1 Sequences of Complex Numbers

Definition 1.1 (Sequence of Complex Numbers)**.** A sequence of $\mathbb{K}$ numbers is a function $z : \mathbb{N} \to \mathbb{C}$, and we denote the sequence by $(z_n)_n = z(n)$.

Definition 1.2 (Convergence of a Sequence)**.** A sequence $(z_n)_n$ of complex numbers is said to *converge* to a complex number L if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N$. The complex number L is called a limit of $(z_n)_n$ and is denoted by $\lim_{n\to\infty} z_n = L$. Moreover, the sequence of $(z_n)_n$ is said to diverge if $(z_n)_n$ is said to *diverge* if $(z_n)_n$ does not converge to any complex number.

Lemma 1.3 (Conditions for Convergence of a Sequence)**.** *Let $(z_n)_n$ be a sequence of complex numbers and let $L \in \mathbb{C}$. Then $(z_n)_n$ converges to L if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N$.*

Lemma 1.4. *Let $(z_n)_n$ be a sequence of complex numbers and let $L \in \mathbb{C}$. Then $(z_n)_n$ converges to L if and only if $(\Re(z_n))_n$ and $(\Im(z_n))_n$ converges to $\Re(L)$ and $\Im(L)$ respectively.*

Corollary 1.5 (Uniqueness of a Limit)**.** *Let $L, K \in \mathbb{C}$ and let $(z_n)_n$ be a sequence of complex numbers. If L and K are limits of $(z_n)_n$, then $L = K$.*

Corollary 1.6 (Convergent Sequences are Bounded)**.** *Let $(z_n)_n$ be a sequence of complex numbers, and let $L \in \mathbb{C}$. If $(z_n)_n$ is a convergent sequence that converges to L, then $(z_n)_n$ is bounded.*

Corollary 1.7 (Algebraic Limit Theorems for Sequences)**.** *Let $L, K \in \mathbb{C}$ and let $(z_n)_n$ and $(w_n)_n$ be sequences of complex numbers that converge to L and K respectively. Then the following are true.*

1. *$(z_n + w_n)_n$ converges to $L + K$.*
2. *$(z_n w_n)$ converges to LK.*
3. *$(\alpha z_n)_n$ converges to αL.*
4. *If $L \neq 0$ and $z_n \neq 0$ for all $n \in \mathbb{N}$, then $\left(\frac{1}{z_n}\right)_n$ converges to $\frac{1}{L}$.*
5. *If $K \neq 0$ and $w_n \neq 0$ for all $n \in \mathbb{N}$, then $\left(\frac{z_n}{w_n}\right)_n$ converges to $\frac{L}{K}$.*
6. *$(\bar{z}_n)_n$ converges to $\bar{L}$.*
7. *$(|z_n|)_n$ converges to $|L|$.*

Definition 1.8 (Cauchy Sequences)**.** A sequence $(z_n)_n$ of complex numbers is said to be *Cauchy* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $n, m \geq N$.

Theorem 1.9 (Cauchy Criterion for Sequences)**.** *A sequence of complex numbers converges if and only if it is Cauchy.*

Definition 1.10 (Increasing and Decreasing Sequences)**.** A sequence of real numbers $(x_n)_n$ is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence of real numbers is called monotone if it is either increasing or decreasing for all $n \in \mathbb{N}$.

Theorem 1.11 (Monotone Convergence Theorem)**.** *Let $(x_n)_n$ be a sequence of real numbers. If $(x_n)_n$ is monotone and bounded, then $(x_n)_n$ converges to L.*

Corollary 1.12. *Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers such that there exists an $N \in \mathbb{N}$ with $x_n \leq y_n$ for all $n \geq N$.*

1. *If $(x_n)_n$ is an increasing sequence, and if $(y_n)_n$ converges, then $(x_n)_n$ converges.*
2. *If $(x_n)_n$ is an increasing sequence, and if $(x_n)_n$ diverges, then $(y_n)_n$ diverges.*
3. *If $(y_n)_n$ is a decreasing sequence, and if $(x_n)_n$ converges, then $(y_n)_n$ converges.*
4. *If $(y_n)_n$ is a decreasing sequence, and if $(y_n)_n$ diverges, then $(x_n)_n$ diverges.*

Theorem 1.13 (Order Limit Theorem)**.** *Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers, and let $L, K \in \mathbb{R}$ such that $(x_n)_n$ and $(y_n)_n$ converges to L and K respectively. If there exists an $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $L \leq K$.*

Theorem 1.14 (Squeeze Theorem)**.** *Let $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ be sequences of real numbers. If there exists an $N \in \mathbb{N}$ such that $x_n \leq y_n \leq z_n$ and $\lim_{n\to\infty} x_n = L = \lim_{n\to\infty} z_n$ for all $n \geq N$, then $\lim_{n\to\infty} y_n = L$.*

2 Series of Complex Numbers

Definition 2.1 (Partial Sums and Series)**.** Let $(z_n)_n$ be a sequence of complex numbers.

1. The sequence of partial sums associated to $(z_n)_n$ is a sequence $(S_N)_N$ defined by

$$S_N = \sum_{k=1}^{N} z_k = z_1 + z_2 + z_3 + \cdots z_N$$

2. The (infinite) series of complex numbers associated to the sequence $(z_n)_n$ is defined by

$$L = \lim_{n\to\infty} S_n$$

Definition 2.2 (Convergence of Series of Complex Numbers)**.** Let $(z_n)_n$ be a sequence of complex numbers and for each $N \in \mathbb{N}$, let $S_N = \sum_{k=1}^{N} z_k$ be the partial sums associated to $(z_n)_n$. The series $\sum_{n=1}^{\infty} z_n$ is said to converge to $L \in \mathbb{C}$ if the sequence $(S_N)_N$ converges to L. The series $\sum_{n=1}^{\infty} z_n$ is said to diverge if the sequence $(S_N)_N$ diverges.

Lemma 2.3 (Algebraic Limit Theorems For Series)**.** *Let $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ be convergent series of complex numbers. Then $\sum_{n=1}^{\infty} z_n + w_n$ converges and*

$$\sum_{n=1}^{\infty} z_n + w_n = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$$

Moreover, for all $\alpha \in \mathbb{C}$, *the series* $\sum_{n=1}^{\infty} \alpha z_n$ *converges and*

$$\sum_{n=1}^{\infty} \alpha z_n = \alpha \sum_{n=1}^{\infty} z_n$$

Theorem 2.4 (Cauchy Criterion for Series). *A series* $\sum_{n=1}^{\infty}$ *converges if and only if for all* $\epsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that*

$$\left| \sum_{k=N}^{m} z_k \right| < \epsilon$$

for all $m \geq N$.

Corollary 2.5 (Property of Convergence). *If a series* $\sum_{n=1}^{\infty} z_n$ *of complex numbers converge, then* $\lim_{n\to\infty} z_n = 0$.

Theorem 2.6 (Convergence of Series of Nonnegative Real Numbers). *Let* $(x_n)_n$ *be a sequence of real numbers with* $x_n \geq 0$ *for all* $n \in \mathbb{N}$.

1. *The series* $\sum_{n=1}^{\infty} x_n$ *converges if and only if there exists an* $M \in \mathbb{R}$ *such that* $\sum_{k=1}^{N} x_k \leq M$ *for all* $N \in \mathbb{N}$.
2. *If* $\sum_{k=1}^{N} x_k \leq M$ *for all* $N \in \mathbb{N}$, *then* $\sum_{n=1}^{\infty} x_k \leq M$.
3. *If* $\sum_{n=1}^{\infty} x_n$ *converges, then for all* $\epsilon > 0$, *there exists a* $N_0 \in \mathbb{N}$ *such that* $\sum_{k=N}^{\infty} x_k < \epsilon$ *for all* $N > N_0$.

Definition 2.7 (Absolutely Convergent Series). A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge *absolutely* if $\sum_{n=1}^{\infty} |z_n|$ converges.

Theorem 2.8 (Criterion for Absolutely Convergent Series). *If* $\sum_{n=1}^{\infty} z_n$ *is an absolutely convergent series of complex numbers, then* $\sum_{n=1}^{\infty} z_n$ *converges. Moreover,*

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|$$

Theorem 2.9 (Comparison Test). *Let* $(x_n)_n$ *and* $(y_n)_n$ *be sequences of real numbers such that* $0 \leq x_n \leq y_n$ *for all* $n \in \mathbb{N}$. *Then*

1. *If* $\sum_{n=1}^{\infty} y_n$ *converges, then* $\sum_{n=1}^{\infty} x_n$ *converges.*
2. *If* $\sum_{n=1}^{\infty} x_n$ *diverges, then* $\sum_{n=1}^{\infty} y_n$ *diverges.*

Theorem 2.10 (Integral Test). *If* $f : [1, \infty) \to [0, \infty)$ *be a non-increasing function and* $x_n = f(n)$ *for all* $n \in \mathbb{N}$, *then* $\sum_{n=1}^{\infty} x_n$ *converges if and only if* $\int_1^{\infty} f(x)dx$ *converges.*

Corollary 2.11 (p-Series Test). *The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$.

Theorem 2.12 (Ratio Test). *Let* $(x_n)_n$ *be a sequence of real numbers such that* $x_n > 0$ *for all* $n \in \mathbb{N}$. *Suppose for* $r > 0$, $r = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ *exists. Then*

(a) $\sum_{n=1}^{\infty} x_n$ *converges if* $0 < r < 1$

(b) $\sum_{n=1}^{\infty} x_n$ *diverges if* $r > 1$.

Theorem 2.13 (Root Test)**.** *Let* $(x_n)_n$ *be a sequence of real numbers such that* $x_n > 0$ *for all* $n \in \mathbb{N}$. *Suppose for* $r > 0$, $r = \lim_{n\to\infty} \sqrt[n]{x_n}$ *exists. Then*

(a) $\sum_{n=1}^{\infty} x_n$ *converges if* $0 < r < 1$

(b) $\sum_{n=1}^{\infty} x_n$ *diverges if* $r > 1$.

Theorem 2.14 (Leibniz's Theorem)**.** *Let* $(x_n)_n$ *be a non-increasing sequence of non-negative real numbers such that* $\lim_{n\to\infty} x_n = 0$. *Then* $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ *converges.*

Definition 2.15 (Conditionally Convergent Series)**.** A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge *conditionally* if $\sum_{n=1}^{\infty} z_n$ converges but does not converge absolutely.

3 Rearrangements of Series

Definition 3.1 (Rearrangements of Series)**.** A series $\sum_{n=1}^{\infty} w_n$ is said to be a rearrangement of the series $\sum_{n=1}^{\infty} z_n$ if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $w_n = z_{\sigma(n)}$ for all $n \in \mathbb{N}$.

Theorem 3.2. *Let* $\sum_{n=1}^{\infty} x_n$ *be a conditionally convergent series of real numbers. For any* $L \in \mathbb{R}$, *there exists a bijection* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{n=1}^{\infty} x_{\sigma(n)} = L$

Theorem 3.3. *Let* $\sum_{n=1}^{\infty} z_n$ *be an absolutely convergent series of complex numbers. For all bijections* $\sigma : \mathbb{N} \to \mathbb{N}$, *the series* $\sum_{n=1}^{\infty} z_{\sigma(n)}$ *converges absolutely* $\sum_{n=1}^{\infty} z_{\sigma(n)} = \sum_{n=1}^{\infty} z_n$.

4 Continuity of Complex-Valued Functions

Definition 4.1 (Continuity at a Point)**.** Let $U \subseteq \mathbb{C}$. A function $f : U \to \mathbb{C}$ is said to be *continuous at a point* $z_0 \in U$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $z \in U$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$. Moreover, f is said to be *continuous on* U if f is continuous at every point in U.

Lemma 4.2. *Let* $U \subseteq \mathbb{C}$, *let* $z_0 \in U$, *and let* $f : U \to \mathbb{C}$. *Then* f *is continuous at* z_0 *if and only if whenever* $(z_n)_n$ *is a sequence in* U *that converge to* z_0, *we have* $\lim_{n\to\infty} f(z_n) = f(z_0)$.

Proposition 4.3. *Let* $U, V \subset \mathbb{C}$, *let* $z_0 \in U$, *let* $f : U \to V$ *and let* $g : V \to \mathbb{C}$. *If* f *is continuous at* z_0 *and* g *is continuous at* $f(z_0)$, *then* $g \circ f$ *is continuous at* z_0.

Lemma 4.4. *Let* $U \subseteq \mathbb{C}$ *and let* $f : U \to \mathbb{C}$ *and* $g : U \to \mathbb{C}$ *be continuous functions. Then the following are true.*

1. *The function* $f + g : U \to \mathbb{C}$ *defined by* $(f + g)(z) = f(z) + g(z)$ *for all* $z \in U$ *is continuous.*
2. *The function* $fg : U \to \mathbb{C}$ *defined by* $(fg)(z) = f(z)g(z)$ *for all* $z \in U$ *is continuous.*
3. *For all* $\alpha \in \mathbb{C}$, *the function* $\alpha f : U \to \mathbb{C}$ *defined by* $(\alpha f)(z) = \alpha f(z)$.

4. *If $f(z) \neq 0$ for all $z \in U$, the function $\frac{1}{f} : U \to \mathbb{C}$ defined by $\left(\frac{1}{f}\right)(z) = \frac{1}{f(z)}$ for all $z \in U$ is continuous.*
5. *The function $\bar{f} : U \to \mathbb{C}$ defined by $\bar{f}(z) = \overline{f(z)}$ for all $z \in U$ is continuous.*
6. *The function $|f| : U \to \mathbb{C}$ defined by $|f|(z) = |f(z)|$ for all $z \in U$ is continuous.*

Definition 4.5 (Real and Imaginary Functions)**.** Let $U \subseteq \mathbb{C}$ and let $f : U \to \mathbb{C}$. The real and imaginary parts of f are the functions $\Re(f)$, $\Im(f) : U \to \mathbb{R}$ respectively where

$$(\Re(f))(z) = \Re(f(z)) = \frac{f(z) + \overline{f(z)}}{2} \qquad (\Im(f))(z) = \Im(f(z)) = \frac{f(z) - \overline{f(z)}}{2i}$$

for all $z \in U$.

Lemma 4.6. *Let $U \subset \mathbb{C}$ and let $f : U \to \mathbb{C}$. Then f is continuous if and only if $\Re(f)$ and $\Im(f)$ are continuous real-valued functions.*

5 Continuity of Sequences and Series of Functions

Definition 5.1 (Pointwise Convergence)**.** Let $U \subseteq \mathbb{C}$. A sequence $(f_n)_n$ of complex-valued functions on U is said to *converge pointwise* on U to $f : U \to \mathbb{C}$ if

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for all $z \in U$. The definition of the limit means that for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon$$

for all $n \geq N$.

Definition 5.2 (Pointwise Convergence of Series of Functions)**.** Let $U \subseteq \mathbb{C}$ and for each $n \in \mathbb{N}$, let $f_n : U \to \mathbb{C}$. The series $\sum_{n=1}^{\infty} f_n$ is said to *converge pointwise* on U if $\sum_{n=1}^{\infty} f_n(z)$ converges for each $z \in U$. Moreover, the function $f : U \to \mathbb{C}$ defined by

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

for all $z \in U$ is called the *(pointwise) sum of* $(f_n)_n$ and is denoted by $\sum_{n=1}^{\infty} f_n$.

Definition 5.3 (Uniform Convergence)**.** Let $U \subseteq \mathbb{C}$. A sequence $(f_n)_n$ of complex-valued functions on U is said to *converge uniformly* on U to $f : U \to \mathbb{C}$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$ for all $z \in U$ whenever $n \geq N$. That is,

$$\sup_{z\in U} |f_n(z) - f(z)| < \epsilon$$

Theorem 5.4. *Let $U \subseteq \mathbb{C}$, let $z_0 \in U$, and let $(f_n)_n$ be a sequence of complex-valued functions on U that converge uniformly on U to $f : U \to \mathbb{C}$. If each f_n is continuous at z_0, then f is continuous at z_0. Consequently, a uniform limit of continuous functions is continuous.*

Definition 5.5 (Uniform Bounded Functions)**.** Let $U \subseteq \mathbb{C}$ and let $(f_n)_n$ be a sequence of complex-valued functions on U. Then $(f_n)_n$ is uniformly bounded on U if there exists an M such that

$$|f_n(z)| \leq M$$

for all $z \in U$ and $n \in \mathbb{N}$.

Proposition 5.6. *Let I be a bounded closed interval on $\mathbb{R}$ and let $(f_n)_n$ be a sequence of complex-valued continuous functions on I that converge uniformly on I to $f : I \to \mathbb{C}$. Then $(f_n)_n$ is uniformly bounded.*

Proposition 5.7. *Let I be a bounded closed interval in $\mathbb{R}$ and let $(f_n)_n$ and $(g_n)_n$ be sequences of complex-valued functions on I that converge uniformly on I to $f : I \to \mathbb{C}$ and $g : I \to \mathbb{C}$ respectively. Then the following are true.*

1. *$(f_n + g_n)_n$ converges uniformly to $f + g$ on I.*
2. *$(f_n g_n)_n$ converges uniformly to fg on I.*
3. *If $(a_n)_n$ is a sequence of complex numbers that converges to $\alpha \in \mathbb{C}$, then $(\alpha_n f_n)_n$ converges uniformly to αf on I.*
4. *$(\bar{f}_n)_n$ converges uniformly to $\bar{f}$ on I.*

Theorem 5.8. *Let $U \subseteq \mathbb{C}$, let $z_0 \in U$, and let $(f_n)_n$ be a sequence of complex-valued functions on U that converge uniformly on U to $f : U \to \mathbb{C}$. If each f_n is continuous at z_0, then f is continuous at z_0. Consequently, a uniform limit of continuous functions is continuous.*

Definition 5.9. Let $U \subseteq \mathbb{C}$ and $f_n : U \to \mathbb{C}$ be a sequence of complex-valued functions. The sequence $(f_n)_n$ is said to be a Cauchy sequence of a function if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sup_{z \in U} |f_n(z) - f_m(z)| < \epsilon$$

for all $n \geq N$.

Theorem 5.10 (Cauchy Criterion for Uniform Convergence)**.** *A sequence of complex-valued functions converges uniformly on $U \subseteq \mathbb{C}$ if and only if it is a Cauchy sequence of functions.*

Theorem 5.11. *Let $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$ and $(f_n)_n$ be a sequence of complex-valued functions defined on U. Suppose that the series of complex-valued functions $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on U and each f_n is continuous at $z \in U$. Then f is continuous at z.*

Theorem 5.12 (Cauchy Criterion for Uniform Convergence of Series of Functions)**.** *Let $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$, and $(f_n)_n$ be a sequence of complex-valued functions defiend on U. The series of complex-valued functions $\sum_{n=1}^{\infty} f_n$ converges uniformly on U if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that*

$$\sup_{z \in U} \left| \sum_{k=n}^{m} f_k(x) \right| < \epsilon$$

for all $m \geq n \geq N$.

Theorem 5.13 (Weierstrass M-Test)**.** *Let $U \subseteq \mathbb{C}$ and let $(f_n)_n$ be a sequence of complex-valued functions on U. For each $n \in \mathbb{N}$, suppose*

$$0 \leq M_n = \sup_{z \in U} |f_n(z)| < \infty$$

Furthermore, suppose $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely for all $z \in U$ and if $f : U \to \mathbb{C}$ is defined by

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

for all $z \in U$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to f.

6 Uniform Continuity

Definition 6.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{C}$ is said to be uniformly continuous on I if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Theorem 6.2. *Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{C}$ is continuous, then f is uniformly continuous.*

7 Integration of Series of Functions

Theorem 7.1. *Let $(f_n)_n$ be a sequence of real-valued functions, Riemann integrable functions on a closed interval $[a, b]$. If $(f_n)_n$ converges uniformly on $[a, b]$ to $f : [a, b] \to \mathbb{R}$, then f is Riemann integrable and*

$$\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$$

Corollary 7.2. *Let $(f_n)_n$ be a sequence of real-valued Riemann integrable functions on $[a, b]$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f : [a, b] \to \mathbb{R}$, then f is Riemann integrable and*

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx$$