Question 1. We define two operations $\boxplus$ and $\boxtimes$ on $\mathbb{Z}$ as

$$a \boxplus b = a + b - 1$$
$$a \boxtimes b = ab - a - b + 2$$

for $a, b \in \mathbb{Z}$.

(a) Show that $\mathbb{Z}$ together with addition $\boxplus$ and multiplication $\boxtimes$ is a ring.
(b) Determine if this ring is
  (i) A commutative ring
  (ii) A ring with identity
  (iii) An integral domain.
  (iv) A field.

Question 2. Let $\mathbb{Z}_n[i] = \{a + ib : a, b \in \mathbb{Z}_n, i^2 = -1\}$ denote the Gaussian integers modulo n.

(a) Generate the multiplication table of $\mathbb{Z}_n[i]$ for $n = 2, 3, ..., 7$. (Use computer. Submit the tables for only $n = 2, 3$.)
(b) Determine all integers $n \geq 2$ for which $\mathbb{Z}_n[i]$ is an integral domain, hence, a field. (Hint: Fermat's theorem on the sum of squares. You may assume that $a^2 + b^2 = 0 \bmod p$ implies $a = b = 0 \bmod p$.)

Question 3. Let R be a ring. Define the *center of* R to be

$$Z(R) = \{a \in R : ar = ra \text{ for all } r \in R\}$$

Prove that $Z(R)$ is a commutative subring of R.

Question 4. An element a is an *idempotent* if $a^2 = a$.

(a) Prove that the only idempotents in an integral domain are 0 and 1.
(b) Find a ring with an idempotent that is not equal to 0 nor 1.
(c) Let R be a commutative ring with characteristic 2. Prove that the set $S = \{a \in R : a^2 = a\}$ is a subring of R.

Question 5. (a) Let R be a commutative ring with identity. Show that the set of units in R, $U(R)$, is an abelian group under $\times_R$.
(b) Let F be a finite field with n elements. Show that $a^{n-1} = 1$ for all $a \neq 0 \in F$.

Question 6. Let F be a field and let K be a subset of F with at least two elements. Prove that K is a subfield of F if for any $a, b \in K$, $a - b \in K$ and $ab^{-1} \in K$.

Question 7. (a) Let R be a commutative ring with prime characteristic p. Show that for $r, s \in R$,
  (i) $(r + s)^p = r^p + s^p$
  (ii) $(r + s)^{p^m} = r^{p^m} + s^{p^m}$ for all positive integers m.
  (iii) The Frobenius map $x \mapsto x^p$ is a ring homomorphism from R to R.
(b) Give an example of a ring of characteristic 4 and elements r and s such that $(r+s)^4 \neq r^4+s^4$.

Question 8. Let $\phi : R \to S$ be a ring homomorphism. Let $\phi(R) = \{\phi(r) : r \in R\}$. Prove each of the following statements:

(a) $\phi(0_R) = 0_S$
(b) $\phi(-b) = -\phi(b)$ for all $b \in R$
(c) $\phi(R)$ is a subring of S
(d) If R is a commutative subring, then $\phi(R)$ is a commutative subring.
(e) Suppose R and S are rings with identities. If ϕ is onto, then $\phi(1_R) = 1_S$.

(f) If R is a field and $\phi(R) \neq \{0_S\}$ then $\phi(R)$ is a field.

Question 9. Consider the ring $S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ with matrix addition and matrix multiplication. Show that $\phi : \mathbb{C} \to S$ given by

$$\phi(a + ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

is a ring isomorphism.

Question 10. Show that $\mathbb{Z}_3[i]$ is ring isomorphic to $\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$.