Theorem 1 (Weierstrass Approximation Theorem)**.** *Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous, then there exist a sequence of polynomial functions $(p_n)_n$ of polynomials that converge uniformly to f on $[a, b]$. That is, for all $\epsilon > 0$, there exists a polynomial p such that*

$$\sup_{x\in[a,b]} |p(x) - f(x)| < \epsilon$$

for all $x \in [a, b]$.

Proof. We first construct an auxiliary sequence of functions. For $n \in \mathbb{N}$, let

$$c_n = \int_{-1}^{1} (1 - x^2)^n dx$$

and $\phi_n : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\phi_n(x) = \begin{cases} \frac{1}{c_n}(1 - x^2)^n & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

Then ϕ_n is continuous, $\phi_n(x) = \phi_n(x)$ and

$$1 = \int_{-\infty}^{\infty} \phi_n(x)dx = \int_{-1}^{1} \phi_n(x) = 2\int_{0}^{1} \phi_n(x)dx$$

for all $n \in \mathbb{N}$.

Then we next require the following Lemma.

Lemma 2. *For $\delta \in (0, 1)$, the sequence $(\phi_n)_n$ converges uniformly to a constant function $f(x) = 0$ on $(-\infty, \delta] \cup [\delta, \infty)$.*

Proof. Let $\epsilon > 0$ be arbitrary. For each $n \in \mathbb{N}$, we have that

$$c_n = \int_{-1}^{1} (1 - x^2)^n dx = 2\int_{0}^{1} (1 - x^2)^n dx = 2\int_{0}^{1} (1 + x)^n(1 - x)^n dx \geq 2\int_{0}^{1} (1 - x)^n dx = \frac{2}{1 + x}$$

Using the above, for $x \in (-\infty, \delta] \cup [\delta, \infty)$ we can obtain that

$$|\phi_n(x)| = \frac{1}{c_n}(1 - x^2)^n \leq \frac{n + 1}{2}(1 - \delta^2)^n$$

Since $0 < 1 - \delta^2 < 1$, we have that $\lim_{n\to\infty} \frac{n+1}{2}(1 - \delta^2)^n = 0$. That is, there exists an $N \in \mathbb{N}$ such that $|\phi_n(x)| \leq \frac{n+1}{2}(1 - \delta^2)^n < \frac{\epsilon}{2}$ for $n \geq N$. The above allows us to conclude that

$$\sup_{x\in(-\infty,\delta]\cup[\delta,\infty)} |\phi_n(x) - f(x)| = \sup_{x\in(-\infty,\delta]\cup[\delta,\infty)} |\phi_n(x)| < \epsilon$$

for $n \geq N$ and $(\phi_n)_n$ converges uniformly to the constant function $g(x) = 0$ on $(-\infty, \delta] \cup [\delta, \infty)$. □

Now let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$ and define $\hat{f} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{f} = \begin{cases} f(x) & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to see that $\hat{f}$ is continuous on $\mathbb{R}$. For $n \in \mathbb{N}$, let $p_n : [0,1] \to \mathbb{R}$ be the function defined by

$$p_n(x) = \int_{-\infty}^{\infty} \hat{f}(x-t)\phi_n(t)dt = \int_{-\infty}^{\infty} \hat{f}(t)\phi_n(x-t)dt$$

for $x \in [0,1]$. For each $n \in \mathbb{N}$ and $x \in [0,1]$, using the definition of ϕ_n and $\hat{f}$, it is easy to see htat

$$p_n(x) = \int_{-1}^{1} \hat{f}(x-t)\phi_n(t)dt = \int_{0}^{1} f(t)\phi_n(x-t)dt$$

Now we need the following Lemma.

Lemma 3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$ and $(p_n)_n$ is a sequence of polynomial functions defined as above. Then $p_n : [0,1] \to \mathbb{R}$ is a polynomial.*

Proof. For $x \in [0,1]$ and $n \in \mathbb{N}$, using that $p_n(x) = \int_0^1 f(s)\phi_n(s-x)ds$ and noting that

$$\begin{aligned}
\phi_n(s-x) &= \frac{1}{c_n}(1-(s-x)^2)^n \\
&= \frac{1}{c_n}\sum_{k=0}^{n}\binom{n}{k}1^{n-k}(-1)^k(s-x)^{2k} \\
&= \frac{1}{c_n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{\ell=0}^{2k}\binom{2k}{\ell}s^{2k-\ell}(-1)^\ell x^\ell \\
&= \frac{1}{c_n}\sum_{k=0}^{n}\sum_{\ell=0}^{2k}\left[\binom{n}{k}\binom{2k}{\ell}(-1)^{k+\ell}s^{2k-\ell}\right]x^\ell \\
&= \frac{1}{c_n}\sum_{m=0}^{2n} h_m(s)x^m
\end{aligned}$$

we can obtain that

$$p_n(x) = \int_0^1 f(s)\phi_n(s-x)ds = \int_0^1 f(s)\frac{1}{c_n}\sum_{m=0}^{2n} h_m(s)x^m ds = \frac{1}{c_n}\sum_{m=0}^{2n} x^m \int_0^1 f(s)h_m(s)ds = \sum_{m=0}^{2n}\frac{d_m}{c_n}x^m$$

which is a polynomial. □

Now recall from Question 4, Assignment 2,

Proposition 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{n\to-\infty} f(x) = 0 = \lim_{n\to\infty} f(x)$ then f is uniformly continuous on $\mathbb{R}$.*

Next, consider the next Lemma

Lemma 5. *Let $f : [0,1] \to \mathbb{R}$ and $(p_n)_n$ be as in Lemma 3, then $(p_n)_n$ converges uniformly to f on $[0,1]$*

Proof. Let $\epsilon > 0$ be arbitrary. Since $[0,1]$ is compact,

$$M = \sup_{x\in[0,1]} |f(x)| = \sup_{x\in\mathbb{R}} |\hat{f}(x)| < \infty$$

and using Proposition 4, $\hat{f}$ is uniformly continuous on $\mathbb{R}$. Using the above and Lemma 3, there exists $\delta \in (0,1)$ and $N \in \mathbb{N}$ such that

$$|\hat{f}(y) - \hat{f}(x)| < \frac{\epsilon}{6}$$

for all $x, y \in [0,1]$ with $|x-y| < \delta$ and so

$$|\phi_n(t)| < \frac{\epsilon}{12M}$$

for all $|t| \geq \delta$ and $n \geq N$. Let $n \geq N$ and $x \in [0,1]$, noting that

$$\int_{-1}^{1} \phi_n(t)dt = 1$$

it is easy to see that $f(x) = \hat{f}(x) \int_{-1}^{1} \phi_n(t)dt = \int_{-1}^{1} \hat{f}(x)\phi_n(t)dt$. Then,

$$\begin{aligned}
|p_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x-t) - f(x)]\phi_n(t)dt \right| \\
&\leq \int_{-1}^{1} \left|\hat{f}(x-t) - \hat{f}(x)\right| |\phi_n(t)|dt \\
&= \int_{-1}^{-\delta} \left|\hat{f}(x-t) - \hat{f}(x)\right| |\phi_n(t)|dt + \int_{-\delta}^{\delta} \left|\hat{f}(x-t) - \hat{f}(x)\right| |\phi_n(t)|dt \\
&+ \int_{\delta}^{1} \left|\hat{f}(x-t) - \hat{f}(x)\right| |\phi_n(t)|dt
\end{aligned}$$

where

$$\begin{aligned}
&\int_{-1}^{-\delta} \left|\hat{f}(x-t) - \hat{f}(x)\right| |\phi_n(t)|dt < \int_{-\delta}^{\delta} \frac{\epsilon}{6} |\phi_n(t)|dt < \frac{\epsilon}{6}, \text{since } |x-(x-t)| < \delta \\
&\int_{-\delta}^{\delta} \left|\hat{f}(x-t) - \hat{f}(x)\right| |\phi_n(t)|dt < \int_{-1}^{-\delta} \left[|\hat{f}(x-t)| + |\hat{f}(x)|\right] \frac{\epsilon}{12M} dt < (1+\delta)2M\frac{\epsilon}{12M} < \frac{\epsilon}{6} \\
&\int_{\delta}^{1} \left|\hat{f}(x-t) - \hat{f}(x)\right| |\phi_n(t)|dt < \int_{\delta}^{1} \left[|\hat{f}(x-t)| + |\hat{f}(x)|\right] \frac{\epsilon}{12M} dt < (1+\delta)2M\frac{\epsilon}{12M} < \frac{\epsilon}{6}
\end{aligned}$$

Therefore,

$$\sup_{x\in[0,1]} |p_n(x) - f(x)| < \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2} < \epsilon$$

□

Now it all comes down to the proof of the Weierstrass Approximation Theorem, using everything that we have constructed above...which we will finish next class. □

