Question 1. *Determine whether the following sequence of functions converges pointwise on $\mathbb{R}$ and if so, find the limit function $f_n(x) = \frac{x}{n+x}$ for $n \in \mathbb{N}$.*

Solution 1. We claim that the sequence of functions $f_n(x) = \frac{x}{n+x}$ converges pointwise to $f(x) = 0$ on $\mathbb{R}$. To see that this is true, note that

$$f(x) = \lim_{n\to\infty} \frac{x}{n+x} = \lim_{n\to\infty} \frac{\frac{x}{n}}{1+\frac{x}{n}} = 0$$

Hence, the pointwise convergence of $f_n(x)$ is $f(x) = 0$.

Question 2. *Determine whether the series of functions converges uniformly on $[0,1]$: $\sum_{n=1}^{\infty} \frac{x^n}{n}$.*

Solution 2. We claim that the series of functions $\sum_{n=1}^{\infty} \frac{x^n}{n}$ does not converge uniformly on $[0,1]$. To see that this is true, we will use the Weierstrass M-Test. If $f_n(x) = \frac{x^n}{n}$ for $x \in [0,1]$, then

$$\left|\frac{x^n}{n}\right| \leq \frac{1}{n} = M_n$$

Then since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series that diverges, then this implies that the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ does not converge uniformly on $x \in [0,1]$.

Question 3. *Show that the sequence of functions $f_n(x) = x^n$ converges pointwise to the function*

$$f(x) = \begin{cases} 0 & \textit{if } 0 \leq x < 1 \\ 1 & \textit{if } x = 1 \end{cases}$$

on $[0,1]$ but does not converge uniformly.

Solution 3. We want to show that $f_n(x) = x^n$ converges pointwise to the given $f(x)$. For $x = 0$, we have that for each $n \in \mathbb{N}$,

$$f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$$

for $x = 1$, for each $n \in \mathbb{N}$,

$$f(1) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1$$

and for $x \in (0,1)$, for each $n \in \mathbb{N}$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$$

(since x is small and raised to a large number will simply be very small). Therefore, we have that $f_n(x)$ converges pointwise to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

The convergence of $f_n(x)$ is not uniform since each $f_n(x)$ is continuous at $x = 1$ but $f(x)$ is not continuous at $x = 1$. Alternatively, if $x_n = \sqrt[n]{\frac{1}{2}}$ and $\epsilon = \frac{1}{4}$, then $x_n \in (0,1)$

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq |f_n(x_n) - f(x_n)| = \frac{1}{2} > \frac{1}{4} = \epsilon$$

Therefore, the convergence is not uniform.

Question 4. *Let $f_n(x) = \frac{x^n}{n^2}$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Use the Weierstrass M-Test to show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0,1]$.*

Solution 4. We will use the Weierstrass M-Test to show that the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $x \in [0,1]$. We have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2} = M_n$$

Here since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then we have that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[0,1]$.

Question 5. *Show that the sequence of functions $f_n(x) = \frac{nx}{1+n^2x^2}$ converges uniformly to the function $f(x) = 0$ on $[0,1]$.*

Solution 5. We want to show that the sequence of functions $f_n(x) = \frac{nx}{1+n^2x^2}$ converges uniformly to $f(x) = 0$ on $[0,1]$. Since

$$f(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^2} + x^2} = 0$$

Then we have that the pointwise limit is $f(x) = 0$ on $x \in [0,1]$. Now we want to show that the convergence is uniform. Let $\epsilon > 0$ be arbitrary. Since $f_n(x) \to 0$, there exists an $N \in \mathbb{N}$ such that

$$\left|\frac{nx}{1+n^2x^2}\right| \leq \left|\frac{nx}{n^2x^2}\right| = \frac{1}{nx}$$

Then as $n \to \infty$, we have that for $N > \frac{1}{\epsilon}$,

$$\left|\frac{nx}{1+n^2x^2}\right| \leq \frac{1}{nx} < \frac{\epsilon}{n} < \epsilon$$

Therefore, as $\epsilon > 0$ was arbitrary, $f_n(x) = \frac{nx}{1+n^2x^2}$ converges uniformly to $f(x) = 0$ on $[0,1]$.

Question 6. *Determine whether the following series of functions converges uniformly on $\mathbb{R}$:*

$$\sum_{n=1}^{\infty} \frac{1}{n(1+nx^2)}$$

Solution 6. To determine whether the series of functions converges uniformly on $\mathbb{R}$, we can use the Weierstrass M-Test to do so. Since

$$\left|\frac{1}{n(1+nx^2)}\right| \leq \frac{1}{n}$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows that $\sum_{n=1}^{\infty} \frac{1}{n(1+nx^2)}$ does not converges uniformly on $\mathbb{R}$.

Question 7. *Consider the series of functions $\sum_{n=1}^{\infty} \frac{x^n}{n!}$. Use the Weierstrass M-Test to show that the series converges uniformly on any bounded interval $[-a,a]$.*

Solution 7. Using the Weierstrass M-Test,

$$\left|\frac{x^n}{n!}\right| \leq \frac{a^n}{n!} = M_n$$

Since the series $\sum_{n=1}^{\infty} \frac{a^n}{n!} = e^a$ which is a finite number, so it converges for all $x \in [-a,a]$, it follows that $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$ converges uniformly on $x \in [-a,a]$ as well.

