Lesson 4

3.4 Indefinite Integrals

We know the relationship between definite integrals and antiderivatives, but to make things simpler, we say that the integral of a function is the general antiderivative, that is

$$\int f(x)\,dx = F(x) + \underline{C} \quad \text{where } C \text{ is a constant}$$

Indefinite Integral no lower or upper bound. $F'(x) = f(x)$

Example 3.4.1. *Evaluate* $\int (x^3 + 4x)dx$

$$= \int x^3\,dx + \int 4x\,dx = \int x^3\,dx + 4\int x\,dx$$

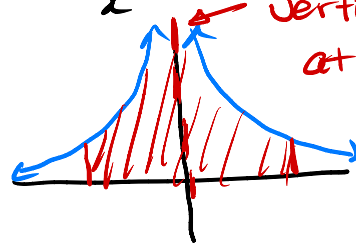
$$= \frac{1}{4}x^4 + 4\cdot\frac{1}{2}x^2 = \frac{1}{4}x^4 + 2x^2 + \underline{C}.$$

The general result of an indefinite integral is another function. It is assumed that the result is valid on a certain interval. For example,

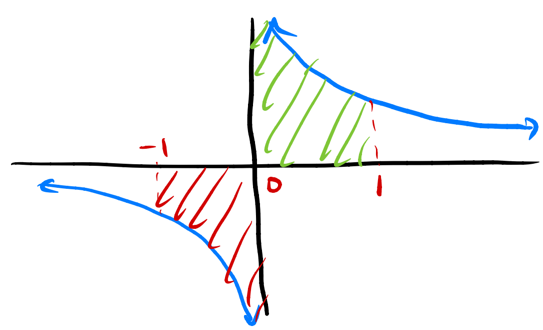
Example 3.4.2. *Compute* $\int \frac{1}{x^2}dx$

$$= \int x^{-2}\,dx = -1x^{-1} + C = -\frac{1}{x} + C.$$

vertical asymptote at $x = 0$.



Example 3.4.3. *Compute* $\int_{-1}^{1} \frac{1}{x}dx$



$$= \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

$$= \Big[\ln|x|\Big]_{-1}^{0} + \Big[\ln|x|\Big]_{0}^{1}$$

Suppose we let $t \to 0$.

$$= \lim_{t\to 0}\Big[\big[\ln|x|\big]_{-1}^{t} + \big[\ln|x|\big]_{t}^{1}\Big]$$

$$= \lim_{t\to 0}\ln|t| - \cancel{\lim_{t\to 0}\ln|-1|} \to 0 + \cancel{\lim_{t\to 0}\ln|1|} \to 0 - \lim_{t\to 0}\ln|t|$$

$$= \lim_{t\to 0}\ln|t| - \lim_{t\to 0}\ln|t| = 0$$

Example 3.4.4. *Compute the following integrals:*

(a) $\int (\sin(x) - \cos(x))dx$ $= \int \sin(x)\,dx - \int \cos(x)\,dx$

$= -\cos(x) - \sin(x) + C.$

(b) $\int \frac{2x^3 - 3x}{x^2} dx$ $= \int \frac{2x^3}{x^2} dx - \int \frac{3x}{x^2} dx = \int 2x\,dx - \int \frac{3}{x} dx$

$= x^2 - 3\ln|x| + C.$

(c) $\int_0^1 \left(x^2 - \frac{1}{\sqrt{1-x^2}}\right) dx$ $= \int_0^1 x^2\,dx - \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

$= \left[\frac{1}{3}x^3\right|_0^1 - \left[\sin^{-1}(x)\right|_0^1 = \frac{1}{3} - (\underbrace{\sin^{-1}(1)}_{\frac{\pi}{2}} - \underbrace{\sin^{-1}(0)}_{0})$

$= \frac{1}{3} - \frac{\pi}{2}.$

(d) $\int_1^3 \frac{x^4 - x\sqrt{x} - 1}{x} dx$ $= \int_1^3 \frac{x^4}{x} dx - \int_1^3 \frac{x\sqrt{x}}{x} dx - \int_1^3 \frac{1}{x} dx$

$= \int_1^3 x^3\,dx - \int_1^3 \sqrt{x}\,dx - \int_1^3 \frac{1}{x}\,dx$

$\int x^{\frac{1}{2}}\,dx = \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1}$

$= \left[\frac{1}{4}x^4\right|_1^3 - \left[\frac{2}{3}x^{\frac{3}{2}}\right|_1^3 - \left[\ln|x|\right|_1^3$

$= \frac{81}{4} - \frac{1}{4} - \left(\frac{2}{\cancel{3}}\cancel{3}\sqrt{3} - \frac{2}{3}\right) - (\ln(3) - \underbrace{\cancel{\ln(1)}}_{0})$

$= 20 - \left(2\sqrt{3} - \frac{2}{3}\right) - \ln(3)$

$= 6\frac{2}{3} - 2\sqrt{3} - \ln(3).$

If you have the rate of change of a quantity, for example, velocity is the rate of change of position, then the integral of the rate of change (velocity) is the net change of the quantity (position).

Example 3.4.5. *The velocity of an object is given by the function $v(t) = t^3 - 4t$. What is the total displacement of the object form 3.0 s to 10.0 s?*

$$\Delta S = \int_3^{10} v(t)dt = \int_3^{10} (t^3 - 4t)\,dt = \left[\frac{1}{4}t^4 - 2t^2\right|_3^{10}$$

$$= \frac{1}{4}10^4 - 2(10)^2 - \left(\frac{3^4}{4} - 2(3)^2\right)$$

$$= 2297.75 \text{ units.}$$

Example 3.4.6. *Air is being blown into a balloon at a rate given by $V'(t) = 3t$ cm^3/s. If the balloon is initially empty, what is the volume of the balloon after 5.0 s?*

$$V(0) = 0 \qquad \frac{dV}{dt} = 3t$$

$$\Delta V = \int_0^5 3t\,dt = \left[\frac{3t^2}{2}\right|_0^5 = \frac{3(5)^2}{2} - 0 = \frac{75}{2} \text{ cm}^3.$$

Example 3.4.7. *Compute each indefinite integral:*

(a) $\int (x^3 + 6x + 1)dx$

$$= \int x^3 dx + \int 6x\,dx + \int dx$$

$$= \frac{1}{4}x^4 + 3x^2 + x + C$$

(b) $\int \frac{\sin(x)}{1 - \sin^2(x)}dx$ $\to \cos^2(x)$

$$= \int \frac{\sin(x)}{\cos^2(x)}\,dx$$

$$= \int \sec(x)\tan(x)\,dx$$

$$= \sec(x) + C$$

3.5 Techniques of Integration

It is sometimes difficult to determine antiderivatives of most functions using only our knowledge of differentiation rules. Therefore, we need a set of tools that we can use to evaluate integrals.

3.5.1 The Substitution Rule

Let's start with a (sort of) simple derivative of the function $y = (\overset{u}{\overbrace{1 + x^2}})^3$

Let $u = 1 + x^2$. $\frac{du}{dx} = 2x$ $y = u^3$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \frac{dy}{du} = 3u^2$$

$$\frac{dy}{dx} = 3u^2 \cdot \frac{du}{dx} = 3u^2(2x) = 6x(1+x^2)^2$$

We can apply this idea to integrals.

Note: you often substitute for what is under the $\sqrt{}$

Example 3.5.1. *Compute* $\int 2x\sqrt{1 + x^2}dx$

Let $u = 1 + x^2$. $\Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$$\int \sqrt{1+x^2}\, 2x\,dx = \int \sqrt{u}\, du$$ ← Integrating with respect to u.

$$= \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

This method of substituting works when we have an integral of the form:

$$\int f(u) \cdot \frac{du}{dx} \cdot dx.$$

Example 3.5.2. *Compute the following integrals using a substitution.*

(a) $\int \tan(x)dx$ $= \int \frac{\sin(x)}{\cos(x)} dx$. let $u = \cos(x) \Rightarrow du = -\sin(x)dx$

$= \int \frac{-du}{u} = -\int \frac{1}{u} du$ $\Rightarrow -du = \sin(x)dx$.

$= -\ln|u| + C = -\ln|\cos(x)| + C$.

$= \ln|\sec(x)| + C$

(b) $\int_1^3 x(4+x^2)^3 dx$ Let $u = 4+x^2$ $du = 2xdx$. $\Rightarrow \frac{1}{2}du = xdx$.

$= \int_5^{13} \frac{1}{2}u^3 du = \frac{1}{2}\left[\frac{1}{4}u^4\right|_5^{13}$, When $x=1$, $u = 5$, When $x=3$ $u = 13$

$= \frac{1}{2}\left(\frac{13^4}{4} - \frac{5^4}{4}\right) =$

(c) $\int x\sqrt{2x-1}dx$ Let $u = 2x-1 \Rightarrow du = 2dx \Rightarrow \frac{1}{2}du = dx$

Note: $x = \frac{u+1}{2}$

$u\sqrt{u} = u^{\frac{3}{2}}$

$= \int \left(\frac{u+1}{2}\right)\sqrt{u}\, du = \int \frac{u\sqrt{u}}{2} du + \int \frac{\sqrt{u}}{2} du$

$\int u^{\frac{3}{2}} du = \frac{1}{\frac{3}{2}+1} u^{\frac{3}{2}+1}$

$= \frac{1}{2} \times \frac{2}{5} u^{\frac{5}{2}} + \frac{1}{2} \times \frac{2}{3} u^{\frac{3}{2}} = \frac{1}{5}u^{\frac{5}{2}} + \frac{1}{3}u^{\frac{3}{2}} + C$.

(d) $\int \frac{x}{\sqrt{x^2-1}} dx$ Let $u = x^2-1 \Rightarrow du = 2xdx \Rightarrow \frac{1}{2}du = xdx$.

$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}\int u^{-\frac{1}{2}} du = \frac{1}{2} \times \frac{1}{-\frac{1}{2}+1} u^{-\frac{1}{2}+1}$

$= \frac{1}{2} \times 2u^{\frac{1}{2}} = u^{\frac{1}{2}} = (x^2-1)^{\frac{1}{2}} + C$

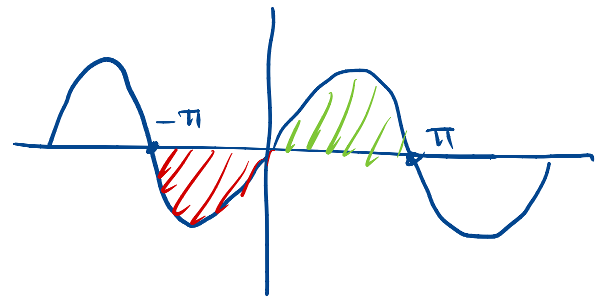
Many functions are symmetric about an axis or about a line. In such cases, we can use this knowledge to simplify definite integrals over certain intervals.

Example 3.5.3. *Evaluate and illustrate graphically* $\int_{-\pi}^{\pi} \sin(x)dx$

$$= \int_{-\pi}^{\pi} \sin(x)dx = \left[-\cos(x)\right|_{-\pi}^{\pi}$$

$$= \left[-\cos(\pi) - (-\cos(-\pi))\right]$$

$$= -(-1) + (-1) = 1 - 1 = 0.$$



Example 3.5.4. *Evaluate and illustrate graphically* $\int_{-2}^{2} x^2 dx$

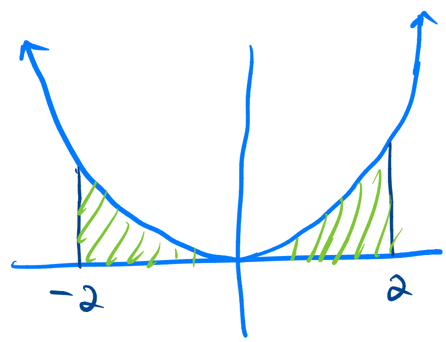
$$\int_{-2}^{2} x^2 dx = \left[\frac{1}{3}x^3\right|_{-2}^{2}$$

$$= \frac{8}{3} - \frac{-8}{3} = \frac{16}{3}$$

$$2 \times \int_0^2 x^2 dx$$

$$= 2 \times \left[\frac{1}{3}x^3\right|_0^2$$

$$= 2 \times \frac{8}{3} = \frac{16}{3}$$



We can make a rule for the integrals of even and odd functions.

For even functions

$$f(x) = f(-x)$$

$$\int_{-t}^{t} f(x)dx = 2 \times \int_0^2 f(x)dx$$

For odd functions

$$f(-x) = -f(x).$$

$$\int_{-t}^{t} f(x)dx = 0.$$

3.5.2 Integration by Parts

Suppose we differentiate a function that is a product of two simpler functions.

$f = uv$

$f' = u'v + uv'$

$f = \int f' = \int u'v + \int uv'$

$uv = \int u'v + \int uv'$

$\int uv' = uv - \int u'v$

$f(x) = g(x)h(x)$

$f'(x) = g'(x)h(x) + g(x)h'(x)$

$f = \int g'(x)h(x)dx + \int g(x)h'(x)dx$

$g(x)h(x) = \int g'(x)h(x)dx + \int g(x)h'(x)\,dx$

$\int g(x)h'(x) = g(x)h(x) - \int g'(x)h(x)dx$

Example 3.5.5. *Evaluate* $\int x\sin(x)dx$

$u = x \quad v' = \sin(x)$

$u' = 1 \quad v = -\cos(x)$

$= x(-\cos(x)) - \int 1\cdot(-\cos(x))dx$

$= -x\cos(x) + \sin(x) + C$

Tabular/DI Method

	D	I
+	x	$\sin(x)$
−	1	$-\cos(x)$
+	0	$-\sin(x)$

Example 3.5.6. *Evaluate* $\int \ln(x)dx = \int 1\cdot\ln(x)dx$

$u = \ln(x) \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

$= x\ln(x) - \int \frac{1}{x}\cdot x\,dx = x\ln(x) - \int dx = x\ln(x) - x + C$

Sometimes it is often useful to use a tabular method for integration by parts.

Example 3.5.7. *Evaluate the integral* $\int x^3e^{-x}dx.$

	D	I
+	x^3	e^{-x}
−	$3x^2$	$-e^{-x}$
+	$6x$	e^{-x}
−	6	$-e^{-x}$
+	0	e^{-x}

$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + C$

Example 3.5.8. *Evaluate* $\int e^x \cos(x)dx$. $= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$

	D	I
+	e^x	$\cos(x)$
−	e^x	$\sin(x)$
+	e^x	$-\cos(x)$

$2\int e^x \cos(x) = e^x \sin(x) + e^x \cos(x)$

$\int e^x \cos(x) dx = \dfrac{e^x \sin(x) + e^x \cos(x)}{2}$

Lesson 5

3.5.3 Trigonometric Integrals

$\cos^2(x) = 1 - \sin^2(x)$

Different combinations of trigonometric functions require different substitutions.

Example 3.5.9. *Evaluate* $\int \cos^3(x)dx$ $= \int \cos^2(x)\cos(x)dx.$

$= \int (1-\sin^2(x))\cos(x)dx.$ Let $u = \sin(x)$ $du = \cos(x)dx.$

$= \int (1-u^2)du = u - \frac{1}{3}u^3 + C = \sin(x) - \frac{1}{3}\sin^3(x) + C.$

$(\sin^2(x))^2 = (1-\cos^2(x))^2$

Example 3.5.10. *Evaluate* $\int \sin^5(x)\cos^2(x)dx$ $= \int \sin^4(x)\sin(x)\cos^2(x)dx$

$= \int (1-\cos^2(x))^2 \sin(x)\cos^2(x)dx$ Let $u = \cos(x)$

$du = -\sin(x)dx$

$= \int (1-u^2)^2 u^2 (-du) = -\int (1-2u^2+u^4)u^2 du$ $-du = \sin(x)dx.$

$= -\int (u^2 - 2u^4 + u^6) du = -\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$

Example 3.5.11. *Evaluate* $\int \cos^2(x)dx$ $= \int \dfrac{1+\cos(2x)}{2}dx$

$= \frac{1}{2}x + \frac{1}{2}\int \cos(2x)dx = \frac{1}{2}x + \frac{1}{2}\times\frac{1}{2}\sin(2x)$

$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$

$\frac{1}{2}$-Angle Formula

1) $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

2) $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

3) $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$

Recall: $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1$

$\cos^2(x) = \dfrac{\cos(2x)+1}{2}$