Lecture 12

Recall: A function $f\begin{pmatrix}x\\y\end{pmatrix}$ is said to be a **homogeneous function of degree α** if there exists a $\lambda \in \mathbb{R}$ such that

$$f\begin{pmatrix}\lambda x\\ \lambda y\end{pmatrix} = \lambda^{\alpha} f\begin{pmatrix}x\\y\end{pmatrix}$$

An exact equation of the form $M\begin{pmatrix}x\\y\end{pmatrix}dx + N\begin{pmatrix}x\\y\end{pmatrix}dy = 0$ is also said to be a homogeneous function of degree α if both $M\begin{pmatrix}x\\y\end{pmatrix}$ and $N\begin{pmatrix}x\\y\end{pmatrix}$ are both homogeneous, and has the property that

$$\frac{dx}{x} + \frac{N\begin{pmatrix}1\\u\end{pmatrix}du}{M\begin{pmatrix}1\\u\end{pmatrix} + uN\begin{pmatrix}1\\u\end{pmatrix}} = 0.$$

Example: Solve $(x^2+y^2)dx + (x^2-xy)dy = 0$.

Solution: Let $M\begin{pmatrix}x\\y\end{pmatrix} = x^2+y^2$ and $N\begin{pmatrix}x\\y\end{pmatrix} = x^2-xy$. Then observe that for some $\lambda \in \mathbb{R}$, that

$$M\begin{pmatrix}\lambda x\\ \lambda y\end{pmatrix} = (\lambda x)^2 + (\lambda y)^2 = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2 M\begin{pmatrix}x\\y\end{pmatrix}$$

and

$$N\begin{pmatrix}\lambda x\\ \lambda y\end{pmatrix} = (\lambda x)^2 - (\lambda x)(\lambda y) = \lambda^2 x^2 - \lambda^2 xy = \lambda^2 N\begin{pmatrix}x\\y\end{pmatrix}$$

so both M and N are homogeneous functions of degree 2.

Let $y = ux$. Then $dy = d(ux) = u\,dx + x\,du$, and so.

$$(x^2+u^2x^2)dx + (x^2-x^2u)(u\,dx + x\,du) = 0.$$

$$\underline{x^2(1+u^2)dx} + \underline{u(x^2-x^2u)dx} + x(x^2-x^2u)du = 0.$$

$$(x^2 + \cancel{x^2u^2} + ux^2 - \cancel{x^2u^2})dx + x^3(1-u)du = 0.$$

$$x^2(1+u)\,dx + x^3(1-u)\,du = 0.$$

$$x^3(1+u)$$

$$\frac{dx}{x} + \frac{1-u}{1+u}\,du = 0.$$

$$\int \frac{dx}{x} + \left[-1 + \frac{2}{1+u}\right]du = \int 0.$$

$$\ln|x| - u + 2\ln|1+u| = c$$

$$\ln|x| - \frac{y}{x} + 2\ln\left|1+\frac{y}{x}\right| = c$$

$$\ln|x| - \frac{y}{x} + \ln\left[\left(\frac{x+y}{x}\right)^2\right] = c$$

$$-\frac{y}{x} + \ln\left(\frac{(x+y)^2}{x}\right) = c$$

$$\ln\left(\frac{(x+y)^2}{x}\right) = c + \frac{y}{x}.$$

$$\frac{(x+y)^2}{x} = e^{c+\frac{y}{x}}$$

still constant

$$(x+y)^2 = x e^{c+\frac{y}{x}}. \qquad \text{Let } c = \ln|\beta|$$

$$(x+y)^2 = x e^{\ln|\beta| + \frac{y}{x}}$$

$$\boxed{(x+y)^2 = \beta x e^{\frac{y}{x}}.}$$

Example: Solve $(x-y)\,dx + x\,dy = 0.$

Solution: Let $M\binom{x}{y} = x-y$ and $N\binom{x}{y} = x$. Then for some $\lambda \in \mathbb{R}$,

$$M\binom{\lambda x}{\lambda y} = \lambda x - \lambda y = \lambda(x-y) = \lambda^1 M\binom{x}{y}$$

and

$$N\binom{\lambda x}{\lambda y} = \lambda x = \lambda^1 N\binom{x}{y}.$$ So M, N are homogeneous functions of degree 1.

Let $y = ux$. Then $dy = x\,du + u\,dx$. Thus,

$$(x - ux)\,dx + x(x\,du + u\,dx) = 0.$$

$$(x - \cancel{ux})\,dx + x^2\,du + \cancel{xu}\,dx = 0$$

$$\underbrace{x\,dx + x^2\,du}_{x^2} = 0 \Rightarrow \frac{dx}{x} + du = 0$$

$$\Rightarrow \int \frac{dx}{x} + \int du = \int 0 \Rightarrow \ln|x| + u = \beta.$$

$$\Rightarrow \ln|x| + \frac{y}{x} = \beta \Rightarrow \frac{y}{x} = \beta - \ln|x| \Rightarrow y = \beta x - x\ln x.$$