LECTURE 2

September 8, 2023

1. Equivalence Relations

In the previous lecture, we started talking about the equivalence relations.

DEFINITION 1.1. Let X be a nonempty set. A binary relation "$\sim$" on X is called an *equivalence relation* if the following properties hold:

- (Reflexivity) For all $x \in X$, $x \sim x$.
- (Symmetry) For all $x, y \in X$, if $x \sim y$, then $y \sim x$.
- (Transitivity) For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

EXAMPLE 1.2. Let $n \in \mathbb{N}$ and let $x, y \in \mathbb{Z}$. We say that x *is congruent to* y *modulo* n *and write* $x \equiv y$ *mod* n if $x - y$ is an integer multiple of n, i.e. $n \mid x - y$, or there exists an integer $k \in \mathbb{Z}$ such that $x - y = kn$.

For example, if we consider $7 \equiv 17 \mod 5$, then this congruence is true since $7 - 17 = -10 = (-2) \cdot 5$.

Using Definition 1.1, we will show that $x \equiv y \mod n$ is an equivalence relation.

First, showing reflexivity, let $x \in \mathbb{Z}$ be arbitrary. Then

$$x - x = 0 \cdot n \Rightarrow x \equiv x \mod n$$

Therefore, congruence modulo n is reflexive.

Next, showing symmetry, let $x, y \in \mathbb{Z}$ be arbitrary. Then there exists a $k \in \mathbb{Z}$ such that

$$x - y = kn \Rightarrow y - x = -k \cdot n \Rightarrow y \equiv x \mod n$$

Therefore, congruence modulo n is symmetric.

Finally, showing transitivity, since $x \equiv y \mod n$, there exists an integer $k_0 \in \mathbb{Z}$ such that $x - y = k_0 n$. Similarly, since $y \equiv z \mod n$, there exists an integer $k_1 \in \mathbb{Z}$ such that $y - z = k_1 n$. Now adding the two equations above, we have

$$x - z = k_0 n + k_1 n = (k_0 + k_1)n \Rightarrow x \equiv z \mod n$$

Therefore, congruence modulo n is transitive, and hence, congruence modulo n is an equivalence relation.

EXAMPLE 1.3. Consider the set $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Define "$\sim$" on X as follows: If $(p, q), (r, s) \in X$, then $(p, q) \sim (r, s)$ if $ps = qr$. This is an equivalence relation.

For example, consider $(3,2) \sim (9,6)$. Then these are equivalent since $3 \cdot 6 = 2 \cdot 9 = 18$.

As an exercise, show that $\sim$ is reflexive and symmetric. We will show that $\sim$ is transitive.

Consider three pairs $(p,q),(r,s),(u,v) \in X$ such that $(p,q) \sim (r,s)$ and $(r,s) \sim (u,v)$. Then this implies that $ps = qr$ and $rv = su$. We want to show that $(p,q) \sim (u,v)$, that is, $pv = qu$. Let us multiply the first equation by v. Then we would have that $pvs = qvr$. Here now, we can replace $rv = su$ and so $pvs = qsu$ and since $s \neq 0$, then we would obtain that $pv = qu$ and therefore, $(p,q) \sim (u,v)$.

2. Equivalence Classes

DEFINITION 2.1. Let X be a nonempty set, and let $\sim$ denote an equivalence relation on X. For $x \in X$, we define the *equivalence class* of x as follows:

$$[x] = \{y \in X : x \sim y\}$$

EXAMPLE 2.2. Take $\mathbb{Z}$ with congruence modulo 5. Say we take $3 \in \mathbb{Z}$, then

$$\begin{aligned}[3] &= \{3, 8, 13, 18, ..., -2, -7, ...\} \\ &= \{k \in \mathbb{Z} : 3 + 5k\}\end{aligned}$$

EXAMPLE 2.3. Take $X = \mathbb{Z} \setminus (\mathbb{Z} \setminus \{0\})$ with the defined $\sim$ in Example 1.3. Then say we take $(2,3)$. We have

$$\begin{aligned}[(2,3)] &= \{(2,3), (-2,-3), (4,6), (-4,-6), ...\} \\ &= \{k \in \mathbb{Z} : (2k, 3k)\} \\ &= \left\{\text{all pairs } (p,q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \text{ such that } \frac{p}{q} = \frac{2}{3}\right\}\end{aligned}$$

THEOREM 2.4. *Let $\sim$ be an equivalence relation on a set X. Then the equivalence classes of $\sim$ form a partition of X. That is,*

1. *$X = \bigcup_{x \in X}[x]$, i.e. the collection of equivalence classes covers the set X.*
2. *If $x, y \in X$, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.*

PROOF. (i) Let $x, \in X$ be arbitrary. Then $[x] \subset X$ implies that $\bigcup_{x \in X}[x] \subset X$. We also show $X \subset \bigcup_{x \in X}[x]$. Let $x_0 \in X$ be arbitrary. Then $x_0 \sim x_0$ implies that $x_0 \in [x_0] \subset \bigcup_{x \in X}[x]$ implies that $X \subset \bigcup_{x \in X}[x]$, and hence $X = \bigcup_{x \in X}[x]$.

(ii) Let $x, y \in X$ be arbitrary. If $[x] \cap [y] = \emptyset$, there is nothing to prove. Otherwise, assume that there exists a point $z_0 \in [x] \cap [y]$. We will show that $[x] = [y]$, that is, $[x] \subset [y]$ and $[y] \subset [x]$. To see that $[x] \subset [y]$, let $w \in [x]$ be arbitrary. Then this implies that $x \sim w$. But $z_0 \in [x]$ as well, and so $x \sim z_0$. By these two relations, $w \sim z_0$ by transitivity. Furthermore, $z_0 \in [y]$ as well and hence, $y \sim z_0$ and hence $w \sim y$ by transitivity once again, implying to $w \in [y]$, as desired. $\square$

EXAMPLE 2.5. Let $X = \mathbb{Z}$ be equipped with the congruence modulo 5. Consider the following equivalence classes:

$$[0] = \{..., -10, -5, 0, 5, 10, ...\}$$
$$[1] = \{..., -9, -4, 1, 6, 11, ...\}$$
$$[2] = \{..., -8, -3, 2, 7, 12, ...\}$$
$$[3] = \{..., -7, -2, 3, 8, 13, ...\}$$
$$[4] = \{..., -6, -1, 4, 9, 14, ...\}$$

We can see that all of these sets are disjoint from each other.

3. The Integers: Mathematical Induction

Let $S(n)$ be a statement about some integers that is either True or False for each n.

EXAMPLE 3.1. For example,

- $S(n) := \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $S(n) := n$ is odd

THEOREM 3.2 (Principal of Mathematical Induction). *Let $S(n)$ be a statement about integers. Assume*

- *There exists an integer $k_0 \in \mathbb{Z}$ such that $S(k_0)$ is true.*
- *For any $k \geq k_0$ if $S(k)$ is true, then $S(k+1)$ must also be true.*

Then $S(n)$ must be true for all $n \geq k_0$.