Section 4.1: Projections, Complementability, and Auerbach Bases (Summary)

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1 Summary

Introduced the definition of a linear projection (something that we have defined early on in Functional Analysis), and also introduced the concept of algebraic linear decomposition of a projection. Usually, if X is a vector space and Y is a subspace of X, then Z is called an algebraic complement of Y in X if and only if $X = Y \oplus Z$. There were a few more concepts introduced, such as what it means to be complemented, and what it means to be a topological complement in X. There was some characterization about Y is complemented of X if and only if there is a topological complement of Y in X. This proof uses the Closed Graph Theorem for one of the implications, and also verifying that $\ker(P)$ and $\mathrm{Range}(P)$ form an algebraic linear decomposition.

The second half of the section introduced the concept about biorthogonal system, which is something similar that was talked about in Functional Analysis. The idea is that if we have a collection of functionals $\{f_1, ..., f_n\}$ of X, then these functionals are said to be biorthogonal to $\{e_1, ..., e_n\}$ if $f_i(e_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. We denote $\{e_i; f_i\}_{i=1}^n$ to be the biorthogonal system in $X \times X^*$. With the concept of the biorthogonal system, this section introduced Auerbach bases, in which the biorthogonal system is Auerbach if $(e_i)_{i=1}^n$ is a Hamel basis of X such that for all $1 \leq i \leq n$, $x_i \in S_X$ and $f_i \in S_{X^*}$. Some reason, the authors of this book use x_i^* to denote the functionals of X, but will use f_i for the regular functionals. As it turns out, every finite-dimensional Banach space admits an Auerbach basis.

2 Questions?

- I kinda half-understand Proposition 4.4. In particular, the part where I am having a bit of a hard time understanding is the statement *the dual of X is isomorphic to $Y^* \oplus Z^*$*. But I think the idea of the proof of this part involves two projections P of X onto Y, and Q of X onto Z, and since $Y^* \simeq P^*[X^*]$ and $Z^* \simeq Q^*[Y^*]$, and use this to show $Y^* \oplus Z^* \simeq P^*[X^*] \oplus Q^*[X^*] = X^*$, where equality occurs when showing that $P^*[X^*]$ and $Q^*[X^*]$ form an algebraic linear decomposition.