Chapter 1

Calculus I Review

Recall: $\sin(A+B)$
$= \sin(A)\cos(B) + \cos(A)\sin(B)$

1.1 Derivatives of Trigonometric Functions

Below are two limits in which they are important for when proving the derivatives of trigonometric functions:

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0$$

We can determine the derivative of $f(x) = \sin(x)$ by using the definition of the derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\left(\sin(x)\boxed{\frac{(\cos(h)-1)}{h}}^{\to 0} + \cos(x)\boxed{\frac{\sin(h)}{h}}^{\to 1}\right)$$

$$= \cos(x)$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Similarly, we can determine that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$. Using these definitions, we can determine the derivatives of the other trigonometric functions.

Example 1.1.1. *Determine the derivative of* $f(x) = \tan(x)$. $= \frac{\sin(x)}{\cos(x)}$

$$f'(x) = \frac{\overbrace{\cos(x)\cos(x)}^{\cos^2(x)}\overbrace{-\sin(x)(-\sin(x))}^{+\sin^2(x)}}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x) \qquad \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

product rule

$(fg)' = f'g + fg'$

Example 1.1.2. *Differentiate the following functions:*

(a) $f(x) = x\sin(x)$

$$f'(x) = 1 \cdot \sin(x) + x \cdot \cos(x)$$
$$= \sin(x) + x\cos(x).$$

(b) $y = \dfrac{\sec(x)}{1+\sec(x)}$

$$\frac{dy}{dx} = \frac{\sec(x)\tan(x)(1+\sec(x)) - \sec(x)\sec(x)\tan(x)}{(1+\sec(x))^2}$$
$$= \frac{\sec(x)\tan(x) + \cancel{\sec^2(x)\tan(x)} - \cancel{\sec^2(x)\tan(x)}}{(1+\sec(x))^2}$$
$$= \frac{\sec(x)\tan(x)}{(1+\sec(x))^2}$$

(c) $y = e^u(\cos(u) + \overset{\text{constant}}{c}u)$ $\quad y = g(u)$

$$\frac{dy}{dx} = e^u(\cos(u)+cu) + e^u(-\sin(u)+c)$$
$$= e^u(\cos(u)+cu-\sin(u)+c)$$

(d) $y = \cos(\overset{\text{constant}}{a^3} + x^3)$

$$\frac{dy}{dx} = -\sin(a^3+x^3) \times 3x^2$$
$$= -3x^2\sin(a^3+x^3).$$

(e) $y = \sin(\sin(\sin(x^3)))$

$$\frac{dy}{dx} = \cos(\sin(\sin(x^3)))\cos(\sin(x^3))\cos(x^3)\,3x^2$$
$$= 3x^2\cos(\sin(\sin(x^3)))\cos(\sin(x^3))\cos(x^3).$$

$y = y(x)$

$xy = 1 \Rightarrow y + x \cdot \frac{dy}{dx} = 0$

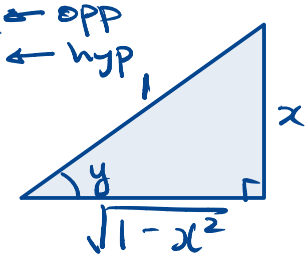
$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$.

1.2 Inverse Trigonometric Functions

We can also determine the derivatives of the inverse functions using implicit differentiation.

Example 1.2.1. *Differentiate* $y = \sin^{-1}(x)$

$\frac{x}{1} \leftarrow \text{opp}$
$ \leftarrow \text{hyp}$



$x = \sin(y)$.

$\frac{d}{dx}(x) = \frac{d}{dx}(\sin(y(x)))$

$1 = \cos(y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$

Analogous, we can also find the derivatives of the inverse trigonometric functions which are provided below.

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}$$

Example 1.2.2. *Differentiate the following functions*

(a) $y = \sin^{-1}(2x+1)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1+(2x+1)^2}} \cdot 2$

$= \frac{2}{\sqrt{1+(2x+1)^2}}$

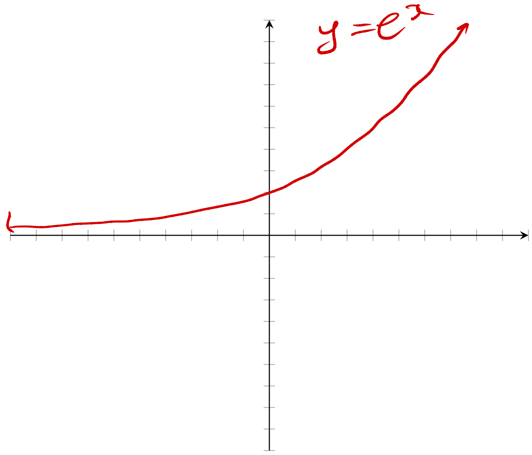
(b) $f(\theta) = \cos^{-1}\left(e^{2\theta}\right)$

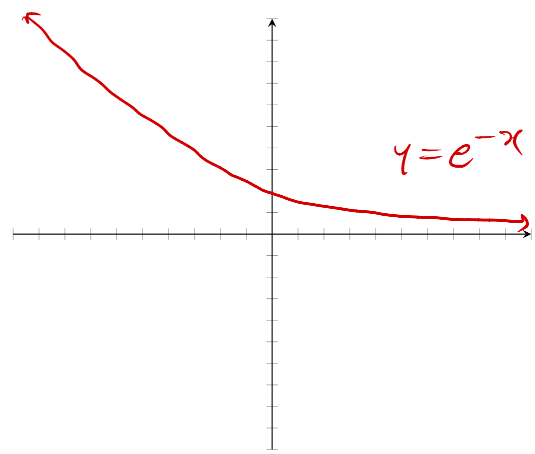
$f'(\theta) = -\frac{1}{\sqrt{1-(e^{2\theta})^2}}\, 2e^{2\theta}$

$= -\frac{2e^{2\theta}}{\sqrt{1-e^{4\theta}}}$

1.3 Hyperbolic Functions

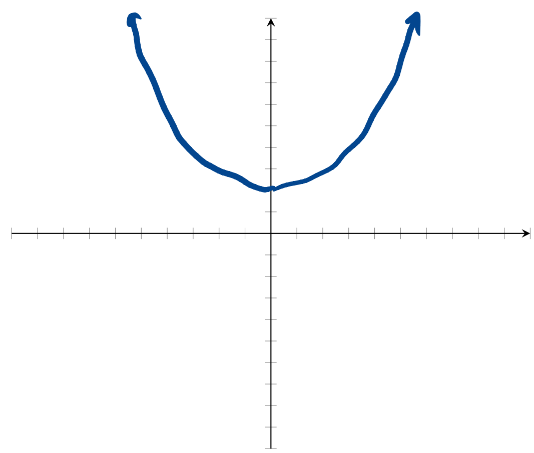
Let's start with the exponential functions $y = e^x$ and $y = e^{-x}$.





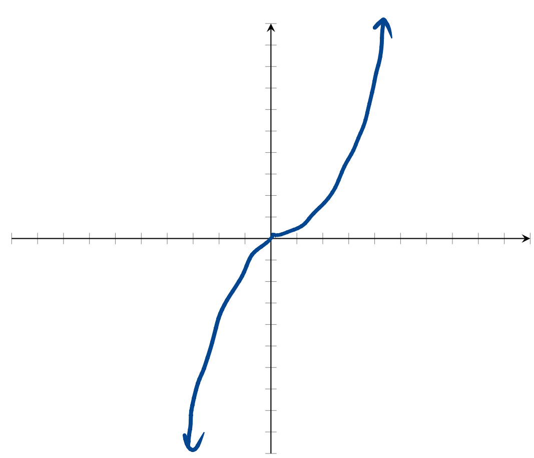
The function $y = \cosh(x)$ is equivalent to the expression

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



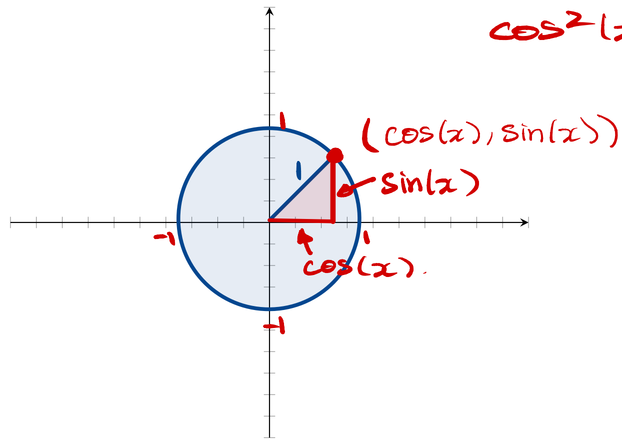
The function $y = \sinh(x)$ is equivalent to the expression

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$



Why are they called hyperbolic trigonometric functions? The trigonometric functions of sine and cosine define a unit circle when used as the x and y coordinates:

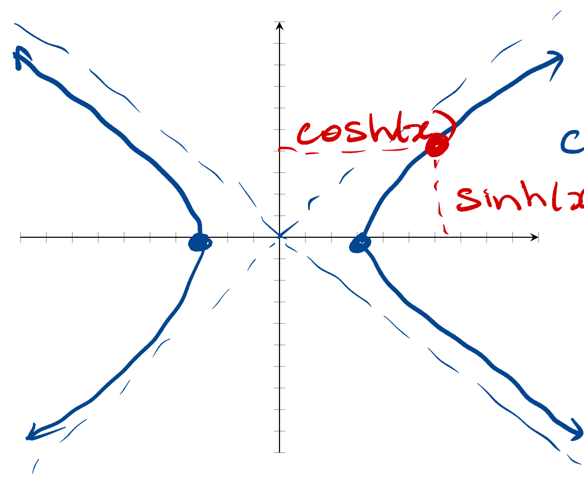
$$\cos^2(x) + \sin^2(x) = 1$$



When the hyperbolic functions are used as the x and y coordinates:

$$(\cosh(x), \sinh(x))$$

$$\cosh^2(x) - \sinh^2(x) = 1$$



Example 1.3.1. *Determine the derivatives of* $y = \cosh(x)$ *and* $y = \sinh(x)$

$$\frac{d}{dx}(\cosh(x)) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(e^x - e^{-x})$$

$$= \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$\frac{d}{dx}(\sinh(x)) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

Hyperbolic Identities

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad 1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \qquad \cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \qquad \frac{d}{dx}(\cosh(x)) = \sinh(x)$$

$$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x) \qquad \frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x)\coth(x)$$

$$\frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x)\tanh(x) \qquad \frac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$$

Inverse Hyperbolic Functions

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right),\ x \in \mathbb{R}$$

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right),\ x > 1$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ x \in (-1,1)$$

Derivative of Inverse Hyperbolic Functions

$$\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}\left(\cosh^{-1}(x)\right) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tanh^{-1}(x)\right) = \frac{1}{1-x^2} \qquad \frac{d}{dx}\left(\operatorname{csch}^{-1}(x)\right) = -\frac{1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1}(x)\right) = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\coth^{-1}(x)\right) = \frac{1}{1-x^2}$$

1.4 L'Hôpital's Rule

Say f, g $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{0}{0}$. does not mean that does not exist.

$\Rightarrow \lim_{x\to 0}\frac{f'(x)}{g'(x)} = L$

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Example 1.4.1. *Evaluate the following limits.*

Indeterminate

(a) $\lim_{x\to 0}\frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0}\frac{e^x}{1} = \lim_{x\to 0} e^x = e^0 = 1.$

(b) $\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \frac{1-\cos(0)}{0^2} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0}\frac{\sin(x)}{2x} \overset{L'H}{=} \lim_{x\to 0}\frac{\cos(x)}{2} = \frac{1}{2}.$

1.5 Rolle's Theorem and The Mean Value Theorem

Theorem 1.5.1 (**Rolle's Theorem**). *If f is a function that is*

(i) continuous on the closed interval $[a, b]$

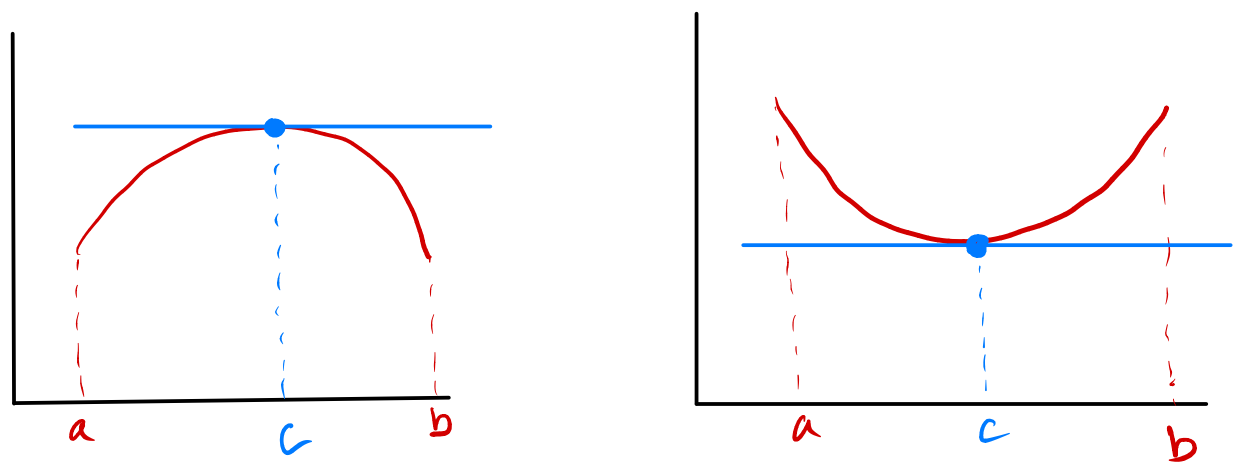
(ii) differentiable on the open interval (a, b)

(iii) $f(a) = f(b)$

Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

There exists a point where the graph is horizontal,

What goes up, must come down (or what goes down, must come up?)



Theorem 1.5.2 (**The Mean Value Theorem**). *Let f be a function that is*

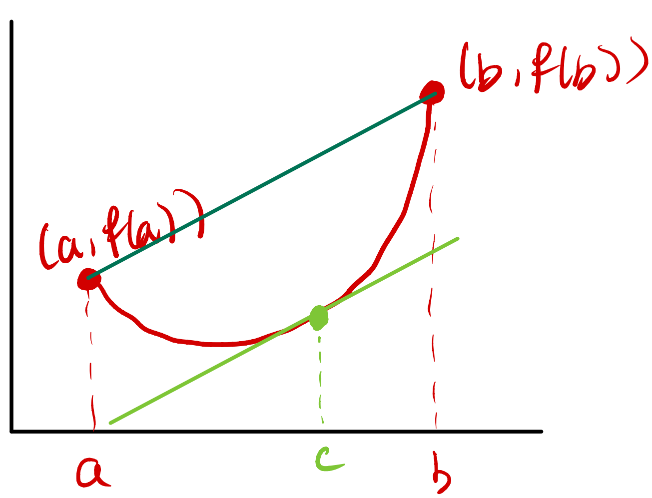
(i) continuous on the closed interval $[a, b]$

(ii) differentiable on the open interval (a, b)

Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

there is a point on the graph that has the same slope as the line from a to b.



1.6 Antiderivatives

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

For example, suppose we want to find the position of an object that has a velocity given by $v(t) = t^2$. We know that the velocity is the derivative of the position, so we can work backwards to determine the position of the object by thinking about the power rule of derivatives.

$s(t) = ?$

$s'(t) = v(t) = t^2$.

Start t^3, derivative $3t^2$

$s(t) = \frac{1}{3}t^3$

Is this the only function that works?

$s(t) = \frac{1}{3}t^3 + 5$ $\quad$ $s(t) = \frac{1}{3}t^3 - 100$.

In general, the general antiderivative of f on an interval I is $F(x) + C$ where C is an arbitrary constant as F is a particular antiderivative of f on the interval I.

Example 1.6.1. *Determine the antiderivatives of the following:*

(a) $f(x) = \cos(x)$

$\frac{d}{dx}(\sin(x)) = \cos(x)$

$F(x) = \sin(x) + C$.

(b) $f(x) = \dfrac{1}{x}$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

$F(x) = \ln(x) + C$.

(c) $f(x) = e^x$

$F(x) = e^x + C$

(d) $f(x) = \dfrac{1}{1+x^2}$

$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$

$F(x) = \tan^{-1}(x) + C$.

Function	Particular Antiderivative	Function	Particular Antiderivative
$f(x) = a$	$F(x) = ax$	$\cos(x)$	$\sin(x)$
$f(x) = x^n$	$\frac{1}{n+1}x^{n+1}$	$\frac{1}{x}$	$\ln(x)$
$\sin(x)$	$-\cos(x)$	e^x	e^x

In some instances we are given additional information that allows us to find the actual function. For example, suppose we have the object that moves with velocity $v(t) = t^2$. If it is known that the object is at a position of 10 units at a time $t = 3$ s, we can determine the exact position for the position of the object.

$s(t) = \frac{1}{3}t^3 + C$

$s(3) = 10$

$10 = \frac{1}{3}(3)^3 + C$

$10 = 9 + C$

$C = 1$

$s(t) = \frac{1}{3}t^3 + 1$

Example 1.6.2. *Find an antiderivative F of f that satisfies the following condition.*

(a) $f(x) = 5x^4 - 2x^5$; $F(0) = 4$

$F(x) = x^5 - \frac{1}{3}x^6 + C$

$4 = 0^5 - 0^6 + C$

$C = 4$

$F(x) = x^5 - \frac{1}{3}x^6 + 4.$

(b) $f(x) = 4 - 3\underline{(1 + x^2)^{-1}}$, $F(1) = 0$

$\frac{3}{1+x^2}$

$F(x) = 4x - 3\tan^{-1}(x) + \frac{3\pi}{4} - 4$

$F(x) = 4x - 3\tan^{-1}(x) + C$

$0 = 4 - 3\tan^{-1}(1) + C$

$0 = 4 - \frac{3\pi}{4} + C$

$C = \frac{3\pi}{4} - 4$

1.7 Series

It is convenient in many situations to express a series using summation notation.

$$\sum_{k=3}^{12} 2k^2 = 2(3)^2 + 2(4)^2 + 2(5)^2 + 2(6)^2 + 2(7)^2 + \cdots + 2(12)^2 = \_\_\_\_$$

Some rules ofr summation notation

(i) $\sum_{k=a}^{b} Cx_k = C\sum_{k=a}^{b} x_k$

(ii) $\sum_{k=a}^{b}(x_k \pm y_k) = \sum_{k=a}^{b} x_k \pm \sum_{k=a}^{b} y_k$

(iii) $\sum_{k=a}^{c} x_k + \sum_{k=c+1}^{b} x_k = \sum_{k=a}^{b} x_k$

$$\sum_{k=2}^{8} k^2 + \sum_{k=9}^{20} k^2 = \sum_{k=2}^{20} k^2$$

Example 1.7.1. *Determine expressions for the following sums*

(a) $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(b) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(c) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(d) $\sum_{k=1}^{n} C = \underbrace{C + C + C + \cdots + C}_{n \text{ times}}$

$$\sum_{k=1}^{n} C = nC$$