Tutorial 2: Mathematical Statements, Biconditional Statements, Logical Equivalence

MATH 1200: Problems, Conjectures, and Proofs

Joe Tran

York University

September 12, 2024

Contents

Announcements

1. First Assignment Due September 19, 2024 at 11:59 PM.

Discussion Prompt 1

Discussion

A Marvel special was aired 4 times. Each airing had a unique day of the week, time of day, and the number of viewers that watch the special. The days on which was aired were Sunday, Tuesday, Friday, and Saturday, the times on which the special was aired were 11 AM, 1 PM, 6 PM, and 10 PM, and the number of viewers each airing the special were 1 billion, 2 billion, 3 billion, and four billion people. Using only the clues below, match each airing time, day of the week, and the number of viewers for each airing of the special.

- The show with 3 billion viewers was on Tuesday
- The program that aired at 6 PM had 1 billion more viewers than the program that aired on Sunday.
- The program that aired at 1 PM had 4 billion viewers.
- Of the program with 3 billion viewers and the program that aired on Saturday, one aired at 11 AM and the other aired at 1 PM.

Discussion Prompt 2

Discussion

Professor Skoufranis sent each of the MATH 1200 TAs to the bookstore with instructions to purchase one item necessary for their TAing duties. Each TA had to pick up a unique item and each item had a unique price. Determine which TA picked up which item and how much they paid.

- The four items were chalk, paper, red pens, and printer ink.
- Felix's purchase cost more than the red pens.
- Blake's purchase cost is \$7
- The \$5 purchase is either Aysa's or was the chalk.
- The chalk was either purchased by Daniel or cost \$6.
- For the paper and red pens, one cost \$4 and the other was purchased by Felix.

Mathematical Statements

During the lecture, we have talked about what it means for a sentence to be a mathematical statement:

Definition

A mathematical statement is a declarative sentence in mathematics that is either true or false.

Example

1. $3 + 5 = 8$ True
2. For all real values of x, $x^2 \geq 0$ True
3. There exists a real number x such that $x^2 = -1$ False
4. The closed interval $[0, 1]$ is finite. False

We recall some logic symbols that have been used in the lecture as well:

Definition

Let p and q be statements.

1. The negation of p, denoted by $\neg p$, means "not p"
2. The logical or of p and q, denoted by $p \vee q$, means "either p or q".
3. The logical and of p and q, denoted by $p \wedge q$, means "both p and q".
4. p implies q, denoted by $p \implies q$, means "if p happens, then q happens".
5. p if and only if q, denoted by $p \iff q$, means "if p happens, then q happens, and vice versa".

Example

Consider the following statements: p being the statement "You get a 100% on the final exam", and q being the statement "You will get an A+ in MATH 1200".

1. $\neg p$: "You do not get a 100% on the final exam."
2. $p \vee q$: "You either get 100% on the final exam or you get an A+ in MATH 1200".
3. $p \wedge q$: "You get 100% on the final exam and you get an A+ in MATH 1200".
4. $p \implies q$: "If you get 100% on the final exam, then you will get an A+ in MATH 1200".
5. $p \iff q$: "You get 100% on the final exam if and only if you get an A+ in MATH 1200".

Some other important implications:

Definition

Let p and q be statements and consider the statement that $p \implies q$:

1. The converse of the statement $p \implies q$ is the statement $q \implies p$.
2. The contrapositive of the statement $p \implies q$ is the statement $\neg q \implies \neg p$.

Example

Consider the same statements as above: p being the statement "You get a 100% on the final exam", and q being the statement "You will get an A+ in MATH 1200", and consider the statement $p \implies q$: "If you get 100% on the final exam, then you will get an A+ in MATH 1200".

1. Converse ($q \implies p$): "If you get an A+ in MATH 1200, then you will get 100% on the final exam".
2. Contrapositive ($\neg q \implies \neg p$): "If you do not get an A+ in MATH 1200, then you did not get 100% on the final exam.

Logical Equivalence

Definition

Two statements p and q are said to be logically equivalent, denoted by $p \equiv q$, if both p and q always share the same truth value.

Here, we recall the truth tables of two statements p and q

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \implies q$	$p \iff q$
T	T	F	T	T	T	T
T	F	F	T	F	F	F
F	T	T	T	F	T	F
F	F	T	F	F	T	T

Example

If p and q are statements, determine whether the statements $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Because the last two columns of the truth table always share the same truth value, the statements $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Example

If p, q, and r are statements, determine if $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

Since the last two columns of the truth table always share the same truth value, the statements are logically equivalent.

Take-Home Exercises

Exercise

Show that $(p \implies q) \wedge (p \implies r)$ and $p \implies (q \wedge r)$ are logically equivalent.

Exercise

Show that $p \iff q$ and $(p \implies q) \wedge (q \implies p)$ are logically equivalent.

Exercise

Explain why $(p \implies q) \implies (r \implies s)$ and $(p \implies r) \implies (q \implies s)$ are not logically equivalent.