Tutorial 6: Complex Numbers

MATH 1200A02: Problems, Conjectures, and Proofs

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Complex Numbers

Definition

The set of complex number is given by

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}, i = \sqrt{-1}\}$$

If $z = a + ib$, then we call a the real part of z, denoted by $\text{Re}(z) = a$ and we call b the imaginary part of z, denoted by $\text{Im}(z) = b$.

Theorem

Let $z, w \in \mathbb{C}$ with $z = a + ib$ and $w = c + id$. Then the following hold:

- $z + w = (a + c) + i(b + d)$
- $z \cdot w = (ac - bd) + i(ad + bc)$
- $\bar{z} = a - ib$ *(Conjugate of z)*
- $|z| = \sqrt{a^2 + b^2}$ *(Modulus of z)*
- $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{a-ib}{a^2+b^2}$, $z \neq 0$
- $\frac{z}{w} = \frac{z\bar{w}}{|w|^2} = \frac{(a+ib)(c-id)}{c^2+d^2}$

Let $z = x + iy \in \mathbb{C}$, where $x, y \in \mathbb{R}$. Suppose $|z| = \sqrt{x^2 + y^2} = r$. Then recall the trigonometric formulas

$$\cos(\theta) = \frac{x}{r} \quad \sin(\theta) = \frac{y}{r}$$

Then

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

and so

$$z = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))$$

This is called the polar form of complex numbers, where $\theta = \arg(z)$ is called the argument of z.

Example

Given $z = 1 + i$, find: (i) $|z|$, (ii) $\arg(z)$, (iii) z in polar form.

Theorem

Let $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$ *and* $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$*, then*

1. $zw = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
2. $\frac{z}{w} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

More generally,

Theorem (De Moivre's Theorem)

Let $z = r(\cos(\theta) + i\sin(\theta))$ *and let* $n \in \mathbb{Z}$*. Then*

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

In order to prove De Moivre's Theorem, we need to use Mathematical Induction

Euler's Formula

If $z = x + iy = r(\cos(\theta) + i\sin(\theta))$, then Euler's formula is given by

$$z = re^{i\theta}$$

in other words,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

is called Euler's formula. It is very useful for solving complex equations.

Example

Solve the following roots of unity.

1. 3rd roots of unity.
2. 4th roots of unity.
3. 5th roots of unity.
4. What can you generalize about the nth root geometrically?

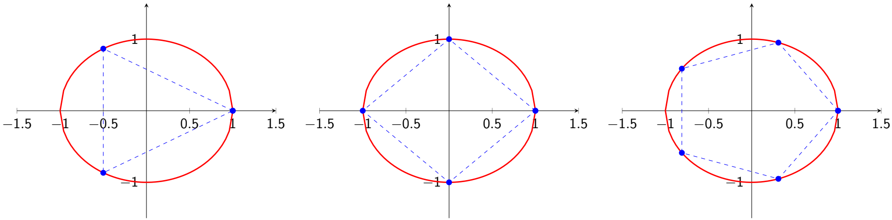

Figure: Third, Fourth, and Fifth Roots of Unity

Tutorial Problems

Exercise 1

Prove the following

1. $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ for $z, w \in \mathbb{C}$.
2. $\overline{z^{-1}} = \bar{z}^{-1}$ for $z \in \mathbb{C} \setminus \{0\}$

Exercise 2

Convert $z = e^3 i$ into polar form, and write it in the form $z = re^{i\theta}$. Also convert $\sqrt{z}$ to standard form.

Exercise 3

Give a complete solution to $x^4 + 16 = 0$.

Midterm Review

So far, we have covered

1. Proof writing
2. Divisibility and Modular Arithmetic
3. Mathematical Induction
4. Complex Numbers