Initial-Value Problems and Boundary-Value Problems

In Section 1.2, we defined an initial-value problem for a general nth-order differential equation. For a linear differential equation an nth order initial value problem is:

- Solve: $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$
- Subject to: $y(x_0) = y_0$, $y'(x_0) = y_1$,..., $y^{(n-1)}(x_0) = y_{n-1}$.

Recall that for a problem such as the above, we seek a function defined on some interval I containing x_0, that satisfies the differential equation and the n initial conditions specified at x_0. We have already seen that in the case of a second-order initial-value problem a solution curve must pass through the point (x_0, y_0) and have slope y_1 at this point.

Existence and Uniqueness

In Section 1.2, we stated a theorem that gave conditions under which the existence and uniqueness of a solution of a first-order initial-value problem were guaranteed. The following theorem gives sufficient conditions for the existence of a unique solution of the problem above.

Theorem 1. *Let $a_n(x), a_{n-1}(x), ..., a_0(x)$ and $g(x)$ be continuous functions on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem exists on the interval and is unique.*

Example 2. The initial-value problem

$$3y''' + 5y'' - y' + 7y = 0, y(1) = 0, y'(1) = 0, y''(1) = 0$$

possesses the trivial solution $y = 0$. Since the third-order equation is linear with constant coefficients, it follows that all the conditions of Theorem 1 are fulfilled. Hence, $y = 0$ is the only solution on any interval containing $x = 1$.

Example 3. The function $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of the initial-value problem

$$y'' - 4y = 12x, y(0) = 4, y'(0) = 1 \qquad \text{(Verify)}$$

Now the differential equation is linear, the coefficients as well as $g(x) = 12x$ are continuous and $a_2(x) = 1 \neq 0$ on any interval I containing $x = 0$. Therefore, Theorem 1 allows us to conclude that the given function is a unique solution on I.

Boundary-Value Problems

Another type of problem consists of solving a linear differential equation of two or greater in which the dependent variable y or its derivatives are specified at different points. A problem such as:

- Solve $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$
- Subject to: $y(a) = y_0$, $y(b) = y_1$

is called a boundary-value problem. The prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called the boundary conditions. A solution of the foregoing problem is a function satisfying the differential equation on some interval I containing both a and b whose graph passes through the two points (a, y_0) and (b, y_1).

Homogeneous Equations

Definition 4. A linear n-th order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be *homogeneous*.

Note that the above definition implies that we require $g(x) = 0$ in order for the above to be a homogeneous equation. When $g(x) \neq 0$, then we call the equation *nonhomogeneous.*

We shall see that to solve a nonhomogeneous linear equation, we need to be able to solve the associated homogeneous equation.

To avoid needless repetition throughout the course, we shall, make the important assumptions when stating definitions and theorems about linear equations. On some common interval I,

- The coefficient functions $a_0, a_2, ..., a_n$ and $g(x)$ are continuous,
- $a_n(x) \neq 0$ for every x in the interval.

Differential Operators

In calculus differentiation is often denoted by the capital letter D, that is, $\frac{dy}{dx} = Dy$. The symbol D is called a *differential operator* because it transforms a differentiable function into another function. That is

$$D : f \to f'$$

For example, $D(\cos 4x) = -4 \sin 4x$ and $D(5x^3 - 6x^2) = 15x^2 - 12x$. Higher order derivatives can be expressed in terms of D in a natural manner:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y \qquad \frac{d^ny}{dx^n} = D^ny$$

where y represents a sufficiently differentiable function. Polynomial expressions involving D, such as $D+3$, D^2+3D-4 and $5x^3D^3-6x^2D^2+4xD+9$ are also differential operators. In general, we define an nth-order differential operator or polynomial operator to be

$$\mathcal{P} = a_n(x)D^n + a_{n-1}D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

As a consequence of two basic properties of differentiation, $D(cf(x)) = cDf(x)$ where c is a constant, and $D[f(x) + g(x)] = Df(x) + Dg(x)$, and so the differential operator $\mathcal{P}$ possesses a linearity property (i.e. it is closed under addition and scalar multiplication). If $\mathcal{P}$ is operating on a linear combination of two differentiable functions is the same as the linear combination of $\mathcal{P}$ operating on the individual functions. In symbols

$$\mathcal{P}[\alpha f(x) + \beta g(x)] = \alpha\mathcal{P}[f(x)] + \beta\mathcal{P}[g(x)]$$

where α and β are constants. With the equation above, we say that the nth order differential operator $\mathcal{P}$ is a linear operator.

Differential Equations

Any linear differential equation can be expressed in terms of the D notation. For example, the differential equation

$$y'' + 5y' + 6y = 5x - 3$$

can be written as

$$D^2y + 5Dy + 6y = 5x - 3$$

or

$$(D^2 + 5D + 6)y = 5x - 3$$

Using the equation above, we can write the linear n-th order differential equation compactly as

$$\mathcal{P}(y) = 0 \qquad \mathcal{P}(y) = g(x)$$

respectively.

Superposition Principle

In the next theorem we shall see that the sum or *superposition principle* of two or more solutions of a homogeneous linear differential equation is also a solution.

Theorem 5. *Let $y_1, y_2, ..., y_k$ be solutions of the homogeneous nth-order differential equation on an interval I. Then the linear combination*

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

where each c_i are arbitrary constants, is also a solution on the interval.

Proof. we prove the case $k = 2$, and it can be proven using mathematical induction for any general k (exercise). Let $\mathcal{P}$ be the differential operator, and let $y_1(x)$ and $y_2(x)$ be solutions of the homogeneous equation $\mathcal{P}(y) = 0$. If we define $y = c_1y_1(x) + c_2y_2(x)$ then by linearity of $\mathcal{P}$, we have

$$\mathcal{P}(y) = \mathcal{P}[c_1y_1(x) + c_2y_2(x)] = c_1\mathcal{P}(y_1) + c_2\mathcal{P}(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

□

Corollary 6. *A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.*

Proof. Exercise. □

Corollary 7. *A homogeneous linear differential equation always possesses the trivial solution $y = 0$.*

Proof. Exercise. □

Example 8. The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. By the superposition principle, the linear combination

$$y = c_1x^2 + c_2x^2 \ln x$$

is also a solution of the equation on the interval.

Definition 9 (Linear Independence of Functions)**.** A set of functions f_1, f_2, ..., f_n on an interval I is said to be *linearly independent* if the only constants for which

$$c_1f_1(x) + c_2f_2(x) + c \cdots + c_nf_n(x) = 0$$

for every x in the interval are $c_1 = c_2 = \cdots = c_n = 0$. If the function is not linearly independent, it is called *linearly dependent*.

Example 10. The set of functions $f_1(x) = \sqrt{x} + 5$, $f_2(x) = \sqrt{x} + 5x$, $f_3(x) = x - 1$ and $f_4(x) = x^2$ is linearly dependent on the interval $(0, \infty)$ since f_2 can be written as a linear combination of f_1, f_3 and f_4. Observe that

$$f_2(x) = 1f_1(x) + 5f_3(x) + 0f_4(x)$$

Solutions of Differential Equations

The question of whether the set of n solutions y_1, y_2,..., y_n of a homogeneous linear nth-order differential equation is linearly independent can be settled somewhat mechanically by using a determinant.

Definition 11 (Wronskian)**.** Suppose each of the functions f_1, f_2, ..., f_n possesses at least $n-1$ derivatives. The determinant

$$\mathbb{W}(f_1, f_2, ..., f_n) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{n-1} & \cdots & f_n^{(n-1)} \end{pmatrix}$$

where the primes denote derivatives, is called the Wronskian of the functions.