MATH 1200 A02 TUTORIAL 4 SUMMARY OF PROOF METHODS

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To prove an assertion with propositions P and Q that translates to $P \to Q$. We can either proceed by directly proving the assertion or by taking an indirect approach, including contradiction or contrapositive.

1. Direct Proofs

Assume P holds true, then proceed to show Q holds true. It is because for the conditional $\to$, with P true, only the truth value true for Q makes the conditional valid.

2. Proof by Contraposition

As the word contraposition conveys "taking a contrary position", we assume the "contrary of Q" to be true, i.e. assume $\neg Q$ true, and proceed to show that the "contrary of P", i.e. $\neg P$ is true, we show

$$\neg Q \to \neg P$$

3. Proof by Contradiction

We aim to get a contradiction of the original assumption, by assuming the hypothesis and the contrary of the conclusion. So we assume $p \wedge \neg Q$ to be true, i.e. P and the contrary of Q to be true. We proceed to find that $\neg P$ to be true, that is,

$$(P \wedge \neg Q) \to \neg P$$

4. Truth Table

Now consider the truth table of those arguments:

P	Q	$\neg P$	$\neg Q$	$P \wedge \neg Q$	$\neg Q \to \neg P$	$(P \wedge \neg Q) \to \neg P$	$P \to Q$
T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

The last three columns are identical in truth values, hence their logical equivalences:

$$P \to Q \equiv \neg Q \to \neg P \equiv (P \wedge \neg Q) \to \neg P$$

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Example. Suppose we want to prove the following proposition: *"If n is an integer and $n+8$ is even, then n is even"*. We will prove it in three different ways.

Direct Proof. Assume that n is an integer and $n + 8$ is even. We want to show that n is an even integer. Since $n + 8$ is an even integer, then there exists an integer $k \in \mathbb{Z}$ such that $n+8 = 2k$. But then $n = 2k-8 = 2(k-4)$. Since $k \in \mathbb{Z}$, it follows that $k - 4 \in \mathbb{Z}$ and thus by the definition of even, n is even. $\square$

Proof by Contraposition. Assume that n is an odd. We want to show that $n + 8$ is odd. Since n is odd, there exists an integer $k \in \mathbb{Z}$ such that $n = 2k + 1$. Then $n + 8 = (2k + 1) + 8 = (2k + 8) + 1$. Since $k \in \mathbb{Z}$, it follows that $2k + 8 \in \mathbb{Z}$, and therefore by the definition of odd, $n + 8$ is also odd. Therefore, since we have proven the contrary, if n is even, then $n + 8$ is even. $\square$

Proof by Contradiction. Assume that n is an integer and $n + 8$ is even. For the sake of contradiction, assume that n is an odd integer. Since $n + 8$ is an even integer, then there exists an integer $k_1 \in \mathbb{Z}$ such that $n + 8 = 2k_1$. Since n is an odd integer, then there exists an integer $k_2 \in \mathbb{Z}$ such that $n = 2k_2 + 1$. But then $n + 8 = (2k_2 + 1) + 8 = 2k_2 + 9 = 2k_1$, and thus, $2(k_1 - k_2) = -9$. Since $k_1, k_2 \in \mathbb{Z}$, then it follows that $k_1 - k_2 \in \mathbb{Z}$ and thus, $2(k_1 - k_2)$ is an even integer, which is a contradiction since we have even on one side, and odd on the other side. Therefore, it must be the case that n is an odd integer. $\square$

5. Tutorial Problems

Exercise 1. Prove the following

(1) Show that for each integer a, if $a \not\equiv 0 \pmod 3$, then $a^2 \equiv 1 \pmod 3$.
(2) Using (1) show that for each natural number n, $\sqrt{3n+2}$ is not a natural number.

Exercise 2. Prove that the product of three consecutive integers is divisible by 3.

Exercise 3. Prove that the difference of a rational number and an irrational number is irrational.