Week 3: Proof Methods I

MATH 1200: Problems, Conjectures, and Proofs

Joe Tran

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Discussion Prompt 1

Prompt 1

Let f be the function where

$$f(x) = \begin{cases} x & \text{if } x < 3 \\ x^2 - 9 & \text{if } x \geq 3 \end{cases}$$

Determine whether the following mathematical statement is true or false. Prove your answer: For all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, $|x - 3| < \delta$ then $|f(x) - f(3)| < \varepsilon$.

Note

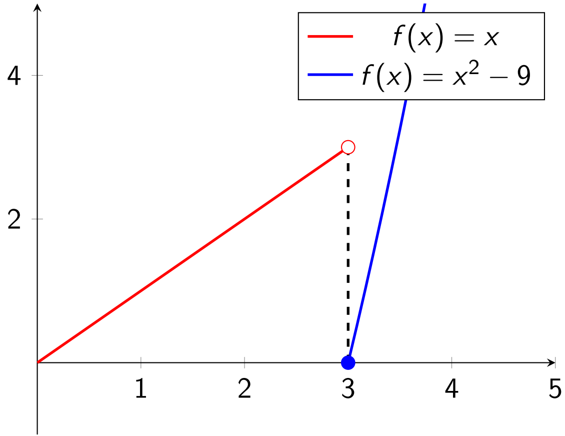
The precise definition of continuity is as follows: For $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.
The negation of this definition, i.e. when the function is not continuous, is: There exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists an x such that $|x - a| < \delta$, but $|f(x) - f(a)| \geq \varepsilon$

Note

These are one of the most important definitions of the limit, in which you will encounter in courses such as MATH 1300, MATH 1310, MATH 2001, MATH 3001, MATH 4011.

Solution

Here we note that $f(3) = 0$. We claim that the statement is false. Try taking $\varepsilon = \frac{1}{2}$, take $x = 2.999$ and fix $\delta > 0$. Note that the condition that $|2.999 - 3| < \delta \implies 0 < 0.001 < \delta$ is satisfied. However, $|f(2.999) - f(3)| = |2.999 - 0| = 2.999 \geq \frac{1}{2}$. Therefore the statement is false.



$f(x) = x$
$f(x) = x^2 - 9$
4
2
1
2
3
4
5

Counterexamples

In the above Discussion Prompt, we have proved that a statement is false by providing a counterexample. A counterexample is a way for you to disprove a statement. Let us consider the following statement.

Example

If n is an even integer, then $n + 3$ is an even integer.

This statement is false. Why? Take $n = 2$ which is an even integer. But then $n + 3 = 2 + 3 = 5$ is not an even integer. In general, to disprove a statement:

1. Find an example that satisfies the initial conditions.
2. Show that this example does not satisfy the conclusion of the statement.

Using Definitions To Prove Statements

Using definitions are a general way to prove some elementary statements. Let's consider the following definitions:

Definition

- An integer n is said to be even if there exists an integer k such that $n = 2k$.
- An integer n is said to be odd if there exists an integer k such that $n = 2k + 1$.

We will work through some examples using these two definitions to prove some statements.

Guidelines for Writing Proofs

Let's work this out together:

Example

Prove that if n is an even integer, then n^2 is also an even integer.

1. Start the proof by stating what you are given as an assumption, and mention what your targetted goal is: For example, you may start the beginning of the proof with "Assume that p is true... We want to show that q is true" Assume that n is an even integer. We want to show that n^2 is also an even integer.
2. Use what you know about p: This may include definitions or facts about the statement p, and apply those definitions or facts. Since n is an even integer, then by definition, there exists an integer k such that $n = 2k$.
3. Remember what your targetted goal is, and apply any operations to the statement p if needed: This may include more definitions or facts that you know. Squaring both sides, we obtain that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
4. Make sure you identify any closure properties: This may include any of the ones that we have talked about in our Discussion Prompt. Since the integers are closed under multiplication and since k is an integer, then $2k^2$ is also an integer.

1. Conclude the proof, using any definitions or facts that you know: This is the part where you conclude the proof based on definitions and facts, and we mention that we have shown that q is true. Therefore, by definition of an even integer, $n^2 = 2(2k)$ is an even integer. This completes the proof.

Effective Tool for First-Timers

Whenever we're starting out with writing proofs in mathematics, using a step-action-reason table is recommended. For example, using the above statement: If n is an even integer, then n^2 is an even integer, we have such a table as follows:

Step	**Action**	**Reason**
1	n is even	Assumption
2	$n = 2k$ for some $k \in \mathbb{Z}$	Definition of even
3	$n^2 = 4k^2 = 2(2k^2)$	Want to show that n^2 is even
4	$2k^2$ is also an integer	Closure properties of integers
5	$n^2 = 2(2k^2)$ is an even integer	Definition of even
6	n^2 is even	Conclusion

Here is what the full proof should look like:

Proof.

Assume that n is an even integer. We want to show that n^2 is an even integer. Since n is even, then by definition of an even integer, there exists an integer k such that $n = 2k$. Then squaring both sides, we get that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Note that since the integers are closed under multiplication and since k is an integer, then $2k^2$ is also an integer. Therefore, by definition of an even integer, $n^2 = 2(2k^2)$ is an even integer. This completes the proof. $\square$

Important Note When Writing Proofs

Remark

Giving examples to a statement is not writing a proof.

For example, if we considered the statement above, taking 4 to be an even integer, then $4^2 = 16$ is also an even integer; this is not a proof!

Divisibility

We may be familiar with the term divisibility. For example, 15 is divisible by 3, or 48 is divisible by 4. But what does this mean in mathematical terms?

Definition

Given two integers a and b with $a > 1$, we say that a divides b, denoted as $a \mid b$, if there exists an integer k such that $b = ka$.

From the above examples, 3 divides 15, i.e. $3 \mid 15$, and 4 divides 48, i.e. $4 \mid 48$.

Proofs Involving Divisibility

Let us consider the following statement:

Example

For $a, b, c \in \mathbb{Z}$ with $a > 1$, if $a \mid b$ and $a \mid c$, then $a \mid bc$.

Is this statement true or false?

Example

For $a, b, c \in \mathbb{Z}$ with $a > 1$, if $a \mid b$ and $a \mid c$, then $a \mid bc$.

The answer is true. Let's prove it using the guidelines mentioned above.

Proof.

1. Assumption: Assume that $a \mid b$ and $a \mid c$.
2. Definition: There exists $k_1, k_2 \in \mathbb{Z}$ such that $b = k_1 a$ and $c = k_2 a$.
3. Facts: Multiplying b and c together gives us $bc = k_1 k_2 a^2 = a(k_1 k_2 a)$.
4. Closure Properties: Since the integers are closed under multiplication, and a, k_1 and k_2 are integers, so $k_1 k_2 a$ is also an integer.
5. Definition and Conclusion: $a \mid bc$, as required.

$\square$

Discussion Prompt 2

Prompt 2

Is the converse true: For $a, b, c \in \mathbb{Z}$ with $a > 1$, if $a \mid bc$, then $a \mid b$ and $a \mid c$.

Statement

For $a, b, c \in \mathbb{Z}$ with $a > 1$, if $a \mid bc$, then $a \mid b$ and $a \mid c$.

Solution

The statement is false. Take $a = 3$, $b = 4$, and $c = 6$. Then $3 \mid 4 \cdot 6 = 24$ is true, but $3 \nmid 4$ and $3 \mid 6$, so the statement is false.