Chapter 2

Complex Numbers $\mathbb{C}$

2.1 The Complex Number

A complex number can be represented by an expression of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. The complex number can be represented by the ordered pair (a, b) and plotted as a point in the plane (called the Argand plane). Thus, the complex number $\underbrace{i = 0}_{a} + \underbrace{1}_{b} \cdot i$ is identified with the point $(0, 1)$.

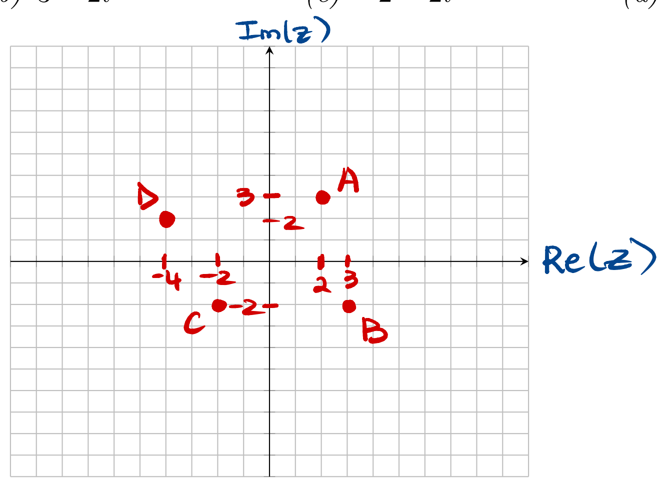
Example 2.1.1. *Plot the following complex numbers:*

(a) $2 + 3i$ (2,3)

(b) $3 - 2i$ (3,-2)

(c) $-2 - 2i$ (-2,-2)

(d) $-4 + 2i$ (-4,2)



The real part of the complex number $a + bi$ is the real number a, denoted by $\Re(a + bi) = a$ Re(a+bi) and the imaginary part is the real number b, denoted by $\Im(a + bi) = b$. For example, the complex number $4 - 3i$ has as its real part 4 and its imaginary part -3. Im(a+bi) = b

We will denote the complex number using the variable z where $z = a + ib$.

2.2 Operations with Complex Numbers

The sum and difference of two complex numbers are defined by adding their real parts and imaginary parts. For example, if $z = a + bi$ and $w = c + di$, then

$$z \pm w = (a+bi) \pm (c+di) = (a \pm c) + (bi \pm di)$$
$$= (a \pm c) + i(b \pm d)$$

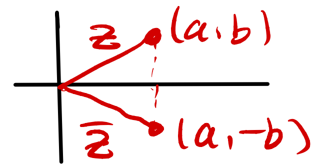
$i = \sqrt{-1} \Rightarrow i^2 = -1$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$z \cdot w = (a+bi)(c+di) = ac + adi + bci + bdi^2.$$
$$= (ac - bd) + (adi + bci)$$
$$= (ac - bd) + i(ad + bc).$$

Example 2.2.1. *Multiply* $(-1+3i)(2-5i)$ $a = -1,\ b = 3,\ c = 2,\ d = -5$

$$(-1+3i)(2-5i)$$
$$= ((-1)(2) - (3)(-5)) + i((-1)(-5) + (3)(2))$$
$$= (-2+15) + i(5+6) = 13 + 11i$$



Division of a complex number is much like rationalizing the denominator of a rational expression. For a complex number $z = a + bi$, we define the complex conjugate to be $\bar{z} = a - bi$. To find the quotient of two complex numbers, we multiply the numerator and denominator by the complex conjugate of the denominator.

Example 2.2.2. *Express the number* $\dfrac{-1+3i}{2+5i}$ *in the form* $a + bi$

$z = 2 + 5i$
$\bar{z} = 2 - 5i$

$$\frac{-1+3i}{2+5i} \times \frac{2-5i}{2-5i} = \frac{(-1+3i)(2-5i)}{2^2 - (5i)^2} = \frac{13+11i}{29}$$

$$= \underbrace{\frac{13}{29}}_{a} + \underbrace{\frac{11}{29}}_{b} i$$

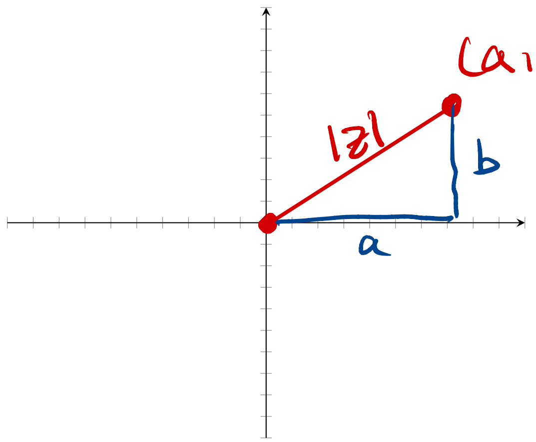
Properties of Complex Conjugate

(i) $\overline{z \pm w} = \bar{z} \pm \bar{w}$

(ii) $\overline{zw} = \bar{z}\bar{w}$

(iii) $\overline{z^n} = \bar{z}^n$

The modulus, or absolute value, $|z|$ of a complex number $z = a + bi$ is its distance from the origin.



(a,b) $(0,0) \to (a,b)$

$z = a + bi$

(a,b)

$|z| = \sqrt{a^2 + b^2}$

$|z|^2 = a^2 + b^2$

Example 2.2.3. *If* $z = a + bi$*, find* $z\bar{z}$ $\quad \bar{z} = a - bi$

$(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 + b^2 = |z|^2$

2.3 Solving Complex Equations

Since $i^2 = -1$, we can think of i as a square root of -1. But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1. We say that i is the principal square root of -1 and write $i = \sqrt{-1}$. In general, if c is a positive number, then

$\in \mathbb{Z}^+$

$i^1 = i \quad i^2 = -1 \quad i^3 = i^2 \cdot i = -i \quad i^4 = i^2 \cdot i^2 = 1 = i^0$

$i^c = \begin{cases} 1 & \text{if } c = 4k \\ i & \text{if } c = 4k+1 \end{cases} \Rightarrow \begin{cases} -1 & \text{if } c = 4k+2 \\ -i & \text{if } c = 4k+3. \end{cases}$

With the convention above, the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 2.3.1. *Solve* $x^2 + x + 1 = 0$ $a=1,\ b=1,\ c=1$

$$x = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

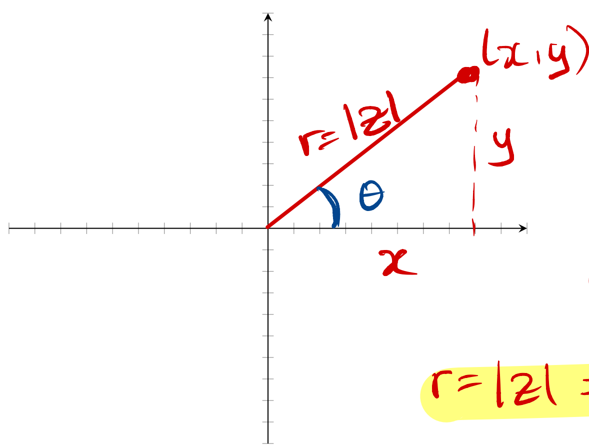
$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + z_0 = 0$$

of degree at least one has a solution among the complex numbers. This is known as the Fundamental Theorem of Algebra.

2.4 Polar Form

The complex number $z = x + yi$ can be considered as a point (x, y) and that any such point can be represented as

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$



$\cos(\theta) = \frac{x}{r}$

$\sin(\theta) = \frac{y}{r}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r = |z| = \sqrt{x^2+y^2}$

By making the substitution above, we have

$$z = r\cos(\theta) + ir\sin(\theta)$$
$$= r(\cos(\theta) + i\sin(\theta))$$

The angle θ is called the argument of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique, meaning that any two arguments of z differ by an integer multiple of 2π.

Example 2.4.1. *Write the following numbers in polar form.*

(a) $z = \overset{x}{1} + \overset{yi}{i}$

$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

(b) $w = \sqrt{3} - i$

$r = |z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{11\pi}{6}$.

$w = 2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$

The polar form of complex numbers gives insight into multiplication and division.

Example 2.4.2. *If* $z = r_1(\cos(\theta) + i\sin(\theta))$ *and* $w = r_2(\cos(\phi) + i\sin(\phi))$, *find*

(a) zw $w = z$

$= r_1 r_2 (\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi))$

$= r_1 r_2 (\cos(\theta)\cos(\phi) + i\sin(\phi)\cos(\theta) + i\sin(\theta)\cos(\phi) + i^2\sin(\theta)\sin(\phi))$

$= r_1 r_2 [\underbrace{(\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi))}_{} + i(\underbrace{\sin(\theta)\cos(\phi) + \sin(\phi)\cos(\theta)}_{\sin(\theta+\phi)})]$

$= r_1 r_2 (\cos(\theta+\phi) + i\sin(\theta+\phi))$

(b) $\frac{z}{w} = \frac{r_1}{r_2} \frac{\cos(\theta) + i\sin(\theta)}{\cos(\phi) + i\sin(\phi)} \times \frac{\cos(\phi) - i\sin(\phi)}{\cos(\phi) - i\sin(\phi)}$ $=1$

$= \frac{r_1}{r_2}(\cos(\theta) + i\sin(\theta))(\cos(\phi) - i\sin(\phi))$

$= \frac{r_1}{r_2}(\cos(\theta)\cos(\phi) - i\sin(\phi)\cos(\theta) + i\sin(\theta)\cos(\phi) - i^2\sin(\theta)\sin(\phi))$

$= \frac{r_1}{r_2}[(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)) + i(\sin(\theta)\cos(\phi) - \sin(\phi)\cos(\theta))]$

$= \frac{r_1}{r_2}(\cos(\theta-\phi) + i\sin(\theta-\phi))$

(c) $\frac{1}{z}$

$= \frac{1}{r_1} \cdot \frac{1}{\cos(\theta) + i\sin(\theta)} \times \frac{\cos(\theta) - i\sin(\theta)}{\cos(\theta) - i\sin(\theta)} = \frac{1}{r_1} \times \frac{\cos(\theta) - i\sin(\theta)}{1}$

$= \frac{1}{r_1}(\cos(\theta) - i\sin(\theta))$.

$z = a + bi$

$\frac{1}{z} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$

$= \frac{\bar{z}}{|z|^2}$

Theorem 2.4.1 (De Moivre's Theorem). *If $z = r(\cos(\theta) + i\sin(\theta))$ and $n \in \mathbb{N}$, then*

$$z^n = [r(\cos(\theta) + i\sin(\theta))]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Example 2.4.3. *Use De Moivre's Theorem to evaluate* $\left(\frac{1}{2} + \frac{i}{2}\right)^{10}$

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$

$\theta = \tan^{-1}\left(\frac{1/2}{1/2}\right) = \frac{\pi}{4}$

$n = 10$

$z = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$\left(\frac{1}{2} + \frac{i}{2}\right)^{10}$

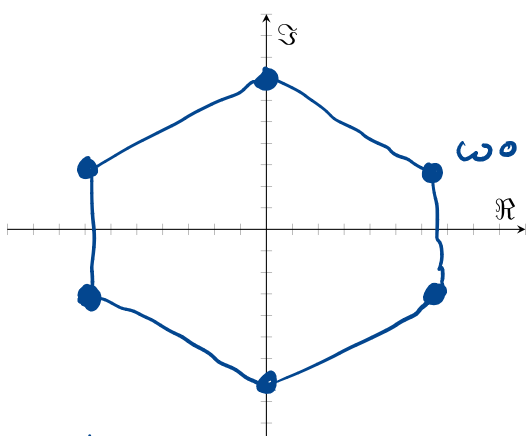
$= \frac{1}{2^5}\left(\cos\left(10 \times \frac{\pi}{4}\right) + i\sin\left(10 \times \frac{\pi}{4}\right)\right)$

Theorem 2.4.2 (Roots of a Complex Number). *Let $z = r(\cos(\theta) + i\sin(\theta))$ and let $n \in \mathbb{N}$. Then z has n distinct nth roots given by*

$$\omega_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$.

Example 2.4.4. *Find the six sixth roots of $z = -8$, and graph these roots in the complex plane.*



ω_0

$r = 8$

$\theta = \pi$

$\omega_k = 8^{\frac{1}{6}}\left(\cos\left(\frac{\pi + 2k\pi}{6}\right) + i\sin\left(\frac{\pi + 2k\pi}{6}\right)\right)$

$k = 0, 1, 2, 3, 4, 5.$

$\omega_0 = 8^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

$\omega_1 = 8^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$\omega_2 = 8^{\frac{1}{6}}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$

$\omega_3 = 8^{\frac{1}{6}}\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$

$\omega_4 = 8^{\frac{1}{6}}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$

$\omega_5 = 8^{\frac{1}{6}}\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$

2.5 Euler's Formula

Euler's Formula is a popular formula in complex numbers and is given by the formula

$$e^{iz} = \cos(z) + i\sin(z)$$

Example 2.5.1. *Evaluate the following numbers using Euler's Formula*

(a) $e^{i\pi} = -1$

$= \cos(\pi) + i\sin(\pi)$

$= -1$

(b) $e^{-1+\frac{i\pi}{2}} = \frac{i}{e}$

$= e^{-1} e^{\frac{i\pi}{2}}$

$= e^{-1}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= ie^{-1} = \frac{i}{e}$

We can use Euler's Formula to easily prove De Moivre Formula.

$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ $\quad r = 1$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$(e^{i\theta})^n = (\cos(\theta) + i\sin(\theta))^n$

$e^{i(n\theta)} = (\cos(\theta) + i\sin(\theta))^n$

$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$