MATH 3021 ALGEBRA I ASSIGNMENT 1

JOE TRAN

Question 1. *Let $n \in \mathbb{N}$, and $a, b, c, d \in \mathbb{Z}$ such that*

$$a \equiv b \ (mod\ n) \quad c \equiv d \ (mod\ n)$$

(i.e. $n \mid a - b$ and $n \mid c - d$. Prove the following:

(a) $a + c \equiv b + d$ *(mod n),*
(b) $a \cdot c \equiv b \cdot d$ *(mod n)*

Question 2. *Use mathematical induction to prove that for every $n \in \mathbb{N}$, $4 \cdot 10^{n+2} + 10^n + 1$ is divisible by 3.*

Question 3. *Use mathematical induction to prove that for any $n \in \mathbb{N}$ and nonnegative real numbers $a_1, a_2, ..., a_n \in \mathbb{R}$,*

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k$$

Question 4.
(a) *Let p and q be prime numbers such that $p \mid q$. Prove that $p = q$.*
(b) *Let $a, b \in \mathbb{N}$ with $a, b \geq 2$ such that $a \mid b$. Prove that there exists a prime number such that $p \mid a$ and $p \mid b$.*

Question 5. *Let $a, b \in \mathbb{N}$ with $a, b \geq 2$. Prove that $\gcd(a, b) \neq 1$ if and only if there exists a prime number such that $p \mid a$ and $p \mid b$.*

Question 6. *It is known that for any $k < m \in \mathbb{N}$, the binomial coefficient $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ is an integer. Let p be a prime number and $1 \leq k < p$*

(a) *Prove that p does not divide any of the numbers $k!$ and $(p - k)!$.*
(b) *Prove that p divides $\binom{p}{k}$.*
(c) *Use the binomial theorem to prove that for any integers x, y, we have*

$$(x + y)^p \equiv x^p + y^p \ (mod\ p)$$

Date: September 15, 2023, Due: September 29, 2023 @ 11:59 PM.