Population Dynamics

If $P(t)$ denotes the size of a population at time t, the model for the exponential growth begins with the assumption that $\frac{dP}{dt} = kP$ for some $k > 0$. In this model, the relative, or specific, growth rate is defined by

$$\frac{\frac{dP}{dt}}{P}$$

is a constant k.

The assumption that the rate at which a population grows (or decreases) is dependent only on the number P present and not on any time-dependent mechanisms, i.e.

$$\frac{\frac{dP}{dt}}{P} = f(P) \qquad \frac{dP}{dt} = Pf(P)$$

The differential equation is called the *density-dependent hypothesis*.

Logistic Equation

Suppose an environment is capable of sustaining no more than a fixed number K of individuals in its population. The quantity K is called the carrying capacity of the environment. Hence, for the function f we have $f(K) = 0$ adn we simply let $f(0) = r$. The simplest assumption that we can make is that $f(P)$ is linear, i.e. $f(P) = \alpha P + \beta$. If we use the conditions $f(0) = r$ and $f(K) = 0$, we find, in turn, $\beta = r$ and $\alpha = -\frac{r}{K}$, and so f takes on the form $f(P) = r - \frac{r}{K}P$ and so

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right)$$

With the constants relabelled, the nonlnear equation is the same as

$$\frac{dP}{dt} = P(a - bP)$$

Method of Solution of the Logistic Equation

One method of solving the equation is separating the variables. Decomposing the left side of $\frac{dP}{P(a-bP)} = dt$ into partial fractions and integrating gives

$$\begin{aligned}
\left(\frac{1/a}{P} + \frac{b/a}{a-bP}\right)dP &= dt \\
\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| &= t + \beta \\
\ln\left|\frac{P}{a-bP}\right| &= at + a\beta \\
\frac{P}{a-bP} &= \beta e^{at}
\end{aligned}$$

It follows that from the last equation

$$P(t) = \frac{a\beta e^{at}}{1 + b\beta e^{at}} = \frac{a\beta}{b\beta + e^{-at}}$$

If $P(0) = P_0$, $P_0 \neq \frac{a}{b}$, we find $\beta = \frac{P_0}{a-bP_0}$ and so

$$P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}$$

Example 1. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

Assuming that no one leaves the campus throughout the duration of the disease, we must solve the initial-value problem

$$\frac{dx}{dt} = kx(1000 - x), x(0) = 1$$

By making the identification $a = 1000k$ and $b = k$, we have immediately that

$$x(t) = \frac{1000k}{k + 999ke^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}$$

Now using the information that $x(4) = 50$, we want to determine the value of k,

$$50 = \frac{1000}{1 + 999e^{-4000k}}$$

We find $-1000k = \frac{1}{4}\ln\frac{19}{999} = -0.9906$. Thus,

$$x(t) = \frac{1000}{1 + 999e^{-0.9906t}}$$

Finally,

$$x(6) = \frac{1000}{1 + 999e^{-5.9436}} = 276$$

Modifications of the Logistic Equation

There are many variations of the logistic equation. For example, the differential equations

$$\frac{dP}{dt} = P(a - bP) - h \qquad \frac{dP}{dt} = P(a - bP) + h$$

could serve, in turn, as models for the population in a fishery where fish are *harvested* or are *restocked* at rate h. When $h > 0$ is a constant, the differential equation above can be readily analyzed qualitatively or solved analytically by separation of variables. The equation could also serve as models of the human population decreased by *emigration* or increased by immigration, respectively.

Chemical Reactions

Suppose that a grams of chemical A are combined with b grams of chemical B. If there are M parts of A and N parts of B formed in the compound and $\chi(t)$ is the number of grams of chemical C formed, then the number of grams of chemical A and the number of grams of chemical B remaining at time t, respectively.

$$a - \frac{M}{M+N}\chi \qquad b - \frac{N}{M+N}\chi$$

The law of mass action states when no temperature change is involed, the rate at which the two substances react is proportional to the product of the amount of A and B that are untransformed (remaining) at time t:

$$\frac{d\chi}{dt} \propto \left(a - \frac{M}{M+N}\chi\right)\left(b - \frac{N}{M+N}\chi\right)$$

If we factor out $\frac{M}{M+N}$ from the first factor and $\frac{N}{M+N}$ from the second and introduce a constant of proportionality $k > 0$,

$$\frac{d\chi}{dt} = k(\alpha - \chi)(\beta - \chi)$$

where $\alpha = a\frac{M+N}{M}$ and $\beta = b\frac{M+N}{N}$.

Example 2. A compound C is formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 g of B is used. It is observed that 30 g of the compound C is formed in 10 min. Determine the amount of C at time t if the rate of the reaction is proportional to the amounts of A and B remaining if initially there are 50 g of A and 32 g of B. How much of the compound C is present at 15 min? Interpret the solution as $t \to \infty$.

Let $\chi(t)$ denote the number of grams of the compound C present at time t. Clearly, $\chi(0) = 0$ and $\chi(10) = 30$ g. Suppose for example, 2 g of compound C is present, we must have used, say a g of A and b g of B so $a + b$ and $b = 4a$. Thus, we must use $a = \frac{2}{5} = 2\frac{1}{5}$ g of chemical A and $b = \frac{8}{5} = 2\frac{4}{5}$ g of B. In general, for χ grams of C, we must use

$$\frac{1}{5}\chi_A \qquad \frac{4}{5}\chi_B$$

The amounts of A and B remaining at time t are then

$$A : 50 - \frac{1}{5}\chi \qquad B : 32 - \frac{4}{5}\chi$$

respectively.

Now we knwo that the rate at which compound C is formed satisfies:

$$\frac{d\chi}{dt} \propto \left(50 - \frac{1}{5}\chi\right)\left(32 - \frac{4}{5}\chi\right)$$

To simplify the subsequent algebra, we factor $\frac{1}{5}$ from the first term and $\frac{4}{5}$ from the second and then introduce the constant of proportionality:

$$\frac{d\chi}{dt} = k(250 - \chi)(40 - \chi)$$

By separation of variables and partial fractions we can write

$$-\frac{\frac{1}{210}}{250 - \chi}d\chi + \frac{\frac{1}{210}}{40 - \chi}d\chi = kdt$$

Integrating gives

$$\ln\left(\frac{250 - \chi}{40 - \chi}\right) = 210kt + \beta \Rightarrow \frac{250 - \chi}{40 - \chi} = \beta e^{210kt}$$

When $t = 0$, $\chi = 0$, so it follows that $\beta = \frac{25}{4}$. Using $\chi = 30$ g at $t = 10$, we find $210k = \frac{1}{10}\ln\left(\frac{88}{25}\right) = 0.1258$. With this information, we solve the last equation for χ:

$$\chi(t) = 1000\frac{1 - e^{-0.1258t}}{25 - 4e^{-0.1258t}}$$

We find that $\chi(15) = 34.78$. The behavior of χ as a function of time shows that as $t \to \infty$, $\chi \to 40$. This means that 40 g of compound C is formed and

$$50 - \frac{1}{5}(40) = 42 \text{ g of } A \qquad 32 - \frac{4}{5}(40) = 0 \text{ g of } B$$

