APPENDIX B

# Cantor's Middle Thirds Set

One of the most important examples to consider in analysis of subset of $\mathbb{R}$ is the following set.

Definition 0.1. Let $P_0 = [0,1]$. Construct $P_1$ from $P_0$ by removing the interval of length $\frac{1}{3}$ from the middle of $P_0$, i.e.

$$P_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]$$

Then construct $P_2$ from $P_1$ by removing the open intervals of length $\frac{1}{3^2}$ from the middle of each closed subinterval of $P_1$. Subsequently, having constructed $P_n$, construct $P_{n+1}$ by removing the open intervals of length $\frac{1}{3^{n+1}}$ from the middle of each of the $2^n$ closed subintervals of $P_n$. The set

$$\mathcal{C} = \bigcap_{n=1}^{\infty} P_n$$

is known as the *Cantor Set*

Remark 0.2. The Cantor set has many interesting properties. Firstly, we note that the Cantor set is closed being the intersection of closed sets.

The following Lemma gives another characterization of the Cantor sets.

Lemma 0.3. *Let $x \in \mathbb{R}$. Then $x \in \mathcal{C}$ if and only if there is a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \{0,2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, i.e. $x \in [0,1]$ and $x$ has a ternary expansion using only 0s and 2s.*

Proof. Suppose $x \in \mathcal{C}$. Hence, $x \in P_n$ for all $n \in \mathbb{N}$. Hence, by the recursive construction of $P_n$, there exists numbers $a_1, a_2, ... \in \{0,2\}$ such that

$$x \in \left[\sum_{k=1}^{n} \frac{a_k}{3^k}, \frac{1}{3^k} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right] \subset P_n$$

for all $n \in \mathbb{N}$. To see that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, we notice that

$$\left| x - \sum_{k=1}^{n} \frac{a_k}{3^k} \right| \leq \left| \left(\frac{1}{3} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right) - \sum_{k=1}^{n} \frac{a_k}{3^k} \right| = \frac{1}{3^n}$$

Therefore, since $\lim_{n\to\infty} \frac{1}{3^n} = 0$, we obtain that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$ as desired.

Conversely, suppose that $x \in \mathbb{R}$ is such that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$. For each $n \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} \frac{a_k}{3^k}$. Hence, by the description of $P_n$, we obtain that $s_n \in P_n$ for all $n$. In fact, we see that $s_m \in P_n$ whenever $m \geq n$. Indeed, if $m \geq n$, then

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k}{3^k} \leq \sum_{k=1}^{m} \frac{a_k}{3^k} = s_m &\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \sum_{k=n+1}^{m} \frac{2}{3^k} \\
&\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{2}{3^{n+1}} \cdot \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{1 - \frac{1}{3}} \\
&= \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{3^n} \\
&\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1}{3^n}
\end{aligned}$$

Since each $P_n$ is a closed set, since $x = \lim_{m\to\infty} s_m$ and since $s_m \in P_n$ whenever $m \geq n$, we obtain that $x \in P_n$ for each $n \in \mathbb{N}$ by the sequential description of closed sets. Hence, $x \in \bigcap_{n=1}^{\infty} P_n = \mathcal{C}$. ☐

COROLLARY 0.4. $|\mathcal{C}| = |\mathbb{R}|$. *In otherwords, $\mathcal{C}$ is uncountable.*

PROOF. To see that $\mathcal{C}$ is uncountable, define $f : \prod_{n=1}^{\infty}\{0, 1\} \to \mathcal{C}$ by

$$f((a_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2a_k}{3^k}$$

Clearly, $f$ is a well-defined injection so $|\mathcal{C}| \geq 2^{|\mathbb{N}|} = |\mathbb{R}|$. Since $\mathcal{C} \subset \mathbb{R}$, we obtain that $|\mathcal{C}| = |\mathbb{R}|$ as desired. ☐