MA Survey Paper Notes

Theory of Banach Spaces

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Chapter 1

Schauder Bases

In this chapter, we shall introduce Schauder bases, an important concept in Banach space theory. Elements of a Banach space with a Schauder basis may be represented as infinite sequences of "coordinates", which is very natural and useful for analytical work. Although not every separable Banach space admits a Schauder basis, basic sequences exist in every infinite-dimensional Banach space and are ideal for the study of linear subspaces and quotients. This notion has proved to be an extremely useful tool in the study of the structure of classical as well as abstract Banach spaces.

1.1 Projections and Complementability, and Auerbach Bases

Definition 1.1.1

Let X be a vector space. A *linear projection* $P : X \to X$ is a mapping such that $P^2 = P$. In this situation, we say that P is a projection from X to $\mathrm{Range}(X)$ parallel to $\ker(P)$.

Associated to a projection $P : X \to X$, there exists an *algebraic direct sum*, or an *algebraic linear decomposition* by

$$X = \mathrm{Range}(P) \oplus \ker(P)$$

On the other hand, if $X = Y_1 \oplus Y_2$ is a linear decomposition of X, then the mappings $P_i : X \to X$ given by $P_i(x) = x_i$, where $x = x_1 + x_2$, $x_i \in Y_i$ for $i = 1, 2$, are linear projections onto Y_i for $i = 1, 2$ and $P_1 + P_2 = \mathrm{id}_X$.

Definition 1.1.2

Let X be a vector space and Y be a subspace of X. Then there exists a subspace Z such that $X = Y \oplus Z$. Such a subspace Z is called an *algebraic complement* of Y in X.

Definition 1.1.3

Let X be a Banach space and Y be a closed subspace of X.

(i) Y is said to be *complemented* in X if there exists a bounded linear projection $P : X \to X$ with $\operatorname{Range}(P) = Y$.

(ii) If λ is a real number, in particular $\lambda \geq 1$, such that $\|P\| \leq 1$, then Y is λ-*complemented* in X.

(iii) Let Y_1 be a closed subspace of X. We say that Y_2 is a *topological complement* in X if $X = Y_1 \oplus Y_2$ and Y_2 is a closed subspace of X.

Remark 1.1.4

Every complemented subspace of a Banach space X is closed, but not every closed subspace of X is complemented. To see the former, note that if $\operatorname{Range}(P) = Y$ for some projection P on X, then $Y = (\operatorname{id} - P)^{-1}(\{0_X\})$.

Proposition 1.1.5

Let Y be a closed subspace of a Banach space X. The following assertions are equivalent:

(a) Y is complemented in X.

(b) There exists a topological complement of Y in X.

Proof.

(a) $\Rightarrow$ (b) Assume that Y is complemented in X. Then by Definition 1.1.3 (i), there exists a bounded linear projection $P : X \to X$ with $\operatorname{Range}(P) = Y$. We claim that $\ker(P)$ is a topological complement of $\operatorname{Range}(P)$ in X. That is, we need to show that $X = \operatorname{Range}(P) \oplus \ker(P)$ and $\ker(P)$ is a closed subspace of X. For the former, we show that $\operatorname{Range}(P)$ and $\ker(P)$ form a linear decomposition as follows:

- Let $x \in \operatorname{Range}(P) \cap \ker(P)$. Then this implies that $x = Px = 0_X$, and therefore, $\operatorname{Range}(P) \cap \ker(P) = \{0_X\}$.

- Let $x \in X$ be arbitrary and put $y \in \operatorname{Range}(P)$ and $z \in \ker(P)$ as $y = Px$ and $z = x - Px$. Then obviously, $y \in \operatorname{Range}(P)$, and

$$Pz = P(x - Px) = Px - PPx = Px - Px = 0_X$$

  so $z \in \ker(P)$. Therefore, $x = Px - (x - Px)$, so we have $X = \operatorname{Range}(P) + \ker(P)$

Therefore, we have shown that $\operatorname{Range}(P)$ and $\ker(P)$ are a linear decomposition of X. Now we need to show that $\ker(P)$ is closed. But this follows immediately from the fact that P is a bounded linear operator, and thus, is continuous, so $\ker(P)$ is closed.

(b) $\Rightarrow$ (a) Assume there exists a topological complement Z of Y in X and let $P : X \to X$ be the projection such that $\text{Range}(P) = Y$ and $\ker(P) = Z$. To show that P is a bounded linear projection onto Y, we will show that $\text{Gr}(P) = \{(x, Px) : x \in X\}$ is closed, so let $(x_n, Px_n)_{n=1}^{\infty}$ be a sequence in $\text{Gr}(P)$ such that $(x_n, Px_n) \to (x, y)$, where $(x, y) \in X \times X$. We want to show that $(x, y) \in \text{Gr}(P)$. Since Y is a closed subspace of X, we have $y \in Y$. Then since $x - Px \in \ker(P)$ for any $x \in X$, then this implies that for each $n \in \mathbb{N}$,

$$P(x_n - Px_n) = Px_n - PPx_n = Px_n - Px_n = 0_X$$

which then implies that $x - y \in Z$. By the uniqueness of the decomposition, $x - y$ is unique, and $y = Px$, and thus, $(x, y) \in \text{Gr}(P)$. Therefore, P is a bounded linear projection onto Y. ■

Proposition 1.1.6

Let X and Y be Banach spaces and let $T : X \to Y$ be an isomorphism. Assume that X_1 is a complemented subspace of X with topological complement X_2, then $T[X_1]$ is a complemented subspace of Y with topological complement $T[X_2]$.

Proof.

Assume that X_1 is a complemented subspace of X with topological complement X_2 and assume that $T : X \to Y$ is an isomorphism. Then clearly, X_1 and X_2 are closed, and so by Definition 1.1.3, there exists a projection $P : X \to X$ with $\text{Range}(P) = X_2$, and that $X = X_1 \oplus X_2$. To see that $T[X_1]$ is a complemented subspace of X with topological complement $T[X_2]$ note that $Q : Y \to Y$ with $\text{Range}(Q) = T[X_2]$ is a bounded linear projection onto $T[X_2]$. To see that $T[X_2]$ is a topological complement of $T[X_1]$, it suffices to check that $Y = T[X_1] \oplus T[X_2]$. But in this case, we have $T[X_1] = \ker(Q)$ and $T[X_2] = \text{Range}(Q)$.

- Let $y \in T[X_1] \cap T[X_2]$. Then $y = QTx = 0_Y$ so $T[X_1] \cap T[X_2] = \{0_Y\}$.
- Fix $y \in Y$, and put $z = Qy$ and $w = y - Qy$, and it is easy to see that $z \in \text{Range}(Q)$ and $w \in \ker(Q)$. Then we have $y = z + w$, so $Y = T[X_1] + T[X_2]$.

Therefore, we have $Y = T[X_1] \oplus T[X_2]$, so $T[X_1]$ is a complemented subspace of Y with topological complement $T[X_2]$. ■

Note that if P is a projection of X onto X_1 with $\ker(P) = X_2$, then $Q = TPT^{-1}$ is a projection of Y onto $T[X_1]$ with $\ker(Q) = T[X_2]$.

Let X be a normed space such that $X = Y \oplus Z$ for complemented subspaces Y and Z. Then X is isomorphic to the direct sum $(Y \oplus Z)_\infty$ of spaces Y and Z, with the norm described by

$$\|(y, z)\|_\infty = \max\{\|y\|, \|z\|\}$$

Proposition 1.1.7

Let X be a Banach space and Y be a closed subspace of X. The following statements hold.

(i) If Y is complete and Z is a complement of Y in X, then X/Y is isomorphic to Z. Moreover, if $Q : X \to X$ is the projection onto Z parallel to Y and $\|Q\| = 1$, then X/Y is isometrically isomorphic to Z.

(ii) The dual X^* is isomorphic to $Y^* \oplus Z^*$; that is, $(Y \oplus Z)^* = Y^* \oplus Z^*$. If $P : X \to X$ is the projection onto Y parallel to Z, then Y^* is isomorphic to $P^*[X^*]$. Moreover, if $\|P\| = 1$, then Y^* is isomorphic to $P^*[X^*]$.

(iii) If Y is a closed subspace of a Hilbert space $\mathcal{H}$ and Z is the orthogonal complement of Y in $\mathcal{H}$, then $\mathcal{H}/Y$ is isometrically isomorphic to Z.

Proof.

We will first prove (i). Let $J|_Z : Z \to X/Y$ be the quotient map restricted to Z be given as $Jz = [z]_Y$, where $[z]_Y$ denotes the equivalence class containing z. From the definition of the norm of X/Y, it follows that $\|J(z)\| \leq \|z\|$ for every $z \in Z$, so J is a bounded linear operator. We claim that $J|_Z$ is one-to-one. Indeed, let $z \in Z$ be such that $Jz = 0_{X/Y}$. Then since $z \in [z]_Y$, we have $z \in Z \cap Y$, and so $z = 0_X$. We also show that $J|_Z$ is onto. Indeed, for any $[x]_Y \in X/Y$, pick $x \in [x]_Y$ and write $x = y + z$, where $y \in Y$ and $z \in Z$. Then $[x]_Y = [z]_Y = Jz$. Therefore, $J|_Z$ is an isomorphism of Z onto X/Y by the open mapping theorem. Now assume that $\|Q\| = 1$. Let $z \in Z$ and if $x \in X$ is an arbitrary element of $[z]_Y$, then $x = y + z$ for some $y \in Y$. Then $\|z\| = \|Qx\| \leq \|x\|$ by the definition of the norm in X/Y, so we have $\|z\| \leq \|[z]_Y\|$. This proves the reverse inequality and so $J|_Z$ is an isometry.

We next prove (ii). Let P be a projection of X onto Y and let Q be the projection of X onto Z. Then $Y^* \simeq P^*[X^*]$ and $Z^* \simeq Q^*[X^*]$. Moreover, from $\mathrm{id}_X = P + Q$ we get $\mathrm{id}_{X^*} = P^* + Q^*$, i.e. $P^*[X^*] + Q^*[X^*] = X^*$. It remains to show that $P^*[X^*] \cap Q^*[X^*] = \{0_{X^*}\}$. Indeed, let $f \in P^*[X^*] \cap Q^*[X^*]$. Then for $y \in Y$, $f(y) = Q^*(f)(y) = f(Qy) = 0$, and for $z \in Z$, $f(z) = P^*(f)(z) = f(Pz) = 0$, so $f = 0$. Since $P^*[X^*]$ and $Q^*[X^*]$ are closed subspaces, it follows from Proposition 1.1.5 that $X^* = P^*[X^*] \oplus Q^*[X^*]$ is a topological direct sum.

Finally, for (iii), if $\mathcal{H}$ is a Hilbert space, then $\|Q\| = 1$ and the conclusion follows from (i). ∎

Definition 1.1.8

Let X be a Banach space and let $\{x_1, ..., x_n\} \subset X$. Functionals $\{f_1, ..., f_n\} \subset X^*$ are called *biorthogonal* to $\{x_1, ..., x_n\}$ if $f_i(e_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. In other words, the set $\{(x_i, f_i)\}_{i=1}^n$ is called a *biorthogonal system* in $X \times X^*$.

Definition 1.1.9

A biorthogonal system $\{(x_i, f_i)\}_{i=1}^n$ is called an *Auerbach basis of* X if $(x_i)_{i=1}^n$ is a Hamel basis of X and $\|x_i\| = \|f_i\| = 1$ for all $1 \leq i \leq n$.

Theorem 1.1.10

Let X be a Banach space and Y be a subspace of X. The following hold:

(i) Every finite-dimensional Banach space X admits an Auerbach basis $\{(x_i, f_i)\}_{i=1}^n$ of X.

(ii) If Y is a finite-dimensional subspace of X, then there exists a projection $P : X \to X$ with $\text{Range}(P) = Y$ and $\|P\| \leq n$.

Proof.

We will first prove the first assertion. Let $\{x_1, ..., x_n\}$ be a Hamel basis of X. For $e_1, ..., e_n \in B_X$, let $J(u_1, ..., u_n)$ be the determinant of the matrix whose jth row is formed by the coordinates of u_j in the basis $\{x_1, ..., x_n\}$. The function $|J|$ is continuous on the compact set $B_x \times \cdots \times B_X$, and so there exists $(e_1, ..., e_n) \in B_X \times \cdots \times B_X$ such that

$$J(e_1, ..., e_n) = \max\{|J(x_1, ..., x_n)| : (x_1, ..., x_n) \in B_X \times \cdots \times B_X\}$$

Since determinants are homogeneous in each coordinate, we have $e_i \in S_X$. As $J(e_1, ..., e_n) \neq 0$, the vectors $\{e_1, ..., e_n\}$ are linearly independent, so they form a basis for X. For each $1 \leq i \leq n$, let $f_i \in X^*$ be defined as

$$f_i(x) = \frac{J(e_1, ..., e_{i-1}, e_{i+1}, ..., e_n)}{J(e_1, ..., e_n)}$$

Then $f_i(e_j) = \delta_{ij}$, so $\{(e_i, f_i)\}_{i=1}^n$ is a biorthogonal system. Moreover, $\|f_j\| \leq 1$ for each $1 \leq j \leq n$, so $\|f_j\| = 1$. Thus, $\{(e_i, f_i)\}_{i=1}^n$ is an Auerbach basis of X.

For the second assertion, let $\{(e_i, f_i)\}_{i=1}^n$ be an Auerbach basis of Y. We extend f_i to norm-one functionals on X by Hahn-Banach. Then we define an operator $P : X \to Y$ by $P(x) = \sum_{i=1}^n f_i(x)e_i$ for $x \in X$. We claim that P is a projection onto Y. Indeed, for any $y = \sum_{i=1}^n \lambda_i e_i$ in Y, we have $\lambda_i = f_i(y)$, and therefore, $P(y) = \sum_{i=1}^n f_i(y)e_i = y$. Finally, if $x \in X$ and $\|x\| \leq 1$, $\|Px\| \leq \sum_{i=1}^n |f_i(x)|\|e_i\| \leq \sum_{i=1}^n 1 = n$. This completes the proof. ∎

Exercise 1.1.11

(i) Let M be a subspace of a vector space X. Show that there exists a linear projection of V onto M, i.e. $P|_M = \text{id}_M$ and $\text{Range}(P) = M$.

(ii) Show that if a linear mapping $P : X \to X$ satisfies $P^2 = P$, then $X = \ker(P) \oplus \text{Range}(P)$. Moreover, $Q = \text{id}_X - P$ is a projection such that $\text{Range}(Q) = \ker(P)$ and $\ker(Q) = \text{Range}(P)$.

Solution.

(i) Let A be a Hamel basis of M and extend the Hamel basis A to a Hamel basis of X given by $A \cup B$ so that $A \cup B$ is a Hamel basis of X. Let $P : X \to X$ be the linear projection defined as follows: for $x \in X$,

$$Px = \begin{cases} x & \text{if } x \in A \\ 0 & \text{if } x \in B \end{cases}$$

Then we claim that $P|_M = \mathrm{id}_M$ and $\mathrm{Range}(P) = M$. Indeed, for the first part, fix $m \in M$. Then since $m \in \langle A \rangle$, as A is a Hamel basis of M, then we obtain $Pm = m$, which shows that $P|_M = \mathrm{id}_M$ as required. To see the latter, fix $m \in M$. We seek a $x \in X$ such that $Px = m$. In particular, since $m \in M$, then we can simply choose $x = m$, since $m \in A \setminus B$, and we are done.

(ii) To see that $X = \ker(P) \oplus \mathrm{Range}(P)$, we need to show thar $\ker(P)$ and $\mathrm{Range}(P)$ is a linear decomposition of X, and that $\ker(P)$ and $\mathrm{Range}(P)$ are closed subspaces of X. Indeed, we first show that $\ker(P)$ and $\mathrm{Range}(P)$ are a linear decomposition of X, so fix $x \in \ker(P) \cap \mathrm{Range}(P)$. Since $x \in \ker(P)$, then we have that $Px = 0$, and since $x \in \mathrm{Range}(P)$, we have that

$$x = Px = 0$$

so $x = 0$. Now to see that $X = \ker(P) + \mathrm{Range}(P)$, fix $x \in X$. Then choose $Px \in \mathrm{Range}(P)$ and $x - Px \in \ker(P)$. It is easy to see that $Px \in \mathrm{Range}(P)$. For the latter, we have

$$P(x - Px) = Px - PPx = Px - Px = 0$$

so $x - Px \in \ker(P)$, and therefore, $x = Px + (x - Px)$. Therefore, we have shown that $\ker(P)$ and $\mathrm{Range}(P)$ are a linear decomposition of X. It is easy to note that $\ker(P)$ and $\mathrm{Range}(P)$ are closed. ■

Exercise 1.1.12

Let P be a bounded linear projection in a Banach space X. Show that P^* is a bounded linear projection in X^*.

Solution.

Assume that P is a bounded linear projection in a Banach space X. Note that since P is a projection, and using the fact that for operators $T, S : X \to X$, $(TS)^* = S^*T^*$, we have that

$$(P^*)^2 = P^*P^* = (PP)^* = (P^2)^* = P^*$$

so P^* is a linear projection. Then since P is bounded, i.e. $\|P\| \leq M$ for some $M \in \mathbb{R}$, this implies that

$$\|P^*\| = \|(P^*)^2\| = \|(P^2)^*\| = \|P^2\| = \|P\| \leq M$$

so P^* is also bounded as desired. ■

Exercise 1.1.13

Let P and Q be projections in a Banach space X. Show that the following are equivalent:

(a) $\text{Range}(P) \subset \text{Range}(Q)$ and $\text{Range}(P^*) \subset \text{Range}(Q^*)$.

(b) $PQ = QP = P$.

(c) $\text{Range}(P) \subset \text{Range}(Q)$ and $\ker(Q) \subset \ker(P)$.

Solution.

(a) $\Rightarrow$ (b) Assume that $\text{Range}(P) \subset \text{Range}(Q)$ and $\text{Range}(P^*) \subset \text{Range}(Q^*)$. Then for any $y \in \text{Range}(P)$, there exists an $x \in X$ such that $y = Px$. But by assumption, this implies that $y \in \text{Range}(Q)$, i.e. $Px = QPx$, which shows that $P = QP$. Furthermore, for any $g \in \text{Range}(P^*)$, there exists an $f \in X^*$ such that $g = P^*f$, but then by assumption, $P^*f = Q^*P^*f = (PQ)^*f$, which implies that $P = PQ$, so we have $PQ = QP = P$.

(b) $\Rightarrow$ (c) Now assume that $PQ = QP = P$. Then clearly, since $P = QP$, we have that for any $x \in X$, $Px = QPx$, i.e. $\text{Range}(P) \subset \text{Range}(Q)$. To see that $\ker(Q) \subset \ker(P)$, let $x \in X$ be such that $Qx = 0$. Then by assumption, we have $PQx = P(0) = 0$, so $Qx \in \text{Range}(P)$, proving (c).

(c) $\Rightarrow$ (a) Assume that $\text{Range}(P) \subset \text{Range}(Q)$ and $\ker(Q) \subset \ker(P)$. It suffices to show that $\text{Range}(P^*) \subset \text{Range}(Q^*)$, so let $g \in \text{Range}(P^*)$ be arbitrary. Then there exists an $f \in X^*$ such that $g = P^*f$, i.e. for all $x \in X$, $g(x) = (P^*f)(x) = f(Px)$. By assumption, since $\text{Range}(P) \subset \text{Range}(Q)$, then this implies that $P = QP$, i.e. for all $x \in X$, $Px = QPx$. Therefore,

$$g(x) = f(Px) = f(QPx) = ((QP)^*f)(x) = Q^*((P^*f)(x))$$

which shows that $g \in \text{Range}(Q^*)$, proving (a). ■

Exercise 1.1.14

Let P be a bounded linear projection of a Banach space X onto $\text{Range}(P)$. Show that for every $x \in X$,

$$\text{dist}(x, \text{Range}(P)) \leq \|x - Px\| \leq (\|P\| + 1)\,\text{dist}(x, \text{Range}(P))$$

Solution.

It is easy to note that $\text{dist}(x, \text{Range}(P)) \leq \|x - Px\|$. On the other hand, fix $y \in \text{Range}(P)$. Write

$$x - Px = x - y + y - Px = x - y + Py - Px$$

which then, we have by the triangle inequality,

$$\|x - Px\| \leq \|x - y\| + \|Py - Px\| = \|x - y\| + \|P\|\|x - y\| = (\|P\| + 1)\|x - y\|$$

as desired. ■

Exercise 1.1.15

Let Y be a subspace of a Banach space X. Show that if there exists a bounded linear projection onto Y, then Y is closed.

Solution.

Let $(y_n)_{n=1}^{\infty}$ be a sequence in Y such that $y_n \to y$ for some $y \in X$. We want to show that $y \in Y$. Let P be a bounded linear projection onto Y. Indeed, since $y_n \to y$, we have $Py_n \to Py$, and so $y_n \to Py$. By the uniqueness of limits, we have $y = Py$, and so $y \in Y$. ∎

Exercise 1.1.16

Assume that Y_1, Y_2 are subspaces of a Banach space X such that $X = Y_1 \oplus Y_2$. Let P_1, P_2 be the associated linear projections onto Y_1, Y_2, respectively so that $P_1 + P_2 = \mathrm{id}_X$. Show that:

(i) $\mathrm{Range}(P_1) = \ker(P_2)$

(ii) P_1 is bounded if and only if P_2 is bounded.

(iii) Both Y_1 and Y_2 are closed if and only if both P_1 and P_2 are bounded.

(iv) Find an example when Y_1 is closed but Y_2 is not.

Solution.

(i) By Exercise 1.1.11, since $P_1 + P_2 = \mathrm{id}_X$, we have $P_1 = \mathrm{id}_X - P_2$ and since P_1 is a projection, $\mathrm{Range}(P_1) = \ker(P_2)$.

(ii) Also, by Exercise 1.1.11, we have $P_2 = \mathrm{id}_X - P_1$, and so, if P_1 is bounded, then

$$\|P_2\| = \|\mathrm{id}_X - P_1\| \leq \|\mathrm{id}_X\| + \|P_1\| < \infty$$

since the identity is always bounded.

(iii) follows from Proposition 1.1.5, since $X = Y_1 \oplus Y_2$, Y_1 is complemented in X, so there exists a bounded linear projection P_1 onto Y_1. Similarly, Y_2 is complemented in X, so there exists a bounded linear projection P_2 onto Y_2. For the converse, if P_1 and P_2 are bounded linear projections onto Y_1 and Y_2, respectively, then Y_1 and Y_2 are complemented, so by Proposition 1.1.5, Y_1 and Y_2 are closed subspaces.

(iv) Consider $Y_1 = \mathbb{R}$, which is closed, and $Y_2 = \ker(f)$ for some discontinuous linear functional f. ∎

Exercise 1.1.17

Let X be a Banach space. Show that X^* is complemented in X^{***}.

Solution.

To see that X^* is complemented in X^{***} we seek a bounded linear projection $P^{***} : X^{***} \to X^{***}$ onto X^*. Indeed, let $P^{***} : X^{***} \to X^*$ be the mapping defined as $P^{***}f = f|_{X^*}$. Then clearly, P^{***} is bounded with $\|P^{***}\| = 1$.

Exercise 1.1.18

(i) Let Y be a complemented subspace of a Banach space X. Let P be the projection of X onto Y. Show that the dual operator P^* is a mapping that extends elements of Y^* to elements in X^*. If $\|P\| = 1$, we get a linear Hahn-Banach extension.

(ii) Let Y be a closed subspace of a reflexive Banach space X. Assume that there exists a bounded operator $E : Y^* \to X^*$ such that $E(f)$ is an extension of f on X with the same norm. Show that Y is complemented in X.

1.2 Basics on Schauder Bases

Definition 1.2.1

Let X be an infinite-dimensional normed space. A sequence $\{e_n\}_{n=1}^\infty$ in X is called a *Schauder basis* of X if for every $x \in X$, there exists unique scalars $(\lambda_n)_{n=1}^\infty$, called the *coordinates of* x, such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n$$

Remark 1.2.2

(i) $\{e_n\}_{n=1}^\infty$ is a linearly independent set in X.

(ii) Every Banach space that has a Schauder basis is separable, as finite rational combinations of elements of the basis form a countable dense set.

(iii) If X is finite-dimensional, then the Schauder bases coincides with Hamel bases. We will write $\{e_n\}_{n=1}^\infty$ and $\langle\{e_n\}_{n=1}^\infty\rangle$, etc. to cover both the finite and infinite-dimensional case.

(iv) The condition in Definition 1.2.1 can be weakened. If every $x \in X$ has a unique decomposition $\sum_{n=1}^\infty \lambda_n e_n$ such that $\sum_{n=1}^N \lambda_n e_n \rightharpoonup x$ as $N \to \infty$, it already implies that $\{e_n\}_{n=1}^\infty$ is a Schauder basis.

Lemma 1.2.3

Let $\{e_n\}_{n=1}^\infty$ be a Schauder basis of a normed space X. The canonical projection P_k satisfy the following:

(i) $\dim(P_k(X)) = k$.

(ii) $P_k P_\ell = P_\ell P_k = P_{\min(k,\ell)}$.

(iii) $P_k(x) \to x$ in X for every $x \in X$.

Conversely, if bounded linear projections $(P_k)_{k=1}^\infty$ in a normed space X satisfy (i)-(iii), then P_k are canonical projections associated with some Schauder basis of X.

Proof.

The set $\{e_n\}_{n\in\mathbb{N}}$ is a linearly independent set in X. Thus, (i) and (ii) are obvious. Property (iii) follows from the definition of the Schauder basis. Conversely, if projections P_k satisfy (i)-(iii), put $P_0 = 0$, and choose $0 \leq e_n \in P_n(X) \cap \ker(P_{n-1})$ for $n \in \mathbb{N}$. Then

$$\begin{aligned} x &= \lim_{n\to\infty} P_n(x) \\ &= \lim_{n\to\infty} (P_n(x) - P_0(x)) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (P_i(x) - P_{i-1}(x)) \\ &= \sum_{n=1}^{\infty} (P_n(x) - P_{n-1}(x)) \\ &= \sum_{n=1}^{\infty} \lambda_n e_n \end{aligned}$$

for some scalars $(\lambda_n)_{n=1}^\infty$ as $\dim(P_n(X)/P_{n-1}(X)) = 1$ for all $n \in \mathbb{N}$.

The uniqueness of $(\lambda_n)_{n=1}^\infty$ follows from the fact that if $x = \sum_{n=1}^\infty \mu_n e_n$, then by the continuity of P_n, we get $P_n(x) = \sum_{i=1}^n \mu_i e_i$ and so $\mu_i e_i = P_i(x) - P_{i-1}(x) = \lambda_i e_i$ for all $n \in \mathbb{N}$. Therefore, $\{e_n\}_{n=1}^\infty$ is a Schauder basis of X and P_n are the projections associated to $\{e_n\}_{n=1}^\infty$. ■

Proposition 1.2.4

Let $\{e_n\}_{n=1}^\infty$ be a Schauder basis of a normed space X with canonical projections $(P_n)_{n=1}^\infty$. Assume that

$$\sup_{n\in\mathbb{N}} \|P_n\| < \infty$$

i.e. P_n is uniformly bounded, then $\{e_n\}_{n=1}^\infty$ is also a Schauder basis of the completion $\mathcal{X}$ of X.

Proof.

We will show that the extensions $\widetilde{P}_n$ of P_n on $\mathcal{X}$ satisfy (i)-(iii) of Lemma 1.2.3. Since $P_n(X)$ is finite-dimensional, it is closed in $\mathcal{X}$ and thus, $\widetilde{P}_n(\mathcal{X}) = P_n(X)$ so (i) follows. (ii) is extended from P_n to $\widetilde{P}_n$ by the continuity of P_n. Since $P_n(x) \to x$ for all x in a dense set X and P_n are uniformly bounded, we have $\widetilde{P}_n(\widetilde{x}) \to \widetilde{x}$ in $\mathcal{X}$, so (iii) is also true. Since $e_n \in P_n(X) \cap \ker(P_{n-1})$ we get $e_n \in \widetilde{P}_n(\mathcal{X}) \cap \ker(\widetilde{P}_{n-1})$ for every n. Therefore, $\widetilde{P}_n$ are canonical projections associated with the Schauder basis $\{e_n\}_{n=1}^\infty$ of $\mathcal{X}$. ■

Lemma 1.2.5

Let $\{e_n\}_{n=1}^\infty$ be a Schauder basis of a Banach space $(X, \|\cdot\|)$. Define $|||\cdot|||$ on X by

$$|||x||| = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n \lambda_i e_i \right\|$$

for $x = \sum_{n=1}^\infty \lambda_n e_n$. Then the following hold:

(i) $|||\cdot|||$ is a norm on X, $\{e_n\}_{n=1}^\infty$ is a Schauder basis of $(X, |||\cdot|||)$, and the canonical projections P_n are uniformly bounded by 1 in $|||\cdot|||$.

(ii) $|||\cdot|||$ is an equivalent norm on X.

Proof.

We first proof the first assertion. It is easy to prove positivity and absolute homogenity. So we prove the triangle inequality, so let $x = \sum_{n=1}^\infty \lambda_n e_n$ and $y = \sum_{n=1}^\infty \mu_n e_n$. Then

$$x + y = \sum_{n=1}^\infty \lambda_n e_n + \sum_{n=1}^\infty \mu_n e_n = \sum_{n=1}^\infty (\lambda_n + \mu_n) e_n$$

and so,

$$\begin{aligned}
|||x+y||| &= \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n (\lambda_i + \mu_i) e_i \right\| \\
&\le \sup_{n\in\mathbb{N}} \left(\left\| \sum_{i=1}^n \lambda_i e_i \right\| + \left\| \sum_{i=1}^n \mu_i e_i \right\| \right) \\
&= \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n \lambda_i e_i \right\| + \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n \mu_i e_i \right\| \\
&= |||x||| + |||y|||
\end{aligned}$$

Therefore, $|||\cdot|||$ is a norm on X. Next, we will show that $\{e_n\}_{n=1}^\infty$ is a Schauder basis on X with $|||\cdot|||$, we check by using Lemma 1.2.3. Indeed, (i) and (ii) are easy to check. To see (iii), note that

$$|||P_m(x) - x||| = \sup_{n\in\mathbb{N}} \|P_n P_m(x) - P_n(x)\| = \sup_{n\ge m} \|P_n(x) - P_m(x)\| \to 0$$

as $m \to \infty$. Finally, for $m \in \mathbb{N}$, we estimate

$$\begin{aligned}
|||P_m||| &= \sup_{|||x|||\le 1} |||P_m(x)||| \\
&= \sup_{|||x|||\le 1} \sup_{n\in\mathbb{N}} \|P_n P_m(x)\| \\
&= \sup_{n\in\mathbb{N}} \sup_{|||x|||\le 1} \|P_n P_m(x)\|
\end{aligned}$$

$$= \sup_{n\in\mathbb{N}} \left\{ \sup\{\|P_n P_m(x)\| : x \text{ with } \sup_{n\in\mathbb{N}} \|P_n(x)\| \le 1\} \right\} \le 1$$

To see that (ii) holds, we will show that X is a Banach space with $|||\cdot|||$, i.e. that $\mathbb{X} \subset X$, where $\mathbb{X}$ is the completion of X in $|||\cdot|||$. By (i), we know that $\{e_n\}_{n=1}^\infty$ is a Schauder basis of $\mathbb{X}$. Given $x \in \mathbb{X}$, there exists a unique sequence of scalars λ_n such that $x = \sum_{n=1}^\infty \lambda_n e_n$, where the convergence is the norm in $|||\cdot|||$. Since $|||\cdot||| \ge \|\cdot\|$ on X, we get that $\sum_{n=1}^\infty \lambda_n e_n$ is Cauchy in $\|\cdot\|$, and thus, convergent to some element $x' \in X$ in the form $\|\cdot\|$. As shown in (i), $\sum_{n=1}^\infty$ converges to x' in the form $|||\cdot|||$. Thus, $x = x'$ in X. This means that X is complete in $|||\cdot|||$. The formal identity mapping $\mathrm{id}_X : (X, |||\cdot|||) \to (X, \|\cdot\|)$ is a linear bijection of a Banach space onto another Banach space which is continuous since $|||\cdot||| \ge \|\cdot\|$. From the Open Mapping Theorem, id_X^{-1} is continuous, which means that $|||\cdot|||$ is an equivalent norm on X. This completes the proof. ■

Theorem 1.2.6

Let $\{e_n\}_{n=1}^\infty$ be a Schauder basis of a Banach space X. The canonical projections P_i associated with $\{e_n\}_{n=1}^\infty$ are uniformly bounded.

We denote the value $\mathfrak{bc}(\{e_n\}_{n=1}^\infty) = \sup_{n\in\mathbb{N}} \|P_n\|$ to be the *basis constant* of $\{e_n\}_{n=1}^\infty$.

Proof.

Using the norm defined in Lemma 1.2.5, $|||P_n||| \le 1$ for all $n \in \mathbb{N}$, and since $|||\cdot|||$ is an equivalent norm on X, the canonical projections are uniformly bounded. ■

Considering the vectors e_n, we have $\|P_n\| \ge 1$, and in particular, $\mathfrak{bc}(\{e_n\}_{n=1}^\infty) \ge 1$.

Definition 1.2.7

Let X be a Banach space and let $\{e_n\}_{n=1}^\infty$ be a Schauder basis of X.

(i) The Schauder basis is said to be *normalized* if $\|e_n\| = 1$ for all $n \in \mathbb{N}$.

(ii) The Schauder basis is said to be *seminormalized* if

$$0 < \inf_{n\in\mathbb{N}} \|e_n\| \le \sup_{n\in\mathbb{N}} \|e_n\| < \infty$$

(iii) The Schauder basis is said to be *monotone* if $\mathfrak{bc}(\{e_n\}_{n=1}^\infty) = 1$, that is, its associated projections satisfy $\|P_n\| = 1$ for all $n \in \mathbb{N}$.

In Lemma 1.2.5, we have basically showed that every Banach space with a Schauder basis can be re-normed in such a way that in the new norm, the Schauder basis becomes monotone, i.e. $\mathfrak{bc}(\{e_n\}_{n=1}^\infty) = 1$.

The following is a consequence of Theorem 1.2.6: If X is a Banach space and $\{e_n\}_{n=1}^\infty$ is a Schauder basis for X, then as it turns out, X can be written as a direct sum of $\langle\{e_i\}_{i=1}^n\rangle$ and $\overline{\langle\{e_i\}_{i=n+1}^\infty\rangle}$, i.e.

$$X = \langle\{e_i\}_{i=1}^n\rangle \oplus \overline{\langle\{e_i\}_{i=n+1}^\infty\rangle}$$

This is indeed true because P_n is continuous and $\mathrm{Range}(\mathrm{id} - P_n) = \ker(P_n)$.

If $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of a Banach space X, then for each $j \in \mathbb{N}$ and $x = \sum_{n=1}^{\infty} \lambda_i e_i$, denote $f_j(x) = \lambda_j$. Then we have

$$\|P_j(x) - P_{j-1}(x)\| = \|f_j(x)e_j\| = |f_j(x)|\|e_j\|$$

and therefore,

$$\|f_j\| = \sup_{x \in B_X} |f_j(x)| = \|e_j\|^{-1}| \sup_{x \in B_X} \|f_j(x)e_j\| \leq 2\|e_j\|^{-1}| \sup_{n \in \mathbb{N}} \|P_n\|$$

which then shows that $f_j \in X^*$, i.e. f_j is bounded for every fixed j.

Definition 1.2.8

The functionals $\{f_n\}_{n=1}^{\infty}$ from above are called the *associated biorthogonal functionals* (or *coordinate functionals*) to the Schauder basis $\{e_n\}_{n=1}^{\infty}$. We say that $\{e_n; f_n\}_{n=1}^{\infty}$ is a Schauder basis of the Banach space X.

For every $x \in X$, this would imply that

$$x = \sum_{n=1}^{\infty} f_n(x) e_n$$

That is, we can take our scalars and replace them with coordinate functionals that depend on x. This is nearly identical to Hamel bases and coordinate functionals on Hamel bases, where we can take any $x \in X$ and write as a finite linear combination of basis elements with scalars being the coordinate functionals that depend on x.

Note that $\{e_n; f_n\}_{n=1}^{\infty}$ is a biorthogonal system and from the above, we deduce that

$$\|e_n\|\|f_n\| \leq 2\, \mathfrak{bc}(\{e_n\}_{n=1}^{\infty})$$

and we can also note that these functionals is separating for X, that is, for any $x \neq y \in X$ there exists $f \in \{f_n\}_{n=1}^{\infty}$ such that $f(x) \neq f(y)$.

Example 1.2.9

(a) If X is a finite-dimensional normed space and B is a Hamel basis of X, then B is a Schauder basis of X. This is because X is finite-dimensional, so it is also a Banach space, and since there are finitely countably many elements in the Hamel basis B, it is also a Schauder basis of X.

(b) Any orthonormal basis $\mathcal{B}$ of a Hilbert space $\mathcal{H}$ is also a Schauder basis

(c) If $X = c_0(\mathbb{N})$ or $X = \ell_p(\mathbb{N})$ for $1 \leq p < \infty$, then the sequence $(e_n)_{n=1}^{\infty}$ is called the *standard basis of* X (sometimes called the *unit vector basis*), and the unit vector basis is a Schauder basis of X.

It is easy to see that (c) is true: Since $c_{00}(\mathbb{N})$ is countable and dense in $\ell_p(\mathbb{N})$, then $\ell_p(\mathbb{N})$ is separable. With the unit vector basis given by $(e_n)_{n=1}^{\infty}$, where

$$e_n = (0, 0, 0, ..., \underset{\substack{\uparrow \\ n\text{th coordinate}}}{1}, 0, 0, ...)$$

and for any $x \in \ell_p(\mathbb{N})$, we claim that there exists unique scalars $(\lambda_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n$$

Indeed, if $x = (x_1, x_2, ...)$, where $x_1, x_2, ... \in \mathbb{R}$, then we can take $\lambda_n = x_n$ for each $n \in \mathbb{N}$ so that we can express x as an infinite linear combination of e_n. That is,

$$x = \sum_{n=1}^{\infty} x_n e_n$$

Now to show the uniqueness, assume that $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ are scalars of x such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n = \sum_{n=1}^{\infty} \mu_n e_n$$

Then this implies that

$$0 = \sum_{n=1}^{\infty} (\lambda_n - \mu_n) e_n$$

Then since the unit vector basis is linearly independent, it follows that for each $n \in \mathbb{N}$, $\lambda_n - \mu_n = 0$, or $\lambda_n = \mu_n$ for each $n \in \mathbb{N}$, hence the scalars are unique.

Example 1.2.10

Let $(x_n)_{n=1}^{\infty}$ be a sequence of distinct points in $[0, 1]$ such that $x_1 = 0$, $x_2 = 1$ and $\overline{\{x_n\}_{n=1}^{\infty}} = [0, 1]$. Define projections $P_n : C([0, 1]) \to C([0, 1])$ by $P_1(f) = f(0)$ and $P_n(f)$ to be the piecewise linear functions with nodes at x_j for $j = 1, 2, ..., n$ and such that $P_n(f)(x_j) = f(x_j)$ for $j = 1, 2, ..., n$. By the uniform continuity of continuous functions on $[0, 1]$, the projections P_n determine a monotone Schauder basis of $C([0, 1])$. Such a basis is called the *Faber-Schauder basis of* $C([0, 1])$.

1.3 Shrinking and Boundedly Complete Bases, Perturbation

Let $\{e_n\}_{n=1}^{\infty}$ be a Schauder basis of X. When we consider $\widehat{e}_n \in X^{**}$, the system $\{(f_n, \widehat{e}_n)\}_{n=1}^{\infty}$ is a biorthogonal system in X^*, where f_n are the coordinate projections.

Proposition 1.3.1

Let $\{e_n; f_n\}_{n=1}^{\infty}$ be a Schauder basis of a Banach space X with the canonical projections P_n. The following hold:

(i) For $n \in \mathbb{N}$, $P_n^*(f) = \sum_{i=1}^n f(e_i) f_i = \sum_{i=1}^n \widehat{e}_i(f) f_i$ for every $f \in X^*$.

(ii) $P_n^*(f) \xrightarrow{*} f$ in X^* for every $f \in X^*$.

(iii) $\{(f_n, \widehat{e}_n)\}_{n=1}^{\infty}$ is a Schauder basis of $\overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$ with canonical projections P_n^*. In particular, $P_n^*(f) \to f$ for every $f \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$.

Proof.

We first prove the first assertion, so fix $n \in \mathbb{N}$ and let $f \in X^*$ be arbitrary. Then for every $x \in X$, we have

$$(P_n^* f)(x) = f(P_n x) = f\left(\sum_{i=1}^n f_i(x) e_i\right) = \sum_{i=1}^n f_i(x) f(e_i) = \sum_{i=1}^n \widehat{e}_i(f) f_i(x) = \left(\sum_{i=1}^n \widehat{e}_i(f) f_i\right)(x)$$

which then implies that

$$P_n^* f = \sum_{i=1}^n \widehat{e}_i(f) f_i$$

as desired.

To prove the second assertion, note that since $f \in X^*$ is bounded and using (i), we have for $x \in X$

$$\lim_{n\to\infty} (P_n^* f)(x) = \lim_{n\to\infty} \sum_{i=1}^n f_i(x) f(e_i) = f\left(\lim_{n\to\infty} \sum_{i=1}^n f_i(x) e_i\right) = f(x)$$

Finally, to prove the last assertion, we verify Lemma 1.2.5. It is easy to note that for any $n, m \in \mathbb{N}$, $P_n^* P_m^* = P_{\min(n,m)}^*$. If $f \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$, then $P_n^*(f) = f$ n sufficiently large, which then implies that

$$\lim_{n\to\infty} \|P_n^* f - f\| = 0$$

Since $\|P_n\| = \|P_n^*\|$ it follows that $\{P_n^*\}$ are uniformly bounded and we apply Lemma 1.2.5 and Proposition 1.2.4. ∎

Definition 1.3.2

Let X be a Banach space and $\{e_n; f_n\}_{n=1}^{\infty}$ be a Schauder basis of X.

(i) The Schauder basis is said to be *shrinking* if $\overline{\langle \{f_n\}_{n=1}^{\infty} \rangle} = X^*$.

(ii) The Schauder basis is said to be *boundedly complete* if $\sum_{n=1}^{\infty} \lambda_n e_n$ converges whenever the scalars are such that

$$\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n \lambda_i e_i \right\| < \infty$$

Remark 1.3.3

Every normalized shrinking basis $\{e_n\}_{n=1}^{\infty}$ has the property that $e_n \rightharpoonup 0$ since $\lim_{n\to\infty} f_k(e_n) = 0$ for every $k \in \mathbb{N}$. But there exists a Banach space with a normalized non-shrinking Schauder basis $\{e_n\}_{n=1}^{\infty}$ such that $e_N \rightharpoonup 0$ in X.

Proposition 1.3.4

Let X be a Banach space and let $\{e_n; f_n\}_{n=1}^{\infty}$ be a Schauder basis of X with the canonical projections P_n. The following assertions are equivalent:

(a) $\{e_n; f_n\}_{n=1}^{\infty}$ is a shrinking.

(b) $\{f_n\}_{n=1}^{\infty}$ is a Schauder basis of X^*.

(c) If $E = \{e_n\}_{n=1}^{\infty}$, then $\lim_{n\to\infty} \|f|_{E\setminus\{e_1,\ldots,e_n\}}\| = 0$ for every $f \in X^*$.

Proof.

We will prove the following directions: (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (b) follows from Proposition 1.3.1.

(b) $\Rightarrow$ (a) If the projections $\{P_n^*\}$ generate a Schauder basis of X^*, then $P_n^* f \to f$ for all $f \in X^*$, which then implies that $X^* = \overline{\langle \{f_n\}_{n=1}^{\infty}\rangle}$.

(a) $\Leftrightarrow$ (c) Note that if P is a bounded linear projection of a Banach space onto Range(P), then

$$\sup_{f\in P[B_X]} f = \sup_{x\in B_X} \|(P^* f)(x)\| = \|P^* f\|$$

which then implies that

$$B_{\mathrm{Range}(P)} \subset P[B_X] \subset \|P\| B_X \cap \mathrm{Range}(P) \subset \|P\| B_{\mathrm{Range}(P)}$$

Therefore, for every $f \in X^*$, we obtain

$$\begin{aligned}
\|f|_{(\mathrm{id}_X - P_n)(x)}\| &= \sup\{f(x) : x \in B_{(\mathrm{id}_X - P_n)(x)}\} \\
&\le \sup\{f(x) : x \in (\mathrm{id}_X - P_n)[B_X]\} \\
&\le \sup\{f(x) : x \in \|P_n\| + 1) B_{(\mathrm{id}_X - P_n)[X]}\}
\end{aligned}$$

and so

$$\|f|_{(\mathrm{id}_X - P_n)[X]}\| \le \|f - P_n^* f\| \le (\|P_n\| + 1) \|f|_{(\mathrm{id}_X - P_n)[X]}\|$$

thus, $\{e_n\}_{n=1}^{\infty}$ is shrinking if and only if $\|f|_{(\mathrm{id}_X - P_n)[X]}\| \to 0$ for all $f \in X^*$. ∎

Example 1.3.5

For $1 < p < \infty$ consider $\ell_p(\mathbb{N})$ with the unit vector basis given by $\{e_n\}_{n=1}^{\infty}$ where for each $n \in \mathbb{N}$,

$$e_n = (0, 0, 0, ..., \underset{\substack{\uparrow \\ n\text{th coordinate}}}{1}, 0, 0, ...)$$

Note that $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis for $\ell_p(\mathbb{N})$. We claim that the Schauder basis for $\ell_p(\mathbb{N})$ is shrinking. To see this, we will use Proposition 1.3.4 (c) to show this. Indeed, recall that $(\ell_p(\mathbb{N}))^* \equiv \ell_q(\mathbb{N})$ with the canonical map $\Phi : \ell_q(\mathbb{N}) \to (\ell_p(\mathbb{N}))^*$ defined as follows: for $x \in \ell_p(\mathbb{N})$, define $\Phi x : \ell_p(\mathbb{N}) \to \mathbb{R}$ where

$$(\Phi x)(y) = \sum_{n=1}^{\infty} x_n y_n$$

for $y \in \ell_p(\mathbb{N})$. Then observe that for $x \in \ell_p(\mathbb{N})$ such that $\|x\|_p = 1$, we have

$$\begin{aligned}
\lim_{N\to\infty} |(\Phi x)(y)| &= \lim_{N\to\infty} \left| \sum_{n=N}^{\infty} x_n y_n \right| \\
&\leq \lim_{N\to\infty} \left(\sum_{n=N}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=N}^{\infty} |y_n|^q \right)^{\frac{1}{q}} \\
&= \lim_{N\to\infty} \left(\sum_{n=N}^{\infty} |y_n|^q \right)^{\frac{1}{q}} \\
&= 0
\end{aligned}$$

Therefore, by Proposition 1.3.4, $\{e_n; f_n\}_{n=1}^{\infty}$ is shrinking.

Example 1.3.6

In a similar situation, we can consider $c_0(\mathbb{N})$ with the same standard basis given by $\{e_n\}_{n=1}^{\infty}$ where for each $n \in \mathbb{N}$,

$$e_n = (0, 0, 0, ..., \underset{\substack{\uparrow \\ n\text{th coordinate}}}{1}, 0, 0, ...)$$

Note that $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis for $c_0(\mathbb{N})$.

Example 1.3.7

For $1 \leq p < \infty$, the standard basis for $\ell_p(\mathbb{N})$ is boundedly complete for $p \in [1, \infty)$. Indeed, for any $x \in \ell_p(\mathbb{N})$, we have that for unique scalars $(\lambda_n)_{n=1}^{\infty}$

$$\sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| < \infty$$

which implies that

$$\lim_{N \to \infty} \sum_{n=1}^{N} \lambda_n e_n \to \sum_{n=1}^{\infty} \lambda_n e_n = x$$

Example 1.3.8

Considering $c_0(\mathbb{N})$, the standard basis for $c_0(\mathbb{N})$ is not boundedly complete, as demonstrated by the sequence $(1, ..., 1, 0, ...)$, i.e. as

$$\sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| = n < \infty$$

But $\sum_{n=1}^{\infty} \lambda_n e_n$ does not converge.

Proposition 1.3.9

Let $\{e_n; f_n\}_{n=1}^{\infty}$ be a Schauder basis of a Banach space X. If $\{e_n\}_{n=1}^{\infty}$ is shrinking, then the map $T(\widehat{x}) = (\widehat{x}(f_n))_{n=1}^{\infty} = (f_n(x))_{n=1}^{\infty}$ is an isomorphism of X^{**} onto the space of all sequences $(\lambda_n)_{n=1}^{\infty}$ such that

$$|||(\lambda_n)_{n=1}^{\infty}||| = \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^{N} \lambda_n e_n \right\| < \infty$$

Moreover, if $\{e_n\}_{n=1}^{\infty}$ is monotone, then T is an isometry.

Proof.

It is easy to prove that $|||\cdot|||$ defines a norm on the vector space of $(\lambda_n)_{n=1}^{\infty}$ such that $|||(\lambda_n)_{n=1}^{\infty}||| < \infty$. Let $K = \mathfrak{bc}(\{e_n\}_{n=1}^{\infty})$ and let P_n be the canonical projections associated with $\{e_n\}_{n=1}^{\infty}$. For any $x \in X$, $f \in X^*$ and $\widehat{x} \in X^{**}$, we have that

$$P_n^* f = \sum_{i=1}^{n} f(e_i) f_i$$

and

$$(\widehat{P}_n\widehat{x})(f) = \sum_{i=1}^{n} \widehat{x}(f_i)f(e_i)$$

so we can write

$$\widehat{P}_n\widehat{x} = \sum_{i=1}^{n} \widehat{x}(f_i)e_i$$

Therefore,

$$|||T\widehat{x}||| = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} \widehat{x}(f_i)e_i \right\| = \sup_{n\in\mathbb{N}} \left\| \widehat{P}_n\widehat{x} \right\| \leq K\|\widehat{x}\|$$

which shows that T is a bounded linear operator with $\|T\| \leq K$.

Now suppose that $(\lambda_n)_{n=1}^{\infty}$ such that $|||(\lambda_n)_{n=1}^{\infty}||| < \infty$. Since X^* is separable (as $\{e_n\}_{n=1}^{\infty}$ is shrinking) and $\{\sum_{i=1}^{n} \lambda_i e_i\}$ is bounded in X^{**}, there exists a w^*-cluster point $\widehat{x}$ of $\{\sum_{i=1}^{n} \lambda_i e_i\}$ such that $\widehat{x}(f_i) = \lambda_i$. Moreover,

$$\|\widehat{x}\| \leq \limsup_{n\to\infty} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| \leq |||(\lambda_n)_{n=1}^{\infty}|||$$

Therefore, $T\widehat{x} = (\lambda_n)_{n=1}^{\infty}$ and $\|T\widehat{x}\| \geq \|\widehat{x}\|$, and this completes the proof. ∎

Theorem 1.3.10

Let $\{e_n; f_n\}_{n=1}^{\infty}$ be a Schauder basis of a Banach space X. If $\{e_n\}_{n=1}^{\infty}$ is boundedly complete, then $X \simeq \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^*$.

Proof.

Let P_n denote the canonical embedding associated with $\{e_n\}_{n=1}^{\infty}$, let $Y = \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$, and define $J : X \to Y^*$ by the mapping

$$(Jx)(y) = y(x)$$

then J is a bounded linear operator.

We will show that J is an isomorphism of X onto Y^*, so fix $x \in X$. Then for every $y \in Y$, we have

$$|(Jx)(y)| = |y(x)| \leq \|y\|\|x\|$$

and so $\|Jx\| \leq \|x\|$. On the other hand, for $n \in \mathbb{N}$, find $f \in S_{X^*}$ such that $f(P_n x) = \|P_n x\|$. Since $P_n^*[X^*] = \langle \{f_i\}_{i=1}^{n} \rangle$, we have that $P_n^* f \in Y$ and $\|P_n^* f\| \leq K = \mathfrak{bc}(\{e_n\}_{n=1}^{\infty})$. In particular,

$$(J(P_n x))(P_n^* f) = (P_n^* f)(P_n x) = f(P_n^* x) = f(P_n x) = \|P_n x\|$$

and therefore,

$$\|J(P_n x)\| \geq J(P_n x)\left(\frac{P_n^* f}{\|P_n^* f\|}\right) = \frac{1}{\|P_n^* f\|}\|P_n x\| \geq \frac{1}{K}\|P_n x\|$$

By the continuity of J, we have that

$$\frac{1}{K}\|x\| \leq \|Jx\| \leq \|x\|$$

for all $x \in X$.

Now we will show that J maps X onto Y^*. Indeed, observe that $\{f_n; J(e_n)\}_{n=1}^{\infty}$ is a Schauder basis of Y, let $\widetilde{P}_n$ denote its canonical projections. Then $\widetilde{P}_n^* g \xrightarrow{*} g$ in Y^* and

$$\sup_{n\in\mathbb{N}} \|\widetilde{P}_n^*\| = \sup_{n\in\mathbb{N}} \|\widetilde{P}_n\| \leq K < \infty$$

so for every $g \in Y^*$ and $n \in \mathbb{N}$,

$$\left\| J\left(\sum_{i=1}^{n} g(f_i)e_i\right)\right\| = \left\|\sum_{i=1}^{n} g(f_i)J(e_i)\right\| = \|\widetilde{P}_n^* g\| \leq K\|g\|$$

Therefore, we have

$$\left\|\sum_{i=1}^{n} g(f_i)e_i\right\| \leq K\left\| J\left(\sum_{i=1}^{n} g(f_i)e_i\right)\right\| \leq K^2\|g\|$$

Since the basis $\{e_n\}_{n=1}^{\infty}$ is boundedly complete by assumption, the series $\sum_{n=1}^{\infty} g(f_n)e_n$ is convergent in X to some $x \in X$. Since J is continuous, we have

$$Jx = \lim_{n\to\infty} J\left(\sum_{i=1}^{n} g(f_i)e_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} g(f_i)J(e_i) = \lim_{n\to\infty} \widetilde{P}_n^* g$$

in the norm topology of Y^*. But since $\widetilde{P}_n^* g \xrightarrow{*} g$, then $g = Jx$, so J is onto. This completes the proof. ∎

The following theorem is a very important theorem in which allows us to relate reflexivity with shrinking and boundedly complete Schauder bases.

Theorem 1.3.11: James

Let X be a Banach space with Schauder basis $\{e_n\}_{n=1}^{\infty}$. The following are equivalent:

(a) X is reflexive.

(b) $\{e_n\}_{n=1}^{\infty}$ is both shrinking and boundedly complete.

Before we prove Theorem 1.3.11, let us revisit Examples 1.3.5, 1.3.6, 1.3.7, and 1.3.8.

Example 1.3.12: (Example 1.3.5 and 1.3.7)

Consider the space $\ell_p(\mathbb{N})$ for $1 \leq p < \infty$. It was shown in Example 1.3.5 that $\ell_p(\mathbb{N})$ is shrinking for $1 < p < \infty$ and boundedly complete for $1 \leq p < \infty$ in Example 1.3.7. Indeed, $\ell_1(\mathbb{N})$ is not shrinking but it is boundedly complete. In particular this is because $\ell_1(\mathbb{N})$ is not reflexive. However, for $1 < p < \infty$, $\ell_p(\mathbb{N})$ is reflexive, so the Schauder basis $\{e_n\}_{n=1}^{\infty}$ (which is the unit vector basis), is shrinking and boundedly complete.

Example 1.3.13: (Example 1.3.6 and 1.3.8)

Consider the space $c_0(\mathbb{N})$. It was shown that in Example 1.3.6 that $c_0(\mathbb{N})$ is shrinking, but in Example 1.3.8, it was shown that $c_0(\mathbb{N})$ is not boundedly complete. Indeed, this is because $c_0(\mathbb{N})$ is not reflexive, i.e.

$$(c_0(\mathbb{N}))^{**} \equiv (\ell_1(\mathbb{N}))^* \equiv \ell_\infty(\mathbb{N})$$

which is not $c_0(\mathbb{N})$, so $c_0(\mathbb{N})$ is not reflexive, thus, either not shrinking or boundedly complete (in this case, is not boundedly complete.)

Proof.

(a) $\Rightarrow$ (b) Assume that X is reflexive. Then the weak and weak-$*$ topologies coincide. Then by Proposition 1.3.1, for any $f \in X^*$, $P_n^* f \overset{*}{\rightharpoonup} f$ which implies that $P_n^* f \rightharpoonup f$ in X^*. This implies that, by Mazur's Theorem,

$$X^* = \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{w} = \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$$

and $\{e_n\}_{n=1}^{\infty}$ is a shrinking basis of X. Now we need to show that $\{e_n\}_{n=1}^{\infty}$ is boundedly complete. By Proposition 1.3.9, the space $X^{**} \simeq Y$, where

$$Y = \left\{ (\lambda_n)_{n=1}^{\infty} : |||(\lambda_n)_{n=1}^{\infty}||| = \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| < \infty \right\}$$

Under this correspondence, $X \subset X^{**}$ corresponds to $Z = \{(\lambda_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} \lambda_n e_n \text{ converges}\}$. Since X is reflexive, we get $Y = Z$, so $\{e_n\}_{n=1}^{\infty}$ is boundedly complete.

(b) $\Rightarrow$ (a) This direction is easy; if $\{e_n\}_{n=1}^{\infty}$ is shrinking and boundedly complete, then $Y = Z$ by the above and so $X = X^{**}$, and thus, X is reflexive. ∎

Definition 1.3.14

Let X be a Banach space. A sequence $(e_n)_{n=1}^{\infty}$ is called a *basic sequence* in X if $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of $\overline{\langle \{e_n\}_{n=1}^{\infty} \rangle}$. A basic sequence $(e_n)_{n=1}^{\infty}$ is called *shrinking* (resp. *boundedly complete*) if it is a shrinking (resp. boundedly complete) basis of $\overline{\langle \{e_n\}_{n=1}^{\infty} \rangle}$.

Proposition 1.3.15: Banach

Let X be a Banach space and $(e_n)_{n=1}^{\infty}$ be a sequence of nonzero vectors. The following are equivalent:

(a) $(e_n)_{n=1}^{\infty}$ is basic.

(b) There exists $K > 0$ such that for all $n < m$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$,

$$\left\|\sum_{i=1}^{n} \lambda_i e_i\right\| \leq K \left\|\sum_{i=1}^{m} \lambda_i e_i\right\|$$

Moreover, the smallest such K is equal to $\mathfrak{bc}(\{e_n\}_{n=1}^{\infty})$.

Proof.

(a) $\Rightarrow$ (b) This direction is obvious since for any canonical projection P_n

$$\left\|\sum_{i=1}^{n} \lambda_i e_i\right\| = \left\|P_n\left(\sum_{i=1}^{m} \lambda_i e_i\right)\right\| \leq \|P_n\| \left\|\sum_{i=1}^{m} \lambda_i e_i\right\| \leq \mathfrak{bc}(\{e_n\}_{n=1}^{\infty}) \left\|\sum_{i=1}^{m} \lambda_i e_i\right\|$$

(b) $\Rightarrow$ (a) Assume there exists $K > 0$ such that for all $n < m$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$,

$$\left\|\sum_{i=1}^{n} \lambda_i e_i\right\| \leq K \left\|\sum_{i=1}^{m} \lambda_i e_i\right\|$$

for all λ_i and $n \leq m$. Define the canonical projections P_n on $\langle\{e_n\}_{n=1}^{\infty}\rangle$ as follows:

$$P_n\left(\sum_{i=1}^{m} \lambda_i e_i\right) = \sum_{i=1}^{n} \lambda_i e_i$$

for all $n < m$ and scalars λ_i. Observe that P_n have norm at most K. We check that P_n satisfies (i)-(iii) of Lemma 1.2.3 so that by Proposition 1.2.4, $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of $\overline{\langle\{e_n\}_{n=1}^{\infty}\rangle}$ and $\mathfrak{bc}(\{e_n\}_{n=1}^{\infty}) \leq K$. ■

Not every separable Banach space admits a Schauder basis, but we have the following.

Theorem 1.3.16: Mazur

Let X be a Banach space. The following hold.

(i) If X is infinite-dimensional, then X contains a basic sequence.

(ii) If X^* is separable, then X contains a shrinking basic sequence.

In order to prove Theorem 1.3.16, we require the following lemma.

Lemma 1.3.17

Let X be an infinite-dimensional Banach space and let Y be a finite-dimensional subspace of X. For every $\varepsilon > 0$, there exists $x \in S_X$ such that

$$\|y\| \leq (1 + \varepsilon)\|y + \lambda x\|$$

for every $y \in Y$ and $\lambda \in \mathbb{R}$.

Proof.

Fix $\varepsilon \in (0,1)$ and let $(y_i)_{i=1}^m$ be an $\varepsilon/2$-net in S_Y. For $1 \leq i \leq m$, choose $f_i \in S_{X^*}$ with $f_i(y_i) = 1$. Since X is infinite-dimensional, there exists $x \in S_X$ such that $f_i(x) = 0$ for every $1 \leq i \leq m$. We claim that x has the desired property. Indeed, fix $y \in S_Y$ and choose $1 \leq i \leq m$ such that $\|y_i - y\| < \varepsilon/2$. Let $\lambda \in \mathbb{R}$ be arbitrary. Then see that

$$\|y + \lambda x\| \geq \|y_i + \lambda x\| - \frac{\varepsilon}{2} \geq f_i(y_i + \lambda x) - \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} \geq \frac{1}{1+\varepsilon}$$

Thus, for any nonzero $y \in Y$ and $\lambda \in \mathbb{R}$,

$$\left\| \frac{y}{\|y\|} + \frac{\lambda}{\|y\|} x \right\| \geq \frac{1}{1+\varepsilon}$$

and this completes the proof. ■

Proof.

(of Theorem 1.3.16) We first prove (i), so fix $\varepsilon > 0$, choose $\varepsilon_n > 0$ with $\prod_{n=1}^{\infty}(1+\varepsilon_n) \leq 1+\varepsilon$. Let $x_1 \in S_X$ be arbitrary. Then by Lemma 1.3.17, construct inductively a sequence $(x_n)_{n=2}^{\infty}$ in S_X such that for all $n \in \mathbb{N}$,

$$\|y\| \leq (1+\varepsilon_n)\|y + \lambda x_{n+1}\|$$

for all $y \in \langle x_1, ..., x_n \rangle$. By induction, for $n < m$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}$,

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \leq \prod_{i=n}^{m-1}(1+\varepsilon_i) \left\| \sum_{i=1}^{m} \lambda_i x_i \right\| \leq (1+\varepsilon) \left\| \sum_{i=1}^{m} \lambda_i x_i \right\|$$

By Proposition 1.3.15, $(x_n)_{n=1}^{\infty}$ is a basic sequence and $\mathfrak{bc}(\{x_n\}_{n=1}^{\infty}) \leq 1+\varepsilon$. Note that $\|P_n\| \leq \prod_{i=n}^{\infty}(1+\varepsilon_i) \to 1$ as $n \to \infty$.

Now we show (ii). We will modify Lemma 1.3.17 and then the proof will be complete. Indeed, let $D = \{f_n : n \in \mathbb{N}\}$ be a $\|\cdot\|$-dense countable subset of X^*. At the nth step in the construction of the basic sequence $(x_n)_{n=1}^{\infty}$ and for ε_n, consider in X^* not only the (finite) set $\{g_i : 1 \leq i \leq m\}$ but also the set $D_n = \{f_i : 1 \leq i \leq n\}$ in order to find

$$x_{n+1} \in \bigcap_{i=1}^{m} \ker(g_i) \cap \bigcap_{i=1}^{n} \ker(f_i) \cap S_X$$

Now an appeal to Proposition 1.3.4 gives us the conclusion: Indeed, for $f \in X^*$ and $\varepsilon > 0$, find $N \in \mathbb{N}$ such that $\|f_N - f\| < \varepsilon$. Then $\|f_N|_{\overline{\langle\{x_n\}_{n \geq N_1}\rangle}}\| = 0$ and thus, $\|f|_{\overline{\langle\{x_n\}_{n \geq N_1}\rangle}}\| < \varepsilon$ for all $N_1 > N$. This proves that the basic sequence is shrinking. ■

It is not known whether every separable Banach space X contains a closed subspace Y such that both Y and X/Y have a Schauder basis. We note that if X is separable and nonreflexive, there exists a nonreflexive closed subspace Y of X such that Y has a Schauder basis.

Example 1.3.18

Consider $\ell_1(\mathbb{N})$ which separable and nonreflexive and let Y be the subspace of $\ell_1(\mathbb{N})$ given by

$$Y = \{x \in \ell_1(\mathbb{N}) : x_{2n} = 0 \text{ for all } n \in \mathbb{N}\}$$

Then we note that Y is a closed, but not reflexive, subspace of $\ell_1(\mathbb{N})$. Indeed, Y is not reflexive since $\ell_1(\mathbb{N})$ is not reflexive. Y is also closed, since

$$Y = \bigcap_{k=1}^{\infty} \{x \in \ell_1(\mathbb{N}) : x_{2k} = 0\}$$

and for each $Y_k = \{x \in \ell_1 : x_{2k} = 0\}$ is closed since it is the kernel of the continuous linear functionals given by the map $x \mapsto x_{2k}$. Thus, since Y is nonreflexive and closed subspace of X, then there exists a Schauder basis for Y. One example of a Schauder basis for Y could be $\{e_{2n-1}\}_{n=1}^{\infty}$ (the unit vector basis for Y) since for any $y = (y_1, 0, y_3, 0, ...)$, we can write

$$y = \sum_{n=1}^{\infty} y_{2n-1} e_{2n-1} = \sum$$

Definition 1.3.19

Let $(e_n)_{n=1}^{\infty}$ be a basic sequence in a Banach space X and let $(\widetilde{e}_n)_{n=1}^{\infty}$ be a basic sequence in a Banach space Y. We say that $(e_n)_{n=1}^{\infty}$ is *equivalent* to $\{\widetilde{e}_n\}_{n=1}^{\infty}$ if for all $(\lambda_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ scalars, $\sum_{n=1}^{\infty} \lambda_n e_n$ converges if and only if $\sum_{n=1}^{\infty} \lambda_n \widetilde{e}_n$ converges.

Proposition 1.3.20

Let X and Y be Banach spaces, let $(e_n)_{n=1}^{\infty}$ be a basic sequence in X, and $(\widetilde{e}_n)_{n=1}^{\infty}$ be a sequence in Y. The following assertions are equivalent:

(a) $(\widetilde{e}_n)_{n=1}^{\infty}$ is a basic sequence that is equivalent to $(e_n)_{n=1}^{\infty}$.

(b) There exists an isomorphism T of $\overline{\langle \{e_n\}_{n=1}^{\infty} \rangle}$ onto $\overline{\langle \{\widetilde{e}_n\}_{n=1}^{\infty} \rangle}$ such that $Te_n = \widetilde{e}_n$ for all $n \in \mathbb{N}$.

(c) There exists $A, B > 0$ such that for all $\lambda_1, ..., \lambda_n \in \mathbb{R}$,

$$\frac{1}{A} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| \leq \left\| \sum_{i=1}^{n} \lambda_i \widetilde{e}_i \right\| \leq B \left\| \sum_{i=1}^{n} \lambda_i e_i \right\|$$

Proof.

(a) ⇒ (b) Let T be the map $\overline{\langle\{e_n\}_{n=1}^\infty\rangle}$ into $\overline{\langle\{\widetilde{e}_n\}_{n=1}^\infty\rangle}$ by

$$T\left(\sum_{n=1}^\infty \lambda_n e_n\right) = \sum_{n=1}^\infty \widetilde{e}_n$$

From the equivalence of $\{e_n\}_{n=1}^\infty$ and $\{\widetilde{e}_n\}_{n=1}^\infty$, we have that T is well-defined, one-to-one, and onto $\overline{\langle\{e_n\}_{n=1}^\infty\rangle}$. We now show that T has a closed graph. Indeed, if $x^k = \sum_{n=1}^\infty \lambda_n^k e_n$ converge to $x = \sum_{n=1}^\infty \lambda_n e_n$, and $Tx^k = \sum_{n=1}^\infty \lambda_n^k \widetilde{e}_n$ converges to $\sum_{n=1}^\infty b_n \widetilde{e}_n$, then by the continuity of the coordinate functionals, we have $a_n^k \to a_n$ and in the same way $a_n^k \to b_n$ for all $n \in \mathbb{N}$. In particular, for every $n \in \mathbb{N}$, $a_n = b_n$, and so $Tx^k \to \sum_{n=1}^\infty \lambda_n \widetilde{e}_n = Tx$. By the Closed Graph Theorem, T is continuous, and by the Open Mapping Theorem, T^{-1} is continuous as well.

(b) ⇒ (c) This follows easily with $B = \|T\|$ and $A = \|T^{-1}\|$.

(c) ⇒ (a) For $n < m$ and $\lambda_1, ..., \lambda_m$, we have

$$\left\|\sum_{i=1}^n \lambda_i \widetilde{e}_i\right\| \leq C_1 C_2 \,\mathfrak{bc}(\{e_n\}_{n=1}^\infty) \left\|\sum_{i=1}^m \lambda_i \widetilde{e}_i\right\|$$

so by Proposition 1.3.15, $\{\widetilde{e}_n\}_{n=1}^\infty$ is a basic sequence. From (iii), we have that $\sum_{n=1}^\infty \lambda_n \widetilde{e}_n$ is Cauchy if and only if $\sum_{n=1}^\infty \lambda_n e_n$ is Cauchy. ∎

The following result is sometimes called the *small perturbation lemma*.

Theorem 1.3.21: Small Perturbation Lemma

Let $\{e_n\}_{n=1}^\infty$ be a basic sequence in a Banach space X and let $\{f_n\}_{n=1}^\infty$ be the coefficient functionals of the basis $\{e_n\}_{n=1}^\infty$ of $\overline{\langle\{e_n\}_{n=1}^\infty\rangle}$. Assume that $\{\widetilde{e}_n\}_{n=1}^\infty$ is a sequence in X such that

$$\sum_{n=1}^\infty \|e_n - \widetilde{e}_n\|\|f_n\| = C < 1$$

Then the following hold

(i) $\{\widetilde{e}_n\}_{n=1}^\infty$ is a basic sequence in X equivalent to $\{e_n\}_{n=1}^\infty$.

(ii) If $\overline{\langle\{e_n\}_{n=1}^\infty\rangle}$ is complemented in X, then so is $\overline{\langle\{\widetilde{e}_n\}_{n=1}^\infty\rangle}$.

(iii) If $\{e_n\}_{n=1}^\infty$ is a Schauder basis of X, then so is $\{\widetilde{e}_n\}_{n=1}^\infty$. Moreover, let $\{\widetilde{f}_n\}_{n=1}^\infty$ be the coefficient functionals of the basis $\{\widetilde{e}_n\}_{n=1}^\infty$ of X. Then $\overline{\langle\{f_n\}_{n=1}^\infty\rangle} = \overline{\langle\{\widetilde{f}_n\}_{n=1}^\infty\rangle}$.

Proof.

We first prove the first assertion. For each $n \in \mathbb{N}$, we extend f_n to functionals on X of the same norm. For any $x \in X$, we have that

$$\sum_{n=1}^\infty \|f_n(x)(e_n - \widetilde{e}_n)\| \leq \|x\| \sum_{n=1}^\infty \|f_n\|\|e_n - \widetilde{e}_n\| = C\|x\|$$

and so the map

$$Sx = \sum_{n=1}^{\infty} f_n(x)(e_n - \widetilde{e}_n)$$

defines a bounded linear operator from X into X with $\|S\| \le C < 1$. Let $T = \mathrm{id}_X - S$. Then we have $\|x - Tx\| = \|Sx\| \le C\|x\|$, and so $\|Tx\| \ge (1-C)\|x\|$. Since $1 - C > 0$, T is an isomorphic embedding and thus, $\mathrm{Range}(T)$ is closed. We claim that $\mathrm{Range}(T) = X$. Assume for a contradiction that $\mathrm{Range}(T) \neq X$. Since $C < 1$, then there exists $x \in S_X$ such that $\mathrm{dist}(x, \mathrm{Range}(T)) > C$, which contradicts $\|x - Tx\| \le C$. So T is an isomorphism of X onto X. Then using $Te_n = \widetilde{e}_n$, we obtain that T maps $\overline{\{e_n\}_{n=1}^{\infty}}$ onto $\overline{\{\widetilde{e}_n\}_{n=1}^{\infty}}$, proving (i).

To see that the second assertion holds, note that it follows from the proof of (i) using T and Proposition 1.1.6 and the note following this proposition.

To see the last assertion, let $T : X \to X$ be the isomorphism from (i). Since $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of X, we have

$$X = \mathrm{Range}(T) = T[\overline{\{e_n\}_{n=1}^{\infty}}] = \overline{\{\widetilde{e}_n\}_{n=1}^{\infty}}$$

so $\{\widetilde{e}_n\}_{n=1}^{\infty}$ is a Schauder basis of X. For some $n \in \mathbb{N}$, denote

$$g_k = \sum_{j=1}^{k} \widetilde{f}_n(e_j) f_j$$

for $k \in \mathbb{N}$. Then $g_k \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$ so by Proposition 1.3.1, $g_k \xrightarrow{*} \widetilde{f}_n$. For $x \in B_X$, we then have

$$\widetilde{f}_n(x) = \sum_{j=1}^{\infty} \widetilde{f}_n(e_j) f_j(x)$$

so for $k \ge n$, we estimate

$$|(\widetilde{f}_n - g_k)(x)| = \left| \sum_{j=k+1}^{\infty} \widetilde{f}_n(e_j) f_j(x) \right| = \left| \sum_{j=k+1}^{\infty} \widetilde{f}_n(e_j - \widetilde{e}_j) f_j(x) \right| \le \|\widetilde{f}_n\| \sum_{j=k+1}^{\infty} \|e_j - \widetilde{e}_j\| \|f_j\| \to 0$$

as $k \to \infty$. Since the estimate is independent of $x \in B_X$, we get that $\|\widetilde{f}_n - g_k\| \to 0$ as $k \to \infty$ and thus, $\widetilde{f}_n \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$, so $\overline{\langle \{\widetilde{f}_n\}_{n=1}^{\infty} \rangle} \subset \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$. For the other inclusion, note that $(T^{-1})^*(f_n) = \widetilde{f}_n$, so

$$\|\widetilde{f}_n\| \le \|(T^{-1})^*\| \|f_n\|$$

for all $n \in \mathbb{N}$, so the series $\sum_{n=1}^{\infty} \|\widetilde{f}_n\| \|e_n - \widetilde{e}_n\|$ converges and we can reverse the roles of e_n and $\widetilde{e}_n$ in the previous part, obtaining $\overline{\langle \{\widetilde{f}_n\}_{n=1}^{\infty} \rangle} \supset \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}$. ∎

1.4 Block Bases, Bessaga-Pełczyński Selection Principle

Definition 1.4.1

Let X be a Banach space and let $\{e_n\}_{n=1}^{\infty}$ be a basic sequence in X. A sequence of nonzero vectors $\{u_j\}_{j=1}^{\infty}$ of the form

$$u_j = \sum_{i=p_j+1}^{p_{j+1}} \lambda_i e_i$$

where $\lambda_{p_j+1}, ..., \lambda_{p_{j+1}} \in \mathbb{R}$ and $p_1 < p_2 < \cdots$, is called a *block basic sequence* of $\{e_n\}_{n=1}^{\infty}$.

Note that a block basic sequence of $\{e_n\}_{n=1}^{\infty}$ is a basic sequence with basis constant not greater than $\mathfrak{bc}(\{e_n\}_{n=1}^{\infty})$. For $k \leq \ell$, see that

$$\begin{aligned}
\left\|\sum_{j=1}^{k} \alpha_j u_j\right\| &= \left\|\sum_{j=1}^{k} \alpha_j \sum_{i=p_j+1}^{p_{j+1}} \lambda_i e_i\right\| \\
&= \left\|\sum_{j=1}^{k} \sum_{i=p_j+1}^{p_{j+1}} \alpha_j \lambda_i e_i\right\| \\
&\leq \mathfrak{bc}(\{e_n\}_{n=1}^{\infty}) \left\|\sum_{j=1}^{\ell} \sum_{i=p_j+1}^{p_{j+1}} \alpha_j \lambda_i e_i\right\| \\
&= \mathfrak{bc} \left\|\sum_{j=1}^{\ell} \alpha_j u_j\right\|
\end{aligned}$$

The following result is often used when investigating subspaces of a given space.

Theorem 1.4.2: Pełczyński

Let X be a Banach space and let $\{e_n\}_{n=1}^{\infty}$ be a Schauder basis of X. If Y is an infinite-dimensional closed subspace of X, then Y contains an infinite-dimensional closed subspace Z with a Schauder basis $\{\widetilde{e}_n\}_{n=1}^{\infty}$ such that if $\{u_j\}_{j=1}^{\infty}$ is a block basic sequence of $\{e_n\}_{n=1}^{\infty}$, then $\{\widetilde{e}_n\}_{n=1}^{\infty}$ is equivalent to $\{u_j\}_{j=1}^{\infty}$.

Proof.

Let $K = \mathfrak{bc}(\{e_n\}_{n=1}^{\infty})$ and for $p \in \mathbb{N}$, define W_p to be the finite codimensional subspace of X defined as

$$W_p = \left\{ x \in X : x = \sum_{i=p+1}^{\infty} \lambda_i e_i \right\} = \overline{\langle \{e_i\}_{i=p+1}^{\infty} \rangle}$$

Then $W_p \cap Y$ is infinite-dimensional and since W_p is of finite codimension, there exists a nontrivial element $y \in S_Y \cap W_p$. The idea now is to construct the two equivalent basic sequences by induction.

- Fix $y_1 = \sum_{n=1}^{\infty} a_n^{(1)} e_n \in Y$ with $\|y_1\| = 1$. Then choose $p_1 \in \mathbb{N}$ such that for
$$u_1 = \sum_{n=1}^{p_1} a_n^{(1)} e_n \in X$$
we have $\|y_1 - u_1\| < \frac{1}{4K}$.

- Fix $y_2 = \sum_{n=!}^{\infty} a_n^{(2)} e_n \in S_Y \cap W_{p_1}$. Then choose $p_2 \in \mathbb{N}$ such that for
$$u_2 = \sum_{n=p_1+1}^{p_2} a_n^{(2)} e_n$$
we have $\|y_2 - u_2\| < \frac{1}{8K}$.

- Continue inductively, so that for $k \in \mathbb{N}$, fix $y_k = \sum_{n=1}^{\infty} a_n^{(k)} e_n \in S_Y \cap W_{p_{k-1}}$. Then choose $p_k \in \mathbb{N}$ such that for
$$u_k = \sum_{n=p_{k-1}+1}^{p_k} a_n^{(k)} e_n$$
we have $\|y_k - u_k\| < \frac{1}{2^{k+1}K}$.

Once we have constructed such a sequence, we obtain that $\{u_j\}_{j=1}^{\infty}$ is a block basic sequence of $\{e_n\}_{n=1}^{\infty}$. Then since
$$\sum_{j=1}^{\infty} \|y_j - u_j\| < \sum_{j=1}^{\infty} \frac{1}{2^{j+1}K} = \frac{1}{2K}$$
and also, if f_j denotes the coordinate functionals of $\{u_j\}_{j=1}^{\infty}$, we have by Theorem 1.3.21,
$$\|f_j\| \leq 2\,\mathfrak{bc}(\{u_j\}_{j=1}^{\infty}) \leq 2K$$
then $\{y_n\}_{n=1}^{\infty}$ is a basic sequence of Y equivalent to $\{u_j\}_{j=1}^{\infty}$, so $Z = \overline{\langle \{y_n\}_{n=1}^{\infty} \rangle}$ has the desired property. ■

The above procedure of finding vectors with almost successive supports is called the *sliding hump argument*. An easy modification provides the following.

Corollary 1.4.3: Bessaga-Pełczyński Selection Principle

Let X be a Banach space and let $\{e_n\}_{n=1}^{\infty}$ be a Schauder basis of X. Let $\{u_j\}_{j=1}^{\infty}$ be its corresponding block basic sequence. If $(x_n)_{n=1}^{\infty}$ is a sequence such that

(i) $\inf_{n\in\mathbb{N}} \|x_n\| > 0$

(ii) $x_n \rightharpoonup 0$

then some subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ is a basic sequence equivalent to $\{u_j\}_{j=1}^{\infty}$.

Theorem 1.4.4: Johnson-Rosenthal

Let X be a separable Banach space. Then X has a quotient with a Schauder basis.

Proof.

Here, $\overline{\langle L\rangle}^{\|\cdot\|}$ will denote the norm-closed linear span of L. Let $(y_k)_{k=1}^{\infty}$ be a sequence in S_{X^*} such that $y_k \xrightarrow{*} 0$. Let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence of positive numbers less than 1 such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, and thus, $\prod_{n=1}^{\infty}(1-\varepsilon_n)^{-1} < \infty$.

By the compactness of the unit ball on a finite-dimensional space, and using the fact that X is separable, let $(k_n)_{n=1}^{\infty}$ be an increasing sequence of positive numbers and let $(F_n)_{n=1}^{\infty}$ be finite subsets of S_X such that $F_1 \subset F_2 \subset \cdots$ and $\langle \bigcup_{n=1}^{\infty} F_n\rangle$ is dense in X, and that for each $n \in \mathbb{N}$,

(i) For each $f \in \overline{\langle \{y_{k_n}\}_{n=1}^{\infty}\rangle}^{\|\cdot\|*}$ with $\|f\| = 1$, there exists an $x \in F_n$ such that

$$|y(x) - f(x)| < \frac{\varepsilon_n \|y\|}{3}$$

for all $y \in \overline{\langle \{y_{k_n}\}_{n=1}^{\infty}\rangle}^{\|\cdot\|}$.

(ii) $|y_{k_{n+1}}(x)| < \frac{\varepsilon_n}{3}$ for all $x \in F_n$.

We want to show that $(y_{k_n})_{n=1}^{\infty}$ is a basic sequence, i.e. $(y_{k_n})_{n=1}^{\infty}$ is a Schauder basis of $\overline{\langle \{y_{k_n}\}_{n=1}^{\infty}\rangle}$. For it, fix $n \in \mathbb{N}$, and let $\lambda_1, ..., \lambda_n \in \mathbb{R}$ be such that

$$\left\| \sum_{i=1}^{n} \lambda_i y_{k_i} \right\| = 1$$

Choose $f \in \overline{\langle \{y_{k_n}\}_{n=1}^{\infty}\rangle}^{\|\cdot\|*}$ such that

$$f\left(\sum_{i=1}^{n} \lambda_i y_{k_i}\right) = \|f\| = 1$$

and choose $x \in F_n$ such that (i) is satisfied for such an f.

Then we have that

$$\left| \left(\sum_{i=1}^{n} \lambda_i y_{k_i}\right)(x) \right| \geq 1 - \frac{\varepsilon_n}{3}$$

So for any $\lambda \in \mathbb{R}$, we have two possible cases:

- Case 1: If $\lambda \leq 2$, we have that

$$\begin{aligned} \left\| \sum_{i=1}^{n} \lambda_i y_{k_i} + \lambda y_{k_{n+1}} \right\| &\geq \left| \sum_{i=1}^{n} \lambda_i y_{k_i}(x) + \lambda y_{k_{n+1}}(x) \right| \\ &\geq 1 - \frac{\varepsilon_n}{3} - \lambda \frac{\varepsilon_n}{3} \\ &\geq 1 - \frac{\varepsilon_n}{3} - 2\frac{\varepsilon_n}{3} \end{aligned}$$

$$= 1 - \varepsilon_n$$

- Case 2: If $\lambda > 2$, then

$$\left\| \sum_{i=1}^{n} \lambda_i y_{k_i} + \lambda y_{k_{n+1}} \right\| \geq \left| \sum_{i=1}^{n} \lambda_i y_{k_i}(x) + \lambda y_{k_{n+1}}(x) \right| \geq 1$$

In particular, we obtain that

$$\left\| \sum_{i=1}^{n} \lambda_i y_{k_i} \right\| \leq \frac{1}{1-\varepsilon_n} \left\| \sum_{i=1}^{n+1} \lambda_i y_{k_i} \right\|$$

holds for any $\lambda_1, ..., \lambda_{n+1} \in \mathbb{R}$. By induction, we have that for any $k \in \mathbb{N}$ and $\lambda_1, ..., \lambda_{n+k}$,

$$\left\| \sum_{i=1}^{n} \lambda_i y_{k_i} \right\| \leq \left(\prod_{j=1}^{n+k-1} \frac{1}{1-\varepsilon_n} \right) \left\| \sum_{i=1}^{n+k} \lambda_i y_{k_i} \right\|$$

and so $(y_{k_n})_{n=1}^{\infty}$ is a basic sequence.

Let $(f_n)_{n=1}^{\infty}$ denote the functionals in $\overline{\langle \{y_{k_n}\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$ biorthogonal to $(y_{k_n})_{n=1}^{\infty}$, that is, for all $1 \leq i, j$, we have $f_i(y_{k_j}) = \delta_{ij}$. Define the projections $P_n : \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|} \to \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$ by

$$P_m f = \sum_{i=1}^{m} f(y_{k_i}) f_i$$

for $f \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$. Then we have that

$$\|P_m\| \leq \prod_{n=m}^{\infty} \frac{1}{1-\varepsilon_n} < \infty$$

and as $\varepsilon_n \to 0$, we have that $\|P_m\| \to 1$, which shows that $(f_n)_{n=1}^{\infty}$ is a Schauder basis of $\overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$.

To complete the proof, we need to show that $\overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$ is a quotient of X, that is, we seek a bounded linear operator T from X onto $\overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$. For that, let $T : X \to \overline{\langle \{y_{k_n}\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$ be defined as follows: for $x \in X$,

$$(Tx)(y) = y(x)$$

for $y \in \overline{\langle \{y_{n_k}\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$. We need to show that $\text{Range}(T) = \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$. Indeed, we show by double inclusion. For some $n \in \mathbb{N}$, let $x \in F_n$ so that by (ii), we have

$$\sum_{i=1}^{\infty} |y_{k_i}(x)| < \infty$$

and so

$$(Tx)(y) = \sum_{i=1}^{\infty} y_{k_i}(x) f_i \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$$

Since the $\langle \bigcup_{n=1}^{\infty} F_n \rangle$ is dense in X and T is a bounded linear operator, we have Range$(T) \subset \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$.

Next, we require the following claim:

Claim: For every $g \in \langle \{f_n\}_{n=1}^{\infty} \rangle \cap S_{X^*}$, and for any $\varepsilon > 0$, there exists an $x \in S_X$ such that $\|Tx - g\| < 4\varepsilon$.

Given the claim, by the Open Mapping Theorem, we obtain Range$(T) = \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$, and we are done. To prove the claim, let $0 < \varepsilon < 1$ and let $N \in \mathbb{N}$ be such that $\sum_{j=n}^{\infty} \varepsilon_j < \varepsilon$ and $\|P_n\| \leq 1 + \varepsilon$ for all $n \geq N + 1$. Fix $n \geq N + 1$ and define $\|f\|_1 = \|f|_{\overline{\langle \{y_{k_n}\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}}\|$ for $f \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$. Observe that $\|f\|_1 \leq \|f\| \leq \|P_n\| \|f\|_1 \leq 2\|f\|_1$ for all $f \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|}$.

Now fix $g \in \overline{\langle \{f_n\}_{n=1}^{\infty} \rangle}^{\|\cdot\|} \cap S_{X^*}$ and put $f = \frac{g}{\|g\|_1}$. Choose $x \in F_n$ such that (i) is satisfied for such an f. Then by (i)

$$\left\| \sum_{j=1}^{n} y_{k_j}(x) f_j - f \right\|_1 \leq \frac{\varepsilon_n}{3}$$

and so we have

$$\left\| \sum_{j=1}^{n} y_{k_i}(x) f_j - f \right\| \leq \frac{2\varepsilon_n}{3} < \frac{2\varepsilon}{3}$$

Moreover, since $\|f_j\| = \|P_j - P_{j-1}\| \leq 4$ for all $j \geq n + 1$, by (ii),

$$\left\| \sum_{j=n+1}^{\infty} y_{k_j}(x) f_j \right\| \leq 4 \sum_{j=n}^{\infty} \frac{\varepsilon_j}{3} < \frac{4\varepsilon}{3}$$

Then

$$\|Tx - f\| \leq \left\| \sum_{j=n+1}^{\infty} y_{k_j}(x) f_j \right\| + \left\| \sum_{j=1}^{n} y_{k_i}(x) f_j - f \right\| \leq \frac{4\varepsilon}{3} + \frac{2\varepsilon}{3} = 2\varepsilon$$

Moreover, since as above,

$$1 = \|g\| \leq \|P_n\| \|g\|_1 \leq (1 + \varepsilon) \|g\|_1$$

so $\|g\|_1 \geq \frac{1}{1+\varepsilon}$, and

$$\|f - g\| = \left\| \frac{g}{\|g\|_1} - g \right\| = \|g\| \left(\frac{1}{\|g\|_1} - 1 \right) \leq \varepsilon$$

Therefore,

$$\|Tx - g\| \leq \|Tx - f\| + \|f - g\| \leq 2\varepsilon + \varepsilon = 3\varepsilon$$

This completes the proof of the Theorem. ■

Corollary 1.4.5: Johnson-Rosenthal

Let X be a Banach space such that X^* is separable, and assume that Y is an infinite dimensional subspace of X^* with separable dual Y^*. Then Y has an infinite-dimensional reflexive subspace.

Proof.

By Mazur's Theorem (Theorem 1.3.16) (ii), by assumption, since Y^* is separable, then Y contains a shrinking basic sequence given by $(y_k)_{k=1}^\infty$ of S_Y. Then since Y^* is separable, we also have that $y_k \rightharpoonup 0$ in Y and, thus, $y_k \rightharpoonup 0$ in X^*, meaning that $y_k \overset{*}{\rightharpoonup} 0$ in X^*. By the proof of Theorem 1.4.4 and using Corollary 1.4.3, there exists a subsequence $(y_{k_n})_{n=1}^\infty$ of $(y_k)_{k=1}^\infty$ that is a boundedly complete basic sequence, that is,

$$\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n \lambda_i y_{k_i} \right\| < \infty \implies \sum_{n=1}^\infty \lambda_n y_{k_n} \text{ converges}$$

and since $(y_k)_{k=1}^\infty$ is a shrinking basic sequence, then so is $(y_{k_n})_{n=1}^\infty$. Therefore, $\overline{\langle \{y_{k_n}\}_{n=1}^\infty \rangle}^{\|\cdot\|}$ is reflexive by Theorem 1.3.11. ∎

Corollary 1.4.6: Johnson-Rosenthal

Assume that X is an infinite-dimensional Banach space such that X^{**} is separable. The following hold.

(i) Every infinite-dimensional subspace of X contains an infinite-dimensional reflexive subspace.

(ii) Every infinite-dimensional subspace of X^* contains an infinite-dimensional reflexive subspace.

Proof.

Note that (ii) follows from Corollary 1.4.5. To prove (i), assume that Y is an infinite-dimensional subspace of X. Then Y is also an infinite-dimensional subspace of X^{**}. Moreover, Y^* is separable since X^* is separable, so by Corollary 1.4.5, Y contains an infinite-dimensional reflexive subspace. ∎

For the remainder of this section, we will introduce the notion of finite-dimensional decomposition (FDD for short).

Definition 1.4.7

Let X be a Banach space. A collection of finite-dimensional subspace $\{X_n\}_{n=1}^\infty$ is a finite-dimensional decomposition of X, if every $x \in X$ has a unique representation of the form

$$x = \sum_{n=1}^{\infty} x_n$$

with $x_n \in X_n$ for every $n \in \mathbb{N}$.

In general when we consider vector spaces, we had $X = Y + Z$ for some subspaces Y and Z of X. In a similar manner, when X is a Banach space, we can also say that $\{X_n\}_{n=1}^{\infty}$ is an FDD if $X = \sum_{n=1}^{\infty} X_n$, where

$$\sum_{n=1}^{\infty} X_n = \left\{ \sum_{n=1}^{\infty} x_n : x_n \in X_n \text{ for all } n \in \mathbb{N} \right\}$$

Proposition 1.4.8

If $\dim(X_n) = 1$ for every $n \in \mathbb{N}$, that is, if $X_n = \langle x_n \rangle$ for some $x_n \in X_n$, then the following are equivalent:

(a) $\{X_n\}_{n=1}^{\infty}$ is an FDD for a Banach space X.

(b) $(x_n)_{n=1}^{\infty}$ is a Schauder basis of X.

Proof.

Indeed, from the above, if $\dim(X_n) = \langle x_n \rangle$ for each $n \in \mathbb{N}$, then the above implies that

$$X = \sum_{n=1}^{\infty} X_n = \sum_{n=1}^{\infty} \langle x_n \rangle$$

Which gives us the equivalence immediately. ■

If $\{X_n\}_{n=1}^{\infty}$ is an FDD for a Banach space X, define projections P_n on X by

$$P_n \left(\sum_{i=1}^{\infty} x_i \right) = \sum_{i=1}^{n} x_i$$

The following result is the FDD version of Lemma 1.2.3.

Proposition 1.4.9

(i) If $\{X_n\}_{n=1}^{\infty}$ is an FDD for a Banach space X, then

$$\sup_{n \in \mathbb{N}} \|P_n\| < \infty$$

(ii) If $(P_n)_{n=1}^{\infty}$ is a sequence of finite rank projections on X such that

(a) $P_n P_m = P_{\min(m,n)}$

(b) $\lim_{n\to\infty} P_n x = x$ for all $x \in X$

then $\{X_n\}_{n=1}^{\infty}$ determines a unique FDD on X by putting $X_1 = P_1$ and $X_n = (P_n - P_{n-1})(X)$ for $n \geq 2$.

Definition 1.4.10

An FDD $\{X_n\}_{n=1}^{\infty}$ of a Banach space X is called *shrinking* if the associated projections satisfy

$$\lim_{n\to\infty} \|P_n^* f - f\| = 0$$

for all $f \in X^*$.

The following definition will be used to prove the following theorem. We introduce the definition of Markushevich bases.

Definition 1.4.11

Let X be a Banach space. A biorthogonal system $\{x_i; f_i\}_{i\in I}$ in X is called a *Markushevich basis of* X if $X = \overline{\langle \{x_i\}_{i\in I}\rangle}$ and $\{f_i\}_{i\in I}$ separates the points in X. A Markushevich basis $\{x_i, f_i\}_{i\in I}$ is called *shrinking* if $\overline{\langle \{f_i\}_{i\in I}\rangle} = X^*$.

Sometimes, a Markushevich basis $\{x_i, f_i\}_{i\in I}$ will be denoted by $\{x_i\}_{i\in I}$ if the set of functional coefficients is understood.

Clearly, every Schauder basis of a Banach space X is a Markushevich basis of X. An example of a Markushevich basis that is not a Schauder basis is the sequence of trigonometric polynomials $\{e^{2\pi int}\}_{n\in\mathbb{Z}}$ in the space $\widetilde{C}([0,1])$ of complex continuous functions on $[0,1]$ whose values at 0 and 1 are equal, with the sup-norm.

Theorem 1.4.12: Johnson-Rosenthal

(i) If X is a separable Banach space, then there exists a subspace Y of X such that Y and X/Y have an FDD.

(ii) If X^* is separable, then Y may be chosen so that both Y and X/Y have a shrinking FDD.

Proof.

(i) Let $\{x_n; f_n\}$ be a 1-norming Markushevich basis for X, i.e. a biorthogonal system with $X = \overline{\langle \{x_n\}_{n=1}^{\infty}\rangle}$ and $\overline{\langle \{f_n\}_{n=1}^{\infty}\rangle}$ a 1-norming subspace of X^*. Let $\sigma_1 \subset \sigma_2 \subset \cdots$ and $\Delta_1 \subset \Delta_2 \subset \cdots$ be finite subsets such that $\sigma = \bigcup_{n=1}^{\infty} \sigma_n$ and $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$ are complementary infinite subsets of $\mathbb{N}$ and that for each $n \in \mathbb{N}$,

(i) If $f \in \overline{\langle \{f_i\}_{i\in\Delta_n}\rangle}$, there exists an $x \in \overline{\langle \{x_i\}_{i\in\Delta_n\cup\sigma_{n+1}}\rangle}$ so that $\|x\| = 1$ and

$$|f(x)| > \left(1 - \frac{1}{n+1}\right)\|f\|$$

(ii) If $x \in \overline{\langle \{x_i\}_{i\in\sigma_n}\rangle}$, there exists an $f \in \overline{\langle \{f_i\}_{i\in\sigma_n\cup\Delta_n}\rangle}$ such that $\|f\| = 1$ and

$$|f(x)| > \left(1 - \frac{1}{n+1}\right)\|x\|$$

For $n \in \mathbb{N}$, let $S_n : X \to X$ and $T_n : X \to X$ be defined by

$$S_n x = \sum_{i\in\sigma_n} f_i(x)x_i \qquad T_n x = \sum_{i\in\Delta_n} f_i(x)x_i$$

Then for $n \in \mathbb{N}$,

$$\|T_n^*|_{\{x_i : i\in\sigma_{n+1}\}^\perp}\| \le 1 + \frac{1}{n} \qquad \|S_n^*|_{\{f_i : i\in\Delta_n\}} \le 1 + \frac{1}{n}$$

Indeed, to see the first inequality, suppose $y \in \{x_i\}^\perp_{i\in\sigma_{n+1}}$. Let $x \in \overline{\langle \{x_i\}_{i\in\Delta_n\cup\sigma_{n+1}}\rangle}$ so that $\|x\| = 1$ and

$$\|T_n^* y(x)\| \ge \left(1 - \frac{1}{n+1}\right)\|T_n^* y\|$$

Since $y \in \{x_i\}^\perp_{i\in\sigma_{n+1}}$, $y(x) = T_n^* y(x)$ and so

$$\|y\| \ge |y(x)| \ge \left(1 - \frac{1}{n+1}\right)\|T_n^* y\|$$

and so

$$\|T_n^* y\| \le \left(1 - \frac{1}{n}\right)\|y\|$$

The other inequality follows similarly. Let $Y = \overline{\langle \{x_i\}_{i\in\sigma}\rangle}$. We claim that

$$Y^\perp = \overline{\langle \{f_i\}_{i\in\Delta}\rangle}^{w^*}$$

The only nontrivial conclusion is $Y^\perp \subset \overline{\langle \{f_i\}_{i\in\Delta}\rangle}^{w^*}$. Indeed, let $y \in \{x_i\}^\perp_{i\in\sigma}$. It suffices to show that $T_n^* y \xrightarrow{*} f$ for some $f \in X^*$. Then $T_n^* f = T_n^* y$ for every $n \in \mathbb{N}$ and so $f - y \in \{x_i\}^\perp_{i\in\Delta_n}$ for every $n \in \mathbb{N}$. Thus, $f - y \in \{x_i\}^\perp_{i\in\Delta}$. However, $f - y \in \{x_i\}^\perp_{i\in\sigma}$, so $y = f$ and $T_n^* y \xrightarrow{*} y$. Hence, $Y \subset \overline{\langle \{f_i\}_{i\in\Delta}\rangle}^{w^*}$. Now,

$$\overline{\langle \{x_i\}_{i\in\sigma}\rangle} = \{x_i : i \in \sigma\}^{\perp\top} = (\overline{\langle \{f_i\}_{i\in\Delta}\rangle}^{w^*})^\top = \{f_i\}^\top_{i\in\Delta}$$

We have

$$(X/Y)^* = Y^\perp = \overline{\langle \{f_i\}_{i\in\Delta}\rangle}^{w^*}$$

and $(T_n^*)_{n=1}^\infty$ produce by duality an FDD for X/Y. The sequence of projections $(S_n|_Y)_{n=1}^\infty$ then in turn produce an FDD for Y.

(ii) If X^* is separable and the dual norm is "locally uniformly rotund", which is shared by subspaces and quotients of X, we get there exists a shrinking FDD both in Y and X/Y. ■

The term "locally uniformly rotund" is mentioned below as the following definition.

Definition 1.4.13

The norm $\|\cdot\|$ of a Banach space X is called *locally uniformly rotund* (LUR) if for all $x, x_n \in X$ such that

$$\lim_{n\to\infty}(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0$$

we have

$$\lim_{n\to\infty}\|x_n - x\| = 0$$

If this is the case, we also say that X is a locally uniformly rotund Banach space.

1.5 Unconditional Bases

Recall that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space is *unconditionally convergent* if $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges for all choices of signs $\varepsilon_n = \pm 1$.

Definition 1.5.1

A Schauder basis $\{e_n\}_{n=1}^{\infty}$ of a Banach space X is said to be *unconditional* if for every $x \in X$, its expansion $\sum_{n=1}^{\infty} \lambda_n e_n$ converges unconditionally. A sequence $\{e_n\}_{n=1}^{\infty}$ in a Banach space X is called an *unconditional basic sequence* if it is a unconditional basis of $\overline{\langle\{e_n\}_{n=1}^{\infty}\rangle}$.

Example 1.5.2

For $1 \le p < \infty$, consider $\ell_p(\mathbb{N})$ with Schauder basis given by $\{e_n\}_{n=1}^{\infty}$ where for each $n \in \mathbb{N}$,

$$e_n = (0, 0, 0, ..., \underset{\substack{\uparrow \\ n\text{th}}}{1}, 0, ...)$$

We claim that $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis for $\ell_p(\mathbb{N})$. Indeed, fix $x \in \ell_p(\mathbb{N})$. Then there are scalars $(\lambda_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$. We check that $\sum_{n=1}^{\infty} \varepsilon_n \lambda_n e_n$ converges for all choices $\varepsilon_n = \pm 1$. In particular, we check that the sequence of partial sums is Cauchy. Observe that because $x \in \ell_p(\mathbb{N})$,

$$\|x\|_p^p = \left\|\sum_{n=1}^{\infty} \lambda_n e_n\right\|_p^p = \sum_{i=1}^{\infty}\left|\sum_{n=1}^{\infty} \lambda_n e_n(i)\right| = \sum_{i=1}^{n} |\lambda_i|^p < \infty$$

so for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x\|_p^p = \sum_{i=N+1}^{\infty} |\lambda_i|^p < \delta^p$$

Now, for all $n > m \geq N$, we have

$$\left\|\sum_{i=1}^{n} \varepsilon_i \lambda_i e_i - \sum_{i=1}^{m} \varepsilon_i \lambda_i e_i\right\|_p^p = \left\|\sum_{i=m+1}^{n} \varepsilon_i \lambda_i e_i\right\|_p^p = \sum_{i=m+1}^{n} |\lambda_i|^p \leq \sum_{i=m+1}^{\infty} |\lambda_i|^p < \delta^p$$

which shows that the sequence of partial sums are Cauchy, so $\sum_{n=1}^{\infty} \varepsilon_n \lambda_n e_n$ converges, i.e. $\sum_{n=1}^{\infty} \lambda_n e_n$ converges unconditionally.

Example 1.5.3

Let $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{O}$ is an orthonormal basis for $\mathcal{H}$, then $\mathcal{O}$ is also an unconditional basis.
Indeed, if $\mathcal{O} = \{e_\gamma\}_{\gamma \in \Gamma}$ is an orthonormal basis and $\mathcal{H}$ is separable, then for every $x \in \mathcal{H}$, we get

$$x = \sum_{\gamma \in \Gamma} \langle x, e_\gamma \rangle e_\gamma$$

Apply the same technique as in Example 1.5.2 to get that $\mathcal{O}$ is an unconditional basis, i.e. if $F \subset \Gamma$ is a countable set, say $F = \{e_{\gamma_n}\}_{n=1}^{\infty}$, then since $\mathcal{O}$ on F is unconditional, so is $\mathcal{O}$ on Γ.

Remark 1.5.4

Every basis equivalent to an unconditional basis is also unconditional, as the following example will show.

Example 1.5.5

Let $\{e_n\}_{n=1}^{\infty}$ be the usual unit vector basis for $c_0(\mathbb{N})$. For each $n \in \mathbb{N}$, let $x_n = \sum_{i=1}^{n} e_i$. We check that if $x = \sum_{n=1}^{\infty} \lambda_n e_n$, then $x = \sum_{n=1}^{\infty} \mu_n x_n$, where $\mu_n = \lambda_n - \lambda_{n+1}$. Note that $\sum_{n=1}^{\infty} \mu_n = \lambda_1$, so $\{\mu_n\}_{n=1}^{\infty}$ forms a convergent series. Also,

$$\|x\|_\infty = \sup_{k \in \mathbb{N}} \left|\sum_{n=k}^{\infty} \mu_n\right|$$

On the other hand, for every convergent series $\sum_{n=1}^{\infty} \mu_n$, we have

$$x = \sum_{i=1}^{\infty} \mu_i x_i = \sum_{i=1}^{n} \left(\sum_{n=1}^{\infty} \mu_n\right) e_i \in c_0(\mathbb{N})$$

In conclusion, $(c_0(\mathbb{N}), \|\cdot\|_\infty) \equiv (\mathcal{S}, \|\cdot\|)$, where

$$\mathcal{S} = \left\{ \mu : \|\mu\| = \sup_{k\in\mathbb{N}} \left| \sum_{n=k}^{\infty} \mu_k \right| \right\}$$

The basis $\{x_n\}_{n=1}^\infty$ is called the *summing basis for* $c_0(\mathbb{N})$.

Before proving the following result, we introduce a lemma that will be equivalent to statements being an unconditional basis.

Lemma 1.5.6

There exists an $L > 0$ such that for all $\lambda_1, ..., \lambda_m$ and $\sigma \subset \{1, 2, ..., m\}$,

$$\left\| \sum_{i\in\sigma} \lambda_i e_i \right\| \leq L \left\| \sum_{i=1}^{m} \lambda_i e_i \right\|$$

In particular, we can extend $\{1, 2, ..., m\}$ to $\mathbb{N}$ so

$$\left\| \sum_{i\in\sigma} \lambda_i e_i \right\| \leq L \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|$$

Proof.

Let σ be any subset of $\mathbb{N}$ (possibly infinite). Then by assumption, $\sum_{i\in\sigma} \lambda_i e_i$ is Cauchy. Indeed, for $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m > n \geq N$,

$$\left\| \sum_{i=n+1}^{m} \lambda_i e_i \right\| \leq \frac{\varepsilon}{L}$$

Let $\tau = \sigma \cap \{n, n+1, ..., m\}$, let $\mu_i = \lambda_i$ for all $n \leq i \leq m$, and $\mu_i = 0$ otherwise. By assumption,

$$\left\| \sum_{\substack{i\in\sigma \\ i=n}}^{m} \lambda_i e_i \right\| < \varepsilon$$

Therefore, $\sum_{i\in\sigma} \lambda_i e_i$ is convergent. Passing to the limit, we have proved the desired result. ∎

Proposition 1.5.7

Let $\{e_i\}_{i=1}^\infty$ be a sequence in a Banach space X. The following are equivalent:

(a) $\{e_i\}_{i=1}^\infty$ is an unconditional basic sequence.

(b) There exists a constant $K > 0$ such that for all $\lambda_1, ..., \lambda_m \in \mathbb{R}$ and signs $\varepsilon_n = \pm 1$,

$$\left\| \sum_{i=1}^{m} \varepsilon \lambda_i e_i \right\| \leq K \left\| \sum_{i=1}^{m} \lambda_i e_i \right\|$$

(c) Lemma 1.5.6.

Proof.

We will prove the following directions as follows:

$$\begin{tikzcd} & (a) \arrow[dr, leftrightarrow] & \\ (b) \arrow[ur] \arrow[rr, leftrightarrow] & & (c) \end{tikzcd}$$

That is, we will show (a) ⇒ (c), (c) ⇒ (b), (b) ⇒ (c), and combine (b) and (c) to prove (a).

(a) ⇒ (c) Let $Y = \overline{\langle \{e_i\}_{i=1}^{\infty} \rangle}$. Let $\sigma \subset \mathbb{N}$, and define P_σ from Y into Y by

$$P_\sigma(x) = \sum_{i \in \sigma} \lambda e_i$$

for $x = \sum_{i=1}^{\infty} \lambda_i e_i$. Note that the operator P_σ is well-defined since $\sum_{i \in \sigma} \lambda_i e_i$ converges whenever $\sum_{i=1}^{\infty} \lambda_i e_i$ converges. We now check that P_σ has a closed graph. Indeed, let $x^{(k)} \to x$ in Y for $x^{(k)} = \sum_{i=1}^{\infty} \lambda_i^{(k)} e_i$, $x = \sum_{i=1}^{\infty} \lambda_i e_i$, and $P_\sigma(x^{(k)}) = \sum_{i \in \sigma} \lambda_i^{(k)} e_i \to y = \sum_{i=1}^{\infty} \mu_i e_i$. From the continuity of the biorthogonal functions in Schauder bases, we have $\lambda_i^{(k)} \to \lambda_i$ for every i, and for the same reason, $\lambda_i^{(k)} \to \mu_i$ for every i. Thus, $\mu_i = \lambda_i$ for every i and hence, $P_\sigma(x) = y$, meaning that P_σ has a closed graph, and is thus, continuous.

Consider the family of operators $\{P_\sigma\}_{\sigma \in \Sigma}$ with $\sigma \subset \mathcal{P}(\mathbb{N})$. We claim that for every $x = \sum_{i=1}^{\infty} \lambda_i e_i \in X$, the family $\{P_\sigma(x)\}_{\sigma \in \Sigma}$ is bounded. Indeed, from the unconditionality, we get that for $\varepsilon > 0$, there exists a finite $F \subset \mathbb{N}$ such that

$$\left\| \sum_{i \in A} \lambda_i e_i \right\| < \varepsilon$$

whenever $A \cap F = \emptyset$. From this, the boundedness of $\{P_\sigma(x)\}_{\sigma \in \Sigma}$ for every $x \in X$ follows. Now, by the Uniform Boundedness Principle (Banach-Steinhaus), $\{P_\sigma\}$ are uniformly bounded by some L, proving (ii).

(c) ⇒ (b) Let $\lambda_1, ..., \lambda_m \in \mathbb{R}$ and signs $\varepsilon_i = \pm 1$, we define $\sigma = \{i : \varepsilon_i = 1\}$ and $\tau = \{1, 2, ..., m\} \setminus \sigma$. Then

$$\begin{aligned} \left\| \sum_{i=1}^{m} \varepsilon_i \lambda_i e_i \right\| &= \left\| \sum_{i \in \sigma} \lambda_i e_i - \sum_{i \in \tau} \lambda_i e_i \right\| \\ &\leq \left\| \sum_{i \in \sigma} \lambda_i e_i \right\| + \left\| \sum_{i \in \tau} \lambda_i e_i \right\| \end{aligned}$$

$$\leq 2L \left\| \sum_{i=1}^{m} \lambda_i e_i \right\|$$

Thereby proving (b).

(b) $\Rightarrow$ (c) Let $\lambda_1, ..., \lambda_m \in \mathbb{R}$ and $\sigma \subset \{1, 2, ..., m\}$, we define $\varepsilon_i = 1$ if $i \in \sigma$ and $\varepsilon_i = -1$ for $i \in \{1, 2, ..., m\} \setminus \sigma$. Then

$$\begin{aligned} \left\| \sum_{i\in\sigma} \lambda_i e_i \right\| &= \frac{1}{2} \left\| \sum_{i=1}^{m} (\varepsilon_i \lambda_i e_i + \lambda_i e_i) \right\| \\ &\leq \frac{1}{2} \left\| \sum_{i=1}^{m} \varepsilon_i \lambda_i e_i \right\| + \frac{1}{2} \left\| \sum_{i=1}^{m} \lambda_i e_i \right\| \\ &\leq K \left\| \sum_{i=1}^{m} \lambda_i e_i \right\| \end{aligned}$$

Thereby proving (c).

(b) $\cap$ (c) $\Rightarrow$ (a) Assume that $n < m$ and let $\lambda_1, ..., \lambda_m \in \mathbb{R}$. We use (iii) with $\sigma = \{1, 2, ..., m\}$ to see that $\{e_i\}_{i=1}^{\infty}$ is a basic sequence with basis constant $\mathfrak{bc}(\{e_i\}_{i=1}^{\infty}) \leq L$.

Now let $\sum_{i=1}^{\infty} \lambda_i e_i$ be a convergent series. If $\varepsilon_i = \pm 1$, by (ii), we show that

$$\left\| \sum_{i=n}^{m} \varepsilon_i \lambda_i e_i \right\| \leq 2K \left\| \sum_{i=n}^{m} \lambda_i e_i \right\|$$

Thus, $\sum_{i=1}^{\infty} \varepsilon_i \lambda_i e_i$ is Cauchy, hence convergent. This shows that $\sum_{i=1}^{\infty} \lambda_i e_i$ converges unconditionally. ■

The best possible constant K from Proposition 1.5.7 (ii) is called the *unconditional basis constant* of $\{e_i\}_{i=1}^{\infty}$ and is denoted by $\mathfrak{ubc}(\{e_i\}_{i=1}^{\infty})$. In the proof above, we have shown that $L \leq K$ and $\mathfrak{bc}(\{e_i\}_{i=1}^{\infty}) \leq \mathfrak{ubc}(\{e_i\}_{i=1}^{\infty})$.

A natural question is whether every Banach space contains an unconditional basic sequence. This long-standing problem was answered in the negative by Gowers and Maurey. Recall that a Schauder basis is called *boundedly complete* if $\sum_{i=1}^{\infty} \lambda_n e_n$ converges whenever $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^{n} \lambda_i e_i\| < \infty$.

Theorem 1.5.8: James

Let X be a separable Banach space. If X has an unconditional Schauder basis that is not boundedly complete, then X contains an isomorphic copy of $c_0(\mathbb{N})$.

In the proof, we use the following statement.

Lemma 1.5.9

Let $\{e_i\}_{i=1}^{\infty}$ be an unconditional basic sequence in a Banach space X. Then for all scalars

$(\lambda_i)_{i=1}^\infty$ such that $\sum_{i=1}^\infty \lambda_i e_i$ converges and all bounded sequences of scalars $(\mu_i)_{i=1}^\infty$, we have

$$\left\|\sum_{i=1}^{\infty} \mu_i \lambda_i e_i\right\| \leq \mathfrak{ubc}(\{e_i\}_{i=1}^\infty) \left(\sup_{i\in\mathbb{N}} |\mu_i|\right) \left\|\sum_{i=1}^{\infty} \lambda_i e_i\right\|$$

Proof.

Fix $m \in \mathbb{N}$ and let $f \in S_{X^*}$ be defined by

$$f\left(\sum_{i=1}^{m} \mu_i \lambda_i e_i\right) = \left\|\sum_{i=1}^{m} \mu_i \lambda_i e_i\right\|$$

and define ε_i by $\varepsilon_i = 1$ if $\lambda_i f(e_i) \geq 0$ and $\varepsilon_i = -1$ if $\lambda_i f(e_i) < 0$. Then

$$\begin{aligned} f\left(\sum_{i=1}^{m} \mu_i \lambda_i e_i\right) = \left\|\sum_{i=1}^{m} \mu_i \lambda_i e_i\right\| &\leq \sum_{i=1}^{m} \|\mu_i \lambda_i e_i\| = \sum_{i=1}^{m} f(\mu_i \lambda_i e_i) \leq \sum_{i=1}^{m} |\mu_i \lambda_i f(e_i)| = \sum_{i=1}^{m} |\mu_i||\lambda_i f(e_i)| \\ &\leq \left(\sup_{1\leq i\leq m} |\mu_i|\right) \sum_{i=1}^{m} \varepsilon_i \lambda_i f(e_i) \leq \left(\sup_{1\leq i\leq m} |\mu_i|\right) \|f\| \left\|\sum_{i=1}^{m} \varepsilon_i \lambda_i e_i\right\| \\ &\leq \left(\sup_{1\leq i\leq m} |\mu_i|\right) \mathfrak{ubc}(\{e_i\}_{i=1}^m) \left\|\sum_{i=1}^{m} \lambda_i e_i\right\| \end{aligned}$$

Thus, taking the limit as $m \to \infty$ yields

$$\left\|\sum_{i=1}^{\infty} \mu_i \lambda_i e_i\right\| \leq \left(\sup_{i\in\mathbb{N}} |\mu_i|\right) \mathfrak{ubc}\left(\{e_i\}_{i=1}^\infty\right) \left\|\sum_{i=1}^{m} \lambda_i e_i\right\|$$

This completes the proof. ∎

Proof.

(of Theorem 1.5.8) Let $\{e_i\}_{i=1}^\infty$ be an unconditional basis of X that is not boundedly complete. Then there are scalars $(\lambda_i)_{i=1}^\infty$ such that $\|\sum_{i=1}^n \lambda_i e_i\| \leq 1$ for every $n \in \mathbb{N}$, and $\sum_{i=1}^\infty \lambda_i e_i$ does not converge.

Then by Cauchy's Criterion for Series, there exists $\varepsilon > 0$ and positive integers $p_1 < q_1 < p_2 < q_2$ such that for

$$u_j = \sum_{j=p_j}^{q_i} \lambda_j e_j$$

we have $\|u_j\| \geq \varepsilon$ for every $j \in \mathbb{N}$, however,

$$\left\|\sum_{i=1}^{m} u_i\right\| \leq K \left\|\sum_{i=1}^{\infty} \lambda_i e_i\right\| \leq K$$

where $K = \mathfrak{ubc}(\{e_i\}_{i=1}^\infty)$.

Now by Lemma 1.5.9, for any sequence $(\mu_j)_{j=1}^m$ of scalars, we have

$$\left\|\sum_{j=1}^m \lambda_j u_j\right\| \leq K \sup_{1\leq j\leq m} |\lambda_j| \left\|\sum_{j=1}^m u_j\right\| \leq K^2 \sup_{1\leq j\leq m} |\lambda_j| = K^2\|(\lambda_j)\|_\infty$$

On the other hand, from the unconditionality of $\{e_i\}_{i=1}^\infty$, we have for each $1 \leq i \leq m$,

$$\left\|\sum_{j=1}^m \lambda_j u_j\right\| \geq \frac{1}{K}\|\lambda_i u_i\| \geq \frac{\varepsilon}{K}|\lambda_i|$$

that is, $\frac{\varepsilon}{K}\|(\lambda_j)\|_\infty \leq \left\|\sum_{j=1}^m \lambda_j u_j\right\|$. Thus, $\{u_j\}_{j=1}^\infty$ is equivalent to the canonical basis of $c_0(\mathbb{N})$.

In a similar manner, we are able to obtain an isomorphic copy of $\ell_1(\mathbb{N})$, simply by changing the initial condition that X has an unconditional Schauder basis that is *not shrinking*. Recall that a Schauder basis is said to be shrinking if $(f_i)_{i=1}^\infty$ are functionals on X, and $\overline{\langle\{f_n\}_{i=1}^\infty\rangle} = X^*$.

Theorem 1.5.10: James

Let X be a separable Banach space. if X has an unconditional Schauder basis that is not shrinking, then X contains an isomorphic copy of $\ell_1(\mathbb{N})$. In particular, a separable Banach space with an unconditional Schauder basis contains an isomorphic copy of $\ell_1(\mathbb{N})$ if X^* is nonseparable.

Analyzing the "In particular" part of Theorem 1.5.10, we can observe that since $(\ell_1(\mathbb{N}))^* \equiv \ell_\infty(\mathbb{N})$, which is nonseparable, so on $\ell_1(\mathbb{N})$, we have an unconditional Schauder basis that contains an isomorphic copy of $\ell_1(\mathbb{N})$. Similarly, $L_1([0,1])$ is separable, but $L_\infty([0,1])$ is not separable, and so on $L_1([0,1])$, there exists an unconditional Schauder basis that contains an isomorphic copy of $\ell_1(\mathbb{N})$.

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