Section 4.4: Block Bases, Bessaga-Pelczynski Principle (Summary)

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1 Summary

In the first part of this section, we introduced the concept of a block basic sequence of a basic seqeunce in a Banach space X. A sequence $\{u_j\}_{j=1}^{\infty}$ of the form

$$u_j = \sum_{i=p_j+1}^{p_{j+1}} \lambda_i e_i$$

where $(p_i)_{i=1}^{\infty}$ is an increasing sequence of positive integers, and $\lambda_{p_j+1}, \ldots \in \mathbb{R}$, is called a *block basic sequence* of $\{e_i\}_{i=1}^{\infty}$ (basic block sequence?) By the definition of a block basic sequence, it's basis constant cannot be greater than the basis constant for $\{e_i\}_{i=1}^{\infty}$. Pelczynski's theorem allows us to construct basic block sequences that are equivalent to a Schauder basis of a closed subspace. That is, whenever we have an Banach space X, a Schauder basis $\{e_i\}_{i=1}^{\infty}$ of X, an infinite-dimensional closed subspace of Y, then Y contains an infinite dimensional closed subspace Z with Schauder basis $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$ that is equivalent to a block basic sequence of $\{e_i\}_{i=1}^{\infty}$. The idea behind the proof is to construct this block basic sequence inductively. However, Bessaga-Pelczynski Selection Principle is a modification of Pelczynski's theorem which states that if we have a Banach space X, a Schauder basis $\{e_i\}_{i=1}^{\infty}$, a block basic sequence $\{u_j\}_{j=1}^{\infty}$ of $\{e_i\}_{i=1}^{\infty}$, and if we take a sequence $\{x_i\}_{i=1}^{\infty}$ such that $\inf_{i\in\mathbb{N}} \|x_i\| > 0$ and $x_n \rightharpoonup 0$, then there is a subseqeunce $(x_{i_k})_{k=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ that is a basic sequence equivalent to the basic block sequence.

The next part of this section is mainly theorems from Johnson-Rosenthal. One of the theorems I had some trouble understanding, maybe even still could use some clarification on [!!!]. In particular, one of Johnson-Rosenthal's result is that whenever we have a separable Banach space, then X contains a *quotient* with a Schauder basis. By quotient we say that Y is a quotient of X if there exists an onto bounded linear operator $T : X \to Y$. This proof was pretty long in general. Took some steps to prove this. Some more results by Johnson-Rosenthal include the separability of X^* and infinite dimensional Banach spaces such that X^{**} is separable. For the former, it mentions that if X^* is separable and Y is infinite-dimensional subspace of X^* with separable dual Y^*, then Y admits an infinite-dimensional reflexive subspace. For the latter, there are two results: (i) Every infinite dimensional subspace has an infinite-dimensional reflexive subspace and (ii) every infinite dimensional subspace of X^* contains an infinite-dimensional reflexive subspace. Almost identical to each other, but different structure.

The last part of this section mentions about finite-dimensional decompositions (or FDD for short). Simply speaking, when we have a collection of finite-dimensional subspaces $\{X_n\}_{n=1}^{\infty}$ of a

Banach space X, then every $x \in X$ has a unique representation of the form

$$x = \sum_{n=1}^{\infty} x_n$$

where for each $n \in \mathbb{N}$, $x_n \in X_n$. An observation that I made was the following: Back when we considered algebraic linear decompositions, we say that Y and Z are an algebraic linear decomposition and denote $X = Y \oplus Z$. I am pretty sure this result would hold if X was infinite-dimensional and Y and Z are infinite-dimensional subspaces of X. On the other hand, if I were to compare this for infinite-dimensional subspaces of X, then we could (?) say that $\{X_n\}_{n=1}^{\infty}$ is an FDD of X if

$$X = \sum_{n=1}^{\infty} X_n = \left\{ \sum_{n=1}^{\infty} x_n : x_n \in X_n \ \forall n \in \mathbb{N} \right\}$$

(which I could possibly treat as the infinite-Minkowski sum of vector spaces). One nice thing about this form, is that when $\dim(X_n) = 1$ for all $n \in \mathbb{N}$, i.e. $X_n = \langle x_n \rangle$ for some $x_n \in X_n$, then $\{X_n\}_{n=1}^{\infty}$ is a Schauder basis of X if and only if $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis of X. Indeed, the proof follows immediately since by the above representation, we say that

$$X = \sum_{n=1}^{\infty} X_n = \sum_{n=1}^{\infty} \langle x_n \rangle$$

and it is easily checked that we get both equivalences at the same time. There were some properties of finite rank projections which the proof of is similar to that of the canonical projections back in Section 4.2.

We also had the notion of shrinking FDDs, in which we say an FDD is shrinking if the associated projections satisfy

$$\lim_{n \to \infty} \|P_n^* f - f\| = 0$$

for all $f \in X^*$. The last proof of this section of Johnson-Rosenthal uses the idea of Markushevich bases which are introduced in a later chapter. But the idea with Markushevich bases, whenever we have a separable Banach space, we have some subspace Y of X such that Y and X/Y both have FDDs. Furthermore, whenever X^* is separable, then X/Y and Y could have shrinking FDDs. The proof also used the idea of *locally uniformly rotund* Banach spaces, which is also introduced in a later chapter.

2 Questions?

- Although it was an observation that $X = \sum_{n=1}^{\infty} X_n$, is it theoretically correct? Since we are basically taking the infinite sum of finite-dimensional subspaces, and we are basically treating it as a infinite Minkowski sum of vector spaces.
- Probably need to go over the proof of Johnson-Rosenthal again, that part is still confusing sometimes.
- Based on my current understanding, there are some concepts mentioned in a later chapter,

that is not 4 or 6, so would it be safe to skip some of these concepts?