MATH 1200 Section A (Tutorial 02)
Tutorial 12: Final Exam Review

---

1. Recap

The following will be covered on the final exam:

(1) Propositional logic
(2) Complex numbers*
(3) Mathematical induction**
(4) Equivalence relations *
(5) Properties of functions
(6) Disproving properties of integers*
(7) Cardinality of sets

Announcements

1. Office Hours
   - Wed. Dec. 6 @ 2-4
   - Fri. Dec. 8 @ 2-4
2. Tutorial Survey

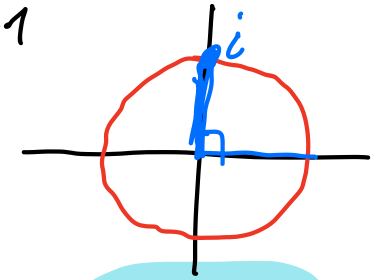
Example 1. *Find all* $z \in \mathbb{C}$ *such that* $\left(\frac{z+1}{z-1}\right)^n = i$

$$\left(\frac{z+1}{z-1}\right)^n = e^{\frac{i\pi}{2} + 2k\pi i}, k = 0, 1, 2, \dots, n-1$$

$$\frac{z+1}{z-1} = \underbrace{e^{\frac{i\pi + 4k\pi i}{2n}}}_{c}$$

$$\frac{z+1}{z-1} = c \Rightarrow z+1 = c(z-1)$$
$$z+1 = cz - c$$
$$z - zc = -1 - c$$
$$z(1-c) = -(1+c)$$

$$z_K = -\frac{1+c}{1-c}$$



Example 2. *Let* R *be the relation on the set of ordered pairs of positive integers such that* $(\underbrace{(a,b)}_{x}, \underbrace{(c,d)}_{y}) \in R$ *if and only if* $a + d = b + c$. *Show that* R *is an equivalence relation.*

1. Reflexive: Let $(a,b) \in \mathbb{N} \times \mathbb{N}$ then $a + b = a + b$ ✓

2. Symmetric: Let $(a,b), (c,d) \in \mathbb{N} \times \mathbb{N}$. Since $a + d = b + c$
$(a,b) \sim (c,d)$, we show that $(c,d) \sim (a,b)$
$\Rightarrow b + c = a + d \Rightarrow c + b = d + a \Rightarrow (c,d) \sim (a,b)$ ✓

3. Transitivity: Let $(a,b), (c,d), (e,f) \in \mathbb{N} \times \mathbb{N}$. $(a,b) \sim (c,d)$
(1) $a + d = b + c$ and since $(c,d) \sim (e,f) \Rightarrow c + f = d + e$ (2)
$a + \cancel{d} = b + \cancel{d} + e - f \Rightarrow a + f = b + e$. $\quad c = d + e - f$
$\Rightarrow (a,b) \sim (e,f)$. ✓

$$f(x) = e^{g(x)} \Rightarrow f'(x) = g'(x)e^{g(x)}$$

Example 3. *Let $f(x) = e^{2x}$, and suppose that $f^{(n)}(x)$ denotes the nth derivative of $f(x)$ with respect to x. Use mathematical induction on n to prove that $f^{(n)}(x) = 2^n e^{2x}$ for all $n \in \mathbb{N}$.*

Base Case: For $n=1$, $f'(x) = (e^{2x})' = (2x)'e^{2x} = 2e^{2x}$

$$f^{(1)}(x) = 2^1 e^{2x} = 2e^{2x}$$

Inductive Step: For $n=k$, assume, $f^{(k)}(x) = 2^k e^{2x}$. We show for $n=k+1$. $f^{(k+1)}(x) = 2^{k+1}e^{2x}$

$f^{(k+1)}(x) = f'(f^{(k)}(x)) = f'(2^k e^{2x}) = 2^k \cdot 2e^{2x} = 2^{k+1}e^{2x}$

Therefore, by POMI, $f^{(n)}(x) = 2^n e^{2x}$ for $n \in \mathbb{N}$.

Example 4. *Consider the sequence of numbers $a_0, a_1, \ldots$, given by $a_0 = 1$, $a_1 = 0$ and for all positive integers $n \geq 2$, $a_n = \frac{a_{n-2}}{4}$. Prove that for all $n \geq 2$*

$$a_n = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}$$

Base Case: $n=2$. $a_2 = \frac{a_{2-2}}{4} = \frac{1}{4}$ ✓

$$a_2 = \left(\frac{1}{2}\right)^{2+1} - \left(-\frac{1}{2}\right)^{2+1}$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} ✓$$

For $n=3$, $a_3 = \frac{a_{3-2}}{4} = \frac{a_1}{4} = \frac{0}{4} = 0$

$$a_3 = \left(\frac{1}{2}\right)^{3+1} - \left(-\frac{1}{2}\right)^{3+1} = 0$$

Inductive Step: For all $2 \leq k \leq n$, if $a_k = \frac{a_{k-2}}{4}$, then

$$a_k = \left(\frac{1}{2}\right)^{k+1} - \left(-\frac{1}{2}\right)^{k+1}$$

We show for $n = k+1$

$$a_{k+1} = \frac{a_{k-1}}{4} \overset{IH}{=} \frac{\left(\frac{1}{2}\right)^k - \left(-\frac{1}{2}\right)^k}{4} = \frac{\frac{1}{2^k} - (-1)^k \cdot \frac{1}{2^k}}{4}$$

By SI, we have shown a_n is true $\forall n \geq 2$

$$= \frac{1}{4} \cdot \frac{1}{2^k} - (-1)^k \cdot \frac{1}{4} \cdot \frac{1}{2^k} = \frac{1}{2^2} \cdot \frac{1}{2^k} - (-1)^{k+2} \cdot \frac{1}{2^2} \cdot \frac{1}{2^k}$$

$$= \left(\frac{1}{2}\right)^{k+2} - \left(-\frac{1}{2}\right)^{k+2} ✓$$

Example 5. *Prove or disprove: If n is an odd positive integer, prove that* $n^2 \equiv 1 \pmod 8$.

Proof: Assume that n is odd, i.e $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$

We show that $8 \mid n^2 - 1 \Rightarrow n^2 - 1 = 8x \ \exists x \in \mathbb{Z}$

Then:

$$(2k+1)^2 - 1 = 4k^2 + 4k = 4\underbrace{k(k+1)} = 8x$$

Case 1: If k is even $k = 2m \ \exists m \in \mathbb{Z}$

$$4(2m)(2m+1) = 8m(2m+1) = 8x \Rightarrow 8 \mid n^2 - 1 \checkmark$$

Case 2: Similar k odd (exercise)

From the above cases we conclude $n^2 \equiv 1 \pmod 8$