Week 6: Mathematical Induction

MATH 1200: Problems, Conjectures, and Proofs

Joe Tran

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Discussion Prompt

Discussion Prompt

Let n be a natural number. Determine a formula in terms of n for the sum of the first n odd natural numbers. For example, when $n = 4$, the sum of the first four odd natural numbers is $1 + 3 + 5 + 7 = 16$.

Hint: Recall that if $a_1, a_2, \ldots, a_n$ is a sequence of integers, then the arithmetic sum of the sequence is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Solution

If the nth natural number is $2n-1$, then we have the sum given by

$$1+3+5+\cdots+(2n-1)$$

So there are n terms, with $a_1 = 1$ and $a_n = 2n-1$, so

$$S_n = \frac{n}{2}(1+(2n-1)) = \frac{n}{2}(2n) = n^2$$

Therefore,

$$1+3+5+\cdots+(2n-1) = n^2$$

We will come back to this example, momentarily. We need to discuss about Mathematical Induction first.

Mathematical Induction

Theorem (Principal of Mathematical Induction)

Let $P(n)$ be the statement for $n \in \mathbb{Z}$. Assume that

- *(Base Case) There exists a $k_0 \in \mathbb{Z}$ such that $P(k_0)$ is true.*
- *(Inductive Step) For any $k \geq k_0$, if $P(k)$ is true, then $P(k+1)$ is also true.*

Then for all $n \geq k_0$, $P(n)$ is true.

Theorem (Strong Induction)

Let $P(n)$ be a statement for $n \in \mathbb{Z}$. Assume that

- *(Base Case) There exists a $k_0 \in \mathbb{Z}$ such that $P(k_0)$ is true.*
- *(Inductive Step) If $k \geq k_0$ is an integer such that $P(k_0)$, $P(k_0+1)$,..., $P(k)$ are all true, then $P(k+1)$ must be true as well.*

Then for all $n \geq k_0$, $P(n)$ is true.

In general, mathematical induction can be used to prove statements that assert $P(n)$ is true for all $n \in \mathbb{N}$, where $P(n)$ is a propositional statement.

A proof by mathematical induction has two parts: If $P(n)$ is some propositional statement for $n \in \mathbb{N}$,

- Base Case: Show that $P(1)$ is true.
- Inductive Step: Show that for all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Template for Proofs Using Mathematical Induction

1. Express the statement that is to be proved in one of the following forms:
   - If b is an is an integer: For all $n \geq b$, $P(n)$
   - If $b = 1$, i.e. $n \in \mathbb{N}$: For all $n \in \mathbb{N}$, $P(n)$
   - If $b = 0$: For every nonnegative integer n, $P(n)$
2. Write out the basis step. Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words inductive step, and state, and clearly identify, the inductive hypothesis of the form: Assume that $P(k)$ is true for any $k \geq b$, where $k \in \mathbb{Z}$.
4. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
5. Prove the statement $P(k+1)$ making use of the assumption $P(k)$.
6. Clearly identify the conclusion of the inductive saying: This completes the inductive step.
7. After completing the basis step and the inductive step, state the conclusion, namely, By mathematical induction, $P(n)$ is true for all integers $n \geq b$.

Common Example Involving Induction

Example

Show that if $n \in \mathbb{N}$, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

Let $P(n)$ be the statement: For all $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Base Case: We need to show that $P(1)$ is true. Indeed, see that

$$1 = \frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1$$

Therefore, the base case has been proven. $\square$

Proof.

Inductive Step: Assume that for all $k \in \mathbb{N}$, $P(k)$ is true, i.e.

$$P(k) : 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \tag{IH}$$

We need to show that $P(k+1)$ is true, i.e.

$$P(k+1) : 1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$\square$

Proof.

Indeed, see that

$$\begin{aligned}\underbrace{1+2+3+\cdots+k}_{IH}+(k+1) &= \frac{k(k+1)}{2}+(k+1)\\ &= \frac{k(k+1)+2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

This completes the inductive step. Therefore, by mathematical induction, the statement for all $n \in \mathbb{N}$, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ is true for every $n \in \mathbb{N}$. □

Discussion Prompt

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Going back to the sum of the first n natural numbers formula:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Use mathematical induction to prove that the statement is true for all $n \in \mathbb{N}$. (The steps will be exactly the same as the previous example!)

Proof.

Let $P(n)$ be the statement: "for all $n \in \mathbb{N}$, $1 + 3 + 5 + \cdots + (2n-1) = n^2$".

Base Case: We need to show that $P(1)$ is true. Indeed, $1 = 1^2 = 1$. Therefore, the base case has been proven.

Inductive Step: Assume that for all $k \in \mathbb{N}$, $P(k)$ is true, i.e. $1 + 3 + 5 + \cdots + (2k-1) = k^2$ (IH). We need to show that $P(k+1)$ is true, i.e. $1 + 3 + 5 + \cdots + (2k+1) = (k+1)^2$. Indeed, see that

$$\underbrace{1 + 3 + 5 + \cdots + (2k-1)}_{IH} + (2k+1) = k^2 + 2k + 1 = (k+1)^2$$

This completes the inductive step. Therefore, by mathematical induction, the statement "for all $n \in \mathbb{N}$, $1 + 3 + 5 + \cdots + (2n-1) = n^2$" is true for every $n \in \mathbb{N}$. $\square$

Let's try this one together:

Example

Use mathematical induction to prove that for all $n \in \mathbb{N}$, $3 \mid n^3 - n$.