**Example 1.** Consider the ring $R = \mathcal{M}_2(\mathbb{Z}_6) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_6 \right\}$ and let $A$ be the matrix given by

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} \in \mathcal{M}_2(\mathbb{Z}_6)$$

Then

$$2 \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} = 2A = A + A = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$$

and also

$$3 \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} = 3A = A + A + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Recall that the additive order of a ring is the smallest integer $n \in \mathbb{N}$ such that for any $r \in R$, $nr = 0$. If no such integer exists, then the additive order is defined to be 0. From Example 1, we see that $3A = 0$ so smallest integer is 3.

**Example 2.** From Example 1, say we take the two matrices $2A$ and $3A$ and we add them together. Then

$$2A + 3A = (A + A) + (A + A + A) = 5A = (2 + 3)A$$

**Definition 1.** The characteristic of a ring $R$ is the least positive integer $n$ such that $nr = 0$ for any $r \in R$. In other words,

$$\text{char}(R) = \min\{n \in \mathbb{N} : nr = 0\}$$

If no such integer exists, then $nr = 0$.

**Example 3.**
- For the set $\mathcal{M}_2(\mathbb{Z}_6)$ has characteristic 6.
- The set $\mathbb{Z}_n$ has characteristic $n$
- The set $\mathbb{Z}$ has characteristic 0.

**Question 1.** *Is it possible to have a finite $R$ to have characteristic 0?*

Not possible. If $\text{char}(R) = 0$, then for every $r \in R$, there exists $n \in \mathbb{N}$ with additive order larger than $n$. Then $\{0r, 1r, 2r, ..., nr\}$ consists of pairwise distinct elements, so that $|R| > n$. But as $n$ was arbitrary, then $R$ is infinite.

**Question 2.** *Is it possible to have an infinite ring with positive characteristic?*

Possible. Take the set $\mathbb{Z}_3[x]$ which is infinite, and it has characteristic 3.

**Lemma 1.** *Let $R$ be a ring with identity $1 \in R$. If 1 has additive order $\infty$, then $\text{char}(R) = 0$. If 1 has additive order $n$, then $\text{char}(R) = n$.*

*Proof.* If 1 has $\infty$ additive order, then there is no such positive integer $n$ such that $nr = 0$. Thus, $\text{char}(R) = 0$. On the other hand, if $n1 = 0$, then for any $r \in R$,

$$nr = r + r + \cdots + r = 1r + 1r + \cdots + 1r = (1 + 1 + \cdots + 1)r = 0r = 0$$

Therefore, $\text{char}(R) = n$. $\square$

**Theorem 1.** *The characteristic of an integral domain is either prime $p$ or 0.*

*Proof.* Let $D$ be an integral domain. If $D$ is an infinite integral domain, then we are done. On the other hand, if $D$ is an integral domain with characteristic $n$, and suppose that $n = ab$ for some $a, b \in D$. Then by using Lemma 1,

$$n1 = 1 + 1 + \cdots + 1 = \underbrace{(1 + 1 + \cdots + 1)}_{a\ times}\underbrace{(1 + 1 + \cdots + 1)}_{b\ times} = 0$$

Then $(a1)(b1) = 0$. Since $D$ is an integral domain that has no zero divisors, then without loss of generality, assume that $a1 = 0$. Then the only possible case in which this could happen is when $a = n$, because $a \leq n$ and $n$ is the smallest positive integer such that $n1 = 0$. $\square$

Observe that the characteristic of a field is either prime $p$ or zero. Indeed, a finite field has positive characteristic.

**Example 4.** Suppose $R$ is a commutative ring with no zero divisors. Show that all nonzero elements have the same additive order.

Let $a, b \in R$ be nonzero elements, and suppose that the additive order of $a$ is $m$ and the additive order of $b$ is $n$. Then without loss of generality, let us assume that $n \leq m$. Then

$$n(ab) = \underbrace{ab + ab + \cdots + ab}_{n\ times} = \underbrace{(a + a + \cdots + a)}_{n\ times} b = 0b = 0$$

and similarly,

$$n(ab) = \underbrace{ab + ab + \cdots + ab}_{n\ times} = a\,\underbrace{(b + b + \cdots + b)}_{n\ times} = a0 = 0$$

Therefore, $n = m$.