overline{K}}$
2. $K' \in \mathcal{M}(K)$ *if and only if* $\overline{K'} \in \mathcal{M}\left(\overline{K}\right)$

Proof. Fix an $m \times n$ tournament K.

(1) Note that for any $b \in B$, we have $K^{-1}(b) = A \setminus \overline{K}(b)$. Indeed, for any $a \in A = A_K = B_{\overline{K}}$,

$$\begin{aligned} a \in K^{-1}(b) &\iff K_{ab} = 1 \\ &\iff 1 - K_{ab} = 0 \\ &\iff \overline{K}_{ba} = 0 \\ &\iff a \notin \overline{K}(b) \end{aligned}$$

This means that for any $b, b' \in B$,

$$\begin{aligned} b \leqslant^{\mathcal{B}}_{K} b' &\iff K^{-1}(b) \supseteq K^{-1}(b') \\ &\iff A \setminus \overline{K}(b) \supseteq A \setminus \overline{K}(b') \\ &\iff \overline{K}(b) \subseteq \overline{K}(b') \\ &\iff b \leqslant^{\mathcal{A}}_{\overline{K}} b' \end{aligned}$$

so $\leqslant^{\mathcal{B}}_{K} = \leqslant^{\mathcal{A}}_{\overline{K}}$.

(2) (âĞŠ) Suppose $K' \in \mathcal{M}(K)$. First we show that $\overline{K'}$ has the chain property. It is sufficient to show that $\leqslant^{\mathcal{B}}_{K'}$ is a total preorder,[1] since part (1) then implies $\leqslant^{\mathcal{A}}_{\overline{K'}}$ is a total preorder and $\overline{K'}$ has the chain property by definition.

Since $\leqslant^{\mathcal{B}}_{K'}$ always has reflexivity and transitivity, we only need to show the totality property. Let $b, b' \in B$ and suppose $b \not\leqslant^{\mathcal{B}}_{K'} b'$. We must show $b' \leqslant^{\mathcal{B}}_{K'} b$, i.e. $(K')^{-1}(b') \supseteq (K')^{-1}(b)$. To that end, let $a \in (K')^{-1}(b)$.

Since $(K')^{-1}(b) \not\supseteq (K')^{-1}(b')$, there is some $\hat{a} \in (K')^{-1}(b')$ with $\hat{a} \notin (K')^{-1}(b)$. That is, $b' \in K'(\hat{a})$ but $b \notin K'(\hat{a})$. Since $b \in K'(a)$, we have $K'(a) \not\subseteq K'(\hat{a})$. By the chain property for K', we get $K'(\hat{a}) \subset K'(a)$. Finally, this means $b' \in K'(\hat{a}) \subseteq K'(a)$, i.e $a \in (K')^{-1}(b')$. This shows $b' \leqslant^{\mathcal{B}}_{K'} b$ as required.

It remains to show that $d(\overline{K}, \overline{K'})$ is minimal. Since every tournament is the dual of its dual, any $n \times m$ chain tournament is of the form $\overline{K''}$ for an $m \times n$ tournament K''. The above argument shows that the chain property is preserved by taking the dual, so that K'' has the chain property also. Since $K' \in \mathcal{M}(K)$, we have $d(K, K'') \geq d(K, K')$. It is easily verified that the Hamming distance is also preserved under duals, so

$$d(\overline{K}, \overline{K'}) = d(K, K') \leq d(K, K'') = d(\overline{K}, \overline{K''})$$

We have shown that $\overline{K'}$ is as close to $\overline{K}$ as any other $n \times m$ tournament with the chain property, which shows $\overline{K'} \in \mathcal{M}\left(\overline{K}\right)$ as required.

(âĞŘ) Suppose $\overline{K'} \in \mathcal{M}\left(\overline{K}\right)$. By the 'only if' statement above, we have $\overline{\overline{K'}} \in \mathcal{M}\left(\overline{\overline{K}}\right)$. But $\overline{\overline{K}} = K$ and $\overline{\overline{K'}} = K'$, so $K' \in \mathcal{M}(K)$ as required. □

Proof of Proposition 6. Let φ be an arbitrary operator satisfying **chain-min**. Then there is a function $\alpha : \mathcal{K} \to \mathcal{K}$ such that $\varphi(K) = (\leqslant^{\mathcal{A}}_{\alpha(K)}, \leqslant^{\mathcal{B}}_{\alpha(K)})$ and $\alpha(K) \in \mathcal{M}(K)$ for all tournaments K. We will construct a new function α', based on α, such that $\alpha'(\overline{K}) = \overline{\alpha'(K)}$.

Let $\ll$ be a total order on the set of all tournaments $\mathcal{K}$.[2] Write

$$T = \{K \in \mathcal{K} \mid K \ll \overline{K}\}$$

Note that since $K \neq \overline{K}$ for all K, exactly one of K and $\overline{K}$ lies in T. Informally, we view the tournaments in T as somehow 'canonical', and those in $\mathcal{K} \setminus T$ as the dual of a canonical tournament. We use this notion to define α':

$$\alpha'(K) = \begin{cases} \alpha(K), & K \in T \\ \overline{\alpha(\overline{K})}, & K \notin T \end{cases}$$

First we claim $\alpha'(K) \in \mathcal{M}(K)$ for all K. Indeed, if $K \in T$ then $\alpha'(K) = \alpha(K) \in \mathcal{M}(K)$ by the assumption on α. Otherwise, $\alpha(\overline{K}) \in \mathcal{M}\left(\overline{K}\right)$, so Lemma 15 part (2) implies $\alpha'(K) = \overline{\alpha(\overline{K})} \in \mathcal{M}\left(\overline{\overline{K}}\right) = \mathcal{M}(K)$.

Next we show $\overline{\alpha'(K)} = \alpha'(\overline{K})$. First suppose $K \in T$. Then $\alpha'(K) = \alpha(K)$ and $\overline{K} \notin T$, so $\alpha'(\overline{K}) = \overline{\alpha(\overline{\overline{K}})} = \overline{\alpha(K)} = \overline{\alpha'(K)}$ as required. Similarly, if $K \notin T$ then

[1] Note that we claim this holds for any K' with the chain property in the body of the paper, but this has not yet been proven.

$\overline{K} \in T$, so $\alpha'(\overline{K}) = \alpha(\overline{K})$, and $\alpha'(K) = \overline{\alpha(\overline{K})} = \overline{\alpha'(\overline{K})}$. Taking the dual of both sides, we get $\overline{\alpha'(K)} = \alpha'(\overline{K})$.

Finally, define a new operator φ' by $\varphi'(K) = (\leqslant^{\mathcal{A}}_{\alpha'(K)}, \leqslant^{\mathcal{B}}_{\alpha'(K)})$. Since $\alpha'(K) \in \mathcal{M}(K)$ for all K, φ' satisfies **chain-min**. Moreover, using Lemma 15 part (1) and the fact that $\overline{\alpha'(K)} = \alpha'(\overline{K})$, for any tournament K and $b, b' \in B$ we have

$$\begin{aligned} b \sqsubseteq^{\varphi'}_{K} b' &\iff b \leqslant^{\mathcal{B}}_{\alpha'(K)} b' \\ &\iff b \leqslant^{\mathcal{A}}_{\overline{\alpha'(K)}} b' \\ &\iff b \leqslant^{\mathcal{A}}_{\alpha'(\overline{K})} b' \\ &\iff b \sqsubseteq^{\varphi'}_{\overline{K}} b' \end{aligned}$$

which shows φ' also satisfies **dual**. □

Next we prove Proposition 7. We will proceed in three stages. First, Lemma 16 shows that if $K(a_1) \subseteq K(a_2)$ and $K' \in \mathcal{M}(K)$ is some closest chain tournament with the reverse inclusion $K'(a_2) \subseteq K'(a_1)$, then swapping a_1 and a_2 in K' yields obtain another closest chain tournament $K'' \in \mathcal{M}(K)$. Next, we show in Lemma 17 that by performing successive swaps in this way, we can find $K' \in \mathcal{M}(K)$ such that $K'(a_1) \subseteq K'(a_2)$ whenever $K(a_1) \subset K(a_2)$ (note the strict inclusion). Finally, we modify this K' in Lemma 18 to additionally satisfy $K'(a_1) = K'(a_2)$ whenever $K(a_1) = K(a_2)$. This shows that there always exist an element of $\mathcal{M}(K)$ extending the neighbourhood-subset relation $\leqslant^{\mathcal{A}}_{K}$, and consequently it is possible to satisfy **chain-min** and **mon** simultaneously.

Definition 26. *Let K be a tournament and $a_1, a_2 \in A$. We denote by $\mathsf{swap}(K; a_1, a_2)$ the tournament obtained by swapping the a_1 and a_2-th rows of K, i.e.*

$$[\mathsf{swap}(K; a_1, a_2)]_{ab} = \begin{cases} K_{a_1,b}, & a = a_2 \\ K_{a_2,b}, & a = a_1 \\ K_{a,b}, & a \notin \{a_1, a_2\} \end{cases}$$

Lemma 16. *Suppose $K(a_1) \subseteq K(a_2)$ and $K' \in \mathcal{M}(K)$ is such that $K'(a_2) \subseteq K'(a_1)$. Then $\mathsf{swap}(K'; a_1, a_2) \in \mathcal{M}(K)$.*

Proof. Write $K'' = \mathsf{swap}(K'; a_1, a_2)$. It is clear that K'' has the chain property since K' does. Since $K' \in \mathcal{M}(K)$, we have $d(K, K'') \geq d(K, K')$. We will show that $d(K, K'') \leq d(K, K')$ also, which implies $d(K, K'') = d(K, K') = m(K)$ and thus $K'' \in \mathcal{M}(K)$.

To that end, observe that for any tournament $\hat{K}$,

$$d(K, \hat{K}) = \sum_{a \in A} |K(a) \triangle \hat{K}(a)|$$

[2] Note that $\mathcal{K}$ is countable, so such an order can be easily constructed. Alternatively, one could use the axiom of choice and appeal to the well-ordering theorem to obtain $\ll$.

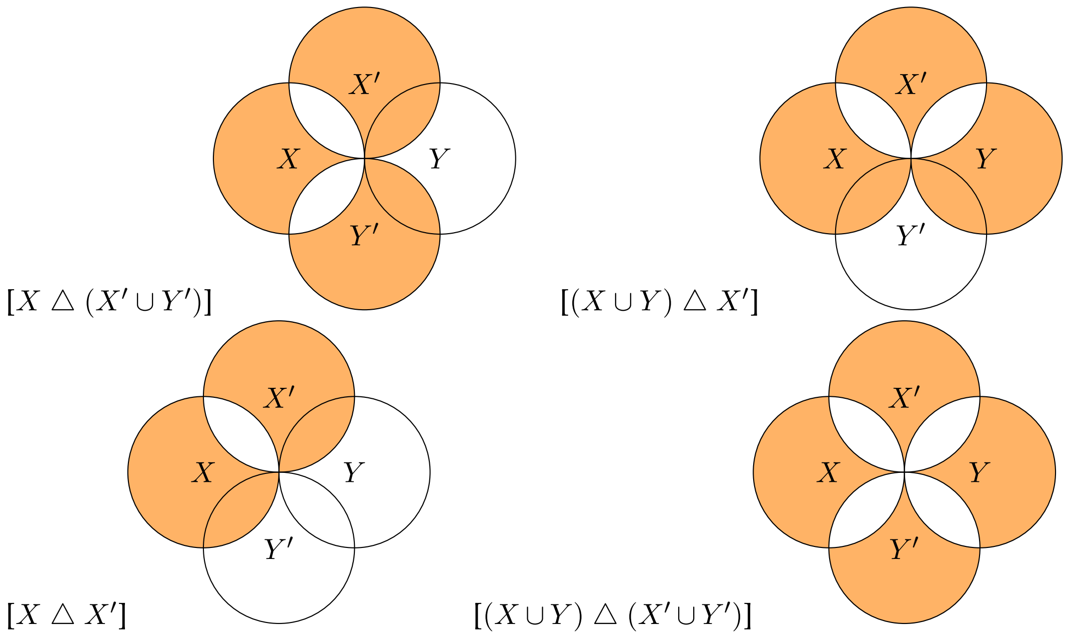

Figure B.1: Depictions of the sets in Equation (B.1)

Noting that $K'(a) = K''(a)$ for $a \notin \{a_1, a_2\}$ and $K''(a_1) = K'(a_2)$, $K''(a_2) = K'(a_1)$, we have

$$\begin{aligned} d(K, K') - d(K, K'') &= \sum_{i \in \{1,2\}} \big(|K(a_i) \triangle K'(a_i)| - |K(a_i) \triangle K''(a_i)| \big) \\ &= |K(a_1) \triangle K'(a_1)| - |K(a_1) \triangle K'(a_2)| \\ &\quad + |K(a_2) \triangle K'(a_2)| - |K(a_2) \triangle K'(a_1)| \end{aligned}$$

To simplify notation, write $X = K(a_1)$, $X' = K'(a_2)$, $Y = K(a_2) \setminus K(a_1)$ and $Y' = K'(a_1) \setminus K'(a_2)$ so that

$$\begin{aligned} K(a_1) &= X; & K(a_2) &= X \cup Y \\ K'(a_1) &= X' \cup Y'; & K'(a_2) &= X' \end{aligned}$$

and $X \cap Y = X' \cap Y' = \emptyset$. Rewriting the above we have

$$\begin{aligned} d(K, K') - d(K, K'') &= |K(a_1) \triangle K'(a_1)| + |K(a_2) \triangle K'(a_2)| \\ &\quad - |K(a_1) \triangle K'(a_2)| - |K(a_2) \triangle K'(a_1)| \\ &= |X \triangle (X' \cup Y')| + |(X \cup Y) \triangle X'| \\ &\quad - |X \triangle X'| - |(X \cup Y) \triangle (X' \cup Y')| \end{aligned} \tag{B.1}$$

Each of the symmetric differences in Eq. (B.1) are depicted in Figure B.1. Note that each of these sets can be expressed as a union of the 8 disjoint subsets of $X \cup Y \cup X' \cup Y'$ shown in the figure. Expanding the symmetric differences in Eq. (B.1) and consulting Figure B.1, it can be seen that most terms cancel out, and in fact we are left with

$$d(K, K') - d(K, K'') = 2|Y \cap Y'| \geq 0$$

This shows that $d(K, K'') \leq d(K, K')$, and the proof is complete. □

Notation. *For a relation R on a set X and $x \in X$, write*

$$U(x, R) = \{y \in X \mid x \mathrel{R} y\}$$

$$L(x, R) = \{y \in X \mid y \mathrel{R} x\}$$

for the upper- and lower-sets of x respectively.

Lemma 17. *For any tournament K there is $K' \in \mathcal{M}(K)$ such that for all $a \in A$:*

$$U(a, <^{\mathcal{A}}_{K}) \subseteq U(a, \leqslant^{\mathcal{A}}_{K'})$$

That is, $K(a) \subset K(a')$ implies $K'(a) \subseteq K'(a')$ for all $a, a' \in A$.

Proof. Write $A = \{a_1, \ldots, a_m\}$, ordered such that $|L(a_1, \leqslant^{\mathcal{A}}_{K})| \leq \cdots \leq |L(a_m, \leqslant^{\mathcal{A}}_{K})|$. We will show by induction that for each $0 \leq i \leq m$ there is $K_i \in \mathcal{M}(K)$ such that:

$$1 \leq j \leq i \implies U(a_j, <^{\mathcal{A}}_{K}) \subseteq U(a_j, \leqslant^{\mathcal{A}}_{K_i}) \qquad (*)$$

The result follows by taking $K' = K_m$.

The case $i = 0$ is vacuously true, and we may take K_0 to be an arbitrary member of $\mathcal{M}(K)$. For the inductive step, suppose $(*)$ holds for some $0 \leq i < m$. If $U(a_{i+1}, <^{\mathcal{A}}_{K}) = \emptyset$ then we may set $K_{i+1} = K_i$, so assume that $U(a_{i+1}, <^{\mathcal{A}}_{K})$ is non-empty. Take some $\hat{a} \in \min(U(a_{i+1}, <^{\mathcal{A}}_{K}), \leqslant^{\mathcal{A}}_{K_i})$. Then $\hat{a}$ has (one of) the smallest neighbourhoods in K_i amongst those in A with a strictly larger neighbourhood than a_{i+1} in K.

If $K_i(a_{i+1}) \subseteq K_i(\hat{a})$ then we claim $(*)$ holds with $K_{i+1} = K_i$. Indeed, for $j < i+1$ the inclusion in $(*)$ holds since it does for K_i. For $j = i+1$, let $a \in U(a_{i+1}, <^{\mathcal{A}}_{K})$. The definition of $\hat{a}$ implies $K_i(a) \not\subset K_i(\hat{a})$; since K_i has the chain property this means $K_i(\hat{a}) \subseteq K_i(a)$. Consequently $K_i(a_{i+1}) \subseteq K_i(\hat{a}) \subseteq K_i(a)$, i.e. $a \in U(a_{i+1}, \leqslant^{\mathcal{A}}_{K_i}) = U(a_{i+1}, \leqslant^{\mathcal{A}}_{K_{i+1}})$ as required.

For the remainder of the proof we therefore suppose $K_i(a_{i+1}) \not\subseteq K_i(\hat{a})$. The chain property for K_i gives $K_i(\hat{a}) \subset K_i(a_{i+1})$. Since $K_i \in \mathcal{M}(K)$ and $K(a_{i+1}) \subset K(\hat{a})$, we may apply Lemma 16. Set $K_{i+1} = \mathsf{swap}(K_i; a_{i+1}, \hat{a}) \in \mathcal{M}(K)$. The inclusion in $(*)$ is easy to show for $j = i + 1$: if $a \in U(a_{i+1}, <^{\mathcal{A}}_{K})$ then either $a = \hat{a}$ – in which case $K_{i+1}(a_{i+1}) \subset K_{i+1}(a)$ by construction – or $a \neq \hat{a}$ and $K_{i+1}(a_{i+1}) = K_i(\hat{a}) \subseteq K_i(a) = K_{i+1}(a)$. In either case $a \in U(a_{i+1}, \leqslant^{\mathcal{A}}_{K_{i+1}})$ as required.

Now suppose $1 \leq j < i + 1$. First note that due to our assumption on the ordering of $\{a_1, \ldots, a_m\}$, we have $a_j \neq \hat{a}$ (indeed, if $a_j = \hat{a}$ then $K(a_{i+1}) \subset K(a_j)$ and $|L(a_j, <^{\mathcal{A}}_{K})| > |L(a_{i+1}, <^{\mathcal{A}}_{K})|$). Since $a_j \neq a_{i+1}$ also, a_j was not involved in the swapping in the construction of K_{i+1}, and consequently $K_{i+1}(a_j) = K_i(a_j)$. Let $a \in U(a_j, <^{\mathcal{A}}_{K})$. We must show that $K_{i+1}(a_j) \subseteq K_{i+1}(a)$. We consider cases.

Case 1: $a = \hat{a}$. Using the fact that $(*)$ holds for K_i we have

$$K_{i+1}(a_j) = K_i(a_j) \subseteq K_i(\hat{a}) \subset K_i(a_{i+1}) = K_{i+1}(\hat{a})$$

Case 2: $a = a_{i+1}$. Here $K(a_j) \subset K(a_{i+1}) \subset K(\hat{a})$, i.e. $\hat{a} \in U(a_j, <^{\mathcal{A}}_{K})$. Applying the inductive hypothesis again we have

$$K_{i+1}(a_j) = K_i(a_j) \subseteq K_i(\hat{a}) = K_{i+1}(a_{i+1})$$

Case 3: $a \notin \{\hat{a}, a_{i+1}\}$. Here neither a_j nor a were involved in the swap, so $K_{i+1}(a_j) = K_i(a_j) \subseteq K_i(a) = K_{i+1}(a)$.

By induction, the proof is complete. □

Lemma 18. *Let K be a tournament and suppose $K' \in \mathcal{M}(K)$ is such that $U(a, <_K^{\mathcal{A}}) \subseteq U(a, \leqslant_{K'}^{\mathcal{A}})$ for all $a \in A$. Then there is $K'' \in \mathcal{M}(K)$ such that $\leqslant_K^{\mathcal{A}} \subseteq \leqslant_{K''}^{\mathcal{A}}$.*

Proof. Let $A_1, \ldots, A_t \subseteq A$ be the equivalence classes of $\approx_K^{\mathcal{A}}$, the symmetric part of $\leqslant_K^{\mathcal{A}}$. Note that $a \approx_K^{\mathcal{A}} a'$ iff $K(a) = K(a')$, so we can associate each A_i with a neighbourhood $B_i \subseteq B$ such that $K(a) = B_i$ whenever $a \in A_i$.

Our aim is to select a single element from each equivalence class A_i, which we denote by $f(A_i)$, and modify K' to set the neighbourhood of each $a \in A_i$ to $K'(f(A_i))$. To that end, construct a function $f : \{A_1, \ldots, A_t\} \to A$ such that

$$f(A_i) \in \arg\min_{a \in A_i} |B_i \triangle K'(a)| \in A_i$$

Define K'' by $K''_{ab} = K'_{f([a]),b}$, where $[a]$ denotes the equivalence class of a. Then $K''(a) = K'(f([a]))$ for all a.

Next we show that $K'' \in \mathcal{M}(K)$. Note that K'' has the chain property, since $a_1 \leqslant_{K''}^{\mathcal{A}} a_2$ iff $f([a_1]) \leqslant_{K'}^{\mathcal{A}} f([a_2])$, and $f([a_1]), f([a_2])$ are guaranteed to be comparable with respect to $\leqslant_{K'}^{\mathcal{A}}$ since K' has the chain property. To show $d(K, K'')$ is minimal, observe that

$$\begin{aligned} d(K, K'') &= \sum_{a \in A} |K(a) \triangle K''(a)| \\ &= \sum_{i=1}^{t} \sum_{a \in A_i} |B_i \triangle K'(f(A_i))| \end{aligned}$$

By definition of f, we have $|B_i \triangle K'(f(A_i))| \leq |B_i \triangle K'(a)|$ for all $a \in A_i$. Consequently

$$\begin{aligned} d(K, K'') &\leq \sum_{i=1}^{t} \sum_{a \in A_i} |B_i \triangle K'(a)| \\ &= d(K, K') \\ &= m(K) \end{aligned}$$

which implies $K'' \in \mathcal{M}(K)$.

We are now ready to prove the result. Suppose $a \leqslant_K^{\mathcal{A}} a'$ i.e. $K(a) \subseteq K(a')$. If $K(a) = K(a')$ then $[a] = [a']$, so

$$K''(a) = K'(f([a])) = K'(f([a'])) = K''(a')$$

and in particular $K''(a) \subseteq K''(a')$. If instead $K(a) \subset K(a')$, then $K(f([a])) = K(a) \subset K(a') = K(f([a']))$, i.e. $f([a]) <_K^{\mathcal{A}} f([a'])$. By the assumption on K' in the statement of the lemma, this means $f([a]) \leqslant_{K'}^{\mathcal{A}} f([a'])$, and so

$$K''(a) = K'(f([a])) \subseteq K'(f([a'])) = K''(a')$$

In either case $K''(a) \subseteq K''(a')$, i.e. $a \leqslant_{K''}^{\mathcal{A}} a'$. Since a, a' were arbitrary, this shows that $\leqslant_K^{\mathcal{A}} \subseteq \leqslant_{K''}^{\mathcal{A}}$ as required. $\square$

The pieces are now in place to prove Proposition 7

Proof of Proposition 7. For any tournament K, write

$$\mathcal{M}_{\mathsf{mon}}(K) = \{K' \in \mathcal{M}(K) \mid \leqslant_K^{\mathcal{A}} \subseteq \leqslant_{K'}^{\mathcal{A}}\}$$

By Lemma 17 and Lemma 18, $\mathcal{M}_{\mathsf{mon}}(K)$ is non-empty. Let $\ll$ be any total order on the set $\mathcal{K}$ of all tournaments. Define a function $\alpha : \mathcal{K} \to \mathcal{K}$ by

$$\alpha(K) = \min(\mathcal{M}_{\mathsf{mon}}(K), \ll) \in \mathcal{M}_{\mathsf{mon}}(K)$$

Note that the minimum is unique since $\ll$ is a total order. Defining an operator φ by $\varphi(K) = (\leqslant_{\alpha(K)}^{\mathcal{A}}, \leqslant_{\alpha(K)}^{\mathcal{B}})$, we see that φ satisfies **chain-min** and **mon**, as required. $\square$

B.6 Proof of Theorem 14

The following preliminary result is required.

Lemma 19. *Let k and l be integers with $1 \leq k \leq l$. Then*

$$\sum_{i=k}^{l} 2^{-i} < 2^{-(k-1)}$$

Proof. This follows from the formula for the sum of a finite geometric series:

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

which holds for all $r \neq 1$. In this case we have

$$\begin{aligned}
\sum_{i=k}^{l} 2^{-i} &= \sum_{i=0}^{l} 2^{-i} - \sum_{i=0}^{k-1} 2^{-i} \\
&= \sum_{i=0}^{l} \left(\frac{1}{2}\right)^i - \sum_{i=0}^{k-1} \left(\frac{1}{2}\right)^i \\
&= \frac{1-\left(\frac{1}{2}\right)^{l+1}}{1-\left(\frac{1}{2}\right)} - \frac{1-\left(\frac{1}{2}\right)^{k}}{1-\left(\frac{1}{2}\right)} \\
&= 2\left(2^{-k} - 2^{-(l+1)}\right) \\
&= 2^{-(k-1)} - \underbrace{2^{-l}}_{>0} \\
&< 2^{-(k-1)}
\end{aligned}$$

as required. $\square$

Proof of Theorem 14. Let $\trianglelefteq$ be a total order on $\mathbb{N} \times \mathbb{N}$ and let $m, n \in \mathbb{N}$. For $a \in [m]$ and $b \in [n]$, write

$$p(a,b) = 1 + |\{(a',b') \in [m] \times [n] : (a',b') \lhd (a,b)\}|$$

for the 'position' of (a, b) in $\trianglelefteq \restriction ([m] \times [n])$ (where 1 corresponds to the minimal pair). Define w by

$$w(a, b) = 1 + 2^{-p(a,b)}$$

If we abuse notation slightly and view w as an $m \times n$ matrix, we have, by construction, $\text{vec}_{\trianglelefteq}(w) = (1 + 2^{-1}, \dots, 1 + 2^{-mn})$. Noting that $|K_{ab} - K'_{ab}| = [K \oplus K']_{ab}$ for any tournaments K, K', and letting $\bullet$ denote the dot product, it is easy to see that

$$\begin{aligned} d_w(K, K') &= \text{vec}_{\trianglelefteq}(w) \bullet \text{vec}_{\trianglelefteq}(K \oplus K') \\ &= (1 + 2^{-1}, \dots, 1 + 2^{-mn}) \bullet \text{vec}_{\trianglelefteq}(K \oplus K') \\ &= d(K, K') + \boldsymbol{x} \bullet \text{vec}_{\trianglelefteq}(K \oplus K') \end{aligned}$$

where $\boldsymbol{x} = (2^{-1}, \dots, 2^{-mn})$ and $d(K, K')$ is the unweighted Hamming distance. In particular, since $\boldsymbol{x}$ and $\text{vec}_{\trianglelefteq}(K \oplus K')$ are non-negative, we have $d_w(K, K') \geq d(K, K')$.

Now, we will show that for any $m \times n$ tournament K and $K' \in \mathcal{C}_{m,n}$ with $K' \neq \alpha_{\trianglelefteq}(K)$ we have $d_w(K, \alpha_{\trianglelefteq}(K)) < d_w(K, K')$. Since $\alpha_{\trianglelefteq}(K) \in \mathcal{M}(K) \subseteq \mathcal{C}_{m,n}$ by definition, this will show that $\alpha_{\trianglelefteq}(K)$ is the unique minimum in Equation (5.6), as required.

So, let K be an $m \times n$ tournament and $K' \in \mathcal{C}_{m,n}$. To ease notation, write $v = \text{vec}_{\trianglelefteq}(K \oplus \alpha_{\trianglelefteq}(K))$ and $v' = \text{vec}_{\trianglelefteq}(K \oplus K')$. There are two cases.

Case 1: $K' \notin \mathcal{M}(K)$. In this case we have $d(K, K') \geq m(K) + 1$, and

$$\begin{aligned} d_w(K, \alpha_{\trianglelefteq}(K)) &= \underbrace{d(K, \alpha_{\trianglelefteq}(K))}_{=m(K)} + \boldsymbol{x} \bullet v \\ &= m(K) + \sum_{i=1}^{mn} 2^{-i} \cdot \underbrace{v_i}_{\leq 1} \\ &\leq m(K) + \underbrace{\sum_{i=1}^{mn} 2^{-i}}_{<2^{-0}=1} \\ &< m(K) + 1 \\ &\leq d(K, K') \\ &\leq d_w(K, K') \end{aligned}$$

where Lemma 19 was applied in the 4th step. This shows $d_w(K, \alpha_{\trianglelefteq}(K)) < d_w(K, K')$, as required.

Case 2: $K \in \mathcal{M}(K)$. In this case we have

$$\begin{aligned} d(K, \alpha_{\trianglelefteq}(K)) - d(K, K') &= (m(K) + \boldsymbol{x} \bullet v) - (m(K) + \boldsymbol{x} \bullet v') \\ &= \boldsymbol{x} \bullet (v - v') \end{aligned}$$

Now, since $K' \in \mathcal{M}(K)$, v' appears as one of the vectors over which the $\arg\min$ is taken in Equation (5.5). By definition of $\alpha_{\trianglelefteq}$ we therefore know that v strictly precedes v' with respect to the lexicographic order on $\{0, 1\}^{mn}$. Consequently there is $j \geq 1$ such that $v_i = v'_i$ for $i < j$ and $v_j < v'_j$. That is, $v_j = 0$ and $v'_j = 1$. This means

$$\begin{aligned}
d(K, \alpha_{\trianglelefteq}(K)) - d(K, K') &= \boldsymbol{x} \bullet (v - v') \\
&= \sum_{i=1}^{mn} 2^{-i}(v_i - v_i') \\
&= \sum_{i=1}^{j-1} 2^{-i} \underbrace{(v_i - v_i')}_{=0} + \sum_{i=j}^{mn} 2^{-i}(v_i - v_i') \\
&= 2^{-j} \underbrace{(v_j - v_j')}_{=-1} + \sum_{i=j+1}^{mn} 2^{-i} \underbrace{(v_i - v_i')}_{\leq 1} \\
&\leq -2^{-j} + \sum_{i=j+1}^{mn} 2^{-i} \\
&< -2^{-j} + 2^{-j} \\
&= 0
\end{aligned}$$

where Lemma 19 was applied in the second to last step. Again, this shows $d_w(K, \alpha_{\trianglelefteq}(K)) < d_w(K, K')$, and the proof is complete. □

B.7 Proof of Theorem 15

Proof. First we set up some notation. For a total preorder $\preceq$ on a set Z and $z \in Z$, write $[z]_{\preceq}$ for the rank of $\preceq$ containing z, i.e. the equivalence class of z in the symmetric closure of $\preceq$:

$$[z]_{\preceq} = \{z' \in Z \mid z \preceq z' \text{ and } z' \preceq z\}$$

Also note that $\preceq$ can be extended to a total order on the ranks by setting $[z]_{\preceq} \leq [z']_{\preceq}$ iff $z \preceq z'$.

(âĞŠ) We start with the 'only if' statement of the theorem. Suppose φ satisfies **chain-def**, and let K be a tournament. We need to show that $|\mathsf{ranks}(\preceq_K^{\varphi}) - \mathsf{ranks}(\sqsubseteq_K^{\varphi})| \leq 1$.

By chain-definability, there is K' with the chain property such that $a \preceq_K^{\varphi} a'$ iff $K'(a) \subseteq K'(a')$ and $b \sqsubseteq_K^{\varphi} b'$ iff $(K')^{-1}(b) \supseteq (K')^{-1}(b')$. Write

$$\mathcal{X} = \{[a]_{\preceq_K^{\varphi}} \mid a \in A, K'(a) \neq \emptyset\}$$

$$\mathcal{Y} = \{[b]_{\sqsubseteq_K^{\varphi}} \mid b \in B, (K')^{-1}(b) \neq \emptyset\}$$

for the set of ranks in each of the two orders, excluding those who have empty neighbourhoods in K'. Note that $[a]_{\preceq_K^{\varphi}} = [a']_{\preceq_K^{\varphi}}$ if and only if $K'(a) = K'(a')$ (and similar for B).

We will show that $|\mathcal{X}| = |\mathcal{Y}|$. Enumerate $\mathcal{X} = \{X_1, \ldots, X_s\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_t\}$, ordered such that $X_1 < \cdots < X_s$ and $Y_1 < \cdots < Y_t$. First we show $|\mathcal{X}| \leq |\mathcal{Y}|$.

For each $1 \leq i \leq s$, the a_i be an arbitrary element of X_i. Then $a_1 \prec_K^{\varphi} \cdots \prec_K^{\varphi} a_s$, so $\emptyset \subset K'(a_1) \subset \cdots \subset K'(a_s)$. Since these inclusions are strict, we can choose $b_1, \ldots, b_s \in B$ such that $b_1 \in K'(a_1)$ and $b_{i+1} \in K'(a_{i+1}) \setminus K'(a_i)$ for $1 \leq i < s$.

It follows that $a_i \in (K')^{-1}(b_i) \backslash (K')^{-1}(b_{i+1})$, and thus $(K')^{-1}(b_i) \not\subseteq (K')^{-1}(b_{i+1})$. Since K' has the chain property, this means $(K')^{-1}(b_{i+1}) \subset (K')^{-1}(b_i)$, i.e. $b_i \sqsubset^{\varphi}_{K} b_{i+1}$.

We now have $b_1 \sqsubset^{\varphi}_{K} \cdots \sqsubset^{\varphi}_{K} b_s$; a chain of s strict inequalities in $\sqsubseteq^{\varphi}_{K}$. The corresponding ranks $[b_1], \ldots, [b_s]$ are all distinct and lie inside $\mathcal{Y}$. But now we have found $s = |\mathcal{X}|$ distinct elements of $\mathcal{Y}$, so $|\mathcal{X}| \leq |\mathcal{Y}|$ as promised.

Repeating this argument with the roles of $\mathcal{X}$ and $\mathcal{Y}$ interchanged, we find that $|\mathcal{Y}| \leq |\mathcal{X}|$ also, and therefore $|\mathcal{X}| = |\mathcal{Y}|$.

To conclude, note that $\mathsf{ranks}(\preceq^{\varphi}_{K}) \in \{|\mathcal{X}|, |\mathcal{X}| + 1\}$, since there can exist at most one rank which was excluded from $\mathcal{X}$ (namely, those $a \in A$ with $K'(a) = \emptyset$). For identical reasons, $\mathsf{ranks}(\sqsubseteq^{\varphi}_{K}) \in \{|\mathcal{Y}|, |\mathcal{Y}| + 1\}$. Since $|\mathcal{X}| = |\mathcal{Y}|$, it is clear that $\mathsf{ranks}(\preceq^{\varphi}_{K})$ and $\mathsf{ranks}(\sqsubseteq^{\varphi}_{K})$ can differ by at most one, as required.

(âĞŘ) Now we prove the 'if' statement. Let K be a tournament. We have $|\mathsf{ranks}(\preceq^{\varphi}_{K}) - \mathsf{ranks}(\sqsubseteq^{\varphi}_{K})| \leq 1$, and must show there is tournament K' with the chain property such that $\varphi(K) = (\leqslant^{\mathcal{A}}_{K'}, \leqslant^{\mathcal{B}}_{K'})$.

Let $X_1 < \cdots < X_s$ and $Y_1 < \cdots < Y_t$ be the ranks of $\preceq^{\varphi}_{K}$ and $\sqsubseteq^{\varphi}_{K}$ respectively. By hypothesis $|s - t| \leq 1$. Define $g : \{1, \ldots, s\} \to \{0, \ldots, t\}$ by

$$g(i) = \begin{cases} i, & s \in \{t-1, t\} \\ i-1, & s = t+1 \end{cases}$$

Not that the two cases above cover all possibilities, since $|s - t| \leq 1$. For $i \in [s]$, write

$$N_i = \bigcup_{0 \leq j \leq g(i)} Y_j$$

where $Y_0 := \emptyset$. Note that $g(i+1) = g(i) + 1$, and consequently

$$N_{i+1} = \bigcup_{j \leq g(i)+1} Y_j = N_i \cup Y_{g(i)+1} = N_i \cup Y_{g(i+1)}$$

Since $g(i+1) > 0$ we have $Y_{g(i+1)} \neq \emptyset$, and thus $N_{i+1} \supset N_i$ for all $i < s$.

Now, for any $a \in A$, let $p(a) \in [s]$ be the unique integer such that $a \in X_{p(a)}$; such $p(a)$ always exists since $\{X_1, \ldots, X_s\}$ is a partition of A. Note that due to the assumption on the ordering of the X_i, we have $a \preceq^{\varphi}_{K} a'$ if and only if $p(a) \leq p(a')$.

Let K' be the unique tournament such that $K'(a) = N_{p(a)}$ for each $a \in A$. Since $N_1 \subset \cdots \subset N_p$, we have

$$\begin{aligned} a \preceq^{\varphi}_{K} a' &\iff p(a) \leq p(a') \\ &\iff N_{p(a)} \subseteq N_{p(a')} \\ &\iff K'(a) \subseteq K'(a') \\ &\iff a \leqslant^{\mathcal{A}}_{K'} a' \end{aligned} \tag{B.2}$$

i.e. $\preceq^{\varphi}_{K} = \leqslant^{\mathcal{A}}_{K'}$. Since $\preceq^{\varphi}_{K}$ is a total preorder, this shows that K' has the chain property.

It only remains to show that $\sqsubseteq^{\varphi}_{K} = \leqslant^{\mathcal{B}}_{K'}$. First note that if $a \in X_i$ and $b \in Y_j$, the fact that $\{Y_1, \ldots, Y_t\}$ are disjoint implies

$$\begin{aligned} a \in (K')^{-1}(b) &\iff b \in K'(a) = N_i = \bigcup_{0 \leq k \leq g(i)} Y_k \\ &\iff j \leq g(i) \end{aligned}$$

Hence $(K')^{-1}(b)$ only depends on j: every $b \in Y_j$ shares the same neighbourhood M_j, given by

$$M_j = \bigcup_{i\in[s]:\ g(i)\geq j} X_i$$

Note that if $1 \leq j < t$,

$$\begin{aligned} M_j &= \bigcup_{i\in[s]:\ g(i)\geq j} X_i \\ &= \left(\bigcup_{i\in[s]:\ g(i)\geq j+1} X_i \right) \cup \left(\bigcup_{i\in g^{-1}(j)} X_i \right) \\ &= M_{j+1} \cup \bigcup_{i\in g^{-1}(j)} X_i \end{aligned}$$

Since $1 \leq j < t$ we have

$$g^{-1}(j) = \begin{cases} \{j\}, & s \in \{t-1, t\} \\ \{j+1\}, & s = t+1 \end{cases}$$

In particular $g^{-1}(j) \neq \emptyset$, which means $\bigcup_{i\in g^{-1}(j)} X_i \neq \emptyset$ and thus $M_j \supset M_{j+1}$ for all $1 \leq j < t$.

Finally, since $(K')^{-1}(b) = M_j$ for $b \in Y_j$ and $M_1 \supset \cdots \supset M_t$, an argument almost identical to (B.2) shows that $\sqsubseteq^{\varphi}_{K} = \leqslant^{\mathcal{B}}_{K'}$.

We have shown that $\varphi(K) = (\leqslant^{\mathcal{A}}_{K'}, \leqslant^{\mathcal{B}}_{K'})$ and that K' has the chain property, and the proof is therefore complete. □

B.8 Proof that the interleaving procedure eventually terminates

Proposition 8. *Let (f, g) be selection functions. Fix a tournament K and let A_i, B_i $(i \geq 0)$ be as in Definition 25. Then there are $j, j' \geq 1$ such that $A_j = \emptyset$ and $B_{j'} = \emptyset$. Moreover, there is $t \geq 1$ such that both $A_t = B_t = \emptyset$.*

Proof. Suppose $i \geq 0$ and $A_i \neq \emptyset$. Then properties (i) and (ii) for f in Definition 24 imply that $\emptyset \subset f(K, A_i, B_i) \subseteq A_i$, and consequently $A_{i+1} = A_i \setminus f(K, A_i, B_i) \subset A_i$.

Supposing that $A_j \neq \emptyset$ for all $j \geq 0$, we would have $A_0 \supset A_1 \supset A_2 \supset \cdots$ which clearly cannot be the case since each A_j lies inside A which is a finite set. Hence there is $j \geq 1$ such that $A_j = \emptyset$. Moreover, since $A_j \supseteq A_{j+1} \supseteq A_{j+2} \supseteq \cdots$, we have $A_k = \emptyset$ for all $k \geq j$.

An identical argument with g shows that there is $j' \geq 1$ such that $B_{j'} = \emptyset$ and $B_k = \emptyset$ for all $k \geq j'$.

Taking $t = \max\{j, j'\}$, we have $A_t = B_t = \emptyset$ as required. □

B.9 Proof of Theorem 16

Proof. Throughout the proof we will refer to a pair of total preorders $(\preceq, \sqsubseteq)$ as 'chain-definable' if there is a chain tournament K such that $\preceq = \leqslant^{\mathcal{A}}_{K}$ and $\sqsubseteq = \leqslant^{\mathcal{B}}_{K}$.

(âĞŘ) First we prove the 'if' direction. Let $\varphi = \varphi^{\mathsf{int}}_{f,g}$ be an interleaving operator with selection functions (f, g), and fix a tournament K. We will show that $\varphi(K)$ is chain-definable.

As per Proposition 8, let $j, j' \geq 1$ be the minimal integers such that $A_j = \emptyset$ and $B_{j'} = \emptyset$. Then we have $A_0 \supset \cdots \supset A_{j-1} \supset A_j = \emptyset$ and $B_0 \supset \cdots \supset B_{j'-1} \supset B_{j'} = \emptyset$.

Recall that, for $a \in A$, we have by definition $r(a) = \max\{i \mid a \in A_i\}$, which is the unique integer such that $a \in A_{r(a)} \setminus A_{r(a)+1}$. Since $a \preceq^{\varphi}_K a'$ iff $r(a) \geq r(a')$, it follows that the non-empty sets $A_0 \setminus A_1, \ldots, A_{j-1} \setminus A_j$ form the ranks of the total preorder $\preceq^{\varphi}_K$ (that is, the equivalence classes of the symmetric closure $\approx^{\varphi}_K$). Thus, $\preceq^{\varphi}_K$ has j ranks. An identical argument shows that $\sqsubseteq^{\varphi}_K$ has j' ranks.

It follows from Theorem 15 that $\varphi(K)$ is chain-definable if and only if $|j - j'| \leq 1$. If $j = j'$ this is clear. Suppose $j < j'$. Then $A_j = \emptyset$ and $B_j \neq \emptyset$. By property (iii) for g in Definition 24, we have $g(K, A_j, B_j) = g(K, \emptyset, B_j) = B_j$. But this means $B_{j+1} = B_j \setminus g(K, A_j, B_j) = B_j \setminus B_j = \emptyset$. Consequently $j' = j + 1$, and $|j - j'| = |-1| = 1$

If instead $j > j'$, then a similar argument using property (iii) for f in Definition 24 shows that $j = j' + 1$, and we have $|j - j'| = |1| = 1$.

Hence $|j - j'| \leq 1$ in all cases, and $\varphi(K)$ is chain-definable as required.

(âĞŠ) Now for the 'only if' direction. Suppose φ satisfies **chain-def**. We will define f, g such that $\varphi = \varphi^{\mathsf{int}}_{f,g}$. The idea behind the construction is straightforward: since f and g pick off the next-top-ranked As and Bs at each iteration, simply define $f(K, A_i, B_i)$ as the maximal elements of A_i with respect to the existing ordering $\preceq^{\varphi}_K$ (g will be defined similarly). The interleaving algorithm will then select the ranks of $\preceq^{\varphi}_K$ and $\sqsubseteq^{\varphi}_K$ one-by-one; the fact that $\varphi(K)$ is chain-definable ensures that we select *all* the ranks before the iterative procedure ends. The formal details follow.

Fix a tournament K. By Theorem 15, $|\mathsf{ranks}(\preceq^{\varphi}_K) - \mathsf{ranks}(\sqsubseteq^{\varphi}_K)| \leq 1$. Taking $t = \max\{\mathsf{ranks}(\preceq^{\varphi}_K), \mathsf{ranks}(\sqsubseteq^{\varphi}_K)\}$, we can write $X_1, \ldots, X_t \subseteq A$ and $Y_1, \ldots, Y_t \subseteq B$ for the ranks of $\preceq^{\varphi}_K$ and $\sqsubseteq^{\varphi}_K$ respectively, possibly with $X_1 = \emptyset$ if $\mathsf{ranks}(\sqsubseteq^{\varphi}_K) = 1 + \mathsf{ranks}(\preceq^{\varphi}_K)$ or $Y_1 = \emptyset$ if $\mathsf{ranks}(\preceq^{\varphi}_K) = 1 + \mathsf{ranks}(\sqsubseteq^{\varphi}_K)$. Note that $X_i, Y_i \neq \emptyset$ for $i > 1$. Assume these sets are ordered such that $a \preceq^{\varphi}_K a'$ iff $i \leq j$ whenever $a \in X_i$ and $a' \in X_j$ (and similar for the Y_i). Also note that the $X_i \cap X_j = \emptyset$ for $i \neq j$ (and similar for the Y_i).

Now set [3]

$$f(K, A', B') = \begin{cases} \max(A', \preceq^{\varphi}_K), & B' \neq \emptyset \\ A', & B' = \emptyset \end{cases}$$

$$g(K, A', B') = \begin{cases} \max(B', \sqsubseteq^{\varphi}_K), & A' \neq \emptyset \\ B', & A' = \emptyset \end{cases}$$

It is not difficult to see that f and g satisfy the conditions of Definition 24 for selection functions. We claim that for with A_i, B_i denoting the interleaving sets for K and (f, g), for all $0 \leq i \leq t$ we have

$$A_i = \bigcup_{j=1}^{t-i} X_j, \quad B_i = \bigcup_{j=1}^{t-i} Y_j \tag{B.3}$$

[3] Here $\max(Z, \preceq) = \{z \in Z \mid \nexists z' \in Z : z \prec z'\}$, for any set Z and a total preorder $\preceq$ on Z (with strict part $\prec$).

For $i = 0$ this is clear: since $X_1, \ldots, X_t$ contains all ranks of $\preceq_K^\varphi$ we have $\bigcup_{j=1}^{t-0} = X_1 \cup \cdots \cup X_t = A = A_0$ (and similar for B).

Now suppose (B.3) holds for some $0 \leq i < t$. We will show that $f(K, A_i, B_i) = X_{t-i}$ by considering three possible cases, at least one of which must hold.

Case 1: ($A_i \neq \emptyset$, $B_i \neq \emptyset$). Here we have

$$\begin{aligned} f(K, A_i, B_i) &= \max(A_i, \preceq_K^\varphi) \\ &= \max(X_1 \cup \cdots \cup X_{t-i}, \preceq_K^\varphi) \\ &= X_{t-i} \end{aligned}$$

since the X_j form (disjoint) ranks of $\preceq_K^\varphi$ with $X_j \prec X_k$ for $j < k$.

Case 2: ($B_i = \emptyset$). Here we have $\bigcup_{j=1}^{t-i} Y_j = \emptyset$. Since $t - i \geq 1$ and $Y_j \neq \emptyset$ for $j > 1$, it must be the case that $t - i = 1$ and $B_i = Y_1 = \emptyset$. Consequently by the induction hypothesis we have $A_i = \bigcup_{j=1}^{1} X_j = X_1$, and thus

$$\begin{aligned} f(K, A_i, B_i) &= f(K, A_i, \emptyset) \\ &= A_i \\ &= X_1 \\ &= X_{t-i} \end{aligned}$$

Case 3: ($A_i = \emptyset$). By a similar argument as in case 2, we must have $t - i = 1$ and $A_i = X_1 = \emptyset$. Using the fact that $f(K, A_i, B_i) \subseteq A_i$ we get

$$\begin{aligned} f(K, A_i, B_i) &= \underbrace{f(K, \emptyset, B_i)}_{\subseteq \emptyset} \\ &= \emptyset \\ &= X_1 \\ &= X_{t-i} \end{aligned}$$

We have now covered all cases, and have shown that $f(K, A_i, B_i) = X_{t-i}$ must hold. Consequently, using again the fact that the X_j are disjoint,

$$\begin{aligned} A_{i+1} &= A_i \setminus f(K, A_i, B_i) \\ &= \left(\bigcup_{j=1}^{t-i} X_j \right) \setminus X_{t-i} \\ &= \bigcup_{j=1}^{t-(i+1)} X_j \end{aligned}$$

as required. By almost identical arguments we can show that $g(K, A_i, B_i) = Y_{t-i}$, and thus $B_{i+1} = \bigcup_{j=1}^{t-(i+1)} Y_j$ also. By induction, (B.3) holds for all $0 \leq i \leq t$.

It remains to show that $a \preceq_K^\varphi a'$ iff $a \preceq_K^{\varphi_{f,g}^{\text{int}}} a'$ and that $b \sqsubseteq_K^\varphi b'$ iff $b \sqsubseteq_K^{\varphi_{f,g}^{\text{int}}} b'$.

For $a \in A$, let $p(a)$ be the unique integer such that $a \in X_{p(a)}$, i.e. $p(a)$ is the index of the rank of a in the ordering $\preceq_K^\varphi$. Note that we have

$$a \in A_i = X_1 \cup \cdots \cup X_{t-i} \iff t - i \geq p(a)$$

and therefore

$$r(a) = \max\{i \mid a \in A_i\} = \max\{i \mid t - i \geq p(a)\} = t - p(a)$$

Using the fact that $X_i \prec X_j$ for $i < j$, we get

$$\begin{aligned} a \preceq_K^{\varphi^{\text{int}}_{f,g}} a' &\iff r(a) \geq r(a') \\ &\iff t - p(a) \geq t - p(a') \\ &\iff p(a) \leq p(a') \\ &\iff a \preceq_K^{\varphi} a' \end{aligned}$$

A similar argument shows that $b \sqsubseteq_K^{\varphi} b'$ iff $b \sqsubseteq_K^{\varphi^{\text{int}}_{f,g}} b'$ for any $b, b' \in B$. Since K was arbitrary, we have shown that $\varphi = \varphi^{\text{int}}_{f,g}$ as required. □

B.10 Proof of Theorem 17

Proof. Since **chain-min** implies **chain-def**, Theorem 13 implies the existence of an operator with **chain-def** and **dual**, and an operator with **chain-def** and **mon**. Moreover, the trivial operator which ranks all As and Bs equally satisfies **anon** and **IIM**. It only remains to show that there is an operator satisfying both **chain-def** and **pos-resp**.

To that end, for any tournament K, define K' by

$$K'_{ab} = \begin{cases} 1, & b \leq |K(a)| \\ 0, & b > |K(a)| \end{cases}$$

Note that $K'(a) = \{1, \ldots, |K(a)|\}$ for $|K(a)| > 0$. Consequently $K'(a) \subseteq K'(a')$ iff $|K(a)| \leq |K(a)|$. We see that K' has the chain property, and the operator φ defined by $\varphi(K) = (\leqslant^{A}_{K'}, \leqslant^{B}_{K'})$ satisfies **chain-def**. In particular, $a \preceq_K^{\varphi} a'$ iff $|K(a)| \leq |K(a')|$.

To show **pos-resp**, suppose $a \preceq_K^{\varphi} a'$ and $K_{a',b} = 0$ for some $a, a' \in A$ and $b \in B$. Write $\hat{K} = K + \mathbf{1}_{a',b}$.

Since $a \preceq_K^{\varphi} a'$ implies $|K(a)| \leq |K(a')|$, we have $|\hat{K}(a')| = 1 + |K(a')| > |K(a)| = |\hat{K}(a)|$, and therefore $a \prec_{\hat{K}}^{\varphi} a'$ as required for **pos-resp**. □

B.11 Proof of Theorem 18

Proof. For contradiction, suppose there is an operator φ satisfying the stated axioms. Consider

$$K = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and two tournaments obtained by removing a single 1 entry:

$$K_1 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{0} \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & \mathbf{0} \end{bmatrix}$$

Now, **anon** in K_1 gives $1 \approx_{K_1}^{\varphi} 2$ (e.g. take $\sigma = (1\,2), \pi = \mathrm{id}_B$). In particular, $1 \preceq_{K_1}^{\varphi} 2$, so **pos-resp** implies $1 \prec_K^{\varphi} 2$. A similar argument with K_2 shows that $3 \approx_{K_2}^{\varphi} 4$ and $3 \prec_K^{\varphi} 4$.

On the other hand, applying **anon** to K directly with $\sigma = (2\ 3)$ and $\pi = (1\ 2)$, we see that $2 \approx^\varphi_K 3$. The ranking of A is thus fully determined as $1 \prec 2 \approx 3 \prec 4$. In particular, $\mathsf{ranks}(\preceq^\varphi_K) = 3$.

But now considering the dual tournament $\overline{K} = [\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{smallmatrix}]$ and applying permutations $\sigma = (1\ 2)$ and $\pi = (2\ 3)$, we obtain $1 \approx^\varphi_{\overline{K}} 2$ by **anon**, i.e. the A ranking in $\overline{K}$ is flat. By **dual** this implies the B ranking in K is flat, i.e. $\mathsf{ranks}(\sqsubseteq^\varphi_K) = 1$. We see that $\mathsf{ranks}(\preceq^\varphi_K)$ and $\mathsf{ranks}(\sqsubseteq^\varphi_K)$ differ by 2, contradicting **chain-def** according to Theorem 15. $\square$

B.12 Proof of Theorem 19

We require a preliminary result providing sufficient conditions for an interleaving operator $\varphi^{\mathsf{int}}_{f,g}$ to satisfy various axioms.

Lemma 20. *Let $\varphi = \varphi^{int}_{f,g}$ be an interleaving operator.*

1. *If for any tournament K, $A' \subseteq A$, $B' \subseteq B$ and for any pair of permutations $\sigma : A \to A$ and $\pi : B \to B$ we have*
$$f(\pi(\sigma(K)), \sigma(A'), \pi(B')) = \sigma(f(K, A', B'))$$
$$g(\pi(\sigma(K)), \sigma(A'), \pi(B')) = \pi(g(K, A', B'))$$
*then φ satisfies **anon**.*

2. *If for any tournament K and $A' \subseteq A$, $B \subseteq B$ we have*
$$g(K, A', B') = f(\overline{K}, B', A')$$
*then φ satisfies **dual**.*

3. *If for any tournament K, $A' \subseteq A$, $B' \subseteq B$ and $a, a' \in A'$ we have*
$$K(a) \subseteq K(a') \implies a \notin f(K, A', B') \text{ or } a' \in f(K, A', B')$$
*then φ satisfies **mon**.*

Proof. We take each statement in turn.

1. Let K be a tournament. For brevity, write $K' = \pi(\sigma(K))$. Let us write A_i, B_i and A'_i, B'_i $(i \geq 0)$ for the sets defined in Definition 25 for K and K' respectively. We claim that for all $i \geq 0$:
$$A'_i = \sigma(A_i), \quad B'_i = \pi(B_i) \tag{B.4}$$
For $i = 0$ this is trivial since $A'_0 = A = \sigma(A) = \sigma(A_0)$ since σ is a bijection. The fact that $B'_0 = \pi(B_0)$ is shown similarly.

Suppose that (B.4) holds for some $i \geq 0$. Then applying our assumption on f:
$$\begin{aligned} A'_{i+1} &= A'_i \setminus f(K', A'_i, B'_i) \\ &= \sigma(A_i) \setminus f(K', \sigma(A_i), \pi(B_i)) \\ &= \sigma(A_i) \setminus \sigma(f(K, A_i, B_i)) \\ &= \sigma(A_i \setminus f(K, A_i, B_i)) \\ &= \sigma(A_{i+1}) \end{aligned}$$

(note that $\sigma(X) \setminus \sigma(Y) = \sigma(X \setminus Y)$ holds for any sets X, Y due to injectivity of σ). Using the assumption on g we can show that $B'_{i+1} = \pi(B_{i+1})$ in a similar manner. Therefore, by induction, (B.4) holds for all $i \geq 0$. This means that for any $a \in A$ we have

$$\sigma(a) \in A'_i \iff \sigma(a) \in \sigma(A_i) \iff a \in A_i$$

and therefore, with r_K and $r_{K'}$ denoting the functions $A \to \mathbb{N}_0$ defined in Definition 25 for K and K' respectively,

$$\begin{aligned} r_{K'}(\sigma(a)) &= \max\{i \mid \sigma(a) \in A'_i\} \\ &= \max\{i \mid a \in A_i\} \\ &= r_K(a) \end{aligned}$$

From this it easily follows that $a \preceq^{\varphi}_K a'$ iff $\sigma(a) \preceq^{\varphi}_{K'} \sigma(a')$, i.e. φ satisfies **anon**.

2. Once again, fix a tournament K and let A_i, B_i and A'_i, B'_i denote the sets from Definition 25 for K and $\overline{K}$ respectively. It is easy to show by induction that the assumption on f and g implies $A'_i = B_i$ and $B'_i = A_i$ for all $i \geq 0$. This means that for any $b \in B_K$:

$$\begin{aligned} s_K(b) &= \max\{i \mid b \in B_i\} \\ &= \max\{i \mid b \in A'_i\} \\ &= r_{\overline{K}}(b) \end{aligned}$$

which implies $b \sqsubseteq^{\varphi}_K b'$ iff $b \preceq^{\varphi}_{\overline{K}} b'$, as required for **dual**.

3. Let K be a tournament and $a, a' \in A$ such that $K(a) \subseteq K(a')$. We must show that $a \preceq^{\varphi}_K a'$.

   Suppose otherwise, i.e. $a' \prec^{\varphi}_K a$. Then $r(a') > r(a)$. Note that by definition of r, we have $a \in A_{r(a)} \setminus A_{r(a)+1} = f(K, A_{r(a)}, B_{r(a)})$. Since $r(a') \geq r(a) + 1$ and $A_{r(a)} \supseteq A_{r(a)+1} \supseteq A_{r(a)+2} \supseteq \cdots$, we get $a' \in A_{r(a)+1} \subseteq A_{r(a)}$. In particular, $a' \notin f(K, A_{r(a)}, B_{r(a)})$.

   Piecing this all together, we have $a, a' \in A_{r(a)}$, $K(a) \subseteq K(a')$, $a \in f(K, A_{r(a)}, B_{r(a)})$ and $a' \notin f(K, A_{r(a)}, B_{r(a)})$. But this directly contradicts our assumption on f, so we are done.

$\square$

Proof of Theorem 19. We take each axiom in turn. Let f and g be the selection functions corresponding to φ_{CI} from Example 6.

chain-def. Since φ_{CI} is an interleaving operator, **chain-def** follows from Theorem 16.

anon. Let K be a tournament and let $\sigma : A \to A$ and $\pi : B \to B$ be bijective mappings. Write $K' = \pi(\sigma(K))$. We will show that the conditions on f and g in Lemma 20 part (1) are satisfied.

Let $A' \subseteq A$ and $B' \subseteq B$. We have

$$\begin{aligned} f(K', \sigma(A'), \pi(B')) &= \arg\max_{\hat{a} \in \sigma(A')} |K'(\hat{a}) \cap \pi(B')| \\ &= \sigma(\arg\max_{a \in A'} |K'(\sigma(a)) \cap \pi(B')|) \end{aligned}$$

where we make the 'substitution' $a = \sigma^{-1}(\hat{a})$. Using the defintion of $K' = \pi(\sigma(K))$ it is easily seen that $K'(\sigma(a)) = \pi(K(a))$. Also, since π is a bijection we have $\pi(X) \cap \pi(Y) = \pi(X \cap Y)$ for any sets X and Y, and $|\pi(X)| = |X|$. Thus

$$\begin{aligned} f(K', \sigma(A'), \pi(B')) &= \sigma(\arg\max_{a \in A'} |K'(\sigma(a)) \cap \pi(B')|) \\ &= \sigma(\arg\max_{a \in A'} |\pi(K(a)) \cap \pi(B')|) \\ &= \sigma(\arg\max_{a \in A'} |\pi(K(a) \cap B')|) \\ &= \sigma(\arg\max_{a \in A'} |K(a) \cap B'|) \\ &= \sigma(f(K, A', B')) \end{aligned}$$

as required. The result for g follows by a near-identical argument. Thus φ_{CI} satisfies **anon** by Lemma 20 part (1).

dual. Fix a tournament K and let $A' \subseteq A$, $B' \subseteq B$. Note that for $b \in B'$ we have

$$\begin{aligned} |K^{-1}(b) \cap A'| &= |(A \setminus \overline{K}(b)) \cap A'| \\ &= |A' \setminus \overline{K}(b)| \\ &= |A'| - |\overline{K}(b) \cap A'| \end{aligned}$$

Consequently

$$\begin{aligned} g(K, A', B') &= \arg\min_{b \in B'} |K^{-1}(b) \cap A'| \\ &= \arg\min_{b \in B'} \left(|A'| - |\overline{K}(b) \cap A'|\right) \\ &= \arg\max_{b \in B'} |\overline{K}(b) \cap A'| \\ &= f(\overline{K}, B', A') \end{aligned}$$

and, by Lemma 20 part (2), φ_{CI} satisfies **dual**.

mon. Once again, we use Lemma 20. Let K be a tournament and $A' \subseteq A$, $B' \subseteq B$. Suppose $a, a' \in A'$ with $K(a) \subseteq K(a')$. We need to show that either $a \notin f(K, A', B')$ or $a' \in f(K, A', B')$

Suppose $a \in f(K, A', B')$. Then $a \in \arg\max_{\hat{a} \in A'} |K(\hat{a}) \cap B'|$, so $|K(a) \cap B'| \geq |K(a') \cap B'|$. On the other hand $K(a) \cap B' \subseteq K(a') \cap B'$, so $|K(a) \cap B'| \leq |K(a') \cap B'|$. Consequently $|K(a) \cap B'| = |K(a') \cap B'|$, and so $a' \in f(K, A', B')$. This shows the property required by Lemma 20 part (3) is satisfied, and thus φ_{CI} satisfies **mon**.

pos-resp. We have show that φ_{CI} satisfies **chain-def**, **anon** and **dual**; due to impossibility result of Theorem 18, φ_{CI} cannot satisfy **pos-resp**.

IIM. Write

$$K_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Note that the first and second rows of each tournament are identical, so **IIM** would imply $1 \preceq_{K_1}^{\varphi_{\mathsf{CI}}} 2$ iff $1 \preceq_{K_2}^{\varphi_{\mathsf{CI}}} 2$. However, it is easily verified that $1 \prec_{K_1}^{\varphi_{\mathsf{CI}}} 2$ whereas $2 \prec_{K_2}^{\varphi_{\mathsf{CI}}} 1$. Therefore φ_{CI} does not satisfy **IIM**. $\square$