Theorem 7.1. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. For any $L \in \mathbb{R}$, there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\sigma(n)} = L$*

Proof. For each $n \in \mathbb{N}$, let

$$x_n^+ = \begin{cases} x_n & \text{if } x_n \geq 0 \\ 0 & \text{if } x_n < 0 \end{cases} \qquad x_n^- = \begin{cases} 0 & \text{if } x_n \geq 0 \\ x_n & \text{if } x_n < 0 \end{cases}$$

Hence, for all $n \in \mathbb{N}$,

$$x_n = x_n^+ + x_n^- \qquad |x_n| = x_n^+ - x_n^-$$

If both $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ converged, then $\sum_{n=1}^{\infty} |x_n|$ would converge since

$$\sum_{k=1}^{N} |x_k| = \sum_{k=1}^{N} x_k^+ - \sum_{k=1}^{N} x_k^-$$

for all $N \geq 1$. However, since $\sum_{n=1}^{\infty} |x_n|$ does not converge, it must be the case that at least one of $\sum_{n=1}^{\infty} x_n^+$ diverges. Moreover, since $\sum_{n=1}^{\infty} x_n$ converges and

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{N} x_k^+ + \sum_{k=1}^{N} x_k^-$$

for all $N \in \mathbb{N}$, if one of $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ converged then both would need to converge thereby contradicting what was just demonstrated. Hence, both $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ diverges.

Let $(\alpha_n)_n$ denote the sequence of all non-negative terms from $(x_n)_n$ listed in the same order they appear, and let $(\beta_n)_n$ denote the sequence of all negative terms from $(x_n)_n$ listed in the order they appear in. Since

$$\sum_{n=1}^{\infty} \alpha_n \qquad \sum_{n=1}^{\infty} \beta_n$$

diverge by what was demonstrated above, then

$$\sup\left(\left\{\sum_{k=1}^{N} \alpha_k : N \in \mathbb{N}\right\}\right) = \infty \qquad \inf\left(\left\{\sum_{k=1}^{N} \beta_k : N \in \mathbb{N}\right\}\right) = -\infty$$

Fix $L \in \mathbb{R}$. To find a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that $\sum_{n=1}^{\infty} x_{\sigma(n)} = L$. Our goal is to add α_ks up to the point where we obtain a number just larger than L, then add (thereby decreasing the value) β_ks up to the point where we obtain a number just smaller than L, then add α_ks up to the point where we obtain a number just larger than L, then add (thereby decreasing the value) β_ks up to the point where we obtain a number just smaller than L, and so on. This procedure will create a rearrangement of the series so that the partial sums will be within x_n of L for some increasingly large n. Therefore, since $\sum_{n=1}^{\infty} x_n$ converges, so $\lim_{n\to\infty} x_n = 0$, so this rearrangement will converge to L.

First note since the above supremum is infinity that there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{N} \alpha_k > L$$

Choose $N_1 \in \mathbb{N}$ to be the *smallest* $N \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_k > L$. Therefore, since $\alpha_k \geq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=1}^{N_1} \alpha_k > L \geq \sum_{k=1}^{N} \alpha_k$$

for all $N < N_1$. Let

$$T_1 = \sum_{k=1}^{N_1} \alpha_k$$

and note $0 < T_1 - L$.

Next, since the above infimum is negative infinity, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{N} \beta_k < L - T_1$$

Choose $M_1 \in \mathbb{N}$ to be the smallest $N \in \mathbb{N}$ such that $\sum_{k=1}^{N} \beta_k < L - T_1$. Therefore since $\beta_k \leq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=1}^{M_1} < L - T_1 \leq \sum_{k=1}^{N} \beta_k$$

for all $N < M_1$. Let

$$R_1 = \sum_{k=1}^{M_1} \beta_k$$

Notice that the above inequalities imply that

$$0 < L - T_1 - R_1 = (L - T_1) - R_1 \leq \sum_{k=1}^{M_1 - 1} \beta_k - \sum_{k=1}^{M_1} \beta_k = -\beta_{M_1}$$

Next, since the above supremum is infinity, there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=N_1+1}^{N} \alpha_k > L - T_1 - R_1$$

Choose $N_2 \in \mathbb{N}$ to be the smallest $N \in \mathbb{N}$ such that $N > N_1$ and $\sum_{k=N_1+1}^{N} \alpha_k > L - T_1 - R_1$. Therefore, since $\alpha_k \geq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=N_1+1}^{N_2} \alpha_k > L - T_1 - R_1 \geq \sum_{k=N_1+1}^{N} \alpha_k$$

for all $N \in \{N_1 + 1, ..., N_2 - 1\}$. Let

$$T_2 = \sum_{k=N_1+1}^{N_2} \alpha_k$$

Notice that the above inequalities and the fact that $\alpha_k \geq 0$ for all $k \in \mathbb{N}$ imply that

$$0 \leq L - T_1 - R_1 - \sum_{k=N_1+1}^{N} \alpha_k \leq L - T_1 - R_1 \leq \beta_{M_1}$$

for all $N \in \{N_1 + 1, ..., N_2 - 1\}$ and

$$0 < T_2 - (L - T_1 - R_1) \leq \sum_{k=N_1+1}^{N_2} \alpha_k - \sum_{k=N_1+1}^{N_2 - 1} \alpha_k = \alpha_{N_2}$$

Once more for clarity, since $T_1 + R_1 + T_2 - L > 0$ and the above infimum is negative infinity, there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=M_1+1}^{N} \beta_k < L - T_1 - R_1 - T_2$$

Choose $M_2 \in \mathbb{N}$ to be the smallest $N \in \mathbb{N}$ such that $N > M_1$ and $\sum_{k=M_1+1}^{N} \beta_k < L - T_1 - R_1 - T_2$. Therefore, since $\beta_k \leq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=M_1+1}^{M_2} \beta_k < L - T_1 - R_1 - T_2 \leq \sum_{k=M_1+1}^{N} \beta_k$$

for all $N \in \{M_1 + 1, ..., M_2 - 1\}$. Let

$$R_2 = \sum_{k=M_1+1}^{M_2} \beta_k$$

Notice that the above inequalities and the fact that $\beta_k \leq 0$ for all $k \in \mathbb{N}$ imply that

$$0 \geq L - T_1 - R_1 - T_2 - \sum_{k=M_1+1}^{N} \beta_k \geq L - T_1 - R_1 - T_2 > -\alpha_{N_2}$$

for all $N \in \{M_1 + 1, ..., M_2 - 1\}$ and

$$0 < (L - T_1 - R_1 - T_2) - R_2 \leq \sum_{k=M_1+1}^{M_2-1} \beta_k - \sum_{k=M_1+1}^{M_2} \beta_k = -\beta_{M_2}$$

By repeating the procedure, there exist strictly increasing sequences $(N_j)_j$ and $(M_j)_j$ so that if

$$T_j = \sum_{k-N_{j-1}+1}^{N_j} \alpha_k \qquad R_j = \sum_{k=M_{j-1}+1}^{M_j} \beta_k$$

then for all $\ell \geq 1$, we have that

$$0 \leq L - \sum_{j=1}^{\ell} (T_j + R_j) - \sum_{k=N_j+1}^{N} \alpha_k \leq -\beta_{M_\ell}$$

for all $N \in \{N_\ell + 1, ..., N_{\ell+1} - 1\}$,

$$0 < -L + T_{\ell+1} + \sum_{j=1}^{\ell} (T_j + R_j) \leq \alpha_{N_{\ell+1}}$$

and

$$0 \geq L - T_{\ell+1} - \sum_{j=1}^{\ell} (T_j + R_j) - \sum_{k=M_j+1}^{N} \beta_k > -\alpha_{N_{\ell+1}}$$

for all $N \in \{M_\ell + 1, ..., M_{\ell+1} - 1\}$ and

$$0 < L - \sum_{j=1}^{\ell+1} (T_j + R_j) \leq -\beta_{M_{\ell+1}}$$

Since $(N_j)_j$ and $(M_j)_j$ are strictly increasing functions, we see that

$$\alpha_1, ..., \alpha_{N_1}, \beta_1, ..., \beta_M, \alpha_{N_1+1}, ..., \alpha_{N_2}, \beta_{M_1+1}, ..., \beta_{M_2}, \alpha_{N_2+1}, ..., \alpha_{N_3}, ..., \beta_{M_2+1}, ..., \beta_{M_3}, ...$$

is a rearrangement of $\sum_{n=1}^{\infty} x_n$. Moreover by construction, the partial sums of this rearrangement are within either α_{N_ℓ} or $-\beta_{M_\ell}$ of L for progressively large ℓ at every step fo the construction. Since $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$. Therefore, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, so the partial sums of this rearrangement converge to L as desired. □

We are done this chapter, no need to come back to this.

Series of Functions: Continuity of Complex-Valued Functions

Definition 7.2. Let $U \subseteq \mathbb{C}$. A function $f : U \to \mathbb{C}$ is said to be continuous at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $z \in \Omega$ and $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \epsilon$. Moreover, f is said to be continuous on U if f is continuous at every point in U.

Lemma 7.3. *Let $U \subseteq \mathbb{C}$, let $z_0 \in U$ and let $f : U \to \mathbb{C}$. Then f is continuous at z_0 if and only if whenever $(z_n)_n$ is a sequence in U that converges to z_0, we have $\lim_{n\to\infty} f(z_n) = f(z_0)$.*

Proof. ($\Rightarrow$) Suppose f is continuous at z_0. Let $(z_n)_n$ be a sequence in U that converge to z_0. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $z \in U$ and $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \epsilon$. Since $(z_n)_n$ converges to z_0, there exists an $N \in \mathbb{N}$ such that $|z_n - z_0| < \delta$ for all $n \geq N$. Hence, for all $n \geq N$, we have that $|z_n - z_0| < \delta$ so $|f(z_n) - f(z_0)| < \epsilon$. Therefore, $\lim_{n\to\infty} f(z_n) = f(z_0)$.

($\Leftarrow$) Suppose that f is not continuous at z_0. Therefore, there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists a $z \in U$ such that $|z - z_0| < \delta$ but $|f(z) - f(z_0)| \geq \epsilon_0$. Hence, for all $n \in \mathbb{N}$, there exists a $z_n \in U$ such that $|z_n - z_0| < \frac{1}{n}$ but $|f(z_n) - f(z_0)| \geq \epsilon_0$. Thus, $(z_n)_n$ is a sequence in U that converges to z_0 such that $(f(z_n))_n$ does not converge to $f(z_0)$ since $|f(z_n) - f(z_0)| \geq \epsilon_0$ for all $n \in \mathbb{N}$, which fails for $\epsilon = \epsilon_0$. □

Proposition 7.4. *Let $U \subseteq \mathbb{C}$, let $z_0 \in U$, let $f : U \to \mathbb{C}$ and let $g : \mathbb{C} \to \mathbb{C}$. If f is continuous at z_0 and g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0.*

Proof. Let $(z_n)_n$ be an arbitrary sequence in U that converges to z_0. Since f is continuous at z_0 and $(z_n)_n$ converges to z_0, $(f_n(z_0))_n$ converges to $f(z_0)$ by Lemma 7.3. Similarly, since g is continuous at $f(z_0)$ and $(f_n(z_0))_n$ converges to $f(z_0)$, $(g(f(z_n)))_n$ converges to $g(f(z_0))$ by Lemma 7.3. Therefore, since $(z_n)_n$ was arbitrary, Lemma 7.3 implies that $g \circ f$ is continuous at z_0. □