Question 1 (22 points total). *Let $X \subset \mathbb{C}$, $x \in X$, $f : \mathbb{C} \to \mathbb{C}$, $(x_n)_n$ be a sequence of complex numbers on X and $(f_n)_n$ be a sequence of complex-valued functions defined on X.*

(a) *(3 pts.) Define what it means for f to be continuous at x.*

(b) *(4 pts.) Define what it means for $(x_n)_n$ to converge to x.*

(c) *(4 pts.) Define what it means for $(f_n)_n$ to converge uniformly to f on X.*

(d) *(11 pts.) Assume that each f_n is continuous on X, $(f_n)_n$ converges uniformly on to f on X and $(x_n)_n$ converges to x. Prove that $(f_n(x_n))_n$ converges to $f(x)$.*

Question 2 (20 points total). *Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers. Determine whether the following mathematical statements are true or false. Prove your answers.*

(a) *(9 pts.) If $x_n \leq y_n$ for all $n \in \mathbb{N}$ and $(y_n)_n$ is a convergent sequence, then $(x_n)_n$ is a convergent sequence.*

(b) *(11 pts.) If $(x_n)_n$ is increasing, $x_n \leq y_n$ for all $n \in \mathbb{N}$ and $(y_n)_n$ is a convergent sequence, then $(x_n)_n$ is a convergent sequence.*

Question 3 (25 points total). *Let $X \subset \mathbb{C}$, $(f_n)_n$ and $(g_n)_n$ be sequences of real-valued functions defined on X and $(s_n)_n$ the sequence of partial sums of the sequence $(f_n)_n$, that is, $s_n = f_1 + f_2 + \cdots + f_n$. Assume that*

(i) $\sup_{x \in X} |s_n(x)| \leq M$ *for all $n \in \mathbb{N}$.*

(ii) *$g_{n+1}(x) \leq g_n(x)$ for all $n \in \mathbb{N}$ and $x \in X$.*

(iii) *$(g_n)_n$ converges uniformly to the constant function $g(x) = 0$ on X.*

Prove that the series $\sum_{i=1}^{\infty} f_i g_i$ converges uniformly on X.

Question 4 (33 points total). *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series such that $(\sqrt[n]{|a_n|})_n$ is bounded, R be the radius of convergence and $f : (-R, R) \to \mathbb{R}$ be defined by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-R, R)$. Suppose that there exists $x_n \in (-R, R)$ such that $(x_n) \to 0$, $x_n \neq 0$ and $f(x_n) = 0$ for $n \in \mathbb{N}$.*

(a) *(10 pts.) Show that $f(0) = 0$.*

(b) *(13 pts.) Show that f is differentiable on $(-R, R)$ and estimate $f'(0)$. (Hint: Use the Mean Value Theorem)*

(c) *(10 pts.) Show that $f(x) = 0$ for $x \in (-R, R)$.*

Question 5 (BONUS, 10 points total). *Determine whether the following mathematical statements are true or false. Prove your answers.*

(a) *(6 pts.) If $\sum_{i=1}^{\infty} x_i$ is an absolutely convergent series, then $\sum_{i=1}^{\infty} x_i^2$ converges.*

(b) *(4 pts.) If $\sum_{i=1}^{\infty} x_i$ and $\sum_{i=1}^{\infty} y_i$ are convergent series of real numbers, then $\sum_{i=1}^{\infty} x_i y_i$ converges.*