LECTURE 1

September 7, 2023

1. Introduction

DEFINITION 1.1. A *metric* is the notion of the distance.

EXAMPLE 1.2. In $\mathbb{R}$, if we take two points x and y, then the *usual distance* is given by

$$d(x, y) = |x - y|$$

where d denotes the distance and $|\cdot|$ denotes the absolute value.

EXAMPLE 1.3. In $\mathbb{R}^n$, if we take two points x and y, then the *Euclidean distance* is given by

$$d_2(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

Let us recall the following definitions from MATH 2001 and MATH 3001.

DEFINITION 1.4. A sequence $(x_n)_{n \in \mathbb{N}}$ is said to *converge to x* if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

DEFINITION 1.5. A point $x \in U \subset \mathbb{R}^n$ is said to be an *interior point* if there exists a ball of radius $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ such that for all y, $|x - y| < \varepsilon$.

DEFINITION 1.6. A function f is said to be *continuous at a point* if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. The function is said to be *continuous on a set* if f is continuous at every point in the set.

DEFINITION 1.7. We say that x is the *limit point* of A if for all $\varepsilon > 0$ there exists a point $a \in A$ such that $|x - a| < \varepsilon$.

THEOREM 1.8. *Let $f : [0, 1] \to [0, 1]$ be a function such that*

(i) *$x \in [0, 1]$ is arbitrary.*
(ii) *f is a contraction if there exists $r < 1$ such that for all x, y, $f(x) - f(y) \leq r|x - y|$.*

EXERCISE 1.9. Contractions are (uniformly) continuous.

PROOF OF THEOREM 1.8. Since $[0, 1]$ is compact, then $f([0, 1])$ is also compact, and hence it is closed. Let $K_0 = [0, 1]$ and let $K_1 = f([0, 1])$. Then we have that $K_{n+1} = f(K_n) \subset K_n$ (Verify). Define the diameter of K_n by $\mathrm{diam}(K_n)$. Then $\bigcap_{n\in\mathbb{N}} K_n = \{x\}$ which implies that $x \in K_n$ and therefore $f(x) \in K_{n+1}$. $\square$

2. Metric Spaces

DEFINITION 2.1. Let X be a nonempty set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ such that

(i) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
(ii) For all $x, y \in X$, $d(x, y) = d(y, x)$.
(iii) (Triangle Inequality) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A *metric space* is a pair (X, d) where X is a nonempty set, and d is the metric on X.

EXAMPLE 2.2. Define $d_T : \mathbb{R}^2 \to [0, \infty)$ by

$$d_T(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

Then d_T is a metric on $\mathbb{R}^2$. d_T is called the *Taxicab metric*.

EXAMPLE 2.3. Define $d_\infty : \mathbb{R}^2 \to [0, \infty)$ by

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Then d_∞ is a metric on $\mathbb{R}^2$. To see that d_∞ is a metric, let us prove the triangle inequality (the first and second items are trivial). For $i \in \{1, 2\}$, we have

$$d_\infty(x, y) = |x_i - y_i| \leq |x_i - z_i| + |z_i - y_i| = d_\infty(x, z) + d_\infty(z, y)$$

EXAMPLE 2.4. Let $d : \mathbb{R}^2 \to [0, \infty)$ be a the function defined by

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_1 \neq y_1 \\ |x_2 - y_2| & \text{if } x_1 = y_1 \end{cases}$$

Then d is not a metric on $\mathbb{R}$.

EXAMPLE 2.5. Let $X \subset \mathbb{R}$, let $f : X \to \mathbb{R}$ and let $d_f : \mathbb{R} \to [0, \infty)$ be defined by

$$d_f(x, y) = |f(x) - f(y)|$$

Then d_f is called a *pseudometric* on X. If f is one-to-one, then d_f is a metric.