Section 4.5: Unconditional Bases (Summary)

Joe Tran

January 20, 2025

1 Summary

In this section, we were introduced about the concept of unconditional convergence and unconditional bases. We say that a Schauder basis $\{e_i\}_{i=1}^{\infty}$ of a Banach space X to be *unconditional* if for every $x = \sum_{i=1}^{\infty} \lambda_i e_i \in X$, $\sum_{i=1}^{\infty} \varepsilon_i \lambda_i e_i$ converges for all possible choices of $\varepsilon_i = \pm 1$ i.e. $\sum_{i=1}^{\infty} \lambda_i e_i$ converges unconditionally. There were a couple of nice examples involving unconditional bases, and the worked-out examples that I did were for the $\ell_p(\mathbb{N})$ and Hilbert spaces. There was also another nice example involving the summing basis for $c_0(\mathbb{N})$, where we have $\{x_n\}_{n=1}^{\infty}$ given by $x_n = \sum_{i=1}^{n} e_i$ and we have $x = \sum_{n=1}^{\infty} \lambda_n e_n = \sum_{n=1}^{\infty} (\lambda_n - \lambda_{n+1}) x_n$.

With unconditional bases, permutations in a way to have a role, since there could be a lot of choices for ε_i for each $i \in \mathbb{N}$, and pretty much, any finite permutation can be extended to a countable permutation, in which there is a Lemma where if you have a finite subset $\sigma \subseteq \{1, ..., m\}$, we have some $L > 0$ and scalars $\lambda_1, ..., \lambda_m$ such that $\left\|\sum_{i \in \sigma} \lambda_i e_i\right\| \leq L \left\|\sum_{i=1}^{m} \lambda_i e_i\right\| \leq L \left\|\sum_{i=1}^{\infty} \lambda_i e_i\right\|$.

There are also some very important theorems that were mentioned in this section. For example, we could have some sort of relationship between unconditional bases, and shrinking and boundedly complete bases introduced in Section 4.3. In particular, if we have a separable Banach space with an unconditional Schauder basis, and either (i) the unconditional basis is not shrinking or (ii) the unconditional basis is not boundedly complete, then either (i) X contains an isomorphic copy of $c_0(\mathbb{N})$, or (ii) X contains an isomorphic copy of $\ell_1(\mathbb{N})$. In fact, we can combine both of these results into one, where we also relate to reflexivity. Indeed, we note that X is reflexive if and only if X contains no isomorphic copy of $\ell_1(\mathbb{N})$ or $c_0(\mathbb{N})$.

The section concludes by introducing the James space $\mathcal{J}$, which consists all sequences of real numbers x such that $\lim_{i \to \infty} x_i = 0$ and $\|x\|_{\mathcal{J}} < \infty$, where

$$\|x\|_{\mathcal{J}} = \sup_{n_1 < \cdots < n_k} \left(\sum_{i=1}^{k} (x_{n_i} - x_{n_{i+1}})^2 \right)^{\frac{1}{2}} < \infty$$

In better notation,

$$\mathcal{J} = \left\{ x \in c_0(\mathbb{N}) : \sup_{n_1 < \cdots < n_k} \left(\sum_{i=1}^{k} (x_{n_i} - x_{n_{i+1}})^2 \right)^{\frac{1}{2}} < \infty \right\}$$

As it turns out, the James space $\mathcal{J}$ is a Banach space with $\|\cdot\|_{\mathcal{J}}$, but it is not reflexive. Indeed, we have that $\langle\{(1,1,...)\}\rangle$ is a topological complement of $\mathcal{J}$ in $\mathcal{J}^{**}$, i.e.

$$\mathcal{J}^{**} = \mathcal{J} \oplus \langle\{(1,1,1,...)\}\rangle$$