MATH 4011 METRIC SPACES ASSIGNMENT 1

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Question 1. *Show that each of the following are metrics.*

(a) $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ *for* $x, y \in X$.

(b) $s(x,y) = \|x-y\|_1 = \sum_{i=1}^n |x_i - y_i|$ *for* $x, y \in \mathbb{R}^n$.

(c) $m(x,y) = \|x-y\|_\infty = \max_{1 \le i \le n} |x_i - y_i|$ *for* $x, y \in \mathbb{R}^n$.

(d) $h(x,y) = \begin{cases} \|x-y\|_2 & \text{if } x = ry \text{ for some } r \in \mathbb{R} \\ \|x\|_2 + \|y\|_2 & \text{otherwise} \end{cases}$, *for* $x, y \in \mathbb{R}^n$.

(e) *If* d *is a metric on* X*, define* $\bar{d}$ *by*

$$\bar{d}(x,y) = \begin{cases} d(x,y) & \text{if } d(x,y) < 1 \\ 1 & \text{if } d(x,y) \ge 1 \end{cases}$$

Then $\bar{d}$ *is a metric on* X.

Question 2. *We proved in detail that for any* $x \in \mathbb{R}^n$ *and any* $1 \le p < q$, *that*

$$\|x\|_q \le \|x\|_p \tag{*}$$

Write out a detailed proof of the case $n = 2$, $p = 2$ *and* $p = 3$.

Question 3. *Use the inequality (*) from above to prove that for all* $1 \le p < q \le \infty$, $\ell_p \subset \ell_q$.

Question 4. *Recall the proof from Calc II or Real Analysis II that*

$$s = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \in \ell_p \setminus \ell_1$$

for all $p > 1$. *Also show that for all* $1 \le p < q \le \infty$ *that* ℓ_p *is a proper subset of* ℓ_q.

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