APPENDIX A

# Young, Hölder, and Minkowski

This appendix chapter will dedicate to the proofs of Young's, Minkowski's, and Hölder's Inequalities that we should be familiar with in Lecture 2 Section 3. The purpose of this section is to understand how Young's Inequality implies Hölder's Inequality and furthermore implies Minkowski's inequality. Furthermore, note that we will be using $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 0.1. Let $p \in (1, \infty)$. The unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ is called the conjugate of $p$. Note we consider $\infty$ to be the conjugate of 1 and 1 to be the conjugate of $\infty$.

THEOREM 0.2 (Young's Inequality). *Let $a, b \geq 0$ and let $p, q \in (1, \infty)$ be conjugates. Then $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.*

PROOF. Notice $1 = \frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq}$ implies $p + q - pq = 0$. Hence, $q = \frac{p}{p-1}$. Fix $b \geq 0$. Notice if $b = 0$, the inequality easily holds. Thus we will assume that $b > 0$. Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \frac{1}{p}x^p + \frac{1}{q}b^q - bx$. Then $f(0) > 0$ and $\lim_{x\to\infty} f(x) = \infty$ as $p > 1$ so $x^p$ grows faster than $x$. We claim that $f(x) \geq 0$ for all $x \in [0, \infty)$ thereby proving the inequality. Notice $f$ is differentiable on $[0, \infty)$ with

$$f'(x) = x^{p-1} - b$$

Therefore, $f'(x) = 0$ if and only if $x = b^{\frac{1}{p-1}}$. Moreover, it is elementary to see that from the derivative that $f$ has a local minimum at $x = b^{\frac{1}{p-1}}$ and thus $f$ has a global minimum at $b^{\frac{1}{p-1}}$ due to boundary conditions. Therefore, since

$$f\left(b^{\frac{1}{p-1}}\right) = \frac{1}{p}b^{\frac{p}{p-1}} + \frac{1}{q}b^q - b^{1+\frac{1}{p-1}} = \frac{1}{p}b^q + \frac{1}{q}b^q - b^q = 0$$

we obtain that $f(x) \geq 0$ for all $x \in [0, \infty)$ as desired. $\square$

Using Young's Inequality, we have a stepping stone towards the triangle inequality.

THEOREM 0.3 (Hölder's Inequality). *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in \mathbb{K}$*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

Proof. Let $\alpha = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ and let $\beta = (\sum_{i=1}^n |y_i|^q)^{\frac{1}{q}}$. Then it is clear that $\alpha = 0$ implies $x_i = 0$ for all $1 \leq i \leq n$ which implies that $\sum_{i=1}^n |x_i y_i| = 0$ and thus the inequality will hold in this case. Similarly, if $\beta = 0$, then the inequality holds. Hence, we may assume that $\alpha, \beta > 0$.

Since $\alpha, \beta > 0$ we obtain that

$$\begin{aligned}
\sum_{i=1}^n |x_i y_i| &= \alpha\beta \sum_{i=1}^n \left|\frac{x_i}{\alpha}\right| \left|\frac{y_i}{\beta}\right| \\
\text{(Young's (Theorem 0.2))} \qquad &\leq \alpha\beta \left( \sum_{i=1}^n \frac{1}{p} \left|\frac{x_i}{\alpha}\right|^p + \frac{1}{q} \left|\frac{y_i}{\beta}\right|^q \right) \\
&= \alpha\beta \left( \frac{1}{p\alpha^p} \sum_{i=1}^n |x_i|^p + \frac{1}{q\beta^q} \sum_{i=1}^n |y_i|^q \right) \\
&= \alpha\beta \left( \frac{1}{p} + \frac{1}{q} \right) \\
&= \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}
\end{aligned}$$

as desired. □

Note that Hölder's Inequality has the following trivial extension.

Corollary 0.4. *For any $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in \mathbb{K}$*

$$\sum_{i=1}^n |x_i y_i| \leq \|(x_1, x_2, ..., x_n)\|_1 \|(y_1, y_2, ..., y_n)\|_\infty$$

Finally, Hölder's Inequality enables us to prove the triangle inequality for the $p$-norm.

Theorem 0.5 (Minkowski's Inequality). *Let $p \in (1, \infty)$. For any $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in \mathbb{K}$*

$$\left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

PROOF. Choose $q \in (1,\infty)$ so that $\frac{1}{p}+\frac{1}{q}=1$. Thus, $q=\frac{p}{p-1}$. Since $p \in (1,\infty)$ notice by Hölder's Inequality that

$$\begin{aligned}
\sum_{i=1}^{n}|x_i+y_i|^p &= \sum_{i=1}^{n}|x_i+y_i||x_i+y_i|^{p-1} \\
&\leq \sum_{i=1}^{n}(|x_i|+|y_i|)|x_i+y_i|^{p-1} \\
&= \sum_{i=1}^{n}|x_i||x_i+y_i|^{p-1}+\sum_{i=1}^{n}|y_i||x_i+y_i|^{p-1} \\
&\leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}(|x_i+y_i|^p)^q\right)^{\frac{1}{q}}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}(|x_i+y_i|^p)^q\right)^{\frac{1}{q}} \\
&= \left(\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{\frac{1}{q}}
\end{aligned}$$

If $\sum_{i=1}^{n}|x_i+y_i|^p=0$, the result follows trivially. Otherwise, we may divide both sides of the equation by $(\sum_{i=1}^{n}|x_i+y_i|^p)^{\frac{1}{q}}$ so that

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{\frac{1}{p}}=\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1-\frac{1}{q}}\leq\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}$$

as desired. □

EXAMPLE 0.6. Let $p \in [1,\infty)$. Let $\ell_p(\mathbb{N})$ denote all sequences $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ such that $\sum_{n=1}^{\infty}|x_n|^p<\infty$, or,

$$\ell_p(\mathbb{N})=\left\{(x_n)_{n\in\mathbb{N}}:\sum_{n=1}^{\infty}|x_n|^p<\infty\right\}$$

Then $\ell_p(\mathbb{N})$ is a normed linear space with norm $\|\cdot\|_p:\ell_p(\mathbb{N})\to[0,\infty)$ defined by

$$\|(x_n)_{n\in\mathbb{N}}\|_p=\left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}}$$

It is elementary to see that $\|\cdot\|$ is well defined and satisfies the first two properties of a norm. To see that $\|\cdot\|_p$ satisfies the Triangle Inequality, we note that Minkowski's Inequality (Theorem 0.5 imply that

$$\left(\sum_{n=1}^{m}|x_n+y_n|^p\right)^{\frac{1}{p}}\leq\left(\sum_{n=1}^{m}|x_n|^p\right)^{\frac{1}{p}}+\left(\sum_{n=1}^{m}|y_n|^p\right)^{\frac{1}{p}}$$

for all $m\in\mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are in $\ell_p(\mathbb{N})$. By taking the limit as $m\to\infty$, we obtain the triangle inequality and the fact that if $x,y\in\ell_p(\mathbb{N})$ then $x+y\in\ell_p(\mathbb{N})$.

EXAMPLE 0.7. Let $\ell_\infty(\mathbb{N})$ denote all sequences $(x_n)_{n\in\mathbb{N}}$ of $\mathbb{K}$ such that $\sup_{n\in\mathbb{N}}|x_n| < \infty$. Then $\ell_\infty(\mathbb{N})$ is a normed linear space with $\|\cdot\|_\infty : \ell_\infty(\mathbb{N}) \to [0,\infty)$ defined by

$$\|(x_n)_{n\in\mathbb{N}}\|_p = \sup_{n\in\mathbb{N}}|x_n|$$

It is elementary to see that $\|\cdot\|_\infty$ is a well-defined norm, which we call the sup-norm or $\infty$-norm.

REMARK 0.8. It is not difficult to see that if $p, q \in [1,\infty]$, and $p < q$, then $\ell_p(\mathbb{N}) \subsetneq \ell_q(\mathbb{N})$. Indeed, if $(x_n)_{n\in\mathbb{N}} \in \ell_p$, then $\sum_{n=1}^\infty |x_n|^p < \infty$ so $(x_n)_{n\in\mathbb{N}}$ is bounded and $\sum_{n=1}^\infty |x_n|^q < \infty$ for all $q \in (p,\infty)$. To see that the inclusion is strict, notice that $\left(\frac{1}{n^{\frac{1}{p}}}\right)_{n\in\mathbb{N}} \notin \ell_p(\mathbb{N})$ but is in $\ell_q(\mathbb{N})$ for all $p < q$.

We have alternate versions for Hölder's Inequality presented below that uses the $\ell_p$ norms.

THEOREM 0.9 (Hölder's Inequality). *Let $p, q \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $(x_n)_{n\in\mathbb{N}} \in \ell_p$ and $(y_n)_{n\in\mathbb{N}} \in \ell_q$, then $(x_ny_n)_{n\in\mathbb{N}} \in \ell_1$ and*

$$\|(x_ny_n)_{n\in\mathbb{N}}\| \leq \|(x_n)_{n\in\mathbb{N}}\|_p \|(y_n)_{n\in\mathbb{N}}\|_q$$

Instead of having sequences on $p$-norms, we can also have continuous functions on $p$-norms. We replace their sums with their generalization, namely integrals. Note that we denote $\mathscr{C}[a,b]$ to be the set of all continuous functions on the closed interval $[a,b]$.

DEFINITION 0.10. For $p \in [1,\infty)$, define $\|\cdot\|_p : \mathscr{C}[a,b] \to [0,\infty)$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$$

for all $f \in \mathscr{C}[a,b]$.

We also have the equivalent forms of Hölder's Inequality and Minkowski's Inequality for continuous functions. We won't prove these as they are similar to the ones we have proven earlier.

THEOREM 0.11 (Hölder's Inequality). *Let $p, q \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g \in \mathscr{C}[a,b]$, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

THEOREM 0.12 (Minkowski's Inequality). *Let $p \in [1,\infty)$. If $f, g \in \mathscr{C}[a,b]$, then*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Taking a step further from Definition 0.10, we can replace continuous functions on an interval $\mathscr{C}[a,b]$ by *bounded functions on* $[a,b]$, denoted by $\mathscr{B}[a,b]$ and *Riemann integrable functions on* $[a,b]$ , denoted by $\mathscr{R}[a,b]$.

DEFINITION 0.13. For $p > 1$, define $\|\cdot\|_p : \mathscr{B}[a,b] \to [0,\infty)$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$$

for all $f \in \mathscr{B}[a,b]$.

DEFINITION 0.14. For $p > 1$, define $\|\cdot\|_p : \mathscr{R}[a,b] \to [0,\infty)$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$$

for all $f \in \mathscr{R}[a,b]$.