Unconditional Bases

Definition: Let X be a Banach space. A series $\sum_{n=1}^{\infty} x_n$ is **unconditionally convergent** if $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges for all choices of signs $\varepsilon_n = \pm 1$.

Definition: Let X be a Banach space and let $\{e_n\}_{n=1}^{\infty}$ be a Schauder basis for X.

(i) We say that $\{e_n\}_{n=1}^{\infty}$ is an **unconditional basis** if for every $x \in X$, its expansion $x = \sum_{n=1}^{\infty} \lambda_n e_n$ converges unconditionally.

(ii) We say that $\{e_n\}_{n=1}^{\infty}$ is a **unconditional basic sequence** if $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis of $\overline{\langle \{e_n\}_{n=1}^{\infty} \rangle}$.

Example: For $1 \le p < \infty$, consider $\ell_p(\mathbb{N})$ with Schauder basis given by $\{e_n\}_{n=1}^{\infty}$ where for each $n \in \mathbb{N}$,

$$e_n = (0, 0, 0, \ldots, \underset{\substack{\uparrow \\ n\text{th}}}{1}, 0, \ldots)$$

We claim that $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis for $\ell_p(\mathbb{N})$. Let $x \in \ell_p(\mathbb{N})$. Then there are scalars $(\lambda_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$. We check that $\sum_{n=1}^{\infty} \varepsilon_n \lambda_n e_n$ converges for all choices $\varepsilon_n = \pm 1$. In particular, we check that the sequence of partial sums is Cauchy. Observe that because $x \in \ell_p(\mathbb{N})$,

$$\|x\|_p^p = \left\| \sum_{n=1}^{\infty} \lambda_n e_n \right\|_p^p = \sum_{i=1}^{\infty} \left| \sum_{n=1}^{\infty} \lambda_n e_n(i) \right| = \sum_{i=1}^{\infty} |\lambda_i|^p < \infty$$

so for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x\|_p^p = \sum_{i=N+1}^{\infty} |\lambda_i|^p < \varepsilon^p$$

Now, for all $n > m \geq N$, we have

$$\left\| \sum_{i=1}^{n} \varepsilon_i \lambda_i e_i - \sum_{i=1}^{m} \varepsilon_i \lambda_i e_i \right\|_p^p = \left\| \sum_{i=m+1}^{n} \varepsilon_i \lambda_i e_i \right\|_p^p = \sum_{i=m+1}^{n} |\lambda_i|^p$$

$$\leq \sum_{i=m+1}^{\infty} |\lambda_i|^p < \varepsilon^p$$

which shows that the sequence of partial sums are Cauchy, so $\sum_{n=1}^{\infty} \varepsilon_n \lambda_n e_n$ converges, i.e. $\sum_{n=1}^{\infty} \lambda_n e_n$ converges unconditionally.

Example: Let $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{O}$ is an orthonormal basis for $\mathcal{H}$, then $\mathcal{O}$ is also an unconditional basis.

Indeed, if $\mathcal{O} = \{e_\gamma\}_{\gamma \in \Gamma}$ is an orthonormal basis and $\mathcal{H}$ is separable then for every $x \in \mathcal{H}$, we get

$$x = \sum_{\gamma \in \Gamma} \langle x, e_\gamma \rangle e_\gamma .$$

Apply the same technique as previous example to get that $\mathcal{O}$ is an unconditional basis, i.e. if $F \subset \Gamma$ is a countable subset, say $F = \{e_{\gamma_n}\}_{n=1}^{\infty}$, then because $\mathcal{O}$ on F is unconditional, so is $\mathcal{O}$ on Γ.

Remark: Every basis equivalent to an unconditional basis is also unconditional.

Example: Let $\{e_n\}_{n=1}^{\infty}$ be the usual unit vector basis for $c_0(\mathbb{N})$. For each $n \in \mathbb{N}$, let $x_n = \sum_{i=1}^{n} e_i$. We check that if $x = \sum_{n=1}^{\infty} \lambda_n e_n$, then $x = \sum_{n=1}^{\infty} \mu_n x_n$, where $\mu_n = \lambda_n - \lambda_{n+1}$. Note that $\sum_{n=1}^{\infty} \mu_n = \lambda_1$, so $\{\mu_n\}_{n=1}^{\infty}$ forms a convergent series. Also

$$\|x\|_{\infty} = \sup_{k \in \mathbb{N}} \left| \sum_{n=k}^{\infty} \mu_n \right|$$

On the other hand, for every convergent series $\sum_{n=1}^{\infty} \mu_n$ we have

$$x = \sum_{i=1}^{\infty} \mu_i x_i = \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} \mu_n \right) e_i \in c_0(\mathbb{N})$$

Conclusion: $(c_0(\mathbb{N}), \|\cdot\|_{\infty}) \equiv (S, \|\cdot\|)$ where

$$S = \left\{ \mu : \|\mu\| = \sup_{k \in \mathbb{N}} \left| \sum_{n=k}^{\infty} \mu_k \right| \right\}.$$

The basis $\{x_n\}_{n=1}^{\infty}$ is called the summing basis for $c_0(\mathbb{N})$.

Proposition: Let $\{e_n\}_{n=1}^{\infty}$ be a sequence in a Banach space X. TFAE:

(i) $\{e_n\}_{n=1}^{\infty}$ is an unconditional basic sequence.

(ii) There exists $K > 0$ s.t. $\forall \lambda_1, \dots, \lambda_m$ and signs $\varepsilon_n = \pm 1$

$$\left\| \sum_{i=1}^{m} \varepsilon_i \lambda_i e_i \right\| \le K \left\| \sum_{n=1}^{\infty} \lambda_n e_n \right\|$$

(iii) There exists $L > 0$ s.t. $\forall \lambda_1, \dots, \lambda_m$ and $\sigma \subset \{1, \dots, m\}$

$$\left\| \sum_{i \in \sigma} \lambda_i e_i \right\| \le L \left\| \sum_{i=1}^{m} \lambda_i e_i \right\|$$

In particular, we can extend $\{1, \dots, m\}$ to $\mathbb{N}$ so

$$\left\| \sum_{i \in \sigma} \lambda_i e_i \right\| \le L \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|$$

Proof (iii): Let σ be any subset of $\mathbb{N}$, possibly infinite. Then by assumption of (iii) $\sum_{i\in\sigma} \lambda_i e_i$ is Cauchy: For $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall m > n \geq N$,
$\left\| \sum_{i=n+1}^{m} \lambda_i e_i \right\| \leq \frac{\varepsilon}{L}$. Let $\tau = \sigma \cap \{n, \dots, m\}$, let $\mu_i = \lambda_i$ for all $n \leq i \leq m$, and $\mu_i = 0$ otherwise. By assumption,

$$\left\| \sum_{\substack{i\in\sigma \\ i=n}}^{m} \lambda_i e_i \right\| < \varepsilon \quad \Rightarrow \quad \sum_{i\in\sigma} \lambda_i e_i \text{ convergent.}$$

Passing to the limit we get the claim.