Autonomous First-Order Differential Equations

We previously divided the class of ordinary differential equations: linear and nonlinear. We now consider briefly another kind of classification of ordinary differential equations, a classification that is of particular importance in the qualitative investigation of differential equations. An ordinary differential equation in which the independent variable does not appear explicitly is said to be *autonomous*. If the symbol x denotes the independent variable, then an autonomous first-order differential equation can be written as $f(y, y') = 0$ or

$$\frac{dy}{dx} = f(y)$$

For example, the ordinary differential equation

$$\frac{dy}{dx} = 1 + y^2$$

is autonomous and the equation

$$\frac{dy}{dx} = 0.2xy$$

is non-autonomous.

Many differential equations encountered in applications or equations that are models of physical laws that do not change over time are autonomous. If t represents the time and t represents the independent variable,

$$\frac{dA}{dt} = kA \qquad \frac{dx}{dt} = kx(n+1-x) \qquad \frac{dT}{dt} = k(T - T_m) \qquad \frac{dA}{dt} = 6 - \frac{1}{100}A$$

where k, n and T_m are all constants, shows that each equation is time independent. Indeed, all of the first-order differential equations here are autonomous.

Critical Points

The zeros of the function $f(y)$ are of special importance. We say that a real number α is a critical point of the autonomous differential equation if it is a zero of f, that is $f(\alpha) = 0$. A critical point is also called an *equilibrium point* or *stationary point*. Now when we substitute the constant function $y(x) = \alpha$, then both sides of the equation are zero. Then this implies that *if α is a critical point, then $y(x) = \alpha$ is a constant solution of the autonomous differential equation.*

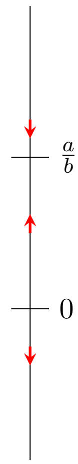
Example 1. The differential equation

$$\frac{dP}{dt} = P(a - bP)$$

where a and b are positive constants, then the normal form $\frac{dP}{dt} = f(P)$ with t and P playing the parts x and y, is autonomous.

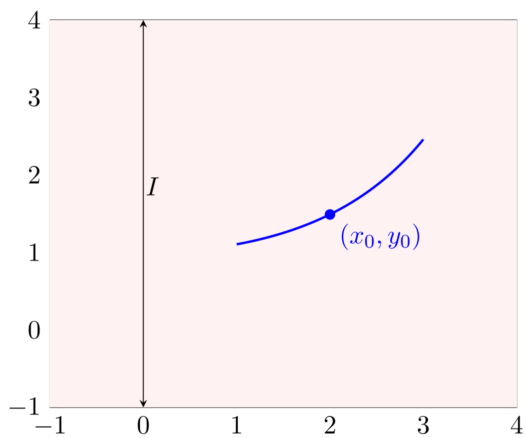
From $f(P) = P(a - bP) = 0$, we see that 0 and $\frac{a}{b}$ are critical points of the equation, so the equilibrium solutions are $P(t) = 0$ and $P(t) = \frac{a}{b}$. By putting the critical points on a vertical line, we divide the line into three intervals defined by $P \in (-\infty, 0)$, $P \in (0, \frac{a}{b})$ and $P \in (\frac{a}{b}, \infty)$. The table below explains the diagram.

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	$-$	DEC	Down
$(0, \frac{a}{b})$	$+$	INC	Up
$(\frac{a}{b,\infty})$	$-$	DEC	Down

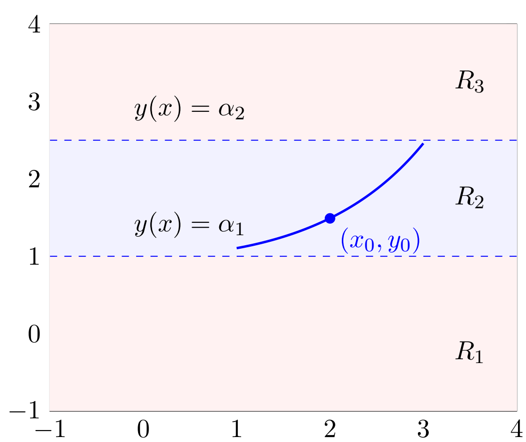


Solution Curves

Without solving an autonomous differential equation, we can usually say a great deal about its solution curves. Since the function f is independent of the variable x, we may consider f defined for $x \in \mathbb{R}$, or for $0 \leq x < \infty$. Also, since f and f' are continuous functions of y on some interval I of the y-axis, the fundamental result in Section 1.2 hold in some horizontal strip or a region R in the xy-plane corresponding to I, and so through any point (x_0, y_0) in R, there passes only one solution curve.



The graphs of the equilibrium solutions $y(x) = \alpha_1$ and $y(x) = \alpha_2$ are horizontal lines, and these lines partition the region R into three subregions R_1, R_2 and R_3.



Here are some conclusions that we can draw about a nonconstant solution $y(x)$:

- If (x_0, y_0) is in a subregion R_i for $i = 1, 2, 3$, and $y(x)$ is a solution whose graph passes through this point, then $y(x)$ remains in the subregion R_i for all x. The solution $y(x)$ in R_2 is bounded below by α_1 and bounded above by α_2, that is, $y(x) \in (\alpha_1, \alpha_2)$ for all x. The solution curve stays within R_2 for all x because the graph of a nonconstant solution cannot cross the graph of either equilibrium solution.

- By continuity of f we must have either $f(y) > 0$ or $f(y) < 0$ for all x in a subregion R_i. In other words, $f(y)$ cannot change signs in a subregion.

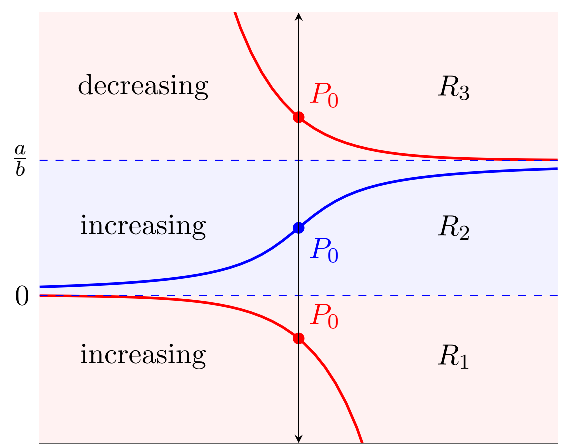
- Since $\frac{dy}{dx} = f(y(x))$ is either positive or negative in a subregion R_i a solution $y(x)$ is strictly monatomic, i.e. $y(x)$ is either increasing or decreasing in the subregion R_i. Therefore, $y(x)$ cannot be oscillatory, nor can it have a relative extremum.

- If $y(x)$ is *bounded above* by a critical point α_1, then the graph of $y(x)$ must approach the graph of the equilibrium solution $y(x) = \alpha_1$. If $y(x)$ is *bounded*, then its graph $y(x)$ must approach the graphs of the equilibrium solutions $y(x) = \alpha_1$ and $y(x) = \alpha_2$. If $y(x)$ is *bounded below* by a critical point then the graph of $y(x)$ must approach the graph of the equilibrium solution $y(x) = \alpha_2$.

Example 2. In Example 1, the three examples determined on the P-axis or phase line by the critical points 0 and $\frac{a}{b}$ now correspond in the tP-plane to three subregions defined by

$$R_1 : P \in (-\infty, 0) \qquad R_2 : P \in \left(0, \frac{a}{b}\right) \qquad R_3 : P \in \left(\frac{a}{b}, \infty\right)$$

where $t \in \mathbb{R}$.



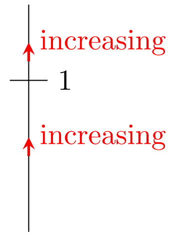
If $P(0) = P_0$ is an initial value, then in R_1, R_2, and R_3 we have, respectively, the following:

- For $P_0 < 0$, $P(t)$ is bounded above. Since $P(t)$ is decreasing, $P(t)$ decreases without bound for increasing t, and so $P(t) \to 0$ as $t \to -\infty$. This means that the negative t-axis, the graph of the equilibrium solution $P(t) = 0$, is a horizontal asymptote for a solution curve.

- For $P_0 \in \left(0, \frac{a}{b}\right)$, P_0 is bounded. Since $P(t)$ is increasing, $P(t) \to \frac{a}{b}$ as $t \to \infty$ and $P(t) \to 0$ as $t \to -\infty$. The graphs of the two equilibrium solutions, $P(t) = 0$ and $P(t) = \frac{a}{b}$, are horizontal lines that are horizontal asymptotes for any solution curve starting in this subregion.

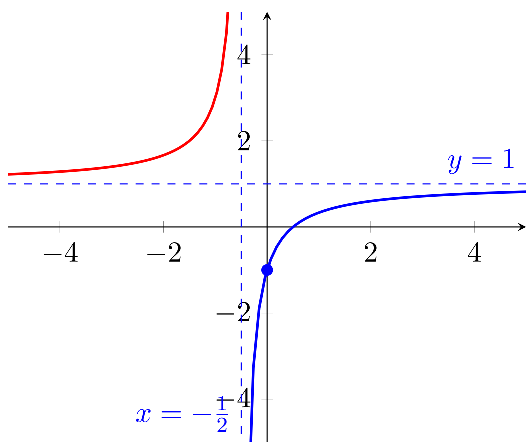
- For $P_0 \in \left(\frac{a}{b}, \infty\right)$, $P(t)$ is bounded below. Since $P(t)$ is decreasing, $P(t) \to \frac{a}{b}$ as $t \to \infty$. The graph of the equilibrium solution $P(t) = \frac{a}{b}$ is a horizontal asymptote for the solution curve.

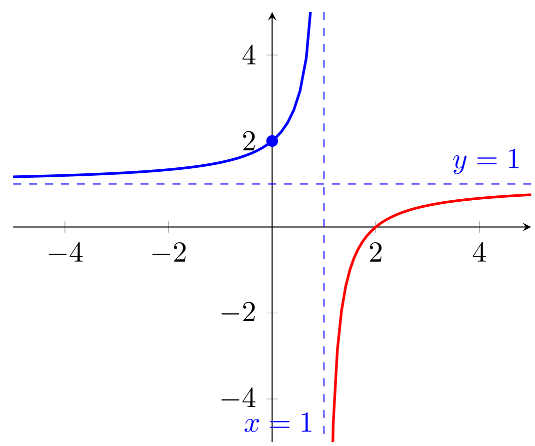
Example 3. The autonomous equation $\frac{dy}{dx} = (y-1)^2$ possesses the single critical point 1. From the phase portrait shown below,



and we can conclude that a solution $y(x)$ is an increasing function in the subregions defined by $y \in (-\infty, 1)$ and $y \in (1, \infty)$, where $x \in \mathbb{R}$. For an initial condition $y(0) = y_0 < 1$ a solution $y(x)$ is increasing and bounded by 1, and so $y(x) \to 1$ as $x \to \infty$; for $y(0) = y_0 > 1$ a solution $y(x)$ is increasing and unbounded.

Now $y(x) = 1 - \frac{1}{x+c}$ is a one-parameter family of solutions of the differential equation. A given initial condition determines a missing value for c. For the initial conditions, say $y(0) = -1 < 1$ and $y(0) = 2 > 1$, we find in turn that $y(x) = 1 - \frac{1}{x+\frac{1}{2}}$ and $y(x) = 1 - \frac{1}{x-1}$. As shown below, the graph of each of these rational function possesses a vertical asymptote.





Note that the solutions of the initial value problems

$$\frac{dy}{dx} = (y-1)^2, y(0) = -1 \qquad \frac{dy}{dx} = (y-1)^2, y(0) = 2$$

are defined on special intervals. The two solutions are, respectively,

$$y(x) = 1 - \frac{1}{x+\frac{1}{2}}, -\frac{1}{2} < x < \infty \qquad y(x) = 1 - \frac{1}{x-1}, -\infty < x < 1$$

Attractors and Repellers

Suppose that $y(x)$ is a nonconstant solution of the autonomous differential equation and that α is a critical point of the differential equation. There are basically three types of behavior that $y(x)$ can exhibit near α: asymptotically stable, unstable, or semi-stable.

