LECTURE 0

Preliminaries

1. Elements of Set Theory

DEFINITION 1.1. A binary relation "$\preceq$" on a set X is called a *partial order* if it satisfies the following:

(i) (Reflexivity) For all $x \in X$, $x \preceq x$
(ii) (Antisymmetric) For all $x, y \in X$, if $x \preceq y$ and $y \preceq x$, then $x = y$.
(iii) (Transitive) For all $x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq x$.

The pair $(X, \preceq)$ is called a *partially ordered set* (pos).

EXAMPLE 1.2.
- Let $(\mathbb{R}, \leq)$. This has the additional property that for all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$. That is, all pairs of elements are *comparable*. A partially ordered set of all pairs of elements of which are comparable is called a *totally ordered space*.
- For a set X, $(\mathcal{P}(X), \subseteq)$ is a partially ordered set that is not totally ordered, unless X is a singleton. Indeed, if $a \neq b \in X$, then for $A = \{a\}$ and $B = \{b\}$ and $A, B \in \mathcal{P}(X)$, but $A \nsubseteq B$ and $B \nsubseteq A$, then A and B are *incomparable*.

DEFINITION 1.3. Let $(X, \preceq)$ be a partially ordered set. A totally order subset $\mathcal{C}$ of X is called a chain, i.e. for every $x, y \in \mathcal{C}$, either $x \preceq y$ or $y \preceq x$.

EXAMPLE 1.4. Take $(\mathcal{P}(\mathbb{R}), \subseteq)$. Define

$$\mathcal{C} = \{[0, x] : x \in \mathbb{R}, x \geq 0\}$$

This is indeed a chain.

DEFINITION 1.5. Let $(X, \preceq)$ be a partially ordered set and $A \subseteq X$.
- A $x_0 \in X$ is called an *upper bound of* A if for all $x \in A$, $x \preceq x_0$.
- A $x_0 \in A$ is called the *maximum of* A if for all $x \in A$, $x \preceq x_0$.
- A $x_0 \in A$ is called a *maximal element of* A if for every $x \in A$ that is comparable to x_0, $x \preceq x_0$.

EXAMPLE 1.6. Consider $X = \mathcal{P}(\mathbb{R})$ with inclusion and

$$\mathcal{A} = \{A \subseteq \mathbb{R} : \text{for some } \alpha \in \mathbb{R}, A \subseteq [\alpha, \alpha + 1]\} \subseteq \mathcal{P}(\mathbb{R})$$

Then for every $\alpha \in \mathbb{R}$, the set $A_\alpha = [\alpha, \alpha + 1]$ is a maximal element of $\mathcal{A}$. Note that
- $\mathcal{A}$ has many different maximal elements.
- $\mathcal{A}$ has no maximum.

- $\mathcal{A}$ has an upper bound, namely $\mathbb{R}$.

The following is one of the most important tools in proving abstract theorems in pure mathematics.

THEOREM 1.7 (Zorn's Lemma). *Let $(X, \preceq)$ be a partially ordered set with $X \neq \emptyset$. Assume that every chain $\mathcal{C} \subseteq X$ has some upper bound. Then X has some maximal element x_0.*

Zorn's Lemma (Theorem 1.7) is derived from the *axiom of choice* and is, in fact, equivalent to it. We will not prove it, but will use it during class. Generally, it is ued to prove the existence of objects that are difficult (or impossible) to construct concretely.

Another important set-theoretic tool is the Well-Ordering Principle, that can be proved using Zorn's Lemma (Theorem 1.7). We will do this, after giving the appropriate definitions.

DEFINITION 1.8. Let $(X, \preceq)$ be a partially ordered set.

- For a subset $A \subseteq X$, an $x_0 \in A$ is called a *minimum* if for every $x \in A$, $x_0 \preceq x$.
- $(X, \preceq)$ is called *well-ordered* if every nonempty subset $A \subseteq X$ has a minimum.

Note that a minimum of A needs to be in A.

REMARK 1.9. A well-ordered partially ordered set is always totally ordered. It is a standard theorem that $(\mathbb{N}, \preceq)$ is well-ordered, but many other usual partially ordered sets are not, i.e. $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$ are not well-ordered. We will, however, prove the following.

THEOREM 1.10 (Well-Ordering Principle). *Every nonempty set X has a well-ordering, i.e. there is a partial order $\preceq$ on X such that $(X, \preceq)$ is well-ordered.*

DEFINITION 1.11. Let $(X, \preceq)$ be a partially ordered set and $\emptyset \neq A \subseteq B$. We say that A is an initial segment of B if for all $a \in A$ and $b \in B$ such that $b \preceq a$, we have $b \in A$.

For example, in $(\mathbb{N}, \leq)$, $\{3, 5, 8\}$ is an initial segment of $\{3, 5, 8, 16, 32\}$, but *not* one of $\{3, 5, 6, 8, 9\}$.

PROOF. Let X be a fixed nonempty set. Consider

$$\mathcal{A} = \{(A, \preceq) : \emptyset \neq A \subseteq X \text{ and } \preceq \text{ is a well-ordering of } A\}$$

This is a nonempty set, i.e. for all $x \in A$, $A = \{x\}$ may be equipped with a trivial well-ordering.

We will define a partial order on $\mathcal{A}$ that satisfies the assumptions of Zorn's Lemma (Theorem 1.7).

For $(A_1, \preceq_1), (A_2, \preceq_2) \in \mathcal{A}$, we will write $(A_1, \preceq_1) \leq (A_2, \preceq_2)$ if

(i) $A_1 \subseteq A_2$

(ii) $\preceq_2|_{A_1}=\preceq_1$, i.e. $\preceq_2$ restricted to A_1 coincides with $\preceq_1$.
(iii) A_1 is an initial segment of $(A_2, \preceq_2)$.

This is a well-defined partial order. We show that any chain has an upper bound.

Let $(A_i, \preceq_i)_{i\in I}$ be a chain in $(\mathcal{A}, \leq)$. Define $A = \bigcup_{i\in I} A_i$ and $\preceq = \bigcup_{i\in I} \preceq_i$ (this is formally correct, but what it means is for all $x, y \in A$, pick $i \in I$ such that $x, y \in A_i$ and let $x \preceq y$ if and only if $x \preceq_i y$).

Let us show that $(A, \preceq) \in \mathcal{A}$. It is standard to check that $\preceq$ is a partial order on A and this only uses properties (i) and (ii) of $\leq$.

To prove that $\preceq$ is a well-ordering of A, we will also use (iii). To that end, let $\emptyset \neq B \subseteq A$. Pick $i_0 \in I$ such that $B \cap A_{i_0} \neq \emptyset$ and let $b_0 = \min_{\preceq_{i_0}}(B \cap A_{i_0})$. To that end, let $b \in B$ and we take two cases.

Case 1: If $b_0 \in A_{i_0}$, then $b_0 \preceq_{i_0} b$ and so $b_0 \preceq b$.

Case 2: If $b \notin A_{i_0}$. Take $i_1 \in I$ such that $b \in A_i$. Since $(A_i, \preceq_i)_{i\in I}$ is a chain, necessarily $(A_{i_0}, \preceq_{i_0}) \leq (A_{i_1}, \preceq_{i_1})$ (the other direction is impossible since $b \in A_{i_1} \setminus A_{i_0}$).

Then A_{i_0} is an initial segment of A_{i_1} with respect to $\preceq_{i_1}$ and since $\preceq_{i_1}$ is a well-ordering, either $b_0 \preceq b$ or $b \preceq b_0$. But the second is impossible since then $b \in A_{i_0}$.

Since $(\mathcal{A}, \leq)$ satisfies the assumptions of Zorn's Lemma, it has a maximal element $(A_0, \preceq_0)$. We claim that $A_0 = X$ and thus $\preceq_0$ is a well-ordering on X. If not, take $x_0 \in X \setminus A_0$ and define $A_1 = A_0 \cup \{x_0\}$. Extend $\preceq_0$ to an $\preceq_1$ on A_1 such that for all $x \in A_1$, $x \preceq_1 x_0$. Then it is straightforward to verify that $(A_1, \preceq_1) \in \mathcal{A}$ and $(A_0, \preceq_0) \leq (A_1, \preceq_1)$ which contradicts the maximality of $(A_0, \preceq_0)$. $\square$

Zorn's Lemma has many other applications. A typical application is the proof of the existence of a basis in every vector space. We sketch the proof, but feel free to fill in the details. Be warned that some familiarity with linear algebra is required.

Definition 1.12. A subset A of a vector space X is called *linearly independent* if for any finite choice of pairwise different $x_1, x_2, ..., x_n \in A$ we have for scalars $\lambda_1, ..., \lambda_n$, if $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$, then

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

Definition 1.13. For a subset A of a vector space X, we define the *linear span of* A by

$$\langle A \rangle = \bigcap \{Y : Y \text{ is a linear subspace of } X \text{ and } A \subseteq Y\}$$

Then $\langle A \rangle$ is a linear subspace of X.

Definition 1.14. A subset A of a vector space X is called a *basis* of X if

- A is linearly independent
- $\langle A \rangle = X$.

THEOREM 1.15. *Every vector space X that is non-trivial (i.e. it contains some nonzero element x) has a basis.*

SKETCH OF PROOF. Define the set

$$\mathcal{A} = \{A : A \text{ is a linearly independent subset of } X\}$$

and endow it with inclusion. Then $(\mathcal{A}, \subseteq)$ is a partially ordered set. We show that it satisfies the assumptions of Zorn's Lemma. First, $\mathcal{A} \neq \emptyset$ because for $0 \neq x \in X$, $A = \{x\}$ is linearly independent and thus $A \in \mathcal{A}$. Next take a chain $\mathcal{C}$ in $\mathcal{A}$, i.e. a family of linearly independent subsets of X that compare to one another.

CLAIM 1.16. *$B = \bigcup \mathcal{C}$ is linearly independent and thus an upper bound for $\mathcal{C}$.*

(Prove the claim using the fact that $\mathcal{C}$ is a chain)
By Zorn's Lemma, there exists a maximal element A_0 in $\mathcal{A}$.

CLAIM 1.17. *$\langle A_0 \rangle = X$ and thus A_0 is a Hamel basis for X.*

(Prove the claim by contradiction. Assume $\langle A_0 \rangle \subsetneq X$, take $x_0 \in X \setminus A_0$ and prove that $A_0 \cup \{x_0\} \in \mathcal{A}$. This would be absurd because then A_0 would not be maximal.) $\square$