Section 4.3: Shrinking and Boundedly Complete Bases, Perturbation (Summary)

Joe Tran

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1 Summary

The section started with an introduction of a biorthogonal system in X^*. Recall that if $\{e_i\}_{i=1}^\infty$ is a Schauder basis of X and we have $\{f_i\}_{i=1}^\infty$ to be the coordinate functionals associated to $\{e_i\}_{i=1}^\infty$, then we can move things around and take our biorthogonal system from X given by $\{e_i; f_i\}_{i=1}^\infty$, and transform the biorthogonal system on X^*, where in this case, we consider $\widehat{e}_n \in X^{**}$ and $f_n \in X^*$. In other words, our biorthogonal system in X looked like something from $X \times X^*$. A biorthogonal system on X^* would look something like $X^* \times \widehat{X}$ or $X^* \times X^{**}$ depending on whether X is reflexive or not. But we would have $\{f_i; \widehat{e}_i\}_{i=1}^\infty$ as our biorthogonal system on X^*. With the biorthogonal systems in X^*, the section introduced a proposition that had to deal with canonical projections on X and the dual operators of canonical projections on X^*.

We introduced one of the main concepts of this section, that being a shrinking Schauder basis and a boundedly complete Schauder basis. The part that really needs the coordinate functionals is when we consider shrinking Schauder bases. A Schauder basis $\{e_i; f_i\}_{i=1}^\infty$ is said to be *shrinking* if and only if $\overline{\langle \{f_i\}_{i=1}^\infty \rangle} = X^*$, and a Schauder basis is said to be *boundedly complete* if and only if whenever the scalars $(\lambda_i)_{i=1}^\infty$ are such that $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^n \lambda_i e_i\| < \infty$, then $\sum_{i=1}^\infty \lambda_i e_i$ converges.

The text mentioned some examples of some classical spaces that are or aren't shrinking, and some classical spaces that are or aren't boundedly complete. Expanded detail of these examples have been noted down in my notes, and we note that $\ell_p(\mathbb{N})$ for $1 < p < \infty$ with the usual standard basis is shrinking. In a similar manner, $c_0(\mathbb{N})$ is also shrinking. In the case where $1 \leq p < \infty$, we have that $\ell_p(\mathbb{N})$ is boundedly complete, but $c_0(\mathbb{N})$ is not boundedly complete.

There are some concepts about having a shrinking Schauder basis and a map defined by $T\widehat{x} = (\widehat{x}(f_i))_{i=1}^\infty = (f_i(x))_{i=1}^\infty$ is an isomorphism of X^{**} onto the space of all sequences $\lambda = (\lambda_i)_{i=1}^\infty$ such that

$$|||\lambda||| = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n \lambda_i e_i \right\| < \infty$$

and we have isometry if the Schauder basis is monotone. We have one for boundedly complete as well. But in this case, we have that $X \simeq \overline{\langle \{f_i\}_{i=1}^\infty \rangle}^*$, in which we take the canonical mapping $J : X \to Y^*$, where $Y = \overline{\langle \{f_i\}_{i=1}^\infty \rangle}$ by $(Jx)(y) = y(x)$. This map J is a well-defined bounded linear operator, and is an isomorphism of X onto Y^*.

James's Theorem introduces a very handy tool that relates reflexivity, shrinking, and boundedly complete Schauder bases. We have that X is reflexive if and only if $\{e_n\}_{n=1}^\infty$ are both shrinking and boundedly complete. Going back to the examples above, we noted that $\ell_1(\mathbb{N})$ is not shrinking since $(\ell_1(\mathbb{N}))^{**} \neq \ell_1(\mathbb{N})$, and similarly, we noted that $c_0(\mathbb{N})$ is not boundedly complete, since $(c_0(\mathbb{N}))^{**} \neq c_0(\mathbb{N})$.

The next part of this section introduced the concept of a basic sequence of a Banach space. A basic sequence is simply a Schauder basis, but instead of being a Schauder basis of X, it is a Schauder basis of $\overline{\langle \{e_i\}_{i=1}^{\infty} \rangle}$. Similar terminologies for shrinking and boundedly complete basic sequences. Banach's Theorem introduces a classification for basic sequences, that we have a basic seqeunce $\{e_i\}_{i=1}^{\infty}$ if and only if there is a $K > 0$ such that for all $n < m$ and scalars $\lambda_1, ..., \lambda_m \in \mathbb{R}$,

$$\left\| \sum_{i=1}^{n} \lambda_i e_i \right\| \leq K \left\| \sum_{i=1}^{m} \lambda_i e_i \right\|$$

In this case, the smallest possible K we could have is the basis constant $\mathrm{bc}(\{e_i\}_{i=1}^{\infty})$. Mazur's theorem after that talks about infinite-dimensional Banach spaces, where if we have a Banach space, (i) every infinite-dimensional Banach space has a basic seqeunce, and (ii) if X^* is separable, then X contains a shrinking basic sequence. The proof took some steps to prove.

The text then mentioned that we do not know if every separable Banach space X contains a closed subspace Y such that both Y and X/Y have a Schauder basis. However, we could note that if X is separable and nonreflexive, then there is a nonreflexive closed subspace Y of X such that Y has a Schauder basis of X. This was not included in the text, but I was able to come up with an example that demonstrates this statement. Indeed, we consider $\ell_1(\mathbb{N})$, which is separable and nonreflexive, and we can consider Y to be

$$Y = \bigcap_{n=1}^{\infty} \{x \in \ell_1(\mathbb{N}) : x_{2n} = 0\}$$

in which Y is closed but not reflexive since $\ell_1(\mathbb{N})$ is not reflexive. The closed follows since Y is simply the intersection of kernels of the continuous linear functionals given by the map $x \mapsto x_{2n}$. An example (?) of a Schauder basis could be $\{e_{2n-1}\}_{n=1}^{\infty}$ for Y.

Similar in the sense of having equivalent norms, we also have the notion of having equivalent basic sequences. Indeed, if say $\{e_i\}_{i=1}^{\infty}$ is a basic sequence in a Banach space X and $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$ is a Banach space for Y, then $\{e_i\}_{i=1}^{\infty}$ and $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$ are equivalent if and only if for all scalars $(\lambda_i)_{i=1}^{\infty}$, we have that $\sum_{i=1}^{\infty} \lambda_i e_i$ converges if and only if $\sum_{n=1}^{\infty} \lambda_i \boldsymbol{e}_i$ converges. Not a usual notation in the text, but will denote that two basic sequences are equivalent by $\{e_i\} \sim \{\boldsymbol{e}_i\}$. A nice characterization of two basic sequences are equivalent is as follows: $\{e_i\}_{i=1}^{\infty} \sim \{\boldsymbol{e}_i\}_{i=1}^{\infty}$ if and only if there exists an isomorphism $T : \overline{\langle \{e_i\}_{i=1}^{\infty} \rangle} \to \overline{\langle \{\boldsymbol{e}_i\}_{i=1}^{\infty} \rangle}$ such that $Te_i = \boldsymbol{e}_i$ for all $n \in \mathbb{N}$.

To conclude this section, we have the Small Perturbation Lemma, where we have a basic sequence $\{e_i\}_{i=1}^{\infty}$ in a Banach space X, sequence of coefficient functionals $\{f_i\}_{i=1}^{\infty}$ of the basis $\{e_i\}_{i=1}^{\infty}$ of $\overline{\langle \{e_i\}_{i=1}^{\infty} \rangle}$. If we have a sequence $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$ of X such that $\sum_{i=1}^{\infty} \|e_i - \boldsymbol{e}_i\| \|f_i\| = C < 1$, then two three results: (i) $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$ is basic in X and $\{\boldsymbol{e}_i\}_{i=1}^{\infty} \sim \{e_i\}_{i=1}^{\infty}$, (ii) if $\overline{\langle \{e_i\}_{i=1}^{\infty} \rangle}$ is complemented in X, then so is $\overline{\langle \{\boldsymbol{e}_i\}_{i=1}^{\infty} \rangle}$, and (iii) If $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis of X, so is $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$.

2 Question?

- So given the definition of equivalent basic sequences, does the operator "$\sim$" define an equivalence relation on all basic sequences of a Banach space X? This would be a good exercise to check.