LECTURE 4

September 19, 2023

1. Cantor Sets and Cantor's Middle Thirds

See Appendix B for notes on the Cantor Middle Thirds.

2. Function Spaces

We can build normed spaces, and therefore metric spaces, out of sets of functions. Let X be a set. The set of all bounded functions $f : X \to \mathbb{R}$ constitutes a vector space $\mathscr{B}(X)$ over $\mathbb{R}$. The sum of two functions and scalar multiplication are defined pointwise. That is, for all $f, g \in \mathscr{B}(X)$ and $\lambda \in \mathbb{K}$,

$$(f + \lambda g)(x) = f(x) + \lambda g(x)$$

We define a norm on $\mathscr{B}(X)$ by letting

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

The distance between two bounded functions with domain X is given by

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

As we saw, as soon as we have a metric we can express the notions of convergence and limit of a sequence. Let $(f_n)_{n\in\mathbb{N}}$ denote a sequence in $\mathscr{B}(X)$ and let $g \in \mathscr{B}(X)$. Then the claim that $\lim_{n\to\infty} f_n(x) = g(x)$ amounts to saying that

$$\lim_{n\to\infty} \sup_{x \in X} |f_n(x) - g(x)| = 0$$

This is stronger than simply requiring that $\lim_{n\to\infty} f_n(x) = g(x)$ should be all $x \in X$. What we want is the following: for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$\sup_{x \in X} |f_n(x) - g(x)| < \varepsilon$$

holds for all $n \geq N$.

We can extend the notion of function spaces and sequence of functions to metric spaces. Most of the time, when we are talking about distance of functions, we will use the notation $\rho(f, g)$.

Definition 2.1. Let $X \subset \mathbb{R}$, and suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions. We say that $(f_n)_{n\in\mathbb{N}}$ *converges pointwise* to f if for all $x \in X$ and for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$. Similarly, we say that $(f_n)_{n\in\mathbb{N}}$ *converges uniformly* to f if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\sup_{x\in X} \rho(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$.

3. Balls in a Metric Space

Definition 3.1. Let $X \subset \mathbb{R}$ let (X, d) be a metric space. We define the open balls and closed balls in X. Let $x_0 \in X$ and $\varepsilon > 0$,

(1) The *open ball* with center x_0 and radius ε is the set
$$B(x_0, \varepsilon) = \{x \in X : d(x_0, x) < \varepsilon\}$$
(2) The *closed ball* with center x_0 and radius ε is the set
$$\bar{B}(x_0, \varepsilon) = \{x \in X : d(x_0, x) \leq \varepsilon\}$$