MATH 3022 ALGEBRA II: LECTURE 1

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1. Recall From MATH 3021

Definition 1. Let G be a set and binary operation $* : G \times G \to G$. The pair $(G, *)$ is called a group if

(1) (Associative) For all $x, y, z \in G$,
$$x * (y * z) = (x * y) * z$$
(2) (Identity) There exists an $e \in G$ such that for all $x \in G$,
$$x * e = e * x = x$$
(3) (Inverse) For every $x \in G$, there exists an inverse $x^{-1} \in G$ such that
$$x * x^{-1} = x^{-1} * x = e$$

Definition 2. A group $(G, *)$ is called an abelian group if for every $x, y \in G$,
$$x * y = y * x$$

Example 1. Below are examples of groups.

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$, $(\mathbb{C}, \cdot)$
- $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$
- $(\mathbb{Z}_n, +_n)$, $(n\mathbb{Z}, +)$
- $(\mathbb{F}[x], +)$ (we will study this when talking about polynomials)[1]
- S_n, D_n, A_n

Remark 1. The examples $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_n$, $n\mathbb{Z}$, $\mathbb{F}[x]$ and $\mathcal{M}_n(\mathbb{F})$ are all abelian groups under addition. Furthermore, we have a type of "multiplication" operation for these examples. This second operation called "multiplication" will be used in Ring Theory.

Question 1. *What are the properties of "multiplication" for the sets in Remark 1?*

Let S be any of the sets in Remark 1. We require that

(1) Associativity: For all $x, y, z \in S$, $x(yz) = (xy)z$.
(2) Distributive: For all $x, y, z \in S$, $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$

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[1]The notation $\mathbb{F}$ means the field of $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Definition 3. Let R be a set with two binary operations $+$ and $\cdot$. Then the set R is called a ring if the following are satisfied:

(1) For all $x, y \in R$,
$$x + y = y + x$$
(2) For all $x, y, z \in R$,
$$(x + y) + z = x + (y + z)$$
(3) There exists a $0 \in R$ such that for all $x \in R$,
$$x + 0 = 0 + x = x$$
(4) For every $x \in R$, there exists an element $-x \in R$ such that
$$x + (-x) = 0$$
(5) For all $x, y, z \in R$, $(xy)z = x(yz)$
(6) For all $x, y, z \in R$, $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$

Example 2. The following are examples of rings.

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_n$, $n\mathbb{Z}$ with the usual addition $+$ and multiplication $\cdot$
- $\mathcal{M}_n(\mathbb{F})$ with matrix addition $+$ and multiplication $\cdot$.
- $\mathcal{C}([a, b])$ with usual addition $+$ and multiplication $\cdot$
- $\mathbb{Z}_n$ with modulo addition $+_n$ and modulo multiplication $\cdot_n$

Note that $GL_n(\mathbb{F})$ is actually not a ring, because it is not closed under addition. In particular, if $A \in GL_n(\mathbb{F})$, then $-A \in GL_n(\mathbb{F})$, but $A + (-A) = 0 \notin GL_n(\mathbb{F})$.

Further Readings: Chapter 16, pg. 190-192