LECTURE 2

September 11, 2023

1. Open Sets in Metric Spaces

DEFINITION 1.1. Let (X, d) be a metric space.

- For $x_0 \in X$ and $\epsilon > 0$ define the open ball centered around x_0 with radius ϵ as
$$B(x_0, \epsilon) = \{y \in X : d(x_0, y) < \epsilon\}$$
- A subset $U \subset X$ is called an open set if for every $x_0 \in U$, there exists an $\epsilon > 0$ (that depends on x_0 and U) such that $B(x_0, \epsilon) \subset U$.

EXAMPLE 1.2. In $\mathbb{R}^2$, the open unit disk defined by
$$U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$
is an open set. (Verify this formally)

EXAMPLE 1.3. In $\mathbb{R}^2$, the closed unit disk defined by
$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$
is *not* an open set.

EXERCISE 1.4. In a metric space (X, d), for any $x_0 \in X$ and $\epsilon > 0$, the open ball $B(x_0, \epsilon)$ is an open set.

PROOF. We want to show that $B(x_0, \epsilon)$ is an open set. Indeed, let $y \in B(x_0, \epsilon)$ be arbitrary. We are looking for a $\delta > 0$ such that $B(y, \delta) \subset B(x_0, \epsilon)$. Take $\delta = \epsilon - d(x_0, y)$. (Use the triangle inequality to show that $\delta > 0$ and $B(y, \delta) \subset B(x_0, \epsilon)$). $\square$

EXERCISE 1.5. Let (X, d) be a metric space. Prove the following.

(i) X is open, and the $\emptyset$ is open. (Prove $\emptyset$ is open by using contradiction, assuming that there is a point and $\epsilon > 0$ such that $B(x_0, \epsilon) < \epsilon$ but there are no such points in $\emptyset$ so we have a contradiction)
(ii) If U and V are open sets, then $U \cap V$ is open.
(iii) If $(U_\alpha)_{\alpha \in I}$ is an arbitrary collection of open subsets of X, then $\bigcup_{\alpha \in I} U_\alpha$ is open.

Note that from (ii), by using induction, for any $n \in \mathbb{N}$, and $U_1, U_2, ..., U_n$ is open, then $\bigcap_{i=1}^{n} U_i$ is open.

Proof of (ii). Let U and V be open. Take any $x \in U \cap V$. since $x \in U$ and U is open, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Similarly, since $x \in V$ and V is open, there exists a $\delta > 0$ such that $B(x, d) \subset V$. Take $\eta = \min\{\epsilon, \delta\} > 0$ and show $B(x, \eta) \subset U \cap V$. (Formalize this) □

Theorem 1.6. *Let (X, d) be a metric space, let $x_0 \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The following are equivalent.*

(i) $\lim_{n\to\infty} x_n = x_0$
(ii) *For every open set U such that $x_0 \in U$, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.*

Proof. To show that (i) implies (ii), take U to be an open set with $x_0 \in U$. We want to find an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. (Formalize this)

To show that (ii) implies (i), let $\epsilon > 0$ be arbitrary, we want to find an $N \in \mathbb{N}$ such that $d(x_0, x_n) < \epsilon$ for all $n \geq N$. Take $U = B(x_0, \epsilon)$. (Formalize this) □

Theorem 1.7. *Let (X, d) and (Y, ρ) be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:*

(i) *$f : (X, d) \to (Y, \rho)$ is continuous.*
(ii) *For every V open subset of Y, $f^{-1}(V)$ is open in X.*

Proof. (i) $\Rightarrow$ (ii): Assume that $f : (X, d) \to (Y, \rho)$ be continuous. Let $V \subset Y$ be an arbitrary subset of Y. We will show that $f^{-1}(V) = U$ is open in X. Take an arbitrary point $x \in U$, we seek a $\delta > 0$ such that $B(x, \delta) \subset U$. Since f is continuous everywhere in X, f is continuous at x. Since $x \in U = f^{-1}(V)$, and so $f(x) \in V$. Furthermore, since V is open, there exists an $\epsilon > 0$ such that $B(f(x), \epsilon) \subset V$. By continuity of f at x, there exists a $\delta > 0$ such that for all $z \in X$ with $d(x, z) < \delta$ implies that $\rho(f(x), f(z)) < \epsilon$. We will now show that for this $\delta > 0$, $B(x, \delta) \subset U$. Take $z_0 \in B(x, \delta)$. We show that $B(x, \delta) \subset U$, i.e. $d(x, z) < \delta$ therefore, $\rho(f(z), f(x)) < \epsilon$ therefore $f(z) \in B(f(x), \epsilon) \subset V$ which implies $f(z) \in V$ and so $z \in f^{-1}(V) = U$.

(ii) $\Rightarrow$ (i): Take $x \in X$ and $\epsilon > 0$. We want to find a $\delta > 0$ such that for all $z \in X$ with $d(x, z) < \delta$, we have $\rho(f(x), f(z)) < \epsilon$. Take the open set $V = B(f(x), \epsilon)$ which is open in Y. By assumption, $U = f^{-1}(V)$ is open in X. Then $x \in U$ because $f(x) \in V$ and therefore, there exists a $\delta > 0$ such that $B(x, \delta) \subset U$. Take $z \in X$ such that $d(x, z) < \delta$, i.e. $z \in B(x, \delta) \subset U$, which implies that $f(z) \in V = B(f(x), \epsilon)$ which implies that $\rho(f(x), f(z)) < \epsilon$ as desired. □

2. Topology

Definition 2.1. Let X be a nonempty set and let $\mathcal{T}$ be a collection of subsets of X, i.e. $\mathcal{T} \subset \mathcal{P}(X)$. If $\mathcal{T}$ satisfies the following, then we call $\mathcal{T}$ a topology on X and $(X, \mathcal{T})$ is called a topological space.

(i) $\emptyset, X \in \mathcal{T}$
(ii) If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.
(iii) If $(U_i)_{i\in I}$ is an arbitrary collection of members of $\mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$

If $\mathcal{T}$ is a topology on X, then the members of $\mathcal{T}$ are called $\mathcal{T}$-open sets (or just open sets)

Remark 2.2. If $\mathcal{T}$ is a topology on X, then for any $n \in \mathbb{N}$ and $U_1, U_2, ..., U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

Example 2.3. For a metric space (X, d) denote

$$\mathcal{T}_d = \{U \subset X : U \text{ is an open set with respect to } d\}$$

This is called the topology induced by the metric d. A topology $\mathcal{T}$ on a set X for which there exists a metric d on X with $\mathcal{T} = \mathcal{T}_d$ is called a metrizable topology.

Example 2.4. For any $X \neq \emptyset$, define $\mathcal{T} = \{\emptyset, X\}$ is a topology called the trivial topology on X. It is elementary to see that $\mathcal{T}$ is a topology.

Example 2.5. For any nonempty set $X \neq \emptyset$, $\mathcal{T} = \mathcal{P}(X)$ is a topology on X, called the discrete topology on X.

Exercise 2.6. Let (X, d) be a discrete metric space. Show that $\mathcal{T}_d = \mathcal{P}(X)$. In particular, the discrete topology on any set is metrizable.

Remark 2.7. If $(X, \mathcal{T})$ is a metrizable topology, there may be different metrics d, d' on X such that $\mathcal{T} = \mathcal{T}_d = \mathcal{T}_{d'}$. For example, on $\mathbb{Z}$ with the usual metric $d(x, y) = |x - y|$, $\mathcal{T}_d = \mathcal{P}(\mathbb{Z})$ (also given by the discrete metric).

Definition 2.8. A toplogical space $(X, \mathcal{T})$ is called *Hausdorff* if for any $x, y \in X$ with $x \neq y$, there exists disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.

Example 2.9. Let $X \neq \emptyset$ with $|X| \geq 2$. Then the trivial topology $\mathcal{T} = \{\emptyset, X\}$ is *not* Hausdorff.

Proposition 2.10. *Any metrizable topological space is Hausdorff.*

Proof. If $(X, \mathcal{T})$ is metrizable, there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$. Take $x \neq y \in X$. Take $r = d(x, y)$. Take $U = B\left(x, \frac{r}{2}\right)$ and $V = B\left(y, \frac{r}{2}\right)$ are containing x and y, respectively. We claim that $U \cap V = \emptyset$. If not, then there exists a $z \in U \cap V$ such that

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r$$

This is a contradiction. □

Example 2.11. Let X be a nonempty set, and let $\mathcal{T}$ to be defined by

$$\mathcal{T} = \{A \subset X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$$

(i) This is a topology on X, called the cofinite topology.
(ii) If X is infinite, then $\mathcal{T}$ is not metrizable.