We have saw that whenever $f(x, y)$ and $\frac{\partial f}{\partial y}$ satisfy certain continuity conditions, qualitative questions about existence and uniqueness of solutions can be answered. We would like to ask the following:

- How does a solution behave near a certain point?
- How does a solution behave as $x \to \infty$?

Recall from calculus that: a derivative $\frac{dy}{dx}$ of a differentiable function $y = f(x)$ gives the slope of tangent lines at points on its graphs.

Slope

Because a solution $y = y(x)$ of a first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is necessarily a differentiable function on its interval I of definition, it must also be continuous on I. Thus, the corresponding solution curve on I must have no breaks and must possess a tangent line at each point $(x, y(x))$. The function f in the normal form is called the *slope function* or *rate function*. The slope of the tangent line at $(x, y(x))$ on a solution curve is the value of the first derivative $\frac{dy}{dx}$ at this point and we know that this is the value of the slope function $f(x, y(x))$.

Suppose that (x, y) represents any point in a region of the xy-plane over which the function f is defined. The value $f(x, y)$ that the function f assigns to the point represents the slope of a line or, as we shall envision it, a line element called a *lineal element*. For example, consider the equation

$$\frac{dy}{dx} = 0.2xy = f(x, y)$$

At, say the point $(2, 3)$, the slope of a lineal element is $f(2, 3) = 0.2(2)(3) = 1.2$.

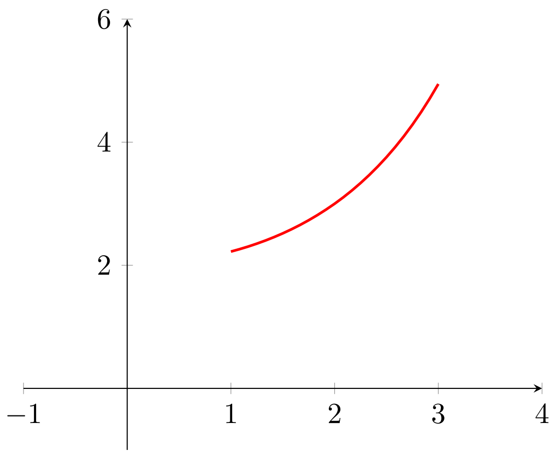


Figure 1: Lineal Element at a Point

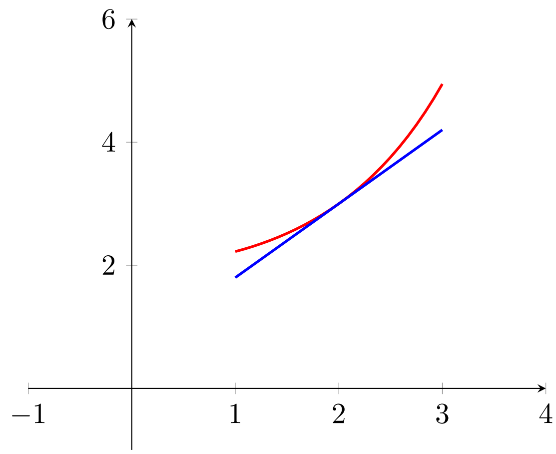


Figure 2: Lineal Element is Tangent to Solution Curve that Passes Through the Point

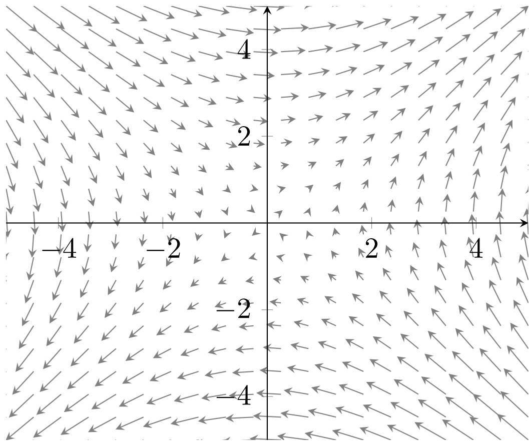
Direction Field

If we systematically evaluate f over a rectangular grid of points in the xy-plane and draw a line element at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all these line elements is called a *direction field* or a *slope field* of the differential equation $\frac{dy}{dx} = f(x, y)$.

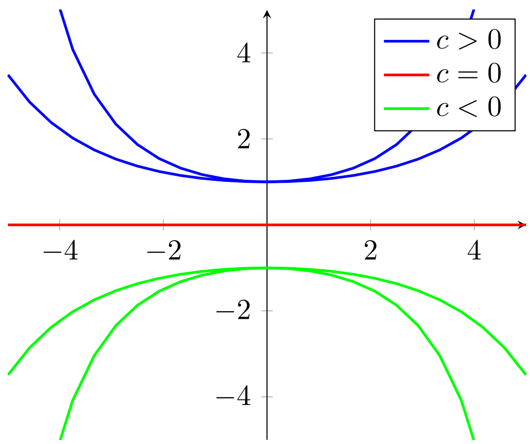
Example 1. The direction field for the differential equation

$$\frac{dy}{dx} = 0.2xy$$

is given below.



Notice that any point along the x-axis and the y-axis, the slopes are $f(x, 0) = 0$ and $f(0, y) = 0$, respectively, so the lineal elements are horizontal. Moreover, observe in the first quadrant that for a fixed value of x, the values of $f(x, y) = 0.2xy$ increases as y increases; similarly, for a fixed y, the values of $f(x, y) = 0.2xy$ increase as x increases. This means that as both x and y increase, the lineal elements almost become vertical and have positive slope.



Example 2. Use a direction field to sketch an approximate solution curve for the initial-value problem $\frac{dy}{dx} = \sin y$ for $y(0) = -\frac{3}{2}$.