MATH 3022 ALGEBRA II: ASSIGNMENT 2

JOE TRAN

Question 1. *Let I and J be ideals of a ring R with $J \subset I$.*

(a) *Prove that $I/J = \{a + J : a \in I\}$ is an ideal in the quotient ring R/J.*
(b) *Prove the Third Isomorphism Theorem for Rings.*

Question 2 (5 pts.). (a) *Let R be a commutative ring with identity. Show that if $\{0_R\}$ and R are the only ideals of R, then R is a field.*
(b) *Let F be a field. Use (a) to show that $F[x]$ is not a field.*

Question 3 (5 pts.). *Show that the principal ideal $\langle x - 1 \rangle$ in $\mathbb{Z}[x]$ is prime but not maximal.*

Question 4. *Let $I_0 = \{f(x) \in \mathbb{Z}[x] : f(0) = 0\}$. Show that for any positive integer n, there exists a sequence of ideals $I_1, ..., I_n$ satisfying*

$$I_0 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n \subsetneq \mathbb{Z}[x]$$

Question 5 (5 pts.). *Let R be an integral domain. Assume that the division algorithm always holds in $R[x]$. Prove that R is a field.*

Question 6. *We have the following definition:*

Definition 1. Let a be a nonzero element in a field F. The *multiplicative order of a* is the least positive integer k where $a^k = 1_F$.

Prove that for any positive integer n, a field F can have at most a finite number of elements of multiplicative order at most n.

Question 7. (a) *Prove that for every prime p,*

$$x^{p-1} - 1 = (x-1)(x-2)\cdots(x-(p-1))$$

in $\mathbb{Z}_p[x]$.
(b) *Prove Wilson's Theorem: For integers $n \geq 2$, $(n-1)! = n - 1 \bmod n$ if and only if n is a prime.*

Question 8 (10 pts.). *Let p be a prime.*

(a) *Show that there are $\frac{p(p+1)}{2}$ reducible polynomials over $\mathbb{Z}_p$ of the form $x^2 + ax + b$.*
(b) *Determine the number of irreducible polynomials over $\mathbb{Z}_p$ of the form $x^2 + ax + b$.*

Date: February 12, 2024.

(c) *Show that there exists a field of order* p^2*, for every prime* p.

(d) *Construct a finite field with four elements. Give the addition and multiplication table of your field.*

Question 9. (a) *Prove the Rational Root Theorem: Let* $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0 \in \mathbb{Z}[x]$ *with* $c_n \neq 0$. *If* $f\left(\frac{a}{b}\right) = 0$ *for some relatively prime integers* a *and* b*, then* $a \mid c_0$ *and* $b \mid c_n$.

(b) *Use (a) to prove that if* r *is a real number such that* $r + \frac{1}{r}$ *is an odd integer, then* $r \notin \mathbb{Q}$.

Question 10 (5 pts.)**.** *Either prove that* $f(x) = 3x^5 - 4x^4 + 7x^3 + 16x^2 - 2$ *is irreducible over* $\mathbb{Q}$ *or factor it into a product of irreducible factors in* $\mathbb{Q}[x]$.

Question 11 (Bonus 6 pts.)**.** *Complete the questions specified on Page 5 of Test 1.*