Week 4: Proof Methods II

MATH 1200: Problems, Conjectures, and Proofs

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Contents

Question

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Club Infinity is planning to host a MATH 1200 study session one week before your midterm date (October 31, 2024), most likely on a weekend, online on Zoom. This is for Sections A and C. Would you be interested in attending this study session?

Recall From Last Week

Recall:

- We learned how to use counterexamples to disprove mathematical statements.
- We learned how to apply definitions to prove mathematical statements: Definitions including odd, even, and divides.
- We looked at an example involving ε-δ from calculus (Note: this is an optional concept in 1200, but required in 1300).

Discussion Prompt

Discussion Prompt

Prove the following statement: If n and m are integers and n is even, then mn is even.

Proof.

Assume that $n, m \in \mathbb{Z}$ such that n is an even integer. Since n is even, there exists an integer k such that $n = 2k$. Then

$$nm = (2k)m = 2(km)$$

Since the integers are closed under multiplication, km is an integer. Therefore, by definition of an even integer, nm is even. $\square$

More Examples of Direct Proofs

Example

If x and y are nonnegative real numbers, then $\sqrt{xy} \leq \frac{x+y}{2}$.

Recall

For any $x, y \in \mathbb{R}$, $(x-y)^2 \geq 0$

Proof.

Let $x, y \in \mathbb{R}$ be nonnegative. Then $(\sqrt{x} - \sqrt{y})^2 \geq 0$ means that $(\sqrt{x})^2 - 2\sqrt{x}\sqrt{y} + (\sqrt{y})^2 \geq 0$, or $x - 2\sqrt{xy} + y \geq 0$. Then $x + y \geq 2\sqrt{xy}$. Dividing both sides by 2 gives us $\frac{x+y}{2} \geq \sqrt{xy}$. Therefore, the statement is proved. $\square$

Proof by Contradiction

If we consider q to be a statement, the statement that $q \wedge \neg q$ is a contradiction. We can prove a statement p to be true, if we can show that $\neg p \implies (q \wedge \neg q)$ is true. This is a proof method known as proof by contradiction.

Definition (Rational Number)

A number x is said to be a rational number if there exists integers $a, b \in \mathbb{Z}$ with $b \neq 0$, such that $x = \frac{a}{b}$.

Typical Example

Example

$\sqrt{2}$ is irrational.

Proof.

Let p be the proposition that "$\sqrt{2}$ is irrational". In order to prove by contradiction, we assume that $\neg p$ is true, which is the statement that "$\sqrt{2}$ is not irrational"; or "$\sqrt{2}$ is rational". We will show that assuming $\neg p$ leads to a contradiction. If $\sqrt{2}$ is rational, then there exists integers a and b with $b \neq 0$ and a and b sharing no common factors such that $\sqrt{2} = \frac{a}{b}$ (this also means that $\frac{a}{b}$ is written in lowest terms). Squaring both sides give $2 = \frac{a^2}{b^2}$ and so $2b^2 = a^2$. It follows that a^2 is an even integer, so a must be even as well. By definition of an even integer, there exists an integer c such that $a = 2c$. So then $2b^2 = (2c)^2 = 4c^2$, which implies that $b^2 = 2c^2$, which shows that b^2 is even, and thus, b is even. But this is a contradiction, because we assumed that a and b share no common factors, but both a and b are even means that 2 divides both a and b. Therefore, $\sqrt{2}$ is irrational. $\square$

In general, when proving a statement of the form $p \implies q$, by contradiction, use the following steps:

1. Assume that p is true.
2. Assume that $\neg q$ is true.
3. Apply what you know about p, i.e. definitions, facts, etc.
4. Apply what you know about $\neg q$, i.e. definitions, facts, etc.
5. Use the steps about p and $\neg q$ to show that $\neg p$ is the contradiction.
6. Therefore, $p \implies q$ must be true.

Discussion Prompt 2

Disucssion Prompt

Give a proof by contradiction for the statement: "If $3n + 2$ is odd, then n is odd" using the following steps:

(a) If p is the statement "$3n + 2$ is odd", and q is the statement "n is odd", what is $\neg q$?

(b) Apply what you know about the statement that "$3n + 2$ is odd", and apply what you know about the statement $\neg q$.

(c) What is the contradiction of this statement? Why is it a contradiction?

Proof.

(a) Assume that $3n+2$ is odd, and assume for a contradiction, that n is not odd; in other words, n is even.

(b) Since $3n+2$ is odd, then there exists an integer k such that $3n+2=2k+1$. Since n is even, then there exists an integer m such that $n=2m$. But then

$$3n+2=3(2m)+2=6m+2=2(3m+1)=2k+1$$

Since the integers are closed under addition and multiplication, $3m+1$ is an integer.

(c) The contradiction comes from the fact that $2(3m+1)=2k+1$, because we have an even integer equal to an odd integer, which is absurd.

Therefore, it must be the case that n is odd. $\square$

Next Week

- Look at Proof by Contrapositive
- Look at Proof by Cases
- Provide a summary of proof methods