Question 1. *Let $(z_n)_n$ be a sequence of complex numbers and $0 < \alpha < 1$ such that $z_n \neq 0$ and $\left|\frac{z_{n+1}}{z_n}\right| \leq \alpha$ for all $n \in \mathbb{N}$. Prove that $(z_n) \to 0$.*

Question 2. *Let $\sum_{n=1}^{\infty} z_n$ be a series of complex numbers.*

(a) *Suppose that there exists an $N \in \mathbb{N}$, $k > 0$ and a convergent series of real numbers $\sum_{n=1}^{\infty} w_n$ such that $w_n \geq 0$ and $|z_n| \leq k w_n$ for all $n \geq N$. Prove that $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.*

(b) *Suppose that there exists $0 < \alpha < 1$ and $N \in \mathbb{N}$ such that $\sqrt[n]{|z_n|} \leq \alpha$ for $n \geq N$. Prove that $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.*

(c) *Suppose that there exists $0 < \alpha < 1$ and $N \in \mathbb{N}$ such that $\left|\frac{z_{n+1}}{z_n}\right| \leq \alpha$ for all $n \geq N$. Prove that $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.*

Question 3. *Let $\sum_{n=1}^{\infty} z_n$ be a series of real numbers ushc that the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ (where $S_n = z_1 + z_2 + \cdots + z_n$, $n \in \mathbb{N}$) is bounded. Let $(w_n)_n$ be a decreasing sequence of real numbers such that $(w_n) \to 0$.*

(a) *First, prove that $\sum_{k=1}^{n} z_k w_k = S_n w_{n+1} - S_0 w_1 + \sum_{k=1}^{n} S_k (w_k - w_{k+1})$.*

(b) *For $n \to \infty$, show that as $n \to \infty$, $S_n w_{n+1} \to 0$ and that $\sum_{n=1}^{\infty} S_n (w_n - w_{n+1})$ converges absolutely.*

Question 4. *Determine whether the following mathematical statements are true or false. Prove your answers.*

(a) *If $\sum_{n=1}^{\infty} z_n$ is a convergent series of real numbers, then $\lim_{n\to\infty} n z_n = 0$.*

(b) *If $\sum_{n=1}^{\infty}$ is an absolutely convergent series, then $\sum_{n=1}^{\infty} z_n^2$.*

(c) *If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ are convergent series of real numbers, then $\sum_{n=1}^{\infty} z_n w_n$ converges.*

(d) *If $\sum_{n=1}^{\infty} z_n^2$ and $\sum_{n=1}^{\infty} w_n^2$ are convergent series of real numbers, then $\sum_{n=1}^{\infty} z_n w_n$ converges.*

Question 5. *Determine whether the each of the following series converges or diverges. Prove your answers.*

(a) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Question 6. *For each of the following sequences $(f_n)_n$ of real-valued functions, find (with proof) the pointwise limit function f of $(f_n)_n$ and determine $(f_n)_n$ (with proof) whether $(f_n)_n$ converges uniformly to f on E.*

(a) *$U = [0, \infty)$ and $f_n(x) = \frac{x^n}{1+x^n}$ for $n \in \mathbb{N}$.*

(b) *$U = [-1, 1]$ and $f_n(x) = x^n (1 - x^2)^n$ for $n \in \mathbb{N}$.*

(c) *$U = [-\pi, \pi]$ and $f_n(x) = n \sin\left(\frac{x}{n}\right)$ for $n \in \mathbb{N}$.*

(d) *$U = \mathbb{R}$ and $f_n(x) = n \sin\left(\frac{x}{n}\right)$ for $n \in \mathbb{N}$.*

Question 7. *Let $U \subset \mathbb{C}$ and $(f_n)_n$ be a sequence of functions such that*

(i) *$f_n : U \to \mathbb{R}$ for $n \in \mathbb{N}$.*

(ii) $f_{n+1}(x) \leq f_n(x)$ *for all* $x \in U$ *and* $n \in \mathbb{N}$.

(iii) $(f_n)_n$ *converges uniformly to the constant function* $f(x) = 0$ *on* U.

Prove that the series $\sum_{n=1}^{\infty} (-1)^n f_n$ *converges uniformly on* E.

Question 8. *Let* $f : \mathbb{R} \to \mathbb{R}$. *Prove that* $\lim_{n\to\infty} f(x) = a$ *if and only if the sequence of functions* $(f_n)_n$ *defined by* $f_n(x) = f(x + n)$ *converges uniformly to the constant function* $f(x) = a$ *on* $U = [0, \infty)$.

Question 9. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $\lim_{n\to-\infty} f(x) = \lim_{n\to\infty} f(x) = 0$. *Prove that* f *is uniformly continuous on* $\mathbb{R}$.

Question 10. *Let* $U, V \subset \mathbb{C}$, $(f_n)_n$ *be a sequence of complex-valued functions defined on* U, $(g_n)_n$ *be a sequence of complex-valued functions defined in* V, $f \in \mathbb{C}^U$ *and* $g \in \mathbb{C}^V$ *such that* g *is uniformly continuous in* V, $f(U) \subset V$ *and* $f_n(U) \subset V$ *for* $n \in \mathbb{N}$.

(a) *If* $(f_n)_n \to f$ *uniformly, prove that* $g \circ f_n \to g \circ f$ *uniformly.*

(b) *If* $(f_n)_n \to f$ *uniformly and* $(g_n) \to g$ *uniformly, prove that* $g_n \circ f_n \to g \circ f$ *uniformly.*

(c) *If* $(g_n) \to g$ *uniformly, is it necessary to assume some condition on* f *to obtain* $g_n \circ f \to g \circ f$*? Explain by providing a proof.*