Lecture 13

Bernoulli Equations

Definition: The differential equation of the form

$$\frac{dy}{dx} + p(x)y(x) = q(x)y^n(x), \quad n \in \mathbb{R}$$

is called the **Bernoulli Equation**.

Special Cases:

(1) When $n=0$, $\frac{dy}{dx} + p(x)y(x) = q(x) \Rightarrow$ Linear.

$$y(x) = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + \beta\right).$$

(2) When $n=1$, $\frac{dy}{dx} + p(x)y(x) = q(x)y(x) \Rightarrow$ Linear.

$$y(x) = \beta e^{-\int (p(x)-q(x))dx}.$$

What if $n>1$? How do we solve this equation?

Method of Solution

Consider $\frac{dy}{dx} + p(x)y(x) = q(x)y^n(x)$ $^{(1)}$ $n>1$. Then let $z = y^{1-n} \Leftrightarrow y(x) = z^{\frac{1}{1-n}}$. Then by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{1-n}z^{\frac{1}{1-n}-1}\frac{dz}{dx}.$$

Substitute to (1) we obtain

$$\frac{1}{1-n}z^{\frac{1}{1-n}-1}\frac{dz}{dx} + p(x)z^{\frac{1}{1-n}} = q(x)z^{\frac{n}{1-n}}$$

$$\frac{1}{1-n}\cancel{z^{\frac{1}{1-n}}}z^{-1}\frac{dz}{dx} + p(x)\cancel{z^{\frac{1}{1-n}}} = q(x)z^{-1}\cancel{z^{\frac{1}{1-n}}}$$

$$\left(\frac{1}{1-n} z^{-1} \frac{dz}{dx} + p(x) = q(x) z^{-1}\right) z(1-n)$$

$$\frac{dz}{dx} + p(x)(1-n)z(x) = q(x)(1-n).$$

Linear equation: $p^*(x) = p(x)(1-n)$ and $q^*(x) = q(x)(1-n)$

and so,

$$z(x) = e^{-\int p^*(x)dx}\left(\int q^*(x) e^{\int p^*(x)dx} dx + \beta\right)$$

$$y^{1-n}(x) = e^{-\int p(x)(1-n)dx}\left(\int q(x)(1-n) e^{\int p(x)(1-n)dx} dx + \beta\right)$$

$$\boxed{y(x) = \left[e^{-\int p(x)(1-n)dx}\left(\int q(x)(1-n) e^{\int p(x)(1-n)dx} dx + \beta\right)\right]^{\frac{1}{1-n}}}$$

For any general solution, we simply need to know three items:

(1) n (2) $p(x)(1-n)$ (3) $q(x)(1-n)$.

Example: Solve $x\frac{dy}{dx} + y = x^2y^2$.

Solution: Rewrite $\frac{dy}{dx} + \frac{1}{x}y = xy^2$. Here, since $n=2$,

$p(x)(1-n) = \frac{1}{x}(1-2) = -\frac{1}{x}$ $q(x) = x(1-2) = -x$.

Therefore,

$$y^{-1}(x) = e^{-\int -\frac{1}{x}dx}\left(\int -x e^{\int -\frac{1}{x}dx} dx + \beta\right)$$

$$= e^{\ln|x|}\left(\int -x e^{-\ln|x|} dx + \beta\right)$$

$$= x\left(-\int dx + \beta\right) = -x^2 + \beta x.$$

$$y^{-1}(x) = -x^2 + \beta x \Rightarrow y(x) = \frac{1}{\beta x - x^2}.$$