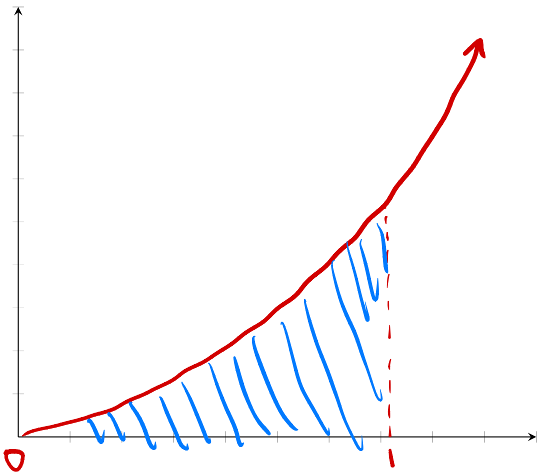
Chapter 3

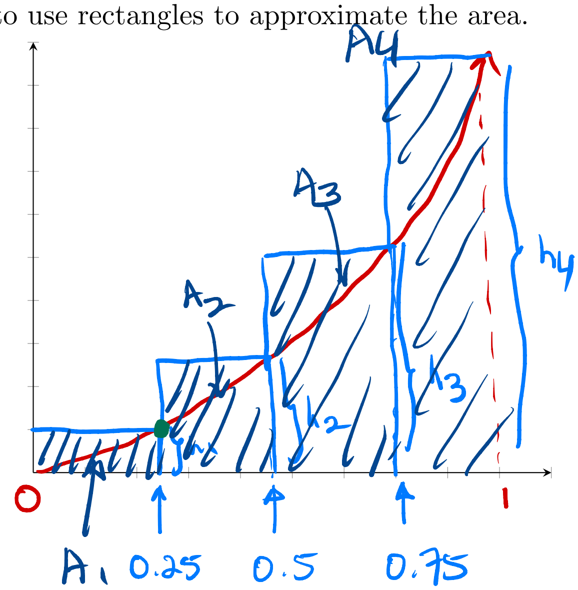
Integrals

3.1 Areas

While areas of certain shapes, such as triangles, rectangles, and even circles are easy to determine, the same is not true for the area under a curve. Consider the graph of a simple parabola



Suppose we need to find the area under the curve from $x = 0$ to $x = 1$. One way we can do this is to use rectangles to approximate the area.



$h_1 = f(0.25) = 0.25^2 = 0.0625$

$h_2 = f(0.5) = 0.5^2 = 0.25$

$h_3 = f(0.75) = 0.75^2 = 0.5625$

$h_4 = f(1) = 1^2 = 1$

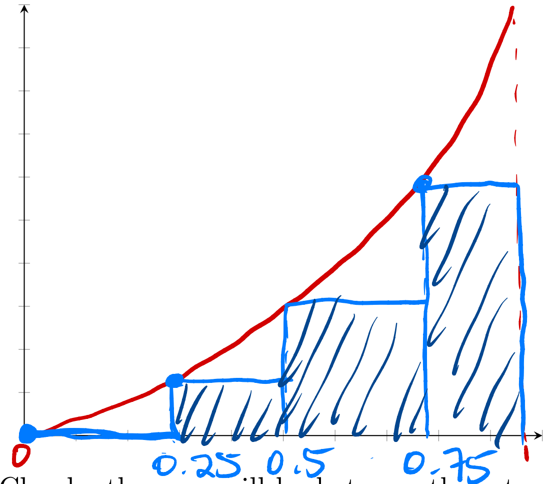
$A_1 = b \times h_1 = 0.25 \times 0.0625$
$= 0.015625$

$A_2 = 0.0625$ $A_3 = 0.140625$

$A_4 = 0.25$

$A = A_1 + A_2 + A_3 + A_4 = 0.46875.$

Now suppose we had used the left edge of each strip to determine the height of each rectangle.



$h_1 = 0$ $h_2 = 0.0625$

$h_3 = 0.25$ $h_4 = 0.5625$

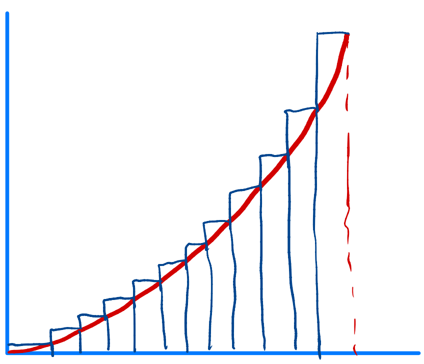
$A_1 = 0$ $A_2 = 0.015625$

$A_3 = 0.0625$ $A_4 = 0.140625$

$A = A_1 + A_2 + A_3 + A_4 = 0.21875.$

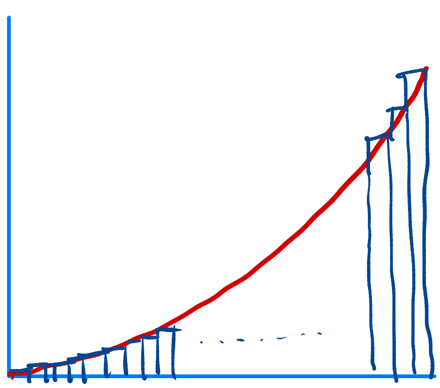
Clearly, the area will be between these two approximations. So how can we make our "guess" even better?

$$\bar{A} = \frac{A(\text{left}) + A(\text{right})}{2} = \frac{0.21875 + 0.46875}{2} = 0.34375.$$



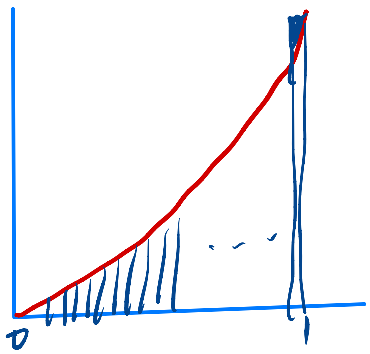
More rectangles

$\Rightarrow$ extra area will be smaller.



Our approximation gets better the more rectangles we draw

If we let n represent the number of rectangles, that is, the number of divisions we make, the approximation of the area should become exact as n approaches infinity.



$\text{base} = \frac{1}{n} \qquad h_1 = ?$

$x_1 = \frac{1}{n} \qquad x_2 = \frac{2}{n} \quad \cdots \quad x_k = \frac{k}{n}$

$h_1 = f(x_1) = \frac{1}{n^2} \qquad h_2 = \frac{4}{n^2} \quad \cdots \quad x_k = \frac{k^2}{n^2}$

$A_1 = \text{base} \times h_1 = \frac{1}{n} \cdot \frac{1}{n^2} = \frac{1}{n^3}$

$A_2 = \frac{1}{n} \times \frac{4}{n^2} = \frac{4}{n^3}$

$\vdots$

$A_k = \frac{1}{n} \times \frac{k^2}{n^2} = \frac{k^2}{n^3}$

$A = A_1 + A_2 + \cdots + A_k + \cdots + A_n$

$= \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{k^2}{n^3}$

For an exact value, we calculate $A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3}$.

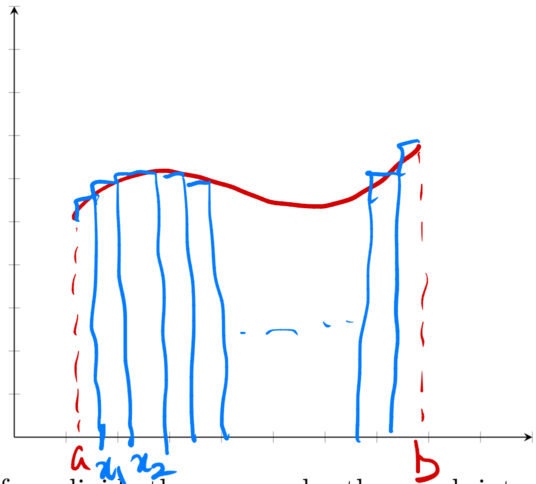
$A = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right)$

$= \lim_{n\to\infty} \frac{1}{n^3} \left(\frac{2n^3 + 3n^2 + n}{6} \right) = \lim_{n\to\infty} \left(\frac{2}{6} + \cancelto{0}{\frac{1}{2n}} + \cancelto{0}{\frac{1}{6n^2}} \right)$

$A = \boxed{\frac{1}{3}} \Leftarrow$ Exact area under $y = x^2$ from $x \in [0, 1]$

We can use this approach to determine the area under graphs of other functions.

Suppose we wish to determine the area under the graph of $y = f(x)$ over the interval $[a, b]$.



$x_1 = a + \text{width}$

$x_2 = a + 2 \times \text{width}$

$\vdots$

$x_k = a + k \times \text{width}$

If we divide the area under the graph into n rectangles of equal width, then the width of each rectangle can be determined as

$$\text{width} = \frac{b-a}{n} \leftarrow \text{length of interval } [a,b]$$

If we use the right edge of each interval to determine the height, we need an expression for the x coordinate

$$x_i^* = a + i\left(\frac{b-a}{n}\right)$$

$$x_1 = a + \frac{b-a}{n}$$

$$x_2 = a + 2 \times \frac{b-a}{n}$$

$$x_k = a + k \times \frac{b-a}{n}$$

The height can then be given as

$$h_k = f(x_k^*)$$

And the total area will be the sum of the areas of the rectangles.

$$A = A_1 + A_2 + \cdots + A_k + \cdots + A_n$$

$$= \text{width} \times f(x_1^*) + \text{width} \times f(x_2^*) + \cdots + \text{width} \times f(x_n^*)$$

$$= \frac{b-a}{n}\left(f\left(a + \frac{b-a}{n}\right)\right) + \frac{b-a}{n}\left(f\left(a + 2\frac{b-a}{n}\right)\right) +$$

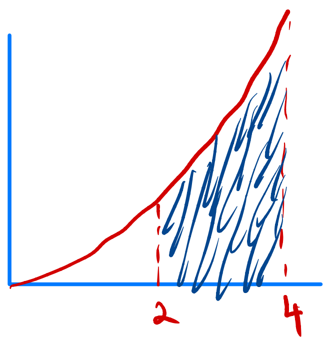
$$\cdots + \frac{b-a}{n}\left(f\left(a + k\frac{b-a}{n}\right)\right) + \cdots$$

It is more convenient to use sigma notation to express this sum.

$$A = \sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f(x_k^*)$$

$$x_k^* = a + k \times \frac{b-a}{n}.$$

Example 3.1.1. *Determine the area under the graph of the function $y = x^2$ from $x = 2$ to $x = 4$.*



$$\text{width} = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$$

$$x_k^* = 2 + k\frac{2}{n} = 2 + \frac{2k}{n}$$

$$f(x_k^*) = \left(2 + \frac{2k}{n}\right)^2 = 4 + \frac{8k}{n} + \frac{4k^2}{n^2}$$

$$A = \sum_{k=1}^{n} \frac{2}{n}\left(4 + \frac{8k}{n} + \frac{4k^2}{n^2}\right) = \sum_{k=1}^{n}\left(\frac{8}{n} + \frac{16k}{n^2} + \frac{8k^2}{n^3}\right)$$

$$= \frac{8}{n}\sum_{k=1}^{n} 1 + \frac{16}{n^2}\sum_{k=1}^{n} k + \frac{8}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{8}{n} \times n + \frac{16}{n^2} \times \frac{n(n+1)}{2} + \frac{8}{n^3} \times \frac{n(n+1)(2n+1)}{6}$$

$$= 8 + \frac{8}{n^2} \times \frac{n^2+n}{1} + \frac{4}{3n^3}(2n^3 + 3n^2 + n)$$

$$= 8 + \frac{8}{\cancel{n^2}}\cancel{n^2}\left(1 + \frac{1}{n}\right) + \frac{4}{3\cancel{n^3}}\cancel{n^3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= 8 + 8\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$A = \lim_{n\to\infty}\left(8 + 8\left(1 + \cancelto{0}{\frac{1}{n}}\right) + \frac{4}{3}\left(2 + \cancelto{0}{\frac{3}{n}} + \cancelto{0}{\frac{1}{n^2}}\right)\right)$$

$$= 8 + 8 + \frac{8}{3} = \frac{56}{3}$$

The limit of the sum that we defined earlier is the definite integral of f from a to b and can be written as

$$\lim_{n\to\infty}\sum_{k=1}^{n}\overbrace{\left(\frac{b-a}{n}\right)}^{\Delta x} f(x_k^*) = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$= \overset{\text{interval}}{\underset{\text{sum}}{\boxed{\int_a^b}}}\ \overset{\text{height}}{f(x)}\ \underset{\text{width}}{dx}$$

If the limit exists, we say that f is integrable on $[a, b]$.

This expression gives us a way to determine the area under a graph by dividing into subintervals. If the graph is above the x-axis, the areas will be positive, and if the graph of the function is below the x-axis, the areas will be negative. If we therefore take the definite integral of a function that has parts above and below the x-axis, it gives the difference between the areas on the positive side and the areas on the negative side.

Example 3.1.2. *Determine the integral of the function $x^2 - 1$ from $x = 0$ to $x = 2$.*

$$\int_0^2 (x^2-1)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x \qquad \Delta x = \frac{2}{n} \quad x_k^* = \frac{2k}{n}$$

$$f(x_k^*) = \left(\frac{2k}{n}\right)^2 - 1 = \frac{4k^2}{n^2} - 1$$

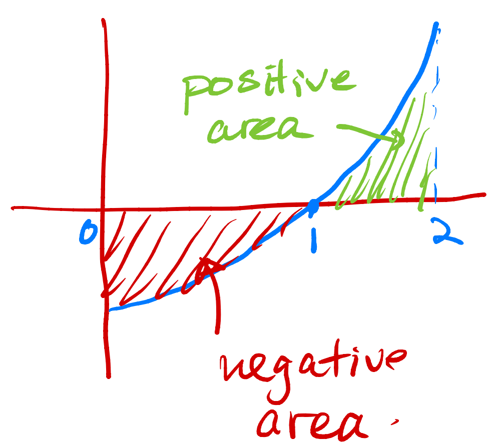
$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}-1\right)\frac{2}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{8k^2}{n^3}-\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{8}{n^3}\sum_{k=1}^{n}k^2 - \frac{2}{n}\sum_{k=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\left[\frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} - \frac{2}{n}\times n\right]$$

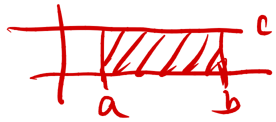
$$= \lim_{n\to\infty}\left[\frac{8}{n^3}\times\frac{2n^3+3n^2+n}{6} - 2\right]$$

$$= \frac{8}{3} - 2 = \frac{2}{3}$$



Since the definite integral is based on summation notation, it has similar properties:

(i) $\int_a^b c\,dx = c(b-a)$



(ii) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

(iii) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx.$

(iv) $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

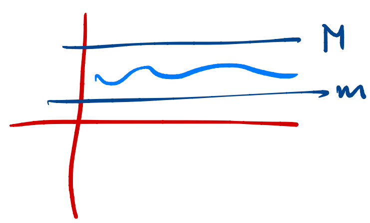
(v) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

(vi) If $f(x) \geq 0$ for $x \in [a,b]$

$\int_a^b f(x)\,dx \geq 0$

(vii) If $f(x) \geq g(x)$ for $x \in [a,b]$

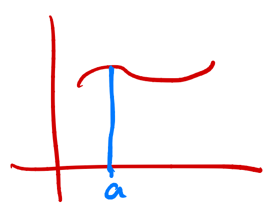
$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$



(viii) If $m \leq f(x) \leq M$ for $x \in [a,b]$

$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

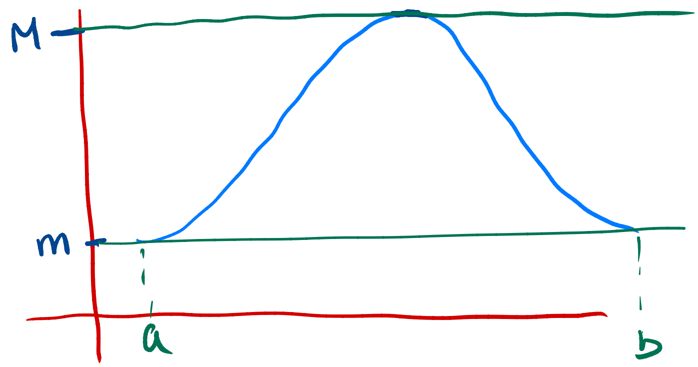
(ix) $\int_a^a f(x)\,dx = 0$



(x) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

3.2 Extreme Value Theorem and Squeeze Theorem

Theorem 3.2.1 (**Extreme Value Theorem**). *If f is continuous on $[a,b]$, then there exists $u, v \in [a,b]$ such that $f(u) = m$ and $f(v) = M$ where m and M are the absolute extrema of f of $[a,b]$.*

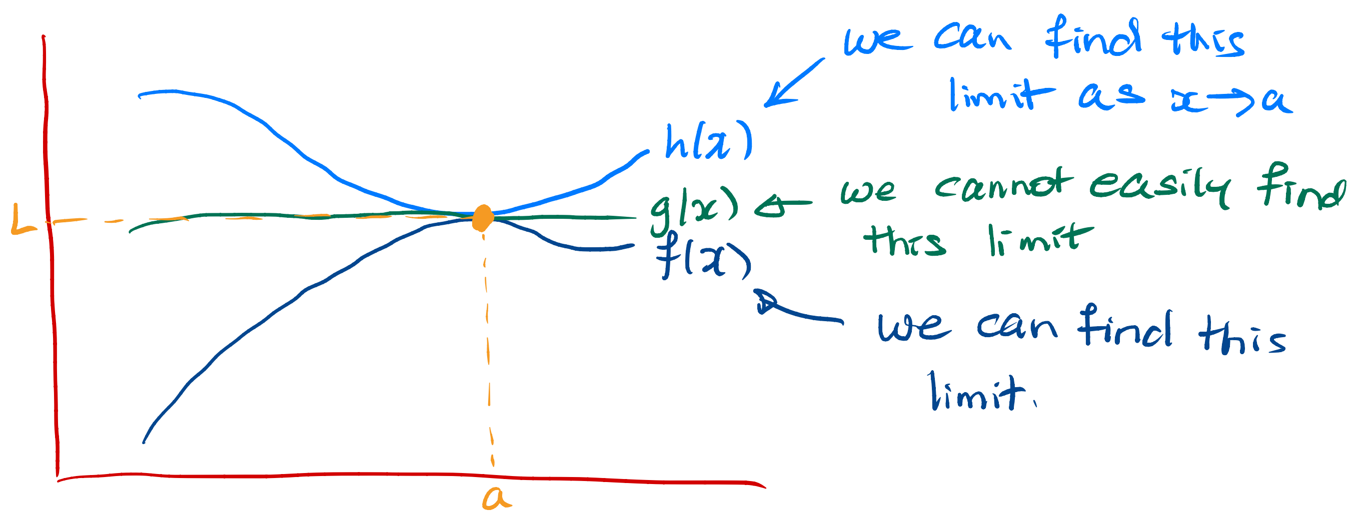


Theorem 3.2.2 (**Squeeze Theorem**). *If $f(x) \le g(x) \le h(x)$ when x is near a and if*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$



$$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0$$

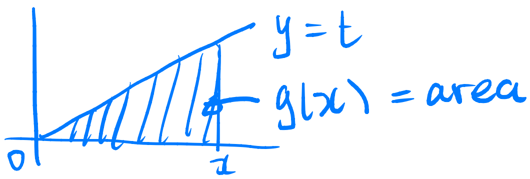
$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0$$

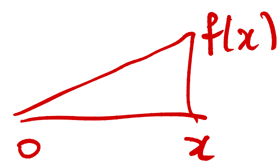
3.3 Fundamental Theorem of Calculus

Suppose we have a function $g(x)$ that is the area of $y = t$ from 0 to x.



Then $g(x)$ can be written as

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt$$



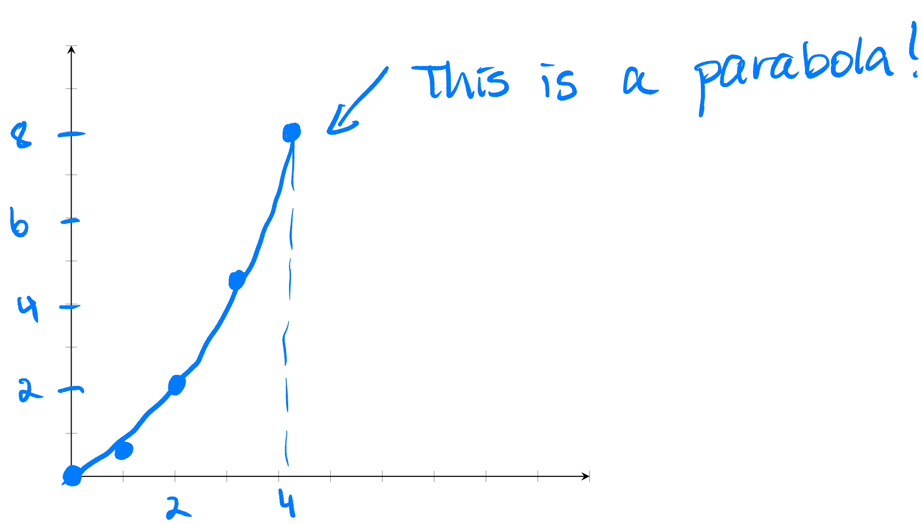
If we compute $g(0)$, $g(1)$, $g(2)$, $g(3)$ and $g(4)$, then graph $y = g(x)$

$$g(0) = \int_0^0 t\,dt = \frac{1}{2}(0)(0) = 0 \quad \text{(by property (ix))}$$

$$g(1) = \int_0^1 t\,dt = \frac{1}{2}(1)(1) = \frac{1}{2}$$

$$g(2) = \int_0^2 t\,dt = \frac{1}{2}(2)(2) = 2$$

$$g(3) = \frac{9}{2} \qquad g(4) = 8$$

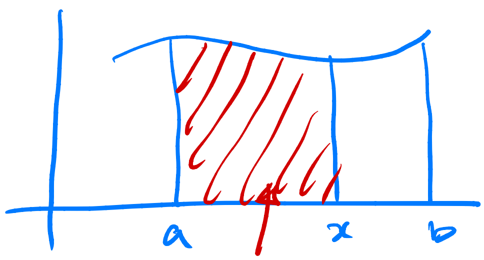


This is the Fundamental Theorem of Calculus, Part I.

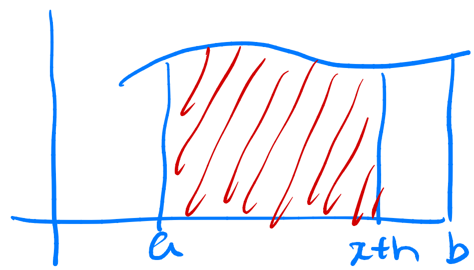
Theorem 3.3.1 (Fundamental Theorem of Calculus I). *If f is continuous on $[a, b]$, and g is defined by*

$$g(x) = \int_a^x f(t)dt$$

for $a \leq x \leq b$, then g is continuous on $[a, b]$ and is differentiable on (a, b) with $g'(x) = f(x)$.

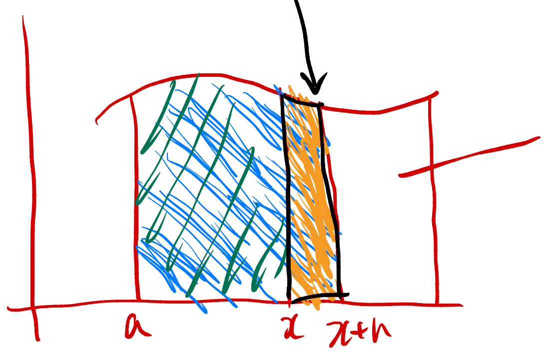


Area $= g(x)$



Area $= g(x+h)$

$g(x+h) - g(x)$



we can approximate this area with a rectangle

as $h \to 0$, $x + h \to x$

and $f(x+h) \to f(x)$

$$\lim_{h \to 0} f(x+h) = f(x)$$

$$\lim_{h \to 0} [g(x+h) - g(x)] = \lim_{h \to 0} f(x) h$$

$$\boxed{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}} = \lim_{h \to 0} f(x)$$

$$g'(x) = f(x)$$

Example 3.3.1. *Determine the derivative of the following functions:*

(a) $g(x) = \int_2^x 2t^3 dt$

$$g'(x) = \frac{d}{dx}\int_2^x 2t^3 dt = 2x^3$$

(b) $h(x) = \int_x^4 \sqrt{t-3}dt$ $= -\int_4^x \sqrt{t-3}\, dt$

$$h'(x) = -\frac{d}{dx}\int_4^x \sqrt{t-3}\, dt = \sqrt{x-3}$$

(c) $p(x) = \int_{-4}^{x^2} (2t-4)^3 dt$

$f(t) \Rightarrow f(x^2)$

$$p'(x) = \frac{d}{dx}\int_{-4}^{x^2} (2t-4)^3 dt = (2x^2-4)^3\, 2x$$

There's also the Fundamental Theorem of Calculus Part II

Theorem 3.3.2 (**Fundamental Theorem of Calculus II**). *If f is continuous on the interval $[a, b]$ and F is the antiderivative of f such that $F' = f$, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 3.3.2. *Find the area under each curve for each specified interval*

(a) $y = x^2$ from $[0, 1]$

$$F(x) = \frac{1}{n+1} x^{n+1}$$

$$A = \int_0^1 x^2 \, dx = \left[\frac{1}{3} x^3 \right|_0^1 = F(1) - F(0)$$

$$= \frac{1}{3} 1^3 - \frac{1}{3} 0^3 = \boxed{\frac{1}{3}}$$

(b) $y = e^x$ from $[0, 1]$

$$A = \int_0^1 e^x \, dx = F(1) - F(0) = e^1 - e^0 = e - 1$$