Example 1. Suppose the logistic model is modified as

$$\frac{dP}{dt} = P(bP - a)$$

(a) If $a > 0$ and $b > 0$ then show by means of a phase portrait that depending on the initial condition $P(0) = P_0$ the mathematical model could include a doomsday scenario, i.e. $P(t) \to \infty$ as $t \to \infty$ or an extinction scenario, i.e. $P(t) \to 0$ as $t \to \infty$.

(b) Solve the initial-value problem for $a = 0.1$, $b = 0.0005$ and $P(0) = 3000$. Then show that this model predicts a doomsday for the population in a finite time T.

(c) Solve the differential equation in (b) subject to the initial condition $P(0) = 100$. Show that this model predicts extinction for population as $t \to \infty$.

Solution:

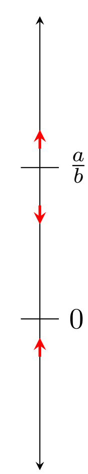
(a) Given that $\frac{dP}{dt} = P(bP - a) = f(P)$, so the equilibrium points are for when $P(bP - a) = 0$, so

$$P^* = 0, P^* = \frac{a}{b} \Rightarrow P(t) = 0, P(t) = \frac{a}{b}$$

We have the table given as follows:

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	$+$	INC	$\uparrow$
$(0, \frac{a}{b})$	$-$	DEC	$\downarrow$
$(\frac{a}{b}, \infty)$	$+$	INC	$\uparrow$

So the phase line is



$P^* = 0$ is an attractor for $P_0 \in \left(0, \frac{a}{b}\right)$ and $P^* = \frac{a}{b}$ is a repeller for $P_0 > \frac{a}{b}$.

So if $P_0 < \frac{a}{b}$ then $P(t) \to 0$ as $t \to \infty$. So there is extinction. If $P_0 > \frac{a}{b}$, then $P(t) \to \infty$ as $t \to \infty$, i.e. a doomsday scenario.

(b) Solving $\frac{dP}{dt} = P(bP - a)$ by separation of variables, we have

$$P(t) = \frac{a}{b - \beta e^{at}}$$

But when $P(0) = P_0 = 300$,

$$\beta = \frac{300b - a}{300}$$

In particular, when $a = 0.1$ and $b = 0.0005$,

$$P(t) = \frac{30}{0.15 - 0.05e^{0.1t}} = \frac{600}{3 - e^{0.1t}}$$

This population does not go to extinction to a finite time. Extinction in finite time means there exists a finite time $t = T$ such that $P(T) = 0$. Hence, the above equation has no solution where $P(T) = 0$.

However, this population shows doomsday scenario in a finite time. That is, there exists a finite time T such that

$$\lim_{t \to T} P(t) = \infty$$

In this case, $P(t) = \frac{600}{3-e^{0.1t}} \to \infty$ in finite time if and only if the denominator $3 - e^{0.1t} = 0$, or $t = 10 \ln 3$. Thus, there exists a doomsday in finite time, since $P(t) \to \infty$ as $t \to 10 \ln 3$.

(c) Given that $P(0) = 100$ and use information from (b),

$$P(t) = \frac{a}{b - \beta e^{at}}$$

and when $P(0) = 100$, $P(t) = \frac{100a}{100b-(100b-a)e^{at}}$. Once again, using $a = 0.1$, $b = 0.0005$, we have

$$P(t) = \frac{10}{0.05 + 0.05e^{0.1t}} = \frac{200}{1 + e^{0.1t}}$$

So

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{200}{1 + e^{0.1t}} = 0$$

Thus, there exists extinction for large t.

Example 2. If a constant number h of fish are harvested from a fishery per unit time, then a model for the population $P(t)$ of the fishery at a time t is given by

$$\frac{dP}{dt} = P(a - bP) - h, P(0) = P_0$$

where a, b, h, P_0 are positive constants.

(a) Find the equilibrium points and equilibrium solution.

(b) Solve this initial value problem for $a = 5$, $b = 1$, $h = 4$.

(c) Suppose $a = 5$, $b = 1$ and $h = 4$ as above. Use (b) to determine the long term behavior of the population for the cases $P_0 \in (0, 1)$, $P_0 \in (1, 4)$ and $P_0 \in (4, \infty)$.

(d) Use the information in (b) and (c) to determine whether the fish population becomes extinct in finite time T. If so, find the T.

Solution:

(a) The critical points are when $P(a-bP)-h=0$, so

$$\begin{aligned} aP - bP^2 - h &= 0 \\ -bP^2 + aP - h &= 0 \\ P &= \frac{-a \pm \sqrt{a^2 - 4(-b)(-h)}}{2(-b)} \\ P &= \frac{-a \pm \sqrt{a^2+4bh}}{-2b} \end{aligned}$$

So

$$P^* = \frac{-a+\sqrt{a^2+4bh}}{-2b} \qquad P^* = \frac{-a-\sqrt{a^2+4bh}}{-2b}$$

are critical points. So the equilibrium solutions are

$$P(t) = \frac{-a+\sqrt{a^2+4bh}}{-2b} \qquad P(t) = \frac{-a-\sqrt{a^2+4bh}}{-2b}$$

For $a=5$, $b=1$, and $h=4$,

$$\frac{dP}{dt} = P(5-P)-4$$

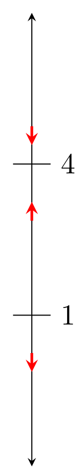
In this, the critical points are

$$P(5-P)-4 = 0 \Rightarrow -(P-4)(P-1) = 0$$

So $P=1$ and $P=4$. So equilibrium solutions are $P(t)=1$ and $P(t)=4$. The table below shows that

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(0,1)$	$-$	DEC	$\downarrow$
$(1,4)$	$+$	INC	$\uparrow$
$(4,\infty)$	$-$	DEC	$\downarrow$

So the phase portrait is



We start at 0 since $P_0 > 0$ is given.

Observe that for $P_0 > 4$ and $P_0 \in (1,4)$, the population is 4 as t increases. For $P_0 \in (0,1)$, the population decreases to 0 in finite time T.

(b) For $a = 5$, $b = 1$ and $h = 4$,

$$P(t) = \frac{4(P_0 - 1) - (P_0 - 4)e^{-3t}}{(P_0 - 1) - (P_0 - 4)e^{-3t}}$$

(c) As seen from (a), for $P_0 > 4$ and $P_0 \in (1, 4)$, the population is 4 as t increases. For $P_0 \in (0, 1)$, the population decreases to 0 in finite time T.

$$\lim_{t\to\infty} P(t) = 0 \qquad (P_0 \in (0,1))$$
$$\lim_{t\to\infty} P(t) = 4 \qquad (P_0 \in (1,4))$$
$$\lim_{t\to\infty} P(t) = 4 \qquad (P_0 \in (4,\infty))$$

(d) Since there exists a time T such that $P_0 \in (0, 1)$, or $P(t) = 0$, we find this T.

$$\frac{4(P_0 - 1) - (P_0 - 4)e^{-3T}}{(P_0 - 1) - (P_0 - 4)e^{-3T}} = 0$$
$$T = \frac{1}{3} \ln\left(\frac{1}{4}\frac{P_0 - 4}{P_0 - 1}\right)$$