# MATH 3001: Real Analysis II
# Series of Functions

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Last Updated: February 27, 2023

# Contents


**1 Complex Numbers** **5**

**2 Differentiation in $\mathbb{C}$** **15**

**3 Integration on $\mathbb{C}$** **23**

**4 Fubini's Theorem** **31**

**5 Leibniz Integral Rule** **39**

**6 Laplace's Equation in Polar Coordinates** **41**

**7 Sequences of Complex Numbers** **45**

**8 Series of Complex Numbers** **55**

**9 Rearrangements of Series** **69**

**10 Continuity of Complex-Valued Functions** **77**

**11 Continuity of Sequence and Series of Functions** **81**

**12 Uniform Continuity** **93**

**13 Integration of Series of Functions** **95**

**14 Differentiation of Series of Functions** **101**

**15 Power Series** **103**

# Chapter 1

# Complex Numbers

In this chapter, we will review the basics of complex numbers.

**Definition 1.1.** The complex numbers, denoted by $\mathbb{C}$, is the set

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$$

where $i$ denotes a fixed symbol.

Given a complex number $z = x + iy$, where $x, y \in \mathbb{R}$, the number $x$ is called the *real part* of $z$ and is denoted by $\Re(z)$ and the number $y$ is called the *imaginary part* of $z$ and is denoted by $\Im(z)$.

The symbol $i$ in a complex number is meant to denote $\sqrt{-1}$. To be specific, we will equip $\mathbb{C}$ with binary operations of addition and multiplication so that $(0+1i)(1+1i) = -1$ so that indeed, $i$ is a complex solution to $x^2 = -1$.

**Definition 1.2.** The binary operations $+ : \mathbb{C}^2 \to \mathbb{C}$ and $\cdot : \mathbb{C}^2 \to \mathbb{C}$ defined by

$$\begin{aligned} (a + ib) + (c + id) &= (a + c) + i(b + d) \\ (a + ib)(c + id) &= (ac - bd) + i(ad + bc) \end{aligned}$$

for all $a, c, b, d \in \mathbb{R}$ are called *complex addition* and *complex multiplication* respectively.

**Example 1.3.** It is not difficult to see that

$$\begin{aligned} (1 + i2) + (3 + i4) &= 4 + 6i \\ (1 + i2)(3 + i4) &= -5 + 10i \end{aligned}$$

Moreover, since

$$i^2 = (0 + i1)(0 + i1) = -1 + 0i$$

we do indeed have that $i$ is a complex solution to $x^2 = -1$. In addition, it is not difficult to see that $-i$ is also a complex solution to $x^2 = -1$.

In order for $\mathbb{C}$ to be as nice to work with as $\mathbb{R}$, we require complex addition and subtraction to have specific properties. To be specific, we want the following.

**Theorem 1.4.** *The set of complex numbers $\mathbb{C}$ together with complex addition and multiplication is a field. That is,*

1. *(Commutativity of Addition) $z + w = w + z$ for all $z, w \in \mathbb{C}$.*
2. *(Associativity of Addition) $z+(w+u) = (z+w)+u$ for all $z, w, u \in \mathbb{C}$.*
3. *(Additive Unit) There exists a $0 \in \mathbb{C}$ such that $z + 0 = z$ for all $z \in \mathbb{C}$.*
4. *(Additive Inverse) For all $z \in \mathbb{C}$, there exists a $-z \in \mathbb{C}$ such that $z + (-z) = 0$.*
5. *(Commutativity of Multiplication) $z \cdot w = w \cdot z$ for all $z \in \mathbb{C}$.*
6. *(Associativity of Multiplication) $z\cdot(w\cdot u) = (z\cdot w)\cdot u$ for all $z, w, u \in \mathbb{C}$.*
7. *(Multiplicative Unit) There exists a $1 \in \mathbb{C}$ such that $1 \cdot z = z$ for all $z \in \mathbb{C}$.*
8. *(Multiplicative Inverse) For all $z \in \mathbb{C} \setminus \{0\}$, there exists a $z^{-1} \in \mathbb{C}$ such that $z^{-1} \cdot z = 1$.*
9. *(Distributive Property) $z(w + u) = z \cdot w + z \cdot u$ for all $z, w, u \in \mathbb{C}$.*

*Proof.* Let $z, w, u \in \mathbb{C}$ be arbitrary. Hence, there exists $a, b, c, d, x, y \in \mathbb{R}$ such that

$$z = a + ib \qquad w = c + id \qquad u = x + iy$$

We will now examine each of the above nine properties for these arbitrary elements of $\mathbb{C}$ and demonstrate the property holds using the analogous property for real numbers.

1. Notice that

$$z + w = (a + c) + i(b + d) = (c + a) + i(d + b) = w + z$$

due to the commutativity of addition of real numbers. This commutativity of addition of complex numbers has been demonstrated.

2. Notice that

$$\begin{aligned} z + (w + u) &= (x_i + ib) + [(c + x) + i(d + y)] \\ &= (a + (c + x)) + i(b + (d + y)) \\ &= ((a + c) + x) + i((b + d) + y) \\ &= ((a + c) + i(b + d)) + (x + iy) \\ &= (z + w) + u \end{aligned}$$

where the third equality holds due to the associativity of addition of real numbers. Thus associativity of addition of complex numbers has been demonstrated.

3. Notice that $0 = 0 + i0$, we have

$$z + 0 = (a + 0) + i(b + 0) = a + ib = z$$

due to the property of the zero element of $\mathbb{R}$. Thus, the complex numbers have an additive unit.

4. Let $-z = (-a) + i(-b)$ where $-a$ and $-b$ are additive inverses of $a, b \in \mathbb{R}$. Then

$$z + (-z) = (a + (-a)) + i(b + (-b)) = 0 + 0i = 0$$

as desired. Thus, the complex numbers have additive inverses.

5. Notice that

$$z \cdot w = (ac - bd) + i(ad + bc) = (ca - db) + (da + cb) = w \cdot z$$

due to the commutativity of addition and multiplication of real numbers. Thus, commutativity of multiplication of complex numbers has been demonstrated.

6. Notice that

$$\begin{aligned}
z\cdot(w\cdot u) &= (a+ib)[(c+id)(x+iy)]\\
&= (a+ib)[(cx-dy)+i(cy+dx)]\\
&= [a(cx-dy)-b(cy+dx)]+i[a(cy+dx)+b(cx-dy)]\\
&= (acx-ady-bcy-bdx)+i(acy+adx+bcx-bdy)\\
&= (acx-bdx-ady-bcy)+i(acy-bdy+adc+bcx)\\
&= [(ac-bd)x-(ad+bc)y]+i[(ac-bd)y+(ad+bc)x]\\
&= [(ac-bd)+i(ad+bc)](x+iy)\\
&= [(a+ib)(c+id)](x+iy)\\
&= (z\cdot w)\cdot u
\end{aligned}$$

where the commutativity and association of addition and distributivity of real numbers have been used. Thus associativity of multiplication of complex numbers has been demonstrated.

7. Notice with $1 = 1 + 0i$ that

$$1\cdot z = (1a-0b)+i(1b+0a) = a+bi = z$$

due to the property of the one element of $\mathbb{R}$. Thus the complex numbers have a multiplicative unit.

8. Assume $z \neq 0$. Thus, $a, b \neq 0$ so $a^2 + b^2 > 0$. Define

$$z^{-1} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$$

which makes sense since $\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \in \mathbb{R}$ since nonzero real numbers have multiplicative inverses. Notice that

$$\begin{aligned}
z^{-1}z &= \left[\left(\frac{a}{a^2+b^2}\right)a - \left(\frac{-b}{a^2+b^2}\right)b\right] + i\left[\left(\frac{a}{a^2+b^2}\right)b + \left(\frac{-b}{a^2+b^2}\right)a\right]\\
&= \left(\frac{a^2+b^2}{a^2+b^2}\right) + i\left(\frac{ab-ba}{a^2+b^2}\right)\\
&= 1+0i = 1
\end{aligned}$$

due to the commutative of multiplication and properties of addition of real numbers. Hence, $z^{-1}$ is indeed the multiplicative inverse of $z$. Thus the existence of multiplicative inverses for nonzero complex numbers has been demonstrated.

9. Notice that

$$\begin{aligned} z(w+u) &= (a+ib)[(c+id)+(x+iy)] \\ &= (a+ib)[(c+x)+i(d+y)] \\ &= [a(c+x)-b(d+y)]+i[a(d+y)+b(c+x)] \\ &= (ac+ac-bd-dy)+i(ad+ay+bc+bx) \\ &= [(ac-bd)+(ax-dy)]+i[(ad+bc)+(ay+bx)] \\ &= [(ac-bd)+i(ad+bc)]+[(ax-dy)+i(ay+bx)] \\ &= [(a+ib)(c+id)]+[(a+ib)(x+iy)] \\ &= z\cdot w+z\cdot u \end{aligned}$$

where commutativity and associativity of addition and distributivity of real numbers have been used. Thus distributivity of complex numbers has been demonstrated.

As all nine properties have now been demonstrated for arbitrary complex numbers, the set of complex numbers together with addition and multiplication are a field. □

*Remark* 1.5. Embedded in the proof of Theorem 1.4 is the formula for the inverse of a nonzero complex number. Indeed, if $z = x + iy$ where $x, y \in \mathbb{R}$ is such that $z \neq 0$, then

$$z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

For example,

$$(3+i4)^{-1} = \frac{3}{25} - i\frac{4}{25}$$

Note we can also write

$$z^{-1} = \frac{x-iy}{x^2+y^2}$$

When dealing with complex numbers, both the numerator and denominator are important common quantities that are worthy of developing further.

**Definition 1.6.** Given a complex number $z = x + iy$ where $x, y \in \mathbb{R}$, the modulus of $z$, denoted by $|z|$ is the quantity

$$|z| = \sqrt{x^2+y^2}$$

**Definition 1.7.** Given a complex number $z = x + iy$ where $x, y \in \mathbb{R}$, the complex conjugate of $z$, denoted by $\bar{z}$ is the quantity

$$\bar{z} = x + (-y)i$$

Using our knowledge of the inverse of a nonzero complex number, we have the following.

**Corollary 1.8.** *If $z \in \mathbb{C} \setminus \{0\}$, then $z^{-1} = \frac{\bar{z}}{|z|^2}$*

**Proposition 1.9.** *For all $z, w \in \mathbb{C}$, the following are true.*

1. $\Re(z) = \frac{z+\bar{z}}{2}$
2. $\Im(z) = \frac{z-\bar{z}}{2i}$
3. $\overline{z+w} = \bar{z} + \bar{w}$
4. $\overline{zw} = \bar{z}\bar{w}$
5. $|z| = \sqrt{z\bar{z}}$
6. $|\Re(z)| \leq |z|$
7. $|\Im(z)| \leq |z|$
8. $|\bar{z}| = |\bar{z}|$
9. $|zw| = |z||w|$
10. *If* $z \neq 0$, *then* $|z^{-1}| = |z|^{-1}$
11. $|z + w| \leq |z| + |w|$
12. $||z| - |w|| \leq |z - w|$

*Proof.* Let $z, w \in \mathbb{C}$ be arbitrary. Thus, there exists $a, b, c, d \in \mathbb{R}$ such that $z = a + ib$ and $w = c + id$.

1. Notice
$$\frac{z + \bar{z}}{2} = \frac{(a + ib) + (a - ib)}{2} = \frac{2a}{2} = a = \Re(z)$$
2. Notice
$$\frac{z - \bar{z}}{2i} = \frac{(a + ib) - (a - ib)}{2i} = \frac{2bi}{2i} = b = \Im(z)$$

3. Notice

$$\begin{aligned}\overline{z+w} &= \overline{(a+c)+i(b+d)} \\ &= (a+c)+i[-(b+d)] \\ &= [a+i(-b)]+[c+i(-d)] \\ &= \bar{z}+\bar{w}\end{aligned}$$

4. Notice

$$\begin{aligned}\overline{zw} &= \overline{ac-bd)+i(ad+bc)} \\ &= (ac-bd)+i[-(ad+bc)] \\ &= [ac-(-b)(-d)]+i[a(-d)+(-b)c] \\ &= [a+i(-b)][c+i(-d)] \\ &= \bar{z}\bar{w}\end{aligned}$$

5. Notice that

$$\begin{aligned}\sqrt{z\bar{z}} &= \sqrt{(a+ib)[a+i(-b)]} \\ &= \sqrt{[a^2-(-b)b]+i[ab+a(-b)]} \\ &= \sqrt{a^2+b^2} = |z|\end{aligned}$$

6. Note that

$$a^2 \le a^2+b^2$$

   and so

$$|\Re(z)| = |a| = \sqrt{a^2} \le \sqrt{a^2+b^2} = |z|$$

7. Note that

$$b^2 \le a^2+b^2$$

   and so

$$|\Im(z)| = |b| = \sqrt{b^2} \le \sqrt{a^2+b^2} = |z|$$

8. Note that

$$|\bar{z}| = |a+i(-b)| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2} = |z|$$

9. Note that

$$zw = (ac - bd) + i(ad + bc)$$

and so

$$\begin{aligned}
|zw| &= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\
&= \sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} \\
&= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\
&= \sqrt{(a^2+b^2)(c^2+d^2)} \\
&= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\
&= |z||w|
\end{aligned}$$

10. From (9), note that

$$|z^{-1}||z| = |z^{-1}z| = 1 = 1$$

so $|z^{-1}| = |z^{-1}|$ as desired.

11. Notice that

$$\begin{aligned}
|z+w|^2 &= (\overline{z+w})(z+w) \\
&= (\bar{z}+\bar{w})(z+w) \\
&= z\bar{z} + \bar{z}w + \bar{w}z + \bar{w}w \\
&= |z|^2 + \bar{z}w + \overline{\bar{z}w} + |w|^2 \\
&= |z|^2 + 2\Re(\bar{z}w) + |w|^2 \\
&\leq |z|^2 + 2|\bar{z}w| + |w|^2 \\
&= |z|^2 + 2|\bar{z}||w| + |w|^2 \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2
\end{aligned}$$

Therefore, taking the square root on both sides of the inequality, the desired result is obtained.

12. First notice that

$$|z| = |(z-w) + w| \leq |z-w| + |w|$$

Therefore,
$$|z| - |w| \leq |z - w|$$
Similarly, notice that
$$\begin{aligned} |w| &= |(w - z) + z| \\ &\leq |w - z| + |z| \\ &= |(-1)(z - w)| + |z| \\ &= |-1||z - w| + |z| \\ &= |z - w| + |z| \end{aligned}$$
Hence,
$$|w| - |z| \leq |z - w|$$
Therefore, combining $|z| - |w| \leq |z - w|$ and $|w| - |z| \leq |z - w|$, we obtain that
$$||z| - |w|| \leq |z - w|$$
as desired.

$\square$

# Chapter 2

# Differentiation in $\mathbb{C}$

In the next two chapters, we will extend the notions of derivatives and integrals of real-valued functions on closed intervals to complex-valued functions. In particular, it will be demonstrated that every elementary notion for derivative and integrals seen in previous courses extend to these complex-valued functions with ease.

As is natural with calculus, we begin with differentiation. To simplify notation, we introduce the following.

**Notation 2.1.** For $a, b \in \mathbb{R}$ with $a < b$, the vector space of all complex-valued functions on $[a, b]$ is denoted by $\mathcal{F}([a, b], \mathbb{C})$. Recall that $f \in \mathcal{F}([a, b], \mathbb{C})$, then the real and imaginary parts of $f$ are the functions $\Re(f) : [a, b] \to \mathbb{R}$ and $\Im(f) : [a, b] \to \mathbb{R}$ defined by

$$\Re(f)(z) = \Re((f(x)) \qquad \Im(f)(x) = \Im((f(x))$$

for all $x \in [a, b]$.

**Definition 2.2.** Let $f \in \mathcal{F}([a, b], \mathbb{C})$ and let $x_0 \in (a, b)$. It is said that $f$ is differentiable at $x_0$ if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists in $\mathbb{C}$. When $f$ is differentiable at $x_0$, the derivative of $f$ at $x_0$ denoted by $f'(x_0)$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \in \mathbb{C}$$

Finally, it is said that $f$ is differentiable on $[a, b]$ if $f$ is differentiable at every point in $(a, b)$ and $f$ is continuous on $[a, b]$. The derivative of $f$ on $(a, b)$ is the function $f' : (a,b) \to \mathbb{C}$ whose value at $x_0 \in (a,b)$ is $f'(x_0)$ as defined above.

Of course, it is possible to develop the properties of derivatives of complex-valued functions on intervals by mirroring the corresponding real-valued results. However, as limits of complex numbers converge if and only if their real and imaginary parts converge, the theory of derivatives of complex-valued functions on intervals reduces down to the theory of derivatives of real-valued functions on intervals.

**Theorem 2.3.** *Let $f \in \mathcal{F}([a,b],\mathbb{C})$, let $u = \Re(f)$, let $v = \Im(f)$, and let $x_0 \in (a,b)$. Then $f$ is differentiable at $x_0$ if and only if $u$ and $v$ are differentiable at $x_0$. When $f$ is differentiable at $x_0$,*

$$f'(x_0) = u'(x_0) + iv'(x_0)$$

*Finally, $f$ is differentiable on $[a,b]$ if and only if $u$ and $v$ are differentiable on $[a,b]$, in which case, $f' = u' + iv'$.*

*Proof.* Since for all $x_0 \in (a,b)$ and $h \in \mathbb{R}$ with $x_0 + h \in [a,b]$, we have that

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{u(x_0+h) - u(x_0)}{h} + i\frac{v(x_0+h) - v(x_0)}{h}$$

it follows that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists if and only if

$$\lim_{h\to 0} \frac{u(x_0+h) - u(x_0)}{h} \qquad \lim_{h\to 0} \frac{v(x_0+h) - v(x_0)}{h}$$

exists. Moreover, this implies that

$$f'(x_0) = u'(x_0) + iv'(x_0)$$

Finally, since $f$ is continuous if and only if $u$ and $v$ are continuous, the result follows. $\square$

It is worthwhile to see that one specific function behaves incredibly well with respect to this definition of differentiation thereby further supporting why this definition of differentiation for complex-valued functions on intervals is desirable.

**Example 2.4.** Let $a, b \in \mathbb{R}$ and $\alpha = a + ib \in \mathbb{C}$ and define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) = e^{\alpha x} = e^{(a+ib)x} = e^{ax} \cos bx + ie^{ax} \sin bx$$

for all $x \in \mathbb{R}$. Then $f$ is differentiable at each point in $\mathbb{R}$ with

$$\begin{aligned} f'(x) &= (e^{ax} \cos bx)' + i(e^{ax} \sin bx)' \\ &= (ae^{ax} \cos bx - be^{ax} \sin bx) + i(ae^{ax} \sin bx + be^{ax} \cos bx) \\ &= (a + ib)e^{ax} \cos bx + (-b + ia)e^{ax} \sin bx \\ &= (a + ib)e^{ax} \cos bx + i(a + ib)e^{ax} \sin bx \\ &= (a + ib)(e^{ax} \cos bx + ie^{ax} \sin bx) \\ &= \alpha f(x) \end{aligned}$$

Hence,

$$(e^{\alpha x})' = \alpha e^{\alpha x}$$

for all $\alpha \in \mathbb{C}$.

Of course, some results, such as the following, immediate import from the theory of derivatives of real-valued functions.

**Corollary 2.5.** *Let $f \in \mathcal{F}([a, b], \mathbb{C})$ and let $x_0 \in (a, b)$. If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.*

*Proof.* Since $f$ is differentiable at $x_0$, then $\Re(f)$ and $\Im(f)$ are differentiable at $x_0$. Therefore, $\Re(f)$ and $\Re(f)$ are continuous at $x_0$, so $f = \Re(f) + i\Im(f)$ is continuous at $x_0$. □

Again, using the results for derivatives of real-valued functions, certain operations behave well with respect to differentiation.

**Corollary 2.6.** *Let $f, g \in \mathcal{F}([a, b], \mathbb{C})$ and let $x_0 \in (a, b)$. If $f$ and $g$ are differentiable at $x_0$, then $f + g$ is differentiable at $x_0$ with*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0)$$

*Proof.* Let $u = \Re(f)$, $v = \Im(f)$, $s = \Re(g)$ and $t = \Im(g)$. Since

$$f + g = (u + iv) + (s + it) = (u + s) + i(v + t)$$

we have that

$$\Re(f + g) = u + s \qquad \Im(f + g) = v + t$$

Therefore, since $f$ and $g$ are differentiable at $x_0$, we know that $u, v, s, t$ are differentiable at $x_0$ and thus $\Re(f+g)$ and $\Im(f+g)$ are differentiable at $x_0$ by results from the real case with

$$\begin{aligned}(f+g)'(x_0) &= (u+s)'(x_0) + i(v+t)'(x_0) \\ &= (u'(x_0) + s'(x_0)) + i(v'(x_0) + t'(x_0)) \\ &= (u'(x_0) + iv'(x_0)) + (s'(x_0) + it'(x_0)) \\ &= f'(x_0) + g'(x_0)\end{aligned}$$

as desired. □

To see that scalar multiplication behaves well with respect to differentiation, it is easier to generalize the product rule first.

**Theorem 2.7** (Product Rule)**.** *Let $f, g \in \mathcal{F}([a,b], \mathbb{C})$ and let $x_0 \in (a,b)$. If $f$ and $g$ are differentiable at $x_0$, then $fg$ is differentiable at $x_0$ with*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

*Proof.* Let $u = \Re(f)$, $v = \Im(f)$, $s = \Re(g)$ and $t = \Im(g)$. Since

$$fg = (u+iv)(s+it) = (us - vt) + i(ut + vs)$$

we have that

$$\Re(fg) = us - vt \qquad \Im(fg) = ut + vs$$

Therefore, since $f$ and $g$ are differentiable at $x_0$, we know that $u, v, s, t$ are differentiable at $x_0$ and thus $\Re(fg)$ and $\Im(fg)$ are differentiable at $x_0$ by the real-valued product rule with

$$\begin{aligned}(fg)'(x_0) &= (us - vt)'(x_0) + i(ut + vs)'(x_0) \\ &= ((us)'(x_0) - (vt)'(x_0)) + i((ut)'(x_0) + (vs)'(x_0)) \\ &= ((u'(x_0)s(x_0) + u(x_0)s'(x_0)) - (v'(x_0)t(x_0) + v(x_0)t'(x_0)) \\ &\quad + i((u'(x_0)t(x_0) + u(x_0)t'(x_0)) + (v'(x_0)s(x_0) + v(x_0)s'(x_0)) \\ &= u'(x_0)(s(x_0) + it(x_0) + s'(x_0)(-t(x_0) + is(x_0)) \\ &\quad + (u(x_0) + iv(x_0))s'(x_0) + (-v(x_0) + iu(x_0))t'(x_0) \\ &= u'(x_0)g(x_0) + iv'(x_0)g(x_0) + f(x_0)s'(x_0) + if(x_0)t'(x_0) \\ &= (u'(x_0) + iv'(x_0))g(x_0) + f(x_0)(s'(x_0) + it'(x_0)) \\ &= f'(x_0)g(x_0) + f(x_0)g'(x_0)\end{aligned}$$

as desired. □

**Corollary 2.8.** *Let $f \in \mathcal{F}([a,b],\mathbb{C})$ and let $x_0 \in (a,b)$. If $f$ is differentiable at $x_0$, and $\alpha \in \mathbb{C}$, then $\alpha f$ is differentiable at $x_0$ with*

$$(\alpha f)'(x_0) = \alpha f'(x_0)$$

*Proof.* Define $g \in \mathcal{F}([a,b],\mathbb{C})$ by $g(x) = \alpha$ for all $x \in [a,b]$. It is elementary to see that based on the definition of the derivative that $g$ is differentiable with $g'(x) = 0$ for all $x \in (a,b)$. Hence, the product rule implies $fg = \alpha f$ is differentiable at $x_0$ with

$$(\alpha f)'(x_0)(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) = \alpha f'(x_0) + 0 = \alpha f'(x_0)$$

as desired. $\square$

Of course, the quotient rule also generalizes.

**Theorem 2.9** (Quotient Rule)**.** *Let $f, g \in \mathcal{F}([a,b],\mathbb{C})$ and let $x_0 \in (a,b)$. If $g$ is differentiable at $x_0$ and $g(x_0) \neq 0$, then $\frac{1}{g}$ is differentiable at $x_0$ with*

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g^2(x_0)}$$

*Therefore, if in addition $f$ is differentiable at $x_0$, then $\frac{f}{g}$ is differentiable at $x_0$ with*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$

*Proof.* Instead of appealing to the real and imaginary parts, it is much easier to return to the definition of the derivative. Suppose $g$ is differentiable at $x_0$ and $g(x_0) \neq 0$. Since $g$ is differentiable at $x_0$, $g$ is continuous at $x_0$ and therefore, since $g(x_0) \neq 0$, there exists a $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x - x_0| < \delta$, then $x \in (a,b)$ and $g(x) \neq 0$. Thus, for all $h \in \mathbb{R}$ with $0 < |h| < \delta$ we have that

$$\frac{\frac{1}{g(x_0+h)} - \frac{1}{g(x_0)}}{h} = \frac{\frac{g(x_0)-g(x_0+h)}{g(x_0)g(x_0+h)}}{h} = \frac{g(x_0) - g(x_0+h)}{g(x_0)g(x_0+h)h}$$

Since $g$ is continuous at $x_0$, we know that

$$\lim_{h\to 0} \frac{1}{g(x_0+h)} = \frac{1}{g(x_0)}$$

Moreover, since $g$ is differentiable at $x_0$, we know that

$$\lim_{h\to 0}\frac{g(x_0)-g(x_0+h)}{h} = -g'(x_0)$$

Hence,

$$\lim_{h\to 0}\frac{\frac{1}{g(x_0+h)}-\frac{1}{g(x_0)}}{h} = -\frac{g'(x_0)}{g^2(x_0)}$$

Thus, $\frac{1}{g}$ is differentiable at $x_0$ with

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g^2(x_0)}$$

Finally, if addition $f$ is differentiable at $x_0$, then $\frac{f}{g} = f\frac{1}{g}$ is differentiable at $x_0$ by the product rule with

$$\begin{aligned}\left(\frac{f}{g}\right)'(x_0) &= f'(x_0)\frac{1}{g(x_0)} + f(x_0)\left(\frac{1}{g}\right)'(x_0)\\ &= \frac{f'(x_0)}{g(x_0)} - f(x_0)\frac{g'(x_0)}{g^2(x_0)}\\ &= \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}\end{aligned}$$

as desired. □

Finally, we have a version of the chain rule for the composition of a real-valued function with a complex-valued function.

**Theorem 2.10** (Chain Rule)**.** *Let $U$ and $V$ be open intervals, let $g : U \to \mathbb{C}$ and let $f : V \to \mathbb{R}$ be such that $f(V) \subseteq U$. Suppose that $a \in V$, $f$ is differentiable at $a$, and $g$ is differentiable at $f(a)$, Then $g \circ f : U \to \mathbb{C}$ is differentiable at $a$ and*

$$(g\circ f)'(a) = g'(f(a))f'(a)$$

*Proof.* Let $u = \Re(g)$ and $v = \Re(g)$. Then

$$(g\circ f)(x) = u(f(x)) + iv(f(x)) = (u\circ f)(x) + i(v\circ f)(x)$$

for all $x \in V$. Therefore, by the chain rule for real-valued functions, $g \circ f$ is differentiable at $a$ with

$$\begin{aligned}
(g \circ f)'(a) &= (u \circ f)'(a) + i(v \circ f)'(a) \\
&= u'(f(a))f'(a) + iv'(f(a))f'(a) \\
&= g'(f(a))f'(a)
\end{aligned}$$

as desired. $\square$

# Chapter 3

# Integration on $\mathbb{C}$

With the basics of differentiation of a complex-valued function on an interval complete, we turn our attention to integration. As differentiation can be done via the real and imaginary parts, the following definition should be no surprise.

**Definition 3.1.** Given $f \in \mathcal{F}([a,b],\mathbb{C})$, it is said that $f$ is Riemann integrable if $\Re(f)$ and $\Im(f)$ are Riemann integrable. When $f$ is Riemann integrable, the complex-valued Riemann integral of $f$ on $[a,b]$ is defined to be the complex number

$$\int_a^b f(x)dx = \int_a^b \Re(f)(x)dx + i\int_a^b \Im(f)(x)dx$$

Using the above definition and by exploiting results in the real-valued settings, we automatically generalize the Fundamental Theorems of Calculus.

**Theorem 3.2** (Fundamental Theorem of Calculus, Part I)**.** *Let $f : [a,b] \to \mathbb{C}$ be continuous and define $F : [a,b] \to \mathbb{C}$ by*

$$F(x) = \int_a^x f(t)dt$$

*for all $x \in [a,b]$. Then $F$ is continuous on $[a,b]$, differentiable on $(a,b)$ and $F'(x) = f(x)$ for all $x \in (a,b)$.*

*Proof.* Let $u = \Re(f)$, $v = \Im(f)$, $U = \Re(F)$ and $V = \Im(F)$. By the definition of the complex-valued Riemann-integral, we have that

$$U(x) = \int_a^x u(t)dt \qquad V(x) = \int_a^x v(t)dt$$

for all $x \in [a, b]$. Since $f$ is continuous, $u$ and $v$ are continuous. Therefore, by the real-valued version of the Fundamental Theorem of Calculus, we obtain that $U$ and $V$ are continuous on $[a, b]$, differentiable on $(a, b)$ and $U'(x) = u(x)$ and $V'(x) = v(x)$ for all $x \in (a, b)$. Thus, it follows from Theorem 2.3 that $F$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $F'(x) = f(x)$ for all $x \in (a, b)$. □

**Theorem 3.3** (Fundamental Theorem of Calculus, Part II)**.** *Let $f, F : [a, b] \to \mathbb{C}$ be such that $f$ is Riemann integrable on $[a, b]$, $F$ is continuous on $[a, b]$, $F$ is differentiable on $(a, b)$, and $F'(x) = f(x)$ for all $x \in (a, b)$. Then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

*Proof.* Let $u = \Re(f)$, $v = \Im(f)$, $U = \Re(F)$ and $V = \Im(F)$. Therefore, $u$ and $v$ are Riemann integrable on $[a, b]$, $U, V$ are continuous on $[a, b]$, $U, V$ are differentiable on $(a, b)$, and $U'(x) = u(x)$ and $V'(x) = v(x)$ for all $x \in (a, b)$ by Theorem 2.3. Therefore, by the real-valued version of the Fundamental Theorem of Calculus, we obtain that

$$\int_a^b u(t)dt = U(b) - U(a) \qquad \int_a^b v(t)dt = V(b) - V(a)$$

Thus,

$$\begin{aligned} \int_a^b f(t)dt &= \int_a^b u(t)dt + i\int_a^b v(t)dt \\ &= [U(b) - U(a)] + i[V(b) - V(a)] \\ &= [U(b) + iV(b)] - [U(a) + iV(a)] = F(b) - F(a) \end{aligned}$$

as desired. □

As with real-valued functions, the Fundamental Theorem of Calculus immediately enables us to integrate any functions we know the antiderivative of. In particular, the exponentials in Example 2.4 are particularly easy yet useful.

**Example 3.4.** Let $a, b \in \mathbb{R}$, let $\alpha = a + ib \in \mathbb{C} \setminus \{0\}$ and define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) = e^{\alpha x} = e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx$$

for all $x \in \mathbb{R}$. Since

$$f'(x) = \alpha e^{\alpha x}$$

for all $x \in \mathbb{R}$, we see that

$$\left(\frac{1}{\alpha} f\right)'(x) = e^{\alpha x}$$

for all $x \in \mathbb{R}$. Thus, the Fundamental Theorem of Calculus implies for all $s, t \in \mathbb{R}$ with $s < t$, that

$$\int_s^t e^{\alpha x} dx = \frac{1}{\alpha} e^{\alpha t} - \frac{1}{\alpha} e^{\alpha s}$$

To conclude this discussion, it is particularly useful to know what operations on Riemann integrable functions produce Riemann integrable functions and how the values of the integrals relate. We begin with the following.

**Proposition 3.5.** *Let $f, g \in \mathcal{F}([a, b], \mathbb{C})$ be Riemann integrable. Then the following are true.*

1. *$f + g$ is Riemann integrable.*
2. *$\bar{f}$ is Riemann integrable.*
3. *$fg$ is Riemann integrable.*
4. *$|f|$ is Riemann integrable.*

*Proof.* Let $u = \Re(f)$, $v = \Im(f)$, $s = \Re(g)$ and $t = \Im(g)$. Since $f$ and $g$ are Riemann integrable, $u, v, s, t$ are Riemann integrable. Therefore, since

$$\begin{aligned}
\Re(f + g) &= u + v \\
\Im(f + g) &= s + t \\
\Re(\bar{f}) &= u \\
\Im(\bar{f}) &= -v \\
\Re(fg) &= us - vt \\
\Im(fg) &= ut + vs \\
|f| &= \sqrt{u^2 + v^2}
\end{aligned}$$

we see that all of these functions are Riemann integrable and thus, $f + g$, $\bar{f}$, $fg$ and $|f|$ are all Riemann integrable. $\square$

Of course, it is unsurprising that the complex integral is linear.

**Proposition 3.6.** *Let $f, g \in \mathcal{F}([a,b],\mathbb{C})$ be Riemann integrable. Then*

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

*moreover, if $\alpha \in \mathbb{C}$, then*

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$$

*Proof.* Let $u = \Re(f)$ and $v = \Im(f)$, $s = \Re(g)$ and $t = \Im(g)$. Since

$$\Re(f+g) = u + s \qquad \Im(f+g) = v + t$$

we obtain by the definition of the complex integral that

$$\begin{aligned}
\int_a^b f(x) + g(x)dx &= \left(\int_a^b u(x) + s(x)dx\right) + i\left(\int_a^b v(x) + t(x)dx\right) \\
&= \left(\int_a^b u(x)dx + \int_a^b s(x)dx\right) + i\left(\int_a^b v(x)dx + \int_a^b t(x)dx\right) \\
&= \left(\int_a^b u(x)dx + i\int_a^b v(x)dx\right) + \left(\int_a^b s(x)dx + i\int_a^b t(x)dx\right) \\
&= \int_a^b f(x)dx + \int_a^b g(x)dx
\end{aligned}$$

as desired.

For the second part of the proof, write $\alpha = c + id$ where $c, d \in \mathbb{R}$. Since

$$\Re(\alpha f) = cu - dv \qquad \Im(\alpha f) = cv + du$$

we obtain by the definition of the complex integral that

$$\begin{aligned}
\int_a^b \alpha f(x)dx &= \left(\int_a^b cu(x) - dv(x)dx\right) + i\left(\int_a^b cv(x) + du(x)dx\right) \\
&= \left(c\int_a^b u(x)dx - d\int_a^b v(x)dx\right) + i\left(c\int_a^b v(x)dx + d\int_a^b u(x)dx\right) \\
&= (c+id)\int_a^b u(x)dx + (-d+ic)\int_a^b v(x)dx \\
&= \alpha\int_a^b u(x)dx + \alpha i\int_a^b v(x)dx \\
&= \alpha\left(\int_a^b u(x)dx + i\int_a^b v(x)dx\right) = \alpha\int_a^b f(x)dx
\end{aligned}$$

as desired. □

**Proposition 3.7.** *Let $f \in \mathcal{F}([a,b],\mathbb{C})$ be Riemann integrable. Then*

$$\int_a^b \overline{f(x)}dx = \overline{\int_a^b f(x)dx}$$

*Proof.* Let $u = \Re(f)$ and $v = \Im(f)$. Since

$$\Re(\bar{f}) = u \qquad \Im(\bar{f}) = -v$$

we obtain by the definition of the complex integral that

$$\begin{aligned}
\int_a^b \overline{f(x)}dx &= \int_a^b u(x)dx + i\int_a^b -v(x)dx \\
&= \int_a^b u(x)dx - i\int_a^b v(x)dx \\
&= \overline{\int_a^b u(x)dx + i\int_a^b v(x)dx} \\
&= \overline{\int_a^b f(x)dx}
\end{aligned}$$

as desired. □

We recall from integrating real-valued functions that integrals of products need not be the product of the integrals. However, integration by parts still works.

**Theorem 3.8** (Integration by Parts)**.** *Let $u, v \in \mathcal{F}([a, b], \mathbb{C})$ be continuously differentiable functions (in this type of situation, we say that $u$ and $v$ are $C^\infty$ functions). Then*

$$\int_a^b u'(x)v(x)dx = (u(b)v(b) - u(a)v(a)) - \int_a^b u(x)v'(x)dx$$

*Proof.* Let $f : [a, b] \to \mathbb{C}$ be defined by

$$f(x) = u(x)v(x)$$

for all $x \in [a, b]$. Since $u$ and $v$ are continuous on $[a, b]$, we obtain that $f$ is continuous on $[a, b]$. Moreover, since $u$ and $v$ are differentiable on $(a, b)$, the product rule implies that

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

for all $x \in (a, b)$. Furthermore, since $u$ and $v$ are $C^\infty$ functions, we see that

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

is Riemann integrable on $[a, b]$. Therefore, the Fundamental Theorem of Calculus II implies that

$$\begin{aligned} u(b)v(b) - u(a)v(a) &= f(b) - f(a) \\ &= \int_a^b f'(x)dx \\ &= \int_a^b u'(x)v(x) + u(x)v'(x)dx \\ &= \int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx \end{aligned}$$

Thus, rearranging this equation, the result follows. □

Finally, the following relates the integrals of a function and its absolute value. However, the as the proof does not follow from the real-valued results, we need to be slightly clever.

**Theorem 3.9.** *Let $f \in \mathcal{F}([a,b],\mathbb{C})$ be Riemann integrable. Then*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

*Proof.* Since

$$\int_a^b f(x)dx \in \mathbb{C}$$

there exists a $z \in \mathbb{C}$ such that $|z| = 1$ and

$$z\int_a^b f(x)dx = \left|\int_a^b f(x)dx\right|$$

(i.e. if $\int_a^b f(x)dx = re^{i\theta}$ for some $r \geq 0$ and $\theta \in [0, 2\pi)$, then take $z = e^{-i\theta}$). Therefore, we have that

$$\begin{aligned} 0 \leq \left|\int_a^b f(x)dx\right| &= z\int_a^b f(x)dx = \int_a^b zf(x)dx \\ &= \int_a^b \Re(zf(x))dx + i\int_a^b \Im(zf(x))dx \end{aligned}$$

However, the inequality implies that it must be true that

$$\int_a^b \Im(zf(x))dx = 0$$

as the only way a real number and a complex number are equal if the imaginary part of the complex number is zero. Therefore, we have that

$$\begin{aligned} \left|\int_a^b f(x)dx\right| &= \int_a^b \Re(zf(x))dx \leq \int_a^b |zf(x)|dx \\ &= \int_a^b |z||f(x)|dx = \int_a^b |f(x)|dx \end{aligned}$$

as desired. $\square$

# Chapter 4

# Fubini's Theorem

In the next three chapters, we will provide proofs to the multivariable calculus results that are used in this course.

We begin with the fundamental result of changing the order of integral signs.

**Theorem 4.1** (Fubini's Theorem). *If $f : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then*

$$y \mapsto \int_a^b f(x,y)dx \qquad x \mapsto \int_c^d f(x,y)dy$$

*are continuous on $[c,d]$ and $[a,b]$ respectively, and*

$$\iint_{[a,b]\times[c,d]} f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Before we get to the proof of Fubini's Theorem (Theorem 4.1), we first desire to present a proof that if $f : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then $f$ is Riemann integrable over $[a,b] \times [c,d]$. To do this, we reimagine the notion of uniform continuity presented in Definition 2.8.1.

**Definition 4.2.** A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be uniformly continuous on $I$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $(x_1, y_1)$, $(x_2, y_2) \in [a,b] \times [c,d]$, $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$, then $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$.

We can generalize the proof of Theorem 2.8.4 to the following.

**Theorem 4.3.** *If $f : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then $f$ is uniformly continuous.*

*Proof.* Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Suppose to the contrary that $f$ is not uniformly continuous. Hence, there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exists $(x_1, y_1), (x_2, y_2) \in [a,b] \times [c,d]$ such that $|x_1 - x_2| < \delta$, $|y_1 - y_2| < \delta$ and $|f(x_1, y_1) - f(x_2, y_2)| \geq \epsilon$. Therefore, for each $n \in \mathbb{N}$, there exist $(x_n, y_n)$, $(x'_n, y'_n) \in [a,b] \times [c,d]$ with $|x_n - x'_n| < \frac{1}{n}$, $|y_n - y'_n| < \delta$ and $|f(x_n, y_n) - f(x'_n, y'_n)| \geq \epsilon$.

Since $[a,b]$ is closed and bounded, the Bolzano-Weierstrass Theorem implies there exists a subsequence $(x_{k_n})_n$ of $(x_n)_n$ that converges to some number $L \in [a,b]$. Subsequently, since $[c,d]$ is closed and bounded, the Bolzano-Weierstrass Theorem implies there exists a subsequence $(y_{m_{k_n}})_n$ of $(y_{k_n})_n$ that converges to some number $K \in [c,d]$. Note $(x_{m_{k_n}})_n$ still converges to $L$ since $(x_{k_n})_n$ does.

Since $f$ is continuous on $[a,b] \times [c,d]$, there exists an $N_1 \in \mathbb{N}$ such that

$$|f(x_{m_{k_n}}, y_{m_{k_n}}) - f(L, K)| < \frac{\epsilon}{2}$$

for all $n \geq N_1$.

Consider the subsequence $(x'_{m_{k_n}})_n$ of $(x'_n)_n$. Notice that for all $n \in \mathbb{N}$ that

$$\begin{aligned} |x'_{m_{k_n}} - L| &\leq |x'_{m_{k_n}} - x_{m_{k_n}}| + |x_{m_{k_n}} - L| \\ &\leq \frac{1}{m_{k_n}} + |x_{m_{k_n}} - L| \\ &\leq \frac{1}{n} + |x_{m_{k_n}} - L| \end{aligned}$$

Therefore, since $\lim_{n\to\infty} |x_{m_{k_n}} - L| = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, we obtain that $\lim_{n\to\infty} x'_{m_{k_n}} = L$. Similarly, $\lim_{n\to\infty} y'_{m_{k_n}} = K$. Since $f$ is continuous this implies there exists an $N_2 \in \mathbb{N}$ such that $|f(x'_{m_{k_n}}, y'_{m_{k_n}}) - f(L, K)| < \frac{\epsilon}{2}$ for all $n \geq N_2$.

Notice if $N = \max\{N_1, N_2\}$, then the above implies that

$$\begin{aligned} |f(x_{m_{k_n}}, y_{m_{k_n}}) - f(x'_{m_{k_n}}, y'_{m_{k_n}})| &\leq |f(x_{m_{k_n}}, y_{m_{k_n}}) - f(L)| + |f(L) - f(x'_{m_{k_n}}, y'_{m_{k_n}})| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

thereby contradicting the fact that $|f(x_{m_{k_n}}, y_{m_{k_n}}) - f(x'_{m_{k_n}}, y'_{m_{k_n}})| \geq \epsilon$. Hence, $f$ is uniformly continuous on $[a,b] \times [c,d]$. □

It is the property of uniform continuity that implies continuous functions are Riemann integrable. The following demonstrates the 2-variable version whereas the 1-variable version follows by similar arguments.

**Corollary 4.4.** *If $f : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then $f$ is Riemann integrable on $[a,b] \times [c,d]$.*

*Proof.* Let $\epsilon > 0$ be arbitrary. Since $f$ is continuous on $[a,b] \times [c,d]$, $f$ is uniformly continuous on $[a,b] \times [c,d]$. Thus, there exists a $\delta > 0$ such that if $(x_1, y_1), (x_2, y_2) \in [a,b] \times [c,d]$ are such that $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$, then

$$|f(x_1, y_1) - f(x_2, y_2)| < \frac{\epsilon}{(b-a)(d-c)}$$

Let $\mathcal{P}$ be any partition of $[a,b] \times [c,d]$ with interval lengths at most $\delta$. Write $\mathcal{P} = \{\{(x_i, y_i)\}_{i=0}^{n}\}_{j=0}^{m}$. where

$$\begin{aligned} a = x_0 < x_1 < x_2 < \cdots < x_n = b \\ c = y_0 < y_1 < y_2 < \cdots < y_n = d \end{aligned}$$

For all $(i,j) \in \{1, 2, ..., n\} \times \{1, 2, ..., m\}$ let

$$\begin{aligned} M_{i,j} &= \sup_{(x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]} f(x,y) \\ m_{i,j} &= \inf_{(x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]} f(x,y) \end{aligned}$$

Then

$$\begin{aligned} U(f, \mathcal{P}) &= \sum_{i=1}^{n} \sum_{j=1}^{m} M_{i,j} (x_i - x_{i-1})(y_j - y_{j-1}) \\ L(f, \mathcal{P}) &= \sum_{i=1}^{n} \sum_{j=1}^{m} m_{i,j} (x_i - x_{i-1})(y_j - y_{j-1}) \end{aligned}$$

Hence

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{i=1}^{n} \sum_{j=1}^{m} (M_{i,j} - m_{i,j})(x_i - x_{i-1})(y_j - y_{j-1}) \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\epsilon}{(b-a)(d-c)} (x_i - x_{i-1})(y_j - y_{j-1}) \\ &= \epsilon \end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, $f$ is Riemann integrable. $\square$

*Proof of Fubini's Theorem (Theorem 4.1).* First, it is necessary to demonstrate that all of the integrals in the statement of the theorem are well-defined.

Since $f$ is Riemann integrable,

$$\iint_{[a,b]\times[c,d]} f(x,y)dA$$

is well defined. Moreover, since

$$x \mapsto f(x, y_0) \qquad y \mapsto f(x_0, y)$$

are continuous on $[a,b]$ and $[c,d]$ respectively for all $x_0 \in [a,b]$ and $y_0 \in [c,d]$ we obtain that

$$\int_a^b f(x, y_0)dx \qquad \int_c^d f(x_0, y)dy$$

are well-defined for all $x_0 \in [a,b]$ and $y_0 \in [c,d]$. Thus, to see that

$$\int_c^d \int_a^b f(x,y)dxdy \qquad \int_a^b \int_c^d f(x,y)dydx$$

are well-defined, it suffices to show that

$$y \mapsto \int_a^b f(x,y)dx \qquad x \mapsto \int_c^d f(x,y)dy$$

are continuous on $[c,d]$ and $[a,b]$ respectively.

To see that the first is continuous, let $\epsilon > 0$ be arbitrary. Since $f$ is uniformly continuous on $[a,b]\times[c,d]$, there exists a $\delta > 0$ such that if $(x_1,y_1),(x_2,y_2) \in [a,b]\times[c,d]$ are such that $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$, then

$$|f(x_1,y_1) - f(x_2,y_2)| < \frac{\epsilon}{b-a}$$

Therefore, for all $y_1, y_2 \in [c,d]$ with $|y_1 - y_1| < \delta$ we have that

$$\begin{aligned}
\left|\int_a^b f(x,y_1)dx - \int_a^b f(x,y_2)dx\right| &= \left|\int_a^b f(x,y_1) - f(x,y_2)dx\right| \\
&\leq \int_a^b |f(x,y_1) - f(x,y_2)|dx \\
&\leq \int_a^b \frac{\epsilon}{b-a}dx \\
&= \epsilon
\end{aligned}$$

Thus, $y \mapsto \int_a^b f(x,y)dx$ is uniformly continuous on $[c,d]$. Similarly, $x \mapsto \int_c^d f(x,y)dy$ is uniformly continuous on $[a,b]$.

To see that

$$\iint_{[a,b]\times[c,d]} f(x,y)da = \int_c^d \int_a^f (x,y)dxdy$$

let $\epsilon > 0$ be arbitrary. Since $f$ is Riemann integrable on $[a,b]\times[c,d]$, there exists a partition $\mathcal{P}$ of $[a,b]\times[c,d]$ such that if $\mathcal{P} = \{\{(x_i,y_i)\}_{i=0}^n\}_{j=0}^m$. where

$$\begin{aligned} a = x_0 < x_1 < x_2 < \cdots < x_n = b \\ c = y_0 < y_1 < y_2 < \cdots < y_n = d \end{aligned}$$

For all $(i,j) \in \{1,2,...,n\}\times\{1,2,...,m\}$ let

$$\begin{aligned} M_{i,j} &= \sup_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x,y) \\ m_{i,j} &= \inf_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x,y) \end{aligned}$$

then with

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{i=1}^n \sum_{j=1}^m M_{i,j}(x_i - x_{i-1})(y_j - y_{j-1}) \\ L(f,\mathcal{P}) &= \sum_{i=1}^n \sum_{j=1}^m m_{i,j}(x_i - x_{i-1})(y_j - y_{j-1}) \end{aligned}$$

we have that

$$L(f,\mathcal{P}) \leq \iint_{[a,b]\times[c,d]} f(x,y)dA \leq U(f,\mathcal{P}) < L(f,\mathcal{P}) + \epsilon$$

Since for all $(i,j) \in \{1,2,...,n\}\times\{1,2,...,m\}$ we have

$$m_{i,j} \leq f(x,y) \leq M_{i,j}$$

for all $(x,y) \in [x_{i-1},x_i]\times[y_{j-1},y_j]$ we obtain that

$$m_{i,j}(x_i - x_{i-1}) \leq \int_{x_{i-1}}^{x_i} f(x,y)dx \leq M_{i,j}(x_i - x_{i-1})$$

for all $y \in [y_{j-1}, y_j]$ and $(i,j) \in \{1,2,...,n\} \times \{1,2,...,m\}$. Therefore, for any fix $j \in \{1,2,...,m\}$ and $y \in [y_{j-1}, y_j]$ we obtain by summing over $i$ that

$$\begin{aligned}\sum_{i=1}^{n} m_{i,j}(x_i - x_{i-1}) &\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x,y)dx \\ &= \int_a^b f(x,y)dx \\ &\leq \sum_{i=1}^{n} M_{i,j}(x_i - x_{i-1})\end{aligned}$$

Thus, by integrating all three expressions over $y \in [y_{j-1}, y_j]$ for all $j \in \{1,2,...,m\}$ we obtain that

$$\begin{aligned}\sum_{i=1}^{n} m_{i,j}(x_i - x_{i-1})(y_j - y_{j-1}) &\leq \int_{y_{j-1}}^{y_j} \int_a^b f(x,y)dxdy \\ &\leq \sum_{i=1}^{n} M_{i,j}(x_i - x_{i-1})(y_j - y_{j-1})\end{aligned}$$

Thus by summing all over $j \in \{1,2,...,m\}$ we obtain that

$$\begin{aligned}L(f,\mathcal{P}) &= \sum_{j=1}^{m}\sum_{i=1}^{n} m_{i,j}(x_i - x_{i-1})(y_j - y_{j-1}) \\ &\leq \sum_{j=1}^{m} \int_{y_{j-1}}^{y_j} \int_a^b f(x,y)dxdy \\ &= \int_c^d \int_a^b f(x,y)dxdy \\ &\leq \sum_{j=1}^{m}\sum_{i=1}^{n} M_{i,j}(x_i - x_{i-1})(y_j - y_{j-1}) \\ &= U(f,\mathcal{P})\end{aligned}$$

Therefore, since both

$$\iint_{[a,b]\times[c,d]} f(x,y)dA \qquad \int_c^d \int_a^b f(x,y)dxdy$$

are in the interval $[L(f,\mathcal{P}),U(f,\mathcal{P})]$ and $U(f,\mathcal{P})-L(f,\mathcal{P})<\epsilon$, we obtain that

$$\left|\iint_{[a,b]\times[c,d]} f(x,y)dA - \int_c^d \int_a^b f(x,y)dxdy\right| < \epsilon$$

Therefore, as $\epsilon > 0$ was arbitrary, we obtain that

$$\iint_{[a,b]\times[c,d]} f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy$$

The proof that

$$\iint_{[a,b]\times[c,d]} f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx$$

is similar. $\square$

# Chapter 5

# Leibniz Integral Rule

Another useful result from multivariable calculus is the ability to differentiate under the integral sign.

**Theorem 5.1** (Leibniz Integral Rule)**.** *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that $\frac{\partial f}{\partial y}$ is continuous on $[a, b] \times [c, d]$. Then*

$$\frac{d}{dy}\left(\int_a^b f(x, y)dx\right) = \int_a^b \frac{\partial f}{\partial y}(x, y)dx$$

*on $(c, d)$.*

*Proof.* Since $\frac{\partial f}{\partial y}$ is continuous on $[a, b] \times [c, d]$, $f$ is continuous on $[a, b] \times [c, d]$ and thus both integrals in the statement of the theorem is well-defined.

To see the desired result, fix $y_0 \in (c, d)$. Consider the function $F : [c, d] \to \mathbb{R}$ defined by

$$F(t) = \int_c^t \int_a^b \frac{\partial f}{\partial y}(x, y)dxdy$$

for all $t \in [c, d]$. Note $F$ is well-defined since $\frac{\partial f}{\partial y}$ is continuous and

$$y \mapsto \int_a^b \frac{\partial f}{\partial y}(x, y)dx$$

is continuous by Fubini's Theorem (Theorem 4.1). Moreover, by the Fundamental Theorem of Calculus, we know that

$$F'(t) = \int_a^b \frac{\partial f}{\partial y}(x, t)dx$$

for all $t \in (c, d)$. Thus it suffices to show that

$$\left(\frac{dy}{dx}\left(\int_a^b f(x,y)dx\right)\right)(y_0) = F'(y_0)$$

Notice if $h > 0$ is such that $y_0+h \in [c, d]$, then by Fubini's Theorem (Theorem 4.1) and the Fundamental Theorem of Calculus, we have that

$$\begin{aligned}
\frac{F(y_0+h)-F(y_0)}{h} &= \frac{1}{h}\left(\int_c^{y_0+h}\int_a^b \frac{\partial f}{\partial y}(x,y)dxdy - \int_c^{y_0}\int_a^b \frac{\partial f}{\partial y}(x,y)dxdy\right) \\
&= \frac{1}{h}\left(\int_{y_0}^{y_0+h}\int_a^b \frac{\partial f}{\partial y}(x,y)dxdy\right) \\
&= \frac{1}{h}\int_a^b\int_{y_0}^{y_0+h} \frac{\partial f}{\partial y}(x,y)dydx \\
&= \frac{1}{h}\int_a^b f(x,y_0+h) - f(x,y_0)dx \\
&= \frac{1}{h}\left(\int_a^b f(x,y_0+h)dx - \int_a^b f(x,y_0)dx\right)
\end{aligned}$$

Thus,

$$\lim_{h\searrow 0}\frac{1}{h}\left(\int_a^b f(x,y_0+h)dx - \int_a^b f(x,y_0)dx\right) = \lim_{h\searrow 0}\frac{F(y_0+h)-F(y_0)}{h} = F'(y_0)$$

Since similar arguments show

$$\lim_{h\nearrow 0}\frac{1}{h}\left(\int_a^b f(x,y_0+h)dx - \int_a^b f(x,y_0)dx\right) = F'(y_0)$$

the result follows. $\square$

# Chapter 6

# Laplace's Equation in Polar Coordinates

One result necessary for applications to the steady-state heat equation is the following version of the Laplacian in polar coordinates.

**Theorem 6.1.** *If $f$ is a continuous function on the closed unit disk centred at the origin that is $C^2$ on the open unit centred at the origin, then using polar coordinates*

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial f^2}{\partial \theta^2}$$

*Proof.* Recall if $(x, y)$ is a point in the closed unit disk and

$$x = r\cos\theta \qquad y = r\sin\theta$$

are the polar coordinates, then

$$\frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

To show the desired formula holds, let us compute

$$\frac{\partial^2 f}{\partial r^2} \qquad \frac{\partial f}{\partial r} \qquad \frac{\partial f^2}{\partial \theta^2}$$

using the Chain Rule. Indeed

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 f}{\partial r^2} &= \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\right) \\
&= \cos\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial x} + \sin\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial y} \\
&= \cos\theta\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial f}{\partial f}\frac{\partial y}{\partial r}\right) \\
&= \cos\theta\left(\cos\theta\frac{\partial^2 f}{\partial x^2} + \sin\theta\frac{\partial^2 f}{\partial y\partial x}\right) + \sin\theta\left(\cos\theta\frac{\partial^2 f}{\partial x\partial y} + \sin\theta\frac{\partial^2 f}{\partial y^2}\right) \\
&= \cos^2\theta\frac{\partial^2 f}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Similarly,

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 f}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\right) \\
&= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial}{\partial \theta}\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial f}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\frac{\partial f}{\partial y} \\
&= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&- r\sin\theta\frac{\partial f}{\partial y} + r\cos\theta\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x^2} + r\cos\theta\frac{\partial^2 f}{\partial y\partial x}\right) \\
&- r\sin\theta\frac{\partial f}{\partial y} + r\cos\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x\partial y} + r\cos\theta\frac{\partial^2 f}{\partial y^2}\right) \\
&= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial f}{\partial y} + r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} \\
&- 2r\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial f^2}{\partial \theta^2} &= \left(\cos^2\theta \frac{\partial^2 f}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 f}{\partial y^2}\right) \\
&+ \frac{1}{r}\left(\cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\right) \\
&+ \frac{1}{r^2}\left(-r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial f}{\partial y}\right) \\
&+ \frac{1}{r^2}\left(r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} - 2r\cos\theta\sin\theta\frac{\partial^2 f}{\partial x \partial y} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}\right) \\
&= (\cos^2\theta + \sin^2\theta)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \\
&= \Delta f
\end{aligned}$$

as desired. $\square$

# Chapter 7

# Sequences of Complex Numbers

**Definition 7.1** (Sequence of Complex Numbers)**.** A sequence of $\mathbb{K}$ numbers is a function $z : \mathbb{N} \to \mathbb{C}$, and we denote the sequence by $(z_n)_n = z(n)$.

As a series will be a limit of finite sums, it is useful to recall the definition of a limit. Since we will be dealing with complex numbers throughout the course, we will quickly generalize the basic properties of limits of sequences of real numbers in a previous course to the complex setting.

**Definition 7.2** (Convergence of a Sequence)**.** A sequence $(z_n)_n$ of complex numbers is said to *converge* to a complex number $L$ if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N$. The complex number $L$ is called a limit of $(z_n)_n$ and is denoted by $\lim_{n\to\infty} z_n = L$. Moreover, the sequence of $(z_n)_n$ is said to diverge if $(z_n)_n$ is said to *diverge* if $(z_n)_n$ does not converge to any complex number.

With Definition 7.2, we are now able to proceed to prove the following lemma.

**Lemma 7.3** (Conditions for Convergence of a Sequence)**.** *Let $(z_n)_n$ be a sequence of complex numbers and let $L \in \mathbb{C}$. Then $(z_n)_n$ converges to $L$ if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N$.*

*Proof.* ($\Rightarrow$) First, let's assume that $(z_n)_n$ converges to $L$. Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N$, hence by Definition 7.2, the proof is complete.

($\Leftarrow$) Next, let's assume that for all $\epsilon > 0$, and define $\epsilon_0 = \frac{\epsilon}{2}$. Since $\epsilon_0 > 0$, then this implies that there exists a $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon_0$ for all $n \geq N$. Hence, $|z_n - L| < \epsilon < \epsilon$ for all $n \geq N$. Therefore, $(z_n)_n$ converges to $L$ by Definition 7.2. $\square$

Recall that if $z = x + iy$ is a complex number, where $(x, y) \in \mathbb{R}^2$, then the *real* and *imaginary* parts of $z$ are given by

$$\Re(z) = x \qquad \Im(z) = y$$

The next lemma will be very useful for the coming proofs.

**Lemma 7.4.** *Let $(z_n)_n$ be a sequence of complex numbers and let $L \in \mathbb{C}$. Then $(z_n)_n$ converges to $L$ if and only if $(\Re(z_n))_n$ and $(\Im(z_n))_n$ converges to $\Re(L)$ and $\Im(L)$ respectively.*

*Proof.* ($\Rightarrow$) Let's first assume that $(z_n)_n$ converges to $L$. Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N$. Then using the fact that $|\Re(z)| \leq |z|$ and $|\Im(z)| \leq |z|$ for any complex number, we have

$$|\Re(z_n) - \Re(L)| \leq |z_n - L| < \epsilon$$

and

$$|\Im(z_n) - \Im(L)| \leq |z_n - L| < \epsilon$$

Therefore, $(\Re(z_n))_n$ and $(\Im(z_n))_n$ converges to $\Re(L)$ and $\Im(L)$ respectively.

($\Leftarrow$) Next, let's assume that $(\Re(z_n))_n$ and $(\Im(z_n))_n$ converges to $\Re(L)$ and $\Im(L)$ respectively. Then for any $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|\Re(z_n) - \Re(L)| < \frac{1}{\sqrt{2}}\epsilon$$

for all $n \geq N_1$ and

$$|\Im(z_n) - \Im(L)| < \frac{1}{\sqrt{2}}\epsilon$$

for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Hence, for all $n \geq N$, we have

$$\begin{aligned}
|z_n - L| &= \sqrt{(\Re(z_n) - \Re(L))^2 + (\Im(z_n) - \Im(L))^2} \\
&= \sqrt{\left(\frac{1}{\sqrt{2}}\epsilon\right)^2 + \left(\frac{1}{\sqrt{2}}\epsilon\right)^2} \\
&= \sqrt{\frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^2} \\
&= \sqrt{\epsilon^2} \\
&= \epsilon
\end{aligned}$$

Therefore, $(z_n)_n$ converges to $L$. □

With Lemma 7.4, we are now able to prove the uniqueness of the limit for convergent sequences of complex numbers.

**Corollary 7.5** (Uniqueness of a Limit)**.** *Let $L, K \in \mathbb{C}$ and let $(z_n)_n$ be a sequence of complex numbers. If $L$ and $K$ are limits of $(z_n)_n$, then $L = K$.*

*Proof.* We can use Lemma 7.4 to prove this corollary. Suppose $L$ and $K$ are limits of $(z_n)_n$, then Lemma 7.4 implies that $\Re(L)$ and $\Re(K)$ are limits of $(\Re(z_n))_n$ and $\Im(L)$ and $\Im(K)$ are limits of $(\Im(z_n))_n$. Hence, the corresponding result for sequences of real numbers implies that $\Re(L) = \Re(K)$ and $\Im(L) = \Im(K)$, and hence $L = K$ as desired.

Alternatively, suppose $L \neq K$, and suppose that $(z_n)_n$ converges to $L$ and $K$. Let $\epsilon = \frac{|L-K|}{2} > 0$. Then there exists $N_1, N_2 \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N_1$, and $|z_n - K|$ for all $n \geq N_2$. Next, let $N = \max\{N_1, N_2\}$. Hence, for all $n \geq N$, using the *triangle inequality*,

$$|L - K| \leq |L - z_n| + |z_n - K| < \epsilon + \epsilon = 2\epsilon = |L - K|$$

which contradicts our assumption, and so it must be the case that $L = K$. □

**Definition 7.6** (Bounded Sequences)**.** A sequence $(z_n)_n$ is said to be bounded if there exists a $M \in \mathbb{R}$ with $M > 0$ such that

$$|z_n| \leq M$$

for all $n \in \mathbb{N}$ where

$$M = \sup(\{|z_n| : n \in \mathbb{N}\}) < \infty$$

**Corollary 7.7** (Convergent Sequences are Bounded). *Let $(z_n)_n$ be a sequence of complex numbers, and let $L \in \mathbb{C}$. If $(z_n)_n$ is a convergent sequence that converges to $L$, then $(z_n)_n$ is bounded.*

*Proof.* Suppose $(z_n)_n$ converges to $L$, and suppose $\epsilon = 1$. Then by Definition 7.2, there exists a $N \in \mathbb{N}$ such that

$$|z_n - L| < 1$$

Then by the reverse triangle inequality:

$$||z_n| - |L|| \leq |z_n - L| < 1$$

and thus, $|z_n| < |L| + 1$ for all $n \geq N$. Let

$$M = \max(\{|z_1|, |z_2|, ..., |z_{N-1}|, L + 1\})$$

Clearly, $M \in \mathbb{R}$ and $M > 0$. Moreover, $|z_n| \leq M$ for all $n < N$ by construction. Furthermore, the above implies that $|z_n| < |L| + 1 \leq M$ for all $n \geq N$. Hence, $|z_n| \leq M$ for all $n \in \mathbb{N}$. Therefore, $(z_n)_n$ is bounded. $\square$

Now we would like to use the results proved above to prove the convergence laws for sequences.

**Corollary 7.8** (Algebraic Limit Theorems for Sequences). *Let $L, K \in \mathbb{C}$ and let $(z_n)_n$ and $(w_n)_n$ be sequences of complex numbers that converge to $L$ and $K$ respectively. Then the following are true.*

1. *$(z_n + w_n)_n$ converges to $L + K$.*
2. *$(z_n w_n)$ converges to $LK$.*
3. *$(\alpha z_n)_n$ converges to $\alpha L$.*
4. *If $L \neq 0$ and $z_n \neq 0$ for all $n \in \mathbb{N}$, then $\left(\frac{1}{z_n}\right)_n$ converges to $\frac{1}{L}$.*
5. *If $K \neq 0$ and $w_n \neq 0$ for all $n \in \mathbb{N}$, then $\left(\frac{z_n}{w_n}\right)_n$ converges to $\frac{L}{K}$.*
6. *$(\bar{z}_n)_n$ converges to $\bar{L}$.*
7. *$(|z_n|)_n$ converges to $|L|$.*

*Proof.* 1. Assume that $(z_n)_n$ and $(w_n)_n$ converges to $L$ and $K$ respectively. Then for any $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|z_n - L| < \frac{\epsilon}{2}$$

for all $n \geq N_1$ and

$$|w_n - K| < \frac{\epsilon}{2}$$

for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Hence, for all $n \geq N$,

$$|(z_n + w_n) - (L + K)| \leq |z_n - L| + |w_n - K| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $(z_n + w_n)_n$ converges to $L + K$.

2. Assume that $(z_n)_n$ and $(w_n)_n$ converges to $L$ and $K$ respectively. By Corollary 7.7, there exists a $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$. Then for any $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|z_n - L| < \frac{\epsilon}{2(|K| + 1)}$$

for all $n \geq N_1$ and

$$|w_n - K| < \frac{\epsilon}{2(M + 1)}$$

for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Hence, for all $n \geq N$,

$$\begin{aligned}
|z_n w_n - LK| &= |z_n w_n - z_n K + z_n K - LK| \\
&\leq |z_n||w_n - K| + |K||z_n - L| \\
&\leq M\frac{\epsilon}{2(M+1)} + |K|\frac{\epsilon}{2(|K|+1)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore, $(z_n w_n)_n$ converges to $LK$.

3. This immediately follows from (2) by using the constant complex $(\alpha)_n$ in place of $(w_n)_n$.

4. First note since $(z_n)_n$ converges to $L$ and since $L \neq 0$, and thus $|L| \neq 0$ that there exists an $N_1 \in \mathbb{N}$ such that $|z_n - L| < \frac{|L|}{2}$ for all $n \geq N_1$. Hence, by the reverse triangle inequality implies for all $n \geq N_1$, that
$$|L| - |z_n| < \frac{|L|}{2}$$
and thus, $\frac{|L|}{2} < |z_n|$ for all $n \geq N_1$. Therefore, $\frac{1}{|z_n|} \leq \frac{2}{|L|}$ for all $n \geq N_1$. Next, let's assume that $(z_n)_n$ converges to $L$. For $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that
$$|z_n - L| < \frac{|L|^2}{2}\epsilon$$
for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Hence for all $n \geq N$,
$$\begin{aligned}\left|\frac{1}{z_n} - \frac{1}{L}\right| &= \left|\frac{L - z_n}{z_n L}\right| \\ &= |L - z_n|\frac{1}{|z_n|}\frac{1}{|L|} \\ &< \left(\frac{|L|^2}{2}\epsilon\right)\left(\frac{2}{|L|}\right)\frac{1}{|L|} = \epsilon\end{aligned}$$
Therefore, $\left(\frac{1}{z_n}\right)_n$ converges to $\frac{1}{L}$.

5. Assume that $(z_n)_n$ converges to $L$, and assume that if $K \neq 0$ and $K \neq 0$, $(w_n)_n$ converges to $K$. Then there exists a $N_1 \in \mathbb{N}$ such that
$$|z_n - L| < \frac{|K|}{4}\epsilon$$
for all $n \geq N_1$. Note since $K \neq 0$ and so $|K| \neq 0$, then there exists an $N_2 \in \mathbb{N}$ such that $|w_n - K| < \frac{|K|}{2}$ for all $n \geq N_2$. Hence, the reverse triangle inequality implies that
$$|K| - |w_n| < \frac{|K|}{2}$$
for all $n \geq N_2$. Therefore, $\frac{1}{|w_n|} \leq \frac{2}{|K|}$ for all $n \geq N_2$. Now for $\epsilon > 0$, there exists $N_3 \in \mathbb{N}$ such that
$$|w_n - K| < \frac{|K|^2}{4(|L| + 1)}\epsilon$$

for all $n \geq N_3$. Let $N = \max\{N_1, N_2, N_3\}$. Hence for all $n \geq N$, we have

$$\begin{aligned}
\left|\frac{z_n}{w_n} - \frac{L}{K}\right| &= \left|\frac{z_n K - w_n L}{w_n K}\right| \\
&= \left|\frac{z_n K - LK + LK - w_n L}{w_n K}\right| \\
&\leq \frac{|K||z_n - L| + |L||w_n - K|}{|w_n||K|} \\
&< \frac{2}{|K|^2}\left(|K|\frac{|K|}{4}\epsilon + |L|\frac{|K|^2}{4(|L|+1)}\epsilon\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Hence, $\left(\frac{z_n}{w_n}\right)_n$ converges to $\frac{L}{K}$.

6. Assume that $(z_n)_n$ converges to $L$. By Lemma 7.4, if $(z_n)_n$ converges to $L$, then $(\Re(z_n))_n$ and $(\Im(z_n))_n$ converges to $\Re(L)$ and $\Im(L)$ respectively. Hence, by (3), $(-\Im(z_n))_n$ converges to $-\Im(L)$, and therefore by the definition of the complex conjugate, Lemma 7.4 implies $(\bar{z}_n)_n$ converges to $\bar{L}$.

7. Assume that $(z_n)_n$ converges to $L$. For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|z_n - L| < \epsilon$ for all $n \geq N$. Therefore, for all $n \geq N$, by the reverse triangle inequality

$$||z_n| - |L|| \leq |z_n - L| < \epsilon$$

Therefore, $(|z_n|)_n$ converges to $|L|$.

□

**Definition 7.9** (Cauchy Sequences)**.** A sequence $(z_n)_n$ of complex numbers is said to be *Cauchy* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $n, m \geq N$.

**Theorem 7.10** (Cauchy Criterion for Sequences)**.** *A sequence of complex numbers converges if and only if it is Cauchy.*

*Proof.* ($\Rightarrow$) Let $(z_n)_n$ be a sequence of complex numbers. Assume that $(z_n)_n$ converges to $L$. Then for $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|z_n - L| < \frac{\epsilon}{2}$$

for all $n \geq N$. Hence, for all $n, m \geq N$,

$$|z_n - z_m| = |z_n - L + L - z_m| \leq |z_n - L| + |z_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $(z_n)_n$ is Cauchy.

($\Leftarrow$) Suppose $(z_n)_n$ is Cauchy. To see that $(z_n)_n$ converges, we claim that $(\Re(z_n))_n$ and $(\Im(z_n))_n$ are Cauchy sequences of real numbers. For $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $n, m \geq N$. Thus, for all $n, m \geq N$, we have that

$$|\Re(z_n) - \Re(z_m)| = |\Re(z_n - z_m)| \leq |z_n - z_m| < \epsilon$$

and

$$|\Im(z_n) - \Im(z_m)| = |\Im(z_n - z_m)| \leq |z_n - z_m| < \epsilon$$

Therefore, $(\Re(z_n))_n$ and $(\Im(z_n))_n$ are Cauchy sequences of real numbers.

Since every Cauchy sequence of real numbers converges, there exists $x, y \in \mathbb{R}$ such that $(\Re(z_n))_n$ and $(\Im(z_n))_n$ converge to $x$ and $y$ respectively. Hence, $(z_n)_n$ converges to $x + iy$ by Lemma 7.4. $\square$

We will now finish this chapter with some interesting results for sequences of real numbers.

**Definition 7.11** (Increasing and Decreasing Sequences)**.** A sequence of real numbers $(x_n)_n$ is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence of real numbers is called monotone if it is either increasing or decreasing for all $n \in \mathbb{N}$.

With the definition above, comes with a very useful theorem for sequences that we will often see and use in this class.

**Theorem 7.12** (Monotone Convergence Theorem)**.** *Let $(x_n)_n$ be a sequence of real numbers. If $(x_n)_n$ is monotone and bounded, then $(x_n)_n$ converges to $L$.*

*Proof.* Let $(x_n)_n$ be a monotone and bounded sequence of real numbers. Suppose $(x_n)_n$ is increasing, and $(x_n)_n$ is bounded. Then there exists a least upper bound

$$L = \sup(\{x_n : n \in \mathbb{N}\}) < \infty$$

For $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$L - \epsilon < x_N \leq x_n \leq L < L + \epsilon$$

and therefore $|x_n - L| < \epsilon$ for all $n \geq N$. Hence, $(x_n)_n$ is convergent and converges to $L$.

Similarly, suppose $(x_n)_n$ is decreasing, and $(x_n)_n$ is bounded. Then there exists a greatest lower bound

$$L = \inf(\{x_n : n \in \mathbb{N}\}) < \infty$$

For $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$L - \epsilon < L \leq x_n \leq x_N < L + \epsilon$$

and therefore $|x_n - L| < \epsilon$ for all $n \geq N$. Hence, $(x_n)_n$ is convergent and converges to $L$. □

**Corollary 7.13.** *Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers such that there exists an $N \in \mathbb{N}$ with $x_n \leq y_n$ for all $n \geq N$.*

1. *If $(x_n)_n$ is an increasing sequence, and if $(y_n)_n$ converges, then $(x_n)_n$ converges.*

2. *If $(x_n)_n$ is an increasing sequence, and if $(x_n)_n$ diverges, then $(y_n)_n$ diverges.*

3. *If $(y_n)_n$ is a decreasing sequence, and if $(x_n)_n$ converges, then $(y_n)_n$ converges.*

4. *If $(y_n)_n$ is a decreasing sequence, and if $(y_n)_n$ diverges, then $(x_n)_n$ diverges.*

*Proof.* Exercise. □

**Theorem 7.14** (Order Limit Theorem). *Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers, and let $L, K \in \mathbb{R}$ such that $(x_n)_n$ and $(y_n)_n$ converges to $L$ and $K$ respectively. If there exists an $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $L \leq K$.*

*Proof.* Let's assume that $L > K$ instead. Then for $\epsilon = \frac{K-L}{2} > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - L| < \epsilon$$

for all $n \geq N_1$ and

$$|y_n - K| < \epsilon$$

for all $n \geq N_2$. Then

$$L - \epsilon < x < L + \epsilon$$

for all $n \geq N_1$ and

$$K - \epsilon < y < K + \epsilon$$

for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Since $L - \epsilon = \frac{L+K}{2} = K + \epsilon$ for all $n \geq N$, we obtain that

$$y_n < \frac{L+K}{2} < x_n$$

which is a contradiction. Therefore, $L \leq K$. □

Recall that for real-valued functions that if $f(x) \leq g(x) \leq h(x)$ and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

Then we can conclude that $\lim_{x \to a} g(x) = L$ as well. This is known as the Squeeze Theorem. This result is also true for sequences of real numbers.

**Theorem 7.15** (Squeeze Theorem)**.** *Let $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ be sequences of real numbers. If there exists an $N \in \mathbb{N}$ such that $x_n \leq y_n \leq z_n$ and $\lim_{n \to \infty} x_n = L = \lim_{n \to \infty} z_n$ for all $n \geq N$, then $\lim_{n \to \infty} y_n = L$.*

*Proof.* Let's assume that $(x_n)_n$ and $(z_n)_n$ converges to $L$. Then for $\epsilon > 0$, there exists $N_1, N_2$ such that

$$|x_n - L| < \epsilon \Rightarrow L - \epsilon < x_n < L + \epsilon$$

for all $n \geq N_1$ and

$$|z_n - L| < \epsilon \Rightarrow L - \epsilon < z_n < L + \epsilon$$

Let $N = \max\{N_1, N_2\}$. Since $x_n \leq y_n \leq z_n$,

$$L - \epsilon < x_n \leq y_n \leq z_n \leq L + \epsilon$$

for all $n \geq N$, then $|y_n - L| < \epsilon$. Therefore, $\lim_{n \to \infty} y_n = L$. □

# Chapter 8

# Series of Complex Numbers

This chapter is dedicated to the study of series of real and complex numbers. The previous chapter on sequences will be essential for the development of this chapter since a series can be seen as the limit of sequence (of partial sums).

**Definition 8.1** (Partial Sums and Series)**.** Let $(z_n)_n$ be a sequence of complex numbers.

1. The sequence of partial sums associated to $(z_n)_n$ is a sequence $(S_N)_N$ defined by
$$S_N = \sum_{k=1}^{N} z_k = z_1 + z_2 + z_3 + \cdots z_N$$
2. The (infinite) series of complex numbers associated to the sequence $(z_n)_n$ is defined by
$$L = \lim_{n\to\infty} S_n$$

**Definition 8.2** (Convergence of Series of Complex Numbers)**.** Let $(z_n)_n$ be a sequence of complex numbers and for each $N \in \mathbb{N}$, let $S_N = \sum_{k=1}^{N} z_k$ be the partial sums associated to $(z_n)_n$. The series $\sum_{n=1}^{\infty} z_n$ is said to converge to $L \in \mathbb{C}$ if the sequence $(S_N)_N$ converges to $L$. The series $\sum_{n=1}^{\infty} z_n$ is said to diverge if the sequence $(S_N)_N$ diverges.

**Example 8.3** (Geometric Series)**.** Let $z \in \mathbb{C}$ be such that $|z| < 1$. We claim that
$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

To see this, let $N \in \mathbb{N}$ be arbitrary. Notice that the $N$th partial sum is

$$S_N = \sum_{k=0}^{N} z^k$$

so multiplying both sides by $z$,

$$zS_N = z\sum_{k=0}^{N} z_k = \sum_{k=0}^{N} z^{k+1} = \sum_{k=0}^{N+1} z^k$$

Hence, $zS_N - S_N = z^{N+1} - 1$ so

$$S_N = \frac{z^{N+1} - 1}{z - 1}$$

since $z \neq 1$. Therefore, since $|z^{N+1}| = |z|^{N+1}$ is easily seen to converge to 0 as $N$ tends to infinity as $|z| < 1$, we see that $\lim_{N\to\infty} z^{N+1} = 0$ and so

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{z^{N+1} - 1}{z - 1} = \frac{0 - 1}{z - 1} = \frac{1}{1 - z}$$

Thus, $\sum_{n=0}^{\infty} \frac{1}{1-z}$. This series is called a *geometric series.*

Unsurprisingly, convergent series of complex numbers behave well in regards to addition and scalar multiplication as convergent sequences behave well due to Corollary 7.8.

**Lemma 8.4** (Algebraic Limit Theorems For Series)**.** *Let $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ be convergent series of complex numbers. Then $\sum_{n=1}^{\infty} z_n + w_n$ converges and*

$$\sum_{n=1}^{\infty} z_n + w_n = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$$

*Moreover, for all $\alpha \in \mathbb{C}$, the series $\sum_{n=1}^{\infty} \alpha z_n$ converges and*

$$\sum_{n=1}^{\infty} \alpha z_n = \alpha \sum_{n=1}^{\infty} z_n$$

*Proof.* For all $N \in \mathbb{N}$, consider the $N$th partial sums

$$S_N = \sum_{k=1}^{N} z_k \quad T_N = \sum_{k=1}^{N} w_k \quad R_N = \sum_{k=1}^{N} z_k + w_k \quad U_N = \sum_{k=1}^{N} \alpha z_k$$

Since $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ converges, we know that $(S_N)_N$ and $(T_N)_N$ converges and

$$\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} z_n \qquad \lim_{N\to\infty} T_N = \sum_{n=1}^{\infty} w_n$$

Since $R_N = S_N + T_N$ and $U_N = \alpha S_N$, Corollary 7.8 implies $(R_N)_N$ and $(U_N)_N$ converge and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} S_N + \lim_{N\to\infty} T_N$$

and

$$\lim_{N\to\infty} U_N = \alpha \lim_{N\to\infty} S_N$$

Hence, $\sum_{n=1}^{\infty} z_n + w_n$ and $\sum_{n=1}^{\infty} \alpha z_n$ converge and

$$\sum_{n=1}^{\infty} z_n + w_n = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$$

and

$$\sum_{n=1}^{\infty} \alpha z_n = \alpha \sum_{n=1}^{\infty} z_n$$

as desired. □

Like what we have introduced with Cauchy sequences (Theorem 7.10), we also have a Cauchy Criterion for Series.

**Theorem 8.5** (Cauchy Criterion for Series). *A series $\sum_{n=1}^{\infty}$ converges if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$\left| \sum_{k=N}^{m} z_k \right| < \epsilon$$

*for all $m \geq N$.*

*Proof.* For each $N \in \mathbb{N}$, let $S_N = \sum_{k=1}^{N} z_k$.

($\Rightarrow$) Suppose $\sum_{n=1}^{\infty} z_n$ converges. For $\epsilon > 0$, this implies that $(S_N)_N$ converges as well, and therefore, is Cauchy by Theorem 7.10. Hence, there exists an $N_0 \in \mathbb{N}$ such that

$$|S_m - S_k| < \epsilon$$

for all $m, k \geq N_0$. Therefore, if $N = N_0 + 1$, we see that for all $m \geq N > N_0$, that

$$\left|\sum_{k=N}^{m} z_k\right| = \left|\sum_{k=1}^{m} z_k - \sum_{k=1}^{N-1} z_k\right| = |S_m - S_{N-1}| < \epsilon$$

Therefore, the desired statement from the theorem holds.

($\Leftarrow$) Suppose for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\left|\sum_{k=N}^{m} z_k\right| < \epsilon$$

for all $m \geq N$. We claim that $(S_N)_N$ is Cauchy. For $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left|\sum_{k=N}^{m} z_k\right| < \frac{\epsilon}{2}$$

for all $m \geq N$. Hence, for all $n \geq m \geq N$,

$$\begin{aligned}
|S_n - S_m| &= \left|\sum_{k=1}^{n} - \sum_{k=1}^{m} z_k\right| \\
&= \left|\sum_{k=N}^{n} z_k - \sum_{k=N}^{m}\right| \\
&\leq \left|\sum_{k=N}^{n} z_k\right| + \left|\sum_{k=N}^{m} z_k\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore, $(S_N)_N$ is Cauchy.

Since $(S_N)_N$ is Cauchy, $(S_N)_N$ converges by Theorem 7.10. Hence, $\sum_{n=1}^{\infty} z_n$ converges by definition. $\square$

Immediately from the Cauchy Criterion for Series (Theorem 8.5) can be used to show a required property for series to converge.

**Corollary 8.6** (Property of Convergence)**.** *If a series* $\sum_{n=1}^{\infty} z_n$ *of complex numbers converge, then* $\lim_{n\to\infty} z_n = 0$.

*Proof.* Let $\epsilon > 0$. By Cauchy Criterion for Series (Theorem 8.5), there exists an $N \in \mathbb{N}$ such that

$$\left|\sum_{k=N}^{m} z_k\right| < \frac{\epsilon}{2}$$

for all $m \geq N$. Therefore, for all $n \geq N+1$, we have $n$, $n-1 \geq N$ so that

$$|z_n| = \left|\sum_{k=N}^{n} z_k - \sum_{k=N}^{n-1} z_k\right| \leq \left|\sum_{k=N}^{n} z_k\right| + \left|\sum_{k=N}^{n-1} z_k\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $\lim_{n\to\infty} z_n = 0$. □

It is important to point out that the converse of Corollary 8.6 is false. There are sequences $(z_n)_n$ of complex numbers with $\lim_{n\to\infty} z_n = 0$ but $\sum_{n=1}^{\infty} z_n$ still diverges.

**Example 8.7.** Let $z \in \mathbb{C}$ be such that $|z| \geq 1$. We claim that the geometric series $\sum_{n=0}^{\infty} z^n$ does not converge. Note that $|z^n - 0| = |z|^n$ for all $n \in \mathbb{N}$. So either $\lim_{n\to\infty} |z|^n$ diverges to infinity or is equal to 1. In either case, $(z^n)_n$ does not converge to 0 as $n$ tends to infinity, so Corollary 8.6 cannot converge.

Although the Cauchy Criterion for Series (Theorem 8.5) helps us with a theoretical tool for determining when a series of complex numbers converges, it is still quite difficult to use. To aid us in our ability to determine whether or not a series converges, let us deal with specific case of series of nonnegative real numbers.

**Theorem 8.8** (Convergence of Series of Nonnegative Real Numbers)**.** *Let* $(x_n)_n$ *be a sequence of real numbers with* $x_n \geq 0$ *for all* $n \in \mathbb{N}$.

1. *The series* $\sum_{n=1}^{\infty} x_n$ *converges if and only if there exists an* $M \in \mathbb{R}$ *such that* $\sum_{k=1}^{N} x_k \leq M$ *for all* $N \in \mathbb{N}$.

2. *If* $\sum_{k=1}^{N} x_k \leq M$ *for all* $N \in \mathbb{N}$*, then* $\sum_{n=1}^{\infty} x_k \leq M$.

3. *If* $\sum_{n=1}^{\infty} x_n$ *converges, then for all* $\epsilon > 0$*, there exists a* $N_0 \in \mathbb{N}$ *such that* $\sum_{k=N}^{\infty} x_k < \epsilon$ *for all* $N > N_0$*.*

*Proof.* For every $N \in \mathbb{N}$, let $S_N = \sum_{k=1}^{N} x_k$.

1. Since $x_{N+1} \geq 0$ for all $N \in \mathbb{N}$, we see that

$$S_{N+1} = \sum_{k=1}^{N+1} x_k = x_{N+1} + \sum_{k=1}^{N} x_k = x_{N+1} + S_N \geq S_N$$

   Hence, $(S_N)_N$ is a non-decreasing sequence of real numbers. Therefore, the Monotone Convergence Theorem (Theorem 7.12) implies that $(S_N)_N$ converges if and only if $(S_N)_N$ is bounded.

2. As any upper bound of $(S_N)_N$ must be greater than or equal to $\lim_{N\to\infty} S_N$, therefore this statement is true.

3. Let $\epsilon > 0$. By the Cauchy Criterion for Series (Theorem 2.3), there exists an $N_0 \in \mathbb{N}$ such that

$$\sum_{k=N_0}^{m} x_k = \left| \sum_{k=N_0}^{m} x_k \right| < \epsilon$$

   Hence, taking the limit as $m \to \infty$, we obtain that $\sum_{k=N_0}^{\infty} x_k < \epsilon$. Since $\sum_{k=N}^{\infty} x_k$ converges as only the tail matters, and since clearly the partial sums of $\sum_{k=N}^{\infty} x_k$ are bounded above by the partial sums of $\sum_{k=N_0}^{\infty} x_k$ for $N \geq N_0$, the result follows.

□

**Definition 8.9** (Absolutely Convergent Series)**.** A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge *absolutely* if $\sum_{n=1}^{\infty} |z_n|$ converges.

**Theorem 8.10** (Criterion for Absolutely Convergent Series)**.** *If* $\sum_{n=1}^{\infty} z_n$ *is an absolutely convergent series of complex numbers, then* $\sum_{n=1}^{\infty} z_n$ *converges. Moreover,*

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|$$

*Proof.* Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} z_n$ converges absolutely, we know that $\sum_{n=1}^{\infty} |z_n|$ converges. Hence, the Cauchy Criterion for Series (Theorem 8.5) implies that there exists an $N \in \mathbb{N}$ such that

$$\left| \sum_{k=N}^{m} |z_k| \right| < \epsilon$$

for all $m \geq N$. Hence, for all $m \geq N$, we have that

$$\left| \sum_{k=N}^{m} z_k \right| \leq \sum_{k=N}^{m} |z_k| = \left| \sum_{k=N}^{m} |z_k| \right| \leq \epsilon$$

Therefore, the Cauchy Criterion for Series (Theorem 8.5) implies that $\sum_{n=1}^{\infty} z_n$ converges.

Moreover, by Corollary 7.8 part (7) implies that

$$\left| \sum_{n=1}^{\infty} z_n \right| = \lim_{N \to \infty} \left| \sum_{k=1}^{N} z_k \right| \leq \liminf_{N \to \infty} \sum_{k=1}^{N} |z_k| = \sum_{n=1}^{\infty} |z_k|$$

as desired. □

Since absolutely convergent series converge, it is useful to develop a collection of 'tests' that will aid us in determining whether a series of non-negative real numbers converges (thereby aiding in determining whether a series of complex numbers converges absolutely).

**Theorem 8.11** (Comparison Test)**.** *Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. Then*

1. *If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.*
2. *If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ diverges.*

*Proof.* As these two statements are contrapositives of each other, it suffices to prove the first statement. Let's assume that $\sum_{n=1}^{\infty} y_n$ converges. Then Theorem 8.8 implies that there exists an $M \in \mathbb{R}$ such that $\sum_{k=1}^{N} y_k \leq M$ for all $N \in \mathbb{N}$. Since $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$, we obtain that for all $N \in \mathbb{N}$ that

$$\sum_{k=1}^{N} x_k \leq \sum_{k=1}^{N} y_k \leq M$$

Hence, Theorem 8.8 implies that $\sum_{n=1}^{\infty} x_n$ converges. □

**Theorem 8.12** (Integral Test)**.** *If $f : [1, \infty) \to [0, \infty)$ be a non-increasing function and $x_n = f(n)$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.*

*Proof.* Using the definition of Riemann sums, it is not difficult to see that for all $N \in \mathbb{N}$ that

$$\sum_{k=2}^{N} x_k \leq \int_1^N f(x)dx \leq \sum_{k=1}^{N} x_k$$

Therefore, since $f$ is non-increasing and non-negative and thus, $x_n \geq 0$ for all $n \in \mathbb{N}$, we see that

$$\left\{ \sum_{k=1}^{N} x_k : N \in \mathbb{N} \right\}$$

is bounded if and only if

$$\left\{ \int_1^b f(x)dx : b \geq 1 \right\}$$

is bounded. Hence, the result follows from Theorem 8.8 and the analogous result for improper integrals of non-increasing non-negative functions. $\square$

**Corollary 8.13** ($p$-Series Test)**.** *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

*Proof.* First, consider the case $p \leq 0$. In this case, the sequence $\left(\frac{1}{n^p}\right)_n$ does not converge to zero and thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ does not converge by Corollary 8.6.

Otherwise, let's consider for $p > 0$. Notice that the function $f : [1, \infty) \to (0, \infty)$ defined by $f(x) = \frac{1}{x^p}$ is well-defined. Moreover, since $f'(x) = -\frac{p}{x^{p+1}} < 0$ for all $x \in [1, \infty)$, we obtain that $f$ is non-decreasing. Furthermore, notice that for all $b > 0$ that

$$\int_1^b f(x)dx = \begin{cases} \ln b & \text{if } p = 1 \\ \frac{1}{p-1} - \frac{1}{(p-1)b^{p-1}} & \text{if } p \neq 1 \end{cases}$$

Hence, we easily see that

$$\lim_{b \to \infty} \int_1^b f(x)dx$$

exists if and only if $p > 1$ as desired. $\square$

**Theorem 8.14** (Ratio Test)**.** *Let $(x_n)_n$ be a sequence of real numbers such that $x_n > 0$ for all $n \in \mathbb{N}$. Suppose for $r > 0$, $r = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists. Then*

*(a)* $\sum_{n=1}^{\infty} x_n$ *converges if* $0 < r < 1$

*(b)* $\sum_{n=1}^{\infty} x_n$ *diverges if* $r > 1$.

*Proof.* First suppose $r \in (0,1)$. We want to show that the tail of the series is bounded above by a convergent geometric series.

Let $\epsilon = \frac{1-r}{2} > 0$. Since $r = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$, there exists a $K \in \mathbb{N}$ such that

$$\left| \frac{x_{k+1}}{x_k} - r \right| < \epsilon$$

for all $k \geq K$. Therefore,

$$\frac{x_{k+1}}{x_k} < r + \epsilon = r + \frac{1-r}{2} = \frac{1+r}{2}$$

for all $k \geq K$, so

$$x_{k+1} < \left(\frac{1+r}{2}\right) x_k$$

for lal $k \geq K$. A simple induction argument then implies

$$x_k \leq \left(\frac{1+r}{2}\right)^{k-K} x_K$$

for all $k \geq K$. Notice since $\frac{1+r}{2} < \frac{1+1}{2} = 1$ that the geometric series:

$$\sum_{n=1}^{\infty} \left(\frac{1+r}{2}\right)^n$$

converges by Example 8.3. Thus,

$$\sum_{n=K}^{\infty} \left(\frac{1+r}{2}\right)^{k-K} x_K$$

converges and so $\sum_{n=K}^{\infty} x_n$ converges by the Comparison Test (Theorem 8.11). Hence, $\sum_{n=1}^{\infty} x_n$ converges as only the tail of the series matters.

Now suppose $r > 1$. Our goal for this direction is to show that the tail of the series is bounded below by a divergent geometric series.

Let $\epsilon = \frac{r-1}{2} > 0$. Since $r = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$, there exists a $K \in \mathbb{N}$ such that

$$\left| \frac{x_{k+1}}{x_k} - r \right| < \epsilon$$

for all $k \geq K$. Therefore,

$$\frac{x_{k+1}}{x_k} > r - \epsilon = r - \frac{r-1}{2} = \frac{r+1}{2}$$

for all $k \geq K$. A simple induction argument then implies

$$x_k \geq \left( \frac{1+r}{2} \right)^{k-K} x_K$$

for all $k \geq K$. Notice since $\frac{1+r}{2} > \frac{1+1}{2} = 1$, that the geometric series

$$\sum_{n=1}^{\infty} \left( \frac{1+r}{2} \right)^n$$

diverges by Example 8.7. Thus,

$$\sum_{n=K}^{\infty} \left( \frac{1+r}{2} \right)^{k-K} x_K$$

diverges (since $x_K \neq 0$) so $\sum_{n=K}^{\infty} x_n$ diverges by the Comparison Test (Theorem 8.11). Hence, $\sum_{n=1}^{\infty} x_n$ diverges as only the tail of the series matters. $\square$

**Example 8.15.** We claim that the series $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ converges absolutely for all $z \in \mathbb{C}$. Let

$$x_n = \left| \frac{1}{n!} z^n \right| = \frac{|z|^n}{n!}$$

Note for all $n \in \mathbb{N}$ that

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{|z|^{n+1}}{(n+1)!} \frac{n!}{|z|^{n+1}} \right| = \frac{|z|}{n+1}$$

Since

$$\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

We obtain that $\sum_{n=1}^{\infty} x_n$ converges by the Ratio Test (Theorem 8.14) and thus, $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ converges absolutely for all $z \in \mathbb{C}$.

**Theorem 8.16** (Root Test)**.** *Let $(x_n)_n$ be a sequence of real numbers such that $x_n > 0$ for all $n \in \mathbb{N}$. Suppose for $r > 0$, $r = \lim_{n\to\infty} \sqrt[n]{x_n}$ exists. Then*

*(a)* $\sum_{n=1}^{\infty} x_n$ *converges if* $0 < r < 1$

*(b)* $\sum_{n=1}^{\infty} x_n$ *diverges if* $r > 1$.

*Proof.* First, suppose $r \in (0, 1)$. We want to show that the tail of the series is bounded above by a convergent geometric series.

Let $\epsilon = \frac{1-r}{2} > 0$. Since $r = \lim_{n\to\infty} \sqrt[n]{x_n}$, there exists a $K \in \mathbb{N}$ such that

$$|\sqrt[k]{x_k} - r| < \epsilon$$

for all $k \geq K$. Therefore,

$$\sqrt[k]{x_k} < r + \epsilon = r + \frac{1-r}{2} = \frac{1+r}{2}$$

for all $k \geq K$, so

$$x_k < \left(\frac{1+r}{2}\right)^k$$

for all $k \geq K$. Notice since $\frac{1+r}{2} < \frac{1+1}{2} = 1$, that the geometric series

$$\sum_{n=K}^{\infty} \left(\frac{1+r}{2}\right)^n$$

converges by Example 8.3 so $\sum_{n=K}^{\infty} x_n$ converges by the Comparison Test (Theorem 8.11). Hence, $\sum_{n=1}^{\infty} x_n$ converges as only the tail of the series matters.

Next suppose $r > 1$. We want to show that the tail of the series is bounded below by a divergent geometric series.

Let $\epsilon = \frac{r-1}{2} > 0$. Since $r = \lim_{n\to\infty} \sqrt[n]{x_n}$, there exists a $K \in \mathbb{N}$ such that

$$|\sqrt[k]{x_k} - r| < \epsilon$$

for all $k \geq K$. Therefore,

$$\sqrt[k]{x_k} > r - \epsilon = r - \frac{r-1}{2} = \frac{r+1}{2}$$

for all $k \geq K$ so

$$x_k > \left(\frac{1+r}{2}\right)^k$$

for all $k \geq K$. Notice since $\frac{1+r}{2} > \frac{1+1}{2} = 1$, that the geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1+r}{2}\right)^n$$

diverges by Example 8.7 so $\sum_{n=K}^{\infty} x_n$ diverges by the Comparison Test (Theorem 8.11). Hence, $\sum_{n=1}^{\infty} x_n$ diverges as only the tail of the series matters. $\square$

**Theorem 8.17** (Leibniz's Theorem)**.** *Let $(x_n)_n$ be a non-increasing sequence of non-negative real numbers such that $\lim_{n\to\infty} x_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ converges.*

*Proof.* For each $N \in \mathbb{N}$, let $S_N = \sum_{k=1}^{N}(-1)^{k+1}x_k$. First, we claim for all $N \in \mathbb{N}$ that

$$S_{2N} \leq S_{2N+2} \leq S_{2N+3} \leq S_{2N+1}$$

To see the first inequality, notice since $(x_n)_n$ is a non-increasing sequence of non-negative real numbers that $x_{2N+1} - x_{2N+2} \geq 0$ for all $N \in \mathbb{N}$ so

$$\begin{aligned} S_{2N} &\leq (x_{2N+1} - x_{2N+2}) + S_{2N} \\ &= (-1)^{(2N+1)+1}x_{2N+1} + (-1)^{(2N+2)+1}x_{2N+2} + \sum_{k=1}^{2N} x_k \\ &= \sum_{k=1}^{2N+2} (-1)^{k+1}x_k = S_{2N+2} \end{aligned}$$

Similarly, since $(x_n)_n$ is a non-increasing sequence of non-negative real numbers such that $x_{2N+3} - x_{2N+2} \leq 0$ for all $N \in \mathbb{N}$, so

$$\begin{aligned} S_{2N+3} &= \sum_{k=1}^{2N+3} (-1)^{k+1}x_k \\ &= (-1)^{(2N+3)+1}x_{2N+3} + (-1)^{(2N+2)+1}x_{2N+2} + \sum_{k=1}^{2N+1} x_k \\ &= (x_{2N+3} - x_{2N+2}) + S_{2N+1} \leq S_{2N+1} \end{aligned}$$

Finally, since $x_{2N+3} \geq 0$ we obtain that

$$S_{2N+2} = \sum_{k=1}^{2N+2} x_k \leq x_{2N+3} + \sum_{k=1}^{2N+2} x_k = \sum_{k=1}^{2N+3} x_k = S_{2N+3}$$

as desired.

Notice the inequality proved in the above claim shows that $(S_{2N})_N$ is a non-decreasing sequence and $(S_{2N+1})_N$ is non-increasing sequence both of which are bounded below by $S_2$ and bounded above by $S_1$. Hence, the Monotone Convergence Theorem (Theorem 7.12) implies that $(S_{2N})_N$ and $(S_{2N+1})_N$ converge.

Let

$$L = \lim_{N\to\infty} S_{2N} \qquad K = \lim_{N\to\infty} S_{2N+1}$$

Notice that

$$K = L = \lim_{N\to\infty} S_{2N+1} - S_{2N} = \lim_{N\to\infty} \sum_{k=1}^{2N+1} x_k - \sum_{k=1}^{2N} x_k = \lim_{N\to\infty} x_{2N+1} = 0$$

Hence, $L = K$. Therefore, since $(S_{2N})_N$ and $(S_{2N+1})_N$ both converge to $L$, $(S_N)_N$ converges to $L$. Therefore, $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ converges by definition. $\square$

**Example 8.18.** We claim that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but does not converge absolutely. Note that $\lim_{n\to\infty} \frac{1}{n} = 0$. Hence, Leibniz's Theorem (Theorem 8.17) implies that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. However, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by Corollary 8.13 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ does not converge absolutely.

**Definition 8.19** (Conditionally Convergent Series)**.** A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge *conditionally* if $\sum_{n=1}^{\infty} z_n$ converges but does not converge absolutely.

**Example 8.20.** What are the values of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$? In an attempt to solve this question, consider the following: Let

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Notice that

$$
\begin{aligned}
S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots\right) \\
&= \frac{1}{2}S
\end{aligned}
$$

Therefore, $S = 0$. However, also notice that

$$
\begin{aligned}
S &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots \\
&\geq \frac{1}{2} + 0 + 0 + \cdots = \frac{1}{2}
\end{aligned}
$$

How is this possible? Did we just break mathematics?

# Chapter 9

# Rearrangements of Series

In Example 8.20, the issue with the above computation is that when we were trying to evaluate $S$, we rearranged the order of the terms in the series. This may seem valid in the sense that if we are adding up only a finite number of scalars, then we know we can rearrange the order of the terms in the sum due to the associative and commutative properties of addition. However, in Definition 8.2, the partial sums are formed by adding up the terms of the series in a very specific order and then taking the limit of the partial sums. By rearranging the terms of an infinite series, we are in a sense transforming the series we as we are changing the partial sums and thereby modifying the value of the partial sums converges to. In fact, when dealing with a conditional convergent series, we can reorder the series to make the value of the series anything we want!

**Definition 9.1** (Rearrangements of Series)**.** A series $\sum_{n=1}^{\infty} w_n$ is said to be a rearrangement of the series $\sum_{n=1}^{\infty} z_n$ if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $w_n = z_{\sigma(n)}$ for all $n \in \mathbb{N}$.

With Definition 9.1, we are going to prove two hard theorems that are related to the rearrangement of series.

**Theorem 9.2.** *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. For any $L \in \mathbb{R}$, there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\sigma(n)} = L$*

*Proof.* For each $n \in \mathbb{N}$, let

$$x_n^+ = \begin{cases} x_n & \text{if } x_n \geq 0 \\ 0 & \text{if } x_n < 0 \end{cases} \qquad x_n^- = \begin{cases} 0 & \text{if } x_n \geq 0 \\ x_n & \text{if } x_n < 0 \end{cases}$$

Hence, for all $n \in \mathbb{N}$,

$$x_n = x_n^+ + x_n^- \qquad |x_n| = x_n^+ - x_n^-$$

If both $\sum_{n=1}^\infty x_n^+$ and $\sum_{n=1}^\infty x_n^-$ converged, then $\sum_{n=1}^\infty |x_n|$ would converge since

$$\sum_{k=1}^N |x_k| = \sum_{k=1}^N x_k^+ - \sum_{k=1}^N x_k^-$$

for all $N \geq 1$. However, since $\sum_{n=1}^\infty |x_n|$ does not converge, it must be the case that at least one of $\sum_{n=1}^\infty x_n^+$ diverges. Moreover, since $\sum_{n=1}^\infty x_n$ converges and

$$\sum_{k=1}^\infty x_k = \sum_{k=1}^N x_k^+ + \sum_{k=1}^N x_k^-$$

for all $N \in \mathbb{N}$, if one of $\sum_{n=1}^\infty x_n^+$ and $\sum_{n=1}^\infty x_n^-$ converged then both would need to converge thereby contradicting what was just demonstrated. Hence, both $\sum_{n=1}^\infty x_n^+$ and $\sum_{n=1}^\infty x_n^-$ diverges.

Let $(\alpha_n)_n$ denote the sequence of all non-negative terms from $(x_n)_n$ listed in the same order they appear, and let $(\beta_n)_n$ denote the sequence of all negative terms from $(x_n)_n$ listed in the order they appear in. Since

$$\sum_{n=1}^\infty \alpha_n \qquad \sum_{n=1}^\infty \beta_n$$

diverge by what was demonstrated above, then

$$\sup\left(\left\{\sum_{k=1}^N \alpha_k : N \in \mathbb{N}\right\}\right) = \infty \qquad \inf\left(\left\{\sum_{k=1}^N \beta_k : N \in \mathbb{N}\right\}\right) = -\infty$$

Fix $L \in \mathbb{R}$. To find a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that $\sum_{n=1}^\infty x_{\sigma(n)} = L$. Our goal is to add $\alpha_k$s up to the point where we obtain a number just larger than $L$, then add (thereby decreasing the value) $\beta_k$s up to the point where we obtain a number just smaller than $L$, then add $\alpha_k$s up to the point where we obtain a number just larger than $L$, then add (thereby decreasing the value) $\beta_k$s up to the point where we obtain a number just smaller than $L$, and so on. This procedure will create a rearrangement of the series so that the partial sums will be within $x_n$ of $L$ for some increasingly large $n$. Therefore, since

$\sum_{n=1}^{\infty} x_n$ converges, so $\lim_{n\to\infty} x_n = 0$, so this rearrangement will converge to $L$.

First note since the above supremum is infinity that there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{N} \alpha_k > L$$

Choose $N_1 \in \mathbb{N}$ to be the *smallest* $N \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_k > L$. Therefore, since $\alpha_k \geq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=1}^{N_1} \alpha_k > L \geq \sum_{k=1}^{N} \alpha_k$$

for all $N < N_1$. Let

$$T_1 = \sum_{k=1}^{N_1} \alpha_k$$

and note $0 < T_1 - L$.

Next, since the above infimum is negative infinity, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{N} \beta_k < L - T_1$$

Choose $M_1 \in \mathbb{N}$ to be the smallest $N \in \mathbb{N}$ such that $\sum_{k=1}^{N} \beta_k < L - T_1$. Therefore since $\beta_k \leq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=1}^{M_1} < L - T_1 \leq \sum_{k=1}^{N} \beta_k$$

for all $N < M_1$. Let

$$R_1 = \sum_{k=1}^{M_1} \beta_k$$

Notice that the above inequalities imply that

$$0 < L - T_1 - R_1 = (L - T_1) - R_1 \leq \sum_{k=1}^{M_1 - 1} \beta_k - \sum_{k=1}^{M_1} \beta_k = -\beta_{M_1}$$

Next, since the above supremum is infinity, there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=N_1+1}^{N} \alpha_k > L - T_1 - R_1$$

Choose $N_2 \in \mathbb{N}$ to be the smallest $N \in \mathbb{N}$ such that $N > N_1$ and $\sum_{k=N_1+1}^{N} \alpha_k > L - T_1 - R_1$. Therefore, since $\alpha_k \geq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=N_1+1}^{N_2} \alpha_k > L - T_1 - R_1 \geq \sum_{k=N_1+1}^{N} \alpha_k$$

for all $N \in \{N_1 + 1, ..., N_2 - 1\}$. Let

$$T_2 = \sum_{k=N_1+1}^{N_2} \alpha_k$$

Notice that the above inequalities and the fact that $\alpha_k \geq 0$ for all $k \in \mathbb{N}$ imply that

$$0 \leq L - T_1 - R_1 - \sum_{k=N_1+1}^{N} \alpha_k \leq L - T_1 - R_1 \leq \beta_{M_1}$$

for all $N \in \{N_1 + 1, ..., N_2 - 1\}$ and

$$0 < T_2 - (L - T_1 - R_1) \leq \sum_{k=N_1+1}^{N_2} \alpha_k - \sum_{k=N_1+1}^{N_2-1} \alpha_k = \alpha_{N_2}$$

Once more for clarity, since $T_1 + R_1 + T_2 - L > 0$ and the above infimum is negative infinity, there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=M_1+1}^{N} \beta_k < L - T_1 - R_1 - T_2$$

Choose $M_2 \in \mathbb{N}$ to be the smallest $N \in \mathbb{N}$ such that $N > M_1$ and $\sum_{k=M_1+1}^{N} \beta_k < L - T_1 - R_1 - T_2$. Therefore, since $\beta_k \leq 0$ for all $k \in \mathbb{N}$, this implies that

$$\sum_{k=M_1+1}^{M_2} \beta_k < L - T_1 - R_1 - T_2 \leq \sum_{k=M_1+1}^{N} \beta_k$$

for all $N \in \{M_1+1, ..., M_2-1\}$. Let

$$R_2 = \sum_{k=M_1+1}^{M_2} \beta_k$$

Notice that the above inequalities and the fact that $\beta_k \leq 0$ for all $k \in \mathbb{N}$ imply that

$$0 \geq L - T_1 - R_1 - T_2 - \sum_{k=M_1+1}^{N} \beta_k \geq L - T_1 - R_1 - T_2 > -\alpha_{N_2}$$

for all $N \in \{M_1+1, ..., M_2-1\}$ and

$$0 < (L - T_1 - R_1 - T_2) - R_2 \leq \sum_{k=M_1+1}^{M_2-1} \beta_k - \sum_{k=M_1+1}^{M_2} \beta_k = -\beta_{M_2}$$

By repeating the procedure, there exist strictly increasing sequences $(N_j)_j$ and $(M_j)_j$ so that if

$$T_j = \sum_{k-N_{j-1}+1}^{N_j} \alpha_k \qquad R_j = \sum_{k=M_{j-1}+1}^{M_j} \beta_k$$

then for all $\ell \geq 1$, we have that

$$0 \leq L - \sum_{j=1}^{\ell}(T_j + R_j) - \sum_{k=N_j+1}^{N} \alpha_k \leq -\beta_{M_\ell}$$

for all $N \in \{N_\ell + 1, ..., N_{\ell+1} - 1\}$,

$$0 < -L + T_{\ell+1} + \sum_{j=1}^{\ell}(T_j + R_j) \leq \alpha_{N_{\ell+1}}$$

and

$$0 \geq L - T_{\ell+1} - \sum_{j=1}^{\ell}(T_j + R_j) - \sum_{k=M_j+1}^{N} \beta_k > -\alpha_{N_{\ell+1}}$$

for all $N \in \{M_\ell + 1, ..., M_{\ell+1} - 1\}$ and

$$0 < L - \sum_{j=1}^{\ell+1}(T_j + R_j) \leq -\beta_{M_{\ell+1}}$$

Since $(N_j)_j$ and $(M_j)_j$ are strictly increasing functions, we see that

$$\alpha_1, ..., \alpha_{N_1}, \beta_1, ..., \beta_M, \alpha_{N_1+1}, ..., \alpha_{N_2}, \beta_{M_1+1}, ..., \beta_{M_2}, \alpha_{N_2+1}, ..., \alpha_{N_3}, ..., \beta_{M_2+1}, ..., \beta_{M_3}, ...$$

is a rearrangement of $\sum_{n=1}^{\infty} x_n$. Moreover by construction, the partial sums of this rearrangement are within either $\alpha_{N_\ell}$ or $-\beta_{M_\ell}$ of $L$ for progressively large $\ell$ at every step fo the construction. Since $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$. Therefore, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, so the partial sums of this rearrangement converge to $L$ as desired. □

The main appeal of absolutely convergent series over conditionally convergent series is the dictionary between the following and Theorem 9.2.

**Theorem 9.3.** *Let $\sum_{n=1}^{\infty} z_n$ be an absolutely convergent series of complex numbers. For all bijections $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n=1}^{\infty} z_{\sigma(n)}$ converges absolutely $\sum_{n=1}^{\infty} z_{\sigma(n)} = \sum_{n=1}^{\infty} z_n$.*

*Proof.* Let $L = \sum_{n=1}^{\infty} z_n$ and fix a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ for all $N \in \mathbb{N}$, let

$$S_N = \sum_{k=1}^{N} z_k \qquad T_N = \sum_{k=1}^{N} z_{\sigma(k)}$$

To see that $(T_N)_N$ converges to $L$, let $\epsilon > 0$. Since $L = \sum_{n=1}^{\infty} z_n$ there exists an $N_1 \in \mathbb{N}$ such that

$$|S_N - L| < \frac{\epsilon}{2}$$

for all $N \geq N_1$. Moreover, since $\sum_{n=1}^{\infty} z_n$ converges absolutely, Theorem 8.8 implies there exists an $N_2 \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} |z_k| < \frac{\epsilon}{2}$$

for all $N \geq N_2$.

Let $N_0 = \max\{N_1, N_2\}$. Since $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, there exists an $M_0 \in \mathbb{N}$, such that

$$\{1, 2, ..., N_0\} \subseteq \{\sigma(j) : j \in \{1, 2, ..., M_0\}\}$$

Therefore, for all $M \geq M_0$ we see that

$$\begin{aligned} |T_M - S_{N_0}| &= \left| \sum_{k=1}^{M} z_{\sigma(k)} - \sum_{k=1}^{N_0} z_k \right| \\ &= \left| \sum_{\substack{k\in\sigma(\{1,2,...,M\}) \\ k\notin\{1,2,...,N_0\}}} z_k \right| \\ &\leq \sum_{\substack{k\in\sigma(\{1,2,...,M\}) \\ k\notin\{1,2,...,N_0\}}} |z_k| \\ &\leq \sum_{k=N_0+1}^{\infty} |x_k| < \frac{\epsilon}{2} \end{aligned}$$

Hence for all $M \geq M_0$ we see that

$$|T_M - L| \leq |T_M - S_{N_0}| + |S_{N_0} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $\sum_{n=1}^{\infty} z_{\sigma(n)}$ converges to $L$.

To see that $\sum_{n=1}^{\infty} z_{\sigma(n)}$ converges absolutely, note since $\sum_{n=1}^{\infty} z_n$ converges absolutely that $\sum_{n=1}^{\infty} |z_n|$ converges and thus $\sum_{n=1}^{\infty} |z_{\sigma(n)}|$ converges by the first part of the proof. Hence, $\sum_{n=1}^{\infty} z_{\sigma(n)}$ converges absolutely. $\square$

# Chapter 10

# Continuity of Complex-Valued Functions

In order to discuss series of continuous functions, it is useful to recall the notion of a continuous function. We do so in the situation of functions on the complex numbers for use later in the course.

**Definition 10.1** (Continuity at a Point)**.** Let $U \subseteq \mathbb{C}$. A function $f : U \to \mathbb{C}$ is said to be *continuous at a point* $z_0 \in U$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $z \in U$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$. Moreover, $f$ is said to be *continuous on* $U$ if $f$ is continuous at every point in $U$.

**Lemma 10.2.** *Let $U \subseteq \mathbb{C}$, let $z_0 \in U$, and let $f : U \to \mathbb{C}$. Then $f$ is continuous at $z_0$ if and only if whenever $(z_n)_n$ is a sequence in $U$ that converge to $z_0$, we have* $\lim_{n\to\infty} f(z_n) = f(z_0)$.

*Proof.* ($\Rightarrow$) First suppose $f$ is continuous at $z_0$. Let $(z_n)_n$ be a sequence in $U$ that converge to $z_0$. To show that $\lim_{n\to\infty} f(z_n) = f(z_0)$, let $\epsilon > 0$. Since $f$ is continuous at $z_0$, there exists a $\delta > 0$ such that if $z \in U$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$. Since $(z_n)_n$ converges to $z_0$, there exists an $N \in \mathbb{N}$ such that $|z_n - z_0| < \delta$ for all $n \geq N$. Hence, for all $n \geq N$, we have that $|z_n - z_0| < \delta$ so $|f(z_n) - f(z_0)| < \epsilon$. Therefore, $\lim_{n\to\infty} f(z_n) = f(z_0)$.

($\Leftarrow$) Conversely, suppose that $f$ is not continuous at $z_0$. Therefore, there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists a $z \in U$ such that $|z - z_0| < \delta$ but $|f(z) - f(z_0)| \geq \epsilon_0$. Hence, for all $n \in \mathbb{N}$, there exists a $z_n \in U$ such that $|z_n - z_0| < \frac{1}{n}$ but $|f(z_n) - f(z_0)| \geq \epsilon_0$. Thus, $(z_n)_n$ is a sequence in $U$ that converges to $z_0$ such that $(f(z_n))_n$ does not converge to $f(z_0)$ since $|f(z_n) - f(z_0)| \geq \epsilon_0$ for all $n \in \mathbb{N}$. So Definition 7.2 fails for $\epsilon = \epsilon_0$. $\square$

Consequently, a composition of continuous functions is continuous.

**Proposition 10.3.** *Let $U, V \subset \mathbb{C}$, let $z_0 \in U$, let $f : U \to V$ and let $g : V \to \mathbb{C}$. If $f$ is continuous at $z_0$ and $g$ is continuous at $f(z_0)$, then $g \circ f$ is continuous at $z_0$.*

*Proof.* Let $(z_n)_n$ be an arbitrary sequence in $U$ that converges to $z_0$. Since $f$ is continuous at $z_0$ and $(z_n)_n$ converges to $z_0$, $(f_n(z_0))_n$ converges to $f(z_0)$, by Lemma 10.2. Similarly, since $g$ is continuous at $f(z_0)$ and $(f_n(z_0))_n$ converges to $f(z_0)$, $(g_n(f(z_0)))_n$ converges to $g(f(z_0))$ by Lemma 10.2. Therefore, since $(z_n)_n$ was arbitrary, Lemma 10.2 implies that $g \circ f$ is continuous at $z_0$. $\square$

In addition, the following shows that the set of continuous functions on the complex numbers is a vector subspace of the vector space of all complex-valued functions.

**Lemma 10.4.** *Let $U \subseteq \mathbb{C}$ and let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be continuous functions. Then the following are true.*

1. *The function $f + g : U \to \mathbb{C}$ defined by $(f + g)(z) = f(z) + g(z)$ for all $z \in U$ is continuous.*

2. *The function $fg : U \to \mathbb{C}$ defined by $(fg)(z) = f(z)g(z)$ for all $z \in U$ is continuous.*

3. *For all $\alpha \in \mathbb{C}$, the function $\alpha f : U \to \mathbb{C}$ defined by $(\alpha f)(z) = \alpha f(z)$.*

4. *If $f(z) \neq 0$ for all $z \in U$, the function $\frac{1}{f} : U \to \mathbb{C}$ defined by $\left(\frac{1}{f}\right)(z) = \frac{1}{f(z)}$ for all $z \in U$ is continuous.*

5. *The function $\bar{f} : U \to \mathbb{C}$ defined by $\bar{f}(z) = \overline{f(z)}$ for all $z \in U$ is continuous.*

6. *The function $|f| : U \to \mathbb{C}$ defined by $|f|(z) = |f(z)|$ for all $z \in U$ is continuous.*

*Proof.* This immediately follows from Lemma 10.2 by using Corollary 7.8. (Exercise.) $\square$

As with complex numbers, the ability to take the real and imaginary parts of complex-valued functions is important.

**Definition 10.5** (Real and Imaginary Functions)**.** Let $U \subseteq \mathbb{C}$ and let $f : U \to \mathbb{C}$. The real and imaginary parts of $f$ are the functions $\Re(f)$, $\Im(f) : U \to \mathbb{R}$ respectively where

$$(\Re(f))(z) = \Re(f(z)) = \frac{f(z) + \overline{f(z)}}{2} \qquad (\Im(f))(z) = \Im(f(z)) = \frac{f(z) - \overline{f(z)}}{2i}$$

for all $z \in U$.

Notice if $f : U \to \mathbb{C}$, then $f = \Re(f) + i\Im(f)$. Moreover,

$$\Re(f) = \frac{f + \bar{f}}{2} \qquad \Im(f) = \frac{f - \bar{f}}{2i}$$

Thus, Lemma 10.4 implies the following.

**Lemma 10.6.** *Let $U \subset \mathbb{C}$ and let $f : U \to \mathbb{C}$. Then $f$ is continuous if and only if $\Re(f)$ and $\Im(f)$ are continuous real-valued functions.*

*Proof.* If $\Re(f)$ and $\Im(f)$ are continuous, then $f = \Re(f) + i\Im(f)$ is continuous by Lemma 10.4 parts (1) and (3).

Similarly, if $f$ is continuous, then

$$\Re(f) = \frac{f + \bar{f}}{2} \qquad \Im(f) = \frac{f - \bar{f}}{2i}$$

are continuous by Lemma 10.4 parts (1), (3) and (5). $\square$

# Chapter 11

# Continuity of Sequence and Series of Functions

With our reminders of continuous functions out of the way, let us examine the question of whether a series of continuous functions is continuous. To answer this question we note since a series is a limit of its partial sums that a series of functions is a limit of a sequence of functions. Consequently, it is necessary to discuss the limits of sequences of functions and whether a limit of continuous functions is continuous. We begin with the most obvious way to define the limit of a function and its corresponding restriction to a series of functions.

**Definition 11.1** (Pointwise Convergence)**.** Let $U \subseteq \mathbb{C}$. A sequence $(f_n)_n$ of complex-valued functions on $U$ is said to *converge pointwise* on $U$ to $f : U \to \mathbb{C}$ if

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for all $z \in U$. The definition of the limit means that for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon$$

for all $n \geq N$.

**Definition 11.2** (Pointwise Convergence of Series of Functions)**.** Let $U \subseteq \mathbb{C}$ and for each $n \in \mathbb{N}$, let $f_n : U \to \mathbb{C}$. The series $\sum_{n=1}^{\infty} f_n$ is said to *converge pointwise* on $U$ if $\sum_{n=1}^{\infty} f_n(z)$ converges for each $z \in U$. Moreover, the function $f : U \to \mathbb{C}$ defined by

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

for all $z \in U$ is called the *(pointwise) sum of* $(f_n)_n$ and is denoted by $\sum_{n=1}^{\infty} f_n$.

Unfortunately, pointwise limits are not the limits we are looking for as a pointwise limit of continuous functions is not continuous. Thus a pointwise convergent series of continuous functions need not be continuous.

**Example 11.3.** For each $n \in \mathbb{N}$, define $f_n : [0,1] \to [0,1]$ by $f_1(x) = x$ and for $n \geq 2$,

$$f_n(x) = x^n - x^{n-1}$$

for all $x \in [0,1]$. Clearly, $(f_n)_n$ is a sequence of continuous functions on $[0,1]$.

We claim that $\sum_{n=1}^{\infty} f_n$ converges pointwise on $[0,1]$ to the function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

As $f$ is clearly not continuous at $x = 1$, this provides an example of a series of continuous functions that converges pointwise to a function that is discontinuous at a point.

To see this, for each $N \in \mathbb{N}$, let $S_N = \sum_{k=1}^{N} f_k$. Notice that $S_N(x) = x^N$ for all $N \in \mathbb{N}$. Note if $x = 1$, then

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} 1^N = 1$$

whereas if $0 \leq x < 1$, then

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} x^N = 0$$

Hence, the example is complete. Note this also shows that there exist a sequence $(x^n)_n$ of continuous functions that converge pointwise to a function that is discontinuous at a point.

In order to rectify this situation, we simply need to require a stronger form of limit of functions.

**Definition 11.4** (Uniform Convergence)**.** Let $U \subseteq \mathbb{C}$. A sequence $(f_n)_n$ of complex-valued functions on $U$ is said to *converge uniformly* on $U$ to $f : U \to \mathbb{C}$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$ for all $z \in U$ whenever $n \geq N$. That is,

$$\sup_{z \in U} |f_n(z) - f(z)| < \epsilon$$

*Remark* 11.5 (Uniform Convergence $\Rightarrow$ Pointwise Convergence)*.* Suppose $(f_n)_n$ converges uniformly to $f$ on $U$. Then for $z \in U$, we can see that

$$|f_n(z) - f(z)| \leq \sup_{z_0 \in U} |f_n(z_0) - f(z_0)| < \epsilon$$

for $n \geq N$ which allows us to conclude that $(f_n)_n$ converges pointwise to $f$ on $U$.

*Remark* 11.6*.* It is important to point out the difference between pointwise convergence and uniform convergence. The main difference is, given an $\epsilon > 0$, pointwise convergence simply lets us find for each $z \in U$ an $N_z \in \mathbb{N}$ that depends on $z$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N_z$, whereas uniform convergence lets us find an $N \in \mathbb{N}$ that works for every $z \in U$. That is, $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N$ and $z \in U$. More elegantly said uniform convergence lets us find one $N$ to rule all $N_z$, that is:

$$N = \sup_{z \in U}(N_z) < \infty$$

Note this clearly implies if a sequence of functions converges uniformly to $f$, then they converge pointwise to $f$. However, if a sequence of functions converges pointwise, it need not converge uniformly as the following examples shows.

**Example 11.7.** For $n \in \mathbb{N}$, let $f_n : \mathbb{C} \to \mathbb{C}$ be the function defined by $f_n(z) = \frac{z}{n}$. Then $(f_n)_n$ converges pointwise to the function $f(z) = 0$ on $\mathbb{C}$, but the convergence is not uniform. In fact, for $z \in \mathbb{C}$ and $\epsilon > 0$, let $N \in \mathbb{N}$ such that $N > \frac{|z|}{\epsilon}$. Then for $n \geq N$, it is easy to see that

$$|f_n(z) - f(z)| = \left|\frac{z}{n}\right| \leq \frac{|z|}{n} \leq \frac{|z|}{N_0} < \epsilon$$

which allows us to conclude that $(f_n(z))_n$ converges pointwise to $f(z) = 0$.

However, for $\epsilon = \frac{1}{2} > 0$, noting that $f_n(n) = \frac{n}{n} = 1$, we can conclude that

$$\sup_{z \in \mathbb{C}} |f_n(z) - f(z)| \geq |f_n(n) - f(n)| = |1 - 0| = 1 > \epsilon$$

for all $n \in \mathbb{N}$. Therefore, $f_n(z)$ does not converge uniformly to $f$ on $\mathbb{C}$.

**Example 11.8.** For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$. The function defined by $f(z) = z^n$. Then $(f_n)_n$ converges pointwise on $[0,1]$ to the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(z) = \begin{cases} 0 & \text{if } z \in [0,1) \\ 1 & \text{if } z = 1 \end{cases}$$

In fact, for $0 \leq z < 1$, we know that $z^n$ converges to 0 and $1^n = 1$ for $n \in \mathbb{N}$. However, $(f_n)_n$ does not converge uniformly to $f$. To see this, let $\epsilon = \frac{1}{4}$ and $z_n = \sqrt[n]{\frac{1}{2}}$ for $n \in \mathbb{N}$. Then $z_n \in (0,1)$ and $f_n(z_n) = \frac{1}{2}$ which implies that

$$\sup_{z \in [0,1]} |f_n(z) - f(z)| \geq |f_n(z_n) - f(z_n)| = |f_n(z_n)| = \frac{1}{2} > \epsilon$$

for all $n \in \mathbb{N}$

However, the pathology in the above example surrounds the value of the functions at $z = 1$.

**Example 11.9.** Fix $b \in [0,1)$ and consider the function $f_n : [0,b] \to [0,b]$ is defined by $f_n(z) = z^n$ for all $z \in [0,b]$ and $n \in \mathbb{N}$. We claim that $(f_n)_n$ converges uniformly to 0 (the zero function) on $[0,b]$. To see this, let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} b^n = 0$, there exists an $N \in \mathbb{N}$ such that $|b^n| < \epsilon$ for all $n \geq N$. Hence, for all $z \in [0,b]$ and $n \geq N$, we see that

$$|f_n(z) - 0| = z^n \leq b^n < \epsilon$$

Therefore, $(f_n)_n$ converges uniformly to 0 on $[0,b]$.

**Theorem 11.10.** *Let $U \subseteq \mathbb{C}$, let $z_0 \in U$, and let $(f_n)_n$ be a sequence of complex-valued functions on $U$ that converge uniformly on $U$ to $f : U \to \mathbb{C}$. If each $f_n$ is continuous at $z_0$, then $f$ is continuous at $z_0$. Consequently, a uniform limit of continuous functions is continuous.*

*Proof.* Let $\epsilon > 0$. Since $(f_n)_n$ converges to $f$ uniformly on $U$, there exists an $N \in \mathbb{N}$ such that
$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$
for all $n \geq N$ and $z \in U$. Since $f_N$ is continuous at $z_0$, there exists a $\delta > 0$ such that if $z \in U$ and $|z - z_0| < \delta$, then
$$|f_N(z) - f_N(z_0)| < \frac{\epsilon}{3}$$
Hence, for all $z \in U$ such that $|z - z_0| < \delta$, we have
$$\begin{aligned}|f(z) - f(z_0)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$
(where the first and last terms are less than $\frac{\epsilon}{3}$ by uniform convergence and the middle term is less than $\frac{\epsilon}{3}$ by continuity of $f_N$). Therefore, $f$ is continuous at $z_0$. $\square$

To further emphasize the benefits of uniform convergent continuous functions, we desire an analogue of 'every convergent sequence is bounded'. To do so, we define the following notion of boundedness for a sequence of functions.

**Definition 11.11** (Uniform Bounded Functions)**.** Let $U \subseteq \mathbb{C}$ and let $(f_n)_n$ be a sequence of complex-valued functions on $U$. Then $(f_n)_n$ is uniformly bounded on $U$ if there exists an $M$ such that
$$|f_n(z)| \leq M$$
for all $z \in U$ and $n \in \mathbb{N}$.

**Proposition 11.12.** *Let $I$ be a bounded closed interval on $\mathbb{R}$ and let $(f_n)_n$ be a sequence of complex-valued continuous functions on $I$ that converge uniformly on $I$ to $f : I \to \mathbb{C}$. Then $(f_n)_n$ is uniformly bounded.*

*Proof.* Since $I$ is a closed interval, the Extreme Value Theorem implies there exists an $M_n \in \mathbb{N}$ such that
$$|f_n(x)| \leq M_n$$
for all $x \in I$. Moreover, since $(f_n)_n$ converge uniformly to $f$ on $I$, Theorem 11.10 implies that $f$ is continuous on $I$. Hence, the Extreme Value Theorem implies there exists an $M_0 \in \mathbb{R}$ such that $|f(x)| \leq M_0$ for all $x \in I$.

Since $(f_n)_n$ converges uniformly to $f$ on $I$, there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| \leq 1$$

for all $n \geq N$ and $x \in I$ and thus,

$$|f_n(x)| \leq M_0 + 1$$

for all $n \geq N$ and $x \in I$. Therefore, if

$$M = \max\{M_0 + 1, M_1, M_2, ..., M_N\}$$

then

$$|f_n(x)| \leq M$$

for all $x \in I$ and $n \in \mathbb{N}$. Hence $(f_n)_n$ is uniformly bounded. □

Moreover, uniform convergence behaves well with respect to the operations of addition and scalar multiplication. The proofs of these facts are similar to the proof of Corollary 7.8 once combined with Proposition 11.12.

**Proposition 11.13.** *Let $I$ be a bounded closed interval in $\mathbb{R}$ and let $(f_n)_n$ and $(g_n)_n$ be sequences of complex-valued functions on $I$ that converge uniformly on $I$ to $f : I \to \mathbb{C}$ and $g : I \to \mathbb{C}$ respectively. Then the following are true.*

1. *$(f_n + g_n)_n$ converges uniformly to $f + g$ on $I$.*
2. *$(f_n g_n)_n$ converges uniformly to $fg$ on $I$.*
3. *If $(a_n)_n$ is a sequence of complex numbers that converges to $\alpha \in \mathbb{C}$, then $(\alpha_n f_n)_n$ converges uniformly to $\alpha f$ on $I$.*
4. *$(\bar{f}_n)_n$ converges uniformly to $\bar{f}$ on $I$.*

*Proof.* Exercise. □

**Example 11.14.** Let $U \subseteq \mathbb{C}$, $(w_n)_n$ be a convergent sequence of complex numbers, $w = \lim_{n\to\infty} w_n \in \mathbb{C}$ and $u : U \to \mathbb{C}$ be a bounded function. Let $f_n : U \to \mathbb{C}$ and $f : U \to \mathbb{C}$ be the functions defined by

$$f_n(z) = w_n u(z)$$

and

$$f(z) = wu(z)$$

for $z \in U$ and $n \in \mathbb{N}$. Then the sequence of functions $(f_n)_n$ converges uniformly to $f$ on $U$.

In fact, let $M > 0$ such that $|u(z)| \leq M$. Since $w = \lim_{n\to\infty} w_n$, for $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|w_n - w| < \frac{\epsilon}{M}$ for $n \geq N$. Then

$$|f_n(z) - f(z)| = |w_n u(z) - wu(z)| = |u(z)||w_n - w| < M\frac{\epsilon}{M} = \epsilon$$

which allows us to conclude

$$|f_n(z) - f(z)| \leq \sup_{z\in U} |f_n(z) - f(z)| < \epsilon$$

for all $n \geq N$.

**Example 11.15.** Fix $b \in [0,1)$ and consider the function $f_n : [0,b] \to [0,b]$ is defined by $f_n(x) = x^n$ for all $x \in [0,b]$ and $n \in \mathbb{N}$. We claim that $(f_n)_n$ converges uniformly to 0 (the zero function) on $[0,b]$. To see this, let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} b^n = 0$, there exists an $N \in \mathbb{N}$ such that $|b^n| < \epsilon$ for all $n \geq N$. Hence, for all $x \in [0,b]$ and $n \geq N$, we see that

$$|f_n(x) - 0| = x^n \leq b^n < \epsilon$$

which allows us to conclude that

$$\sup_{x\in[0,b]} |f_n(x) - f(x)| < \epsilon$$

Therefore, $(f_n)_n$ converges uniformly to 0 on $[0,b]$.

The following result shows continuity behaves properly when uniform limits are used. It is also useful to point out that the proof of the following result is a very common and incredibly useful argument in analysis known as the three-$\epsilon$ argument.

**Theorem 11.16.** *Let $U \subseteq \mathbb{C}$, let $z_0 \in U$, and let $(f_n)_n$ be a sequence of complex-valued functions on $U$ that converge uniformly on $U$ to $f : U \to \mathbb{C}$. If each $f_n$ is continuous at $z_0$, then $f$ is continuous at $z_0$. Consequently, a uniform limit of continuous functions is continuous.*

*Proof.* Let $\epsilon > 0$. Since $(f_n)_n$ converges to $f$ uniformly on $U$, there exists an $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

for all $n \geq N$ and $z \in U$. Since $f_N$ is continuous at $z_0$, there exists a $\delta > 0$ such that if $z \in U$ and $|z - z_0| < \delta$, then

$$|f_N(z) - f_N(z_0)| < \frac{\epsilon}{3}$$

Hence, for all $z \in U$ such that $|z - z_0| < \delta$, we have

$$\begin{aligned} |f(z) - f(z_0)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

(where the first and last terms are less than $\frac{\epsilon}{3}$ by uniform convergence and the middle term is less than $\frac{\epsilon}{3}$ by continuity of $f_N$). Therefore, $f$ is continuous at $z_0$. □

**Definition 11.17.** Let $U \subseteq \mathbb{C}$ and $f_n : U \to \mathbb{C}$ be a sequence of complex-valued functions. The sequence $(f_n)_n$ is said to be a Cauchy sequence of a function if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sup_{z \in U} |f_n(z) - f_m(z)| < \epsilon$$

for all $n \geq N$.

**Theorem 11.18** (Cauchy Criterion for Uniform Convergence)**.** *A sequence of complex-valued functions converges uniformly on $U \subseteq \mathbb{C}$ if and only if it is a Cauchy sequence of functions.*

*Proof.* Let $f_n : U \to \mathbb{C}$. Suppose $(f_n)_n$ converges uniformly to a function $f : U \to \mathbb{C}$; that is, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sup_{z \in U} |f_n(z) - f(z)| < \frac{\epsilon}{2}$$

for all $n \geq N$. Then for $m, n \geq N$, we have

$$|f_n(z) - f_m(z)| \leq |f_n(z) - f(z)| + |f_m(z) - f(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, as $\epsilon > 0$ was arbitrary, $(f_n)_n$ is a Cauchy sequence of functions.

Now assume that $(f_n)_n$ is a Cauchy sequence of functions. That is, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sup_{w \in U} |f_n(w) - f_m(w)| < \frac{\epsilon}{2}$$

for $m, n \geq N$. Then for $z \in U$, we have that

$$|f_n(z) - f_m(z)| \leq \sup_{w \in U} |f_n(w) - f_m(w)| < \frac{\epsilon}{2} < \epsilon$$

for $m, n \geq N$. That is, the sequence $(f_n(z))_n$ is convergent for all $z \in U$ using Theorem 7.10. For $f : U \to \mathbb{C}$ to be the function defined by

$$f(x) = \lim_{n \to \infty} f_n(x)$$

for $z \in U$. Then for $z \in U$ and $n \geq N$, since $(f_n)_n$ is a Cauchy sequence of functions, we have that

$$0 \leq |f_n(z) - f_m(z)| \leq \sup_{w \in U} |f_n(w) - f_m(w)| < \frac{\epsilon}{2}$$

for all $n, m \geq N$. Then using the fact that $(f_m(z))_m$ converges to $f(x)$, then by Lemma 7.3 and Squeeze Theorem (Theorem 7.15), we can obtain that

$$|f_n(z) - f(z)| < \frac{\epsilon}{2}$$

for all $z \in U$ and $n \geq N$, which allows us to conclude that

$$\sup_{z \in U} |f_n(z) - f(z)| < \epsilon$$

for all $n \geq N$. Therefore, $(f_n(z))_n$ converges to $f$ uniformly on $U$. $\square$

As Theorem 11.10 shows that uniform convergence is the limit we are looking for in order to preserve continuity of functions, we now discuss uniform convergence in the context of series.

From Definition 11.2, we can see that a series of functions is a particular case of limit of sequence of functions. Analogous, we can also obtain the limit of a sequence of functions as a series of functions. In fact, let $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$, and $(f_n)_n$ be a sequence of complex-valued functions such that

$$f(z) = \lim_{n \to \infty} f_n(z)$$

for $z \in U$. Defining $u_1 = f_1$ and $u_n = f_n - f_{n-1}$ for $n \geq 2$, we obtain a sequence of complex-valued functions defined on $U$ such that $(u_n)_n \sum_{k=1}^{n} u_k = f_n$. Then

$$f(z) = \lim_{n \to \infty} f_n(z) = \sum_{n=1}^{\infty} u_n(z)$$

for all $z \in U$.

Using Definition 11.2, Theorem 11.10 and Cauchy Criterion for Uniform Convergence (Theorem 11.18) we can obtain the following results.

**Theorem 11.19.** *Let $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$ and $(f_n)_n$ be a sequence of complex-valued functions defined on $U$. Suppose that the series of complex-valued functions $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$ on $U$ and each $f_n$ is continuous at $z \in U$. Then $f$ is continuous at $z$.*

**Theorem 11.20** (Cauchy Criterion for Uniform Convergence of Series of Functions)**.** *Let $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$, and $(f_n)_n$ be a sequence of complex-valued functions defiend on $U$. The series of complex-valued functions $\sum_{n=1}^{\infty} f_n$ converges uniformly on $U$ if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that*

$$\sup_{z \in U} \left| \sum_{k=n}^{m} f_k(x) \right| < \epsilon$$

*for all $m \geq n \geq N$.*

**Theorem 11.21** (Weierstrass $M$-Test)**.** *Let $U \subseteq \mathbb{C}$ and let $(f_n)_n$ be a sequence of complex-valued functions on $U$. For each $n \in \mathbb{N}$, suppose*

$$0 \leq M_n = \sup_{z \in U} |f_n(z)| < \infty$$

*Furthermore, suppose $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely for all $z \in U$ and if $f : U \to \mathbb{C}$ is defined by*

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

*for all $z \in U$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$.*

*Proof.* Note that for all $z \in U$ that

$$|f_n(z)| \leq M_n$$

SInce $\sum_{n=1}^{\infty} M_n$ converges, $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely for all $z \in U$ by the Comparison Test (Theorem 8.11).

To see that $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$, let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, Theorem 8.8 implies there exists an $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0}^{\infty} M_n < \epsilon$$

Hence for all $N \geq N_0$ and $z \in U$ we see that

$$\begin{aligned}
\left| \sum_{k=1}^{N} f_k(z) - \sum_{k=1}^{\infty} f_k(z) \right| &= \left| \sum_{k=N+1}^{\infty} f_k(z) \right| \\
&\leq \sum_{k=N+1}^{\infty} |f_k(z)| && \text{(Theorem 8.10)} \\
&\leq \sum_{k=N+1}^{\infty} M_k && \text{(by assumption)} \\
&\leq \sum_{k=N_0}^{\infty} M_k < \epsilon
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} f_n$ converges to $f$. $\square$

To demonstrate the power of the Weierstrass $M$-Test (Theorem 11.21), we demonstrate the following series of functions converge uniformly and thus, define continuous functions.

**Example 11.22.** For each $n \in \mathbb{N} \cup \{0\}$, let $f_n : \mathbb{C} \to \mathbb{C}$ be defined by

$$f_n(z) = \frac{z^n}{n!}$$

for all $z \in \mathbb{C}$. Clearly, each $f_n$ is a continuous function such that $\sum_{n=0}^{\infty} f_n(z) = e^z$ for all $z \in \mathbb{C}$.

For each $M \in \mathbb{N}$, let $U_M = \{z \in \mathbb{C} : |z| \leq M\}$. Since

$$|f_n(z)| \leq \frac{M^n}{n!}$$

for all $z \in U_M$ and since $\sum_{n=0}^{\infty} \frac{M^n}{n!} < \infty$ by Example 8.15, the Weierstrass $M$-Test (Theorem 11.21) implies that

$$e^z = \sum_{n=0}^{\infty} f_n(z)$$

is continuous on $U_M$. Since $M \in \mathbb{N}$ was arbitrary, $e^z$ is a continuous function on $\bigcup_{M=1}^{\infty} U_M = \mathbb{C}$.

**Example 11.23.** Consider the function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(x) = e^{ix}$$

where $x \in \mathbb{R}$ and $i = \sqrt{-1}$. Since $f$ is the restriction to $\mathbb{R}$ of a continuous function on $\mathbb{C}$, then by Example 11.22, we obtain that $f$ is continuous on $\mathbb{R}$. Since

$$\Re(f(x)) = \cos x \qquad \Im(f(x)) = \sin x$$

for all $x \in \mathbb{R}$, we obtain that $\cos x$ and $\sin x$ are continuous functions by Lemma 10.6.

# Chapter 12

# Uniform Continuity

This short chapter we will be focusing on the concept of uniform continuity.

**Definition 12.1.** Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{C}$ is said to be uniformly continuous on $I$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

**Example 12.2.** We claim that if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ for all $x \in \mathbb{R}$, then Definition 12.1 fails for $\epsilon = 2$ and thus, $f$ is not uniformly continuous. To see this, let $\delta > 0$ be arbitrary. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ and let $x_n = n$ and $y_n = n + \frac{1}{n}$. Then $|x_n - y_n| < \frac{1}{n} < \delta$ yet

$$|f(x_n) - f(y_n)| = \left| n^2 - \left(n + \frac{1}{n}\right)^2 \right| = 2 + \frac{1}{n^2} \geq 2$$

Hence $f$ is not uniformly continuous on $\mathbb{R}$.

*Remark* 12.3. The above shows that $x^2$ is not uniformly continuous on all of $\mathbb{R}$ as $x^2$ grows too quickly as $x$ tends to infinity. Consequently, one may ask, ”are things much nicer if we restrict to finite intervals?” For closed intervals, yes!

**Theorem 12.4.** *Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{C}$ is continuous, then $f$ is uniformly continuous.*

*Proof.* Let $f : [a, b] \to \mathbb{C}$ be continuous. Suppose to the contrary that $f$ is not uniformly continuous. Hence, there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists $x, y \in [a, b]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Therefore, for each $n \in \mathbb{N}$, there exists $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$.

Since $[a, b]$ is closed and bounded, the Bolzano-Weierstrass Theorem implies there exists a subsequence $(x_{k_n})_n$ of $(x_n)_n$ that converges to some number $L \in [a, b]$. Since $f$ is continuous, there exists an $N_1 \in \mathbb{N}$ such that $|f(x_{k_n}) - f(L)| < \frac{\epsilon}{2}$ for all $n \geq N$.

Consider the subsequence $(y_{k_n})_n$ of $(y_n)_n$. Notice for all $n \in \mathbb{N}$ that

$$|y_{k_n} - L| \leq |y_{k_n} - x_{k_n}| + |x_{k_n} - L| \leq \frac{1}{k_n} + |x_{k_n} - L| \leq \frac{1}{n} + |x_{k_n} - L|$$

Therefore, since $\lim_{n\to\infty} |x_{k_n} - L| = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, we obtain that

$$\lim_{n\to\infty} y_{k_n} = L$$

Since $f$ is continuous this implies that there exists an $N_2 \in \mathbb{N}$ such that $|f(y_{k_n}) - f(L)| < \frac{\epsilon}{2}$ for all $n \geq N_2$.

Notice if $N = \max\{N_1, N_2\}$, then the above implies that

$$|f(x_{k_N}) - f(y_{k_N)})| \leq |f(x_{k_N}) - f(L)| + |f(L) - f(y_{k_N})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

thereby contradicting the fact that $|f(x_{k_N}) - f(y_{k_N})| \geq \epsilon$. Hence, $f$ is uniformly continuous on $[a, b]$. $\square$

# Chapter 13

# Integration of Series of Functions

The existence of the Weierstrass functions puts the idea that series of differentiable functions can be differentiable into great jeopardy. To rectify this situation, we ignore this question and turn to integration.

**Example 13.1.** We claim that there exists a sequence $(f_n)_n$ of real-valued continuous functions on $[0, 1]$ that converge pointwise to a continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)dx \neq \lim_{n\to\infty} \int_0^1 f_n(x)dx$$

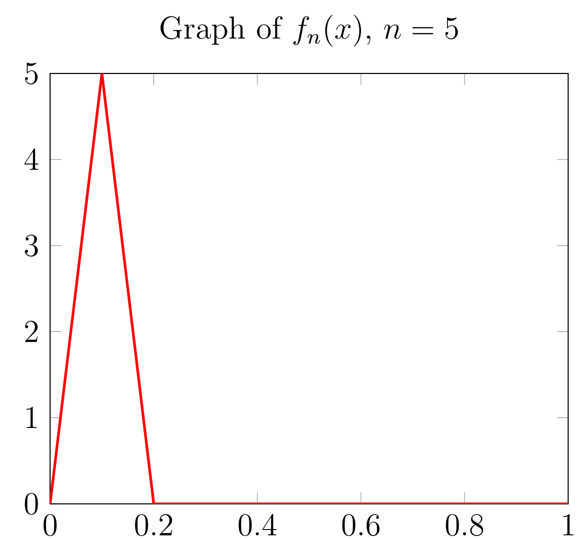
To see this, for each $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 2n^2x & \text{if } 0 \leq x \leq \frac{1}{2n} \\ 2n - 2n^2x & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

In particular, the graph of $f_n$ creates an isosceles triangle with base $\left[0, \frac{1}{n}\right]$, with height $n$, and otherwise is 0. Thus, $f_n$ is continuous and

$$\int_0^1 f_n(x)dx = \frac{1}{2}$$

for all $n \in \mathbb{N}$.




We claim that $(f_n)_n$ converges pointwise to 0 on $[0, 1]$. This will complete the example since

$$\int_0^1 0dx = 0 \neq \frac{1}{2} = \lim_{n\to\infty} \int_0^1 f_n(x)dx$$

To see $(f_n)_n$ converges pointwise to 0, let $x \in [0, 1]$ be arbitrary. If $x = 0$, then $f_n(0) = 0$ for all $n \in \mathbb{N}$, and we clearly see that $(f_n(x))_n$ converges to 0. Otherwise, assume $x > 0$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{n} < x$ for all $n \geq N$. Thus, the definition of $f_n$ implies that $f_n(x) = 0$ for all $n \geq N$ and thus $(f_n(x))_n$ converges to 0.

**Example 13.2.** We claim that there exists a sequence $(f_n)_n$ of real-valued Riemann integrable functions on $[0, 1]$ that converge pointwise to a function $f : [0, 1] \to \mathbb{R}$ that is bounded but not Riemann integrable. To see this, recall that $\mathbb{Q}$ is a countable set. Hence, we can write $\mathbb{Q} \cap [0, 1] = \{r_n : n \in \mathbb{N}\}$. Define $f_n : [0, 1] \to [0, 1]$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, ..., r_n\} \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in [0, 1]$ and define $f : [0, 1] \to [0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

We claim that $(f_n)_n$ converges pointwise to $f$. To see this, we note that $f_n(x) = 0 = f(x)$ for all $x \in [0,1] \setminus \mathbb{Q}$. Otherwise, if $x \in \mathbb{Q} \cap [0,1]$, then $x = r_N$ for some $N \in \mathbb{N}$ and $f_n(x) = 1 = f(x)$ for all $n \geq N$. Hence, $(f_n)_n$ converges pointwise to $f$.

We next claim that $f_n$ is Riemann integrable for all $n \in \mathbb{N}$. To see this, fix $n \in \mathbb{N}$ and let $\epsilon > 0$ be arbitrary. Let $\mathcal{P}_\epsilon$ be a partition of $[0,1]$ formed by taking the endpoints of the open intervals of length $\frac{\epsilon}{n}$ centered at each $r_k$ for $k \leq n$. For any interval in $\mathcal{P}$ that does not contain an $r_k$ for $k \leq n$, the maximal and minimal values of $f_n$ on this interval are both 0. Moreover, the interval of $\mathcal{P}$ containing an $r_k$ with $k \leq n$ is of length at most $\frac{\epsilon}{k}$ and the difference between the maximal and minimal values of $f_n$ on this interval is at most 1. Therefore, as there are $n$ possible $r_k$ for $k \leq n$, we obtain that

$$U(f_n, \mathcal{P}_\epsilon) - L(f_n, \mathcal{P}_\epsilon) \leq n(1-0)\frac{\epsilon}{n} = \epsilon$$

Therefore, as $\epsilon > 0$ was arbitrary, $f_n$ is Riemann integrable for all $n \in \mathbb{N}$.

However, notice for any partition of $[0,1]$ that

$$U(f, \mathcal{P}) = 1 \qquad L(f, \mathcal{P}) = 0$$

Hence, $f$ is not integrable. Therefore, the example is complete.

Of course, the problem with the above two examples is that pointwise convergence generally yields no analytical information about the limit function. As we have seen with continuity, uniform convergence we should consider in analysis. The following results further emphasizes this point.

**Theorem 13.3.** *Let $(f_n)_n$ be a sequence of real-valued functions, Riemann integrable functions on a closed interval $[a,b]$. If $(f_n)_n$ converges uniformly on $[a,b]$ to $f : [a,b] \to \mathbb{R}$, then $f$ is Riemann integrable and*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

*Proof.* To see that $f$ is Riemann integrable, let $\epsilon > 0$ be arbitrary. Since $(f_n)_n$ converges to $f$ uniformly, there exists an $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < \frac{\epsilon}{4(b-a)}$$

for all $x \in [a,b]$. Therefore,

$$f_N(x) - \frac{\epsilon}{4(b-a)} < f(x) < f_N(x) + \frac{\epsilon}{4(b-a)}$$

for all $x \in [a,b]$.

Since $f_N$ is Riemann integrable, there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(f_N,\mathcal{P}) - L(f_N,\mathcal{P}) < \frac{\epsilon}{2}$$

Write $\mathcal{P} = \{t_k\}_{k=0}^{\ell}$ where $a = t_0 < t_1 < \cdots < t_\ell = b$. Then if

$$M_k = \sup_{x\in[t_{k-1},t_k]} f_N(x)$$
$$m_k = \inf_{x\in[t_{k-1},t_k]} f_N(x)$$

we know by the definition of the upper and lower Riemann sums that

$$U(f_N,\mathcal{P}) = \sum_{k=1}^{\ell} M_k(t_k - t_{k-1}) \qquad L(f_N,\mathcal{P}) = \sum_{k=1}^{\ell} m_k(t_k - t_{k-1})$$

Notice for all $x \in [t_{k-1}, t_k]$ that

$$m_k - \frac{\epsilon}{4(b-a)} \leq f_N(x) - \frac{\epsilon}{4(b-a)} < f(x) < f_N(x) + \frac{\epsilon}{4(b-a)} \leq M_k + \frac{\epsilon}{4(b-a)}$$

Therefore,

$$\begin{aligned} U(f,\mathcal{P}) &\leq \sum_{k=1}^{\ell}\left(M_k + \frac{\epsilon}{4(b-a)}\right)(t_k - t_{k-1}) \\ &= \sum_{k=1}^{\ell} M_k(t_k - t_{k-1}) + \sum_{k=1}^{\ell} \frac{\epsilon}{4(b-a)}(t_k - t_{k-1}) \\ &= U(f_N,\mathcal{P}) + \frac{\epsilon}{4} \end{aligned}$$

and

$$\begin{aligned} L(f,\mathcal{P}) &\geq \sum_{k=1}^{\ell}\left(m_k - \frac{\epsilon}{4(b-a)}\right)(t_k - t_{k-1}) \\ &= \sum_{k=1}^{\ell} m_k(t_k - t_{k-1}) - \sum_{k=1}^{\ell} \frac{\epsilon}{4(b-a)}(t_k - t_{k-1}) \\ &= L(f_N,\mathcal{P}) - \frac{\epsilon}{4} \end{aligned}$$

Hence,

$$\begin{aligned}
U(f,\mathcal{P}) - L(f,\mathcal{P}) &\leq \left(U(f_N,\mathcal{P}) + \frac{\epsilon}{4}\right) - \left(L(f_N,\mathcal{P}) - \frac{\epsilon}{4}\right) \\
&= (U(f_N,\mathcal{P}) - L(f_N,\mathcal{P})) + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $f$ is Riemann integrable.

To see that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

let $\epsilon > 0$ be arbitrary. Since $(f_n)_n$ converges to $f$ uniformly, there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$

for all $n \geq N$ and $x \in [a,b]$. Therefore, for all $n \geq N$, we have

$$\begin{aligned}
\left|\int_a^b f_n(x)dx - \int_a^b f(x)\right| &= \left|\int_a^b f_n(x) - f(x)dx\right| \\
&\leq \int_a^b |f_n(x) - f(x)|dx \\
&\leq \int_a^b \frac{\epsilon}{b-a}dx \\
&= \epsilon
\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, the result is complete! $\square$

Theorem 13.3 immediately allows us to integrate uniformly convergent series of Riemann integrable functions term-by-term!

**Corollary 13.4.** *Let $(f_n)_n$ be a sequence of real-valued Riemann integrable functions on $[a,b]$. If $\sum_{n=1}^\infty f_n$ converges uniformly to $f : [a,b] \to \mathbb{R}$, then $f$ is Riemann integrable and*

$$\int_a^b f(x)dx = \sum_{n=1}^\infty \int_a^b f_n(x)dx$$

*Proof.* For each $N \in \mathbb{N}$, define $S_N : [a,b] \to \mathbb{R}$ by

$$S_N(x) = \sum_{k=1}^{N} f_k(x)$$

for all $x \in [a,b]$. Since $f_n$ is Riemann integrable for all $n$, $S_N$ is Riemann integrable. Moreover, since $(S_N)_N$ converges uniformly to $f$ by assumption, Theorem 13.3 implies that $f$ is Riemann integrable and

$$\begin{aligned}
\int_a^b f(x)dx &= \lim_{N\to\infty} \int_a^b S_N(x)dx \\
&= \lim_{N\to\infty} \int_a^b \sum_{k=1}^{N} f_k(x)dx \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} \int_a^b f_k(x)dx \\
&= \sum_{n=1}^{\infty} \int_a^b f_n(x)dx
\end{aligned}$$

as desired. $\square$

# Chapter 14

# Differentiation of Series of Functions

Recall that given a closed interval $[a, b]$ and a function $f : [a, b] \to \mathbb{R}$, we say that $f$ is differentiable on $[a, b]$ if $f$ is continuous on $[a, b]$ and continuous on $(a, b)$.

**Theorem 14.1** (Differentiable Limit Theorem)**.** *Let $(f_n)_n$ be a sequence of real-valued differentiable functions on a closed interval $[a, b]$ that converge pointwise to a function $f : [a, b] \to \mathbb{R}$. If*

1. *$f_n'$ is Riemann integrable (i.e. continuous) for all $n \in \mathbb{N}$*
2. *$(f_n')_n$ converges uniformly on $[a, b]$ to a continuous function $f' : [a, b] \to \mathbb{R}$,*

*Then $f$ is differentiable on $[a, b]$. That is, $(f_n')_n$ converges uniformly to $f'$ on $[a, b]$.*

*Proof.* Notice that for all $x \in [a, b]$ that

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x f'(t)dt$$

Therefore, by the Fundamental Theorem of Calculus, $f$ is differentiable on $[a, b]$. Hence, $(f_n')_n$ converges uniformly to $f'$ on $[a, b]$. $\square$

Of course, we immediate obtain the analogue for series.

**Theorem 14.2.** *Let $(f_n)_n$ be a sequence of real-valued differentiable functions on $[a, b]$. If*

1. $\sum_{n=1}^{\infty} f_n(x)$ *converges for all* $x \in [a, b]$
2. $f_n'$ *is Riemann integrable (i.e. continuous) for all* $n \in \mathbb{N}$ *and*
3. $\sum_{n=1}^{\infty} f_n'$ *converges uniformly on* $[a, b]$ *to a continuous function* $f' : [a, b] \to \mathbb{R}$.

*Then the function* $f : [a, b] \to \mathbb{R}$ *defined by*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

*for all* $x \in [a, b]$ *is differentiable and*

$$\sum_{n=1}^{\infty} f_n'$$

*converges uniformly to* $f'$ *on* $[a, b]$.

*Proof.* For each $N \in \mathbb{N}$, define $S_N : [a, b] \to \mathbb{R}$ by

$$S_N(x) = \sum_{k=1}^{N} f_k(x)$$

for all $x \in [a, b]$. By the first assumption, $(S_N)_N$ converges pointwise to $f$. Moreover, by the second assumption,

$$S_N'(x) = \sum_{k=1}^{N} f_k'(x)$$

is Riemann integrable on $[a, b]$ for all $N \in \mathbb{N}$, and by the third assumption, $(S_N')_N$ converges uniformly to a continuous function on $[a, b]$, then Theorem 14.1 implies that $f$ is differentiable and

$$\sum_{n=1}^{\infty} f_n'$$

converges uniformly to $f'$ on $[a, b]$. $\square$

# Chapter 15

# Power Series

From Corollary 14.2 we can now finally complete our construction of the exponential, sine and cosine functions by describing their derivatives. As all three of these functions have a very specific form, it is useful for other applications to describe a larger collection of functions and derive their properties.

**Definition 15.1.** Given $c \in \mathbb{R}$, a power series centered at $c$ is any series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

where $x$ is a real variable and $(a_n)_n$ is a sequence of real numbers.

Often, we take $c = 0$ when discussing power series as any results that can be done at $c = 0$ can be translated to an arbitrary $c$. However, we will prove the following at an arbitrary without the need to translate.

**Theorem 15.2.** *Let $(a_n)_n$ be a sequence of real numbers and let $c \in \mathbb{R}$. Suppose $x_0 \in \mathbb{R} \setminus \{c\}$ is such that*

$$\sum_{n=0}^{\infty} a_n(x_0-c)^n$$

*converges. Then for any $r \in \mathbb{R}$ with $0 < r < |x_0 - c|$, the series of functions in $x$*

$$\sum_{n=0}^{\infty} a_n(x-c)^n \qquad \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

*converge uniformly and absolutely on $[c-r, c+r]$.*

*Moreover, if $f : [c-r, c+r] \to \mathbb{R}$ is defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

*for all $x \in [c-r, c+r]$, then $f$ is differentiable on $[c-r, c+r]$ with*

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

*for all $x \in (c-r, c+r)$.*

*Proof.* For each $n \in \mathbb{N} \cup \{0\}$, let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = a_n(x-c)^n \qquad g_n(x) = na_n(x-c)^{n-1}$$

for all $x \in \mathbb{R}$. Our goal is to use the Weierstrass $M$-Test (Theorem 11.21) to show that $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=0}^{\infty} g_n$ converge absolutely, uniformly, and define continuous functions.

To begin, notice since

$$\sum_{n=0}^{\infty} a_n(x_0-c)^n$$

converges, Corollary 8.6 implies that

$$\lim_{n\to\infty} a_n(x_0-c)^n = 0$$

Hence, by Corollary 7.7 implies there exists an $M > 0$ such that

$$|a_n(x_0-c)^n| \leq M$$

Therefore, we have for all $n \in \mathbb{N} \cup \{0\}$ and $x \in [c-r, c+r]$ that

$$|a_n(x-c)^n| = |a_n(x_0-c)^n| \left| \frac{(x-c)^n}{(x-c_0)^n} \right| \leq M \left( \frac{r}{|x_0-c|} \right)^n$$

Therefore, since $0 \leq r < |x_0-c|$, we know that the geometric series

$$\sum_{n=0}^{\infty} M \left( \frac{r}{|x_0-c|} \right)^n$$

converges. Hence, since $f$ is continuous for all $n \in \mathbb{N} \cup \{0\}$, the Weierstrass $M$-test (Theorem 11.21) implies that $\sum_{n=0}^{\infty} f_n$ converges uniformly and absolute to $f$ on $[c-r, c+r]$ and $f$ is continuous on $[c-r, c+r]$.

To obtain a similar result for $\sum_{n=0}^{\infty} g_n$, first we claim that if $r_0 = \frac{r}{|x_0 - c|}$ then

$$\sum_{n=0}^{\infty} Mnr_0^{n-1}$$

converges. To see this, if $b_n = Mnr_0^n$ for all $n \in \mathbb{N}$ then

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{M(n+1)|r_0|^n}{Mn|r_0|^{n-1}} = \frac{n+1}{n}|r_0|$$

Hence,

$$\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = |r_0|$$

Therefore, since $|r_0| < 1$ since $0 \leq r < |x_0 - c|$ we obtain that $\sum_{n=0}^{\infty} Mnr_0^n$ converges by the Ratio Test (Theorem 8.14).

Now notice that for all $n \in \mathbb{N}$ and $x \in [c-r, c+r]$ that

$$|na_n(x-c)^{n-1}| = n|a_n(x_0-c)^{n-1}| \left| \frac{(x-c)^{n-1}}{(x-c_0)^{n-1}} \right| \leq nM \left( \frac{r}{|x_0 - c|} \right)^{n-1}$$

Since

$$\sum_{n=1}^{\infty} Mn \left( \frac{r}{|x_0 - c|} \right)^{n-1}$$

converges and since $g_n$ is continuous for all $n \in \mathbb{N} \cup \{0\}$, the Weierstrass $M$-Test (Theorem 11.21) implies that $\sum_{n=0}^{\infty} g_n$ converges uniformly and absolutely to a continuous function $g$ on $[c-r, c+r]$.

Finally, since $\sum_{n=0}^{\infty} f_n$ converges pointwise to $f$, since $f_n' = g_n$ is continuous (and thus Riemann integrable) for all $n \in \mathbb{N} \cup \{0\}$ and since $\sum_{n=0}^{\infty} f_n' = \sum_{n=1}^{\infty} g_n$ converges uniformly to a continuous function on $[c-r, c+r]$, Theorem 14.1 implies that $f$ is differentiable on $[c-r, c+r]$ and

$$f'(x) = \sum_{n=0}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

for all $x \in (c-r, c+r)$ as desired. $\square$

*Remark* 15.3. Given $c \in \mathbb{C}$ and a sequence $(a_n)_n$ of complex numbers is not difficult to verify that the first part of the proof of Theorem 15.2 works for complex power series

$$\sum_{n=0}^{\infty} a_n(z-c)^n \qquad \sum_{n=1}^{\infty} na_n(z-c)^{n-1}$$

where the interval $[c-r, c+r]$ is replaced with the closed disk of radius $r$ centered at $c$. The second part of Theorem 15.2 concerning the second power series is the derivative of the first power series is true, but more complicated to prove since we would need to discuss the derivatives of functions on the complex plane and since we cannot use Theorem 14.1 in its present form as the proof relies heavily on the Fundamental Theorem of Calculus. The details of these results are best left for a complex analysis course (MATH 3410).

With Theorem 15.2 in hand, we can immediately obtain some new convergent series.

**Example 15.4.** Recall for all $x \in (-1, 1)$, the geometric series $\sum_{n=0}^{\infty} x^n$ converges to the function

$$f(x) = \frac{1}{1-x}$$

Hence, Theorem 15.2 implies that $f'(x) = \sum_{n=1}^{\infty} nx^{n-1}$. Hence,

$$\sum_{n=1}^{\infty} nx^n = xf'(x) = \frac{x}{(1-x)^2}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

Of course, Theorem 15.2 finally allows us to completely prove the remaining properties of the exponential, cosine, and sine functions that we desired.

**Corollary 15.5.** *Consider the function $f : \mathbb{R} \to (0, \infty)$ defined by*

$$f(x) = e^x$$

*for all $x \in \mathbb{R}$. Then $f$ is differentiable on its domain and $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Moreover, $f$ is strictly increasing, bijective function.*

*The inverse of $f$ is the function* $\ln : (0,\infty) \to \mathbb{R}$ *is called the natural logarithm. The natural logarithm is differentiable on its domain with* $\ln'(x) = \frac{1}{x}$ *for all* $x \in \mathbb{R}$.

*Proof.* Since

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in \mathbb{R}$, Theorem 15.2 implies that $f$ is differentiable on any closed interval centered at 0 (and thus differentiable on $\mathbb{R}$) with

$$f'(x) = \sum_{n=1}^{\infty} n\frac{1}{n!}x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^{n-1} = e^x$$

as desired.

Notice that

$$f'(x) = e^x > 0$$

for all $x \in \mathbb{R}$. Therefore, $f$ is a strictly increasing continuous function from $\mathbb{R}$ to $(0,\infty)$ and thus has a differentiable inverse $\ln : (0,\infty) \to \mathbb{R}$. Moreover, if $x \in (0,\infty)$ and $\ln x = y$, then $x = e^y$ and

$$\ln'(x) = \frac{1}{f'(y)} = \frac{1}{e^y} = \frac{1}{x}$$

as desired. □

**Example 15.6.** Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

for all $x \in \mathbb{R}$, Theorem 15.2 implies that cos and sin are differentiable with

$$\sin'(x) = \sum_{n=0}^{\infty}(2n+1)\frac{(-1)^n}{(2n+1)!}x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = \cos x$$

and

$$\begin{aligned}\cos'(x) &= \sum_{n=0}^{\infty} 2n\frac{(-1)^n}{(2n)!}x^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}x^{2n-1} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+1)!}x^{2m+1} \\ &= -\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}x^{2m+1} = -\sin x\end{aligned}$$

just as we knew to be true!

*Remark* 15.7. Using Example 15.6, we can derive the known properties of cos and sin. In particular, since $x \in (0,1)$, we know

$$\frac{1}{(4n+1)!}x^{4n+1} - \frac{1}{(4n+3)!}x^{4n+3} > 0$$

for all $n \geq 2$, we obtain that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \geq x - \frac{1}{6}x^3 > 0$$

Hence $\cos'(x) < 0$ for all $x \in (0,1)$, so cosine is a decreasing function on $(0,1)$. Moreover, since $\sin'(x) = \cos x$, which must be positive on some interval around 0, sine must be increasing on $(0,c)$ for some $c > 0$. Thus, the coupling $\cos'(x) = -\sin x$ and $\sin'(x) = \cos x$ imply that cosine must have a first root larger than 0, which is what we call $\frac{\pi}{2}$. From there, we can derive all the special angles for cosine and sine, show the function $e^{i\theta}$ draws out a circle in the complex plane as $\theta$ moves from 0 to $2\pi$ and that the area of the circle is $\pi$ using integrals and trigonometric substitution. That is, our most basic understanding of trigonometry is a bi-product of series of functions.

**Definition 15.8** (Radius of Convergence)**.** Let $(a_n)_n$ be a sequence of real numbers and let $c \in \mathbb{R}$. The *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is

$$R = \sup\left\{|x_0 - c| : x_0 \in \mathbb{R}, \sum_{n=0}^{\infty} a_n(x_0-c)^n \text{ converges}\right\}$$

**Definition 15.9** (Radius of Convergence: Cauchy-Hadamard's Formula)**.** Let $(a_n)_n$ be a sequence of real numbers and let $c \in \mathbb{R}$. The *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

*Remark* 15.10. The Cauchy-Hadamard's Formula is commonly used for Taylor Series and radius of convergence in complex analysis. Although it is used for complex numbers, but the Cauchy-Hadamard's Formula also works for real numbers. Both Definitions 15.8 and 15.9 will give the same result.

*Remark* 15.11. Let $R$ be the radius of convergence of

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

Clearly, $R \in [0, \infty]$ by definition. Moreover, by Theorem 15.2 we know that $\sum_{n=0}^{\infty} a_n(x_0-c)^n$ converges, then so too must $\sum_{n=0}^{\infty} a_n(x-c)^n$ for all $x$ such that $|x-c| < |x_0-c|$. Therefore if $x \in (c-R, c+R)$, then $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges by Theorem 15.2. Moreover, if $x \notin [c-R, c+R]$, then $|x-c| > R$ so $\sum_{n=0}^{\infty} a_n(x-c)^n$ must diverge by definition of the radius of convergences. However, when $x = c-R$ or $x = c+R$, we do not have any information on whether $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges as the following example shows.

**Example 15.12.** For $x \in \mathbb{R}$, consider the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Since $\lim_{n\to\infty} \frac{x^n}{n}$ exists and is zero if and only if $x \in [-1, 1]$, the above series can converge only if $x \in [-1, 1]$ by Corollary 8.6. Moreover, since

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by Example 8.20, the radius of convergence of this power series around 0 is 1. Moreover, this series converges when $x = -1$ whereas, when $x = 1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge by Corollary 8.13. Thus, a power series may or may not converge at the boundary points of its radius of convergence.

**Example 15.13.** Consider the following power series $\sum_{n=0}^{\infty} x^n$. Note that $a_n = 1$ for $n \in \mathbb{N}$, then

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{1}} = \frac{1}{1} = 1$$

Therefore, we can conclude that the power series $\sum_{n=0}^{\infty} x^n$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Note that the power series $\sum_{n=0}^{\infty} x^n$ also diverges for $|x| = 1$, that is, $x = \pm 1$.

**Example 15.14.** Consider the following power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$. Note that

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n}}} = \frac{1}{1} = 1$$

Therefore, we can conclude that the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. For $x = 1$, the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges. However, for $x = -1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-1)^n = -\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**Example 15.15.** Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. Note that $|a_n| = 0$ if $n = 2k$ and $|a_n| = \frac{1}{2k+1}$ if $n = 2k+1$ for $k \in \mathbb{N}$. Therefore, the sequence $(\sqrt[n]{|a_n|})_n$ is not convergent, but

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{1} = 1$$

Then we can conclude that the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. For $x = 1$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} 1^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges, and for $x = -1$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (-1)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ also converges.

**Proposition 15.16.** *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series such that the sequence $(\sqrt[n]{|a_n|})_n$ is bounded and $R$ is the radius of convergence of the series ($R = \infty$ if $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$). Then the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for $0 < r < R$.*

*Proof.* Note that $r < R \Leftrightarrow \limsup_{n\to\infty} r\sqrt[n]{|a_n|} < 1$ which implies that there exists an $N \in \mathbb{N}$ such that $r\sqrt[n]{|a_n|} < 1$ for all $n \geq N$ and the series $\sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} (\sqrt[n]{|a_n|} r)^n$ is convergent. Finally, noting the $|a_n x^n| \leq |a_n| r^n$ for $x \in [-r, r]$ we can conclude that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ using the Weierstrass $M$-Test. $\square$

**Corollary 15.17.** *The function $f : (-R, R) \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$.*

**Theorem 15.18.** *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series such that the sequence $(\sqrt[n]{|a_n|})_n$ is bounded, $R$ be the radius of convergence of the series $f : (-R, R) \to$*

$\mathbb{R}$ *defined by* $f(x) = \sum_{n=0}^{\infty} a_n x^n$. *Then, the function* $f$ *is differentiable on* $(-R, R)$, $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ *and the radius of convergence of* $\sum_{n=0}^{\infty} n a_n x^{n-1}$ *is equal to* $R$.

**Corollary 15.19.** *The function* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is said to be a* $C^{\infty}$ *(infinitely differentiable) function on* $(-R, R)$ *and*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-(k-1))a_n x^{n-k}$$

From Corollary 15.19, we can see that $f^{(k)}(0) = k!a_k$ and so

$$a_k = \frac{f^{(k)}(0)}{k!}$$

and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$