LECTURE 4

September 18, 2023

Recall if $(X, \mathcal{T})$ be a topological space. Let $\mathcal{F} \subset \mathcal{T}$ is called a *subbasis* if

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} V_i : n \in \mathbb{N}, V_1, V_2, ..., V_n \in \mathcal{F} \right\}$$

is a basis for $\mathcal{T}$.

Example 0.1. In $\mathbb{R}$ with the usual topology

$$\mathcal{F} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$$

is a subbasis for $\mathcal{T}$. [Listen back to the lecture later...]

1. Creating Topologies

Theorem 1.1. *Let X be a nonempty set, $\mathcal{W}$ be a collection of subsets of X, then define*

$$\mathcal{T} = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members} \right\} \cup \{X, \emptyset\}$$

The following are equivalent:

1. *$\mathcal{T}$ is a topology on X and $\mathcal{W} \cup \{X\}$ is a basis for $\mathcal{T}$*
2. *For every $A, B \in \mathcal{W}$, and $x \in A \cap B$ there exists a $C \in \mathcal{W}$ with $x \in C$ and $C \subset A \cap B$. In particular, if $\mathcal{F}$ is any collection of subsets of X then the collection*

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} A_i : n \in \mathbb{N}, A_1, A_2, ..., A_n \in \mathcal{F} \right\} \cup \{X\}$$

is a basis for the topology

$$\mathcal{T}(\mathcal{F}) = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members of } \mathcal{B}(\mathcal{F}) \right\} \cup \{X, \emptyset\}$$

called the topology generated by $\mathcal{F}$.

Proof. We will prove (1) $\Rightarrow$ (2): Take $A, B \in \mathcal{W}$ and $x \in A \cap B$. Since $\mathcal{T}$ is a topology and $\mathcal{W} \subset \mathcal{T}$, we have that $A, B \in \mathcal{T}$, $A \cap B \in \mathcal{T}$ as well. Since $\mathcal{W} \cup \{X\}$ is a basis, there exists some $C \in \mathcal{W} \cup \{X\}$ such that $x \in C$ and $C \subset A \cap B$. We claim that $C \in \mathcal{W}$.

Next we will prove (2) $\Rightarrow$ (1): We first show that $\emptyset, X \in \mathcal{T}$, which is obvious by definition. Let $(U_i)_{i\in I}$ be an arbitrary collection of members of $\mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$. If $U, V \in \mathcal{T}$, we need to show that $U \cap V \in \mathcal{T}$. If $U = \emptyset$ or $V = \emptyset$, then $U \cap V = \emptyset$, and if $U = X$ or $V = X$, trivial as well. Without loss of generality, there exists collections $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ of members of $\mathcal{W}$ such that $U = \bigcup_{i\in I} U_i$ and $V = \bigcup_{j\in J} V_j$. Take $x \in U \cap V$. We will find $O_x \in \mathcal{W}$ such that $x \in O_x$ and $O_x \subset U \cap V$. If we find this, we will find that $U \cap V = \bigcup_{x\in U\cap V} O_x \in \mathcal{T}$. Since $x \in U$, then there exists an $i_x \in I$ such that $x \in U_{i_x} \in \mathcal{W}$, and since $x \in V$, then there exists an $j_x \in J$ such that $x \in V_{j_x} \in \mathcal{W}$. Thus, since $x \in U_{i_x} \cap V_{j_x}$. By (2), there exists an $O_x \in \mathcal{W}$ such that $x \in O_x \in \mathcal{W}$ and $O_x \subset U_{i_x} \cap V_{j_x} \subset U \cap V$. $\square$

Remark 1.2. Let X be a nonempty set and $\mathcal{F} \subset \mathcal{P}(X)$ such that $\bigcup \mathcal{F} = X$. Then

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} U_i : n \in \mathbb{N}, U_1, U_2, ..., U_n \in \mathcal{F} \right\}$$

is a basis for the topology

$$\mathcal{T}(\mathcal{F}) = \left\{ \bigcup_{i\in I} U_i : (U_i)_{i\in I} \text{ is a collection of members of } \mathcal{B}(\mathcal{F}) \right\} \cup \{\emptyset\}$$

Recall that if $\mathcal{F} \subset \mathcal{P}(X)$, then $\mathcal{T}(\mathcal{F})$ is called the topology generated by $\mathcal{F}$. If $\bigcup \mathcal{F} = X$, then $\mathcal{F}$ is a subbasis for a topology.

Example 1.3. Let $\mathcal{B}$ be the following collection of subsets of $\mathbb{R}$:

$$\mathcal{B} = \{[a, b) : a < b \in \mathbb{R}\}$$

The topology generated by $\mathcal{B}$ is called the *lower limit topology of* $\mathbb{R}$, denoted by $\mathcal{T}_\ell$. Sometimes $(\mathbb{R}, \mathcal{T}_\ell)$ is called the Sorgenfrey Line.

Exercise 1.4.
(1) Show that $(\mathbb{R}, \mathcal{T}_\ell)$ is Hausdorff.
(2) Show that $\mathcal{B} = \{[a, b) : a < b \in \mathbb{R}\}$ is a basis for $\mathcal{T}_\ell$.
(3) If $\mathcal{T}$ is the usual topology on $\mathbb{R}$, then $\mathcal{T} \subset \mathcal{T}_\ell$.

Remark 1.5. If X is a nonempty set and $\mathcal{F}_1 \subset \mathcal{F}_2$, then $\mathcal{T}(\mathcal{F})_1 \subset \mathcal{T}(\mathcal{F})_2$.

Proposition 1.6. *Let X be a nonempty set and let $\mathcal{F} \subset \mathcal{P}(X)$. Then*

$$\mathcal{T}(\mathcal{F}) = \bigcap \{\mathcal{T} : \mathcal{T} \textit{ is a topology of } X, \mathcal{F} \subset \mathcal{T}\}$$

Thus, $\mathcal{T}(\mathcal{F})$ is the smallest topology on X containing $\mathcal{F}$.

Proof. Denote

$$\mathscr{T} = \{\mathcal{T} : \mathcal{T} \text{ is a topology on } X \text{ and } \mathcal{F} \subset \mathcal{T}\} \neq \emptyset$$

since $\mathcal{T} = \mathcal{P}(X)$ is a topology and $\mathcal{F} \subset \mathcal{T}$, and so $\mathcal{T} \in \mathscr{T}$. Next, denote

$$\mathscr{S} = \bigcap \mathscr{T} = \{A \subset X : \text{for all } \mathcal{T} \in \mathscr{T},\ A \in \mathcal{T}\}$$

Then $\mathscr{S}$ is a topology and $\mathcal{F} \subset \mathscr{S}$. We show that $\mathscr{S} \subset \mathcal{T}(\mathcal{F})$ which is true because $\mathcal{T}(\mathcal{F}) \in \mathscr{T}$.

Secondly, we show that $\mathcal{T}(\mathcal{F}) \subset \mathscr{S}$. Since $\mathcal{F} \cup X$ is a subbasis of $\mathcal{T}(\mathcal{F})$, we get that every member of $\mathcal{T}(\mathcal{F})$ is produced by taking finite intersections and then arbitrary unions of members of $\mathcal{F} \subset \mathscr{S}$, thus it is also a member of $\mathscr{S}$. □

2. Elementary Concepts of Topology

DEFINITION 2.1. Let $(X, \mathcal{T})$ be a topological space.

(1) A subset U of X is called open if $U \in \mathcal{T}$.
(2) A subset F of X is called *closed* if its complement is open, i.e. $X \setminus F$ is open.

EXAMPLE 2.2. In $\mathbb{R}$ with the usual topology,

(1) $(0, 1)$ is an open set.
(2) $[0, 1]$ is an closed set because $\mathbb{R} \setminus [0, 1] = (-\infty, 0) \cup (1, \infty)$ which is open.
(3) The set $\mathbb{R}$ is open and closed, or, a *clopen* set.
(4) The set $\mathbb{Q}$ is neither open or closed.

EXERCISE 2.3. In $(\mathbb{R}, \mathcal{T}_\ell)$, for $a, b \in \mathbb{R}$ with $a < b$, then $[a, b)$ is a clopen set.

PROPOSITION 2.4. *Let $(X, \mathcal{T})$ be a topological space. The following hold:*

(1) *The sets $\emptyset$ and X are closed.*
(2) *If F and G are closed, then $F \cap G$ are closed.*
(3) *If $(F_i)_{i \in I}$ is an arbitrary collection of closed subsets of X, then $\bigcap_{i \in I} F_i$ is closed.*

PROOF. This is an immediate application of definition of a topology and De Morgan's Law. □

PROPOSITION 2.5. *If $(X, \mathcal{T})$ is Hausdorff, then for every $x_0 \in X$, $\{x_0\}$ is closed.*

PROOF. We show that $X \setminus \{x_0\}$ is open. Take $x \in X \setminus \{x_0\}$, i.e. $x \neq x_0$. By Hausdorff, there exists a U, V open and disjoint such that $x \in U$ and $x_0 \in V$. In particular, $x \in U \subset X \setminus \{x_0\}$, therefore, $X \setminus \{x_0\}$ is open. □

DEFINITION 2.6. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. An $x_0 \in X$ is called a

(1) isolated point of A if $x_0 \in A$ and there exists a $U \in \mathcal{T}$ containing x_0 such that $U \cap A = \{x_0\}$.
(2) limit point of A if every $U \in \mathcal{T}$ containing x_0 intersects A, i.e. $U \cap A \neq \emptyset$.
(3) cluster point of A if it is a limit point of $A \setminus \{x_0\}$.

EXAMPLE 2.7. In $\mathbb{R}$, for $A = (0, 1] \cup \{2\}$, 2 is an isolated point and a limit point of A, but not a cluster point of A and any $x \in [0, 1]$ is a limit point and cluster point of A, but not an isolated point of A. Note that $x = 0$ is a limit and cluster point of A, despite not being in A.

PROPOSITION 2.8. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, $A \subset X$ and $x \in X$.*

(1) *x is an isolated point of A if and only if $a \in A$ and there exists $U \in \mathcal{T}$ containing x such that $U \cap A$ is finite.*
(2) *x is a cluster point of A if and only if every for every $U \in \mathcal{T}$ containing x, $U \cap A$ is infinite.*

PROOF. If x is an isolated point of A, then $x \in A$ and there exists $U \in \mathcal{T}$ containing x such that $U \cap A = \{x\}$ which is finite. If on the other hand, $x \in A$, and there exists $U \in \mathcal{T}$ containing x such that $U \cap A = \{x_1, x_2, ..., x_n\}$, where $x_1, x_2, ..., x_n$ are finitely many points. As $x \in U \cap A$ we may assume without loss in generality, that $x_1 = x$. Because $\mathcal{T}$ is Hausdorff, there are pairs of disjoint open sets U_i and V_i with $2 \leq i \leq n$ such that $x_1 \in U_i$ and $x_i \in V_i$. Define $W = U \cap (\bigcap_{i=2}^{n} U_i)$, which contains x_1, but none of the $x_2, ..., x_n$ and is a subset of U. Thus, $x \in W \cap A \subset (U \cap A) \setminus \{x_2, ..., x_n\} = \{x\}$, i.e. $W \cap A = \{x\}$. $\square$