Question 1. *Let* $f_n(x) = \frac{1}{1+n^2x^2}$ *for* $x \in [0,1]$. *Show that* $\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \int_0^1 \lim_{n\to\infty} f_n(x)\ dx$, *but the convergence is not uniform.*

Solution 1. First, on the left hand side, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \lim_{n\to\infty} \int_0^1 \frac{1}{1+(nx)^2}\ dx = \lim_{n\to\infty} \frac{\arctan(n)}{n} = 0$$

On the right hand side, we have

$$\int_0^1 \lim_{n\to\infty} f_n(x)\ dx = \int_0^1 \lim_{n\to\infty} \frac{1}{1+n^2x^2}\ dx = \int_0^1 0\ dx = 0$$

We next claim that the convergence is not uniform. To see that this is true suppose $\epsilon = \frac{1}{4}$ and $x_n = \frac{1}{n}$. Then for all $x_n \in [0,1]$,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq \left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = \left|\frac{1}{1+n^2\frac{1}{n^2}} - 0\right| = \left|\frac{1}{2} - 0\right| = \frac{1}{2} \geq \epsilon$$

Therefore, with the ϵ chosen, we have shown that $f_n(x)$ does not converge uniformly on $[0,1]$.

Question 2. *Let* $f_n(x) = nxe^{-nx}$ *for* $x \in [0,1]$. *Determine whether the sequence* $(f_n)_n$ *converges uniformly on* $[0,1]$ *and if so, compute* $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

Solution 2. First, we claim that the sequence of functions defined by $f_n(x) = nxe^{-nx}$ does not converge uniformly on $[0,1]$. To see that this is true, we first compute the pointwise limit f. For $x = 0$,

$$f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$$

For $x = 1$,

$$f(1) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} ne^{-n} = 0$$

and for $x \in (0,1)$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx} = 0$$

Therefore, the pointwise limit function of $f_n(x)$ is $f(x) = 0$. Next, to show that $f_n(x)$ does not converge uniformly on $[0,1]$, let $\epsilon = \frac{1}{4}$ and $x_n = \frac{1}{n}$, then observe that $x_n \in (0,1)$ and

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq \left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = \left| n\frac{1}{n}e^{-n\frac{1}{n}} - 0\right| = e^{-1} \geq \epsilon$$

Therefore, given our choice of $\epsilon = \frac{1}{4}$, we have shown that the sequence of functions $f_n(x)$ does not converge uniformly on $[0,1]$.

Question 3. *Let* $f_n(x) = \frac{x}{1+nx^2}$ *for* $x \in [0,1]$. *Determine whether the sequence* $(f_n(x))$ *converges uniformly on* $[0,1]$, *and if so, compute* $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

Solution 3. First, we claim that the sequece of functions $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly on $[0,1]$. To see that this is true, we first compute the pointwise limit f. For $x = 0$,

$$f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$$

For $x = 1$,

$$f(1) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{1}{1+n} = 0$$

for $x \in (0,1)$, since $x < 1 + nx^2$, then

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0$$

So we have that the limit function is $f(x) = 0$. Next, to show that the convergence is uniform, note that since $f_n(x)$ is a continuous function on $[0,1]$, and $f_n(x) \to 0$ on $[0,1]$, and since $f(x) = 0$ is continuous on $[0,1]$ we have that $f_n(x)$ converges uniformly on $[0,1]$. So we compute the limit.

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \int_0^1 \frac{1}{1+nx^2}\ dx = \lim_{n\to\infty} \frac{\ln|n+1|}{2n}$$

Question 4. *Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ for $x \in [0,\pi]$. Show that $\lim_{n\to\infty} \int_0^\pi f_n(x)dx = 0$ and the convergence is uniform.*

Solution 4. First, we show that the integral is zero.

$$\lim_{n\to\infty} \int_0^\pi f_n(x)dx = \lim_{n\to\infty} \int_0^\pi \frac{\sin nx}{\sqrt{n}}\ dx = \lim_{n\to\infty} \int_0^\pi \frac{-\cos nx}{n^{\frac{3}{2}}}\ dx = \lim_{n\to\infty} \frac{1-\cos nx}{n^{\frac{3}{2}}} = 0$$

Next to show that the sequence of functions is uniform, we will use Cauchy Criterion for Uniform Convergence. For $n, m \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| = \left|\frac{\sin nx}{\sqrt{n}} - \frac{\sin mx}{\sqrt{m}}\right| \le \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right| \le \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}} \to 0, n, m \to \infty$$

so we can make $\epsilon > 0$ to be anything we want and choose an $N \in \mathbb{N}$ such that the above inequality is true. Therefore, we can then conclude that

$$\sup_{x\in[0,\pi]} |f_n(x) - f(x)| < \epsilon$$

Hence, the convergence is uniform.

Question 5. *Let $f_n(x) = \frac{x^n}{n}$ for $x \in [0,1]$. Determine whether the sequence $(f_n)_n$ converges uniformly on $[0,1]$, and if so, compute $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.*

Solution 5. We claim that the sequence of functions $f_n(x) = \frac{x^n}{n}$ does not converge uniformly on $[0,1]$. Take any $x \in [0,1]$, then the pointwise limit is

$$f(x) = \lim_{n\to\infty} \frac{x^n}{n} = 0$$

for all $n \in \mathbb{N}$. Then to show tht the convergence is not uniform, take $x_n = \sqrt[n]{n}$ and $\epsilon = \frac{1}{4}$, then note that $x_n \in (0,1)$ and

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \ge \left|f_n\left(\sqrt[n]{n}\right) - f\left(\sqrt[n]{n}\right)\right| = \left|\frac{n}{n} - 0\right| = 1 > \epsilon$$

Therefore, given our choice of ϵ, we have shown that the sequence of functions does not converge uniformly on $[0,1]$.

