MATH 1200 Section A (Tutorial 02)
Tutorial 10: Equivalence Relations, Equivalence Classes, Arithmetic Modulo n

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1. Equivalence Relations

Definition 1. If A and B are nonempty sets, a *binary relation from A to B* is a subset of $A \times B$. In other words, a binary relation from A to B is a set R of ordered pairs, where the first element of each ordered pair comes from A and the second element of each ordered pair comes from B.

Notation 1. We use the notation $a \sim b$, or aRb to denote that the element $a \in A$ is related to the element $b \in B$. If $a \sim b$, we denote it as $(a, b) \in R$.

Definition 2. A *relation on a set* A is the relation from A to A.

Example 1. *Let $A = \{1, 2, 3, 4\}$. Which order pairs are in the relation $R = \{(a, b) : a \mid b\}$? List them all.*

(1,1) ✓ (2,1) X (3,1) (4,1)
(1,2) ✓ (2,2) ✓ (3,2) (4,2)
(1,3) ✓ (2,3) X (3,3) ✓ (4,3)
(1,4) ✓ (2,4) ✓ (3,4) (4,4) ✓

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Example 2. *How many potential relations are there on a set with n elements?*

$A = \{1, 2, \ldots, n\}$

2^n

Definition 3. A relation R on a set A is said to be

(1) Reflexive: For all $a \in A$, $(a, a) \in R$, or $a \sim a$.
(2) Symmetric: For all $a, b \in A$, if $a \sim b$, then $b \sim a$.
(3) Transitive: For all $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Definition 4. A binary relation "$\sim$" is said to be an *equivalence relation*, if the binary relation satisfies all three conditions in Definition 3.

Example 3. *Consider the relation defined in Example 1. Determine whether they are reflexive, symmetric, and transitive.*

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

R : ✓

S : X b/c take $(1,2) \in R \Rightarrow 1 \mid 2$. But $2 \nmid 1$.
cannot be symmetric.

T : ✓

$1 \sim 1$, $1 \sim x$
$1 \sim x$

Equiv $\Rightarrow$ RST

Example 4. *Let $R = \{(a,b) \in \mathbb{R} \times \mathbb{R} : a - b \in \mathbb{Z}\}$. Is R an equivalence relation?*

Reflexive: Let $a \in \mathbb{R}$ $(a,a) \in \mathbb{R} \times \mathbb{R} \overset{a \sim a}{\Rightarrow} a - a = 0 \in \mathbb{Z}$ ✓

Symmetric: Let $a, b \in \mathbb{R}$ s.t. $a \sim b \Rightarrow a - b = k \in \mathbb{Z}$

(WTS that $b \sim a$). Multiply by $-1 \Rightarrow b - a = -k \in \mathbb{Z} \Rightarrow b \sim a$.

Transitive: Let $a, b, c \in \mathbb{R}$ s.t $a \sim b$ and $b \sim c$.

$a - b = k \in \mathbb{Z}$ and $b - c = m \in \mathbb{Z}$ then $(a-b)+(b-c) = k+m$

$\Rightarrow a - c = k + m \in \mathbb{Z} \Rightarrow a \sim c$.

Example 5. *Let $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod{n}, n \in \mathbb{N}\}$. Is R an equivalence relation?*

R: $a \equiv a \pmod{n}$ ✓

S: $a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n}$ ✓

T: $a \equiv b \pmod{n}$, $b \equiv c \pmod{n}$

$a \equiv b \equiv c \pmod{n} \Rightarrow a \equiv c \pmod{n}$ ✓

Equiv. Relation.

2. Arithmetic Modulo n

Notation 2. We can define arithmetic operators on $\mathbb{Z}_n$ the set of nonnegative inegers less than n. That is, $\mathbb{Z}_n = \{0, 1, ..., n-1\}$.

- We define addition on $\mathbb{Z}_n$ with the notation $+_n$, where
$$a +_n b = (a + b) \bmod n$$
In other words, the remainder when $a + b$ is divided by n.
- We define multiplication on $\mathbb{Z}_n$ with the notation $\cdot_n$ where
$$a \cdot_n b = (a \cdot b) \bmod n$$
In other words, the remainder when $a \cdot b$ is divided by n.

+ ⊗ $n = 4$

$3 +_4 7 = 10 \bmod 4$
$= 2$

$\mathbb{Z}_n = \{0, \cdots, n-1\}$

Example 6. *List all elements in $\mathbb{Z}_4$*

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

Example 7. *Create an addition table for $\mathbb{Z}_4$ with $+_4$ and $\cdot_4$.*

Try imagining the addition table as sudoku.

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Example 8. *Compute the following:*

(1) $7 +_{11} 9$

$$7 +_{11} 9 = (7+9) \bmod 11 = 16 \bmod 11 = 5$$

(2) $7 \cdot_{11} 9$

$$7 \cdot_{11} 9 = (7 \cdot 9) \bmod 11 = 63 \bmod 11 = 8.$$

3. Equivalence Classes

Definition 5. Let R be an equivalence relation on a set A. The set of all elements that are related to an element $a \in A$ is called an *equivalence class* of a. The equivalence class of a with respect to a relation R is denoted by $[a]$. In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is:

$$[a] = \{s : (a, s) \in R\}$$

Example 9. *What are the equivalence classes of 0, 1, 2, and 3 for congruence modulo 4? List at least five elements in the set, and write a generalized statement for the equivalence relation.*

$[0] = \{\ldots, -4, 0, 4, 8, 12, \ldots\} = \{4k : k \in \mathbb{Z}\}$

$[1] = \{\ldots, -3, 1, 5, 9, 13, \ldots\} = \{4k+1 : k \in \mathbb{Z}\}$

$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\} = \{4k+2 : k \in \mathbb{Z}\}$

$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\} = \{4k+3 : k \in \mathbb{Z}\}$