Week 6: Mathematical Induction

MATH 1200: Problems, Conjectures, and Proofs

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Contents

Mathematical Induction

Theorem (Principal of Mathematical Induction)

Let $P(n)$ be the statement for $n \in \mathbb{Z}$. Assume that

- *(Base Case) There exists a $k_0 \in \mathbb{Z}$ such that $P(k_0)$ is true.*
- *(Inductive Step) For any $k \geq k_0$, if $P(k)$ is true, then $P(k+1)$ is also true.*

Then for all $n \geq k_0$, $P(n)$ is true.

Theorem (Strong Induction)

Let $P(n)$ be a statement for $n \in \mathbb{Z}$. Assume that

- *(Base Case) There exists a $k_0 \in \mathbb{Z}$ such that $P(k_0)$ is true.*
- *(Inductive Step) If $k \geq k_0$ is an integer such that $P(k_0)$, $P(k_0+1)$,..., $P(k)$ are all true, then $P(k+1)$ must be true as well.*

Then for all $n \geq k_0$, $P(n)$ is true.

In general, mathematical induction can be used to prove statements that assert $P(n)$ is true for all $n \in \mathbb{N}$, where $P(n)$ is a propositional statement.

A proof by mathematical induction has two parts: If $P(n)$ is some propositional statement for $n \in \mathbb{N}$,

- Base Case: Show that $P(1)$ is true.
- Inductive Step: Show that for all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Template for Proofs Using Mathematical Induction

1. Express the statement that is to be proved in one of the following forms:
   - If b is an is an integer: For all $n \geq b$, $P(n)$
   - If $b = 1$, i.e. $n \in \mathbb{N}$: For all $n \in \mathbb{N}$, $P(n)$
   - If $b = 0$: For every nonnegative integer n, $P(n)$
2. Write out the basis step. Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words inductive step, and state, and clearly identify, the inductive hypothesis of the form: Assume that $P(k)$ is true for any $k \geq b$, where $k \in \mathbb{Z}$.
4. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
5. Prove the statement $P(k+1)$ making use of the assumption $P(k)$.
6. Clearly identify the conclusion of the inductive saying: This completes the inductive step.
7. After completing the basis step and the inductive step, state the conclusion, namely, By mathematical induction, $P(n)$ is true for all integers $n \geq b$.

Common Example Involving Induction

Example

Show that if $n \in \mathbb{N}$, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

Let $P(n)$ be the statement: For all $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Base Case: We need to show that $P(1)$ is true. Indeed, see that

$$1 = \frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1$$

Therefore, the base case has been proven. $\square$

Proof.

Inductive Step: Assume that for all $k \in \mathbb{N}$, $P(k)$ is true, i.e.

$$P(k) : 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \tag{IH}$$

We need to show that $P(k+1)$ is true, i.e.

$$P(k+1) : 1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$\square$

Proof.

Indeed, see that

$$\begin{aligned}
\underbrace{1+2+3+\cdots+k}_{IH}+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

This completes the inductive step. Therefore, by mathematical induction, the statement for all $n \in \mathbb{N}$, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ is true for every $n \in \mathbb{N}$. □

Let's try this one together:

Example

Use mathematical induction to prove that for all $n \in \mathbb{N}$, $3 \mid n^3 - n$.

Proof.

Let $P(n)$ be the statement: For all $n \in \mathbb{N}$, $3 \mid n^3 - n$.

- **Base Case:** We need to show that $P(1)$ is true. Indeed, see that $3 \mid 1^3 - 1$ implies $3 \mid 0$, which is true, thereby completing the basis step.
- **Inductive Step:** Assume that for all $k \in \mathbb{N}$, $P(k)$ is true, i.e. $3 \mid k^3 - k$, i.e. there exists an integer m such that $k^3 - k = 3m$. We want to show that $P(k+1)$ is true, i.e. $3 \mid (k+1)^3 - (k+1)$. Indeed, by definition, there exists an integer ℓ such that $(k+1)^3 - (k+1) = 3\ell$. Then see that on the left hand side,

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= (k^3 - k) + 3(k^2 + k) \\ &= 3m + 3(k^2 + k) \\ &= 3(m + k^2 + k)\end{aligned}$$

  where $m + k^2 + k \in \mathbb{Z}$, which then implies, by definition, $3 \mid (k+1)^3 - (k+1)$, thereby completing the inductive step. Therefore, for all $n \in \mathbb{N}$, $3 \mid n^3 - n$.

Discussion Prompt

Discussion Prompt

Using the same method as above, prove by mathematical induction that $6 \mid 3n^4 + 2n^3 + 7n$ for all $n \in \mathbb{N}$.

Proof.

Let $P(n)$ be the statement: For all $n \in \mathbb{N}$, $6 \mid 3n^4 + 2n^3 + 7n$.

- **Base Case:** We need to show that $P(1)$ is true. Indeed, $6 \mid 3 \cdot 1^4 + 2 \cdot 1^3 + 7 \cdot 1$ implies $6 \mid 12$, which is true, because $12 = 2 \cdot 6$, thereby completing the basis step.
- **Inductive Step:** Assume that for all $k \in \mathbb{N}$, $P(k)$ is true, i.e. $6 \mid 3k^4 + 2k^3 + 7k$, i.e. there exists an $m \in \mathbb{Z}$ such that $3k^4 + 2k^3 + 7k = 6m$. We want to show that $P(k+1)$ is true, i.e. $6 \mid 3(k+1)^4 + 2(k+1)^3 + 7(k+1)$. By definition, there exists an $\ell \in \mathbb{Z}$ such that $3(k+1)^4 + 2(k+1)^3 + 7(k+1) = 6\ell$. Expanding the left side gives:

$$\begin{aligned} 3k^4 + 14k^3 + 24k^2 + 25k + 12 &= (3k^4 + 2k^3 + 7k) + (12k^3 + 24k^2 + 18k + 12) \\ &= 6m + 6(2k^3 + 4k^2 + 3k + 2) \\ &= 6(m + 2k^3 + 4k^2 + 3k + 2) \end{aligned}$$

  where $m + 2k^3 + 4k^2 + 3k + 2$, which then implies, by definition, $6 \mid 3(k+1)^4 + 2(k+1)^3 + 7(k+1)$, thereby completing the inductive step. Therefore, for all $n \in \mathbb{N}$, $6 \mid 3n^4 + 2n^3 + 7n$.