LECTURE 1

September 6, 2023

- Meetings: MW 1:00 PM (Monday CB 115, Wednesday ACW 205)
- Student Hours: M 4-5 PM on Zoom
- Evaluation:
  - Assignments: 50%
  - Midterm: 20%
  - Final: 30%

Do you have the background for this course? Try the background self-assessment on eClass.

1. Introduction to Topology

DEFINITION 1.1 (Topology). The study of shape without a notion of distance.

EXAMPLE 1.2. Let us consider the following distances. Some of which we may be familiar with.

- On $\mathbb{R}$, the distance between $x, y \in \mathbb{R}$ is the value $d(x, y) = |x - y|$.
- On $\mathbb{R}^n$, the Euclidean distance between two vectors, $x, y \in \mathbb{R}^n$ is

$$d_2(x, y) = \|x - y\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$$

  - For example, in $\mathbb{R}^2$, $d(x, y) = \sqrt{|x_1 - y_1|^2 - |x_2 - y_2|^2}$
- For strings of length n of zeros and ones, $x = x_1x_2...x_n$ and $y = y_1y_2...y_n$ the Hamming distance of x and y is

$$d_H(x, y) = |\{1 \leq i \leq n : x_i \neq y_i\}|$$

  - For example, if $x = 00101$ and $y = 10111$. To compute the distance, we have $d_H(x, y) = 2$, since the first and fourth positions are different.
- For two continuous functions $f, g : [0, 1] \to \mathbb{R}$, the *uniform distance* is given by

$$d_\infty(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$$

DEFINITION 1.3. Let X be a nonempty set. A function $f : X \times X \to [0, \infty)$ is called a *metric* if it satisfies the following properties:

- For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

- For all $x, y \in X$, $d(x,y) = d(y,x)$
- (Triangle Inequality) For all $x, y, z \in X$, $d(x,y) \le d(x,z) + d(z,y)$.

EXERCISE 1.4. Verify that the distances defined in Example 1.2 is are metrics.

NOTATION 1.5. A pair (X,d) where d is a metric on X is called a *metric space*.

REMARK 1.6. There may be different metrics on the same set X.

EXAMPLE 1.7. On $\mathbb{R}^2$, we have the Euclidean metric. Consider for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then define their "taxicab metric" given by

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

This is indeed a metric. (Use Definition 1.3 to verify)

EXAMPLE 1.8. For any set X, we can define the *discrete metric* as follows:

$$d(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{if } x = y \end{cases}$$

This is indeed a metric. (Use Definition 1.3 to verify) A metric space (X,d) where d is the discrete metric is called a *discrete metric space*.

2. Convergence

DEFINITION 2.1. Let (X,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. We say that $(x_n)_{n\in\mathbb{N}}$ *converges to* x (denote it by $x_n \to x$) if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \ge N$. We write $\lim_{n\to\infty} x_n = x$.

EXAMPLE 2.2. Consider the following examples.

- In $\mathbb{R}$, $\lim_{n\to\infty} \frac{1}{n} = 0$.
- In $\mathbb{R}^2$ $\lim_{n\to\infty} \left(\sin\left(\frac{1}{n}\right), \cos\left(\frac{1}{n}\right)\right) = (0,1)$.
- (Exercise) Let (X,d) be a discrete metric space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. Then the following are equivalent:
  (1) $\lim_{n\to\infty} x_n = x$
  (2) There exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \ge N$.

3. Continuity

DEFINITION 3.1. Let (X,d) and (Y,ρ) be metric spaces and let $f : X \to Y$ be a function.

- For a point $x \in X$, we say that f is continuous at x if for $\epsilon > 0$ there is a $\delta > 0$ such that for all $y \in X$, if $d(y,x) < \delta$, then $\rho(f(y), f(x)) < \epsilon$.
- If f is continuous at every point in $x \in X$, then we say that f is continuous on X.

Sometimes we write $f : (X,d) \to (Y,\rho)$ is continuous.

EXAMPLE 3.2. Consider the following examples.

- Let $\mathcal{C}[0,1] = \{$all continuous functions $f : [0,1] \to \mathbb{R}\}$ with the uniform metric d_∞. Define for $t \in [0,1]$, with $d_t : \mathcal{C}[0,1] \to \mathbb{R}$ given by $\delta_t(f) = f(t)$. Then
$$\delta_t : (\mathcal{C}[0,1], d_\infty) \to (\mathbb{R}, d)$$
is continuous. (Hint: fix $f_0 \in \mathcal{C}[0,1]$ and $\epsilon > 0$ and take $\delta = \epsilon$.)
- Consider $H : \mathbb{R} \to \mathbb{R}$ given by
$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 1 \end{cases}$$
This is continuous everywhere, except at $x = 0$.

EXERCISE 3.3. Let (X, d) be a discrete metric space and let (Y, ρ) be a metric space. Then *every* $f : (X, d) \to (Y, \rho)$ is always continuous.

EXERCISE 3.4. Let (X, d) be a metric space. Define $d_1 : (X \times X) \times (X \times X) \to [0, \infty)$ as follows: For (x_1, x_2) and $(y_1, x_2) \in X \times X$, let
$$d_1((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2)$$
Show that d_1 is a metric on $X \times X$ and show that $d : (X \times X, d_1) \to ([0, \infty), d)$ is continuous.

4. Identifying Metric Spaces

DEFINITION 4.1. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called an *isometry* if for all $x_1, x_2 \in X$, $f(x_1), f(x_2) \in Y$,
$$\rho(f(x_1), f(x_2)) = d(x_1, x_2)$$

EXAMPLE 4.2. Take $f : (\mathbb{R}^2, d) \to (\mathbb{R}^3, d_2)$ given by $f((x_1, x_2)) = (x_1, x_2, 0)$.

EXERCISE 4.3. If $f : (X, d) \to (Y, \rho)$ is an isometry, then

- f is continuous.
- f is one-to-one.

DEFINITION 4.4. Let (X, d) and (Y, ρ) be metric spaces. If there exists a surjective isometry $f : (X, d) \to (Y, \rho)$, then we say that (X, d) is *isometric* to (Y, ρ) and write $(X, d) \equiv (Y, \rho)$.

EXERCISE 4.5. If $f : (X, d) \to (Y, \rho)$ is an onto isometry, then f is invertible and $f^{-1} : (Y, \rho) \to (X, d)$ is also an onto isometry. Furthermore, show that "$\equiv$" is an equivalence relation on the class of metric spaces.

EXAMPLE 4.6. In $\mathbb{R}^2$ with Euclidean distance,

- Consider
$$X = \{(x, y) : x^2 + y^2 = 1\}$$
$$Y = \{(x, y) : (x-1)^2 + (y-1)^2 = 1\}$$
Then $X \equiv Y$.

- Consider

$$X = \{(x, y) : y = x^2\} \quad Y = \{(x, y) : x = y^2\}$$

  Then $X \equiv Y$.

DEFINITION 4.7. Let (X, d) and (Y, ρ) be a metric space. A function $f : X \to Y$ is called an *homeomorphism* if

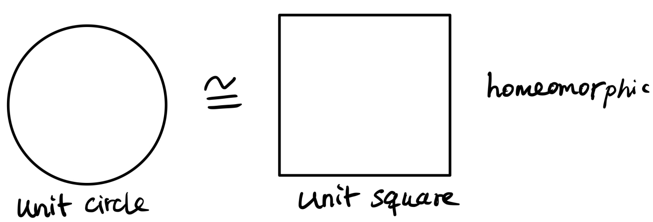
- f is a bijection, in particular, there exists an inverse $f^{-1} : Y \to X$.
- $f : (X, d) \to (Y, \rho)$ is continuous.
- $f^{-1} : (Y, \rho) \to (X, d)$ is continuous.

If such an f exists, we say that (X, d) is *homeomorphic* to (Y, ρ) and write $(X, d) \cong (Y, \rho)$.

EXERCISE 4.8. The binary relation "$\cong$" is an equivalence relation on the class of metric spaces.

EXAMPLE 4.9. On $\mathbb{R}^2$ with Euclidean distance, take $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and take $Y = \left\{(x, y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, a, b > 0\right\}$, then $X \cong Y$ and $f((x, y)) = (ax, by)$.

EXAMPLE 4.10. On $\mathbb{R}^2$, we have the following examples.



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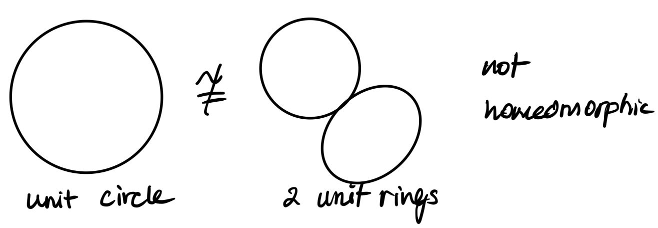

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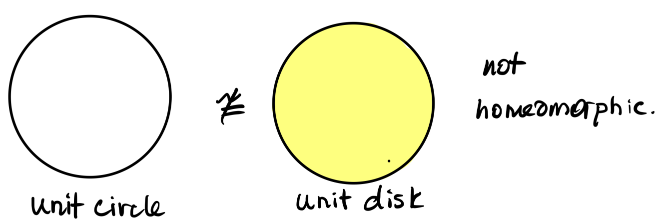

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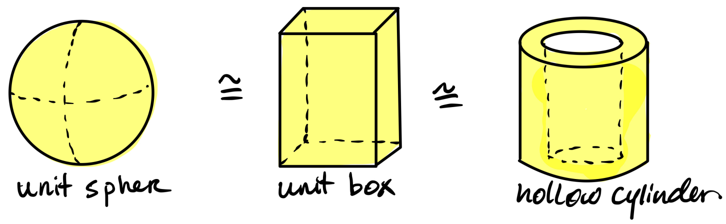
EXAMPLE 4.11. In $\mathbb{R}^3$, the unit sphere is homeomorphic to a unit rectangular box and is also homeomorphic to the hollow cylinder with



EXAMPLE 4.12. The torus is homeomorphic to a "coffee mug".