Example 3.5.8. *Evaluate* $\int e^x \cos(x)dx$. $= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)dx$

	D	I
+	e^x	$\cos(x)$
−	e^x	$\sin(x)$
+	e^x	$-\cos(x)$

$2\int e^x\cos(x) = e^x\sin(x) + e^x\cos(x)$

$\int e^x\cos(x)dx = \frac{e^x\sin(x)+e^x\cos(x)}{2}$

Lesson 5

3.5.3 Trigonometric Integrals

$\cos^2(x) = 1-\sin^2(x)$

Different combinations of trigonometric functions require different substitutions.

Example 3.5.9. *Evaluate* $\int \cos^3(x)dx$ $= \int \cos^2(x)\cos(x)dx$.

$= \int (1-\sin^2(x))\cos(x)dx$. Let $u=\sin(x)$ $du = \cos(x)dx$.

$= \int (1-u^2)du = u - \frac{1}{3}u^3 + C = \sin(x) - \frac{1}{3}\sin^3(x) + C$.

$(\sin^2(x))^2 = (1-\cos^2(x))^2$

Example 3.5.10. *Evaluate* $\int \sin^5(x)\cos^2(x)dx$ $= \int \sin^4(x)\sin(x)\cos^2(x)dx$

$= \int (1-\cos^2(x))^2\sin(x)\cos^2(x)dx$ Let $u=\cos(x)$
$du = -\sin(x)dx$

$= \int (1-u^2)^2u^2(-du) = -\int (1-2u^2+u^4)u^2du$ $-du = \sin(x)dx$.

$= -\int (u^2-2u^4+u^6)du = -\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$

Example 3.5.11. *Evaluate* $\int \cos^2(x)dx$ $= \int \frac{1+\cos(2x)}{2}dx$

$= \frac{1}{2}x + \frac{1}{2}\int\cos(2x)dx = \frac{1}{2}x + \frac{1}{2}\times\frac{1}{2}\sin(2x)$

$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$.

$\frac{1}{2}$-Angle Formula

1) $\cos^2(x) = \frac{1+\cos(2x)}{2}$

2) $\sin^2(x) = \frac{1-\cos(2x)}{2}$

3) $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$

Recall: $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1$

$\cos^2(x) = \frac{\cos(2x)+1}{2}$

We can form rules for integrals of the form $\int \sin^m(x)\cos^n(x)dx$

1. If m is odd
   change $\sin^2(x) \to 1-\cos^2(x)$ and save one $\sin(x)$
   Result: $\int (1-\cos^2(x))^k \cos^n(x)\sin(x)dx$
   substitute $u=\cos(x)$ $du=-\sin(x)dx$.
2. If n is odd.
   change $\cos^2(x) \to 1-\sin^2(x)$ and save one $\cos(x)$
   Result: $\int \sin^m(x)(1-\sin^2(x))^k \cos(x)dx$
   substitute $u=\sin(x)$. $du=\cos(x)dx$.
3. If m and n are even
   use ½ angle identities or if
   $m=n=1$, use double angle for $\sin(2x)$.

Example 3.5.12. *Evaluate* $\int \tan^4(x)\sec^4(x)dx$

$\sec^2(x) = 1+\tan^2(x)$

$= \int \tan^4(x)\sec^2(x)\sec^2(x)\,dx$

Let $u=\tan(x)$
$du = \sec^2(x)dx$.

$= \int \tan^4(x)(1+\tan^2(x))\sec^2(x)dx$

$= \int u^4(1+u^2)du = \int (u^4+u^6)du = \frac{1}{5}\tan^5(x)+\frac{1}{7}\tan^7(x)+C$

Example 3.5.13. *Evaluate* $\int \tan^3(x)\sec^3(x)dx$

$= \int \tan^2(x)\tan(x)\sec^3(x)dx$

Let $u=\sec(x)$
$du=\sec(x)\tan(x)dx$

$= \int (\sec^2(x)-1)\tan(x)\sec^3(x)dx$.

$= \int (\sec^2(x)-1)\tan(x)\sec(x)\sec^2(x)dx$

$= \int (u^2-1)u^2\,du = \int (u^4-u^2)du$

$= \frac{1}{5}\sec^5(x)-\frac{1}{3}\sec^3(x)+C$

We can form rules for integrals of the form $\int \tan^m(x)\sec^n(x)dx$

1. If n is even.
change $\sec^2(x) \to 1+\tan^2(x)$ and save a $\sec^2(x)$.
Result: $\int \tan^m(x)(\sec^2(x))^k \sec^2(x)dx$.
Substitute $u=\tan(x)$ $du=\sec^2(x)dx$.
2. If m is odd
change $\tan^2(x) \to \sec^2(x)-1$ and save a $\sec(x)\tan(x)$
Result: $\int (\tan^2(x))^k \sec^n(x)\sec(x)\tan(x)\,dx$.
substitute $u=\sec(x)$ $du=\sec(x)\tan(x)dx$.

Example 3.5.14. *Evaluate* $\int \sin(4x)\cos(5x)dx$ $A=4x$ $B=5x$

$$=\frac{1}{2}\int(\sin(4x-5x)+\sin(4x+5x))\,dx$$

$$=\frac{1}{2}\int(\sin(-x)+\sin(9x))\,dx$$

$$=\frac{1}{2}\int(-\sin(x)+\sin(9x))\,dx = \frac{1}{2}\left(\cos(x)-\frac{1}{9}\cos(9x)\right)+C$$

When evaluating integrals of the form $\int \sin(mx)\cos(nx)dx$

Identities to use

1) $\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$

2) $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$

3) $\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B)+\cos(A+B))$

Exercise:
prove each identity

Example 3.5.15. *Evaluate the following integrals*

(a) $\int (3e^u + \sec^2(u))du$

$= \int 3e^u\,du + \int \sec^2(u)\,du$

$= 3e^u + \tan(u) + C$

(b) $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

$= \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx$

$= \left[\frac{1}{2}x^2 + 2x + \ln|x|\right]_4^9$

$= \left(\frac{1}{2}(9)^2 + 2(9) + \ln|9|\right) - \left(\frac{1}{2}(4)^2 + 2(4) + \ln|4|\right)$

$=$

(c) $\int \sec(x)dx$

$= \int \sec(x) \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\,dx$

$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx.$

Let $u = \sec(x) + \tan(x)$ $du = (\sec(x)\tan(x) + \sec^2(x))dx$

$= \int \frac{1}{u}\,du = \ln|u| = \ln|\sec(x) + \tan(x)| + C.$

(d) $\int \cos(\ln(x))dx$ Let $u = \ln(x)$ $du = \frac{1}{x}dx$ $x\,du = dx$
$e^u = x$ $e^u du = dx$

$= \int \cos(u)e^u\,du = \int e^u \cos(u)\,du$

$= \frac{e^u \sin(u) + e^u\cos(u)}{2} = \frac{x\sin(\ln(x)) + x\cos(\ln(x))}{2} + C$

(e) $\int \tan^3(x)\,dx = \int \tan^2(x)\tan(x)dx$

$= \int (\sec^2(x) - 1)\tan(x)\,dx$

Let $u = \sec(x)$ $du = \sec(x)\tan(x)dx$

$\frac{du}{\sec(x)} = \frac{du}{u} = \tan(x)dx.$

$= \int (u^2 - 1)\cdot\frac{1}{u}\,du = \int \left(u - \frac{1}{u}\right)du$

$= \frac{1}{2}\sec^2(x) - \ln|\sec(x)| + C$

(f) $\int \sec^3(x)\,dx$

D	I
+ $\sec(x)$	$\sec^2(x)$
− $\sec(x)\tan(x)$	$\tan(x)$

$= \int \sec^2(x)\sec(x)dx$

$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx.$

$= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)dx$

$= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx$

$2\int \sec^3(x)dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|$

$\int \sec^3(x)dx = \frac{\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|}{2} + C$

(g) $\int \sin^3(x)\sqrt{\cos(x)}dx$

$= \int \sin^2(x)\sin(x)\cos^{\frac{1}{2}}(x)dx$

$= \int (1 - \cos^2(x))\cos^{\frac{1}{2}}(x)\sin(x)dx$

Let $u = \cos(x)$ $du = -\sin(x)dx$

$= -\int (1 - u^2)u^{\frac{1}{2}}\,du = \int \left(u^{\frac{5}{2}} - u^{\frac{1}{2}}\right)du$

$= \frac{2}{7}u^{\frac{7}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{7}\cos^{\frac{7}{2}}(x) - \frac{2}{3}\cos^{\frac{3}{2}}(x) + C$

(h) $\int_{\pi/6}^{\pi/2} \cot^2(x)dx$ $\cot^2(x) = \csc^2(x) - 1$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\csc^2(x) - 1)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc^2(x)dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} dx$

$= \left[-\cot(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \left[x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \left(-\cot\left(\frac{\pi}{2}\right) - \left(-\cot\left(\frac{\pi}{6}\right)\right)\right) - \left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

$= (0 + \sqrt{3}) - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}.$

(i) $\int \cos(7\theta)\cos(5\theta)d\theta$

$= \int \frac{1}{2}\left[\cos(7\theta - 5\theta) + \cos(7\theta + 5\theta)\right] d\theta$

$= \frac{1}{2}\int (\cos(2\theta) + \cos(12\theta))\,d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin(2\theta) + \frac{1}{12}\sin(12\theta)\right) + C$

$= \frac{1}{4}\sin(2\theta) + \frac{1}{24}\sin(12\theta) + C$