Linear/Nonlinear Systems

We have seen that a single differential equation can serve as a mathematical model for a single population in an environment. But if there are say, two interacting and perhaps competing species living in the same environment (for example, rabbits and foxes) then a model for their populations $x(t)$ and $y(t)$ might be a system of two first-order differential equations such as:

$$\begin{cases} \frac{dx}{dt} = g_1(t, x, y) \\ \frac{dy}{dt} = g_2(t, x, y) \end{cases}$$

When g_1 and g_2 are linear in the variables x and y, i.e.

$$\begin{cases} g_1(t, x, y) = \beta_1 x + \beta_2 y + f_1(t) \\ g_2(t, x, y) = \beta_3 x + \beta_4 y + f_2(t) \end{cases}$$

where the coefficients β_i could depend on time t, then the system of differential equations is said to be a *linear system*. Otherwise, a system of differential equations that is nonlinear is said to be a *nonlinear system*.

Example 1. Suppose

$$\begin{cases} \frac{dx}{dt} = -0.16x + 0.08xy \\ \frac{dy}{dt} = 4.5y - 0.9xy \end{cases}$$

represents a predator-prey model. Because we are dealing with populations, we have $x(t) \geq 0$ and $y(t) \geq 0$.

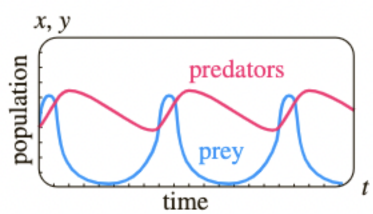

FIGURE 3.3.2 Populations of predators (red) and prey (blue) in Example 1

The initial conditions were $x(0) = 4$ and $y(0) = 4$. The curve in red represents the population $x(t)$ of the predators, and the blue curve is the population $y(t)$ of the prey. Observe that model seems to predict that both $x(t)$ and $y(t)$ are periodic in time.

Competition Models

Now suppose two different species of animals occupy the same ecosystem, not as predator and prey but rather as competitors for the same resources (such as food and living space) in the system. In the absence of the other, let us assume that the rate at which each population grows is given by

$$\frac{dx}{dt} = ax \qquad \frac{dy}{dt} = cy$$

respectively.

Since two species compete, another assumption might be that each of these rates is diminished simply by the influence or existence of another population. Thus, a model for the two populations is given by

$$\begin{cases} \frac{dx}{dt} = ax - by \\ \frac{dy}{dt} = cy - \alpha y \end{cases}$$

where a, b, c and α are constants.

On the other hand, we might assume, that each growth rate should be reduced by a rate proportional to the number of interactions between the two species:

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = cy - \alpha xy \end{cases}$$

Inspection shows that this nonlinear system is similar to the Lotka-Volterra predator-prey model. Finally, it might be more realistic to replace that rates which indicate that the population of each species in isolation grows exponentially, with rates indicating that each population grows logistically:

$$\begin{cases} \frac{dx}{dt} = a_1 x - b_1 x^2 \\ \frac{dy}{dt} = a_2 y - b_2 y^2 \end{cases}$$

With these new rates are decreased by rates proportional to the number of interactions, we obtain another nonlinear model:

$$\begin{cases} \frac{dx}{dt} = a_1 x - b_1 x^2 - c_1 xy = x(a_1 - b_1 x - c_1 y) \\ \frac{dy}{dt} = a_2 y - b_2 y^2 - c_2 xy = y(a_2 - b_2 - c_2 x) \end{cases}$$

where all coefficients are positive. All of these systems are called competition models.