CHAPTER

5 Induction and Recursion

Many mathematical statements assert that a property is true for all positive integers. Examples of such statements are that for every positive integer n: $n! \leq n^n$, $n^3 - n$ is divisible by 3; a set with n elements has 2^n subsets; and the sum of the first n positive integers is $n(n+1)/2$. A major goal of this chapter, and the book, is to provide a thorough understanding of mathematical induction, which is used to prove results of this kind.

Proofs using mathematical induction have two parts. First, they show that the statement holds for the positive integer 1. Second, they show that if the statement holds for a positive integer then it must also hold for the next larger integer. Mathematical induction is based on the rule of inference that tells us that if $P(1)$ and $\forall k(P(k) \rightarrow P(k+1))$ are true for the domain of positive integers, then $\forall n P(n)$ is true. Mathematical induction can be used to prove a tremendous variety of results. Understanding how to read and construct proofs by mathematical induction is a key goal of learning discrete mathematics.

In Chapter 2 we explicitly defined sets and functions. That is, we described sets by listing their elements or by giving some property that characterizes these elements. We gave formulae for the values of functions. There is another important way to define such objects, based on mathematical induction. To define functions, some initial terms are specified, and a rule is given for finding subsequent values from values already known. (We briefly touched on this sort of definition in Chapter 2 when we showed how sequences can be defined using recurrence relations.) Sets can be defined by listing some of their elements and giving rules for constructing elements from those already known to be in the set. Such definitions, called *recursive definitions,* are used throughout discrete mathematics and computer science. Once we have defined a set recursively, we can use a proof method called structural induction to prove results about this set.

When a procedure is specified for solving a problem, this procedure must *always* solve the problem correctly. Just testing to see that the correct result is obtained for a set of input values does not show that the procedure always works correctly. The correctness of a procedure can be guaranteed only by proving that it always yields the correct result. The final section of this chapter contains an introduction to the techniques of program verification. This is a formal technique to verify that procedures are correct. Program verification serves as the basis for attempts under way to prove in a mechanical fashion that programs are correct.

5.1 Mathematical Induction

5.1.1 Introduction

Suppose that we have an infinite ladder, as shown in Figure 1, and we want to know whether we can reach every step on this ladder. We know two things:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we can

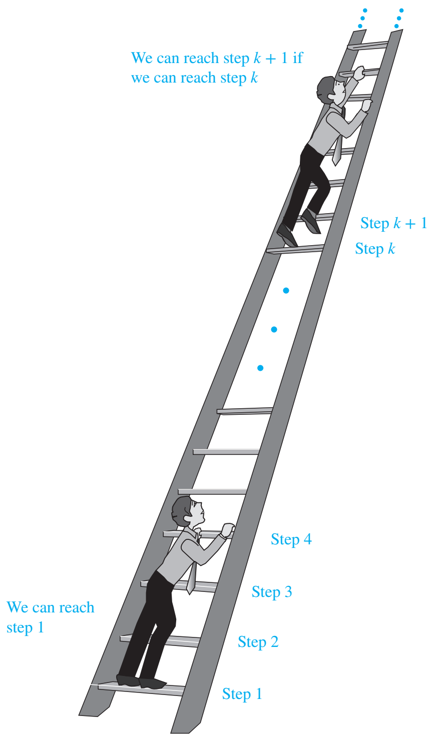


FIGURE 1 **Climbing an infinite ladder.**

reach the fourth rung, the fifth rung, and so on. For example, after 100 uses of (2), we know that we can reach the 101st rung. But can we conclude that we are able to reach every rung of this infinite ladder? The answer is yes, something we can verify using an important proof technique called **mathematical induction**. That is, we can show that the statement that we can reach the nth rung of the ladder is true for all positive integers n.

Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type. As we will see in this section and in subsequent sections of this chapter and later chapters, mathematical induction is used extensively to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

In this section, we will describe how mathematical induction can be used and why it is a valid proof technique. It is extremely important to note that mathematical induction can be used only to prove results obtained in some other way. It is *not* a tool for discovering formulae or theorems.

5.1.2 Mathematical Induction

Assessment

In general, mathematical induction* can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical

*Unfortunately, using the terminology "mathematical induction" clashes with the terminology used to describe different types of reasoning. In logic, **deductive reasoning** uses rules of inference to draw conclusions from premises, whereas **inductive reasoning** makes conclusions only supported, but not ensured, by evidence. Mathematical proofs, including arguments that use mathematical induction, are deductive, not inductive.

induction has two parts, a **basis step**, where we show that $P(1)$ is true, and an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the **inductive hypothesis**. Once we complete both steps in a proof by mathematical induction, we have shown that $P(n)$ is true for all positive integers n, that is, we have shown that $\forall n P(n)$ is true where the quantification is over the set of positive integers. In the inductive step, we show that $\forall k(P(k) \to P(k+1))$ is true, where again, the domain is the set of positive integers.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \land \forall k(P(k) \to P(k+1))) \to \forall n P(n),$$

when the domain is the set of positive integers. Because mathematical induction is such an important technique, it is worthwhile to explain in detail the steps of a proof using this technique. The first thing we do to prove that $P(n)$ is true for all positive integers n is to show that $P(1)$ is true. This amounts to showing that the particular statement obtained when n is replaced by 1 in $P(n)$ is true. Then we must show that $P(k) \to P(k+1)$ is true for every positive integer k. To prove that this conditional statement is true for every positive integer k, we need to show that $P(k+1)$ cannot be false when $P(k)$ is true. This can be accomplished by assuming that $P(k)$ is true and showing that *under this hypothesis* $P(k+1)$ must also be true.

Remark: In a proof by mathematical induction it is *not* assumed that $P(k)$ is true for all positive integers! It is only shown that *if it is assumed* that $P(k)$ is true, then $P(k+1)$ is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

After completing the basis and inductive steps of a proof that $P(n)$ is true for all positive integers n, we know that $P(1)$ is true. That is what is shown in the basis step. We can conclude that $P(2)$ is true, because we know that $P(1)$ is true and from the inductive step we know that $P(1) \to P(2)$. Furthermore, we know that $P(3)$ is true because $P(2)$ is true and we know that $P(2) \to P(3)$ from the inductive step. Continuing along these lines using a finite number of implications, we can show that $P(n)$ is true for any particular positive integer n.

Links

HISTORICAL NOTE The first known use of mathematical induction is in the work of the sixteenth-century mathematician Francesco Maurolico (1494–1575). Maurolico wrote extensively on the works of classical mathematics and made many contributions to geometry and optics. In his book *Arithmeticorum Libri Duo*, Maurolico presented a variety of properties of the integers together with proofs of these properties. To prove some of these properties, he devised the method of mathematical induction. His first use of mathematical induction in this book was to prove that the sum of the first n odd positive integers equals n^2. Augustus De Morgan is credited with the first presentation in 1838 of formal proofs using mathematical induction, as well as introducing the terminology "mathematical induction." Maurolico's proofs were informal and he never used the word "induction." See [Gu10] to learn more about the history of the method of mathematical induction.

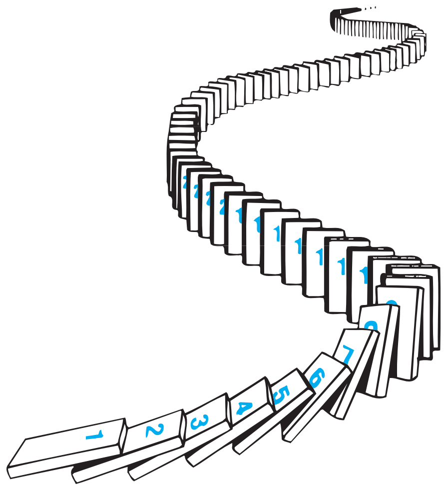


FIGURE 2 **Illustrating how mathematical induction works using dominoes.**

WAYS TO REMEMBER HOW MATHEMATICAL INDUCTION WORKS Thinking of the infinite ladder and the rules for reaching steps can help you remember how mathematical induction works. Note that statements (1) and (2) for the infinite ladder are exactly the basis step and inductive step, respectively, of the proof that $P(n)$ is true for all positive integers n, where $P(n)$ is the statement that we can reach the nth rung of the ladder. Consequently, we can invoke mathematical induction to conclude that we can reach every rung.

Another way to illustrate the principle of mathematical induction is to consider an infinite row of dominoes, labeled $1, 2, 3, \dots, n, \dots$, where each domino is standing up. Let $P(n)$ be the proposition that domino n is knocked over. If the first domino is knocked over—that is, if $P(1)$ is true—and if, whenever the kth domino is knocked over, it also knocks the $(k + 1)$st domino over—that is, if $P(k) \to P(k + 1)$ is true for all positive integers k—then all the dominoes are knocked over. This is illustrated in Figure 2.

5.1.3 Why Mathematical Induction is Valid

Why is mathematical induction a valid proof technique? The reason comes from the well-ordering property, listed in Appendix 1 as an axiom for the set of positive integers, which states that every nonempty subset of the set of positive integers has a least element. So, suppose we know that $P(1)$ is true and that the proposition $P(k) \to P(k + 1)$ is true for all positive integers k. To show that $P(n)$ must be true for all positive integers n, assume that there is at least one positive integer n for which $P(n)$ is false. Then the set S of positive integers n for which $P(n)$ is false is nonempty. Thus, by the well-ordering property, S has a least element, which will be denoted by m. We know that m cannot be 1, because $P(1)$ is true. Because m is positive and greater than 1, $m - 1$ is a positive integer. Furthermore, because $m - 1$ is less than m, it is not in S, so $P(m - 1)$ must be true. Because the conditional statement $P(m - 1) \to P(m)$ is also true, it must be the case that $P(m)$ is true. This contradicts the choice of m. Hence, $P(n)$ must be true for every positive integer n.

Remark: In this book we take the well-ordering property for the positive integers as an axiom. We proved that mathematical induction is a valid proof technique. Instead, we could have taken the principle of mathematical induction as an axiom and proved that the positive integers are well ordered. That is, the well-ordering property for positive integers and the principle of mathematical induction are equivalent. (In Section 5.2 we will present examples of proofs that use the well-ordering property directly. Also, Exercise 41 in that section asks for a proof that the well-ordering property for positive integers is a consequence of the principle of mathematical induction.)

5.1.4 Choosing the Correct Basis Step

Mathematical induction can be used to prove theorems other than those of the form "$P(n)$ is true for all positive integers n." Often, we will need to show that $P(n)$ is true for $n = b,\ b+1,\ b+2, \dots,$ where b is an integer other than 1. We can use mathematical induction to accomplish this, as long as we change the basis step by replacing $P(1)$ with $P(b)$. In other words, to use mathematical induction to show that $P(n)$ is true for $n = b, b+1, b+2, \dots,$ where b is an integer other than 1, we show that $P(b)$ is true in the basis step. In the inductive step, we show that the conditional statement $P(k) \to P(k+1)$ is true for $k = b, b+1, b+2, \dots$. Note that b can be negative, zero, or positive. Following the domino analogy we used earlier, imagine that we begin by knocking down the bth domino (the basis step), and as each domino falls, it knocks down the next domino (the inductive step). We leave it to the reader to show that this form of induction is valid (see Exercise 85).

We will illustrate this notion in Example 3, which states that a summation formula is valid for all nonnegative integers. In this example, we need to prove that $P(n)$ is true for $n = 0, 1, 2, \dots$. So, the basis step in Example 3 will show that $P(0)$ is true.

5.1.5 Guidelines for Proofs by Mathematical Induction

Examples 1–14 will illustrate how to use mathematical induction to prove a diverse collection of theorems. Each of these examples includes all the elements needed in a proof by mathematical induction. We will also present an example of an invalid proof by mathematical induction. Before we give these proofs, we will provide some useful guidelines for constructing correct proofs by mathematical induction.

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b. For statements of the form "$P(n)$ for all positive integers n," let $b = 1$, and for statements of the form "$P(n)$ for all nonnegative integers n," let $b = 0$. For some statements of the form $P(n)$, such as inequalities, you may need to determine the appropriate value of b by checking the truth values of $P(n)$ for small values of n, as is done in Example 6.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step" and state, and clearly identify, the inductive hypothesis, in the form "Assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
4. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
5. Prove the statement $P(k+1)$ making use of the assumption $P(k)$. (Generally, this is the most difficult part of a mathematical induction proof. Decide on the most promising proof strategy and look ahead to see how to use the induction hypothesis to build your proof of the inductive step. Also, be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.)
6. Clearly identify the conclusion of the inductive step, such as by saying "This completes the inductive step."
7. After completing the basis step and the inductive step, state the conclusion, namely, "By mathematical induction, $P(n)$ is true for all integers n with $n \geq b$".

Readers will find it worthwhile to see how the steps described in the template are completed in each of the 14 examples. It will also be useful to follow these guidelines in the solutions of the exercises that ask for proofs by mathematical induction. The guidelines that we presented can be adapted for each of the variants of mathematical induction that we introduce in the exercises and later in this chapter.

5.1.6 The Good and the Bad of Mathematical Induction

You can prove a theorem by mathematical induction even if you do not have the slightest idea why it is true!

An important point needs to be made about mathematical induction before we commence a study of its use. The good thing about mathematical induction is that it can be used to prove a conjecture once it is has been made (and is true). The bad thing about it is that it cannot be used to find new theorems. Mathematicians sometimes find proofs by mathematical induction unsatisfying because they do not provide insights as to why theorems are true. Many theorems can be proved in many ways, including by mathematical induction. Proofs of these theorems by methods other than mathematical induction are often preferred because of the insights they bring. (See Example 8 and the subsequent remark for an example of this.)

5.1.7 Examples of Proofs by Mathematical Induction

Many theorems assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. Mathematical induction is a technique for proving theorems of this kind. In other words, mathematical induction can be used to prove statements of the form $\forall n\, P(n)$, where the domain is the set of positive integers. Mathematical induction can be used to prove an extremely wide variety of theorems, each of which is a statement of this form. (Remember, many mathematical assertions include an implicit universal quantifier. The statement "if n is a positive integer, then $n^3 - n$ is divisible by 3" is an example of this. Making the implicit universal quantifier explicit yields the statement "for every positive integer n, $n^3 - n$ is divisible by 3.")

Links

We will use a variety of examples to illustrate how theorems are proved using mathematical induction. The theorems we will prove include summation formulae, inequalities, identities for combinations of sets, divisibility results, theorems about algorithms, and some other creative results. In this section, and in later sections, we will employ mathematical induction to prove many other types of results, including the correctness of computer programs and algorithms. Mathematical induction can be used to prove a wide variety of theorems both similar to, and also quite different from, the examples here. (For proofs by mathematical induction of many more interesting and diverse results, see the *Handbook of Mathematical Induction* by David Gunderson [Gu11].)

There are many opportunities for errors in induction proofs. We will describe some incorrect proofs by mathematical induction at the end of this section and in the exercises. To avoid making errors in proofs by mathematical induction, try to follow the guidelines for such proofs provided previously in Section 5.1.5.

Look for the $\stackrel{\text{IH}}{=}$ symbol to see where the inductive hypothesis is used.

SEEING WHERE THE INDUCTIVE HYPOTHESIS IS USED To help the reader understand each of the mathematical induction proofs in this section, we will note where the inductive hypothesis is used. We indicate this use in three different ways: by explicit mention in the text, by inserting the acronym IH (for inductive hypothesis) over an equals sign or a sign for an inequality, or by specifying the inductive hypothesis as the reason for a step in a multi-line display.

PROVING SUMMATION FORMULAE We begin by using mathematical induction to prove several summation formulae. As we will see, mathematical induction is particularly well suited for proving that such formulae are valid. However, summation formulae can be proven in other ways. This is not surprising because there are often different ways to prove a theorem. The major disadvantage of using mathematical induction to prove a summation formula is that you cannot use it to derive this formula. That is, you must already have the formula before you attempt to prove it by mathematical induction.

Examples 1–4 illustrate how to use mathematical induction to prove summation formulae. The first summation formula we will prove by mathematical induction, in Example 1, is a closed formula for the sum of the smallest n positive integers.

EXAMPLE 1 Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Extra Examples

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

If you are rusty simplifying algebraic expressions, this is the time to do some reviewing!

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true.

We now look ahead to see how we might be able to prove that $P(k+1)$ holds under the assumption that $P(k)$ is true. We observe that the summation in the left-hand side of $P(k+1)$ is $k+1$ more than the summation in the left-hand side of $P(k)$. Our strategy will be to add $k+1$ to both sides of the equation in $P(k)$ and simplify the result algebraically to complete the inductive step.

We now return to the proof of the inductive step. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n+1)/2$ for all positive integers n. ◀

As we noted, mathematical induction is not a tool for finding theorems about all positive integers. Rather, it is a proof method for proving such results once they are conjectured. In Example 2, using mathematical induction to prove a summation formula, we will both formulate and then prove a conjecture.

EXAMPLE 2 Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$1 = 1, \qquad 1 + 3 = 4, \qquad 1 + 3 + 5 = 9,$$
$$1 + 3 + 5 + 7 = 16, \qquad 1 + 3 + 5 + 7 + 9 = 25.$$

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We need a method to *prove* that this *conjecture* is correct, if in fact it is.

Let $P(n)$ denote the proposition that the sum of the first n odd positive integers is n^2. Our conjecture is that $P(n)$ is true for all positive integers n. To use mathematical induction to prove this conjecture, we must first complete the basis step; that is, we must show that $P(1)$ is true. Then we must carry out the inductive step; that is, we must show that $P(k + 1)$ is true when $P(k)$ is assumed to be true. We now attempt to complete these two steps.

BASIS STEP: $P(1)$ states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1. The basis step is complete.

INDUCTIVE STEP: To complete the inductive step we must show that the proposition $P(k) \to P(k + 1)$ is true for every positive integer k. To do this, we first assume the inductive hypothesis. The inductive hypothesis is the statement that $P(k)$ is true for an arbitrary positive integer k, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

(Note that the kth odd positive integer is $(2k - 1)$, because this integer is obtained by adding 2 a total of $k - 1$ times to 1.)

To show that $\forall k(P(k) \to P(k + 1))$ is true, we must show that if $P(k)$ is true (the inductive hypothesis), then $P(k + 1)$ is true. Note that $P(k + 1)$ is the statement that

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

Before we complete the inductive step, we will take a time out to figure out a strategy. At this stage of a mathematical induction proof it is time to look for a way to use the inductive hypothesis to show that $P(k + 1)$ is true. Here we note that $1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1)$ is the sum of its first k terms $1 + 3 + 5 + \cdots + (2k - 1)$ and its last term $2k - 1$. So, we can use our inductive hypothesis to replace $1 + 3 + 5 + \cdots + (2k - 1)$ by k^2.

We now return to our proof. We find that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + \cdots + (2k - 1)] + (2k + 1) \\ &\overset{\text{IH}}{=} k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

This shows that $P(k + 1)$ follows from $P(k)$. Note that we used the inductive hypothesis $P(k)$ in the second equality to replace the sum of the first k odd positive integers by k^2.

We have now completed both the basis step and the inductive step. That is, we have shown that $P(1)$ is true and the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k. Consequently, by the principle of mathematical induction we can conclude that $P(n)$ is true for all positive integers n. That is, we know that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all positive integers n. ◀

EXAMPLE 3 Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n. ◀

The formula given in Example 3 is a special case of a general result for the sum of terms of a geometric progression (Theorem 1 in Section 2.4). We will use mathematical induction to provide an alternative proof of this formula.

EXAMPLE 4 **Sums of Geometric Progressions** Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

Solution: To prove this formula using mathematical induction, let $P(n)$ be the statement that the sum of the first $n+1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, because

$$\frac{ar^{0+1}-a}{r-1} = \frac{ar-a}{r-1} = \frac{a(r-1)}{r-1} = a.$$

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1}-a}{r-1}.$$

To complete the inductive step we must show that if $P(k)$ is true, then $P(k+1)$ is also true. To show that this is the case, we first add ar^{k+1} to both sides of the equality asserted by $P(k)$. We find that

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \overset{\text{IH}}{=} \frac{ar^{k+1}-a}{r-1} + ar^{k+1}.$$

Rewriting the right-hand side of this equation shows that

$$\begin{aligned}\frac{ar^{k+1}-a}{r-1} + ar^{k+1} &= \frac{ar^{k+1}-a}{r-1} + \frac{ar^{k+2}-ar^{k+1}}{r-1}\\ &= \frac{ar^{k+2}-a}{r-1}.\end{aligned}$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2}-a}{r-1}.$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ must also be true. This completes the inductive argument.

We have completed the basis step and the inductive step, so by mathematical induction $P(n)$ is true for all nonnegative integers n. This shows that the formula for the sum of the terms of a geometric series is correct. ◀

As previously mentioned, the formula in Example 3 is the case of the formula in Example 4 with $a = 1$ and $r = 2$. The reader should verify that putting these values for a and r into the general formula gives the same formula as in Example 3.

PROVING INEQUALITIES Mathematical induction can be used to prove a variety of inequalities that hold for all positive integers greater than a particular positive integer, as Examples 5–7 illustrate.

EXAMPLE 5 Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

Extra Examples

BASIS STEP: $P(1)$ is true, because $1 < 2^1 = 2$. This completes the basis step.

INDUCTIVE STEP: We first assume the inductive hypothesis that $P(k)$ is true for an arbitrary positive integer k. That is, the inductive hypothesis $P(k)$ is the statement that $k < 2^k$. To complete the inductive step, we need to show that if $P(k)$ is true, then $P(k+1)$, which is the statement that $k+1 < 2^{k+1}$ is true. That is, we need to show that if $k < 2^k$, then $k+1 < 2^{k+1}$. To show that this conditional statement is true for the positive integer k, we first add 1 to both sides of $k < 2^k$, and then note that $1 \le 2^k$. This tells us that

$$k + 1 \overset{\text{IH}}{<} 2^k + 1 \le 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

This shows that $P(k+1)$ is true, namely, that $k+1 < 2^{k+1}$, based on the assumption that $P(k)$ is true. The induction step is complete.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n. ◀

EXAMPLE 6 Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \ge 4$. (Note that this inequality is false for $n = 1, 2,$ and 3.)

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: To prove the inequality for $n \ge 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

INDUCTIVE STEP: For the inductive step, we assume that $P(k)$ is true for an arbitrary integer k with $k \ge 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \ge 4$. We must show that under this hypothesis, $P(k+1)$ is also true. That is, we must show that if $2^k < k!$ for an arbitrary positive integer k where $k \ge 4$, then $2^{k+1} < (k+1)!$. We have

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\ &\overset{\text{IH}}{<} 2 \cdot k! && \text{by the inductive hypothesis} \\ &< (k+1)k! && \text{because } 2 < k+1 \\ &= (k+1)! && \text{by definition of factorial function.} \end{aligned}$$

This shows that $P(k+1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.

We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \ge 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \ge 4$. ◀

An important inequality for the sum of the reciprocals of a set of positive integers will be proved in Example 7.

EXAMPLE 7 **An Inequality for Harmonic Numbers** The **harmonic numbers** $H_j, j = 1, 2, 3, \ldots,$ are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{2^k} \geq 1 + \frac{k}{2}$, where k is an arbitrary nonnegative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true. So, assuming the inductive hypothesis, it follows that

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^k\text{th harmonic number} \\
&\overset{\text{IH}}{\geq} \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{because there are } 2^k \text{ terms each} \geq 1/2^{k+1} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{canceling a common factor of } 2^k \text{ in second term} \\
&= 1 + \frac{k+1}{2}.
\end{aligned}$$

This establishes the inductive step of the proof.

We have completed the basis step and the inductive step. Thus, by mathematical induction $P(n)$ is true for all nonnegative integers n. That is, the inequality $H_{2^n} \geq 1 + \frac{n}{2}$ for the harmonic numbers holds for all nonnegative integers n. ◀

Remark: The inequality established here shows that the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is a divergent infinite series. This is an important example in the study of infinite series.

PROVING DIVISIBILITY RESULTS Mathematical induction can be used to prove divisibility results about integers. Although such results are often easier to prove using basic results in number theory, it is instructive to see how to prove such results using mathematical induction, as Examples 8 and 9 illustrate.

EXAMPLE 8 *Extra Examples*

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer. (Note that this is the statement with $p = 3$ of Fermat's little theorem, which is Theorem 3 of Section 4.4.)

Solution: To construct the proof, let $P(n)$ denote the proposition: "$n^3 - n$ is divisible by 3."

BASIS STEP: The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true; that is, we assume that $k^3 - k$ is divisible by 3 for an arbitrary positive integer k. To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(k + 1)$, the statement that $(k + 1)^3 - (k + 1)$ is divisible by 3, is also true. That is, we must show that $(k + 1)^3 - (k + 1)$ is divisible by 3. Note that

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1)\\ &= (k^3 - k) + 3(k^2 + k).\end{aligned}$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3. The second term is divisible by 3 because it is 3 times an integer. So, by part (i) of Theorem 1 in Section 4.1, we know that $(k + 1)^3 - (k + 1)$ is also divisible by 3. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $n^3 - n$ is divisible by 3 whenever n is a positive integer. ◀

Remark: We have included Example 8 as an illustration how a divisibility result can be proved by mathematical induction. However, there are simpler proofs. For example, we can prove that $n^3 - n$ is divisible by 3 for all positive integers n using the factorization $n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1) = (n - 1)n(n + 1)$. Hence, $n^3 - 1$ is divisible by 3 because it is the product of three consecutive integers, one of which is divisible by 3.

Example 9 presents a more challenging proof by mathematical induction of a divisibility result.

EXAMPLE 9 *Extra Examples*

Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Solution: To construct the proof, let $P(n)$ denote the proposition: "$7^{n+2} + 8^{2n+1}$ is divisible by 57."

BASIS STEP: To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer n. We see that $P(0)$ is true because $7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 57$ is divisible by 57. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k + 1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned}
7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3}\\
&= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1}\\
&= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1}\\
&= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.
\end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2} + 8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57 \cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} = 7^{k+3} + 8^{2k+3}$ is divisible by 57. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n. ◀

PROVING RESULTS ABOUT SETS Mathematical induction can be used to prove many results about sets. In particular, in Example 10 we prove a formula for the number of subsets of a finite set and in Example 11 we establish a set identity.

EXAMPLE 10 **The Number of Subsets of a Finite Set** Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets. (We will prove this result directly in several ways in Chapter 6.)

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

BASIS STEP: $P(0)$ is true, because a set with zero elements, the empty set, has exactly $2^0 = 1$ subset, namely, itself.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k, that is, we assume that every set with k elements has 2^k subsets. It must be shown that under this assumption, $P(k + 1)$, which is the statement that every set with $k + 1$ elements has 2^{k+1} subsets, must also be true. To show this, let T be a set with $k + 1$ elements. Then, it is possible to write $T = S \cup \{a\}$, where a is one of the elements of T and $S = T - \{a\}$ (and hence $|S| = k$). The subsets of T can be obtained in the following way. For each subset X of S there are exactly two subsets of T, namely, X and $X \cup \{a\}$. (This is illustrated in Figure 3.) These constitute all the subsets of T and are all distinct. We now use the inductive hypothesis to conclude that S has 2^k subsets, because it has k elements. We also know that there are two subsets of T for each subset of S. Therefore, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This finishes the inductive argument.

Because we have completed the basis step and the inductive step, by mathematical induction it follows that $P(n)$ is true for all nonnegative integers n. That is, we have proved that a set with n elements has 2^n subsets whenever n is a nonnegative integer. ◀

EXAMPLE 11 Use mathematical induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$.

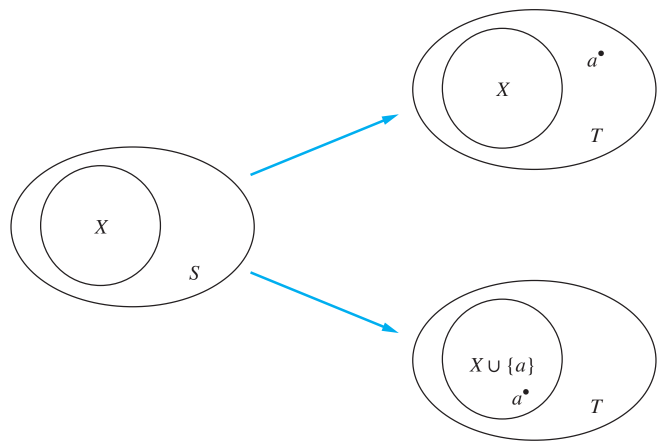


FIGURE 3 Generating subsets of a set with $k + 1$ elements. Here $T = S \cup \{a\}$.

Solution: Let $P(n)$ be the identity for n sets.

BASIS STEP: The statement $P(2)$ asserts that $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is one of De Morgan's laws; it was proved in Example 11 of Section 2.2.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary integer with $k \geq 2$; that is, it is the statement that

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. To carry out the inductive step, we need to show that this assumption implies that $P(k + 1)$ is true. That is, we need to show that if this equality holds for every collection of k subsets of U, then it must also hold for every collection of $k + 1$ subsets of U. Suppose that $A_1, A_2, \ldots, A_k, A_{k+1}$ are subsets of U. When the inductive hypothesis is assumed to hold, it follows that

$$\begin{aligned}
\overline{\bigcap_{j=1}^{k+1} A_j} &= \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}} && \text{by the definition of intersection} \\
&= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}} && \text{by De Morgan's law (where the two sets are } \textstyle\bigcap_{j=1}^{k} A_j \text{ and } A_{k+1}) \\
&\overset{\text{IH}}{=} \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}} && \text{by the inductive hypothesis} \\
&= \bigcup_{j=1}^{k+1} \overline{A_j} && \text{by the definition of union.}
\end{aligned}$$

This completes the inductive step.

Because we have completed both the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true whenever n is a positive integer, $n \geq 2$. That is, we know that

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$. ◀

PROVING RESULTS ABOUT ALGORITHMS Next, we provide an example (somewhat more difficult than previous examples) that illustrates one of many ways mathematical induction is used in the study of algorithms. We will show how mathematical induction can be used to prove that a greedy algorithm we introduced in Section 3.1 always yields an optimal solution.

EXAMPLE 12 Recall the algorithm for scheduling talks discussed in Example 7 of Section 3.1. The input to this algorithm is a group of m proposed talks with preset starting and ending times. The goal is to schedule as many of these lectures as possible in the main lecture hall so that no two talks overlap. Suppose that talk t_j begins at time s_j and ends at time e_j. (No two lectures can proceed in the main lecture hall at the same time, but a lecture in this hall can begin at the same time another one ends.)

Without loss of generality, we assume that the talks are listed in order of nondecreasing ending time, so that $e_1 \leq e_2 \leq \cdots \leq e_m$. The greedy algorithm proceeds by selecting at each stage a talk with the earliest ending time among all those talks that begin no sooner than when the last talk scheduled in the main lecture hall has ended. Note that a talk with the earliest end time is always selected first by the algorithm. We will show that this greedy algorithm is optimal in the sense that it always schedules the most talks possible in the main lecture hall. To prove the optimality of this algorithm we use mathematical induction on the variable n, the number of talks scheduled by the algorithm. We let $P(n)$ be the proposition that if the greedy algorithm schedules n talks in the main lecture hall, then it is not possible to schedule more than n talks in this hall.

BASIS STEP: Suppose that the greedy algorithm managed to schedule just one talk, t_1, in the main lecture hall. This means that no other talk can start at or after e_1, the end time of t_1. Otherwise, the first such talk we come to as we go through the talks in order of nondecreasing end times could be added. Hence, at time e_1 each of the remaining talks needs to use the main lecture hall because they all start before e_1 and end after e_1. It follows that no two talks can be scheduled because both need to use the main lecture hall at time e_1. This shows that $P(1)$ is true and completes the basis step.

INDUCTIVE STEP: The inductive hypothesis is that $P(k)$ is true, where k is an arbitrary positive integer, that is, that the greedy algorithm always schedules the most possible talks when it selects k talks, where k is a positive integer, given any set of talks, no matter how many. We must show that $P(k+1)$ follows from the assumption that $P(k)$ is true, that is, we must show that under the assumption of $P(k)$, the greedy algorithm always schedules the most possible talks when it selects $k+1$ talks.

Now suppose that the greedy algorithm has selected $k+1$ talks. Our first step in completing the inductive step is to show there is a schedule including the most talks possible that contains talk t_1, a talk with the earliest end time. This is easy to see because a schedule that begins with the talk t_i in the list, where $i > 1$, can be changed so that talk t_1 replaces talk t_i. To see this, note that because $e_1 \leq e_i$, all talks that were scheduled to follow talk t_i can still be scheduled.

Once we included talk t_1, scheduling the talks so that as many as possible are scheduled is reduced to scheduling as many talks as possible that begin at or after time e_1. So, if we have scheduled as many talks as possible, the schedule of talks other than talk t_1 is an optimal schedule of the original talks that begin once talk t_1 has ended. Because the greedy algorithm schedules k talks when it creates this schedule, we can apply the inductive hypothesis to conclude that it has scheduled the most possible talks. It follows that the greedy algorithm has scheduled the most possible talks, $k + 1$, when it produced a schedule with $k + 1$ talks, so $P(k + 1)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step. So, by mathematical induction we know that $P(n)$ is true for all positive integers n. This completes the proof of optimality. That is, we have proved that when the greedy algorithm schedules n talks, when n is a positive integer, then it is not possible to schedule more than n talks. ◀

CREATIVE USES OF MATHEMATICAL INDUCTION Mathematical induction can often be used in unexpected ways. We will illustrate two particularly clever uses of mathematical induction here, the first relating to survivors in a pie fight and the second relating to tilings with regular triominoes of checkerboards with one square missing.

EXAMPLE 13

Extra Examples

Odd Pie Fights An odd number of people stand in a yard at mutually distinct distances. At the same time each person throws a pie at their nearest neighbor, hitting this person. Use mathematical induction to show that there is at least one survivor, that is, at least one person who is not hit by a pie. (This problem was introduced by Carmony [Ca79]. Note that this result is false when there are an even number of people; see Exercise 77.)

Solution: Let $P(n)$ be the statement that there is a survivor whenever $2n + 1$ people stand in a yard at distinct mutual distances and each person throws a pie at their nearest neighbor. To prove this result, we will show that $P(n)$ is true for all positive integers n. This follows because as n runs through all positive integers, $2n + 1$ runs through all odd integers greater than or equal to 3. Note that one person cannot engage in a pie fight because there is no one else to throw the pie at.

BASIS STEP: When $n = 1$, there are $2n + 1 = 3$ people in the pie fight. Of the three people, suppose that the closest pair are A and B, and C is the third person. Because distances between pairs of people are different, the distance between A and C and the distance between B and C are both different from, and greater than, the distance between A and B. It follows that A and B throw pies at each other, while C throws a pie at either A or B, whichever is closer. Hence, C is not hit by a pie. This shows that at least one of the three people is not hit by a pie, completing the basis step.

INDUCTIVE STEP: For the inductive step, assume that $P(k)$ is true for an arbitrary odd integer k with $k \geq 3$. That is, assume that there is at least one survivor whenever $2k + 1$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor. We must show that if the inductive hypothesis $P(k)$ is true, then $P(k + 1)$, the statement that there is at least one survivor whenever $2(k + 1) + 1 = 2k + 3$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor, is also true.

So suppose that we have $2(k + 1) + 1 = 2k + 3$ people in a yard with distinct distances between pairs of people. Let A and B be the closest pair of people in this group of $2k + 3$ people. When each person throws a pie at the nearest person, A and B throw pies at each other. We have two cases to consider, *(i)* when someone else throws a pie at either A or B and *(ii)* when no one else throws a pie at either A or B.

Case (i): Because A and B throw pies at each other and someone else throws a pie at either A and B, at least three pies are thrown at A and B, and at most $(2k + 3) - 3 = 2k$ pies are thrown at the remaining $2k + 1$ people. This guarantees that at least one person is a survivor, for if each of these $2k + 1$ people was hit by at least one pie, a total of at least $2k + 1$ pies would have to be thrown at them. (The reasoning used in this last step is an example of the pigeonhole principle discussed further in Section 6.2.)

Case (ii): No one else throws a pie at either A and B. Besides A and B, there are $2k + 1$ people. Because the distances between pairs of these people are all different, we can use the inductive hypothesis to conclude that there is at least one survivor S when these $2k + 1$ people each throws a pie at their nearest neighbor. Furthermore, S is also not hit by either the pie thrown by A or the pie thrown by B because A and B throw their pies at each other, so S is a survivor because S is not hit by any of the pies thrown by these $2k + 3$ people.

We have completed both the basis step and the inductive step, using a proof by cases. So by mathematical induction it follows that $P(n)$ is true for all positive integers n. We conclude that whenever an odd number of people located in a yard at distinct mutual distances each throws a pie at their nearest neighbor, there is at least one survivor. ◀

Links

In Section 1.8 we discussed the tiling of checkerboards by polyominoes. Example 14 illustrates how mathematical induction can be used to prove a result about covering checkerboards with right triominoes, pieces shaped like the letter "L."

EXAMPLE 14 Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, where these pieces cover three squares at a time, as shown in Figure 4.

FIGURE 4 A right triomino.

Solution: Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes. We can use mathematical induction to prove that $P(n)$ is true for all positive integers n.

BASIS STEP: $P(1)$ is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino, as shown in Figure 5.

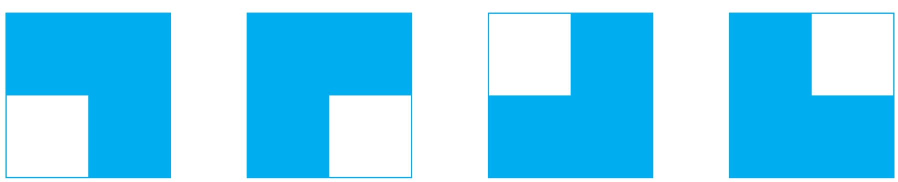


FIGURE 5 Tiling 2 × 2 checkerboards with one square removed.

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(k)$ is true for the positive integer k; that is, it is the assumption that every $2^k \times 2^k$ checkerboard with one square removed can be tiled using right triominoes. It must be shown that under the assumption of the inductive hypothesis, $P(k + 1)$ must also be true; that is, any $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

To see this, consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions. This is illustrated in Figure 6. No square has been removed from three of these four checkerboards. The fourth $2^k \times 2^k$ checkerboard has one square removed, so we now use the inductive hypothesis to conclude that it can be covered by right triominoes. Now temporarily remove the square from each of the other three $2^k \times 2^k$ checkerboards that has the center of the original, larger checkerboard as one of its corners, as shown in Figure 7. By the inductive hypothesis, each of these three $2^k \times 2^k$ checkerboards with a square removed can be tiled by right triominoes. Furthermore, the three squares that were temporarily removed can be covered by one right triomino. Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard can be tiled with right triominoes.

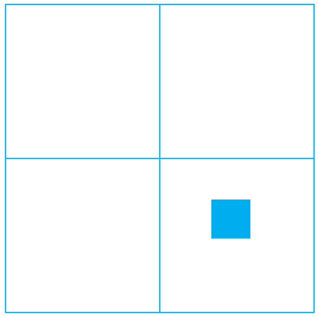


FIGURE 6 **Dividing a $2^{k+1} \times 2^{k+1}$ checkerboard into four $2^k \times 2^k$ checkerboards.**

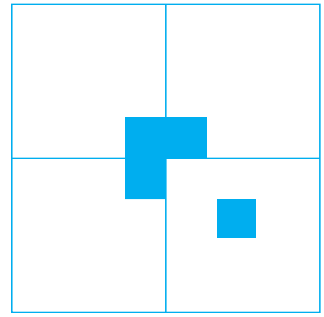


FIGURE 7 **Tiling the $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed.**

We have completed the basis step and the inductive step. Therefore, by mathematical induction $P(n)$ is true for all positive integers n. This shows that we can tile every $2^n \times 2^n$ checkerboard, where n is a positive integer, with one square removed, using right triominoes. ◀

5.1.8 Mistaken Proofs By Mathematical Induction

Consult *Common Errors in Discrete Mathematics* on this book's website for more basic mistakes.

As with every proof method, there are many opportunities for making errors when using mathematical induction. Many well-known mistaken, and often entertaining, proofs by mathematical induction of clearly false statements have been devised, as exemplified by Example 15 and Exercises 49–51. Often, it is not easy to find where the error in reasoning occurs in such mistaken proofs.

To uncover errors in proofs by mathematical induction, remember that in every such proof, both the basis step and the inductive step must be done correctly. Not completing the basis step in a supposed proof by mathematical induction can lead to mistaken proofs of clearly ridiculous statements such as "$n = n + 1$ whenever n is a positive integer." (We leave it to the reader to show that it is easy to construct a correct inductive step in an attempted proof of this statement.) Locating the error in a faulty proof by mathematical induction, as Example 15 illustrates, can be quite tricky, especially when the error is hidden in the basis step.

EXAMPLE 15 Find the error in this "proof" of the clearly false claim that every set of lines in the plane, no two of which are parallel, meet in a common point.

"Proof:" Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. We will attempt to prove that $P(n)$ is true for all positive integers $n \geq 2$.

BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point (by the definition of parallel lines).

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true for the positive integer k, that is, it is the assumption that every set of k lines in the plane, no two of which are parallel, meet in a common point. To complete the inductive step, we must show that if $P(k)$ is true, then $P(k + 1)$ must also be true. That is, we must show that if every set of k lines in the plane, no two of which are parallel, meet in a common point, then every set of $k + 1$ lines in the plane, no two of which are parallel, meet in a common point. So, consider a set of $k + 1$ distinct lines in the plane. By the inductive hypothesis, the first k of these lines meet in a common point

p_1. Moreover, by the inductive hypothesis, the last k of these lines meet in a common point p_2. We will show that p_1 and p_2 must be the same point. If p_1 and p_2 were different points, all lines containing both of them must be the same line because two points determine a line. This contradicts our assumption that all these lines are distinct. Thus, p_1 and p_2 are the same point. We conclude that the point $p_1 = p_2$ lies on all $k + 1$ lines. We have shown that $P(k + 1)$ is true assuming that $P(k)$ is true. That is, we have shown that if we assume that every k, $k \geq 2$, distinct lines meet in a common point, then every $k + 1$ distinct lines meet in a common point. This completes the inductive step.

We have completed the basis step and the inductive step, and supposedly we have a correct proof by mathematical induction.

Solution: Examining this supposed proof by mathematical induction it appears that everything is in order. However, there is an error, as there must be. The error is rather subtle. Carefully looking at the inductive step shows that this step requires that $k \geq 3$. We cannot show that $P(2)$ implies $P(3)$. When $k = 2$, our goal is to show that every three distinct lines meet in a common point. The first two lines must meet in a common point p_1 and the last two lines must meet in a common point p_2. But in this case, p_1 and p_2 do not have to be the same, because only the second line is common to both sets of lines. Here is where the inductive step fails. ◀

Exercises

1. There are infinitely many stations on a train route. Suppose that the train stops at the first station and suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.

2. Suppose that you know that a golfer plays the first hole of a golf course with an infinite number of holes and that if this golfer plays one hole, then the golfer goes on to play the next hole. Prove that this golfer plays every hole on the course.

Use mathematical induction in Exercises 3–17 to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

3. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of a proof that $P(n)$ is true for all positive integers n.

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?

d) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?

e) Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n + 1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof of $P(n)$ for all positive integers n.

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?

d) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?

e) Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3$ whenever n is a nonnegative integer.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.

7. Prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

8. Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

9. a) Find a formula for the sum of the first n even positive integers.

b) Prove the formula that you conjectured in part (a).

10. a) Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

11. a) Find a formula for

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

12. Prove that

$$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^{j}=\frac{2^{n+1}+(-1)^{n}}{3\cdot 2^{n}}$$

whenever n is a nonnegative integer.

13. Prove that $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1} n(n+1)/2$ whenever n is a positive integer.

14. Prove that for every positive integer n, $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

15. Prove that for every positive integer n,

$$1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3.$$

16. Prove that for every positive integer n,

$$\begin{aligned}1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n(n+1)(n+2)\\ = n(n+1)(n+2)(n+3)/4.\end{aligned}$$

17. Prove that $\sum_{j=1}^{n} j^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$ whenever n is a positive integer.

Use mathematical induction to prove the inequalities in Exercises 18–30.

18. Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

a) What is the statement $P(2)$?
b) Show that $P(2)$ is true, completing the basis step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.
c) What is the inductive hypothesis of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1?
d) What do you need to prove in the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1?
e) Complete the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.
f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

19. Let $P(n)$ be the statement that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

where n is an integer greater than 1.

a) What is the statement $P(2)$?
b) Show that $P(2)$ is true, completing the basis step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.
c) What is the inductive hypothesis of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1?
d) What do you need to prove in the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1?
e) Complete the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.
f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

20. Prove that $3^n < n!$ if n is an integer greater than 6.

21. Prove that $2^n > n^2$ if n is an integer greater than 4.

22. For which nonnegative integers n is $n^2 \le n!$? Prove your answer.

23. For which nonnegative integers n is $2n + 3 \le 2^n$? Prove your answer.

24. Prove that $1/(2n) \le [1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)]/(2\cdot 4\cdot \cdots \cdot 2n)$ whenever n is a positive integer.

***25.** Prove that if $h > -1$, then $1 + nh \le (1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

***26.** Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \le na^{n-1}(a-b)$.

***27.** Prove that for every positive integer n,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

28. Prove that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \ge 3$.

In Exercises 29 and 30, H_n denotes the nth harmonic number.

***29.** Prove that $H_{2^n} \le 1 + n$ whenever n is a nonnegative integer.

***30.** Prove that

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

Use mathematical induction in Exercises 31–37 to prove divisibility facts.

31. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

32. Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

33. Prove that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

34. Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

***35.** Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

***36.** Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

***37.** Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.

Use mathematical induction in Exercises 38–46 to prove results about sets.

38. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j.$$

39. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j.$$

40. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).$$

41. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

42. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B.$$

43. Prove that if $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, then

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}.$$

44. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B.$$

45. Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

***46.** Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

In Exercises 47 and 48 we consider the problem of placing towers along a straight road, so that every building on the road receives cellular service. Assume that a building receives cellular service if it is within one mile of a tower.

47. Devise a greedy algorithm that uses the minimum number of towers possible to provide cell service to d buildings located at positions $x_1, x_2, \ldots, x_d$ from the start of the road. [*Hint:* At each step, go as far as possible along the road before adding a tower so as not to leave any buildings without coverage.]

***48.** Use mathematical induction to prove that the algorithm you devised in Exercise 47 produces an optimal solution, that is, that it uses the fewest towers possible to provide cellular service to all buildings.

Exercises 49–51 present incorrect proofs using mathematical induction. You will need to identify an error in reasoning in each exercise.

49. What is wrong with this "proof" that all horses are the same color?
Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color. Consider any $k+1$ horses; number these as horses $1, 2, 3, \ldots, k, k+1$. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Because the set of the first k horses and the set of the last k horses overlap, all $k+1$ must be the same color. This shows that $P(k+1)$ is true and finishes the proof by induction.

50. What is wrong with this "proof"?
"Theorem" For every positive integer n, $\sum_{i=1}^{n} i = (n+\frac{1}{2})^2/2$.

Basis Step: The formula is true for $n = 1$.

Inductive Step: Suppose that $\sum_{i=1}^{n} i = (n+\frac{1}{2})^2/2$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$. By the inductive hypothesis, we have $\sum_{i=1}^{n+1} i = (n+\frac{1}{2})^2/2 + n + 1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n+\frac{3}{2})^2/2 = [(n+1)+\frac{1}{2}]^2/2$, completing the inductive step.

51. What is wrong with this "proof"?
"Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

52. Suppose that m and n are positive integers with $m > n$ and f is a function from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Use mathematical induction on the variable n to show that f is not one-to-one.

***53.** Use mathematical induction to show that n people can divide a cake (where each person gets one or more separate pieces of the cake) so that the cake is divided fairly, that is, in the sense that each person thinks he or she got at least $(1/n)$th of the cake. [*Hint:* For the inductive step, take a fair division of the cake among the first k people, have each person divide their share into what this person thinks are $k + 1$ equal portions, and then have the $(k + 1)$st person select a portion from each of the k people. When showing this produces a fair division for $k + 1$ people, suppose that person $k + 1$ thinks that person i got p_i of the cake, where $\sum_{i=1}^{k} p_i = 1$.]

54. Use mathematical induction to show that given a set of $n + 1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

***55.** A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or two spaces horizontally (in either direction) and one space vertically (in either direction). Suppose that we have an infinite chessboard, made up of all squares (m, n), where m and n are nonnegative integers that denote the row number and the column number of the square, respectively. Use mathematical induction to show that a knight starting at $(0, 0)$ can visit every square using a finite sequence of moves. [*Hint:* Use induction on the variable $s = m + n$.]

56. Suppose that

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where a and b are real numbers. Show that

$$\mathbf{A}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every positive integer n.

57. (*Requires calculus*) Use mathematical induction to prove that the derivative of $f(x) = x^n$ equals nx^{n-1} whenever n is a positive integer. (For the inductive step, use the product rule for derivatives.)

58. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the property $\mathbf{AB} = \mathbf{BA}$. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

59. Suppose that m is a positive integer. Use mathematical induction to prove that if a and b are integers with $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$ whenever k is a nonnegative integer.

60. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ whenever $p_1, p_2, \ldots, p_n$ are propositions.

***61.** Show that

$$[(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_{n-1} \to p_n)] \\ \to [(p_1 \wedge p_2 \wedge \cdots \wedge p_{n-1}) \to p_n]$$

is a tautology whenever $p_1, p_2, \ldots, p_n$ are propositions, where $n \geq 2$.

***62.** Show that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

****63.** Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The **arithmetic mean** of these numbers is defined by

$$A = (a_1 + a_2 + \cdots + a_n)/n,$$

and the **geometric mean** of these numbers is defined by

$$G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$.

64. Use mathematical induction to prove Lemma 3 of Section 4.3, which states that if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where a_i is an integer for $i = 1, 2, 3, \ldots, n$, then $p \mid a_i$ for some integer i.

65. Show that if n is a positive integer, then

$$\sum_{\{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is over all nonempty subsets of the set of the n smallest positive integers.)

***66.** Use the well-ordering property to show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

Basis Step: $P(1)$ and $P(2)$ are true.

Inductive Step: For each positive integer k, if $P(k)$ and $P(k+1)$ are both true, then $P(k+2)$ is true.

67. Show that if $A_1, A_2, \ldots, A_n$ are sets where $n \geq 2$, and for all pairs of integers i and j with $1 \leq i < j \leq n$, either A_i is a subset of A_j or A_j is a subset of A_i, then there is an integer i, $1 \leq i \leq n$, such that A_i is a subset of A_j for all integers j with $1 \leq j \leq n$.

***68.** A guest at a party is a **celebrity** if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party, for if there were two, they would know each other. A particular party may have no celebrity. Your assignment is to find the celebrity, if one exists, at a party, by asking only one type of question—asking a guest whether they know a second guest. Everyone must answer your questions truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the party, then you can find the celebrity, if there is one, with $3(n-1)$ questions. [*Hint:* First ask a question to eliminate one person as a celebrity. Then use the inductive hypothesis to identify a potential celebrity. Finally, ask two more questions to determine whether that person is actually a celebrity.]

Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The **gossip problem** asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals. Exercises 69–71 deal with the gossip problem.

69. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

70. Use mathematical induction to prove that $G(n) \leq 2n - 4$ for $n \geq 4$. [*Hint:* In the inductive step, have a new person call a particular person at the start and at the end.]

****71.** Prove that $G(n) = 2n - 4$ for $n \geq 4$.

***72.** Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them. [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

***73.** Show that if $I_1, I_2, \ldots, I_n$ is a collection of open intervals on the real number line, $n \geq 2$, and every pair of these intervals has a nonempty intersection, that is, $I_i \cap I_j \neq \emptyset$ whenever $1 \leq i \leq n$ and $1 \leq j \leq n$, then the intersection of all these sets is nonempty, that is, $I_1 \cap I_2 \cap \cdots \cap I_n \neq \emptyset$. (Recall that an **open interval** is the set of real numbers x with $a < x < b$, where a and b are real numbers with $a < b$.)

Sometimes we cannot use mathematical induction to prove a result we believe to be true, but we can use mathematical induction to prove a stronger result. Because the inductive hypothesis of the stronger result provides more to work with, this process is called **inductive loading**. We use inductive loading in Exercise 74–76.

74. Show that we cannot use mathematical induction to prove that $\sum_{j=1}^{n} 1/j^2 < 2$ for all positive integers n, but that this inequality is a consequence of the inequality proved by mathematical induction in Exercise 19.

75. Suppose that we want to prove that

$$\sum_{j=1}^{n} j/(j+1)! < 1$$

for all positive integers n.

a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

b) Show that mathematical induction can be used to prove the stronger inequality

$$\sum_{j=1}^{n} j/(j+1)! \leq 1 - 1/(n+1)!$$

for all positive integers n, implying that the weaker inequality is also true.

76. Suppose that we want to prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

b) Show that mathematical induction can be used to prove the stronger inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

for all integers n greater than 1, which, together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

77. Let n be an even integer. Show that it is possible for n people to stand in a yard at mutually distinct distances so that when each person throws a pie at their nearest neighbor, everyone is hit by a pie.

78. Construct a tiling using right triominoes of the 4×4 checkerboard with the square in the upper left corner removed.

79. Construct a tiling using right triominoes of the 8×8 checkerboard with the square in the upper left corner removed.

80. Prove or disprove that all checkerboards of these shapes can be completely covered using right triominoes whenever n is a positive integer.

a) 3×2^n **b)** 6×2^n

c) $3^n \times 3^n$ **d)** $6^n \times 6^n$

***81.** Show that a three-dimensional $2^n \times 2^n \times 2^n$ checkerboard with one $1 \times 1 \times 1$ cube missing can be completely covered by $2 \times 2 \times 2$ cubes with one $1 \times 1 \times 1$ cube removed.

***82.** Show that an $n \times n$ checkerboard with one square removed can be completely covered using right triominoes if $n > 5$, n is odd, and $3 \nmid n$.

83. Show that a 5×5 checkerboard with a corner square removed can be tiled using right triominoes.

***84.** Find a 5×5 checkerboard with a square removed that cannot be tiled using right triominoes. Prove that such a tiling does not exist for this board.

85. Use the principle of mathematical induction to show that $P(n)$ is true for $n = b, b+1, b+2, \ldots$, where b is an integer, if $P(b)$ is true and the conditional statement $P(k) \to P(k+1)$ is true for all integers k with $k \geq b$.

5.2 Strong Induction and Well-Ordering

5.2.1 Introduction

In Section 5.1 we introduced mathematical induction and we showed how to use it to prove a variety of theorems. In this section we will introduce another form of mathematical induction, called **strong induction**, which can often be used when we cannot easily prove a result using mathematical induction. The basis step of a proof by strong induction is the same as a proof of the same result using mathematical induction. That is, in a strong induction proof that $P(n)$ is true for all positive integers n, the basis step shows that $P(1)$ is true. However, the inductive steps in these two proof methods are different. In a proof by mathematical induction, the inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ is also true. In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers j not exceeding k, then $P(k+1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$.

The validity of both mathematical induction and strong induction follow from the well-ordering property in Appendix 1. In fact, mathematical induction, strong induction, and well-ordering are all equivalent principles (as shown in Exercises 41, 42, and 43). That is, the validity of each can be proved from either of the other two. This means that a proof using one of these two principles can be rewritten as a proof using either of the other two principles. Just as it is sometimes the case that it is much easier to see how to prove a result using strong induction rather than mathematical induction, it is sometimes easier to use well-ordering than one of the two forms of mathematical induction. In this section we will give some examples of how the well-ordering property can be used to prove theorems.

5.2.2 Strong Induction

Before we illustrate how to use strong induction, we state this principle again.

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Note that when we use strong induction to prove that $P(n)$ is true for all positive integers n, our inductive hypothesis is the assumption that $P(j)$ is true for $j = 1, 2, \ldots, k$. That is, the inductive hypothesis includes all k statements $P(1), P(2), \ldots, P(k)$. Because we can use all k statements $P(1), P(2), \ldots, P(k)$ to prove $P(k+1)$, rather than just the statement $P(k)$ as in a proof by mathematical induction, strong induction is a more flexible proof technique. Because of this, some mathematicians prefer to always use strong induction instead of mathematical induction, even when a proof by mathematical induction is easy to find.

You may be surprised that mathematical induction and strong induction are equivalent. That is, each can be shown to be a valid proof technique assuming that the other is valid. In particular, any proof using mathematical induction can also be considered to be a proof by strong induction because the inductive hypothesis of a proof by mathematical induction is part of the inductive hypothesis in a proof by strong induction. That is, if we can complete the inductive step of a proof using mathematical induction by showing that $P(k+1)$ follows from $P(k)$ for every positive integer k, then it also follows that $P(k+1)$ follows from all the statements $P(1), P(2), \ldots, P(k)$, because we are assuming that not only $P(k)$ is true, but also more, namely, that the $k-1$ statements $P(1), P(2), \ldots, P(k-1)$ are true. However, it is much more awkward to convert a proof by strong induction into a proof using the principle of mathematical induction. (See Exercise 42.)

Strong induction is sometimes called the **second principle of mathematical induction** or **complete induction**. When the terminology "complete induction" is used, the principle of mathematical induction is called **incomplete induction**, a technical term that is a somewhat unfortunate choice because there is nothing incomplete about the principle of mathematical induction; after all, it is a valid proof technique.

STRONG INDUCTION AND THE INFINITE LADDER To better understand strong induction, consider the infinite ladder in Section 5.1. Strong induction tells us that we can reach all rungs if

1. we can reach the first rung, and
2. for every positive integer k, if we can reach all the first k rungs, then we can reach the $(k+1)$st rung.

That is, if $P(n)$ is the statement that we can reach the nth rung of the ladder, by strong induction we know that $P(n)$ is true for all positive integers n, because (1) tells us $P(1)$ is true, completing the basis step and (2) tells us that $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ implies $P(k+1)$, completing the inductive step.

Example 1 illustrates how strong induction can help us prove a result that cannot easily be proved using the principle of mathematical induction.

EXAMPLE 1 Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Can we prove that we can reach every rung using the principle of mathematical induction? Can we prove that we can reach every rung using strong induction?

Solution: We first try to prove this result using the principle of mathematical induction.

BASIS STEP: The basis step of such a proof holds; here it simply verifies that we can reach the first rung.

ATTEMPTED INDUCTIVE STEP: The inductive hypothesis is the statement that we can reach the kth rung of the ladder. To complete the inductive step, we need to show that if we assume the inductive hypothesis for the positive integer k, namely, if we assume that we can reach the kth rung of the ladder, then we can show that we can reach the $(k+1)$st rung of the ladder. However, there is no obvious way to complete this inductive step because we do not know from the given information that we can reach the $(k+1)$st rung from the kth rung. After all, we only know that if we can reach a rung we can reach the rung two higher.

Now consider a proof using strong induction.

BASIS STEP: The basis step is the same as before; it simply verifies that we can reach the first rung.

INDUCTIVE STEP: The inductive hypothesis states that we can reach each of the first k rungs. To complete the inductive step, we need to show that if we assume that the inductive hypothesis is true, that is, if we can reach each of the first k rungs, then we can reach the $(k+1)$st rung. We already know that we can reach the second rung. We can complete the inductive step by noting that as long as $k \geq 2$, we can reach the $(k+1)$st rung from the $(k-1)$st rung because we know we can climb two rungs from a rung we can already reach, and because $k-1 \leq k$, by the inductive hypothesis we can reach the $(k-1)$st rung. This completes the inductive step and finishes the proof by strong induction.

We have proved that if we can reach the first two rungs of an infinite ladder and for every positive integer k if we can reach all the first k rungs then we can reach the $(k+1)$st rung, then we can reach all rungs of the ladder. ◀

5.2.3 Examples of Proofs Using Strong Induction

Now that we have both mathematical induction and strong induction, how do we decide which method to apply in a particular situation? Although there is no cut-and-dried answer, we can supply some useful pointers. In practice, you should use mathematical induction when it is straightforward to prove that $P(k) \to P(k+1)$ is true for all positive integers k. This is the case for all the proofs in the examples in Section 5.1. In general, you should restrict your use of the principle of mathematical induction to such scenarios. Unless you can clearly see that the inductive step of a proof by mathematical induction goes through, you should attempt a proof by strong induction. That is, use strong induction and not mathematical induction when you see how to prove that $P(k+1)$ is true from the assumption that $P(j)$ is true for all positive integers j not exceeding k, but you cannot see how to prove that $P(k+1)$ follows from just $P(k)$. Keep this in mind as you examine the proofs in this section. For each of these proofs, consider why strong induction works better than mathematical induction.

We will illustrate how strong induction is employed in Examples 2–4. In these examples, we will prove a diverse collection of results. Pay particular attention to the inductive step in each of these examples, where we show that a result $P(k+1)$ follows under the assumption that $P(j)$ holds for all positive integers j not exceeding k, where $P(n)$ is a propositional function.

Before we present these examples, note that we can slightly modify strong induction to handle a wider variety of situations. In particular, we can adapt strong induction to handle cases where the inductive step is valid only for integers greater than a particular integer. Let b be a fixed integer and j a fixed positive integer. The form of strong induction we need tells us that $P(n)$ is true for all integers n with $n \geq b$ if we can complete these two steps:

BASIS STEP: We verify that the propositions $P(b), P(b+1), \ldots, P(b+j)$ are true.

INDUCTIVE STEP: We show that $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq b+j$.

That this alternative form is equivalent to strong induction is left as Exercise 28.

We begin with one of the most prominent uses of strong induction, the part of the fundamental theorem of arithmetic that tells us that every positive integer can be written as the product of primes.

EXAMPLE 2 Show that if n is an integer greater than 1, then n can be written as the product of primes.

Extra Examples

Solution: Let $P(n)$ be the proposition that n can be written as the product of primes.

BASIS STEP: $P(2)$ is true, because 2 can be written as the product of one prime, itself. (Note that $P(2)$ is the first case we need to establish.)

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(j)$ is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k. To complete the inductive step, it must be shown that $P(k+1)$ is true under this assumption, that is, that $k+1$ is the product of primes.

There are two cases to consider, namely, when $k+1$ is prime and when $k+1$ is composite. If $k+1$ is prime, we immediately see that $P(k+1)$ is true. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k+1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b. ◀

Remark: Because 1 can be thought of as an *empty* product of primes, that is, the product of no primes, we could have started the proof in Example 2 with $P(1)$ as the basis step. We chose not to do so because many people find this confusing.

Example 2 completes the proof of the fundamental theorem of arithmetic, which asserts that every nonnegative integer can be written uniquely as the product of primes in nondecreasing order. We showed in Section 4.3 that an integer has at most one such factorization into primes. Example 2 shows there is at least one such factorization.

Next, we show how strong induction can be used to prove that a player has a winning strategy in a game.

EXAMPLE 3 Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

Solution: Let n be the number of matches in each pile. We will use strong induction to prove $P(n)$, the statement that the second player can win when there are initially n matches in each pile.

BASIS STEP: When $n = 1$, the first player has only one choice, removing one match from one of the piles, leaving a single pile with a single match, which the second player can remove to win the game.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for all j with $1 \leq j \leq k$, that is, the assumption that the second player can always win whenever there are j matches, where $1 \leq j \leq k$ in each of the two piles at the start of the game. We need to show that $P(k + 1)$ is true, that is, that the second player can win when there are initially $k + 1$ matches in each pile, under the assumption that $P(j)$ is true for $j = 1, 2, \ldots, k$. So suppose that there are $k + 1$ matches in each of the two piles at the start of the game and suppose that the first player removes r matches $(1 \leq r \leq k)$ from one of the piles, leaving $k + 1 - r$ matches in this pile. By removing the same number of matches from the other pile, the second player creates the situation where there are two piles each with $k + 1 - r$ matches. Because $1 \leq k + 1 - r \leq k$, we can now use the inductive hypothesis to conclude that the second player can always win. We complete the proof by noting that if the first player removes all $k + 1$ matches from one of the piles, the second player can win by removing all the remaining matches. ◀

Using the principle of mathematical induction, instead of strong induction, to prove the results in Examples 2 and 3 is difficult. However, as Example 4 shows, some results can be readily proved using either the principle of mathematical induction or strong induction.

EXAMPLE 4 Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: We will prove this result using the principle of mathematical induction. Then we will present a proof using strong induction. Let $P(n)$ be the statement that postage of n cents can be formed using 4-cent and 5-cent stamps.

We begin by using the principle of mathematical induction.

BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true. That is, under this hypothesis, postage of k cents can be formed using 4-cent and 5-cent stamps. To complete the inductive step, we need to show that when we assume $P(k)$ is true, then $P(k + 1)$ is also true where $k \geq 12$. That is, we need to show that if we can form postage of k cents, then we can form postage of $k + 1$ cents. So, assume the inductive hypothesis is true; that is, assume that we can form postage of k cents using 4-cent and 5-cent stamps. We consider two cases, when at least one 4-cent stamp has been used and when no 4-cent stamps have been used. First, suppose that at least one 4-cent stamp was used to form postage of k cents. Then we can replace this stamp with a 5-cent stamp to form postage of $k + 1$ cents. But if no 4-cent stamps were used, we can form postage of k cents using only 5-cent stamps. Moreover, because $k \geq 12$, we needed at least three 5-cent stamps to form postage of k cents. So, we can replace three 5-cent stamps with four 4-cent stamps to form postage of $k + 1$ cents. This completes the inductive step.

Because we have completed the basis step and the inductive step, we know that $P(n)$ is true for all $n \geq 12$. That is, we can form postage of n cents, where $n \geq 12$ using just 4-cent and 5-cent stamps. This completes the proof by mathematical induction.

Next, we will use strong induction to prove the same result. In this proof, in the basis step we show that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true, that is, that postage of 12, 13, 14, or 15 cents can be formed using just 4-cent and 5-cent stamps. In the inductive step we show how to get postage of $k + 1$ cents for $k \geq 15$ from postage of $k - 3$ cents.

BASIS STEP: We can form postage of 12, 13, 14, and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively. This shows that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true. This completes the basis step.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for $12 \leq j \leq k$, where k is an integer with $k \geq 15$. To complete the inductive step, we assume that we can form postage of j cents, where $12 \leq j \leq k$. We need to show that under the assumption that $P(k+1)$ is true, we can also form postage of $k + 1$ cents. Using the inductive hypothesis, we can assume that $P(k - 3)$ is true because $k - 3 \geq 12$, that is, we can form postage of $k - 3$ cents using just 4-cent and 5-cent stamps. To form postage of $k + 1$ cents, we need only add another 4-cent stamp to the stamps we used to form postage of $k - 3$ cents. That is, we have shown that if the inductive hypothesis is true, then $P(k + 1)$ is also true. This completes the inductive step.

Because we have completed the basis step and the inductive step of a strong induction proof, we know by strong induction that $P(n)$ is true for all integers n with $n \geq 12$. That is, we know that every postage of n cents, where n is at least 12, can be formed using 4-cent and 5-cent stamps. This finishes the proof by strong induction.

(There are other ways to approach this problem besides those described here. Can you find a solution that does not use mathematical induction?) ◀

5.2.4 Using Strong Induction in Computational Geometry

Our next example of strong induction will come from **computational geometry**, the part of discrete mathematics that studies computational problems involving geometric objects. Computational geometry is used extensively in computer graphics, computer games, robotics, scientific calculations, and a vast array of other areas. Before we can present this result, we introduce some terminology, possibly familiar from earlier studies in geometry.

A **polygon** is a closed geometric figure consisting of a sequence of line segments $s_1, s_2, \ldots, s_n$, called **sides**. Each pair of consecutive sides, s_i and s_{i+1}, $i = 1, 2, \ldots, n - 1$, as well as the last side s_n and the first side s_1, of the polygon meet at a common endpoint, called a **vertex**. A polygon is called **simple** if no two nonconsecutive sides intersect. Every simple polygon divides the plane into two regions: its **interior**, consisting of the points inside the curve, and its **exterior**, consisting of the points outside the curve. This last fact is surprisingly complicated to prove. It is a special case of the deceptively simple Jordan curve theorem, an important result with a rich history, which tells us that every simple curve divides the plane into two regions; see [Or00], for example.

A polygon is called **convex** if every line segment connecting, two points in the interior of the polygon lies entirely inside the polygon. (A polygon that is not convex is said to be **nonconvex**.) Figure 1 displays some polygons; polygons (a) and (b) are convex, but polygons (c) and (d) are not. A **diagonal** of a simple polygon is a line segment connecting two nonconsecutive vertices of the polygon, and a diagonal is called an **interior diagonal** if it lies entirely inside the polygon, except for its endpoints. For example, in polygon (d), the line segment connecting a and f is an interior diagonal, but the line segment connecting a and d is a diagonal that is not an interior diagonal.

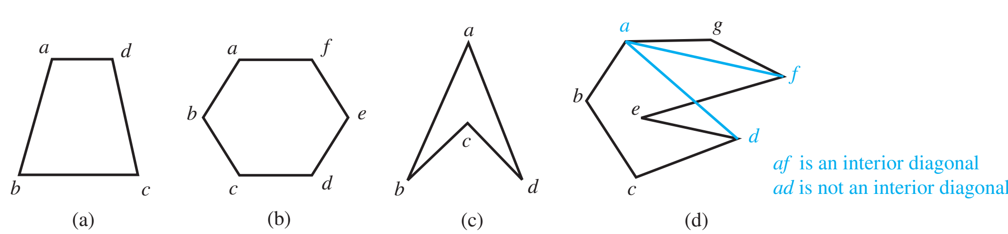


FIGURE 1 Convex and nonconvex polygons.

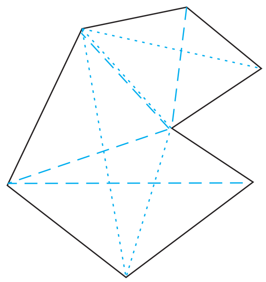


FIGURE 2 Triangulations of a polygon.

One of the most basic operations of computational geometry involves dividing a simple polygon into triangles by adding nonintersecting diagonals. This process is called **triangulation**. Note that a simple polygon can have many different triangulations, as shown in Figure 2. Perhaps the most basic fact in computational geometry is that it is possible to triangulate every simple polygon, as we state in Theorem 1. Furthermore, this theorem tells us that every triangulation of a simple polygon with n sides includes $n - 2$ triangles.

THEOREM 1 A simple polygon with n sides, where n is an integer with $n \geq 3$, can be triangulated into $n - 2$ triangles.

It seems obvious that we should be able to triangulate a simple polygon by successively adding interior diagonals. Consequently, a proof by strong induction seems promising. However, such a proof requires this crucial lemma.

LEMMA 1 Every simple polygon with at least four sides has an interior diagonal.

Although Lemma 1 seems particularly simple, it is surprisingly tricky to prove. In fact, as recently as 30 years ago, a variety of incorrect proofs thought to be correct were commonly seen in books and articles. We defer the proof of Lemma 1 until after we prove Theorem 1. It is not uncommon to prove a theorem pending the later proof of an important lemma.

Proof (of Theorem 1): We will prove this result using strong induction. Let $T(n)$ be the statement that every simple polygon with n sides can be triangulated into $n - 2$ triangles.

BASIS STEP: $T(3)$ is true because a simple polygon with three sides is a triangle. We do not need to add any diagonals to triangulate a triangle; it is already triangulated into one triangle, itself. Consequently, every simple polygon with $n = 3$ has can be triangulated into $n - 2 = 3 - 2 = 1$ triangle.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $T(j)$ is true for all integers j with $3 \leq j \leq k$. That is, we assume that we can triangulate a simple polygon with j sides into $j - 2$ triangles whenever $3 \leq j \leq k$. To complete the inductive step, we must show that when we assume the inductive hypothesis, $P(k + 1)$ is true, that is, that every simple polygon with $k + 1$ sides can be triangulated into $(k + 1) - 2 = k - 1$ triangles.

So, suppose that we have a simple polygon P with $k + 1$ sides. Because $k + 1 \geq 4$, Lemma 1 tells us that P has an interior diagonal ab. Now, ab splits P into two simple polygons Q, with s sides, and R, with t sides. The sides of Q and R are the sides of P, together with the side ab, which is a side of both Q and R. Note that $3 \leq s \leq k$ and $3 \leq t \leq k$ because both Q and R have at least one fewer side than P does (after all, each of these is formed from P by deleting at least two sides and replacing these sides by the diagonal ab). Furthermore, the number of sides of P is two less than the sum of the numbers of sides of Q and the number of sides

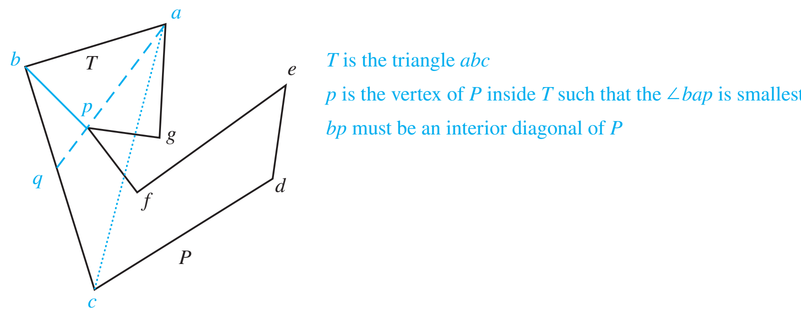


FIGURE 3 **Constructing an interior diagonal of a simple polygon.**

of R, because each side of P is a side of either Q or of R, but not both, and the diagonal ab is a side of both Q and R, but not P. That is, $k + 1 = s + t - 2$.

We now use the inductive hypothesis. Because both $3 \leq s \leq k$ and $3 \leq t \leq k$, by the inductive hypothesis we can triangulate Q and R into $s - 2$ and $t - 2$ triangles, respectively. Next, note that these triangulations together produce a triangulation of P. (Each diagonal added to triangulate one of these smaller polygons is also a diagonal of P.) Consequently, we can triangulate P into a total of $(s - 2) + (t - 2) = s + t - 4 = (k + 1) - 2$ triangles. This completes the proof by strong induction. That is, we have shown that every simple polygon with n sides, where $n \geq 3$, can be triangulated into $n - 2$ triangles. ◁

We now return to our proof of Lemma 1. We present a proof published by Chung-Wu Ho [Ho75]. Note that although this proof may be omitted without loss of continuity, it does provide a correct proof of a result proved incorrectly by many mathematicians.

Proof: Suppose that P is a simple polygon drawn in the plane. Furthermore, suppose that b is the point of P or in the interior of P with the least y-coordinate among the vertices with the smallest x-coordinate. Then b must be a vertex of P, for if it is an interior point, there would have to be a vertex of P with a smaller x-coordinate. Two other vertices each share an edge with b, say a and c. It follows that the angle in the interior of P formed by ab and bc must be less than 180 degrees (otherwise, there would be points of P with smaller x-coordinates than b).

Now let T be the triangle $\triangle abc$. If there are no vertices of P on or inside T, we can connect a and c to obtain an interior diagonal. On the other hand, if there are vertices of P inside T, we will find a vertex p of P on or inside T such that bp is an interior diagonal. (This is the tricky part. Ho noted that in many published proofs of this lemma a vertex p was found such that bp was not necessarily an interior diagonal of P. See Exercise 21.) The key is to select a vertex p such that the angle $\angle bap$ is smallest. To see this, note that the ray starting at a and passing through p hits the line segment bc at a point, say q. It then follows that the triangle $\triangle baq$ cannot contain any vertices of P in its interior. Hence, we can connect b and p to produce an interior diagonal of P. Locating this vertex p is illustrated in Figure 3. ◁

5.2.5 Proofs Using the Well-Ordering Property

The validity of both the principle of mathematical induction and strong induction follows from a fundamental axiom of the set of integers, the **well-ordering property** (see Appendix 1). The well-ordering property states that every nonempty set of nonnegative integers has a least element. We will show how the well-ordering property can be used directly in proofs. Furthermore, it can be shown (see Exercises 41, 42, and 43) that the well-ordering property, the principle of mathematical induction, and strong induction are all equivalent. That is, the validity of each of these three proof techniques implies the validity of the other two techniques. In Section 5.1 we showed that the principle of mathematical induction follows from the well-ordering property. The other parts of this equivalence are left as Exercises 31, 42, and 43.

THE WELL-ORDERING PROPERTY Every nonempty set of nonnegative integers has a least element.

The well-ordering property can be used directly in proofs, as Example 5 illustrates.

EXAMPLE 5 Use the well-ordering property to prove the division algorithm. Recall that the division algorithm states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.

Extra Examples

Solution: Let S be the set of nonnegative integers of the form $a - dq$, where q is an integer. This set is nonempty because $-dq$ can be made as large as desired (taking q to be a negative integer with large absolute value). By the well-ordering property, S has a least element $r = a - dq_0$.

The integer r is nonnegative. It is also the case that $r < d$. If it were not, then there would be a smaller nonnegative element in S, namely, $a - d(q_0 + 1)$. To see this, suppose that $r \geq d$. Because $a = dq_0 + r$, it follows that $a - d(q_0 + 1) = (a - dq_0) - d = r - d \geq 0$. Consequently, there are integers q and r with $0 \leq r < d$. The proof that q and r are unique is left as Exercise 37. ◀

EXAMPLE 6 In a round-robin tournament every player plays every other player exactly once and each match has a winner and a loser. We say that the players $p_1, p_2, \dots, p_m$ form a *cycle* if p_1 beats p_2, p_2 beats p_3, $\dots$, p_{m-1} beats p_m, and p_m beats p_1. Use the well-ordering property to show that if there is a cycle of length m ($m \geq 3$) among the players in a round-robin tournament, there must be a cycle of three of these players.

Solution: We assume that there is no cycle of three players. Because there is at least one cycle in the round-robin tournament, the set of all positive integers n for which there is a cycle of length n is nonempty. By the well-ordering property, this set of positive integers has a least element k, which by assumption must be greater than three. Consequently, there exists a cycle of players $p_1, p_2, p_3, \dots, p_k$ and no shorter cycle exists.

Because there is no cycle of three players, we know that $k > 3$. Consider the first three elements of this cycle, p_1, p_2, and p_3. There are two possible outcomes of the match between p_1 and p_3. If p_3 beats p_1, it follows that p_1, p_2, p_3 is a cycle of length three, contradicting our assumption that there is no cycle of three players. Consequently, it must be the case that p_1 beats p_3. This means that we can omit p_2 from the cycle $p_1, p_2, p_3, \dots, p_k$ to obtain the cycle $p_1, p_3, p_4, \dots, p_k$ of length $k - 1$, contradicting the assumption that the smallest cycle has length k. We conclude that there must be a cycle of length three. ◀

Exercises

1. Use strong induction to show that if you can run one mile or two miles, and if you can always run two more miles once you have run a specified number of miles, then you can run any number of miles.

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.

3. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 8$.

a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$.

b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$?

c) What do you need to prove in the inductive step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$?

d) Complete the inductive step for $k \geq 10$.

e) Explain why these steps show that $P(n)$ is true whenever $n \geq 8$.

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 18$.

a) Show that the statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$.

b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$?

c) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all integers $n \geq 18$?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that $P(n)$ is true for all integers $n \geq 18$.

5. a) Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

6. a) Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

7. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

8. Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

***9.** Use strong induction to prove that $\sqrt{2}$ is irrational. [*Hint:* Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.]

10. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, or any smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

11. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j + 2$, or $4j + 3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer j.

12. Use strong induction to show that every positive integer can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

***13.** A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n - 1$ moves are required to assemble a puzzle with n pieces.

14. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n - 1)/2$.

15. Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is square. [*Hint:* Use strong induction to show that this strategy works. For the first move, the first player chomps all cookies except those in the left and top edges. On subsequent moves, after the second player has chomped cookies on either the top or left edge, the first player chomps cookies in the same relative positions in the left or top edge, respectively.]

***16.** Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is two squares wide, that is, a $2 \times n$ board. [*Hint:* Use strong induction. The first move of the first player should be to chomp the cookie in the bottom row at the far right.]

17. Use strong induction to show that if a simple polygon with at least four sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon.

***18.** Use strong induction to show that when a simple polygon P with consecutive vertices $v_1, v_2, \ldots, v_n$ is triangulated into $n - 2$ triangles, the $n - 2$ triangles can be numbered $1, 2, \ldots, n - 2$ so that v_i is a vertex of triangle i for $i = 1, 2, \ldots, n - 2$.

***19. Pick's theorem** says that the area of a simple polygon P in the plane with vertices that are all lattice points (that is, points with integer coordinates) equals $I(P) + B(P)/2 - 1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of P and on the boundary of P, respectively. Use strong induction on the number of vertices of P to prove Pick's theorem. [*Hint:* For the basis step, first prove the theorem for rectangles, then for right triangles, and finally for all triangles by noting that the area of a triangle is the area of a larger rectangle containing it with the areas of at most three triangles subtracted. For the inductive step, take advantage of Lemma 1.]

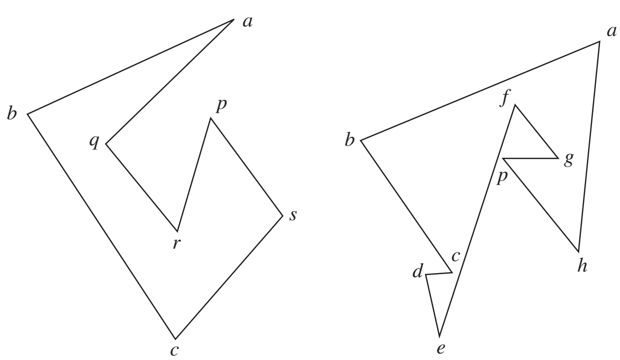
****20.** Suppose that P is a simple polygon with vertices $v_1, v_2, \ldots, v_n$ listed so that consecutive vertices are connected by an edge, and v_1 and v_n are connected by an edge. A vertex v_i is called an **ear** if the line segment connecting the two vertices adjacent to v_i is an interior diagonal of the simple polygon. Two ears v_i and v_j are called **nonoverlapping** if the interiors of the triangles with vertices v_i and its two adjacent vertices and v_j and its two adjacent vertices do not intersect. Prove that every simple polygon with at least four vertices has at least two nonoverlapping ears.

21. In the proof of Lemma 1 we mentioned that many incorrect methods for finding a vertex p such that the line segment bp is an interior diagonal of P have been published. This exercise presents some of the incorrect ways p has been chosen in these proofs. Show, by considering one of the polygons drawn here, that for each of these choices of p, the line segment bp is not necessarily an interior diagonal of P.

a) p is the vertex of P such that the angle $\angle abp$ is smallest.

b) p is the vertex of P with the least x-coordinate (other than b).

c) p is the vertex of P that is closest to b.



Exercises 22 and 23 present examples that show inductive loading can be used to prove results in computational geometry.

***22.** Let $P(n)$ be the statement that when nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two vertices of the polygon are not endpoints of any of these diagonals.

a) Show that when we attempt to prove $P(n)$ for all integers n with $n \geq 3$ using strong induction, the inductive step does not go through.

b) Show that we can prove that $P(n)$ is true for all integers n with $n \geq 3$ by proving by strong induction the stronger assertion $Q(n)$, for $n \geq 4$, where $Q(n)$ states that whenever nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two *nonadjacent* vertices are not endpoints of any of these diagonals.

23. Let $E(n)$ be the statement that in a triangulation of a simple polygon with n sides, at least one of the triangles in the triangulation has two sides bordering the exterior of the polygon.

a) Explain where a proof using strong induction that $E(n)$ is true for all integers $n \geq 4$ runs into difficulties.

b) Show that we can prove that $E(n)$ is true for all integers $n \geq 4$ by proving by strong induction the stronger statement $T(n)$ for all integers $n \geq 4$, which states that in every triangulation of a simple polygon, at least two of the triangles in the triangulation have two sides bordering the exterior of the polygon.

***24.** A stable assignment, defined in the preamble to Exercise 64 in Section 3.1, is called **optimal for suitors** if no stable assignment exists in which a suitor is paired with a suitee whom this suitor prefers to the person to whom this suitor is paired in this stable assignment. Use strong induction to show that the deferred acceptance algorithm produces a stable assignment that is optimal for suitors.

25. Suppose that $P(n)$ is a propositional function. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+2)$ is true.

b) $P(1)$ and $P(2)$ are true; for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

26. Suppose that $P(n)$ is a propositional function. Determine for which nonnegative integers n the statement $P(n)$ must be true if

a) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ is true.

b) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+3)$ is true.

c) $P(0)$ and $P(1)$ are true; for all nonnegative integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

d) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ and $P(n+3)$ are true.

27. Show that if the statement $P(n)$ is true for infinitely many positive integers n and $P(n+1) \to P(n)$ is true for all positive integers n, then $P(n)$ is true for all positive integers n.

28. Let b be a fixed integer and j a fixed positive integer. Show that if $P(b), P(b+1), \ldots, P(b+j)$ are true and $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for every integer $k \geq b+j$, then $P(n)$ is true for all integers n with $n \geq b$.

29. What is wrong with this "proof" by strong induction?

"Theorem" For every nonnegative integer n, $5n = 0$.

Basis Step: $5 \cdot 0 = 0$.

Inductive Step: Suppose that $5j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. Write $k+1 = i+j$, where i and j are natural numbers less than $k+1$. By the inductive hypothesis, $5(k+1) = 5(i+j) = 5i + 5j = 0 + 0 = 0$.

***30.** Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0 = 1$ is true by the definition of a^0.

Inductive Step: Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

***31.** Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

32. Find the flaw with the following "proof" that every postage of three cents or more can be formed using just 3-cent and 4-cent stamps.

Basis Step: We can form postage of three cents with a single 3-cent stamp and we can form postage of four cents using a single 4-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just 3-cent and 4-cent stamps. We can then form postage of $k + 1$ cents by replacing one 3-cent stamp with a 4-cent stamp or by replacing two 4-cent stamps by three 3-cent stamps.

33. Show that we can prove that $P(n, k)$ is true for all pairs of positive integers n and k if we show

a) $P(1, 1)$ is true and $P(n, k) \rightarrow [P(n + 1, k) \wedge P(n, k + 1)]$ is true for all positive integers n and k.

b) $P(1, k)$ is true for all positive integers k, and $P(n, k) \rightarrow P(n + 1, k)$ is true for all positive integers n and k.

c) $P(n, 1)$ is true for all positive integers n, and $P(n, k) \rightarrow P(n, k + 1)$ is true for all positive integers n and k.

34. Prove that $\sum_{j=1}^{n} j(j + 1)(j + 2) \cdots (j + k - 1) = n(n + 1)(n + 2) \cdots (n + k)/(k + 1)$ for all positive integers k and n. [*Hint*: Use a technique from Exercise 33.]

***35.** Show that if $a_1, a_2, \ldots, a_n$ are n distinct real numbers, exactly $n - 1$ multiplications are used to compute the product of these n numbers no matter how parentheses are inserted into their product. [*Hint:* Use strong induction and consider the last multiplication.]

***36.** The well-ordering property can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let S be the set of positive integers of the form $as + bt$, where s and t are integers.

a) Show that S is nonempty.

b) Use the well-ordering property to show that S has a smallest element c.

c) Show that if d is a common divisor of a and b, then d is a divisor of c.

d) Show that $c \mid a$ and $c \mid b$. [*Hint:* First, assume that $c \nmid a$. Then $a = qc + r$, where $0 < r < c$. Show that $r \in S$, contradicting the choice of c.]

e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing that this greatest common divisor is unique.

37. Let a be an integer and d be a positive integer. Show that the integers q and r with $a = dq + r$ and $0 \leq r < d$, which were shown to exist in Example 5, are unique.

38. Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

****39.** Can you use the well-ordering property to prove the statement: "Every positive integer can be described using no more than fifteen English words"? Assume the words come from a particular dictionary of English. [*Hint:* Suppose that there are positive integers that cannot be described using no more than fifteen English words. By well ordering, *the smallest positive integer that cannot be described using no more than fifteen English words* would then exist.]

40. Use the well-ordering property to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$. [*Hint:* Use the Archimedean property, given in Appendix 1, to find a positive integer A with $A > 1/(y - x)$. Then show that there is a rational number r with denominator A between x and y by looking at the numbers $\lfloor x \rfloor + j/A$, where j is a positive integer.]

***41.** Show that the well-ordering property can be proved when the principle of mathematical induction is taken as an axiom.

***42.** Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

***43.** Show that we can prove the well-ordering property when we take strong induction as an axiom instead of taking the well-ordering property as an axiom.

5.3 Recursive Definitions and Structural Induction

5.3.1 Introduction

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called **recursion**. For instance, the picture shown in Figure 1 is produced recursively. First, an original picture is given. Then a process of successively superimposing centered smaller pictures on top of the previous pictures is carried out.