MATH 3022 Algebra II **Homework Assignment 1**

ASSIGNMENT JOE TRAN

Question 1. We define two operations $\boxplus$ and $\boxtimes$ on $\mathbb{Z}$ as

$$a \boxplus b = a + b - 1$$
$$a \boxtimes b = ab - a - b + 2$$

for $a, b \in \mathbb{Z}$.

(a) Show that $\mathbb{Z}$ together with addition $\boxplus$ and multiplication $\boxtimes$ is a ring.
(b) Determine if this ring is
 (i) A commutative ring
 (ii) A ring with identity
 (iii) An integral domain.

Question 2. Let $\mathbb{Z}_n[i] = \{a + ib : a, b \in \mathbb{Z}_n, i^2 = -1\}$ denote the Gaussian integers modulo n.

(a) Generate the multiplication table of $\mathbb{Z}_n[i]$ for $n = 2, 3, ..., 7$.
(b) Determine all integers $n \geq 2$ for which $\mathbb{Z}_n[i]$ is an integral domain, hence, a field.

Question 3. Let R be a ring. Define the *center of* R to be

$$Z(R) = \{a \in R : ar = ra \text{ for all } r \in R\}$$

Prove that $Z(R)$ is a commutative subring of R.

Question 4. An element a is an *idempotent* if $a^2 = a$.

(a) Prove that the only idempotents in an integral domain are 0 and 1.
(b) Find a ring with an idempotent that is not equal to 0 nor 1.
(c) Let R be a commutative ring with characteristic 2. Prove that the set $S = \{a \in R : a^2 = a\}$ is a subring of R.