MATH 4011: Metric Spaces

Joe Tran

Preface

These are the first edition of these lecture notes for MATH 4011 (Metric Spaces). Consequently, there may be several typographical errors. Not every result in these notes will be covered in class. For example, some results will be covered through assignments. However, these notes should be fairly self-contained. If you come across any typos, errors, omissions, or unclear explanations, please feel free to contact me so that I may continually improve these notes.

Contents

Preface 3

Week 1. September 4-8 7

Lecture 1. September 7, 2023 9
- 1. Introduction 9
- 2. Metric Spaces 10

Week 2. September 11-15 11

Lecture 2. September 12, 2023 13
- 1. Metric Spaces 13
- 2. Examples of Metric Spaces 14
- 3. The p-Norms 16

Lecture 3. September 14, 2023 19
- 1. Norms on Vector Spaces 19

Week 3. September 18-22 23

Lecture 4. September 19, 2023 25

Appendix 27

Appendix A. Young, Hölder, and Minkowski 29

Appendix B. Cantor's Middle Thirds Set 33

Appendix. Index 35

Week 1

September 4-8

LECTURE 1

September 7, 2023

1. Introduction

DEFINITION 1.1. A *metric* is the notion of the distance.

EXAMPLE 1.2. In $\mathbb{R}$, if we take two points x and y, then the *usual distance* is given by

$$d(x, y) = |x - y|$$

where d denotes the distance and $|\cdot|$ denotes the absolute value.

EXAMPLE 1.3. In $\mathbb{R}^n$, if we take two points x and y, then the *Euclidean distance* is given by

$$d_2(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$$

Let us recall the following definitions from MATH 2001 and MATH 3001.

DEFINITION 1.4. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to *converge to x* if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

DEFINITION 1.5. A point $x \in U \subset \mathbb{R}^n$ is said to be an *interior point* if there exists a ball of radius $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ such that for all y, $|x - y| < \varepsilon$.

DEFINITION 1.6. A function f is said to be *continuous at a point* if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. The function is said to be *continuous on a set* if f is continuous at every point in the set.

DEFINITION 1.7. We say that x is the *limit point* of A if for all $\varepsilon > 0$ there exists a point $a \in A$ such that $|x - a| < \varepsilon$.

THEOREM 1.8. *Let $f : [0, 1] \to [0, 1]$ be a function such that*

(i) $x \in [0, 1]$ *is arbitrary.*
(ii) f *is a contraction if there exists $r < 1$ such that for all x, y, $f(x) - f(y) \leq r|x - y|$.*

EXERCISE 1.9. Contractions are (uniformly) continuous.

PROOF OF THEOREM 1.8. Since $[0, 1]$ is compact, then $f([0, 1])$ is also compact, and hence it is closed. Let $K_0 = [0, 1]$ and let $K_1 = f([0, 1])$. Then we have that $K_{n+1} = f(K_n) \subset K_n$ (Verify). Define the diameter of K_n by diam(K_n). Then $\bigcap_{n\in\mathbb{N}} K_n = \{x\}$ which implies that $x \in K_n$ and therefore $f(x) \in K_{n+1}$. □

2. Metric Spaces

Definition 2.1. Let X be a nonempty set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ such that

(i) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
(ii) For all $x, y \in X$, $d(x, y) = d(y, x)$.
(iii) (Triangle Inequality) For all $x, y, z \in X$, $d(x, y) \le d(x, z) + d(z, y)$.

A *metric space* is a pair (X, d) where X is a nonempty set, and d is the metric on X.

Example 2.2. Define $d_T : \mathbb{R}^2 \to [0, \infty)$ by

$$d_T(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

Then d_T is a metric on $\mathbb{R}^2$. d_T is called the *Taxicab metric.*

Example 2.3. Define $d_\infty : \mathbb{R}^2 \to [0, \infty)$ by

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Then d_∞ is a metric on $\mathbb{R}^2$. To see that d_∞ is a metric, let us prove the triangle inequality (the first and second items are trivial). For $i \in \{1, 2\}$, we have

$$d_\infty(x, y) = |x_i - y_i| \le |x_i - z_i| + |z_i - y_i| = d_\infty(x, z) + d_\infty(z, y)$$

Example 2.4. Let $d : \mathbb{R}^2 \to [0, \infty)$ be a the function defined by

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_1 \ne y_1 \\ |x_2 - y_2| & \text{if } x_1 = y_1 \end{cases}$$

Then d is not a metric on $\mathbb{R}$.

Example 2.5. Let $X \subset \mathbb{R}$, let $f : X \to \mathbb{R}$ and let $d_f : \mathbb{R} \to [0, \infty)$ be defined by

$$d_f(x, y) = |f(x) - f(y)|$$

Then d_f is called a *pseudometric* on X. If f is one-to-one, then d_f is a metric.

Week 2

September 11-15

LECTURE 2

September 12, 2023

1. Metric Spaces

DEFINITION 1.1. Let X be a nonempty set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ such that

(a) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
(b) For all $x, y \in X$, $d(x, y) = d(y, x)$
(c) (Triangle Inequality) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$

A *metric space* is a pair (X, d) where X is a nonempty set, and d is the metric on X.

DEFINITION 1.2. Let (X, d) be a metric space, let $A \subset X$ be a nonempty subset, and let $x, y \in A$. We define the *diameter of* A by

$$\text{diam}(A) = \sup_{x,y \in A} d(x, y)$$

DEFINITION 1.3. Let (X, d) be a metric space, let $A \subset X$ and let $x \in X$. We say that the *distance from a point to the set* A is the metric

$$d(x, A) = \inf_{a \in A} d(x, a)$$

PROPOSITION 1.4. *Let (X, d) be a metric space, let $A \subset X$ be a nonempty subset of X, and let $x, y \in A$. Then*

$$d(x, A) \leq d(x, y) + d(y, A)$$

PROOF. Let $\varepsilon > 0$ be arbitrary. Using Definition 1.3, there exists a point $z \in A$ such that

$$d(x, z) < d(x, A) + \varepsilon$$

Furthermore,

$$d(x, z) + d(x, y) \geq d(y, z) \geq d(y, A)$$

Hence,

$$d(x, A) + d(x, y) + \varepsilon > d(y, A)$$

that is,

$$d(y, A) - d(x, A) < d(x, y) + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, and z no longer appears in our equation, we have

$$d(y, A) - d(x, A) \leq d(x, y)$$

and by using the interchanging x and y on the left hand side, we would obtain that

$$d(x, A) \leq d(x, y) + d(y, A)$$

as desired. $\square$

Definition 1.5. Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then the sequence $(x_n)_{n\in\mathbb{N}}$ is said to converge to $x \in X$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

2. Examples of Metric Spaces

The most important examples of metrics in multivariable calculus are furnished by the spaces of real coordinate vectors $\mathbb{R}^n$:

Example 2.1. On $\mathbb{R}$ we have $d(x, y) = |x - y|$. Here $|\cdot|$ denotes the absolute value. We will often refer to $(\mathbb{R}, d)$ as the *usual metric space*.

Example 2.2. On $\mathbb{R}^n$, let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d(x, y) = |x - y|$$

Here, $|\cdot|$ denotes the *Euclidean distance* of a vector, i.e.

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Then $(\mathbb{R}^n, d)$ is a metric space. We will eventually see that this example will be related to another example, coming up.

Example 2.3. On $\mathbb{R}^n$, let $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \|x - y\|_1$$

where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ is called the 1-norm. Then $(\mathbb{R}^n, d_1)$ is a metric space.

Example 2.4. On $\mathbb{R}^n$, let $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \|x - y\|_2$$

where $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$ is called thr 2-norm. Then $(\mathbb{R}^n, d_2)$ is a metric space.

From Examples 2.3 and 2.4, we can extend to the following example.

Example 2.5. On $\mathbb{R}^n$, let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d(x, y) = \|x - y\|_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

where $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ is called the p-norm. Then $(\mathbb{R}^n, d)$ is a metric space. Here, p is a fixed real number such that $p \geq 1$.

EXAMPLE 2.6. On $\mathbb{R}^n$, let $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d_\infty(x, y) = \|x - y\|_\infty = \max_{1 \le i \le n} |x_i - y_i|$$

where $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$ is called the sup-norm, or ∞-norm. Then $(\mathbb{R}^n, d_\infty)$ is a metric space.

Let X be a nonempty set and let $d : X \times X \to [0, \infty)$. Sometimes we encounter that d satisfies Definition 1.1 (b) and (c), but not (a) necessarily.

DEFINITION 2.7. Let X be a nonempty set and let $d : X \times X \to [0, \infty)$. Then we say that d is a *pseudometric* on X if d satisfies Definition 1.1(b) and (c), but not (a). That is, when $d(x, x) = 0$ for all $x \in X$, but there are two distinct points x, y such that $d(x, y) = 0$.

EXERCISE 2.8. Let (X, d) be a pseudometric space. Define the relation $x \equiv y$ if and only if $d(x, y) = 0$. Show that $x \equiv y$ is an equivalence relation on X.

DEFINITION 2.9. Let X be a metric space and a pair of nonempty subsets $A, B \subset X$, we define

$$d_H(A, B) = \max\left(\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right)$$

to be the *Hausdorff metric on* X. In fact, the Hausdorff metric is a pseudometric.

EXAMPLE 2.10. Let (X, d_H) be the Hausdorff metric space and let $A, B \subset X$. If $A = (0, 1)$ and $B = [0, 1]$, then calculating the supremums as in Definition **??** we have

$$\sup_{x \in (0,1)} d(x, [0, 1]) = 0$$

and

$$\sup_{x \in [0,1]} d(x, (0, 1)) = 0$$

and so $d_H(A, B) = 0$.

EXERCISE 2.11. If A and B are closed and $d(A, B) = 0$, then $A = B$.

Recall from Linear Algebra that the Euclidean length can be expressed using inner product of two coordinate vectors

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

where $x, y \in \mathbb{R}^n$. Recall as well that $\|x\|_2 = \sqrt{\langle x, x \rangle}$ from Linear Algebra, but is also equivalent to saying $\|x\|_2 = \sqrt{x \cdot x}$ if we consider the dot product on $\mathbb{R}^n$. We will first recall the Cauchy-Schwarz Inequality in which we should remember what it is.

THEOREM 2.12 (Cauchy-Schwarz Inequality). *Let* $x, y \in \mathbb{R}$. *Then* $|\langle x, y\rangle| \leq \|x\|_2\|y\|_2$.

As an application of the Cauchy-Schwarz Inequality is the following.

PROPOSITION 2.13. *For all* $x, y \in \mathbb{R}^n$,

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

PROOF. Let $x, y \in \mathbb{R}^n$ be arbitrary. Then first consider squaring the norm $\|x+y\|^2$, so that

$$\begin{aligned} \|x+y\|_2^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|_2^2 + 2\langle x, y\rangle + \|y\|_2^2 \end{aligned}$$

and so by using Cauchy-Schwarz Inequality (Theorem 2.12, we obtain

$$\|x+y\|_2^2 \leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 = (\|x\|_2 + \|y\|_2)^2$$

and therefore,

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

as desired. $\square$

3. The p-Norms

We have briefly introduced the concept of the p-norm but have not investigated too deeply yet. The p-norms are an important concept for proving three major inequalities: Young's Inequality, Minkowski's Inequality, and Hölder's Inequality.

DEFINITION 3.1. Let $p \in (1, \infty)$. The unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ is called the *conjugate* of p.

NOTATION 3.2. Denote the set of all sequences $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

by $\ell_p(\mathbb{N})$, or ℓ_p or $\ell_p(\mathbb{N}, \mathbb{K})$,

LEMMA 3.3 (Young's Inequality). *Let* $a, b \geq 0$ *and let* $p, q \in (1, \infty)$ *be conjugates. Then* $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.

THEOREM 3.4 (Hölder's Inequality). *Let* $p, q \in (1, \infty)$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $x, y \in \mathbb{K}^n$,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

THEOREM 3.5 (Minkowski's Inequality). *Let* $p \in (1, \infty)$. *For any* $x, y \in \mathbb{K}^n$,

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}$$

LECTURE 3

September 14, 2023

1. Norms on Vector Spaces

NOTATION 1.1. We denote $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 1.2. Let V be a vector space over $\mathbb{K}$. A *norm* on V is a function $\|\cdot\| : V \to [0, \infty)$ such that

(1) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.
(2) For all $x \in V$ and $\alpha \in \mathbb{K}$, $\|\alpha x\| = |\alpha|\|x\|$.
(3) (Triangle Inequality) For all $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

NOTATION 1.3. Let X be any set. Then we denote

$$(\mathbb{R}^X)^* = \{f : X \to \mathbb{R} : f \text{ is a bounded function}\}$$

EXAMPLE 1.4. Let $f \in (\mathbb{R}^X)^*$ be a function. Then

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a norm on X. We will show that the triangle inequality will hold. We show that

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

We claim that it suffices to show that for all $\varepsilon > 0$, $r - \varepsilon < \|f\|_\infty + \|g\|_\infty$. Then $r - \varepsilon$ is not an upper bound for $\{|f(x) + g(x)| : x \in X\}$. Fix $x_0 \in X$ so that $r - \varepsilon < |f(x_0) + g(x_0)| \leq |f(x_0)| + |g(x_0)| \leq \|f\|_\infty + \|g\|_\infty$, as desired.

EXAMPLE 1.5. $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Given a norm on a vector space V, we define a metric

$$d(x, y) = \|x - y\|$$

which we have already seen with some examples. This is always a metric space.

THEOREM 1.6. *Let $x, y \in \mathbb{R}^n$, and let $1 \leq p < q \leq \infty$, then*

$$\|x\|_\infty \leq \|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q$$

PROOF. When $p = 1$ and $q = \infty$, then $\|x\|_\infty \leq \|x\|_1$, clearly,

$$\max_{1 \leq i \leq n} |x_i| \leq \sum_{i=1}^{n} |x_i|$$

For any p and $q = \infty$, then $\|x\|_\infty \leq \|x\|_p$, since

$$\max_{1\leq i\leq n} \left(|x_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

Now let us consider when $p < q$. Consider $\|x\|_q$, which is given by

$$\|x\|_q = \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}}$$

For a p-norm given as above, we want to show that

$$\|x\|_q \leq \|x\|_p \Leftrightarrow \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

Observe that

$$\begin{aligned}
\|x\|_q = \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}} &= \left(\sum_{i=1}^{n} |x_i|^{q-p}|x_i|^p\right)^{\frac{1}{q}} \\
&\leq \left(\max_{1\leq i\leq n} |x_i|^{q-p} \sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{q}} \\
&= \max_{1\leq i\leq n} |x_i|^{\frac{q-p}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{q}} \\
&= \max_{1\leq i\leq n} |x_i|^{\frac{q-p}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}\cdot\frac{p}{q}} \\
&\leq \|x\|_p^{\frac{q-p}{q}} \left(\|x\|_p\right)^{\frac{p}{q}} \\
&= \|x\|_p
\end{aligned}$$

as desired.

Lastly, we want to show that

$$\|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$$

Note that

(Hölder's Inequality)

$$\begin{aligned}
\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \cdot 1\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{n} |x_i|^{p\cdot r}\right)^{\frac{1}{r}} \cdot \left(\sum_{i=1}^{n} 1^s\right)^{\frac{1}{s}} \\
&= \left(\sum_{i=1}^{n} |x_i|^{pr}\right)^{\frac{1}{pr}} \cdot n^{\frac{1}{s}\cdot p}
\end{aligned}$$

Take $r = \frac{q}{r}$ and $s = \frac{q}{q-p}$ $\square$

DEFINITION 1.7. A sequence $(x_n)_{n\in\mathbb{N}}$ on a set $\mathbb{K}^n$ is said to be bounded if there exists some $M \in \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} |x_n| \leq M$$

for all $n \in \mathbb{N}$

NOTATION 1.8. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence. We denote the ℓ_p norm as the set of all sequences in $\mathbb{K}^n$ such that

$$\ell_p = \left\{ (x_n)_{n\in\mathbb{N}} : \sum_{i=1}^{n} |x_i|^p < \infty \right\}$$

Alternatively, if $p = \infty$, then

$$\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is bounded}\}$$

EXAMPLE 1.9. Take $p = 2$, then ℓ_2 is the set of all set of all sequences in $\mathbb{K}^n$ such that

$$\ell_2 = \left\{ (x_n)_{n\in\mathbb{N}} : \sum_{i=1}^{n} |x_i|^2 < \infty \right\}$$

Then the sequence $(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} \in \ell_2$, but $(y_n)_{n\in\mathbb{N}} = \left(\frac{1}{\sqrt{n}}\right)_{n\in\mathbb{N}} \notin \ell_2$.

DEFINITION 1.10. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in ℓ_p Then for any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\|(x_n)_{n\in\mathbb{N}}|_n\|_p \geq \|(x_n)_{n\in\mathbb{N}}\|_p - \varepsilon$, i.e. restricting the sequence $(x_n)_{n\in\mathbb{N}}$ to $(x_1, x_2, ..., x_n)$.

Week 3

September 18-22

LECTURE 4

September 19, 2023

Appendix

APPENDIX A

Young, Hölder, and Minkowski

This appendix chapter will dedicate to the proofs of Young's, Minkowski's, and Hölder's Inequalities that we should be familiar with in Lecture 2 Section 3. The purpose of this section is to understand how Young's Inequality implies Hölder's Inequality and furthermore implies Minkowski's inequality. Furthermore, note that we will be using $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 0.1. Let $p \in (1, \infty)$. The unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ is called the conjugate of p. Note we consider ∞ to be the conjugate of 1 and 1 to be the conjugate of ∞.

THEOREM 0.2 (Young's Inequality). *Let $a, b \geq 0$ and let $p, q \in (1, \infty)$ be conjugates. Then $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.*

PROOF. Notice $1 = \frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq}$ implies $p+q-pq = 0$. Hence, $q = \frac{p}{p-1}$. Fix $b \geq 0$. Notice if $b = 0$, the inequality easily holds. Thus we will assume that $b > 0$. Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \frac{1}{p}x^p + \frac{1}{q}b^q - bx$. Then $f(0) > 0$ and $\lim_{x\to\infty} f(x) = \infty$ as $p > 1$ so x^p grows faster than x. We claim that $f(x) \geq 0$ for all $x \in [0, \infty)$ thereby proving the inequality. Notice f is differentiable on $[0, \infty)$ with

$$f'(x) = x^{p-1} - b$$

Therefore, $f'(x) = 0$ if and only if $x = b^{\frac{1}{p-1}}$. Moreover, it is elementary to see that from the derivative that f has a local minimum at $x = b^{\frac{1}{p-1}}$ and thus f has a global minimum at $b^{\frac{1}{p-1}}$ due to boundary conditions. Therefore, since

$$f\left(b^{\frac{1}{p-1}}\right) = \frac{1}{p}b^{\frac{p}{p-1}} + \frac{1}{q}b^q - b^{1+\frac{1}{p-1}} = \frac{1}{p}b^q + \frac{1}{q}b^q - b^q = 0$$

we obtain that $f(x) \geq 0$ for all $x \in [0, \infty)$ as desired. $\square$

Using Young's Inequality, we have a stepping stone towards the triangle inequality.

THEOREM 0.3 (Hölder's Inequality). *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in \mathbb{K}$*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

PROOF. Let $\alpha = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ and let $\beta = (\sum_{i=1}^n |y_i|^q)^{\frac{1}{q}}$. Then it is clear that $\alpha = 0$ implies $x_i = 0$ for all $1 \leq i \leq n$ which implies that $\sum_{i=1}^n |x_i y_i| = 0$ and thus the inequality will hold in this case. Similarly, if $\beta = 0$, then the inequality holds. Hence, we may assume that $\alpha, \beta > 0$.

Since $\alpha, \beta > 0$ we obtain that

$$\begin{aligned}
\sum_{i=1}^n |x_i y_i| &= \alpha\beta \sum_{i=1}^n \left|\frac{x_i}{\alpha}\right| \left|\frac{y_i}{\beta}\right| \\
\text{(Young's (Theorem 0.2))} \qquad &\leq \alpha\beta \left(\sum_{i=1}^n \frac{1}{p} \left|\frac{x_i}{\alpha}\right|^p + \frac{1}{q}\left|\frac{y_i}{\beta}\right|^q \right) \\
&= \alpha\beta \left(\frac{1}{p\alpha^p} \sum_{i=1}^n |x_i|^p + \frac{1}{q\beta^q} \sum_{i=1}^n |y_i|^q \right) \\
&= \alpha\beta \left(\frac{1}{p} + \frac{1}{q} \right) \\
&= \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}
\end{aligned}$$

as desired. □

Note that Hölder's Inequality has the following trivial extension.

COROLLARY 0.4. *For any $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in \mathbb{K}$*

$$\sum_{i=1}^n |x_i y_i| \leq \|(x_1, x_2, ..., x_n)\|_1 \|(y_1, y_2, ..., y_n)\|_\infty$$

Finally, Hölder's Inequality enables us to prove the triangle inequality for the p-norm.

THEOREM 0.5 (Minkowski's Inequality). *Let $p \in (1, \infty)$. For any $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in \mathbb{K}$*

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

PROOF. Choose $q \in (1, \infty)$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Thus, $q = \frac{p}{p-1}$. Since $p \in (1, \infty)$ notice by Hölder's Inequality that

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i||x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} (|x_i| + |y_i|)|x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \\
&\leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} (|x_i + y_i|^p)^q\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} (|x_i + y_i|^p)^q\right)^{\frac{1}{q}} \\
&= \left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}\right) \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{q}}
\end{aligned}$$

If $\sum_{i=1}^{n} |x_i + y_i|^p = 0$, the result follows trivially. Otherwise, we may divide both sides of the equation by $(\sum_{i=1}^{n} |x_i + y_i|^p)^{\frac{1}{q}}$ so that

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

as desired. □

EXAMPLE 0.6. Let $p \in [1, \infty)$. Let $\ell_p(\mathbb{N})$ denote all sequences $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, or,

$$\ell_p(\mathbb{N}) = \left\{ (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

Then $\ell_p(\mathbb{N})$ is a normed linear space with norm $\|\cdot\|_p : \ell_p(\mathbb{N}) \to [0, \infty)$ defined by

$$\|(x_n)_{n\in\mathbb{N}}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$

It is elementary to see that $\|\cdot\|$ is well defined and satisfies the first two properties of a norm. To see that $\|\cdot\|_p$ satisfies the Triangle Inequality, we note that Minkowski's Inequality (Theorem 0.5 imply that

$$\left(\sum_{n=1}^{m} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{m} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{m} |y_n|^p\right)^{\frac{1}{p}}$$

for all $m \in \mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are in $\ell_p(\mathbb{N})$. By taking the limit as $m \to \infty$, we obtain the triangle inequality and the fact that if $x, y \in \ell_p(\mathbb{N})$ then $x + y \in \ell_p(\mathbb{N})$.

EXAMPLE 0.7. Let $\ell_\infty(\mathbb{N})$ denote all sequences $(x_n)_{n\in\mathbb{N}}$ of $\mathbb{K}$ such that $\sup_{n\in\mathbb{N}}|x_n| < \infty$. Then $\ell_\infty(\mathbb{N})$ is a normed linear space with $\|\cdot\|_\infty : \ell_\infty(\mathbb{N}) \to [0,\infty)$ defined by

$$\|(x_n)_{n\in\mathbb{N}}\|_p = \sup_{n\in\mathbb{N}}|x_n|$$

It is elementary to see that $\|\cdot\|_\infty$ is a well-defined norm, which we call the sup-norm or ∞-norm.

REMARK 0.8. It is not difficult to see that if $p,q \in [1,\infty]$, and $p < q$, then $\ell_p(\mathbb{N}) \subsetneq \ell_q(\mathbb{N})$. Indeed, if $(x_n)_{n\in\mathbb{N}} \in \ell_p$, then $\sum_{n=1}^{\infty}|x_n|^p < \infty$ so $(x_n)_{n\in\mathbb{N}}$ is bounded and $\sum_{n=1}^{\infty}|x_n|^q < \infty$ for all $q \in (p,\infty)$. To see that the inclusion is strict, notice that $\left(\frac{1}{n^{\frac{1}{p}}}\right)_{n\in\mathbb{N}} \notin \ell_p(\mathbb{N})$ but is in $\ell_q(\mathbb{N})$ for all $p < q$.

We have alternate versions for Hölder's Inequality presented below that uses the ℓ_p norms.

THEOREM 0.9 (Hölder's Inequality). *Let $p,q \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $(x_n)_{n\in\mathbb{N}} \in \ell_p$ and $(y_n)_{n\in\mathbb{N}} \in \ell_q$, then $(x_ny_n)_{n\in\mathbb{N}} \in \ell_1$ and*

$$\|(x_ny_n)_{n\in\mathbb{N}}\| \leq \|(x_n)_{n\in\mathbb{N}}\|_p\|(y_n)_{n\in\mathbb{N}}\|_q$$

Instead of having sequences on p-norms, we can also have continuous functions on p-norms. We replace their sums with their generalization, namely integrals. Note that we denote $\mathscr{C}[a,b]$ to be the set of all continuous functions on the closed interval $[a,b]$.

DEFINITION 0.10. For $p \in [1,\infty)$, define $\|\cdot\|_p : \mathscr{C}[a,b] \to [0,\infty)$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$$

for all $f \in \mathscr{C}[a,b]$.

We also have the equivalent forms of Hölder's Inequality and Minkowski's Inequality for continuous functions. We won't prove these as they are similar to the ones we have proven earlier.

THEOREM 0.11 (Hölder's Inequality). *Let $p,q \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f,g \in \mathscr{C}[a,b]$, then*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q$$

THEOREM 0.12 (Minkowski's Inequality). *Let $p \in [1,\infty)$. If $f,g \in \mathscr{C}[a,b]$, then*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

APPENDIX B

Cantor's Middle Thirds Set

One of the most important examples to consider in analysis of subset of $\mathbb{R}$ is the following set.

Definition 0.1. Let $P_0 = [0, 1]$. Construct P_1 from P_0 by removing the interval of length $\frac{1}{3}$ from the middle of P_0, i.e.

$$P_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]$$

Then construct P_2 from P_1 by removing the open intervals of length $\frac{1}{3^2}$ from the middle of each closed subinterval of P_1. Subsequently, having constructed P_n, construct P_{n+1} by removing the open intervals of length $\frac{1}{3^{n+1}}$ from the middle of each of the 2^n closed subintervals of P_n. The set

$$\mathcal{C} = \bigcap_{n=1}^{\infty} P_n$$

is known as the *Cantor Set*

Remark 0.2. The Cantor set has many interesting properties. Firstly, we note that the Cantor set is closed being the intersection of closed sets.

The following Lemma gives another characterization of the Cantor sets.

Lemma 0.3. *Let $x \in \mathbb{R}$. Then $x \in \mathcal{C}$ if and only if there is a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, i.e. $x \in [0, 1]$ and x has a ternary expansion using only 0s and 2s.*

Proof. Suppose $x \in \mathcal{C}$. Hence, $x \in P_n$ for all $n \in \mathbb{N}$. Hence, by the recursive construction of P_n, there exists numbers $a_1, a_2, \ldots \in \{0, 2\}$ such that

$$x \in \left[\sum_{k=1}^{n} \frac{a_k}{3^k}, \frac{1}{3^k} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right] \subset P_n$$

for all $n \in \mathbb{N}$. To see that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, we notice that

$$\left| x - \sum_{k=1}^{n} \frac{a_k}{3^k} \right| \le \left| \left(\frac{1}{3} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right) - \sum_{k=1}^{n} \frac{a_k}{3^k} \right| = \frac{1}{3^n}$$

Therefore, since $\lim_{n\to\infty} \frac{1}{3^n} = 0$, we obtain that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$ as desired.

Conversely, suppose that $x \in \mathbb{R}$ is such that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \{0,2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$. For each $n \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} \frac{a_k}{3^k}$. Hence, by the description of P_n, we obtain that $s_n \in P_n$ for all n. In fact, we see that $s_m \in P_n$ whenever $m \geq n$. Indeed, if $m \geq n$, then

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k}{3^k} \leq \sum_{k=1}^{m} \frac{a_k}{3^k} = s_m &\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \sum_{k=n+1}^{m} \frac{2}{3^k} \\
&\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{2}{3^{n+1}} \cdot \frac{1-\left(\frac{1}{3}\right)^{m-n}}{1-\frac{1}{3}} \\
&= \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1-\left(\frac{1}{3}\right)^{m-n}}{3^n} \\
&\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1}{3^n}
\end{aligned}$$

Since each P_n is a closed set, since $x = \lim_{m\to\infty} s_m$ and since $s_m \in P_n$ whenever $m \geq n$, we obtain that $x \in P_n$ for each $n \in \mathbb{N}$ by the sequential description of closed sets. Hence, $x \in \bigcap_{n=1}^{\infty} P_n = \mathcal{C}$. ☐

COROLLARY 0.4. $|\mathcal{C}| = |\mathbb{R}|$. *In otherwords, $\mathcal{C}$ is uncountable.*

PROOF. To see that $\mathcal{C}$ is uncountable, define $f : \prod_{n=1}^{\infty}\{0,1\} \to \mathcal{C}$ by

$$f((a_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2a_k}{3^k}$$

Clearly, f is a well-defined injection so $|\mathcal{C}| \geq 2^{|\mathbb{N}|} = |\mathbb{R}|$. Since $\mathcal{C} \subset \mathbb{R}$, we obtain that $|\mathcal{C}| = |\mathbb{R}|$ as desired. ☐

Index

p-norm, 14
1-norm, 14
2-norm, 14

continuity, 9
contraction, 9
converge to x, 9
convergence, 14

diameter, 13

Euclidean distance, 9

Hölder's Inequality, 16
Hausdorff, 15

interior point, 9

limit point, 9

metric, 9, 10, 13
metric space, 10, 13
Minkowski's Inequality, 17

pseudometric, 10, 15

sup-norm, 15

Taxicab metric, 10

Young's Inequality, 16