Growth and Decay

The initial-value problem

$$\frac{dx}{dt} = kx, x(t_0) = x_0$$

where k is a constant of proportionality, serves as a model for diverse phenomena involving either growth or decay. Knowing the population at some arbitrary initial time t_0, we can use the solution above to predict the population in the future. That is, when $t > t_0$. The constant of proportionality k can be determined from the solution of the initial-value problem, using a subsequent measurement of x at a time $t_1 > t_0$.

Example 1. A culture has P_0 number of bacteria. At $t = 1$, the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at t, determine the time necessary for the number of bacteria to triple.

We first solve the differential equation. With $t_0 = 0$, the initial condition is $P(0) = P_0$. We then use the empirical observation that $P(1) = \frac{3}{2}P_0$ to determine the constant of proportionality k.

Notice that the differential equation $\frac{dP}{dt} = kP$ is separable and linear. When it is put in the standard form of a linear first-order differential equation

$$\frac{dP}{dt} - kP = 0$$

we see that by inspection the integrating factor is e^{-kt}. Solving the equation we have

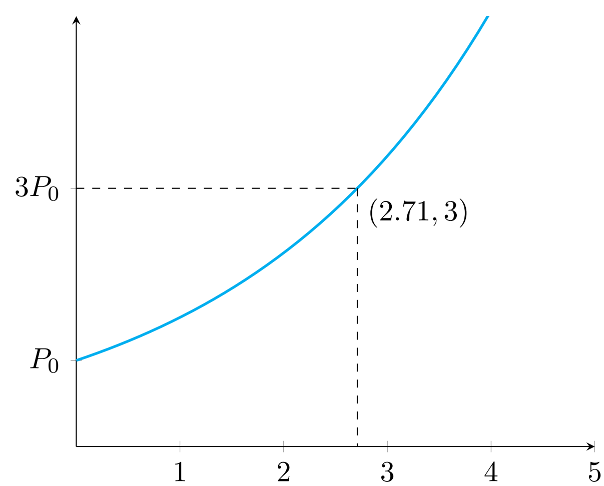
$$P(t) = P_0e^{kt}$$

At $t = 1$, we have

$$\frac{3}{2}P_0 = P_0e^k \Rightarrow k = \ln\left(\frac{3}{2}\right) = 0.4055$$

and so $P(t) = P_0e^{0.4055t}$. To determine the time at which the number of bacteria has tripled, we solve $3P_0 = P_0e^{0.4055t}$ for t. It follows that $0.4055t = \ln 3$ or

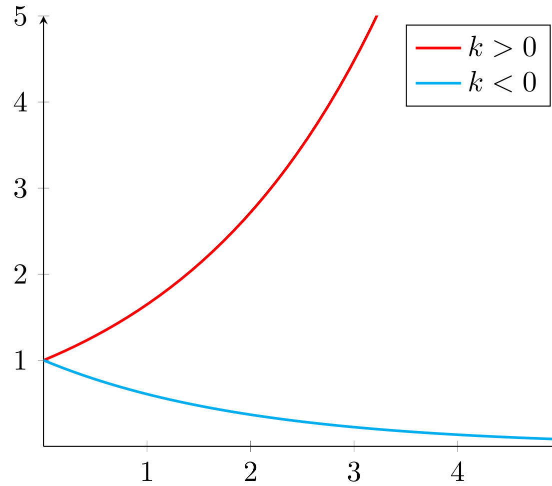
$$t = \frac{\ln 3}{0.4055} = 2.71 \text{ h}$$



There are different versions of constant of proportionality that we consider:

- If $k > 0$, then we say that the constant of proportionality is a growth constant.

- If $k < 0$, then we say that the constant of proportionality is a decay constant.



Half-Life

In physics, the *half-life* is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute into the atoms of another element. The longer the half-life of a substance, the more stable it is.

Example 2. A breeder reactor converts relatively stable uranium-238 into the isotope plutonium-239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Let $A(t)$ denote the amount of plutonium remaining at time t. We solve the initial-value problem (like in Example 1)

$$\frac{dA}{dt} = kA, A(0) = A_0 \Rightarrow A(t) = A_0 e^{kt}$$

If 0.043% of the atoms of A_0 has disintegrated, then 99.957% of the substance remains. To find the decay of constant k, we use

$$0.99957A_0 = A_0 e^{15k} \Rightarrow k = \frac{1}{15}\ln 0.99957 = -0.00002867$$

Hence,

$$A(t) = A_0 e^{-0.00002867t}$$

Now the half-life is the corresponding value of time at which $A(t) = \frac{1}{2}A_0$. Solving for t gives $\frac{1}{2}A_0 = A_0 e^{-0.00002867t}$ or $\frac{1}{2} = e^{-0.00002867t}$. The last equation yields

$$t = \frac{\ln 2}{0.00002867} = 24180$$

Newton's Law of Cooling/Warming

The mathematical formulation of Newton's Empirical Law of cooling/warming of an object is given by the linear first-order differential equation

$$\frac{dT}{dt} = k(T - T_m)$$

where k is the constant of proportionality, $T(t)$ is the temperature of the object for $t > 0$ and T_m is the ambient temperature. That is, the temperature of the medium around the object. Here, T_m will be constant.

Example 3. When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later, its temperature is 200°F. How long will it take for the cake to cool off to a room temperature of 70°F?

We make the identification that $T_m = 70$. We then must solve the initial-value problem

$$\frac{dT}{dt} = k(T - 70), T(0) = 300$$

and determine the value of k so that $T(3) = 200$. As the equation is separable,

$$\frac{1}{T - 70} dT = kdt$$

yields $\ln|T - 70| = kt + \beta$, and so $T = 70 + \beta e^{kt}$. When $t = 0$, $T = 300$, so $300 = 70 + \beta$ gives $\beta = 230$, therefore, $T = 70 + 273e^{kt}$. Finally, the measurement $T(3) = 200$ leads to $e^{3k} = \frac{13}{23}$ or $k = \frac{1}{3}\ln\left(\frac{13}{23}\right) = -0.19018$. Thus,

$$T(t) = 70 + 230e^{-0.19018t}$$

Mixtures

The mixing of two fluids sometimes gives rise to a linear first-order differential equation. We assume that the rate $A'(t)$ at which the amount of salt in the mixing tank changes was a net rate:

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt}) = R_{in} - R_{out}$$

Example 4. A large tank held 300 gal of a brine solution. Salt was entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min; it mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. he concentration of the salt in the inflow was 2 lb/gal, so salt was entering at a rate of $R_{in} = 2 \times 3 = 6$ lb/min and leaving the tank at a rate of $R_{out} = A/300 \times 3 = A/100$ lb/min. If 50 lbs of salt were dissolved initially at 300 gal, how much salt is in the tank after a long time?

To find the amount of salt $A(t)$ in the tank at time t, we solve the initial-value problem

$$\frac{dA}{dt} + \frac{1}{100}A = 6, A(0) = 50$$

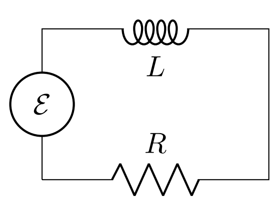
Note here that the side condition is the initial amount of salt $A(0) = 50$ in the tank and not the initial amount of liquid in the tank. Solving the linear equation we obtain

$$A(t) = 600 - 550e^{-\frac{t}{100}}$$

As $t \to \infty$, $A(t) \to 600$.

Series Circuits

For a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor $L\frac{di}{dt}$ and the voltage drop across the resistor iR is the same as the impressed voltage $\mathcal{E}(t)$ on the circuit.

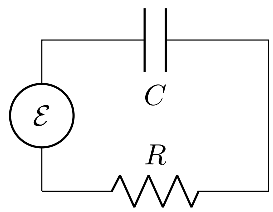


Thus, we obtain the linear differential equation

$$L\frac{di}{dt} + Ri = \mathcal{E}(t)$$

where L and R are known as the inductance and resistance, respectively. The current $i(t)$ is also called the response of the system.

The voltage drop across a capacitor with capacitance C is given by $\frac{q(t)}{C}$ where q is the charge on the capacitor.



Hence, for the series circuit shown above, Kirchhoff's second law gives

$$Ri = \frac{1}{C}q = \mathcal{E}(t)$$

But i and q are related by $i = \frac{dq}{dt}$, so

$$R\frac{dq}{dt} + \frac{1}{C}q = \mathcal{E}(t)$$

Example 5. A 12-V battery is connected to a series circuit in which the inductance is 0.5 H and the resistance is 10 Ω. Determine the current i if the initial current is zero.

We solve the equation

$$0.5\frac{di}{dt} + 10i = 12$$

subject to $i(0) = 0$. First, we multiply the differential equation by 2 and read off the integrating factor e^{20t}. Then solving the equation, we obtain

$$i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}$$