Bases in Classical Spaces.

Theorem: Let X be a Banach space. If X^* has a subspace isomorphic to $c_0(\mathbb{N})$, then X has a complemented subspace isomorphic to $\ell_1(\mathbb{N})$. In particular, X^* has a subspace isomorphic to $\ell_\infty(\mathbb{N})$.

Proof:

<u>Step 1:</u> Let $T: c_0(\mathbb{N}) \to X^*$. Then $T^*: X^{**} \to c_0(\mathbb{N})^*$ but $c_0(\mathbb{N})^* \equiv \ell_1(\mathbb{N})$, so $T^*: X^{**} \to \ell_1(\mathbb{N})$.

<u>Step 2:</u> Because $\overline{B_X}^{w^*} = B_{X^{**}}$ (by Goldstein), there exists a $K>0$ such that

- $\|x_n\| \leq K$ for all $n \in \mathbb{N}$.
- $(T^* x_n)(e_n) = 1$ $\forall n \in \mathbb{N}$,
- $\sum_{i=1}^{n-1} |(T^* x_n)(e_i)| < \frac{1}{n}$ $\forall n \in \mathbb{N}$, where e_i are the standard unit vectors of $c_0(\mathbb{N})$.

<u>Step 3:</u> Claim[1]: Let $X = c_0(\mathbb{N})$ or $X = \ell_p(\mathbb{N})$ for $1 \leq p < \infty$. If $\{u_j\}_{j=1}^\infty$ is a normalized block basic sequence of the standard basis $\{e_i\}_{i=1}^\infty$, then

(i) $\{u_j\} \sim \{e_i\}$

(ii) $\overline{\langle \{u_j\} \rangle} \equiv X$

(iii) $\exists P : X \to \overline{\langle \{u_j\} \rangle}$ with $\|P\| = 1$.

<u>Claim 2:</u> Let X be a Banach space. TFAE

(a) X does not contain an isomorphic copy of $c_0(\mathbb{N})$

(b) If $(x_n)_{n=1}^{\infty}$ is a sequence in X such that the set $S = \{ \sum_{i=1}^{n} \varepsilon_i x_i : \varepsilon_i = \pm 1, n \in \mathbb{N} \}$ is bounded, then S is $\|\cdot\|$-relatively compact.

(c) If $(x_n)_{n=1}^{\infty}$ is a sequence in X such that the set $S = \{ \sum_{i=1}^{n} \varepsilon_i x_i : \varepsilon_i = \pm 1, n \in \mathbb{N} \}$ is bounded, then $\sum_{i=1}^{n} x_i$ is unconditionally convergent.

Step 4: By the Claims above, $\{T^* x_n\}_{n=1}^{\infty}$ has a subsequence $\{T^* x_{n_k}\}$ that is equivalent to the standard basis of $\ell_1(\mathbb{N})$, and whose span is complemented in $\ell_1(\mathbb{N})$ by a projection P. Then for some $M > 0$, and $(\lambda_n)_{n=1}^{\infty} \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$

$$\left\| \sum_{k=1}^{\infty} \lambda_k x_{n_k} \right\| \leq K \sum_{k=1}^{\infty} |\lambda_k| \leq KM \left\| \sum_{k=1}^{\infty} \lambda_k T^* x_{n_k} \right\|$$
$$\leq KM \|T^*\| \left\| \sum_{k=1}^{\infty} \lambda_k x_{n_k} \right\|$$

Therefore, $T^* : \overline{\langle \{x_{n_k}\} \rangle} \to \overline{\langle \{T^* x_{n_k}\} \rangle}$ is an isomorphism, so $\overline{\langle \{x_{n_k}\} \rangle} \simeq \ell_1(\mathbb{N})$ and

$Q = (T^*)^{-1} P T^*$ is a projection from X onto $\overline{\langle \{x_{n_k}\} \rangle}$

Finally, $Q^*[X^*] \simeq Q[X^*] = \ell_1(\mathbb{N})^* = \ell_\infty(\mathbb{N})$.

Proof of Claim 1: We show the proof for $\ell_p(\mathbb{N})$.

Step 1: Let $u_j = \sum_{i=p_j+1}^{p_{j+1}} \lambda_i e_i$ with $\sum_{i=p_j+1}^{p_{j+1}} |\lambda_i|^p = 1$ for $j \in \mathbb{N}$. Then

$$\left\| \sum_{j=1}^{m} a_j u_j \right\|^p = \sum_{j=1}^{m} \sum_{i=p_j+1}^{p_{j+1}} |a_j|^p |\lambda_i|^p = \sum_{j=1}^{m} |a_j|^p \sum_{i=p_j+1}^{p_{j+1}} |\lambda_i|^p$$
$$= \sum_{j=1}^{m} |a_j|^p = \left\| \sum_{j=1}^{m} a_j e_j \right\|^p.$$

Step 2: $T\left(\sum_{j=1}^{m} a_i u_j\right) = \sum_{j=1}^{m} a_j e_j$ is an isometry.

Step 3: For each $j \in \mathbb{N}$, let $f_j \in \langle \{e_{p_j+1}, \dots, e_{p_{j+1}}\} \rangle < \ell_p(\mathbb{N})^*$ s.t. $\|f_j\| = f_j(u_j) = 1$. Then $f_i(u_j) = 0 \ \forall i \neq j$ and the one-to-one operator $P : X \to \overline{\langle \{u_j\} \rangle}$ defined by

$$Px = \sum_{j=1}^{\infty} f_j(x) u_j$$

is a linear projection of X onto $\overline{\langle \{u_j\} \rangle}$.

Step 4: For $x = \sum_{i=1}^{\infty} a_i e_i \in X$, we have

$$|f_j(x)|^p \le \sum_{i=p_j+1}^{p_{j+1}} |\lambda_i|^p \quad \forall j \in \mathbb{N} \quad \text{bec. } \|f_j\| = 1$$

thus,

$$\begin{aligned} \|Px\|^p &= \sum_{j=1}^{\infty} |f_j(x) u_j|^p \\ &= \sum_{j=1}^{\infty} |f_j(x)|^p \underbrace{\sum_{i=p_j+1}^{p_{j+1}} |\lambda_i|^p}_{\to 1} \\ &= \sum_{j=1}^{\infty} |f_j(x)|^p \\ &\le \sum_{j=1}^{\infty} \sum_{i=p_j+1}^{p_{j+1}} |a_i|^p \\ &= \|x\|^p \end{aligned}$$

why?

$$\sum_{j=1}^{\infty} \|f_j\| \, \|x\|^p = \sum_{j=1}^{\infty} \|x\|^p = \sum_{j=1}^{\infty} |a_j|^p$$

$\Rightarrow \|P\| = 1$.

Corollary: Let $X = c_0(\mathbb{N})$ or $X = \ell_p(\mathbb{N})$, $1 \le p < \infty$. If Y is a infinite-dimensional subspace of X. Then Y contains a subspace Z s.t. $Z \simeq X$ and complemented in X.

Proof: Combine Theorem 4.26 and above proposition.

Theorem (Pełczyński Decomposition Method): Let X and Y be Banach spaces such that X is isomorphic to a complemented subspace of Y and Y is isomorphic to a complemented subspace of X. Assume that <u>either</u>

(i) $X \simeq X \oplus X$ or $Y \simeq Y \oplus Y$

(ii) $X \simeq \left(\sum_{n=1}^{\infty} X\right)_{c_0}$ or $X \simeq \left(\sum_{n=1}^{\infty} X\right)_{\ell_\infty}$ or $X \simeq \left(\sum_{n=1}^{\infty} X\right)_{\ell_p}$ for some $1 \leq p < \infty$.

Then $X \simeq Y$.

Proof: Put $Y \simeq X \oplus E$, $X \simeq Y \oplus F$.

(i) $Y \simeq X \oplus E \simeq (X \oplus X) \oplus E \simeq X \oplus Y$.

$$X \simeq Y \oplus F \simeq (Y \oplus Y) \oplus F \simeq Y \oplus X$$

so $X \simeq Y$.

(ii) $X \simeq X \oplus X$ and $Y \simeq X \oplus E \oplus X \simeq X \oplus X \oplus E$ $\simeq X \oplus Y$. Also,

$$\left(\sum X\right)_{\ell_p} \simeq \left(\sum Y \oplus F\right)_{\ell_p} \simeq \left(\sum Y\right)_{\ell_p} \oplus \left(\sum F\right)_{\ell_p}$$

if $X \simeq \left(\sum X\right)_{\ell_p}$

$$\begin{aligned}
X \simeq \left(\sum X\right)_{\ell_p} &\simeq \left(\sum Y\right)_{\ell_p} \oplus \left(\sum F\right)_{\ell_p} \\
&\simeq Y \oplus \left(\sum Y\right)_{\ell_p} \oplus \left(\sum F\right)_{\ell_p} \\
&\simeq Y \oplus \left(\sum Y \oplus F\right)_{\ell_p} \\
&\simeq Y \oplus \left(\sum X\right)_{\ell_p} \\
&\simeq Y \oplus X \\
&\simeq Y.
\end{aligned}$$

Corollary: Let $X = c_0(\mathbb{N})$ or $X = \ell_p(\mathbb{N})$, $1 \leq p < \infty$. If Y is an infinite-dimensional complemented subspace of X, then $Y \simeq X$.

Proof: by above theorem. $Y \supseteq Z \simeq X$ and Z is complemented in X (hence, complemented in Y). So X is isomorphic to a complemented subspace of X and Y is a complemented subspace of X.

- If $X = c_0(\mathbb{N})$, then $X \simeq (\sum X)_{c_0}$.
- If $X = \ell_p(\mathbb{N})$, then $X \simeq (\sum X)_{\ell_p}$

Use (ii) of above theorem, proof done.