Math 1200 Tutorial 7: Midterm Review

Joe Tran

October 23, 2024

What we have covered so far:

- True or false statements
- Definitions: Odd, even, divides
- Proof Methods: Direct, contrapositive, contradiction, cases, counterexamples
- Mathematical induction

Question 1. Let $P(x, y) :=$ "$x + 2y = xy$" where the domain of x and y are the real numbers. Determine the truth value of the following statements:

(i) $\forall x \exists y : P(x, y)$

(ii) $\exists y \exists x : P(x, y)$

(iii) $\exists y \forall x : \neg P(x, y)$

(i) Let $x \in \mathbb{R}$ be arbitrary and let $y = \frac{1}{2}x$. Then

$$x + 2 \cdot \frac{1}{2}x = x \cdot \frac{1}{2}x$$

$$x + x = \frac{1}{2}x^2$$

$$2x = \frac{1}{2}x^2$$

Therefore the statement is false.

(ii) Let $x = 0$ and $y = 0$. Then

$$0 + 2 \cdot 0 = 0 \cdot 0$$

$$0 = 0$$

Therefore, the statement is true.

(iii) Note $\neg P(x, y) :=$ "$x + 2y \neq xy$"

Let $y = 1$ and $x \in \mathbb{R}$ be arbitrary. Then

$$x + 2 \cdot 1 \neq x \cdot 1$$

$$x + 2 \neq x$$

Therefore, the statement is true.

Question 2. Prove that if x is odd and y is even, then $3x + 2y$ is odd.

Assume that x is odd and y is even. Then by definition of odd, there exists a $k_1 \in \mathbb{Z}$ such that $x = 2k_1 + 1$. By definition of even, there exists a $k_2 \in \mathbb{Z}$ such that $y = 2k_2$. Then

$$\begin{aligned} 3x + 2y &= 3(2k_1 + 1) + 2(2k_2) \\ &= 6k_1 + 3 + 4k_2 \\ &= 2(3k_1 + 2k_2 + 1) + 1 \end{aligned}$$

By the closure of $\mathbb{Z}$, $3k_1 + 2k_2 + 1 \in \mathbb{Z}$, so by definition of odd, $3x + 2y$ is odd.

Question 3. Prove that if $a \mid b$ and $b \mid a$, where $a, b \in \mathbb{Z}$, then either $a = b$ or $a = -b$.

Assume that $a|b$ and $b|a$. Then by definition of divides, there exists $k_1, k_2 \in \mathbb{Z}$ such that

$$b = k_1 a$$
$$a = k_2 b.$$

But then $b = k_1 a = k_1(k_2 b) = (k_1 k_2) b$. We consider the following cases:

- If $b = 0$, then it is easy to see equality.
- If $b \neq 0$, then we can write $k_1 k_2 = 1$, so we have the following possibilities:
  - If $k_1 = k_2 = 1$, then we have $a = b$
  - If $k_1 = k_2 = -1$, then we have $a = -b$.

Therefore, we have shown if $a|b$ and $b|a$, then $a = b$ or $a = -b$.

Question 4. Prove by mathematical induction: $\forall n \in \mathbb{Z}, n \geq 0$

$$2 + 2(-7) + 2(-7)^2 + \cdots + 2(-7)^n = \frac{1-(-7)^{n+1}}{4}$$

Let $P(n) :=$ " $2 + 2(-7) + 2(-7)^2 + \cdots + 2(-7)^n = \dfrac{1-(-7)^{n+1}}{4}$ "

Base Case: We show $P(0)$ is true.

$LS = 2(-7)^0 = 2 \cdot 1 = 2.$

$RS = \dfrac{1-(-7)^{0+1}}{4} = \dfrac{1-(-7)}{4} = \dfrac{8}{4} = 2.$

Therefore, because $LS = RS$, $P(0)$ is true, thereby proving the base case.

Inductive Step: Assume that for all $k \geq 0$, $P(k)$ is true, i.e.

$$2 + 2(-7) + 2(-7)^2 + \cdots + 2(-7)^k = \frac{1-(-7)^{k+1}}{4} \quad (IH)$$

We want to show that $P(k+1)$ is true, i.e.

$$2 + 2(-7) + \cdots + 2(-7)^{k+1} = \frac{1-(-7)^{k+2}}{4}$$

Indeed,

$$2 + 2(-7) + \cdots + 2(-7)^k + 2(-7)^{k+1}$$

$$\stackrel{IH}{=} \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1}$$

$$= \frac{1-(-7)^{k+1} + 8(-7)^{k+1}}{4}$$

$$= \frac{1-(-7)^1(-7)^{k+1}}{4} = \frac{1-(-7)^{k+2}}{4}$$

Therefore by mathematical induction, $P(n)$ is true for all $n \geq 0$.

Question 5. Consider the sequence of numbers $x_0, x_1, x_2, ...$ given by

$$x_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = 1 \\ \frac{x_{k-2}}{4} & \text{if } k \geq 2 \end{cases}$$

Prove that for all integers $n \in \mathbb{N}$,

$$x_n = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}$$

Let y_n be the sequence defined by

$$y_n = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}$$

Let x_n be the sequence defined recursively by $x_k = \frac{x_{k-2}}{4}$ for $k \geq 2$ with initial conditions $x_0 = 1$ and $x_1 = 0$.

We want to show that for all $n \geq 0$, $x_n = y_n$, by second principle of mathematical induction.

<u>Base Case:</u> We show $x_0 = y_0$ and $x_1 = y_1$.

For $n=0$, $x_0 = 1$ and $y_0 = \left(\frac{1}{2}\right)^{0+1} - \left(-\frac{1}{2}\right)^{0+1} = \frac{1}{2} + \frac{1}{2} = 1$

For $n=1$, $x_1 = 0$ and $y_0 = \left(\frac{1}{2}\right)^{1+1} - \left(-\frac{1}{2}\right)^{1+1} = \frac{1}{4} - \frac{1}{4} = 0$.

Therefore, the base case is proved.

<u>Inductive Step:</u> Assume for all $k \geq 1$, $x_j = y_j$ for all $0 \leq j \leq k$. (IH) We want to show $x_{k+1} = y_{k+1}$.

Indeed,

$$x_{k+1} = \frac{x_{k-1}}{4} \overset{IH}{=} \frac{y_{k-1}}{4} = \frac{1}{4}\left(\left(\frac{1}{2}\right)^k - \left(-\frac{1}{2}\right)^k\right)$$

$$= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^k - \left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^{k+2} - \left(-\frac{1}{2}\right)^{k+2} = y_{k+1}$$

Question 5. Consider the sequence of numbers $x_0, x_1, x_2, \ldots$ given by

$$x_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = 1 \\ \frac{x_{k-2}}{4} & \text{if } k \geq 2 \end{cases}$$

Prove that for all integers $n \in \mathbb{N}$,

$$x_n = \left(\frac{1}{2}\right)^{n+1}$$

By the second principle of mathematical induction,

$$x_n = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}$$

for all $n \in \mathbb{Z}$, $n \geq 0$.