Section 4.2: Basics on Schauder Bases (Summary)

Joe Tran

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1 Summary

This section introduced the notion of Schauder bases, in which a countable collection of a Banach space is said to be a Schauder basis if x can be approximated by an infinite linear combination. That is, $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis if every $x \in X$, there are unique scalars $(\lambda_i)_{i=1}^{\infty}$ such that x be written as an infinite linear combination $x = \sum_{i=1}^{\infty} \lambda_i e_i$. As it turns out, every Schauder basis of a Banach space is linearly independent. Indeed, we could just take finite linear combinations and show that they are linearly independent. An important property that should be noted is that every Banach space that has a Schauder basis, is always separable. Otherwise, if X is a Banach space, that may not be separable, then it may not contain a Schauder basis.

The definition of a Schauder basis can be weakened to consider weak convergence. Indeed, if $x \in X$ has a unique decomposition $\sum_{n=1}^{\infty} \lambda_i e_i$, such that $\sum_{n=1}^{N} \rightharpoonup x$ as $N \to \infty$, then $\{e_i\}_{i=1}^{\infty}$ is automatically a Schauder basis. Defined the nth canonical projections $P_n : X \to X$ by the map $P_n x = \sum_{i=1}^{n} \lambda_i x_i$ and we note that $P_n x \to x$ as $n \to \infty$. We also note that if $(P_n)_{n=1}^{\infty}$ are canonical projections on normed space X and Schauder basis $\{e_i\}_{i=1}^{\infty}$, are uniformly bounded, i.e. $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$, then the Schauder basis on X given by $\{e_i\}_{i=1}^{\infty}$, is also a Schauder basis on the completion $\mathscr{X}$ of X.

When we have a Schauder basis $\{e_i\}_{n=1}^{\infty}$ on a Banach space X, we introduce a constant, known as the basis constant of $\{e_i\}_{i=1}^{\infty}$, and it is denoted by $\mathrm{bc}(\{e_i\}_{i=1}^{\infty})$, where $\mathrm{bc}(\{e_n\}_{n=1}^{\infty}) = \sup_{n\in\mathbb{N}} \|P_n\|$. There are some terminologies and concepts associated to this basis constant, including the definition of a normalized, seminormalized, and monotone Schauder bases, where we have normalized if $\|e_n\|$ for all $n \in \mathbb{N}$, seminormalized if $0 < \inf_{n\in\mathbb{N}} \|e_n\| \le \sup_{n\in\mathbb{N}} \|e_n\| < \infty$, and monotone if $\mathrm{bc}(\{e_i\}_{i=1}^{\infty}) = 1$.

We related Schauder bases and functionals in such a way that if we have a Schauder basis $\{e_i\}_{i=1}^{\infty}$, then a collection of linear functionals $\{f_i\}_{i=1}^{\infty}$[1] are called the *associated biorthogonal functionals*, or simply the *coordinate functionals*. Thus, if we have a Schauder basis $\{e_i\}_{i=1}^{\infty}$ and coordinate functionals $\{f_i\}_{i=1}^{\infty}$, then the system $\{e_i; f_i\}_{i=1}^{\infty}$ is a Schauder basis of the Banach space X. In the case where in a vector space X and we had a Hamel basis B, then we had that

$$x = \sum_{b\in B} f_b(x) e_b$$

where $B = \{e_b\}_{b\in B}$ and f_b are the coordinate functionals on X. In this case, if we have a Banach

[1] The text uses e_i^* to denote the associated element of the Schauder basis

space X with a Schauder basis $\{e_n\}_{n=1}^{\infty}$, then X is separable, and for any $x \in X$ and functionals $\{f_i\}_{i=1}^{\infty}$ we can write

$$x = \sum_{i=1}^{\infty} f_i(x) e_i$$

that is we treat $f_i(x)$ as the coefficients of x.

Classic examples of Schauder bases include the standard bases for $c_0(\mathbb{N})$ and $\ell_p(\mathbb{N})$ for $1 \leq p < \infty$. We do not have a Schauder basis for $\ell_\infty(\mathbb{N})$ since $\ell_\infty(\mathbb{N})$ is not even separable. Also, any orthonormal basis $\mathcal{O}$ of a Hilbert space $\mathcal{H}$ is also a Schauder basis. Indeed, say we have $\mathcal{O} = \{e_i\}_{i=1}^{\infty}$. On $\mathcal{H}$, we have inner products that define our coefficients. That is, for any $x \in \mathcal{H}$,

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$$

where we identify $f_i(x) = \langle x, e_i \rangle$ for each i. Also, every finite-dimensional Banach space has a Schauder basis. Other examples of Schauder bases include the Faber-Schauder basis of $C([0,1])$.

2 Questions?

- In this section, the author mentioned a consequence for canonical projections associated to a Schauder basis: *If X is a Banach space and $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis for X, then we could write X as an algebraic linear decomposition of $\langle \{e_i\}_{i=1}^{n} \rangle$ and $\overline{\langle \{e_i\}_{i=n+1}^{\infty} \rangle}$.* The part where I get confused about this, is, why do we consider the closure of the tail of the span of the Schauder basis, and why is it a topological complement $\langle \{e_i\}_{i=1}^{n} \rangle$?