Lecture 3 Exercise

Corollary 3.1: Let $\{x_n\}_n$ and $\{y_n\}_n$ be sequences of real numbers such that there exists an $N \in \mathbb{N}$ with $x_n \leq y_n$ for all $n \geq N$, then

(i) Suppose $\{x_n\}_n$ is an increasing sequence

(1) If $\{y_n\}_n$ converges, then $\{x_n\}_n$ converges.

(2) If $\{x_n\}_n$ diverges, then $\{y_n\}_n$ diverges.

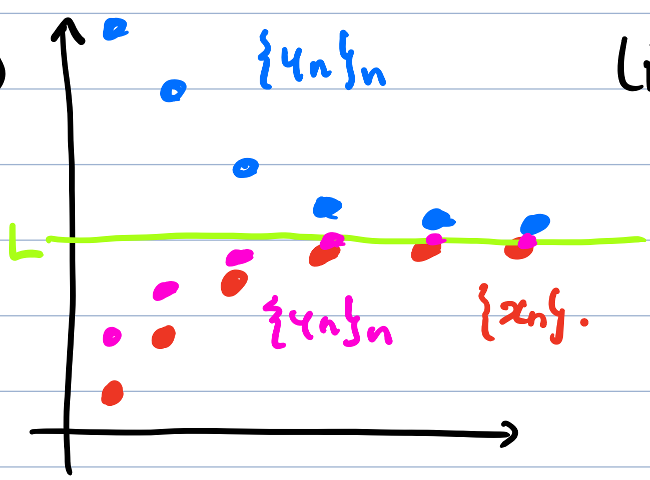
(ii) Suppose $\{y_n\}_n$ is a decreasing sequence.

(1) If $\{x_n\}_n$ converges, then $\{y_n\}_n$ converges.

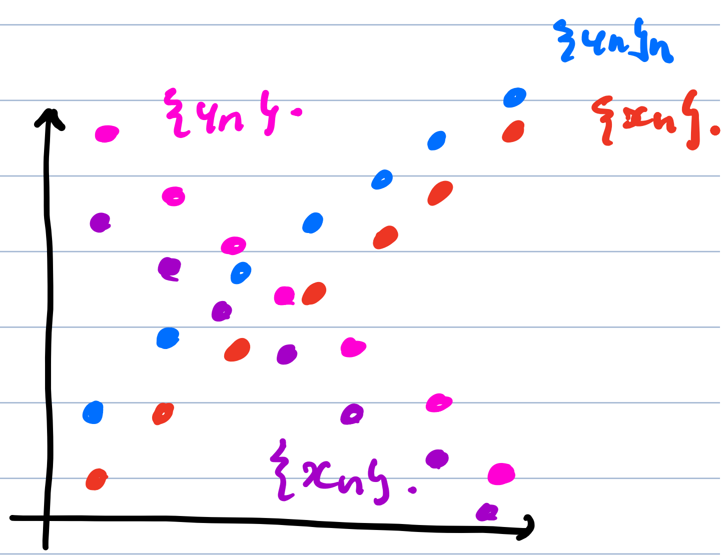
(2) If $\{y_n\}_n$ diverges, then $\{x_n\}_n$ diverges.

Proof by Picture:

(i)



(ii)



Proof: (Only proving (i) as the proof is the exact same).

(i) Suppose $\{x_n\}_n$ is an increasing sequence. Suppose $\{y_n\}_n$ is a convergent sequence. If we assume that $\{y_n\}_n$ is increasing, convergent, sequence, then there exists a $U \in \mathbb{R}$ such that

$$U = \sup(\{y_n : n \in \mathbb{N}\}) < \infty.$$

Let $\varepsilon > 0$ be arbitrary. Then there exists a $N \in \mathbb{N}$ such that

$$U - \varepsilon \leq y_N \leq U$$

Then for all $n \geq N$, we have that

$$U - \varepsilon \leq y_N \leq y_n \leq U \leq U + \varepsilon.$$

Since $x_n \leq y_n$ and $\{x_n\}_n$ is increasing for all $n \geq N$,

$$U - \varepsilon \leq x_n \leq y_n \leq U \leq U + \varepsilon.$$

Which implies that $|x_n - U| < \varepsilon$.

Hence, $\{x_n\}_n$ is also a convergent sequence

Now suppose $\{y_n\}_n$ is a decreasing, convergent sequence. Then there exists a $L \in \mathbb{R}$ such that

$$L = \inf(\{y_n : n \in \mathbb{N}\}) < \infty$$

Then let $\varepsilon > 0$. There exists a $N \in \mathbb{N}$ such that

$$L \leq y_N \leq L + \varepsilon.$$

Then for all $n \geq N$, we have that

$$L \leq y_n \leq y_N \leq L + \varepsilon.$$

Since $x_n \leq y_n$ and $\{x_n\}_n$ is an increasing sequence,

$$L - \varepsilon \leq L \leq x_n \leq y_n \leq L + \varepsilon$$

which implies that $|x_n - L| < \varepsilon$. Hence, the sequence $\{x_n\}_n$ is also convergent.

(2) Immediate by contrapositive. $\square$.