We saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

is a linear combination $y = \beta_1 y_1 + \beta_2 y_2$ where y_1 and y_2 are solutions that constitute linearly independent set on some interval I.

Reduction of Order

Suppose that y_1 denotes a nontrivial solution of the above and that y_1 is defined on the interval I. We want to find a second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on I. If y_1 and y_2 are linearly independent, then their quotient $\frac{y_2}{y_1}$ is nonconstant on I, that $y_2(x) = u(x)y_1(x)$ The function $u(x)$ can be found by substituting $y_2(x) = u(x)y_1(x)$. This method is called reduction of order, since we need to solve a linear-first order differential equation to find u.

Example 1. Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

Solution: If $y = u(x)y_1(x) = u(x)e^x$, then the product rule gives

$$y' = ue^x + e^x\frac{du}{dx}, y'' = ue^x + 2e^x\frac{du}{dx} + e^x\frac{d^2u}{dx^2}$$

and so

$$y'' - y = e^x\left(\frac{d^2u}{dx^2} + 2\frac{du}{dx}\right) = 0$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order in w. Then solving the equation we have

$$y(x) = -\frac{1}{2}\beta_1 e^{-x} + \beta_2 e^x$$

Let $\beta_1 = -2$ and $\beta_2 = 0$, we obtain that $y_2 = e^{-x}$. Since $W(e^x, e^{-x}) \neq 0$ for all x, the solutions are linearly independent on $(-\infty, \infty)$. In other words, the basis $\mathcal{B} = \{e^x, e^{-x}\}$ are linearly independent.

General Solution

Suppose we divide by $a_2(x)$ to put $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ by in the standard form:

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous on some interval I. Let us suppose further that $y_1(x)$ is a known solution of the above on I and that $y_1(x) \neq 0$ for all $x \in I$. If we define $y = u(x)y_1(x)$, it follows that

$$y' = uy_1' + y_1u' \qquad y'' = uy_1'' + 2y'u' + y_1u''$$

and so

$$y'' + py' + qy = u(y_1'' + py_1' + qy_1) + y_1u'' + (2y_1' + py_1)u' = 0$$

This implies that we must have

$$y_1u'' + (2y' + py_1)u' = 0 \qquad y_1w' + (2y_1' + py_1)w = 0$$

where we have let $w = u'$. Observe that the equation above is linear and separable. Separating the variables and integrating we obtain

$$\frac{1}{w}dw + 2\frac{y_1'}{y_1}dx + pdx = 0$$
$$\ln|wy_1^2| = -\int p(x)dx + \beta$$
$$wy_1^2 = \beta e^{-\int p(x)dx}$$

We solve the last equation for w, use $w = u'$ and integrate again.

$$u = \beta_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}dx + \beta$$

let $\beta_1 = 1$ and $\beta_2 = 0$, we find from $y = u(x)y_1(x)$ that a second solution of the equation is

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx$$

The basis formed

$$\mathcal{B} = \{y_1, y_2\} = \left\{ y_1(x), y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx \right\}$$

has linearly independent elements.

Example 2. The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval $(0, \infty)$.

Solution: From the standard form of the equation

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

and so

$$y_2(x) = x^2\int \frac{e^{\int \frac{3}{x}dx}}{x^4}dx = x^2\int \frac{1}{x}dx = x^2\ln(x)$$

The general solution on the interval $(0, \infty)$ is given by $y = \beta_1x^2 + \beta_2x^2\ln(x)$ and note that the elements of the basis $\mathcal{B} = \{x^2, x^2\ln(x)\}$ are linearly independent.

Homogeneous Linear Equations with Constant Coefficients

We will see that the foregoing procedure can produce exponential solutions for homogeneous linear higher-order differential equations,

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = 0$$

where a_k are real constants and $a_n \neq 0$.

Auxiliary Equations

We begin by considering the special case of the second-order equation

$$a_2y'' + a_1y' + a_0y = 0$$

where a_k are constants. If we try to find a solution of the form $y = e^{\alpha x}$, then after substitution of $y' = \alpha e^{\alpha x}$ and $y'' = \alpha^2 e^{\alpha x}$, we have

$$e^{\alpha x}(a_2\alpha^2 + a_1\alpha + a_0) = 0$$

We know that $e^{\alpha x} \neq 0$ for all x. So it is evident to solve the equation

$$a_2\alpha^2 + a_1\alpha + a_0 = 0$$