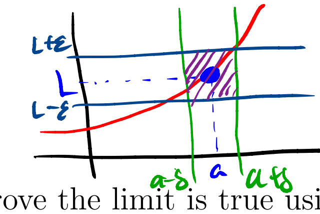
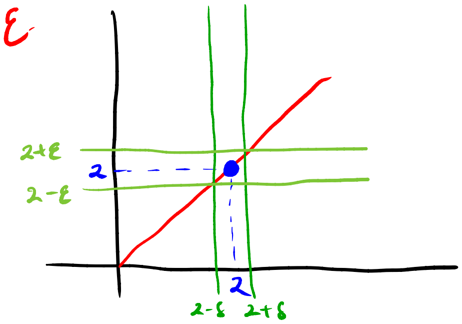


The Definition of the Limit

Whenever we are given a limit $\lim_{x \to a} f(x) = L$, we often wish to prove the limit is true using the definition of the limit...which is often not an easy task for a lot of people. We will have a look at some examples in which the steps of proving the limit should hopefully be clarified for everyone. Let's have a look at the first limit.

Definition 1. If $\lim_{x \to a} f(x) = L$, then we say that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Example 1. Prove that $\lim_{x \to 2} x = 2$. show $|x-2|<\varepsilon$



(1) let $\varepsilon > 0$ be arbitrary. If $\lim_{x \to 2} x = 2$, then there exists a $\delta > 0$ such that if $|x-2| < \delta$, then

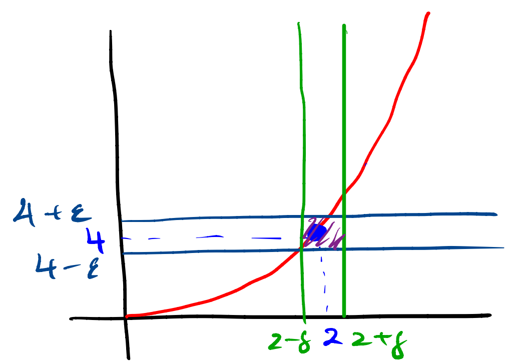
(2) $|f(x) - L| = |x - 2| < \delta$.

(3) Choose $\delta = \varepsilon$. Then

$$|x-2| < \delta = \varepsilon$$

(4) $|x-2| < \varepsilon. \Rightarrow \lim_{x \to 2} x = 2.$ ∎

Example 2. Prove that $\lim_{x \to 2} x^2 = 4$.



let $\varepsilon > 0$ be arbitrary. Since $\lim_{x \to 2} x^2 = 4$, then there exists a $\delta > 0$ such that if $|x-2| < \delta$, then

$$|f(x) - L| = |x^2 - 4| = |x-2||x+2|$$
$$< \delta |x+2|$$

$\delta = \min\{1, \frac{\varepsilon}{5}\}$.

Let $\delta = 1$. Then

$$|x-2| < 1. \Rightarrow \underset{+4}{-1} < x - \underset{+4}{2} < \underset{+4}{1}. \Rightarrow 3 < x+2 < 5$$

Now

let $\delta = \frac{\varepsilon}{5}$

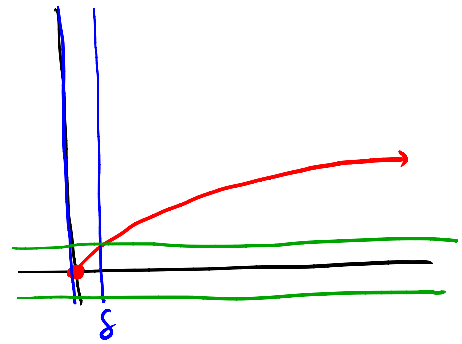
$$|x-2||x+2| < \delta |x+2| < 5\delta = 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

$|x^2 - 4| < \varepsilon. \Rightarrow \lim_{x \to 2} x^2 = 4.$ ∎

Now let's have a look at one sided limits.

Definition 2. If $\lim_{x\to a^+} f(x) = L$, then we say that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Definition 3. If $\lim_{x\to a^-} f(x) = L$, then we say that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \varepsilon$.



Example 3. Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{x\to 0^+} \sqrt{x} = 0$, there exists a $\delta > 0$ such that if $x - 0 = x < \delta$, then $\sqrt{x} < \sqrt{\delta}$, and so

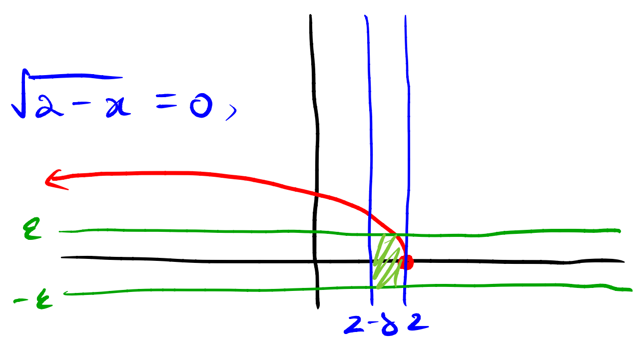
$$|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta}$$

Choose $\delta = \varepsilon^2$.

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

$$\Rightarrow |\sqrt{x} - 0| < \varepsilon \Rightarrow \lim_{x\to 0^+} \sqrt{x} = 0. \qquad \blacksquare$$

Example 4. Prove that $\lim_{x\to 2^-} \sqrt{2 - x} = 0$.



Let $\varepsilon > 0$ be arbitrary. Since $\lim_{x\to 2^-} \sqrt{2-x} = 0$, there exists a $\delta > 0$ such that if $-\delta < x - 2 \Rightarrow 2 - x < \delta$

then $\Rightarrow \sqrt{2-x} < \sqrt{\delta}$

$$|f(x) - L| = |\sqrt{2-x} - 0| = \sqrt{2-x} < \sqrt{\delta}$$

Choose $\delta = \varepsilon^2$. So that

$$\sqrt{2-x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

$$\Rightarrow |\sqrt{2-x} - 0| < \varepsilon \Rightarrow \lim_{x\to 2^-} \sqrt{2-x} = 0 \qquad \blacksquare$$

For infinite limits, we have a similar definition, but instead of $\varepsilon > 0$, we use other variables like M here.

Definition 4. If $\lim_{x\to a} f(x) = \infty$, then we say that for all $M > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $f(x) > M$.

Definition 5. If $\lim_{x\to a} f(x) = -\infty$, then we say that for all $M < 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $f(x) < M$.

Example 5. Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Example 6. Prove that $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$.

Now let's have a look at a finite limit but x is approaching both ∞ or $-\infty$.

Definition 6. If $\lim\limits_{x\to\infty} f(x) = L$, then we say that for all $\varepsilon > 0$, there exists a $N > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > N$.

Definition 7. If $\lim\limits_{x\to-\infty} f(x) = L$, then we say that for all $\varepsilon > 0$, there exists a $N < 0$ such that $|f(x) - L| < \varepsilon$ for all $x < N$.

Example 7. Prove that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

Example 8. Prove that $\lim\limits_{x\to-\infty} \dfrac{1}{x^2} = 0$.