LECTURE 3

September 13, 2023

1. Topological Spaces

Recall in the previous lecture, we introduced the topological space. We say that a toplogical space $(X, \mathcal{T})$ is called

(1) *Hausdorff* if for every $x, y \in X$ with $x \neq y$, there exists disjoint $U, V \in \mathcal{T}$ with $x \in U$ and $y \in V$.
(2) *Metrizable* if there exists a metric d on X with $\mathcal{T} = \mathcal{T}_d$ is called a metrizable topology.

EXAMPLE 1.1. Let X be a nonempty set and define

$$\mathcal{T} = \{A \subset X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$$

(1) This is a topology on X, called the *cofinite topology*.
(2) If X is infinite, then $\mathcal{T}$ is not Hausdorff, and thus, not metrizable.

Let's verify (ii). We will in fact show that if $U, V \in \mathcal{T}$, both nonempty then $U \cap V \neq \emptyset$. Because $U \neq \emptyset$, then $X \setminus U$ is finite and similarly, $X \setminus V$ is finite and so $X \setminus (U \cap V)$ which is finite. So $U \cap V \neq \emptyset$.

EXAMPLE 1.2. Let X be a nonempty set and define

$$\mathcal{T} = \{A \subset X : X \setminus A \text{ is countable}\} \cup \{\emptyset\}$$

(1) This is a topology on X, called the cocountable topology.
(2) If X is uncountable, $\mathcal{T}$ is not Hausdorff and this not metrizable.

EXAMPLE 1.3. Let $(X, \mathcal{T})$ be a topologyical space and let $A \subset X$ with $A \neq \emptyset$. Define

$$\mathcal{T}|_A = \{U \cap A : U \in \mathcal{T}\}$$

This is the topology on A, called the relative topology induced by $\mathcal{T}$.

EXERCISE 1.4. Verify that $(A, \mathcal{T}|_A)$ is a topological space.

EXERCISE 1.5. If (X, d) is a metric space and $\emptyset \neq A \subset X$, we may define a metric $d|_A$ on A by restricting d on $A \times A$ (for $x, y \in A$, $d|_A(x, y) = d(x, y)$). Show that $(A, \mathcal{T}_{d|_A}) = (A, \mathcal{T}_d|_A)$,

EXERCISE 1.6. If $(X, \mathcal{T})$ is a Hausdorff topological space and let $\emptyset \neq A \subset X$, then $(A, \mathcal{T}|_A)$ is also Hausdorff.

EXERCISE 1.7. If $(X, \mathcal{T})$ is Hausdorff and X is infinite, then there exists an infinite sequence of nonempty pairwise disjoint open subsets of X. Deduce that the cardinality of $\mathcal{T}$ is at least the continuum.

2. Basis of a Topology

PROPOSITION 2.1. *Let (X, d) be a metric space. Then every open subset of X can be expressed as a union of open balls.*

PROOF. Indeed, if U is open and nonempty, then for all $x \in U$, there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset U$. Therefore, $U = \bigcup_{x \in U} B(x, \varepsilon_x)$. (Exercise) $\square$

DEFINITION 2.2. Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B} \subset \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every nonempty $U \in \mathcal{T}$ can be expressed as a union of members of $\mathcal{B}$, i.e. there exists $(U_i)_{i \in I}$ in $\mathcal{B}$ such that $U = \bigcup_{i \in I} U_i$.

EXAMPLE 2.3. If (X, d) is a metric space, then

$$\mathcal{B} = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$$

is a basis for $\mathcal{T}_d$.

EXAMPLE 2.4. If $(X, \mathcal{T})$ is a discrete topological space, i.e. $\mathcal{T} = \mathcal{P}(X)$, then

$$\mathcal{B} = \{\{x\} : x \in X\}$$

is a basis for $\mathcal{T}$.

EXAMPLE 2.5. Take $\mathbb{N}$ with the cofine topology. Then show

$$\mathcal{B} = \{\{m\} \cup \{n, n+1, ...\} : n, m \in \mathbb{N}\}$$

This is a basis for the cofinite topology.

PROPOSITION 2.6. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subset \mathcal{T}$. The following are equivalent:*

(1) *$\mathcal{B}$ is a basis for $\mathcal{T}$.*
(2) *For every $x \in X$, $U \in \mathcal{T}$ with $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U$.*

PROOF. (1) $\Rightarrow$ (2): Assume that $\mathcal{B}$ is a basis and take $x \in X$ and $U \in \mathcal{T}$ containing $x \in U$. By assumption, there exists $(V_i)_{i \in I}$ in $\mathcal{B}$ such that $U = \bigcup_{i \in I} V_i$. Since $x \in U$, there exists $i_0 \in I$ such that $x \in V_{i_0}$. Then because $V_{i_0} \subset U$, it is the member of $\mathcal{B}$ we are looking for.

(2) $\Rightarrow$ (1): Take $\emptyset \neq U \in \mathcal{T}$. By assumption, for each $x \in U$, there exists a $V_x \in \mathcal{B}$ such that $x \in V_x$ and, and $V_x \subset U$. Then $U = \bigcup_{x \in U} V_x$. $\square$

EXAMPLE 2.7. In any metric space (X, d),

$$\mathcal{B} = \{B(x, q) : x \in X, q \in \mathbb{Q} \cap (0, \infty)\}$$

This is a basis for $\mathcal{T}_d$. (Verify)

EXAMPLE 2.8. In $\mathbb{R}$ with the usual topology,

$$\mathcal{B} = \{(q, r) : q < r \in \mathbb{Q}\}$$

is a basis. (Verify)

PROPOSITION 2.9. *Let X be a set with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The following are equivalent*

(1) *$\mathcal{T}_1 \subset \mathcal{T}_2$ ($\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ or $\mathcal{T}_2$ is finer than $\mathcal{T}_1$)*
(2) *For every $V \in \mathcal{B}_1$ and $x \in V$, there exists a $U \in \mathcal{B}_2$ with $x \in U$ and $U \subset V$*

EXERCISE 2.10. Let X be a nonempty set and d_1, d_2 be metrics on X such that for all $x, y \in X$, $d_1(x,y) \leq d_2(x,y)$. Then $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$. For example, on $C[0,1] = \{\text{all } f : [0,1] \to \mathbb{R}\}$ continuous take $d_\infty(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$ and $d_1(f,g) = \int_0^1 |f(t) - g(t)| dt$. Examine whether $\mathcal{T}_d$ and $\mathcal{T}_\infty$ compare and if yes, how.

EXAMPLE 2.11. On $\mathbb{R}^n$ define $d_\infty(x,y)$ for $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ by taking

$$d_\infty = \max_{1\leq i\leq n} |x_i - y_i|$$

Then $\mathcal{B} = \{(a_1, , b_1) \times \cdots \times (a_n, b_n) : a_1 < b_1 \in \mathbb{R}, a_2 < b_2 \in \mathbb{R}^2, ...a_n < b_n \in \mathbb{R}\}$ is a basis for d_∞.

Also if $\mathcal{T}$ denotes the Euclidean topology on $\mathbb{R}^n$ and show $\mathcal{T}_{d_\infty} = \mathcal{T}$

DEFINITION 2.12. Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{F} \subset \mathcal{T}$ is called a *subbasis* for $\mathcal{T}$ if the collection

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} V_i : n \in \mathbb{N}, V_1, ..., V_n \in \mathcal{F} \right\}$$

is a basis for $\mathcal{T}$.

EXAMPLE 2.13. In $\mathbb{R}^n$, denote $\mathcal{F}$ to be

$$\mathcal{F} = \bigcup_{i=1}^{n} \{\mathbb{R} \times \cdots \times (a,b) \times \mathbb{R} \times \mathbb{R} : a < b \in \mathbb{R}\}$$

Then $\mathcal{B}(\mathcal{F})$ contains the family $\mathcal{B}$ from the previous example and thus $\mathcal{F}$ is called a subbasis of $\mathbb{R}^n$

Note that anything that is labelled in blue are stuff that weren't mentioned during the lecture or requires clarification.

THEOREM 2.14. *Let X be a nonempty set and let $B \subset \mathcal{P}(X)$ be such that*

(1) *If $x \in X$, then there exists a $B \in \mathcal{B}$ such that $x \in B$*
(2) *If $x \in X$ and $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$ then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$, $B_3 \subset B_1$ and $B_3 \subset B_2$.*

Let $\mathcal{T}_\mathcal{B}$ be the set of all subsets U of X such that for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Then $\mathcal{T}_\mathcal{B}$ is a topology on X such that $\mathcal{B} \subset \mathcal{T}$.

PROOF. To see that $\mathcal{T}_{\mathcal{B}}$ is a topology, we must verify the three properties of a topology. Clearly, $\emptyset \in \mathcal{T}_{\mathcal{B}}$. To see that $X \in \mathcal{T}_{\mathcal{B}}$, recall that for each $x \in X$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x$. As $B_x \subset X$ by definition, we obtain that $X \in \mathcal{T}_{\mathcal{B}}$ by the definition of $\mathcal{T}_{\mathcal{B}}$.

Next, suppose that $(U_i)_{i\in I}$ is a collection of elements of $\mathcal{T}_{\mathcal{B}}$. To see that $\bigcup_{i\in I} U_i \in \mathcal{T}_{\mathcal{B}}$, let $x \in \bigcup_{i\in I} U_i$ be arbitrary. Then there exists a $i_0 \in I$ such that $x \in U_{i_0}$. Since $U_{i_0} \in \mathcal{T}_{\mathcal{B}}$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U_{i_0}$. Hence, $B \subset U \subset \bigcup_{i\in I} U_i$. As $x \in \bigcup_{i\in I} U_i$ was arbitrary, we obtain that $\bigcup_{i\in I} U_i \in \mathcal{T}_{\mathcal{B}}$ by definition.

Finally, suppose $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$. To see that $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{B}}$, let $x \in U_1 \cap U_2$ be arbitrary. Hence, $x \in U_1$ and $x \in U_2$ so as $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$, there exists $B_k \in \mathcal{B}$ (either $k = 1$ or 2) such that $x \in B_k$ and $B_k \subset U_k$. By applying (2) for another time, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset B_k$. Hence, $B \in \mathcal{B}$, $x \in B$, and $B \subset B_k \subset U_k$ so that $B \subset U_1 \cap U_2$. Therefore, as $x \in X$ was arbitrary, $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{B}}$ as desired.

Finally, the fact that $\mathcal{B} \subset \mathcal{T}_{\mathcal{B}}$ follows from the definition of $\mathcal{T}_{\mathcal{B}}$. $\square$

DEFINITION 2.15. Let X be a nonempty set. A *basis for a topology on* X is a collection of subsets $\mathcal{B} \subset \mathcal{P}(X)$ such that

(1) If $x \in X$, then there exists a $B \in \mathcal{B}$ such that $x \in B$
(2) If $x \in X$ and $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1$ and $B_3 \subset B_2$.

EXAMPLE 2.16. Let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$. We claim that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. To see this, it suffices to verify the two defining properties of being a basis from Definition 2.15. To begin, notice if $x \in \mathbb{R}$, $x \in [x, x + 1) \in \mathcal{B}$. Hence, the first property is satisfied. To see the second property, let $[a_1, b_1), [a_2, b_2) \in \mathcal{B}$ and $x \in \mathbb{R}$ such that $x \in [a_1, b_1) \cap [a_2, b_2)$ be arbitrary. Let $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$ and let $B = [a, b)$. Since $x \in [a_1, b_1) \cap [a_2, b_2)$, we see that $a \leq x < b$, so $B \in \mathcal{B}$ and $x \in B$. Furthermore by construction, $B \subset [a_1, b_1) \cap [a_2, b_2)$. Hence, since $[a_1, b_1), [a_2, b_2) \in \mathcal{B}$ and $x \in \mathbb{R}$ were arbitrary, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

THEOREM 2.17. *Let X be a nonempty set and let $\mathcal{B}$ be a basis for a topology on X. Then*

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{B\in\mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\}$$

PROOF. Nice snce $\mathcal{T}_{\mathcal{B}}$ is a topology and since $\mathcal{B} \subset \mathcal{T}_{\mathcal{B}}$, we know that for all $\mathcal{B}_0 \subset \mathcal{B}$ that $\bigcup_{B\in\mathcal{B}_0} B$ is a union of elements of $\mathcal{T}_{\mathcal{B}}$ and thus in $\mathcal{T}_{\mathcal{B}}$. Hence,

$$\left\{ \bigcup_{B\in\mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\} \subset \mathcal{T}_{\mathcal{B}}$$

To show the other inclusion, let $U \in \mathcal{T}_{\mathcal{B}}$ be arbitrary. By the definition of $\mathcal{T}_{\mathcal{B}}$, for each $x \in U$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. Hence, we see that

$$U = \bigcup_{x \in U} B_x$$

Therefore, as $U \in \mathcal{T}_{\mathcal{B}}$ was arbitrary, we obtain that

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{B \in \mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\}$$

as desired. □

Definition 2.18. Let $(X, \mathcal{T})$ be a topological space. A set $\mathcal{B} \subset \mathcal{P}(X)$ is said to be a *basis for* $(X, \mathcal{T})$ if $\mathcal{B}$ is a basis for a topology on X and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$.

Remark 2.19. Of course, $\mathcal{B} \subset \mathcal{T}_{\mathcal{B}}$ for any basis of a topology $\mathcal{B}$, for a set $\mathcal{B} \subset \mathcal{P}(X)$ to be a basis for a topology $\mathcal{T}$, it must be the case that $\mathcal{B} \subset \mathcal{T}$. Furthermore, by Theorem 2.14 and Theorem 2.17, we see that if $\mathcal{B}$ is a basis for $(X, \mathcal{T})$, then

(1) A set $U \subset X$ is open if and only if for every $x \in U$ there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$
(2) The open sets in $(X, \mathcal{T})$ are exactly the union of elements of $\mathcal{B}$.

Example 2.20. Let X be a nonempty set and let $\mathcal{T}$ be the discrete topology on X. Then

$$\mathcal{B} = \{\{x\} : x \in X\}$$

is a basis for $(X, \mathcal{T})$. Indeed, $\mathcal{B} \subset \mathcal{T}$ as $\mathcal{T}$ is the discrete topology. Next, clearly the first property from Definition 2.15 holds and the second property also holds since the only was $x \in X$ and $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$ is if $B_1 = B_2 = \{x\} \in \mathcal{B}$. Hence, $\mathcal{B}$ is a basis for $(X, \mathcal{T})$.

Example 2.21. Let (X, d) be a metric space. Then the set $\mathcal{B}$ of all open balls forms a basis for $(X, \mathcal{T}_d)$. Indeed clearly, $\mathcal{B} \subset \mathcal{T}$ and if $x \in X$ then $B(x, 1) \in \mathcal{T}_d$ so the first property of Definition 2.15 is satisfied. To see the second property of Definition 2.15 is satisfied, let $x \in X$ and $B_1, B_2 \in \mathcal{B}$ be arbitrary such that $x \in B_1 \cap B_2$. Then there exists points $x_1, x_2 \in X$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $B_1 = B(x_1, \varepsilon_1)$ and $B_2 = B(x_2, \varepsilon_2)$. Thus, as $x \in B_1 \cap B_2$, we see that

$$d(x, x_1) < \varepsilon_1 \quad d(x, x_2) < \varepsilon_2$$

Let

$$\varepsilon = \min\{\varepsilon_1 - d(x, x_1), \varepsilon_2 - d(x, x_2)\}$$

Then $\varepsilon > 0$. We see that $B(x, \varepsilon) \subset B(x_1, \varepsilon_1)$ and $B(x, \varepsilon) \subset B(x_2, \varepsilon_2)$. Hence, as $x \in X$ and $B_1, B_2 \in \mathcal{B}$ were arbitrary, the second property of Definition 2.15 is satisfied, so $\mathcal{B}$ is a basis for $(X, \mathcal{T}_d)$.

EXAMPLE 2.22. Let (X, d) be a metric space and let $\varepsilon > 0$. The set $\mathcal{B}$ of all open balls with radius at most ε forms a basis for $(X, \mathcal{T}_d)$. Indeed the proof is identical to that of Example 2.21 with the additional restraint that all radii involved are at most ε.

PROPOSITION 2.23. *Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{B} \subset \mathcal{T}$ has the property that for all $U \in \mathcal{T}$ and for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Then $\mathcal{B}$ is a basis for $(X, \mathcal{T})$.*

PROOF. To see that $\mathcal{B}$ is a basis for a topology on X, we will simply verify the two properties of Definition 2.15. Let $x \in X$ be arbitrary. Then as $X \in \mathcal{T}$, the assumptions of the proposition imply there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset X$. Hence, as $x \in X$ was arbitrary, the first assumption of Definition 2.15 is satisfied.

To see the second property of Definition 2.15 holds, let $x \in X$ and $B_1, B_2 \in \mathcal{B}$ be such that $x \in B_1 \cap B_2$ be arbitrary. As $\mathcal{B} \subset \mathcal{T}$, we see that $B_1, B_2 \in \mathcal{T}$ and thus $B_1 \cap B_2 \in \mathcal{T}$. Therefore, the assumptions of the proposition there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. Hence, as $x \in X$ and $B_1, B_2 \in \mathcal{B}$ were arbitrary, the second property of Definition 2.15 are satisfied. Thus, $\mathcal{B}$ is a basis for a topology on X.

To see that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$, we first note that $\mathcal{B} \subset \mathcal{T}$ and as $\mathcal{T}$ is closed under unions, Theorem 2.17 imply that

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{B \in \mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\} \subset \mathcal{T}$$

Conversely, if $U \in \mathcal{T}$, then the assumptions of the proposition imply that $U \in \mathcal{T}_{\mathcal{B}}$ by Definition 2.15. Hnece $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ as desired. $\square$

COROLLARY 2.24. *Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{B} \subset \mathcal{T}$ has the property that for every $U \in \mathcal{T}$ there exists a subset $\mathcal{B}_0 \subset \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{B}_0} B$. Then $\mathcal{B}$ is a basis for $(X, \mathcal{T})$.*

PROOF. To prove this result, we will verify that the assumption of Proposition 2.23 holds. To see this, let $U \in \mathcal{T}$ and $x \in U$ be arbitrary. Then by the assumptions of this corollary, there exists a subset $\mathcal{B}_0 \subset \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{B}_0} B$. Hence as $x \in U$, there exists a $B_x \in \mathcal{B}_0$ such that $x \in B_x$ and $B_x \subset \bigcup_{B \in \mathcal{B}_0} B = U$. Therefore, as $U \in \mathcal{T}$ and $x \in U$ were arbitrary, the assumption of Proposition 2.23 holds. Hence, the result follows. $\square$

DEFINITION 2.25. Let $(X, \mathcal{T})$ be a topological space. A *subbasis* for $(X, \mathcal{T})$ is a ocllection of subsets $\mathcal{B}_{\mathcal{S}} \subset \mathcal{T}$ such that the set of all finite intersections of elements of $\mathcal{B}_{\mathcal{S}}$ is a basis for $(X, \mathcal{T})$. For a set $\mathcal{B}_{\mathcal{S}}$ to be a subbasis on some topology $\mathcal{T}$ on X, it is necessary that

$$X = \bigcup_{S \in \mathcal{B}_{\mathcal{S}}} S$$

Theorem 2.26. *Let X be a nonempty set and let $\mathcal{B}_{\mathcal{S}} \subset \mathcal{P}(X)$ be such that*

$$X = \bigcup_{S \in \mathcal{B}_{\mathcal{S}}} S$$

Let $\mathcal{B} \subset \mathcal{P}(X)$ be the set of all finite intersections of elements of $\mathcal{B}_{\mathcal{S}}$. Then $\mathcal{B}$ is a basis for a topology on X for which $\mathcal{B}_{\mathcal{S}}$ is a subbasis.

Proof. To see that $\mathcal{B}$ is a basis for a topology on X, we only need to check the two conditions from Definition 2.15. For the first, let $x \in X$ be arbitrary. Since $X = \bigcup_{S \in \mathcal{B}_{\mathcal{S}}} S$, there exists an $S_x \in \mathcal{B}_{\mathcal{S}}$ such that $x \in S_x$. As $\mathcal{B}_{\mathcal{S}} \subset \mathcal{B}$, the first property of being a basis holds for $\mathcal{B}$.

For the second property, let $x \in X$ and $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$ be arbitrary. Since $B_1, B_2 \in \mathcal{B}$, B_1 and B_2 are the finite intersections of elements of $\mathcal{B}_{\mathcal{S}}$. Hence, $B_1 \cap B_2$ is a finite intersection of elements of $\mathcal{B}_{\mathcal{S}}$ so that $B_1 \cap B_2 \in \mathcal{B}$. Therefore, as $x \in X$ and $B_1, B_2 \in \mathcal{B}$ were arbitrary, the second property of being a basis holds for a topology on X. The fact that $\mathcal{B}_{\mathcal{S}}$ is a subbasis is a subbasis for $\mathcal{T}_{\mathcal{B}}$ is then trivial. $\square$

3. Creating Topologies

Theorem 3.1. *Let X be a nonempty set, $\mathcal{W}$ be a collection of susets of X, then define*

$$\mathcal{T} = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members} \right\} \cup \{X, \emptyset\}$$

The following are equivalent:

1. *$\mathcal{T}$ is a topology on X and $\mathcal{W} \cup \{X\}$ is a basis for $\mathcal{T}$*
2. *For every $A, B \in \mathcal{W}$, and $x \in A \cap B$ there exists a $C \in \mathcal{W}$ with $x \in C$ and $C \subset A \cap B$. In particular, if $\mathcal{F}$ is any collection of subsets of X then the collection*

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} A_i : n \in \mathbb{N}, A_1, A_2, ..., A_n \in \mathcal{F} \right\} \cup \{X\}$$

is a basis for the topology

$$\mathcal{T}(\mathcal{F}) = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members of } \mathcal{B}(\mathcal{F}) \right\} \cup \{X, \emptyset\}$$

called the topology generated by $\mathcal{F}$.

Proof. (2) $\Rightarrow$ (1): We take $\mathcal{F} \subset \mathcal{P}(X)$ and show that for any $A, B \in \mathcal{B}(\mathcal{F})$, $A \cap B \in \mathcal{B}(\mathcal{F})$. This easily implies that $\mathcal{B}(\mathcal{F})$ satisfies assumption (2) and is a basis for $\mathcal{T}(\mathcal{F})$. As $A \in \mathcal{B}(\mathcal{F})$, there are $A_1, A_2, ..., A_n \in \mathcal{F}$ such that $A = \bigcap_{i=1}^{n} A_i$ and similarly, there are $B_1, B_2, ..., B_m$ with $B = \bigcap_{j=1}^{m} B_j$. Therefore, $A \cap B = (\bigcap_{i=1}^{n} A_i) \cap \left(\bigcap_{j=1}^{m} B_j\right)$, which is in $\mathcal{B}(\mathcal{F})$ as finite intersection of members of $\mathcal{F}$. $\square$

We will prove (1) $\Rightarrow$ (2) next time.