Week 5: Proof Methods III

MATH 1200: Problems, Conjectures, and Proofs

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Contents

Recap From Last Week

Last week, we...

- Prove mathematical statements directly.
- Prove mathematical statements using a proof by contradiction.
- Worked on examples related to direct proofs and proof by contradiction.

Proof by Cases - Discussion Prompt

A proof by cases must cover all possible cases that arise in a theorem. We will illustrate proof by cases with the following discussion prompt:

Discussion Prompt 1

Prove that a and b are real numbers, then

$$|a + b| \leq |a| + |b|$$

(This is an important inequality called the 'Triangle Inequality') There should be at least four cases that we could consider. Identify these cases.

Proof.

We consider the following cases as follows:

- **Case 1:** Assume that either $a = 0$ or $b = 0$. If $a = 0$, then we have $|0 + b| = |b|$ and $|0| + |b| = |b|$, so $|0 + b| = |0| + |b|$. Similar if we consider when $b = 0$.
- **Case 2:** Assume that $a, b > 0$. Then we have $|a| = a$ and $|b| = b$, so $|a + b| = a + b$ and $|a| + |b| = a + b$, so $|a + b| = |a| + |b|$.
- **Case 3:** Assume that $a, b < 0$. Then this implies that $-a, -b > 0$, and as $a + b < 0$, we have $|a + b| = -(a + b) = (-a) + (-b)$ but also $|a| + |b| = (-a) + (-b)$, so $|a + b| = |a| + |b|$.

$\square$

Proof.

- **Case 4:** Assume that $a > 0$ and $b < 0$. Then we have $|a| + |b| = a + (-b)$. We consider the following subcases:
  - Subcase 1: If $a + b = 0$, then $|a + b| = 0$, and since $b < 0$, we have $-b > 0$, so $a + (-b) > 0 + (-b) = -b > 0$. So since $|x| + |y| = x + (-y)$, we must have $|a| + |b| > |a + b|$.
  - Subcase 2: If $a + b > 0$, then $|a + b| = a + b$. Since $b < 0$, then $-b > 0 > b$ and so $a + (-b) > a + b$. Therefore, we also get $|a| + |b| > |a + b|$.
  - Subcase 3: If $a + b < 0$, then $|a + b| = -(a + b)$. Since $a > 0$, we have $a > 0 > -a$, so $a + (-b) > (-a) + (-a)$, and so $|a| + |b| > |a + b|$.

$\square$

Example

Prove that if x is a real number, then

$$|x-1|+|x+1| \geq 1+|x|$$

Consider the following cases:

- If $x = 1$
- If $x > 1$
- If $x < 1$

Proof

- If $x = 1$, then

$$|1-1| + |1+1| \geq 1 + |1| \implies 2 \geq 2 \quad \checkmark$$

- If $x > 1$, then $x + 1 > 0$ and $x - 1 > 0$, so $|x+1| = x + 1$ and $|x-1| = x - 1$, so

$$|x-1| + |x+1| = x - 1 + x + 1 = 2x \geq 2 \quad \checkmark$$

- If $x < 1$, then $x - 1 < 0$ and $x + 1 > 0$, so $|x-1| = -(x-1)$ and $|x+1| = x + 1$, and so

$$|x-1| + |x+1| = -(x-1) + x + 1 = 2 \geq 2 \quad \checkmark$$

Question

But what about cases such as $x = -1$, $x < -1$ and $x > -1$?

Without Loss of Generality (WLOG)

The idea to shorten the proof for any redundant cases is a method of without loss of generality. In general, the idea is that if we assert that we prove one case of a theorem, no additional argument is required to prove the other cases. That is, the other cases will follow by making straightforward changes to the argument.

Proof by Contraposition

We will explore another proof method known as Proof by Contraposition. Consider the statement $P \implies Q$. Then the contrapositive of this statement will be $\neg Q \implies \neg P$. To prove a statement by contraposition,

1. Assume that $\neg Q$ is true.
2. Apply what you know about $\neg Q$, i.e. definitions, facts, etc.
3. Use the steps about $\neg Q$ to show that $\neg P$ is true.
4. Therefore $P \implies Q$ must be true.

Example

Use a proof by contraposition to prove the following statement: If n is an integer and $3n+2$ is odd, then n is odd.

Proof.

To prove by contrapositive, assume that n is an even integer. We want to show that $3n+2$ is even. By definition of an even integer, there exists an integer k such that $n = 2k$. Then by substituting to $3n+2$, we obtain

$$3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

Since the integers are closed under addition and multiplication, $r = 3k + 1$ is an integer, so $2r$ is an even integer, which means that $3n + 2$ is an even integer. Therefore, since we proved the contrapositive, we have shown that if $3n + 2$ is even, then n is even. $\square$

Example (Example From Last Tutorial)

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using a proof by contraposition.

Proof.

To prove by contrapositive, assume that n is an odd integer. We want to show that $n^3 + 5$ is even. Since n is an odd integer, then there exists an integer k such that $n = 2k + 1$. Then by substituting to $n^3 + 5$, we obtain

$$(2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 1 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$$

Since the integers are closed under addition and multiplication, $r = 4k^3 + 6k^2 + 3k + 3$ is an integer, so $2r$ is an even integer, which means that $n^3 + 5$ is an even integer. Therefore, since we proved the contrapositive, we have shown that if $n^3 + 5$ is odd, then n is even. □