LECTURE 2

September 12, 2023

1. Metric Spaces

DEFINITION 1.1. Let X be a nonempty set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ such that

(a) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
(b) For all $x, y \in X$, $d(x, y) = d(y, x)$
(c) (Triangle Inequality) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$

A *metric space* is a pair (X, d) where X is a nonempty set, and d is the metric on X.

DEFINITION 1.2. Let (X, d) be a metric space, let $A \subset X$ be a nonempty subset, and let $x, y \in A$. We define the *diameter of* A by

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y)$$

DEFINITION 1.3. Let (X, d) be a metric space, let $A \subset X$ and let $x \in X$. We say that the *distance from a point to the set* A is the metric

$$d(x, A) = \inf_{a \in A} d(x, a)$$

PROPOSITION 1.4. *Let (X, d) be a metric space, let $A \subset X$ be a nonempty subset of X, and let $x, y \in A$. Then*

$$d(x, A) \leq d(x, y) + d(y, A)$$

PROOF. Let $\varepsilon > 0$ be arbitrary. Using Definition 1.3, there exists a point $z \in A$ such that

$$d(x, z) < d(x, A) + \varepsilon$$

Furthermore,

$$d(x, z) + d(x, y) \geq d(y, z) \geq d(y, A)$$

Hence,

$$d(x, A) + d(x, y) + \varepsilon > d(y, A)$$

that is,

$$d(y, A) - d(x, A) < d(x, y) + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, and z no longer appears in our equation, we have

$$d(y, A) - d(x, A) \leq d(x, y)$$

and by using the interchanging x and y on the left hand side, we would obtain that

$$d(x, A) \leq d(x, y) + d(y, A)$$

as desired. $\square$

DEFINITION 1.5. Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then the sequence $(x_n)_{n\in\mathbb{N}}$ is said to converge to $x \in X$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

2. Examples of Metric Spaces

The most important examples of metrics in multivariable calculus are furnished by the spaces of real coordinate vectors $\mathbb{R}^n$:

EXAMPLE 2.1. On $\mathbb{R}$ we have $d(x, y) = |x - y|$. Here $|\cdot|$ denotes the absolute value. We will often refer to $(\mathbb{R}, d)$ as the *usual metric space*.

EXAMPLE 2.2. On $\mathbb{R}^n$, let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d(x, y) = |x - y|$$

Here, $|\cdot|$ denotes the *Euclidean distance* of a vector, i.e.

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Then $(\mathbb{R}^n, d)$ is a metric space. We will eventually see that this example will be related to another example, coming up.

EXAMPLE 2.3. On $\mathbb{R}^n$, let $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \|x - y\|_1$$

where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ is called the 1-norm. Then $(\mathbb{R}^n, d_1)$ is a metric space.

EXAMPLE 2.4. On $\mathbb{R}^n$, let $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \|x - y\|_2$$

where $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$ is called thr 2-norm. Then $(\mathbb{R}^n, d_2)$ is a metric space.

From Examples 2.3 and 2.4, we can extend to the following example.

EXAMPLE 2.5. On $\mathbb{R}^n$, let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d(x, y) = \|x - y\|_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

where $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ is called the p-norm. Then $(\mathbb{R}^n, d)$ is a metric space. Here, p is a fixed real number such that $p \geq 1$.

EXAMPLE 2.6. On $\mathbb{R}^n$, let $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d_\infty(x, y) = \|x - y\|_\infty = \max_{1 \le i \le n} |x_i - y_i|$$

where $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$ is called the sup-norm, or ∞-norm. Then $(\mathbb{R}^n, d_\infty)$ is a metric space.

Let X be a nonempty set and let $d : X \times X \to [0, \infty)$. Sometimes we encounter that d satisfies Definition 1.1 (b) and (c), but not (a) necessarily.

DEFINITION 2.7. Let X be a nonempty set and let $d : X \times X \to [0, \infty)$. Then we say that d is a *pseudometric* on X if d satisfies Definition 1.1(b) and (c), but not (a). That is, when $d(x, x) = 0$ for all $x \in X$, but there are two distinct points x, y such that $d(x, y) = 0$.

EXERCISE 2.8. Let (X, d) be a pseudometric space. Define the relation $x \equiv y$ if and only if $d(x, y) = 0$. Show that $x \equiv y$ is an equivalence relation on X.

DEFINITION 2.9. Let X be a metric space and a pair of nonempty subsets $A, B \subset X$, we define

$$d_H(A, B) = \max\left(\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right)$$

to be the *Hausdorff metric on* X. In fact, the Hausdorff metric is a pseudo-metric.

EXAMPLE 2.10. Let (X, d_H) be the Hausdorff metric space and let $A, B \subset X$. If $A = (0, 1)$ and $B = [0, 1]$, then calculating the supremums as in Definition **??** we have

$$\sup_{x \in (0,1)} d(x, [0, 1]) = 0$$

and

$$\sup_{x \in [0,1]} d(x, (0, 1)) = 0$$

and so $d_H(A, B) = 0$.

EXERCISE 2.11. If A and B are closed and $d(A, B) = 0$, then $A = B$.

Recall from Linear Algebra that the Euclidean length can be expressed using inner product of two coordinate vectors

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

where $x, y \in \mathbb{R}^n$. Recall as well that $\|x\|_2 = \sqrt{\langle x, x \rangle}$ from Linear Algebra, but is also equivalent to saying $\|x\|_2 = \sqrt{x \cdot x}$ if we consider the dot product on $\mathbb{R}^n$. We will first recall the Cauchy-Schwarz Inequality in which we should remember what it is.

THEOREM 2.12 (Cauchy-Schwarz Inequality). *Let $x, y \in \mathbb{R}$. Then* $|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$.

As an application of the Cauchy-Schwarz Inequality is the following.

PROPOSITION 2.13. *For all $x, y \in \mathbb{R}^n$,*

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$$

PROOF. Let $x, y \in \mathbb{R}^n$ be arbitrary. Then first consider squaring the norm $\|x + y\|^2$, so that

$$\begin{aligned} \|x + y\|_2^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|_2^2 + 2\langle x, y\rangle + \|y\|_2^2 \end{aligned}$$

and so by using Cauchy-Schwarz Inequality (Theorem 2.12, we obtain

$$\|x + y\|_2^2 \leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 = (\|x\|_2 + \|y\|_2)^2$$

and therefore,

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$$

as desired. □

3. The p-Norms

We have briefly introduced the concept of the p-norm but have not investigated too deeply yet. The p-norms are an important concept for proving three major inequalities: Young's Inequality, Minkowski's Inequality, and Hölder's Inequality.

DEFINITION 3.1. Let $p \in (1, \infty)$. The unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ is called the *conjugate* of p.

NOTATION 3.2. Denote the set of all sequences $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

by $\ell_p(\mathbb{N})$, or ℓ_p or $\ell_p(\mathbb{N}, \mathbb{K})$,

LEMMA 3.3 (Young's Inequality). *Let $a, b \geq 0$ and let $p, q \in (1, \infty)$ be conjugates. Then $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.*

THEOREM 3.4 (Hölder's Inequality). *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For $x, y \in \mathbb{K}^n$,*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

THEOREM 3.5 (Minkowski's Inequality). *Let* $p \in (1, \infty)$. *For any* $x, y \in \mathbb{K}^n$,

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$