Review Session 5 – Solutions

1 Solutions

1.1 Previous Core Competency Problems

Problem 1 (2018 Summer Practice, # 11)**.** Suppose that $X_1, \ldots, X_n \overset{i.i.d.}{\sim} Ber(\lambda/n)$.

(a) What is the distribution of $\sum_{i=1}^n X_i$.

(b) Compute $\lim_{n\to\infty} \mathbb{P}(\sum_{i=1}^n X_i = k)$, where k is any fixed nonnegative integer, and hence show that $\sum_{i=1}^n X_i$ converges in distribution to a random variable Y.

(c) Compute $\mathbb{E}[Y(Y-1)]$, where Y is as in part (b).

Solution

(a) $\sum_{i=1}^n X_i \sim$ Binomial$(n, \lambda/n)$.

(b) We claim $Y = \sum_{i=1}^n X_i \xrightarrow{\text{d}}$ Poisson(λ). Indeed, we have

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^n X_i = k\right) = \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{k!n^k} \cdot \frac{\lambda^k}{(1-\lambda/n)^k}(1-\lambda/n)^n = \frac{\lambda^k}{k!} \lim_{n\to\infty}(1-\lambda/n)^n = \frac{\lambda^k}{k!}e^{-\lambda}.$$

(c) $\mathbb{E}[Y(Y-1)] = \mathbb{E}[Y^2] - \mathbb{E}[Y] = \lambda + \lambda^2 - \lambda = \lambda^2$.

Problem 2 (2018 Summer Practice, # 16)**.** Farmers in the Hudson Valley pack apples into bags of approximately 10 pounds, but due to the variation in apples the actual weight varies. We may model the weight of a bag as uniformly distributed in $[9.5, 10.5]$ and independent of other bags. The farmers load 1200 bags onto a truck with maximal admissible load of 13000 pounds. Find a simple approximation to the probability that the truck is overloaded, expressed in terms of the Normal distribution.

Solution

Let W_i be the weight of the i-th bag, so that $\mathbb{E}[W_i] = 10$ and $\mathrm{Var}(W_i) = 1/12$. Then, by CLT:

$$\frac{\overline{W}_{1200} - 10}{1/\sqrt{12}}\sqrt{1200} \xrightarrow{\text{d}} \mathcal{N}(0,1).$$

Thus,

$$\mathbb{P}\left(\sum_{i=1}^{1200} W_i > 13000\right) = \mathbb{P}\left(\frac{\overline{W} - 10}{1/\sqrt{12}}\sqrt{1200} > 100\right) \approx \mathbb{P}(\mathcal{N}(0,1) > 100)$$

Problem 3 (2018 Summer Practice, # 19)**.** suppose for every $n \geq 1$, A_n is a real symmetric matrix of size $n \times n$, whose eigenvalues $(\lambda_1, \ldots, \lambda_n)$ satisfies the following properties:

(i) $\max_{i=1}^n |\lambda_i| \overset{n\to\infty}{\to} 0$.

(ii) $\sum_{i=1}^n \lambda_i^2 = 1$.

Find the asymptotic distribution of $\sum_{i,j=1}^n A_n(i,j)X_iX_j$, where $\{X_i\}_{i\geq 1}$ is a sequence of i.i.d. $\mathcal{N}(0,1)$.

Solution

By the eigendecomposition of A_n, it suffices to compute the limiting distribution of $S_n := \sum_{i=1}^n X_i^2\lambda_i$. We will proceed via Lyapunov CLT for the triangular array $\{X_i\lambda_i\}_{i,n}$. Condition (ii) gives us that the normalization constant is $\sum_{i=1}^n \lambda_i^2 = 1$ for all $n \in \mathbb{N}$. Next, we verify the third-moment condition:

$$\lim_n \sum_{i=1}^n \mathbb{E}[|X_i^2\lambda_i - \lambda_i|^3] = \lim_n \sum_{i=1}^n |\lambda_i|^3 \mathbb{E}[|X^2-1|^3] \le (\max_i |\lambda_i|) \cdot \mathbb{E}[|X^2-1|^3] \overset{n\to\infty}{\longrightarrow} 0$$

Thus, by Lyapunov CLT, $S_n - \sum_{i=1}^n \lambda_i \xrightarrow{\mathsf{d}} \mathcal{N}(0, \mathrm{Var}(X^2))$.

Problem 4 (2018 September, # 3)**.** Suppose that, for $n \ge 1$, X_n is a random variable taking values in $\{1/n, 2/n, \dots, n/n\}$ with equal probability $1/n$.

(i) Show that X_n converges in distribution, as $n \to \infty$? What is its weak limit?

(ii) Let $f : [0,1] \to \mathbb{R}$ be defined as $f(x) = x\sin(x)$, for $x \in [0,1]$. Using the above or otherwise, show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx.$$

Solution

(i) We will show this by an mgf computation. We have the mgf of X_n is

$$\sum_{k=1}^n \frac{1}{n}\cdot e^{t\cdot k/n} = \begin{cases} \frac{1}{n}\left(\frac{e^{t\cdot\left(\frac{n+1}{n}\right)}-1}{e^{t/n}-1}\right) & t \ne 0 \\ 1 & t = 0 \end{cases}.$$

Next, we take the limit as $n \to \infty$. Suppose $t \ne 0$. We have $\lim_n e^{t\cdot\left(\frac{n+1}{n}\right)} - 1 = e^t - 1$. Next, via L'Hôpital's rule:

$$\lim_n n(e^{t/n}-1) = \lim_n \frac{e^{t/n}-1}{1/n} = \lim_n \frac{-\frac{t}{n^2}e^{t/n}}{-1/n^2} = \lim_n te^{t/n} = t.$$

Thus, the mgf of X_n goes to $(e^t-1)/t$ for $t \ne 0$ and 1 for $t = 0$. This is precisely the mgf of a Unif$([0,1])$, whence $X_n \xrightarrow{\mathsf{d}}$ Unif$([0,1])$.

(ii) f is a bounded, continuous function on $[0,1]$. Thus, by portmanteau theorem,

$$\lim_n \mathbb{E}[f(X_n)] = \mathbb{E}_{X\sim \text{Unif}([0,1])}[f(X)]),$$

which is the desired result.

> **Remark 1.1.** (ii) also follows from just the definition of the Riemann integral $\int_0^1 f(x)\,dx$ (which of course exists for continuous f).

Problem 5 (2018 September, # 7)**.** Suppose $X_1, \dots, X_n$ are i.i.d. with $\mathbb{P}(X_i = \pm 1) = \frac{1}{2}$. Define

$$Y_i := \prod_{j=1}^i X_j, \qquad \text{for } i = 1, \dots, n.$$

(i) Find the joint distribution of (Y_1, Y_2).

(ii) Derive the limiting distribution of $\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i$.

Solution

Working out the various cases, we see that (Y_1, Y_2) takes values $(\pm 1, \pm 1)$ uniformly. We have each Y_i is ± 1 with probability $1/2$ and, by induction, the $\{Y_i\}_{i=1}^n$'s are mutually independent since $\mathbb{P}(Y_{n+1}|Y_1, \ldots, Y_n) = \mathbb{P}(Y_{n+1})$. Thus, by CLT, $\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \xrightarrow{\mathsf{d}} \mathcal{N}(0, 1/4)$.

Problem 6 (2019, May # 2). Let $Z_1, \ldots, Z_n$ be i.i.d. random variables with density f. Suppose that (i) $\mathbb{P}(Z_i > 0) = 1$, and (ii) f is continuous on $[0, \epsilon)$, for some $\epsilon > 0$. Let $\lambda := f(0)$. Let

$$X_n = n \min\{Z_1, \ldots, Z_n\}.$$

Show that X_n converges in distribution, and find the limiting distribution.

Solution

We have for $c > 0$:

$$\mathbb{P}(X_n > c) = \prod_{i=1}^n \mathbb{P}(Z_i > c/n) = (1 - F(c/n))^n.$$

Next, we consider $\lim_n \log((1 - F(c/n)^n) = \lim_n n \log(1 - F(c/n))$. By L'Hôpital's rule:

$$\lim_n \frac{\log(1 - F(c/n))}{1/n} = \lim_{x \to \infty} \frac{\frac{f(c/x) \cdot c/x^2}{1 - F(c/x)}}{-1/x^2} = \lim_{x \to \infty} \frac{-f(c/x) \cdot c}{1 - F(c/x)}.$$

Since f is continuous near 0, so is its antiderivative with $\lim_{x \to \infty} F(c/x) = 0$. Thus, the above RHS is equal to $-f(0) \cdot c = -\lambda \cdot c$, meaning $\mathbb{P}(X_n > c) \to e^{-\lambda \cdot c}$. Thus, $X_n \xrightarrow{\mathsf{d}} \exp(\lambda)$.

Problem 7 (2019 May, # 8). Suppose you have a quadratic form $\mathbf{X}_n^T A_n \mathbf{X}_n$, where $\mathbf{X}_n \sim N_n(\mathbf{0}_{n \times 1}, \mathbf{I}_{n \times n})$, and A_n is a symmetric $n \times n$ matrix with 0 on the diagonal. Let $(\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n})$ denote the eigenvalues of A_n, and let $\|\lambda\|_{2,n} := \sqrt{\sum_{i=1}^n \lambda_{i,n}^2}$ denote the ℓ_2-norm of the eigenvalues.

(a) If $\lim\limits_{n \to \infty} \frac{\max_{i=1,\ldots,n} |\lambda_{i,n}|}{\|\lambda\|_{2,n}} = 0$, show that $T_n := \frac{1}{\|\lambda\|_{2,n}} \mathbf{X}_n^T A_n \mathbf{X}_n \xrightarrow{\mathsf{d}} N(0,1)$ as $n \to \infty$.
[Hint: You may use Lyapunov's[1] CLT. Note that the trace of a square matrix is the sum of its eigenvalues.]

(b) If

$$\lim_{n \to \infty} \frac{\lambda_{1,n}}{\|\lambda\|_{2,n}} = 1,$$

show that $T_n \xrightarrow{\mathsf{d}} \chi_1^2 - 1$.

Solution

(a) By an eigendecomposition of A_n, we can write

$$\frac{1}{\|\lambda\|_{2,n}} \mathbf{X}_n^T A_n \mathbf{X}_n \stackrel{\mathsf{d}}{=} \frac{1}{\|\lambda\|_{2,n}} \sum_{i=1}^n X_{i,n}^2 \cdot \lambda_{i,n}.$$

Next, since $\mathrm{Tr}(A_n) = 0 = \sum_{i=1}^n \lambda_{i,n}$, we have $\mathbb{E}\left[\frac{1}{\|\lambda\|_{2,n}} \mathbf{X}_n^T A_n \mathbf{X}_n\right] = 0$. Finally, for the sake of using Lyapunov's CLT, we last need the third moment condition which amounts to bounding:

$$\lim_n \frac{1}{\|\lambda\|_{2,n}^3} \sum_{i=1}^n |\lambda_{i,n}|^3 \leq \left(\frac{\max_i |\lambda_{i,n}|}{\|\lambda\|_{2,n}}\right) \cdot \frac{\sum_{i=1}^n \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2} \xrightarrow{n \to \infty} 0.$$

[1] **Lyapunov's CLT**: Suppose that $\{Z_1, Z_2, \ldots\}$ is a sequence of independent random variables such that Z_i has finite expected value μ_i and variance σ_i^2. Define $s_n^2 := \sum_{i=1}^n \sigma_i^2$. If $\lim_{n \to \infty} \frac{1}{s_n^3} \sum_{i=1}^n \mathbb{E}[|Z_i - \mu_i|^3] = 0$ is satisfied, then $\frac{1}{s_n} \sum_{i=1}^n (Z_i - \mu_i) \xrightarrow{\mathsf{d}} N(0,1)$.

Thus, by Lyapunov CLT $T_n \xrightarrow{\mathsf{d}} \mathcal{N}(0, \mathrm{Var}(X_{1,1}^2)) = \mathcal{N}(0,2)$.

(b) First, we write

$$T_n = X_{n,1}^2 \cdot \frac{\lambda_1}{\|\lambda\|_{2,n}} + \sum_{i=2}^{n} \frac{X_{n,i}^2 \cdot \lambda_i}{\|\lambda\|_{2,n}}.$$

The first term on the RHS goes to χ_1^2 in distribution by Slutsky. It suffices to show the second term on the RHS goes to -1. In fact, we claim this will follow from:

$$\frac{1}{\|\lambda\|_{2,n}} \left(\sum_{i=2}^{n} X_{n,i}^2 \cdot \lambda_i - \mathbb{E}[X_{n,i}^2] \cdot \lambda_i \right) \xrightarrow{\mathsf{P}} 0. \tag{1}$$

To verify (1), we have for any fixed $\epsilon > 0$, by Chebyshev:

$$\begin{aligned} \mathbb{P}\left(\frac{1}{\|\lambda\|_{2,n}} \left| \sum_{i=2}^{n} X_{n,i}^2 \cdot \lambda_{i,n} - \mathbb{E}[X_{n,i}^2] \cdot \lambda_{i,n} \right| \geq \epsilon \right) &\leq \frac{\mathrm{Var}\left(\sum_{i=2}^{n} X_{n,i}^2 \cdot \lambda_{i,n}\right)}{\|\lambda\|_{2,n}^2 \cdot \epsilon^2} \\ &= \frac{1}{\|\lambda\|_{2,n}^2 \cdot \epsilon^2} \sum_{i=2}^{n} \mathrm{Var}(X_{n,i}^2) \cdot \lambda_{i,n}^2. \end{aligned}$$

Then, it suffices to show $\frac{\sum_{i=2}^{n} \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2} \to 0$. But, this follows immediately from realizing

$$1 = \lim_n \frac{\lambda_{1,n}^2}{\|\lambda\|_{2,n}^2} + \frac{\sum_{i=2}^{n} \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2} = 1 + \lim_n \frac{\sum_{i=2}^{n} \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2}.$$

Finally, we have that $\sum_{i=1}^{n} \lambda_{i,n} = 0$ for all $n \in \mathbb{N}$ gives us

$$\frac{1}{\|\lambda\|_{2,n}} \sum_{i=2}^{n} \mathbb{E}[X_{n,i}^2] \cdot \lambda_{i,n} = \sum_{i=2}^{n} \frac{\lambda_{i,n}}{\|\lambda\|_{2,n}} \to -1.$$

Putting everything together, we conclude $T_n \xrightarrow{\mathsf{d}} \chi_1^2 - 1$.

> **Remark 1.2.** Note that our use of Chebyshev to establish (1) can be considered a strong version of Law of Large Numbers for triangular arrays (e.g., see Theorem 2.2.4 in Durrett's *Probability: Theory and Examples*).

Problem 8 (2019 May, # 9). Let $Y_n = \prod_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are i.i.d. nonnegative non-degenerate random variables with mean $\mathbb{E}(X_i) = 1$. Prove that $Y_n \xrightarrow{\mathsf{P}} 0$ as $n \to \infty$ when: (i) $\mathbb{P}(X_1 = 0) > 0$, and (ii) $\mathbb{P}(X_1 = 0) = 0$.

Solution

(i) For $\epsilon > 0$, we have

$$\mathbb{P}(Y_n > \epsilon) \leq \mathbb{P}(\forall i \in [n] : X_i > 0) = \mathbb{P}(X_1 > 0)^n.$$

However, since $\mathbb{P}(X_1 > 0) < 1$, the RHS above goes to 0 as $n \to \infty$. Thus, $Y_n \xrightarrow{\mathsf{P}} 0$.

(ii) We may now assume WLOG that $X_i > 0$ everywhere. Consider the transformation $X \mapsto \log(X)$. We have, by Jensen:

$$\mathbb{E}[\log(X_1)] < \log \mathbb{E}[X_1] = 0.$$

Thus, SLLN gives $\log(Y_n) = \sum_{i=1}^{n} \log(X_i) \xrightarrow{\text{a.s.}} -\infty$, meaning $Y_n \xrightarrow{\mathsf{P}} 0$.

Problem 9 (2019 May, # 10). Let $f_{X,Y}(x,y)$ be a bivariate density and let $(X_1, Y_1), \ldots, (X_N, Y_N)$ be i.i.d. $f_{X,Y}$. Let $w(\cdot)$ be an

arbitrary probability density function. Let
$$\hat{f}_X(x) = \frac{1}{N}\sum_{i=1}^{N} \frac{f_{X,Y}(x,Y_i)w(X_i)}{f_{X,Y}(X_i,Y_i)}.$$
Show that, for any $x \in \mathbb{R}$, $\hat{f}_X(x) \xrightarrow{\mathsf{P}} f_X(x)$, where f_X is the marginal density of X_1.

Solution

It suffices to show
$$\mathbb{E}_{X_1,Y_1}\left[\frac{f_{X,Y}(x,Y_1)\cdot w(X_1)}{f_{X,Y}(X_1,Y_1)}\right] = f_X(x),$$
whence the result will follow from LLN. Indeed, we have
$$\begin{aligned}\mathbb{E}_{X_1,Y_1}\left[\frac{f_{X,Y}(x,Y_1)\cdot w(X_1)}{f_{X,Y}(X_1,Y_1)}\right] &= \int f_{X,Y}(x,Y_1)\cdot w(X_1)\, d(X_1,Y_1) \\ &= \int f_{X,Y}(x,Y_1)\int w(X_1)\, dX_1\, dY_1 \\ &= \int f_{X,Y}(x,Y_1)\, dY_1 \\ &= f_X(x),\end{aligned}$$
where choosing the order of integration is justified by Tonelli's theorem.

Problem 10 (2019 September, # 6)**.** Suppose that $X_1, X_2, \dots$ are i.i.d. having an exponential distribution with mean 1. Show that
$$\frac{\max_{1\le k\le n} X_k}{\log n} \xrightarrow{\mathsf{P}} 1 \text{ as } n\to\infty$$
where $\xrightarrow{\mathsf{P}}$ denotes convergence in probability.

Solution

We have for $\epsilon \in (0,1)$:
$$\begin{aligned}\mathbb{P}\left(\left|\frac{\max\limits_{1\le k\le n} X_k}{\log(n)} - 1\right| > \epsilon\right) &= \mathbb{P}\left(\max_{1\le k\le n} X_k \ge (1+\epsilon)\log(n)\right) + \mathbb{P}\left(\max_{1\le k\le n} \le (1-\epsilon)\cdot\log(n)\right) \\ &\le n\cdot e^{-(1+\epsilon)\log(n)} + (1-e^{-(1-\epsilon)\cdot\log(n)})^n \\ &= \frac{n}{n^{\epsilon+1}} + \left(1 - \frac{1}{n^{1-\epsilon}}\right)^n\end{aligned}$$
However, both of these last terms go to 0 as $n\to\infty$ (the second limit can be computed with l'Hôpital's rule). Thus, we've shown the definition of $\max_{1\le k\le n} X_k/\log(n) \xrightarrow{\mathsf{P}} 1$.

Problem 11 (2020 May, # 2)**.** Let $X_1, X_2, \dots, X_n$ denote n independent and identically distributed observations from Uniform$(0,1)$. We order these observations according to their distance from $x = 0.75$ and call the ordered ones $X^x_{(1)}, X^x_{(2)}, \dots, X^x_{(n)}$. Note that $X^x_{(1)}$ and $X^x_{(n)}$ are, respectively, the closest and farthest observations from $x = 0.75$.

(i) Prove that $X^x_{(1)}$ converges to 0.75 in probability.

(ii) What does $X^x_{(n)}$ converge to in probability? Prove your answer.

Solution

(i)

$$\mathbb{P}(|X^x_{(1)} - 0.75| > \epsilon) = \prod_{i=1}^{n} \mathbb{P}(|X_i - 0.75| > \epsilon) = \prod_{i=1}^{n} (1-2\epsilon) \overset{n\to\infty}{\longrightarrow} 0.$$

(ii) We claim $X^x_{(n)} \xrightarrow{\mathsf{P}} 0$. Indeed, for $\epsilon < 0.75$, we have $X^x_{(n)} > \epsilon \implies \forall i \in [n] : X_i > \epsilon$ so that

$$\mathbb{P}(|X^x_{(n)}| > \epsilon) \leq \prod_{i=1}^{n} \mathbb{P}(X_i > \epsilon) = (1-\epsilon)^n \overset{n\to\infty}{\longrightarrow} 0$$

Problem 12 (2020 September, # 2). Suppose that $X_1, \dots, X_{2n}$ are i.i.d. $U[0,1]$. Let $Y_i = X_{2i-1} + X_{2i}$ for $1 \leq i \leq n$.

(a) Find the limiting distribution of Y_1.

(b) Find the limiting distribution of $\sqrt{n}(2 - Y_{(n)})$ as $n \to \infty$.

Solution

(a) Y_1 has cdf

$$\mathbb{P}(X_1 + X_2 \leq t) = \begin{cases} \int_0^t \mathbb{P}(X_1 \leq t - x)\, dx = \int_0^t (t-x)\, dx = t^2/2 & t \in [0,1] \\ \int_{t-1}^1 \mathbb{P}(X_1 \leq t-x)\, dx + \int_0^{t-1} 1\, dx = 2t - t^2/2 - 1 & t \in [1,2] \end{cases}.$$

(b) For $w > 0$ and n large enough we have, using part (a),

$$\begin{aligned} \mathbb{P}(\sqrt{n}(2 - Y_{(n)}) > w) &= \mathbb{P}\left(Y_{(n)} < 2 - \frac{w}{\sqrt{n}}\right) \\ &= \left(2\left(2 - \frac{w}{\sqrt{n}}\right) - \frac{1}{2}\left(2 - \frac{w}{\sqrt{n}}\right)^2 - 1\right)^n \\ &= \left(1 - \frac{w^2}{2n}\right)^n. \end{aligned}$$

This last expression goes to $e^{-w^2/2}$ as $n \to \infty$. Thus, the limiting distribution of $\sqrt{n}(2 - Y_{(n)})$ has cdf $F(w) = 1 - e^{-w^2/2}$ for $w > 0$.

Problem 13 (2021 September, # 5). Suppose $\{\xi_i\}_{i\geq 0}$ are i.i.d. $\mathcal{N}(0,1)$ random variables. Find the constant c such that

$$\frac{\max_{1\leq i\leq n} X_i}{\sqrt{\log(n)}} \xrightarrow{\mathsf{P}} c,$$

for each of the following three cases where $\{X_i\}_{i\geq 1}$ is defined.

(i) $X_i = \xi_i$ for $i \geq 1$.

(ii) $X_i = \xi_i + \xi_0$ for $i \geq 1$.

(iii) $X_i = \frac{\xi_i + \xi_{i-1}}{\sqrt{2}}$ for $i \geq 1$.

Solution

(i) We first write the cdf of $\max_i \xi_i/\sqrt{\log(n)}$. This is

$$\mathbb{P}\left(\frac{\max_i \xi_i}{\sqrt{\log(n)}} \le x\right) = \mathbb{P}(\xi_0 \le x\sqrt{\log(n)})^n = \Phi(x\sqrt{\log(n)})^n,$$

where $\Phi(\cdot)$ is the standard normal cdf. First, if $x \le 0$, then we see that

$$\lim_n \Phi(x\sqrt{\log(n)})^n \le \lim_n (1/2)^n = 0.$$

So, it suffices to compute $\lim_n \Phi(x\sqrt{\log(n)})^n$ for $x > 0$. We apply L'Hopital's rule to $\log(\Phi(x\sqrt{\log(n)})^n)$ (let $\phi(\cdot)$ be the standard normal pdf) giving:

$$\begin{aligned}
\lim_n \frac{\log(\Phi(x\sqrt{\log(n)}))}{1/n} &= \lim_n \frac{\frac{1}{\Phi(x\sqrt{\log(n)})}\cdot \phi(x\sqrt{\log(n)})\cdot \frac{x}{\sqrt{\log(n)}}\cdot\left(\frac{1}{2}\right)\cdot\frac{1}{n}}{-1/n^2}\\
&= \lim_n -\frac{n^{1-x^2/2}}{2\sqrt{\log(n)}}\cdot\frac{x}{\Phi(x\sqrt{\log(n)})}\\
&= \begin{cases} -\infty & x \in (0,\sqrt{2})\\ 0 & x \ge \sqrt{2}\end{cases}
\end{aligned}$$

Thus, $\lim_n \Phi(x\sqrt{\log(n)}) \to \mathbf{1}\{x \ge \sqrt{2}\}$ which is the cdf of the constant $\sqrt{2}$. Thus, $c = \sqrt{2}$ for (i).

(ii) $\frac{\max_i \xi_i + \xi_0}{\sqrt{\log(n)}} = \frac{\xi_0}{\sqrt{\log(n)}} + \frac{\max_i \xi_i}{\sqrt{\log(n)}} \xrightarrow{\mathsf{P}} \sqrt{2}$ by (i) and the fact that $\xi_0/\sqrt{\log(n)} \xrightarrow{\mathsf{P}} 0$.

(iii) We have

$$\begin{aligned}
\mathbb{P}\left(\max_i X_i \le x\sqrt{\log(n)}\right) &= \mathbb{P}\left(-\max_i X_i \ge -x\sqrt{\log(n)}\right)\\
&= \mathbb{P}\left(\min_i X_i \ge -x\sqrt{\log(n)}\right).
\end{aligned}$$

where we replace each X_i with $-X_i$ by symmetry at the last step. Next, if $\min_i X_i \ge -x\sqrt{\log(n)}$, then for each $i \in [n]$, $\frac{\xi_i+\xi_{i-1}}{\sqrt{2}} \ge -x\sqrt{\log(n)}$ which means that for each $i \in [n]$ either $\xi_i \ge -x\sqrt{\log(n)/2}$ or $\xi_{i-1} \ge -x\sqrt{\log(n)/2}$. In particular, this is true for every odd $i \in [n]$. Thus, we can bound the RHS probability above by:

$$\begin{aligned}
\mathbb{P}\left(i \in 1,\ldots,\lfloor n/2\rfloor : \max(\xi_{2i},\xi_{2i+1}) \ge -x\sqrt{\log(n)/2}\right) &\le \left(2\cdot\mathbb{P}\left(\mathcal{N}(0,1) \ge -x\sqrt{\log(n)/2}\right)\right)^{\lfloor n/2\rfloor}\\
&\le (2 - 2\Phi(-x\sqrt{\log(n)/2}))^{\lfloor n/2\rfloor}.
\end{aligned}$$

We again apply L'Hopital to find the limit of this last expression. To get rid of the annoying $\lfloor n/2\rfloor$ we'll replace the exponent $\lfloor n/2\rfloor$ with the smaller $n/4$ which only makes the above bound larger (and as we shall see will be suitable for our end results). We have

$$\begin{aligned}
\lim_n \log((2-2\Phi(-x\sqrt{\log(n)/2}))^{n/4}) &= \lim_n \frac{\log(2-2\Phi(-x\sqrt{\log(n)/2}))}{4/n}\\
&= \lim_n \frac{(2-2\Phi(-x\sqrt{\log(n)/2}))^{-1}\cdot(-2\phi(-x\sqrt{\log(n)/2}))\cdot\left(-\frac{x/\sqrt{2}}{\sqrt{\log(n)/2}}\right)\cdot\frac{1}{2n}}{-4/n^2}\\
&\propto -\frac{n^{1-x^2/4}\cdot x}{\sqrt{\log(n)}}\cdot\frac{1}{1-\Phi(-x\sqrt{\log(n)/2})}.
\end{aligned}$$

This last expression goes to $-\infty$ for $x \in [0,2)$ and goes to 0 for $x = 2$. Thus, taking $\exp(\cdot)$, we find that

$$\lim_n \mathbb{P}\left(\max_i X_i \le x\sqrt{\log(n)}\right) \le \begin{cases} 0 & x \in [0,2)\\ 1 & x = 2\end{cases}.$$

On the other hand, $\max_i \frac{\xi_i}{\sqrt{2}} \le \frac{x}{2}\sqrt{\log(n)} \implies \max_i X_i \le x\sqrt{\log(n)}$ so that

$$\mathbb{P}\left(\max_i X_i \le x\sqrt{\log(n)}\right) \ge \mathbb{P}\left(\max_i \xi_i \le x\sqrt{\log(n)/2}\right) = \mathbb{P}\left(\frac{\max_i \xi_i}{\sqrt{\log(n)}} \le x/\sqrt{2}\right) \overset{n\to\infty}{\longrightarrow} \mathbf{1}\{x \ge 2\},$$

where the last part follows from part (i). Putting these upper and lower bounds on the cdf of $\max_i X_i/\sqrt{\log(n)}$ together, we see that $\frac{\max_i X_i}{\sqrt{2\log(n)}} \overset{\mathsf{P}}{\to} 2$.

Problem 14 (2021 September, # 7)**.** Let $X_1, X_2, \dots, X_n \overset{\text{i.i.d.}}{\sim} F$ (F denotes the CDF). Our goal is to estimate $\gamma = F(0) + 2F(1)$. We employ the following estimate

$$\hat{\gamma} = \frac{1}{n}\left(\sum_{i=1}^n \mathbf{1}\{X_i \le 0\} + 2\sum_{i=1}^n \mathbf{1}\{X_i \le 1\}\right),$$

where $\mathbf{1}\{\cdot\}$ dneotes the indicator function.

(i) Calculate $\mathbb{E}[\hat{\gamma}]$.

(ii) What is the limiting distribution of $\sqrt{n}(\hat{\gamma} - \gamma)$? Justify your answer.

Solution

(i) $\mathbb{E}[\hat{\gamma}] = F(0) + 2F(1) = \gamma$.

(ii) By CLT, $\sqrt{n}(\hat{\gamma} - \gamma) \overset{\mathsf{d}}{\to} \mathcal{N}(0, \mathrm{Var}(\mathbf{1}\{X \le 0\} + 2\mathbf{1}\{X \le 1\}))$. This variance is

$$\mathbb{E}[(\mathbf{1}\{X \le 0\} + 2\mathbf{1}\{X \le 1\})^2] - \gamma^2 = 5F(0) + 4F(1) - (F(0) + 2F(1))^2.$$

Problem 15 (2021 September, # 8)**.** Answer the following questions.

(i) Suppose that $(X_n, Y_n) \overset{\mathsf{d}}{\to} \mathcal{N}(0, \Sigma)$ in distribution with $\Sigma = [2, 1; 1, 1]$. What does $(X_n - Y_n)^2$ converge in distribution? Prove your answer.

(ii) Suppose that $(X_n, \sqrt{n}Y_n) \overset{\mathsf{d}}{\to} \mathcal{N}(0, \Sigma)$ in distribution with $\Sigma = [2, 1; 1, 1]$. What does $(X_n - Y_n)^2$ converge to in distribution? Prove your answer.

(iii) Let $X_n \overset{\mathsf{P}}{\to} 1$. For each X_n, we pick Y_n uniformly at random from the internet $[0, X_n]$. What does Y_n converge to in distribution? Prove your answer.

Solution

(i) Let $(Z_1, Z_2) \sim \mathcal{N}(\mathbf{0}_2, \Sigma)$. Then, by continuous mapping theorem, $X_n - Y_n \overset{\mathsf{d}}{\to} Z_1 - Z_2 \sim \mathcal{N}(0,1)$ by Cramer-Wold (where $\mathrm{Var}(Z_1 - Z_2) = 2 + 1 - 2 = 1$). Thus, again by continuous mapping theorem, $(X_n - Y_n)^2 \overset{\mathsf{d}}{\to} \chi_1^2$.

(ii) By Cramer-Wold, we have $\sqrt{n}\cdot Y_n \overset{\mathsf{d}}{\to} \mathcal{N}(0,1)$ which implies $Y_n \overset{\mathsf{d}}{\to} 0$ by Slutsky. Then, again by Slutsky, $X_n - Y_n \overset{\mathsf{d}}{\to} \mathcal{N}(0,2)$ so that $(X_n - Y_n)^2 \overset{\mathsf{d}}{\to} 2\cdot\chi_1^2$.

(iii) I'll assume $X_n > 0$ a.s. for all n or else the problem doesn't make sense. We claim $Y_n \overset{\mathsf{d}}{\to} \mathrm{Unif}([0,1])$. Fix $y \in [0,1]$ and let $G(x) = \frac{y\cdot\mathbf{1}\{y\in[0,x]\}}{x}$ which is the cdf of $Y_n|X_n = x$ at y. Then, note that $G(y)$ is a bounded and a.s. continuous function. Thus, by Portmanteau theorem, $X_n \overset{\mathsf{d}}{\to} 1$ implies $\mathbb{E}[G(X_n)] \to G(1)$. But, $G(1)$ is just the $\mathrm{Unif}([0,1])$ cdf evaluated at y. Thus, we've shown convergence of cdfs $\mathbb{E}[G(X_n)] = F_{Y_n}(y) \to y\cdot\mathbf{1}\{y \in [0,1]\}$ for all $y \in [0,1]$ meaning $Y_n \overset{\mathsf{d}}{\to} \mathrm{Unif}([0,1])$.