Question 1. *(a) Prove that the sequence defined by $x_1 = 3$ and*

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know that $\lim_{n\to\infty} x_n$ exists, explain why $\lim_{n\to\infty} x_{n+1}$ must also exist and equal to the same value.

(c) Take the limit of each side of the recursive equation in (a) to explicitly compute $\lim_{n\to\infty} x_n$.

Solution 1. Let $x_1 = 3$ and $x_{n+1} = \frac{1}{4-x_n}$.

(a) We will prove that $(x_n)_n$ converges using mathematical induction and the monotone convergence theorem. For the base case, we have $x_1 = 3$, and note that $x_2 = \frac{1}{4-x_1} = \frac{1}{4-3} = 1$. We claim that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. We have that

$$\frac{1}{4 - x_{n+1}} \leq \frac{1}{4 - x_{n+2}} \Rightarrow 4 - x_{n+1} \geq 4 - x_{n+2} \Rightarrow -x_{n+1} \geq x_{n+2} \Rightarrow x_{n+1} \leq x_{n+2}$$

Therefore, we have shown that x_n is monotonically increasing for all $n \in \mathbb{N}$. Then, note that since $x_n < 3$ and $x_{n+1} > 0$ for all $n \in \mathbb{N}$, we have that x_n is bounded. Therefore, by the monotone convergence theorem, $(x_n)_n$ converges.

(b) The limit of a sequence is the same as the limit of the sequence of the next term because the value of the convergence does not change whenever we adjust the index by one.

(c) Let $x = \lim_{n\to\infty} x_n = \frac{1}{4-x}$. We solve for x.

$$x = \frac{1}{4 - x} \Rightarrow x(4 - x) = 1 \Rightarrow 4x - x^2 = 1 \Rightarrow x^2 - 4x + 1 = 0$$

Using the Quadratic Formula,

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

Note that $x = 2 + \sqrt{3} > 3$, so we reject this solution. Therefore, the converging limit is $x = 2 - \sqrt{3}$.

Question 2. *(a) Show that*

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Solution 2. (a) Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$. We claim that $(x_n)_n$ is a convergent sequence. We will use mathematical induction and the monotone convergence theorem to show this. For the base case we have $x_1 = \sqrt{2}$, and note that $x_2 = \sqrt{2 + x_1} = \sqrt{2 + \sqrt{2}}$. We claim that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. We have that

$$\sqrt{2 + x_n} \leq \sqrt{2 + x_{n+1}} \Rightarrow 2 + x_n \leq 2 + x_{n+1} \Rightarrow x_n \leq x_{n+1}$$

So we have shown that x_n is monotonically increasing for all $n \in \mathbb{N}$. Then note that since $x_n < 2$ and $x_n > 0$ we have that $(x_n)_n$ is bounded. Therefore, by the monotone convergence theorem, $(x_n)_n$ converges.

Let $x = \lim_{n\to\infty} x_n = \sqrt{2 + x}$. We solve for x.

$$x = \sqrt{2 + x} \Rightarrow x^2 = 2 + x \Rightarrow x^2 - x - 2 = 0$$

Then we have that $x = -1$ and $x = 2$. We reject $x = -1$, since $x_n > 0$ for all $n \in \mathbb{N}$. Therefore, the converging limit is $x = 2$.

(b) Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$. We claim that the sequence $(x_n)_n$ converges. We will use mathematical induction and the monotone convergence theorem to show this. For the base case, we have $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2x_1} = \sqrt{2\sqrt{2}}$, so we have that $x_1 \leq x_2$. Then we claim that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. We have

$$\sqrt{2x_n} \leq \sqrt{2x_{n+1}} \Rightarrow 2x_n \leq 2x_{n+1} \Rightarrow x_n \leq x_{n+1}$$

Therefore, we have shown that $(x_n)_n$ is monotonically increasing. Since $x_n > 0$ and $x_n < 2$, we have that $(x_n)_n$ is bounded. So by the monotone convergence theorem, $(x_n)_n$ converges.

Let $x = \lim_{n\to\infty} x_n = \sqrt{2x}$. We solve for x.

$$x = \sqrt{2x} \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0$$

so we have $x = 0$ and $x = 2$. Since $x_n > 0$, the converging limit of $(x_n)_n$ is $x = 2$.

Question 3 (Calculating Square Roots). *Let $x_1 = 2$ and define*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim_{n\to\infty} x_n = \sqrt{2}$.

(b) Modify the sequence $(x_n)_n$ so that it converges to $\sqrt{c}$.

Solution 3. (a) We will show by mathematical induction that $x_n^2 \geq 2$. Since $x_1^2 \geq 2$, the base case is proven. Assume that $x_n^2 \geq 2$. We want to show that $x_{n+1}^2 \geq 2$.

$$x_{n+1}^2 = \frac{1}{4}\left(x_n + \frac{2}{x_n}\right)^2 = \frac{1}{4}\left(\frac{x_n^2 + 2}{x_n}\right)^2 \geq \frac{1}{4}\left(\frac{x_n + 2}{\sqrt{2}}\right)^2$$

Here, since $x_n^2 \geq 2$, then $x_n^2 + 2 \geq 4$. Therefore,

$$\frac{1}{4}\left(\frac{4}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2 \geq 2$$

Therefore, we have shown that $x_n^2 \geq 2$. To show that $x_n - x_{n+1} \geq 0$, we use the fact that $x_n \geq 0$, and so

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = x_n - \frac{1}{2}x_n - \frac{1}{x_n} \geq 0$$

Since $(x_n)_n \to x$, let $x^2 = 2$. Then $x = \pm\sqrt{2}$. Reject $x = -\sqrt{2}$. So $x = \sqrt{2}$, as required.

(b) Let $x_1 = c$ and define the sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

Solve $x^2 = c$, and find that $x = \sqrt{c}$ is the converging limit of $(x_n)_n$.

Question 4 (Arithmetic–Geometric Mean)**.** *(a) Explain why* $\sqrt{xy} \leq \frac{x+y}{2}$ *for any two positive real numbers* x *and* y*. (The geometric mean is always less than the arithmetic mean)*

(b) Now let $0 \leq x_1 \leq y_1$ *and define*

$$x_{n+1} = \sqrt{x_n y_n} \qquad y_{n+1} = \frac{x_n + y_n}{2}$$

Show that $\lim_{n\to\infty} x_n$ *and* $\lim_{n\to\infty} y_n$ *both exist and are equal.*

Solution 4. (a) Since $\sqrt{xy} \leq \frac{x+y}{2}$, note that

$$xy \leq \frac{(x+y)^2}{4} \Rightarrow 4xy \leq (x+y)^2 \Rightarrow (x+y)^2 - 4xy \geq 0 \Rightarrow x^2 - 2xy + y^2 \geq 0 \Rightarrow (x-y)^2 \geq 0$$

(b) Let $x = \lim_{n\to\infty} x_{n+1}$ and let $y = \lim_{n\to\infty} y_{n+1}$. Then if

$$x = \lim_{n\to\infty} x_{n+1} = \sqrt{xy} \qquad y = \lim_{n\to\infty} y_{n+1} = \frac{x+y}{2}$$

For the first equation, we have

$$x^2 = xy \Rightarrow x^2 - xy = 0 \Rightarrow x(x-y) = 0$$

So we have $x = 0$ or $x = y$. Similarly, for the second equation, we have

$$2y = x + y \Rightarrow x = y$$

So the only valid solution would be $x = y$. Therefore, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ as desired.

Question 5. *Give an example of each of the following, or argue that such a request is impossible.*

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded sequence.

Solution 5. (a) True. Let $x_n = \frac{(-1)^n}{n}$. Then the sequence is Cauchy the sequence is convergent, but it is oscillating so it is not monotonic.

(b) False. A convergent sequence is said to be Cauchy. All convergent sequences are Cauchy. If the sequence is not bounded, it does not converge, so it is not Cauchy.

Question 6. *If $(z_n)_n$ and $(w_n)_n$ are Cauchy sequences, then one easy way to prove that $(z_n + w_n)_n$ is Cauchy is to use the Cauchy Criterion. Since $(z_n)_n$ and $(w_n)_n$ must be convergent, and the Algebraic Limit Theorem then implies that $(z_n + w_n)_n$ is convergent and hence Cauchy.*

(a) Give a direct argument that $(z_n + w_n)_n$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(z_n w_n)_n$.

Solution 6. (a) Let $\epsilon > 0$ be arbitrary. Since $(z_n)_n$ converges to $L \in \mathbb{C}$, $(z_n)_n$ is a Cauchy sequence and there exists an $N_1 \in \mathbb{N}$ such that

$$|z_n - z_m| < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Similarly, since $(w_n)_n$ converges to $K \in \mathbb{C}$, $(w_n)_n$ is a Cauchy sequence and there exists an $N_2 \in \mathbb{N}$ such that

$$|w_n - w_m| < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,

$$|(z_n + w_n) - (z_m - w_m)| \leq |z_n + z_m| + |w_n - w_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, as $\epsilon > 0$ was arbitrary, $(z_n + w_n)_n$ is a Cauchy sequence.

(b) Let $\epsilon > 0$ be arbitrary. Since $(z_n)_n$ converges to L, $(z_n)_n$ is a Cauchy sequence and there exists an $N_1 \in \mathbb{N}$ such that

$$|z_n - z_m| < \frac{\epsilon}{2M_2}$$

for all $n \geq N_1$. Similarly, since $(w_n)_n$ converges to K, $(w_n)_n$ is a Cauchy sequence and there exists an $N_2 \in \mathbb{N}$ such that

$$|w_n - w_m| < \frac{\epsilon}{2M_1}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,

$$\begin{aligned}
|z_n w_n - z_m w_m| &\leq |z_n w_n - z_n w_m| + |z_n w_m - z_m w_m| \\
&= |z_n||w_n - w_m| + |w_m||z_n - z_m| \\
&< |z_n|\frac{\epsilon}{2M_2} + |w_m|\frac{\epsilon}{2M_1} \\
&\leq M_2\frac{\epsilon}{2M_2} + M_1\frac{\epsilon}{2M_1} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, $(z_n w_n)_n$ is a Cauchy sequence.

Question 7. *Decide whether each of the following series converges or diverges.*

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

Solution 7. (a) Let $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$. We claim that the series converges. To see this, we will use the Comparison Test. Since

$$\frac{1}{2^n+n} \leq \frac{1}{2^n}$$

for all $n \in \mathbb{N}$, and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, therefore, the sum $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ also converges.

(b) Let $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$. We claim that the series converges. To see this, we will use the Comparison Test. Since

$$\frac{\sin n}{n^2} \leq \frac{1}{n^2}$$

for all $n \in \mathbb{N}$ and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, the sum $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ also converges.

(c) Let $\sum_{n=1}^{\infty} x_n = 1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$. We claim that the series diverges. Since

$$x_n = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$$

The terms never get smaller than $\frac{1}{2}$ for all $n \in \mathbb{N}$. Therefore, the series diverges.

Question 8. *Give an example of each or explain why the request is impossible by referencing the proper theorem(s).*

(a) *Two series $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ that both diverge but where $\sum_{n=1}^{\infty} z_n w_n$ converges.*

(b) *A convergent series $\sum_{n=1}^{\infty} z_n$ and a bounded sequence $(w_n)_n$ such that $\sum_{n=1}^{\infty} z_n w_n$ diverges.*

(c) *Two sequences $(z_n)_n$ and $(w_n)_n$ where $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty}(z_n + w_n)$ both converge but $\sum_{n=1}^{\infty} w_n$ diverges.*

(d) *A sequence $(z_n)_n$ satisfying $0 \leq z_n \leq \frac{1}{n}$ where $\sum_{n=1}^{\infty}(-1)^n z_n$ diverges.*

Solution 8. (a) True. We can let $z_n = w_n = \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, the series $\sum_{n=1}^{\infty} z_n w_n$ converges.

(b) True. Let $z_n = \frac{(-1)^n}{n}$ and $w_n = (-1)^n$. Then $\sum_{n=1}^{\infty} z_n w_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(c) False. By the Algebraic Sum Property, $\sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty}(z_n + w_n)$, or

$$\sum_{n=1}^{\infty}(z_n + w_n) - \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} w_n$$

The left hand side is convergent, but the right hand side is divergent, so it is not possible.

(d) True. The sequence

$$z_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

diverges for the same reason the harmonic series diverges.

Question 9. (a) *Show that if $z_n > 0$ with $\lim_{n\to\infty} n z_n = L$ with $L \neq 0$, the series $\sum_{n=1}^{\infty} z_n$ diverges.*

(b) *Assume $z_n > 0$ and $\lim_{n\to\infty} n^2 z_n$ exists. Show that $\sum_{n=1}^{\infty} z_n$ converges.*

Solution 9. (a) Suppose $\lim_{n\to\infty} nz_n = L \neq 0$. Then let $\epsilon = \frac{1}{2}$. We have that $nz_n \in \left(L - \frac{1}{2}, L + \frac{1}{2}\right)$, implying that $z_n > \frac{L}{2n}$. The series $\sum_{n=1}^{\infty} z_n$ diverges since

$$\sum_{n=1}^{\infty} z_n > \sum_{n=1}^{\infty} \frac{L}{2n}$$

diverges as it is a multiple of the harmonic series.

(b) Let $L = \lim_{n\to\infty} n^2 z_n$, then let $\epsilon = L$ be such that $n^2 z_n \in (0, 2L)$, implying that $0 \leq z_n \leq \frac{2L}{n^2}$ and so $\sum_{n=1}^{\infty} z_n$ converges by the comparison test with $\sum_{n=1}^{\infty} \frac{2L}{n^2}$.

Question 10. *Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.*

(a) If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then $\sum_{n=1}^{\infty} z_n^2$ converges absolutely.

(b) If $\sum_{n=1}^{\infty} z_n$ converges and $(w_n)_n$ converges, then $\sum_{n=1}^{\infty} z_n w_n$ converges.

(c) If $\sum_{n=1}^{\infty} z_n$ converges conditionally, then $\sum_{n=1}^{\infty} n^2 z_n$ diverges.

Solution 10. (a) True. Since $\sum_{n=1}^{\infty} z_n$ converges absolutely, $\lim_{n\to\infty} z_n = 0$, and so this implies that $(z_n^2) \to 0$ eventually, so $z_n^2 \leq |z_n|$, implying that by the Comparison Test, $\sum_{n=1}^{\infty} z_n^2$ converges absolutely.

(b) False. Let $z_n = \frac{(-1)^n}{\sqrt{n}}$ and $w_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\sum_{n=1}^{\infty} z_n$ converges conditionally and $(w_n) \to 0$. However, if $z_n w_n = \frac{1}{n}$, then

$$\sum_{n=1}^{\infty} z_n w_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by the p-series test.

(c) True. Assume that $\sum_{n=1}^{\infty} z_n$ converges conditionally. Suppose otherwise that $\sum_{n=1}^{\infty} n^2 z_n$ converges. Then this implies that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|n^2 z_n| < \epsilon$ for all $n \in \mathbb{N}$. Then $|z_n| < \frac{\epsilon}{n^2}$, which implies that $|z_n|$ is absolutely convergent by the comparison test. This contradicts the assumption that $\sum_{n=1}^{\infty} z_n$ converges conditionally. Hence, $\sum_{n=1}^{\infty} n^2 z_n$ must diverge.

Question 11 (Ratio Test)**.** *Given a series $\sum_{n=1}^{\infty} z_n$ with $z_n \neq 0$, the Ratio Test states that if $(z_n)_n$ satisfies*

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = r < 1$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an $n \geq N$ implies $|z_{n+1}| \leq |z_n| r'$.

(b) Why does $|z_N| \sum_{n=1}^{\infty} (r')^n$ converge?

(c) Now show that $\sum_{n=1}^{\infty} |z_n|$ converges, and conclude that $\sum_{n=1}^{\infty} z_n$ converges.

Solution 11. (a) Let $r' \in (r, 1)$. Since $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = r$, then there exists an $N \in \mathbb{N}$ such that

$$\left|\frac{z_{n+1}}{z_n} - r\right| < \epsilon$$

Let $\epsilon = r' - r$ be such that

$$r - \epsilon \le \frac{z_{n+1}}{z_n} \le r + \epsilon = r + r' - r = r'$$

So

$$\left|\frac{z_{n+1}}{z_n}\right| \le r' \Rightarrow |z_{n+1}| \le |z_n| r'$$

(b) We claim that for all $n \in \mathbb{N}$, that

$$|z_n| \le |z_{n-1}|(r')^1 \le |z_{n-2}|(r')^2 \le \cdots \le |z_k|(r')^{n-k} \le \cdots \le |z_1|(r')^{n-1}$$

We will use mathematical induction to show that this inequality is true. We will show that $|z_n| \le |z_1|(r')^{n-1}$, which then can be generalized for any $N \in \mathbb{N}$. For the base case where $n = 1$, we have

$$|z_1| \le |z_1|(r')^{1-1} = |z_1|$$

Therefore, the base case is proven. Next, we assume that $|z_n| \le |z_1|(r')^{n-1}$. We want to show that $|z_{n+1}| \le |z_1|(r')^{(n+1)-1}$. We have

$$|z_{n+1}| \le |z_1|(r')^n$$

Then for any $N \in \mathbb{N}$,

$$|z_n| \le |z_N|(r')^{n-N}$$

We can then write

$$\sum_{k=N}^{n} |z_k| \le |z_N| \sum_{k=0}^{n-1} (r')^k$$

which converges as the term on the right hand side is a Geometric series with $|r'| < 1$ and $|z_N|$ is constant.

(c) By the Comparison Test, $\sum_{n=1}^{\infty} z_n$ converges absolutely.

Question 12 (Summation by Parts). *Let $(z_n)_n$ and $(w_n)_n$ be sequences, let $s_n = z_1 + z_2 + \cdots + z_n$ and set $s_0 = 0$. Use the observation that $z_k = s_k - s_{k-1}$ to show that*

$$\sum_{k=m}^{n} z_k w_k = s_n w_{n+1} - s_{m-1} w_m + \sum_{k=m}^{n} s_k (w_k - w_{k+1})$$

Solution 12. If $z_k = s_k - s_{k-1}$, then

$$\sum_{k=m}^{n} z_k w_k = \sum_{k=m}^{n} (s_k - s_{k-1}) w_k = \sum_{k=m}^{n} w_k s_k - \sum_{k=m}^{n} w_k s_{k-1}$$

Observe for the second summation that

$$\begin{aligned}\sum_{k=m}^{n} w_k s_{k-1} &= w_m s_{m-1} + w_{m+1} s_m + \cdots + w_n s_{n-1} + w_{n+1} s_n - w_{n+1} s_n \\ &= w_m s_{m-1} - w_{n+1} s_n + \sum_{k=m}^{n} w_{k+1} s_k\end{aligned}$$

Therefore,

$$\sum_{k=m}^{n} w_k s_k - \left(w_m s_{m-1} - w_{n+1} s_n + \sum_{k=m}^{n} w_{k+1} s_k \right) = w_{n+1} s_n - w_m s_{m-1} + \sum_{k=m}^{n} s_k (w_k - w_{k+1})$$

as required.

Question 13 (Abel's Test). *Abel's Test for convergence states that if the series* $\sum_{k=1}^{\infty} z_n$ *converges, and if* $(w_k)_k$ *is a sequence satisfying*

$$w_1 \geq w_2 \geq w_3 \geq \cdots \geq 0$$

then the series $\sum_{k=1}^{\infty} z_k w_k$ *converges.*

(a) *Use Question 12 to show that*

$$\sum_{k=1}^{n} z_k w_k = s_n w_{n+1} + \sum_{k=1}^{n} s_k(w_k - w_{k+1})$$

where $s_n = z_1 + z_2 + \cdots + z_n$.

(b) *Use the Comparison Test to argue that* $\sum_{k=1}^{\infty} s_k(w_k - w_{k+1})$ *converges absolutely, and show how this leads directly to a proof of Abel's Test.*

Solution 13. (a) From Question 12, we have that

$$\sum_{k=m}^{n} z_k w_k = s_n w_{n+1} - s_{m-1} w_m + \sum_{k=m}^{n} s_k(w_k - w_{k+1})$$

Adjusting the indices we have that

$$\sum_{k=1}^{n} z_k w_k = s_n w_{n+1} + s_0 w_m + \sum_{k=1}^{n} s_k(w_k - w_{k+1})$$

and since $s_0 = 0$,

$$\sum_{k=1}^{n} z_k w_k = s_n w_{n+1} + \sum_{k=1}^{n} s_k(w_k - w_{k+1})$$

(b) First, note that as $n \to \infty$, $s_n w_{n+1}$ converges since w_{n+1} will eventually be constant. We first want to show that $\sum_{k=1}^{\infty} s_k(w_k - w_{k+1})$ is bounded. There exists an $M \in \mathbb{R}$ with $M > 0$ such that $|s_k| \leq M$ and

$$\left|\sum_{k=1}^{\infty} s_k(w_k - w_{k+1})\right| \leq \sum_{k=1}^{\infty} |s_k|(w_k - w_{k+1}) \leq M \sum_{k=1}^{\infty}(w_k - w_{k+1})$$

Note that the series on the right side is simply the telescopic series, i.e.

$$\sum_{k=1}^{\infty}(w_k - w_{k+1}) = (w_1 - w_2) + (w_2 - w_3) + \cdots = w_1$$

and so

$$M \sum_{k=1}^{\infty}(w_k - w_{k+1}) = M w_1$$

Therefore, by the Monotone convergence theorem, since $(w_k)_k$ is decreasing and bounded for all $k \in \mathbb{N}$, the series

$$\sum_{k=1}^{\infty} s_k(w_k - w_{k+1})$$

converges absolutely. Then by the Algebraic Limit Theorem, the sum of two convergent series is a convergent series, this proves Abel's Test.

Question 14. *(a) Define a sequence of functions on* $\mathbb{R}$ *by*

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f *be the pointwise limit of* f_n. *Is each* f_n *continuous at zero? Does* $f_n \to f$ *uniformly on* $\mathbb{R}$*? Is* f *continuous at zero?*

(b) Repeat this exercise using the sequence of functions

$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

(c) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n-1} \\ 0 & \text{otherwise} \end{cases}$$

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem.

Solution 14. (a) Each f_n is continuous at zero. $f_n \not\to f$ uniformly on $\mathbb{R}$, and f is not continuous at zero.

(b) Each g_n is continuous at zero. $g_n \to g$ may be uniform or not on $\mathbb{R}$, and g is continuous at zero. Since $|g_n(x) - g(x)| < \frac{1}{n}$ for all $n \in \mathbb{N}$ and for all x, if $N > \frac{1}{\epsilon}$, we have that for all $n \geq N$ and for all $x \in \mathbb{R}$,

$$|g_n(x) - g(x)| < \epsilon$$

so $\sup_{x\in\mathbb{R}} |g_n(x) - g(x)| < \epsilon$ and $(g_n) \to g$ uniformly.

(c) Each h_n is continuous at zero. $h_n \not\to h$ uniformly on $\mathbb{R}$, so h is not continuous at zero. To show non-uniform convergence, if $x_n = \frac{1}{n}$ and $\epsilon = \frac{1}{n}$, then

$$\left| h_n\left(\frac{1}{n}\right) - h\left(\frac{1}{n}\right) \right| = 1 - \frac{1}{n} \geq \epsilon$$

Therefore, no matter how large n is, it is not possible to make $|h_n(x) - h(x)| < \frac{1}{2}$ for all x, so $h_n \not\to h$ uniformly.

Question 15. *For each* $n \in \mathbb{N}$ *and* $x \in [0, \infty)$, *let*

$$g_n(x) = \frac{x}{1 + x^n} \qquad h_n(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ nx & \text{if } 0 \leq x < \frac{1}{n} \end{cases}$$

Answer the following questions for $(g_n)_n$ *and* $(h_n)_n$.

(a) Find the pointwise limit on $[0, \infty)$.

(b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution 15. (a) For the sequence of functions $(g_n)_n$, we have

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

For the sequence of functions $(h_n)_n$ we have

$$h(x) = \lim_{n\to\infty} h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(b) Clearly, g_n and h_n are continuous functions for each $n \in \mathbb{N}$, but g and h are not.

(c) Over $1 \leq x < \infty$ $h_n(x) = h(x) = 1$ for all $n \in \mathbb{N}$, so $|h_n(x) - h(x)| = 0$ for all $1 \leq x < \infty$, so h_n converges uniformly.

Let $0 \leq t < 1$. Suppose $g_n(x) \to x$ uniformly over $0 \leq x < t$. We have

$$\left| \frac{x}{1+x^n} - x \right| = \left| \frac{x^{n+1}}{1+x^n} \right| < |t^{n+1}| < \epsilon$$

for $n > \log_t \epsilon$.

Question 16. *Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that f_n converges uniformly to f. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.*

Solution 16. To show that f_n uniformly converges to f, let $\epsilon > 0$ be arbitrary. Since f is uniformly continuous on $\mathbb{R}$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Then if $\left|\left(x + \frac{1}{n}\right) - x\right| = \frac{1}{n} < \delta$, we have

$$\left| f\left(x + \frac{1}{n}\right) - f(x) \right| = |f_n(x) - f(x)| < \epsilon$$

for all $n \in \mathbb{N}$ and so

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| < \epsilon$$

and so f_n uniformly converges to f.

To show that the above proposition is false when f is not uniformly continuous, let $f(x) = x^2$. Then f is not uniformly continuous on $\mathbb{R}$ and so

$$|f_n(x) - f(x)| = \left| \left(x + \frac{1}{n}\right)^2 - x^2 \right| = \left| x^2 + \frac{x}{n} + \frac{1}{n^2} - x^2 \right| = \left| \frac{2}{n} + \frac{1}{n^2} \right|$$

then for an arbitrary large x, $(f_n) \to \infty$ and so does not uniformly converge.

Question 17. *Assume that $(f_n)_n$ and $(g_n)_n$ are uniformly convergent sequences of functions.*

(a) Show that $(f_n + g_n)_n$ is a uniformly convergent sequence of functions.

(b) Give an example to show that the product $(f_n g_n)_n$ may not converge uniformly.

(c) *Prove that if there exists an $M \in \mathbb{R}$ with $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_ng_n)_n$ converge uniformly.*

Solution 17. (a) To show that $(f_n + g_n)_n$ is a uniformly convergent sequence of functions, let $\epsilon > 0$ be arbitrary. Since $(f_n)_n$ is a uniformly convergent sequence of functions, there exists an $N_1 \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \leq |f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for all $x \in \mathbb{R}$ and $n \geq N_1$. Similarly, since $(g_n)_n$ is a uniformly convergent sequence of functions, there exists an $N_2 \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}} |g_n(x) - g(x)| \leq |g_n(x) - g(x)| < \frac{\epsilon}{2}$$

Then let $N = \max\{N_1, N_2\}$ be such that for all $n \geq N$,

$$\begin{aligned}
\sup_{x\in\mathbb{R}} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &\leq |(f_n(x) + g_n(x)) - (f(x) + g(x))| \\
&\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, we have shown that $(f_n + g_n)_n$ converges uniformly.

(b) Let $f_n(x) = x = f(x)$ and $g_n(x) = x + \frac{1}{n}$. Suppose there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, the Cauchy Criterion gives us

$$|f_ng_n - f_mg_m| = \left|x\left(\frac{1}{n} - \frac{1}{m}\right)\right|$$

Making x large makes the error blow up regardless how large N is, so f_ng_n does not converge uniformly.

(c) To show that $(f_ng_n)_n$ converges uniformly, let $\epsilon > 0$. Since $(f_n) \to f$ and $|f_n(x)| \leq M$, there exists an $N_1 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2M}$$

for all $n \geq N_1$ and $x \in \mathbb{R}$. Similarly, since $(g_n) \to g$ and $|g_n(x)| \leq M$, there exists an $N_2 \in \mathbb{N}$ such that

$$|g_n(x) - g(x)| < \frac{\epsilon}{2M}$$

for all $n \geq N_2$ and $x \in \mathbb{R}$. Then let $N = \max\{N_1, N_2\}$ be such that for all $n \geq N$,

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_ng(x) - f(x)g(x)| \\
&\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\
&\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, $(f_ng_n)_n$ converges uniformly.

Question 18. *Consider the sequence of functions defined by*

$$g_n(x) = \frac{x^n}{n}$$

(a) *Show $(g_n)_n$ converges uniformly on $[0,1]$ and find $g(x) = \lim_{n\to\infty} g_n(x)$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$.*

(b) *Now show that (g_n') converges on $[0,1]$. Is the convergence uniform? Set $h(x) = \lim_{n\to\infty} g_n'(x)$ and compare h and g'. Are they the same?*

Solution 18. (a) First, we find the pointwise limit of $g_n(x)$. We have

$$g(x) = \lim_{n\to\infty} \frac{x^n}{n} = 0$$

Note that since $0 \le x \le 1$, so $0 \le x^n \le 1$ and since x is small, $(x^n) \to 0$ as $n \to \infty$. To show the convergence is uniform, there exists an $N \in \mathbb{N}$ such that

$$|g_n(x) - g(x)| = \left|\frac{x^n}{n}\right| < \epsilon$$

any $n \ge N = \frac{1}{\epsilon}$ will force $|g_n(x)| < \epsilon$. $g(x) = 0$ is differentiable at zero.

(b) If $g_n(x) = \frac{x^n}{n}$, then $g_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1}$. Since $x \in [0,1]$, x is small, and so $(g_n') \to 0$ if $0 \le x < 1$ and $(g_n') \to 1$ if $x = 1$. However, since

$$h(x) = \lim_{n\to\infty} g_n'(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

h and g' are not the same, so they cannot be uniformly convergent.

Question 19. *Consider the sequence of functions*

$$f_n(x) = \frac{x}{1+nx^2}$$

(a) *Find the points on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove $(f_n)_n$ converges uniformly on $\mathbb{R}$. What is the limit function?*

(b) *Let $f(x) = \lim_{n\to\infty} f_n(x)$. Compute f_n' and find all the values of x for which $f'(x) = \lim_{n\to\infty} f_n'(x)$.*

Solution 19. (a) To find the maximum and minimum values, we can use the first derivative test to determine where the critical points of $f_n(x)$ are.

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} = 0 \Rightarrow 1 - nx^2 = 0 \Rightarrow nx^2 = 1 \Rightarrow x^2 = \frac{1}{n} \Rightarrow x = \pm\frac{1}{\sqrt{n}}$$

Therefore, at $x = \frac{1}{\sqrt{n}}$ and $x = -\frac{1}{\sqrt{n}}$, we have

$$f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{\sqrt{n}}}{1+n\left(\frac{1}{\sqrt{n}}\right)^2} = \frac{\frac{1}{\sqrt{n}}}{1+n\frac{1}{n}} = \frac{\frac{1}{\sqrt{n}}}{1+1} = \frac{1}{2\sqrt{n}}$$

and

$$f_n\left(-\frac{1}{\sqrt{n}}\right) = \frac{-\frac{1}{\sqrt{n}}}{1+n\left(-\frac{1}{\sqrt{n}}\right)^2} = \frac{-\frac{1}{\sqrt{2}}}{1+n\frac{1}{n}} = \frac{-\frac{1}{\sqrt{n}}}{2} = -\frac{1}{2\sqrt{n}}$$

Therefore, $f_n(x)$ is bounded, i.e. $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$. Take $n \to \infty$, and we see that $f_n(x) \to 0$. The limit function is zero.

(b) Since $f(x) = f'(x) = 0$, we have that

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{1+2nx^2+n^2x^4} = \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{1}{n}x^2}{\frac{1}{n^2} + \frac{2}{n}x^2 + x^4} = 0$$

therefore, $f'(x) = \lim_{n\to\infty} f_n'(x)$ everywhere.

Question 20. *Let*

$$h_n(x) = \frac{\sin nx}{\sqrt{n}}$$

Show that $h_n \to 0$ on $\mathbb{R}$ but that the sequence of derivatives $(h_n')_n$ diverges for every $x \in \mathbb{R}$.

Solution 20. We can first find the maximum and minimum values. We use the first derivative test to determine where the critical points of $h_n(x)$ are.

$$h_n'(x) = \frac{n\cos nx}{\sqrt{n}} = \sqrt{n}\cos nx = 0 \Rightarrow \cos nx = 0 \Rightarrow nx = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{2n}, \frac{3\pi}{2n}$$

Therefore, at $x = \frac{\pi}{2n}$ and $x = \frac{3\pi}{2n}$, we have

$$h_n\left(\frac{\pi}{2n}\right) = \frac{\sin n\frac{\pi}{2n}}{\sqrt{n}} = \frac{\sin\frac{\pi}{2}}{\sqrt{n}} = \frac{1}{\sqrt{n}}$$

and

$$h_n\left(\frac{3\pi}{2n}\right) = \frac{\sin n\frac{3\pi}{2n}}{\sqrt{n}} = \frac{\sin\frac{3\pi}{2}}{\sqrt{n}} = -\frac{1}{\sqrt{n}}$$

Therefore, $|h_n(x)| \leq \frac{1}{\sqrt{n}}$, and as $n \to \infty$, $h_n(x) \to 0$.

To show that $(h_n')_n$ diverges, note that

$$\lim_{n\to\infty} \sqrt{n}\cos nx = \infty$$

so $(h_n')_n$ diverges for all $x \in \mathbb{R}$.

Question 21. *Let*

$$g_n(x) = \frac{nx+x^2}{2n}$$

and set $g(x) = \lim_{n\to\infty} g_n(x)$. Show that g is differentiable in two ways.

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$, and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives $(g_n')_n$ converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim_{n\to\infty} g_n'(x)$.

(c) *Repeat (a) and (b) for the sequence* $f_n(x) = \frac{nx^2+1}{2n+x}$.

Solution 21. (a) First, we find the pointwise convergence of $(g_n)_n$.

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{nx + x^2}{2n} = \lim_{n\to\infty} \frac{x + \frac{x^2}{n}}{2} = \frac{x}{2}$$

Then $g'(x) = \frac{1}{2}$.

(b) Since $g_n'(x) = \frac{n+2x}{2n}$, and since $|x| \le M$, let $\epsilon > 0$ be arbitrary. Then there exists an $N \in \mathbb{N}$ such that

$$\left|\frac{n+2x}{2n} - \frac{1}{2}\right| = \left|\frac{n+2x-n}{2n}\right| = \left|\frac{x}{n}\right| \le \frac{M}{n} < \frac{M}{\frac{M}{\epsilon}} = \epsilon$$

for all $n \ge N$ and $x \in [-M, M]$. Therefore, as $\epsilon > 0$ was arbitrary, $g_n'(x)$ converges to $g'(x)$.

(c) Using the method in part (a), we find the pointwise convergence of $(f_n)_n$.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx^2+1}{2n+x} = \lim_{n\to\infty} \frac{x^2 + \frac{1}{n}}{2 + \frac{x}{n}} = \frac{x^2}{2}$$

Then $f'(x) = x$.

Using the method in part (b), since

$$f_n'(x) = \frac{nx^2 + 4n^2x - 1}{(2n+x)^2}$$

and since $|x| \le M$, let $\epsilon > 0$ be arbitrary. Then there exists an $N \in \mathbb{N}$ such that

$$\left|\frac{nx^2 + 4n^2x - 1}{(2n+x)^2} - x\right| = \left|\frac{nx^2 + 4n^2x - 1 - x(2n+x)^2}{(2n+x)^2}\right| \le \frac{M^3 + 3nM^2 + 1}{4n^2 - 4Mn} < \epsilon$$

which approaches zero as $n \to \infty$ since x is independent, and so (f_n') converges uniformly over $[-M, M]$.

Question 22. *Prove an example or explain why the request is impossible. Take the domain of the functions to be all of* $\mathbb{R}$.

(a) *A sequence* $(f_n)_n$ *of nowhere differentiable functions with* $f_n \to f$ *uniformly and* f *is everywhere differentiable.*

(b) *A sequence* $(f_n)_n$ *of differentiable functions such that* $(f_n')_n$ *converges uniformly but the original sequence* $(f_n)_n$ *does not converge for any* $x \in \mathbb{R}$.

(c) *A sequence* $(f_n)_n$ *of differentiable functions such that both* $(f_n)_n$ *and* $(f_n')_n$ *converge uniformly but* $f = \lim_{n\to\infty} f_n$ *is not differentiable at some point.*

Solution 22. (a) True. Let $f_n(x) = \frac{g(x)}{n}$ where $g(x)$ is a bounded function. Then $(f_n(x))_n \to 0$ as $n \to \infty$.

(b) True. Let $f_n(x) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. Then $f_n'(x) = 0$ which converges uniformly to zero, but the original $(f_n)_n$ does not converge at all as the values are going between 0 and 1.

(c) False. If each f_n is differentiable at every point of $x \in \mathbb{R}$ then each f_n must be continuous for all $x \in \mathbb{R}$, and since $(f_n)_n$ and $(f_n')_n$ both converge uniformly, the pointwise limit must also exist.

Question 23. *Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.*

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $(g_n)_n$ converges uniformly to zero.

(b) If $0 \le f_n(x) \le g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Solution 23. (a) True. Applying Cauchy Criterion $n = m+1$, $|g_n(x)| < \epsilon$ for any $\epsilon > 0$ and so $g_n(x) \to 0$.

(b) True. By Cauchy Criterion,

$$\left|\sum_{k=m+1}^{n} f_k(x)\right| = \sum_{k=m+1}^{n} f_k(x) \le \sum_{k=m+1}^{n} g_k(x) = \left|\sum_{k=m+1} g_k(x)\right| < \epsilon$$

Question 24. *(a) Prove that*

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every $-1 \le x < 1$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Solution 24. (a) By the Weierstrass M-Test,

$$\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2} = M_n$$

and since $\sum_{n=1}^{\infty} M_n$ converges by the Weierstrass M-Test, $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ also converges.

(b) Let x_0 be fixed and consider the open interval $(-b, b) \subset [-1, 1)$ where $-b < |x_0| < b$. Then if $M_n = \frac{b^n}{n}$, $M_n > \frac{x_0^n}{n}$ in a neighborhood of x_0, hence f is continuous at x_0.

Question 25. *Let*

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?

Solution 25. (a) If $f_k(x) = \frac{\sin kx}{k^3}$, then

$$|f_k'(x)| = \left|\frac{\cos kx}{k^2}\right| \le \frac{1}{k^2}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, then by the Weierstrass M-Test, $\sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$ converges uniformly. Then as $k \to \infty$, we have that $(f_k') \to 0$, so by the differentiable limit theorem, $f(x)$ is differentiable and $f'(x) = \sum_{k=1}^{\infty} f_k'(x)$. Since $f_k'(x)$ converges uniformly, $f'(x)$ is continuous.

(b) Not twice differentiable. Taking the derivative again we have

$$|f_k''(x)| = \left|\frac{\sin kx}{k}\right| \leq \frac{1}{k}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the Weierstrass M-Test would not work.

Question 26. *Let*

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$$

(a) Show that h is a continuous function defined on all of $\mathbb{R}$.

(b) Is h differentiable. If so, is the derivative function h' continuous?

Solution 26. (a) Let $h_n(x) = \frac{1}{x^2+n^2}$. By the Weierstrass M-Test, since

$$\left|\frac{1}{x^2+n^2}\right| \leq \frac{1}{n^2} = M_n$$

and since $\sum_{n=1}^{\infty} M_n$ converges by the p-series test, it follows that $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly and hence h is continuous on $\mathbb{R}$.

(b) If $h_n(x) = \frac{1}{x^2+n^2}$, then

$$|h_n'(x)| = \left|\frac{2x}{(x^2+n^2)^2}\right| < \frac{2x}{n^4}$$

Then for $b > 0$, we have an interval $(-b, b)$ so that

$$|h_n'(x)| \leq \frac{2b}{n^4} = M_n$$

Hence, by the differentiable limit theorem, and the Weierstrass M-Test, h' is continuous and h differentiable.

Question 27. *Find suitable coefficients $(a_n)_n$ so that the resulting power series $\sum_{n=0}^{\infty} a_n x^n$ has the given properties, or explain why such a request is impossible.*

(a) Converges for every value $x \in \mathbb{R}$.

(b) Diverges for every value of $x \in \mathbb{R}$.

(c) Converges for all $x \in [-1, 1]$ and diverges off this set.

Solution 27. (a) $a_n = \frac{1}{n!}$, then $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is simply the power series for e^x that converges for every value of x.

(b) Impossible. $x = 0$ will always converge.

(c) $a_n = \frac{1}{n^2}$. The radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ is

$$R = \frac{1}{\limsup_{n\to\infty} \left|\frac{1}{n^2}\right|^{\frac{1}{n}}} = 1$$

So the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ is defined for $x \in [-1, 1]$ and undefined for anywhere else.

Question 28. *Previous work on the geometric series justifies the formula*

$$\frac{1}{1-x} = 1 + x + x^2 + x^2 + x^3 + x^4 + \cdots \qquad |x| < 1$$

Use the results about the power series to find the values for $\sum_{n=1}^{\infty} \frac{n}{2^n}$ *and* $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Solution 28. If $\sum_{n=0}^{\infty} a_n x^n$ for $a_n = 1$, $\sum_{n=0}^{\infty} x^n$. Then differentiating the series, we have

$$\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)^2}$$

Then

$$\sum_{n=1}^{\infty} nx^n = \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

For $x = \frac{1}{2}$, we have

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{(1-\frac{1}{2})^2} - \frac{1}{1-\frac{1}{2}} = 4 - 2 = 2$$

Similarly, for the second series, differentiating the series

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} nx^{n-1}\right)' &= \sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=1}^{\infty}(n+1)nx^{n-1} \\
&= \sum_{n=1}^{\infty} n^2x^{n-1} + \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)^2x^n + \sum_{n=0}^{\infty}(n+1)x^n \\
&= \sum_{n=0}^{\infty} n^2x^n + 2\sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} n^2x^n + 3\sum_{n=0}^{\infty} nx^n + 2\sum_{n=0}^{\infty} x^n = \frac{2}{(1-x)^3}
\end{aligned}$$

Then

$$\sum_{n=1}^{\infty} n^2x^n = \frac{2}{(1-x)^3} - 3\sum_{n=0}^{\infty} nx^n - 2\sum_{n=0}^{\infty} x^n$$

for $x = \frac{1}{2}$, we have

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{2}{(1-\frac{1}{2})^3} - 3 \times 2 - 2 \times 2 = 16 - 6 - 4 = 6$$

Question 29. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with* $a_n \neq 0$ *and assume*

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{(Ratio Test)}$$

exists.

(a) *Show that if* $L \neq 0$, *then the series converges for all* $x \in \left(-\frac{1}{L}, \frac{1}{L}\right)$.

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbb{R}$.

(c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} \sup_{k\geq n} \left| \frac{a_{k+1}}{a_k} \right|$$

Solution 29. (a) Since the ratio test converges whenever $L < 1$, let $b_n = a_n x^n$. Then

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|L < 1$$

Then

$$|x| < \frac{1}{L} \Rightarrow x \in \left(-\frac{1}{L}, \frac{1}{L}\right)$$

(b) When $L = 0$, we simply have $x \in \left(-\frac{1}{0}, \frac{1}{0}\right) \Rightarrow x \in (-\infty, \infty)$ and since $0 < 1$ for all x, the series converges still for $L = 0$.

(c) Since $\left(\sup_{k\geq n} \left|\frac{a_{k+1}}{a_k}\right|\right)_n$ converges to L', for every $\epsilon > 0$,

$$\left| \frac{a_{k+1}}{a_k} \right| < M = L' + \epsilon$$

once $k > N$ for some $N \in \mathbb{N}$. Then by the same logic as above, and the ratio test, the radius of convergence is still $\frac{1}{M}$, and since ϵ was arbitrary, this is effectively a radius of convergence of $\frac{1}{L}$.