Theorem: Let R be a commutative ring with identity $1_R \neq 0_R$. Then is a prime ideal iff R/P is an integral domain.

Definition: A prime ideal P of a commutative ring R with identity $1_R \neq 0_R$ is a proper ideal of R such that if $ab \in P$, then either $a \in P$ or $b \in P$.

Proof: $(\Rightarrow)$ Assume that $P \trianglelefteq R$ is a prime ideal and that for $a, b \in R$ $(a+P)(b+P) = 0_R + P$. We want to show that R/P has no zero divisors. Indeed,

$$(a+P)(b+P) = ab+P = 0_R + P \Rightarrow ab \in P. \Rightarrow a \in P \text{ or } b \in P$$

$$\Rightarrow a+P = 0_R \text{ or } b+P = 0_R \Rightarrow R/P \text{ has no zero divisors and}$$

so R/P is an integral domain.

$(\Leftarrow)$. Assume that R/P is an integral domain. We want to show that $P \trianglelefteq R$. Let $a, b \in R$ such that $ab \in P$. Then

$$ab + P = (a+P)(b+P) = 0+P. \Rightarrow a+P = 0+P \text{ or } b+P = 0+P$$

bec. R/P has no zero divisors. so $a \in P$ or $b \in P$, and thus, $P \trianglelefteq R$ is a prime ideal.

Maximal Ideals

Definition: A maximal ideal of a commutative ring is a proper ideal of R such that $\exists I \trianglelefteq R$ satisfying the property that if $M \subset I \subset R$, then $I = M$ or $I = R$.

Example: If $R = \mathbb{Z}[x]$, then $\langle x \rangle \subsetneq \langle x, 2 \rangle \subsetneq \mathbb{Z}[x]$, so $\langle x \rangle$ is not the maximal ideal of $\mathbb{Z}[x]$

Example: If $R = \mathbb{Q}[x]$, then is $\langle x \rangle$ the maximal ideal of $\mathbb{Q}[x]$?

Note that $2 \in \langle x, 2 \rangle$ and so $\frac{1}{2} \cdot 2 \in \langle x \rangle \Rightarrow p(x) \cdot 1 \in \langle x, 2 \rangle$ $\forall p(x) \in \mathbb{Q}[x] \Rightarrow \langle x, 2 \rangle = \mathbb{Q}[x]$.

Suppose $I \trianglelefteq \mathbb{Q}[x]$ satisfying $\langle x \rangle \subset I \subset \mathbb{Q}[x]$. Assume otherwise that $\langle x \rangle \subsetneq I$. We want to show that $I = \mathbb{Q}[x]$. Bec. $\langle x \rangle \subsetneq I$, I contains a polynomial with nonzero constant term

$$p(x) = \sum_{i=1}^{n} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in I$$

$$\underline{- \quad a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x \quad} \in \langle x \rangle \subset I$$

$$a_0$$

$a \in I \Rightarrow a^{-1} a \in I \Rightarrow f(x) \cdot 1 \in I \Rightarrow I = \mathbb{Q}[x]$ $\forall f(x) \in \mathbb{Q}[x]$.

Since $a_0 \neq 0 \Rightarrow \frac{1}{a_0} \in \mathbb{Q}[x] \Rightarrow \frac{1}{a_0} \cdot a_0 = 1 \in I \Rightarrow p(x) \cdot 1 \in I$ $\forall p(x) \in \mathbb{Q}[x] \Rightarrow I = \mathbb{Q}[x]$. Therefore, $\langle x \rangle$ is the maximal ideal of $\mathbb{Q}[x]$.

Theorem: Let R be a commutative ring with identity and $M \trianglelefteq R$. Then M is a maximal ideal of R iff R/M is a field.

Examples:

- $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is a field iff n is prime, then $n\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ iff n is prime.
- $\langle x^2+1 \rangle$ is a maximal ideal of $\mathbb{R}[x]$. Indeed, we want to show that $\mathbb{R}[x]/\langle x^2+1 \rangle$ is a field. Let $\phi : \mathbb{R}[x] \to \mathbb{C}$ given by $\phi(f(x)) = f(i)$. Then ϕ is a ring homomorphism. (Verify)

Now by the First Isomorphism Theorem, $f(i) = 0$ if and only if

$x^2+1 \mid f(x)$ if and only if $f(x) \in \langle x^2+1 \rangle$. Thus $\ker(\phi) = \langle x^2+1 \rangle$.

Now we show that $\phi(\mathbb{R}[x]) = \mathbb{C}$. Let $a+ib \in \mathbb{C}$ where $a, b \in \mathbb{R}$.

Then $\phi(a+bx) = a+ib$. By the First Isomorphism Theorem,

$\mathbb{R}[x]/\langle x^2+1 \rangle \simeq \mathbb{C}$. By a known theorem (16.35), $\langle x^2+1 \rangle$ is a

maximal ideal of $\mathbb{R}[x]$