Question 1. *Let $f_n(x) = \frac{\sin nx}{n}$ for $x \in [0,1]$. Show that $(f_n)_n$ converges uniformly to a continuous function on $[0,1]$, and that the limit function is differentiable. Moreover, show that $(f_n')_n$ does not converge uniformly on $[0,1]$.*

Solution 1. First, we want to determine the pointwise limit function of the sequence $f_n(x) = \frac{\sin nx}{n}$. For any fixed $x \in [0,1]$, we have that

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{n} = 0$$

Next, we need to show that this convergence is uniform. For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left|\frac{\sin nx}{n} - 0\right| = \left|\frac{\sin nx}{n}\right| \leq \frac{1}{n} < \epsilon$$

for all $n \geq N > \frac{1}{\epsilon}$, and hence, the convergence is uniform. Since $f_n(x)$ is continuous on $[0,1]$, and $f_n(x)$ is uniformly converges to $f(x) = 0$, it follows that f is continuous on $[0,1]$.

Next, we find $(f_n')_n$,

$$f_n'(x) = \cos nx$$

Then the pointwise limit function is

$$f'(x) = \lim_{n\to\infty} \cos nx = \text{diverges}$$

Hence, since the pointwise limit function does not exist, $(f_n')_n$ does not converge uniformly on $[0,1]$.

Question 2. *Let $f_n(x) = \frac{x^n}{n}$ for $x \in [0,1]$. Determine whether the sequence $(f_n)_n$ converges uniformly on $[0,1]$, and if so, show that the function is differentiable. Moreover, show that $(f_n')_n$ converges uniformly on $[0,1]$.*

Solution 2. We have shown previously that the sequence of functions $f_n(x) = \frac{x^n}{n}$ is not uniformly convergent on $[0,1]$. Hence, the function is not differentiable. For $(f_n')_n$,

$$f_n'(x) = x^{n-1}$$

and since $x \in [0,1]$, as $n \to \infty$, we have

$$f'(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} x^{n-1} = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Hence, $f'(x)$ is not uniform on $[0,1]$, since it is not continuous at $x = 1$.

Question 3. *Let $f_n(x) = \frac{\cos nx}{n}$ for $x \in [0,1]$. Show that $(f_n)_n$ converges uniformly to a continuous function on $[0,1]$ and that the limit function is differentiable. Moreover, show that $(f_n')_n$ does not converge uniformly on $[0,1]$ and that the function is not differentiable.*

Solution 3. To show that $(f_n)_n$ converges uniformly to a continuous function on $[0,1]$, we first determine the pointwise limit. For a fix $x \in [0,1]$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\cos nx}{n} = \lim_{n\to\infty} 0 = 0$$

To show that this convergence is uniform, let $\epsilon > 0$ be arbitrary. Then there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| = \left|\frac{\cos nx}{n}\right| \leq \frac{1}{n} < \epsilon$$

for every $n \geq N > \frac{1}{\epsilon}$. Hence, as $\epsilon > 0$ was arbitrary, $f_n(x)$ converges uniformly on $[0, 1]$ as required. Therefore since $f_n(x)$ uniformly converges on $[0, 1]$, then f is continuous on $[0, 1]$. Since f is continuous on $[0, 1]$, then f is differentiable at $(0, 1)$.

Next, we want to show that $(f_n')_n$ does not converge uniformly on $[0, 1]$. First, we have

$$f_n'(x) = \frac{-n \sin nx}{n} = -\sin nx$$

Then

$$f'(x) = \lim_{n\to\infty} -\sin nx = \text{ diverges}$$

Hence, since the pointwise limit function does not exist, $(f_n')_n$ does not converge uniformly on $[0, 1]$, and hence, not continuous and differentiable.

Question 4. *Let* $f_n(x) = \frac{\ln(1+nx)}{n}$ *for* $x \in [0, 1]$*. Determine whether the sequence* $(f_n)_n$ *converges uniformly on* $[0, 1]$*, and if so, show that the limit function is differentiable. Moreover, show that* $(f_n')_n$ *converges uniformly on* $[0, 1]$*.*

Solution 4. First, we find the pointwise limit function of $f_n(x)$. For $x = 0$, we have

$$f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \frac{\ln(1)}{n} = 0$$

and for when $x = 1$,

$$f(1) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{\ln(1+n)}{n} = 0$$

Then for $x \in (0, 1)$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\ln(1+nx)}{n} = 0$$

Hence, $f_n(x)$ converges pointwise to the function $f(x) = 0$. We claim that the sequence of functions $f_n(x)$ does not converge uniformly on $[0, 1]$. To see this, for the sake of contradiction, let $\epsilon > 0$ be arbitrary. Then there exists an $N \in \mathbb{N}$ such that

$$\left| \frac{\ln(1+nx)}{n} - 0 \right| \leq \frac{\ln(1+n)}{n} < \epsilon$$

In particular, if $x = 1$ and $\epsilon = \frac{1}{2}$,

$$\frac{\ln(1+n)}{n} < \frac{1}{2} \Rightarrow \ln(1+n) < \frac{n}{2} \Rightarrow 1 + n < e^{\frac{n}{2}}$$

But as $n \to \infty$, $e^{\frac{n}{2}} \to \infty$ as well, which is a contradiction, and while $1 + n$ is bounded above. Therefore, there does not exist any N such that $f_n(x) = \frac{\ln(1+nx)}{n}$ converge uniformly.

Question 5. *Let* $f_n(x) = \frac{1}{n} \sin(nx^2)$ *for* $x \in [0, 1]$*. Show that* $(f_n)_n$ *converges uniformly to a continuous function on* $[0, 1]$ *and that the limit function is differentiable. Moreover, show that* $(f_n')_n$ *cannot converge uniformly on* $[0, 1]$*.*

Solution 5. First, we find the pointwise limit function of $f_n(x)$. We have for $x \in [0, 1]$ and $n \to \infty$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx^2}{n} = 0$$

Hence, $f_n(x)$ converges pointwise to the function $f(x) = 0$. We want to determine whether the sequence of functions converge uniformly on $[0, 1]$. For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left|\frac{\sin nx^2}{n} - 0\right| = \left|\frac{\sin nx^2}{n}\right| \leq \frac{1}{n} < \epsilon$$

for all $n \geq N > \frac{1}{\epsilon}$. Hence, as $\epsilon > 0$ was arbitrary, $f_n(x)$ converges uniformly to $f(x) = 0$ on $[0, 1]$.

Next, we want to show that $(f_n')_n$ converges uniformly on $[0, 1]$. Since

$$f_n'(x) = 2x \cos nx^2$$

and

$$f(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} 2x \cos nx^2 = \text{diverges}$$

$(f_n')_n$ cannot converge uniformly on $[0, 1]$.