LECTURE 3

September 14, 2023

1. Norms on Vector Spaces

NOTATION 1.1. We denote $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 1.2. Let V be a vector space over $\mathbb{K}$. A *norm* on V is a function $\|\cdot\| : V \to [0, \infty)$ such that

(1) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.
(2) For all $x \in V$ and $\alpha \in \mathbb{K}$, $\|\alpha x\| = |\alpha|\|x\|$.
(3) (Triangle Inequality) For all $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

NOTATION 1.3. Let X be any set. Then we denote

$$(\mathbb{R}^X)^* = \{f : X \to \mathbb{R} : f \text{ is a bounded function}\}$$

EXAMPLE 1.4. Let $f \in (\mathbb{R}^X)^*$ be a function. Then

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a norm on X. We will show that the triangle inequality will hold. We show that

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

We claim that it suffices to show that for all $\varepsilon > 0$, $r - \varepsilon < \|f\|_\infty + \|g\|_\infty$. Then $r - \varepsilon$ is not an upper bound for $\{|f(x) + g(x)| : x \in X\}$. Fix $x_0 \in X$ so that $r - \varepsilon < |f(x_0) + g(x_0)| \leq |f(x_0)| + |g(x_0)| \leq \|f\|_\infty + \|g\|_\infty$, as desired.

EXAMPLE 1.5. $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Given a norm on a vector space V, we define a metric

$$d(x, y) = \|x - y\|$$

which we have already seen with some examples. This is always a metric space.

THEOREM 1.6. *Let $x, y \in \mathbb{R}^n$, and let $1 \leq p < q \leq \infty$, then*

$$\|x\|_\infty \leq \|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q$$

PROOF. When $p = 1$ and $q = \infty$, then $\|x\|_\infty \leq \|x\|_1$, clearly,

$$\max_{1 \leq i \leq n} |x_i| \leq \sum_{i=1}^{n} |x_i|$$

For any p and $q = \infty$, then $\|x\|_\infty \leq \|x\|_p$, since

$$\max_{1 \leq i \leq n} (|x_i|^p)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Now let us consider when $p < q$. Consider $\|x\|_q$, which is given by

$$\|x\|_q = \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}}$$

For a p-norm given as above, we want to show that

$$\|x\|_q \leq \|x\|_p \Leftrightarrow \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Observe that

$$\begin{aligned}
\|x\|_q = \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} &= \left(\sum_{i=1}^n |x_i|^{q-p} |x_i|^p \right)^{\frac{1}{q}} \\
&\leq \left(\max_{1 \leq i \leq n} |x_i|^{q-p} \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{q}} \\
&= \max_{1 \leq i \leq n} |x_i|^{\frac{q-p}{q}} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{q}} \\
&= \max_{1 \leq i \leq n} |x_i|^{\frac{q-p}{q}} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p} \cdot \frac{p}{q}} \\
&\leq \|x\|_p^{\frac{q-p}{q}} (\|x\|_p)^{\frac{p}{q}} \\
&= \|x\|_p
\end{aligned}$$

as desired.

Lastly, we want to show that

$$\|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q$$

Note that

(Hölder's Inequality)

$$\begin{aligned}
\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \cdot 1 \right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^n |x_i|^{p \cdot r} \right)^{\frac{1}{r}} \cdot \left(\sum_{i=1}^n 1^s \right)^{\frac{1}{s}} \\
&= \left(\sum_{i=1}^n |x_i|^{pr} \right)^{\frac{1}{pr}} \cdot n^{\frac{1}{s} \cdot p}
\end{aligned}$$

Take $r = \frac{q}{r}$ and $s = \frac{q}{q-p}$ $\square$

DEFINITION 1.7. A sequence $(x_n)_{n\in\mathbb{N}}$ on a set $\mathbb{K}^n$ is said to be bounded if there exists some $M \in \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} |x_n| \leq M$$

for all $n \in \mathbb{N}$

NOTATION 1.8. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence. We denote the ℓ_p norm as the set of all sequences in $\mathbb{K}^n$ such that

$$\ell_p = \left\{ (x_n)_{n\in\mathbb{N}} : \sum_{i=1}^{n} |x_i|^p < \infty \right\}$$

Alternatively, if $p = \infty$, then

$$\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is bounded}\}$$

EXAMPLE 1.9. Take $p = 2$, then ℓ_2 is the set of all set of all sequences in $\mathbb{K}^n$ such that

$$\ell_2 = \left\{ (x_n)_{n\in\mathbb{N}} : \sum_{i=1}^{n} |x_i|^2 < \infty \right\}$$

Then the sequence $(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} \in \ell_2$, but $(y_n)_{n\in\mathbb{N}} = \left(\frac{1}{\sqrt{n}}\right)_{n\in\mathbb{N}} \notin \ell_2$.

DEFINITION 1.10. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in ℓ_p Then for any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\|(x_n)_{n\in\mathbb{N}}|_n\|_p \geq \|(x_n)_{n\in\mathbb{N}}\|_p - \varepsilon$, i.e. restricting the sequence $(x_n)_{n\in\mathbb{N}}$ to $(x_1, x_2, ..., x_n)$.