MATH 4081: Topology

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Preface

These are the first edition of these lecture notes for MATH 4081 (Topology I). Consequently, there may be several typographical errors. Not every result in these notes will be covered in class. For example, some results will be covered through assignments. However, these notes should be fairly self-contained. If you come across any typos, errors, omissions, or unclear explanations, please feel free to contact me so that I may continually improve these notes.

Contents

Week 1

September 4-8

LECTURE 0

Preliminaries

1. Elements of Set Theory

DEFINITION 1.1. A binary relation "$\preceq$" on a set X is called a *partial order* if it satisfies the following:

(i) (Reflexivity) For all $x \in X$, $x \preceq x$
(ii) (Antisymmetric) For all $x, y \in X$, if $x \preceq y$ and $y \preceq x$, then $x = y$.
(iii) (Transitive) For all $x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq x$.

The pair $(X, \preceq)$ is called a *partially ordered set* (pos).

EXAMPLE 1.2.
- Let $(\mathbb{R}, \leq)$. This has the additional property that for all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$. That is, all pairs of elements are *comparable*. A partially ordered set of all pairs of elements of which are comparable is called a *totally ordered space*.
- For a set X, $(\mathcal{P}(X), \subseteq)$ is a partially ordered set that is not totally ordered, unless X is a singleton. Indeed, if $a \neq b \in X$, then for $A = \{a\}$ and $B = \{b\}$ and $A, B \in \mathcal{P}(X)$, but $A \not\subseteq B$ and $B \not\subseteq A$, then A and B are *incomparable*.

DEFINITION 1.3. Let $(X, \preceq)$ be a partially ordered set. A totally order subset $\mathcal{C}$ of X is called a chain, i.e. for every $x, y \in \mathcal{C}$, either $x \preceq y$ or $y \preceq x$.

EXAMPLE 1.4. Take $(\mathcal{P}(\mathbb{R}), \subseteq)$. Define

$$\mathcal{C} = \{[0, x] : x \in \mathbb{R}, x \geq 0\}$$

This is indeed a chain.

DEFINITION 1.5. Let $(X, \preceq)$ be a partially ordered set and $A \subseteq X$.
- A $x_0 \in X$ is called an *upper bound of* A if for all $x \in A$, $x \preceq x_0$.
- A $x_0 \in A$ is called the *maximum of* A if for all $x \in A$, $x \preceq x_0$.
- A $x_0 \in A$ is called a *maximal element of* A if for every $x \in A$ that is comparable to x_0, $x \preceq x_0$.

EXAMPLE 1.6. Consider $X = \mathcal{P}(\mathbb{R})$ with inclusion and

$$\mathcal{A} = \{A \subseteq \mathbb{R} : \text{for some } \alpha \in \mathbb{R}, A \subseteq [\alpha, \alpha + 1]\} \subseteq \mathcal{P}(\mathbb{R})$$

Then for every $\alpha \in \mathbb{R}$, the set $A_\alpha = [\alpha, \alpha + 1]$ is a maximal element of $\mathcal{A}$. Note that
- $\mathcal{A}$ has many different maximal elements.
- $\mathcal{A}$ has no maximum.

- $\mathcal{A}$ has an upper bound, namely $\mathbb{R}$.

The following is one of the most important tools in proving abstract theorems in pure mathematics.

THEOREM 1.7 (Zorn's Lemma). *Let $(X, \preceq)$ be a partially ordered set with $X \neq \emptyset$. Assume that every chain $\mathcal{C} \subseteq X$ has some upper bound. Then X has some maximal element x_0.*

Zorn's Lemma (Theorem 1.7) is derived from the *axiom of choice* and is, in fact, equivalent to it. We will not prove it, but will use it during class. Generally, it is ued to prove the existence of objects that are difficult (or impossible) to construct concretely.

Another important set-theoretic tool is the Well-Ordering Principle, that can be proved using Zorn's Lemma (Theorem 1.7). We will do this, after giving the appropriate definitions.

DEFINITION 1.8. Let $(X, \preceq)$ be a partially ordered set.

- For a subset $A \subseteq X$, an $x_0 \in A$ is called a *minimum* if for every $x \in A$, $x_0 \preceq x$.
- $(X, \preceq)$ is called *well-ordered* if every nonempty subset $A \subseteq X$ has a minimum.

Note that a minimum of A needs to be in A.

REMARK 1.9. A well-ordered partially ordered set is always totally ordered. It is a standard theorem that $(\mathbb{N}, \preceq)$ is well-ordered, but many other usual partially ordered sets are not, i.e. $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$ are not well-ordered. We will, however, prove the following.

THEOREM 1.10 (Well-Ordering Principle). *Every nonempty set X has a well-ordering, i.e. there is a partial order $\preceq$ on X such that $(X, \preceq)$ is well-ordered.*

DEFINITION 1.11. Let $(X, \preceq)$ be a partially ordered set and $\emptyset \neq A \subseteq B$. We say that A is an initial segment of B if for all $a \in A$ and $b \in B$ such that $b \preceq a$, we have $b \in A$.

For example, in $(\mathbb{N}, \leq)$, $\{3, 5, 8\}$ is an initial segment of $\{3, 5, 8, 16, 32\}$, but *not* one of $\{3, 5, 6, 8, 9\}$.

PROOF. Let X be a fixed nonempty set. Consider

$$\mathcal{A} = \{(A, \preceq) : \emptyset \neq A \subseteq X \text{ and } \preceq \text{ is a well-ordering of } A\}$$

This is a nonempty set, i.e. for all $x \in A$, $A = \{x\}$ may be equipped with a trivial well-ordering.

We will define a partial order on $\mathcal{A}$ that satisfies the assumptions of Zorn's Lemma (Theorem 1.7).

For $(A_1, \preceq_1), (A_2, \preceq_2) \in \mathcal{A}$, we will write $(A_1, \preceq_1) \leq (A_2, \preceq_2)$ if

(i) $A_1 \subseteq A_2$

(ii) $\preceq_2|_{A_1} = \preceq_1$, i.e. $\preceq_2$ restricted to A_1 coincides with $\preceq_1$.

(iii) A_1 is an initial segment of $(A_2, \preceq_2)$.

This is a well-defined partial order. We show that any chain has an upper bound.

Let $(A_i, \preceq_i)_{i\in I}$ be a chain in $(\mathcal{A}, \leq)$. Define $A = \bigcup_{i\in I} A_i$ and $\preceq = \bigcup_{i\in I} \preceq_i$ (this is formally correct, but what it means is for all $x, y \in A$, pick $i \in I$ such that $x, y \in A_i$ and let $x \preceq y$ if and only if $x \preceq_i y$).

Let us show that $(A, \preceq) \in \mathcal{A}$. It is standard to check that $\preceq$ is a partial order on A and this only uses properties (i) and (ii) of $\leq$.

To prove that $\preceq$ is a well-ordering of A, we will also use (iii). To that end, let $\emptyset \neq B \subseteq A$. Pick $i_0 \in I$ such that $B \cap A_{i_0} \neq \emptyset$ and let $b_0 = \min_{\preceq_{i_0}}(B \cap A_{i_0})$. To that end, let $b \in B$ and we take two cases.

Case 1: If $b_0 \in A_{i_0}$, then $b_0 \preceq_{i_0} b$ and so $b_0 \preceq b$.

Case 2: If $b \notin A_{i_0}$. Take $i_1 \in I$ such that $b \in A_i$. Since $(A_i, \preceq_i)_{i\in I}$ is a chain, necessarily $(A_{i_0}, \preceq_{i_0}) \leq (A_{i_1}, \preceq_{i_1})$ (the other direction is impossible since $b \in A_{i_1} \setminus A_{i_0}$).

Then A_{i_0} is an initial segment of A_{i_1} with respect to $\preceq_{i_1}$ and since $\preceq_{i_1}$ is a well-ordering, either $b_0 \preceq b$ or $b \preceq b_0$. But the second is impossible since then $b \in A_{i_0}$.

Since $(\mathcal{A}, \leq)$ satisfies the assumptions of Zorn's Lemma, it has a maximal element $(A_0, \preceq_0)$. We claim that $A_0 = X$ and thus $\preceq_0$ is a well-ordering on X. If not, take $x_0 \in X \setminus A_0$ and define $A_1 = A_0 \cup \{x_0\}$. Extend $\preceq_0$ to an $\preceq_1$ on A_1 such that for all $x \in A_1$, $x \preceq_1 x_0$. Then it is straightforward to verify that $(A_1, \preceq_1) \in \mathcal{A}$ and $(A_0, \preceq_0) \leq (A_1, \preceq_1)$ which contradicts the maximality of $(A_0, \preceq_0)$. $\square$

Zorn's Lemma has many other applications. A typical application is the proof of the existence of a basis in every vector space. We sketch the proof, but feel free to fill in the details. Be warned that some familiarity with linear algebra is required.

Definition 1.12. A subset A of a vector space X is called *linearly independent* if for any finite choice of pairwise different $x_1, x_2, ..., x_n \in A$ we have for scalars $\lambda_1, ..., \lambda_n$, if $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$, then

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

Definition 1.13. For a subset A of a vector space X, we define the *linear span of* A by

$$\langle A \rangle = \bigcap \{Y : Y \text{ is a linear subspace of } X \text{ and } A \subseteq Y\}$$

Then $\langle A \rangle$ is a linear subspace of X.

Definition 1.14. A subset A of a vector space X is called a *basis* of X if

- A is linearly independent
- $\langle A \rangle = X$.

THEOREM 1.15. *Every vector space X that is non-trivial (i.e. it contains some nonzero element x) has a basis.*

SKETCH OF PROOF. Define the set

$$\mathcal{A} = \{A : A \text{ is a linearly independent subset of } X\}$$

and endow it with inclusion. Then $(\mathcal{A}, \subseteq)$ is a partially ordered set. We show that it satisfies the assumptions of Zorn's Lemma. First, $\mathcal{A} \neq \emptyset$ because for $0 \neq x \in X$, $A = \{x\}$ is linearly independent and thus $A \in \mathcal{A}$. Next take a chain $\mathcal{C}$ in $\mathcal{A}$, i.e. a family of linearly independent subsets of X that compare to one another.

CLAIM 1.16. *$B = \bigcup \mathcal{C}$ is linearly independent and thus an upper bound for $\mathcal{C}$.*

(Prove the claim using the fact that $\mathcal{C}$ is a chain)

By Zorn's Lemma, there exists a maximal element A_0 in $\mathcal{A}$.

CLAIM 1.17. *$\langle A_0 \rangle = X$ and thus A_0 is a Hamel basis for X.*

(Prove the claim by contradiction. Assume $\langle A_0 \rangle \subsetneq X$, take $x_0 \in X \setminus A_0$ and prove that $A_0 \cup \{x_0\} \in \mathcal{A}$. This would be absurd because then A_0 would not be maximal.) $\square$

LECTURE 1

September 6, 2023

- Meetings: MW 1:00 PM (Monday CB 115, Wednesday ACW 205)
- Student Hours: M 4-5 PM on Zoom
- Evaluation:
  - Assignments: 50%
  - Midterm: 20%
  - Final: 30%

Do you have the background for this course? Try the background self-assessment on eClass.

1. Introduction to Topology

Definition 1.1 (Topology). The study of shape without a notion of distance.

Example 1.2. Let us consider the following distances. Some of which we may be familiar with.

- On $\mathbb{R}$, the distance between $x, y \in \mathbb{R}$ is the value $d(x, y) = |x - y|$.
- On $\mathbb{R}^n$, the Euclidean distance between two vectors, $x, y \in \mathbb{R}^n$ is
$$d_2(x, y) = \|x - y\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$
  - For example, in $\mathbb{R}^2$, $d(x, y) = \sqrt{|x_1 - y_1|^2 - |x_2 - y_2|^2}$
- For strings of length n of zeros and ones, $x = x_1 x_2 ... x_n$ and $y = y_1 y_2 ... y_n$ the Hamming distance of x and y is
$$d_H(x, y) = |\{1 \leq i \leq n : x_i \neq y_i\}|$$
  - For example, if $x = 00101$ and $y = 10111$. To compute the distance, we have $d_H(x, y) = 2$, since the first and fourth positions are different.
- For two continuous functions $f, g : [0, 1] \to \mathbb{R}$, the *uniform distance* is given by
$$d_\infty(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$$

Definition 1.3. Let X be a nonempty set. A function $f : X \times X \to [0, \infty)$ is called a *metric* if it satisfies the following properties:

- For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

- For all $x, y \in X$, $d(x,y) = d(y,x)$
- (Triangle Inequality) For all $x, y, z \in X$, $d(x,y) \leq d(x,z) + d(z,y)$.

EXERCISE 1.4. Verify that the distances defined in Example 1.2 is are metrics.

NOTATION 1.5. A pair (X,d) where d is a metric on X is called a *metric space*.

REMARK 1.6. There may be different metrics on the same set X.

EXAMPLE 1.7. On $\mathbb{R}^2$, we have the Euclidean metric. Consider for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then define their "taxicab metric" given by

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

This is indeed a metric. (Use Definition 1.3 to verify)

EXAMPLE 1.8. For any set X, we can define the *discrete metric* as follows:

$$d(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x = y \end{cases}$$

This is indeed a metric. (Use Definition 1.3 to verify) A metric space (X,d) where d is the discrete metric is called a *discrete metric space*.

2. Convergence

DEFINITION 2.1. Let (X,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. We say that $(x_n)_{n\in\mathbb{N}}$ *converges to* x (denote it by $x_n \to x$) if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} x_n = x$.

EXAMPLE 2.2. Consider the following examples.

- In $\mathbb{R}$, $\lim_{n\to\infty} \frac{1}{n} = 0$.
- In $\mathbb{R}^2$ $\lim_{n\to\infty} \left(\sin\left(\frac{1}{n}\right), \cos\left(\frac{1}{n}\right)\right) = (0,1)$.
- (Exercise) Let (X,d) be a discrete metric space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. Then the following are equivalent:
  (1) $\lim_{n\to\infty} x_n = x$
  (2) There exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

3. Continuity

DEFINITION 3.1. Let (X,d) and (Y,ρ) be metric spaces and let $f : X \to Y$ be a function.

- For a point $x \in X$, we say that f is continuous at x if for $\epsilon > 0$ there is a $\delta > 0$ such that for all $y \in X$, if $d(y,x) < \delta$, then $\rho(f(y), f(x)) < \epsilon$.
- If f is continuous at every point in $x \in X$, then we say that f is continuous on X.

Sometimes we write $f : (X,d) \to (Y,\rho)$ is continuous.

EXAMPLE 3.2. Consider the following examples.

- Let $\mathcal{C}[0,1] = \{\text{all continuous functions } f : [0,1] \to \mathbb{R}\}$ with the uniform metric d_∞. Define for $t \in [0,1]$, with $d_t : \mathcal{C}[0,1] \to \mathbb{R}$ given by $\delta_t(f) = f(t)$. Then
$$\delta_t : (\mathcal{C}[0,1], d_\infty) \to (\mathbb{R}, d)$$
is continuous. (Hint: fix $f_0 \in \mathcal{C}[0,1]$ and $\epsilon > 0$ and take $\delta = \epsilon$.)
- Consider $H : \mathbb{R} \to \mathbb{R}$ given by
$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 1 \end{cases}$$
This is continuous everywhere, except at $x = 0$.

EXERCISE 3.3. Let (X, d) be a discrete metric space and let (Y, ρ) be a metric space. Then *every* $f : (X, d) \to (Y, \rho)$ is always continuous.

EXERCISE 3.4. Let (X, d) be a metric space. Define $d_1 : (X \times X) \times (X \times X) \to [0, \infty)$ as follows: For (x_1, x_2) and $(y_1, x_2) \in X \times X$, let
$$d_1((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2)$$
Show that d_1 is a metric on $X \times X$ and show that $d : (X \times X, d_1) \to ([0, \infty), d)$ is continuous.

4. Identifying Metric Spaces

DEFINITION 4.1. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called an *isometry* if for all $x_1, x_2 \in X$, $f(x_1), f(x_2) \in Y$,
$$\rho(f(x_1), f(x_2)) = d(x_1, x_2)$$

EXAMPLE 4.2. Take $f : (\mathbb{R}^2, d) \to (\mathbb{R}^3, d_2)$ given by $f((x_1, x_2)) = (x_1, x_2, 0)$.

EXERCISE 4.3. If $f : (X, d) \to (Y, \rho)$ is an isometry, then

- f is continuous.
- f is one-to-one.

DEFINITION 4.4. Let (X, d) and (Y, ρ) be metric spaces. If there exists a surjective isometry $f : (X, d) \to (Y, \rho)$, then we say that (X, d) is *isometric* to (Y, ρ) and write $(X, d) \equiv (Y, \rho)$.

EXERCISE 4.5. If $f : (X, d) \to (Y, \rho)$ is an onto isometry, then f is invertible and $f^{-1} : (Y, \rho) \to (X, d)$ is also an onto isometry. Furthermore, show that "$\equiv$" is an equivalence relation on the class of metric spaces.

EXAMPLE 4.6. In $\mathbb{R}^2$ with Euclidean distance,

- Consider
$$X = \{(x, y) : x^2 + y^2 = 1\}$$
$$Y = \{(x, y) : (x - 1)^2 + (y - 1)^2 = 1\}$$
Then $X \equiv Y$.

- Consider
$$X = \{(x, y) : y = x^2\} \quad Y = \{(x, y) : x = y^2\}$$
Then $X \equiv Y$.

Definition 4.7. Let (X, d) and (Y, ρ) be a metric space. A function $f : X \to Y$ is called an *homeomorphism* if

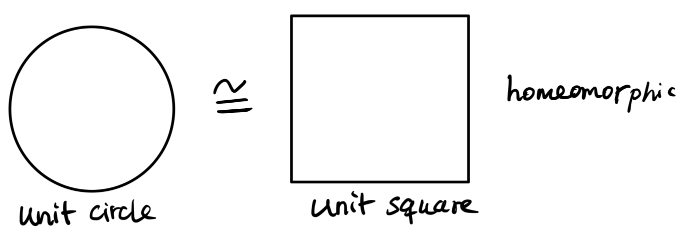
- f is a bijection, in particular, there exists an inverse $f^{-1} : Y \to X$.
- $f : (X, d) \to (Y, \rho)$ is continuous.
- $f^{-1} : (Y, \rho) \to (X, d)$ is continuous.

If such an f exists, we say that (X, d) is *homeomorphic* to (Y, ρ) and write $(X, d) \cong (Y, \rho)$.

Exercise 4.8. The binary relation "$\cong$" is an equivalence relation on the class of metric spaces.

Example 4.9. On $\mathbb{R}^2$ with Euclidean distance, take $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and take $Y = \left\{(x, y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, a, b > 0\right\}$, then $X \cong Y$ and $f((x, y)) = (ax, by)$.

Example 4.10. On $\mathbb{R}^2$, we have the following examples.



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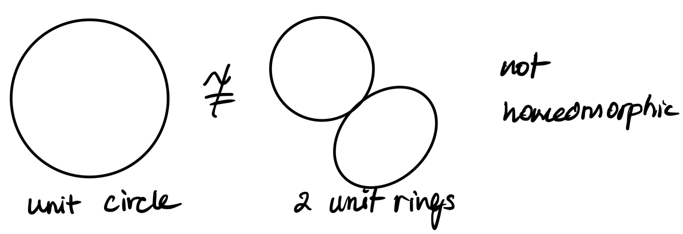

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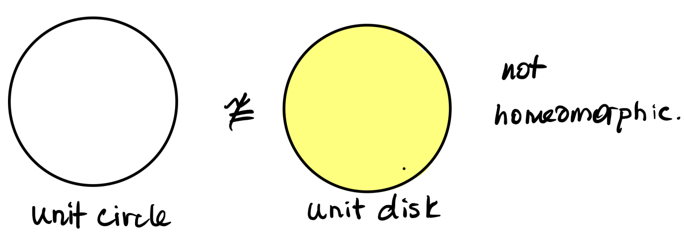

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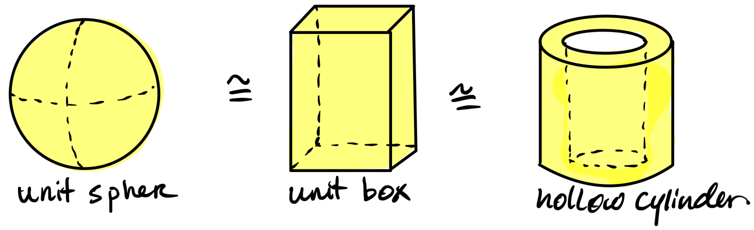
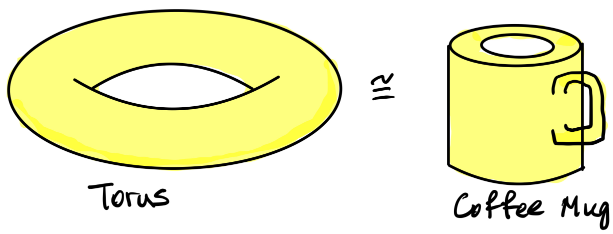
Example 4.11. In $\mathbb{R}^3$, the unit sphere is homeomorphic to a unit rectangular box and is also homeomorphic to the hollow cylinder with



Example 4.12. The torus is homeomorphic to a "coffee mug".



Week 2

September 11-15

LECTURE 2

September 11, 2023

1. Open Sets in Metric Spaces

DEFINITION 1.1. Let (X, d) be a metric space.

- For $x_0 \in X$ and $\epsilon > 0$ define the open ball centered around x_0 with radius ϵ as
$$B(x_0, \epsilon) = \{y \in X : d(x_0, y) < \epsilon\}$$
- A subset $U \subset X$ is called an open set if for every $x_0 \in U$, there exists an $\epsilon > 0$ (that depends on x_0 and U) such that $B(x_0, \epsilon) \subset U$.

EXAMPLE 1.2. In $\mathbb{R}^2$, the open unit disk defined by
$$U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$
is an open set. (Verify this formally)

EXAMPLE 1.3. In $\mathbb{R}^2$, the closed unit disk defined by
$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$
is *not* an open set.

EXERCISE 1.4. In a metric space (X, d), for any $x_0 \in X$ and $\epsilon > 0$, the open ball $B(x_0, \epsilon)$ is an open set.

PROOF. We want to show that $B(x_0, \epsilon)$ is an open set. Indeed, let $y \in B(x_0, \epsilon)$ be arbitrary. We are looking for a $\delta > 0$ such that $B(y, \delta) \subset B(x_0, \epsilon)$. Take $\delta = \epsilon - d(x_0, y)$. (Use the triangle inequality to show that $\delta > 0$ and $B(y, \delta) \subset B(x_0, \epsilon)$). $\square$

EXERCISE 1.5. Let (X, d) be a metric space. Prove the following.

(i) X is open, and the $\emptyset$ is open. (Prove $\emptyset$ is open by using contradiction, assuming that there is a point and $\epsilon > 0$ such that $B(x_0, \epsilon) < \epsilon$ but there are no such points in $\emptyset$ so we have a contradiction)
(ii) If U and V are open sets, then $U \cap V$ is open.
(iii) If $(U_\alpha)_{\alpha \in I}$ is an arbitrary collection of open subsets of X, then $\bigcup_{\alpha \in I} U_\alpha$ is open.

Note that from (ii), by using induction, for any $n \in \mathbb{N}$, and $U_1, U_2, ..., U_n$ is open, then $\bigcap_{i=1}^{n} U_i$ is open.

Proof of (ii). Let U and V be open. Take any $x \in U \cap V$. since $x \in U$ and U is open, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Similarly, since $x \in V$ and V is open, there exists a $\delta > 0$ such that $B(x, d) \subset V$. Take $\eta = \min\{\epsilon, \delta\} > 0$ and show $B(x, \eta) \subset U \cap V$. (Formalize this) □

Theorem 1.6. *Let (X, d) be a metric space, let $x_0 \in X$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. The following are equivalent.*

(i) $\lim_{n \to \infty} x_n = x_0$
(ii) *For every open set U such that $x_0 \in U$, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.*

Proof. To show that (i) implies (ii), take U to be an open set with $x_0 \in U$. We want to find an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. (Formalize this)

To show that (ii) implies (i), let $\epsilon > 0$ be arbitrary, we want to find an $N \in \mathbb{N}$ such that $d(x_0, x_n) < \epsilon$ for all $n \geq N$. Take $U = B(x_0, \epsilon)$. (Formalize this) □

Theorem 1.7. *Let (X, d) and (Y, ρ) be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:*

(i) *$f : (X, d) \to (Y, \rho)$ is continuous.*
(ii) *For every V open subset of Y, $f^{-1}(V)$ is open in X.*

Proof. (i) $\Rightarrow$ (ii): Assume that $f : (X, d) \to (Y, \rho)$ be continuous. Let $V \subset Y$ be an arbitrary subset of Y. We will show that $f^{-1}(V) = U$ is open in X. Take an arbitrary point $x \in U$, we seek a $\delta > 0$ such that $B(x, \delta) \subset U$. Since f is continuous everywhere in X, f is continuous at x. Since $x \in U = f^{-1}(V)$, and so $f(x) \in V$. Furthermore, since V is open, there exists an $\epsilon > 0$ such that $B(f(x), \epsilon) \subset V$. By continuity of f at x, there exists a $\delta > 0$ such that for all $z \in X$ with $d(x, z) < \delta$ implies that $\rho(f(x), f(z)) < \epsilon$. We will now show that for this $\delta > 0$, $B(x, \delta) \subset U$. Take $z_0 \in B(x, \delta)$. We show that $B(x, \delta) \subset U$, i.e. $d(x, z) < \delta$ therefore, $\rho(f(z), f(x)) < \epsilon$ therefore $f(z) \in B(f(x), \epsilon) \subset V$ which implies $f(z) \in V$ and so $z \in f^{-1}(V) = U$.

(ii) $\Rightarrow$ (i): Take $x \in X$ and $\epsilon > 0$. We want to find a $\delta > 0$ such that for all $z \in X$ with $d(x, z) < \delta$, we have $\rho(f(x), f(z)) < \epsilon$. Take the open set $V = B(f(x), \epsilon)$ which is open in Y. By assumption, $U = f^{-1}(V)$ is open in X. Then $x \in U$ because $f(x) \in V$ and therefore, there exists a $\delta > 0$ such that $B(x, \delta) \subset U$. Take $z \in X$ such that $d(x, z) < \delta$, i.e. $z \in B(x, \delta) \subset U$, which implies that $f(z) \in V = B(f(x), \epsilon)$ which implies that $\rho(f(x), f(z)) < \epsilon$ as desired. □

2. Topology

Definition 2.1. Let X be a nonempty set and let $\mathcal{T}$ be a collection of subsets of X, i.e. $\mathcal{T} \subset \mathcal{P}(X)$. If $\mathcal{T}$ satisfies the following, then we call $\mathcal{T}$ a topology on X and $(X, \mathcal{T})$ is called a topological space.

(i) $\emptyset, X \in \mathcal{T}$
(ii) If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.
(iii) If $(U_i)_{i\in I}$ is an arbitrary collection of members of $\mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$

If $\mathcal{T}$ is a topology on X, then the members of $\mathcal{T}$ are called $\mathcal{T}$-open sets (or just open sets)

REMARK 2.2. If $\mathcal{T}$ is a topology on X, then for any $n \in \mathbb{N}$ and $U_1, U_2, ..., U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

EXAMPLE 2.3. For a metric space (X, d) denote

$$\mathcal{T}_d = \{U \subset X : U \text{ is an open set with respect to } d\}$$

This is called the topology induced by the metric d. A topology $\mathcal{T}$ on a set X for which there exists a metric d on X with $\mathcal{T} = \mathcal{T}_d$ is called a metrizable topology.

EXAMPLE 2.4. For any $X \neq \emptyset$, define $\mathcal{T} = \{\emptyset, X\}$ is a topology called the trivial topology on X. It is elementary to see that $\mathcal{T}$ is a topology.

EXAMPLE 2.5. For any nonempty set $X \neq \emptyset$, $\mathcal{T} = \mathcal{P}(X)$ is a topology on X, called the discrete topology on X.

EXERCISE 2.6. Let (X, d) be a discrete metric space. Show that $\mathcal{T}_d = \mathcal{P}(X)$. In particular, the discrete topology on any set is metrizable.

REMARK 2.7. If $(X, \mathcal{T})$ is a metrizable topology, there may be different metrics d, d' on X such that $\mathcal{T} = \mathcal{T}_d = \mathcal{T}_{d'}$. For example, on $\mathbb{Z}$ with the usual metric $d(x, y) = |x - y|$, $\mathcal{T}_d = \mathcal{P}(\mathbb{Z})$ (also given by the discrete metric).

DEFINITION 2.8. A toplogical space $(X, \mathcal{T})$ is called *Hausdorff* if for any $x, y \in X$ with $x \neq y$, there exists disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.

EXAMPLE 2.9. Let $X \neq \emptyset$ with $|X| \geq 2$. Then the trivial topology $\mathcal{T} = \{\emptyset, X\}$ is *not* Hausdorff.

PROPOSITION 2.10. *Any metrizable topological space is Hausdorff.*

PROOF. If $(X, \mathcal{T})$ is metrizable, there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$. Take $x \neq y \in X$. Take $r = d(x, y)$. Take $U = B\left(x, \frac{r}{2}\right)$ and $V = B\left(y, \frac{r}{2}\right)$ are containing x and y, respectively. We claim that $U \cap V = \emptyset$. If not, then there exists a $z \in U \cap V$ such that

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r$$

This is a contradiction. □

EXAMPLE 2.11. Let X be a nonempty set, and let $\mathcal{T}$ to be defined by

$$\mathcal{T} = \{A \subset X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$$

(i) This is a topology on X, called the cofinite topology.
(ii) If X is infinite, then $\mathcal{T}$ is not metrizable.

LECTURE 3

September 13, 2023

1. Topological Spaces

Recall in the previous lecture, we introduced the topological space. We say that a toplogical space $(X, \mathcal{T})$ is called

(1) *Hausdorff* if for every $x, y \in X$ with $x \neq y$, there exists disjoint $U, V \in \mathcal{T}$ with $x \in U$ and $y \in V$.
(2) *Metrizable* if there exists a metric d on X with $\mathcal{T} = \mathcal{T}_d$ is called a metrizable topology.

EXAMPLE 1.1. Let X be a nonempty set and define

$$\mathcal{T} = \{A \subset X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$$

(1) This is a topology on X, called the *cofinite topology.*
(2) If X is infinite, then $\mathcal{T}$ is not Hausdorff, and thus, not metrizable.

Let's verify (ii). We will in fact show that if $U, V \in \mathcal{T}$, both nonempty then $U \cap V \neq \emptyset$. Because $U \neq \emptyset$, then $X \setminus U$ is finite and similarly, $X \setminus V$ is finite and so $X \setminus (U \cap V)$ which is finite. So $U \cap V \neq \emptyset$.

EXAMPLE 1.2. Let X be a nonempty set and define

$$\mathcal{T} = \{A \subset X : X \setminus A \text{ is countable}\} \cup \{\emptyset\}$$

(1) This is a topology on X, called the cocountable topology.
(2) If X is uncountable, $\mathcal{T}$ is not Hausdorff and this not metrizable.

EXAMPLE 1.3. Let $(X, \mathcal{T})$ be a topologyical space and let $A \subset X$ with $A \neq \emptyset$. Define

$$\mathcal{T}|_A = \{U \cap A : U \in \mathcal{T}\}$$

This is the topology on A, called the relative topology induced by $\mathcal{T}$.

EXERCISE 1.4. Verify that $(A, \mathcal{T}|_A)$ is a topological space.

EXERCISE 1.5. If (X, d) is a metric space and $\emptyset \neq A \subset X$, we may define a metric $d|_A$ on A by restricting d on $A \times A$ (for $x, y \in A$, $d|_A(x, y) = d(x, y)$). Show that $(A, \mathcal{T}_{d|_A}) = (A, \mathcal{T}_d|_A)$,

EXERCISE 1.6. If $(X, \mathcal{T})$ is a Hausdorff topological space and let $\emptyset \neq A \subset X$, then $(A, \mathcal{T}|_A)$ is also Hausdorff.

EXERCISE 1.7. If $(X, \mathcal{T})$ is Hausdorff and X is infinite, then there exists an infinite sequence of nonempty pairwise disjoint open subsets of X. Deduce that the cardinality of $\mathcal{T}$ is at least the continuum.

2. Basis of a Topology

PROPOSITION 2.1. *Let (X, d) be a metric space. Then every open subset of X can be expressed as a union of open balls.*

PROOF. Indeed, if U is open and nonempty, then for all $x \in U$, there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset U$. Therefore, $U = \bigcup_{x \in U} B(x, \varepsilon_x)$. (Exercise) $\square$

DEFINITION 2.2. Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B} \subset \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every nonempty $U \in \mathcal{T}$ can be expressed as a union of members of $\mathcal{B}$, i.e. there exists $(U_i)_{i \in I}$ in $\mathcal{B}$ such that $U = \bigcup_{i \in I} U_i$.

EXAMPLE 2.3. If (X, d) is a metric space, then

$$\mathcal{B} = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$$

is a basis for $\mathcal{T}_d$.

EXAMPLE 2.4. If $(X, \mathcal{T})$ is a discrete topological space, i.e. $\mathcal{T} = \mathcal{P}(X)$, then

$$\mathcal{B} = \{\{x\} : x \in X\}$$

is a basis for $\mathcal{T}$.

EXAMPLE 2.5. Take $\mathbb{N}$ with the cofine topology. Then show

$$\mathcal{B} = \{\{m\} \cup \{n, n+1, ...\} : n, m \in \mathbb{N}\}$$

This is a basis for the cofinite topology.

PROPOSITION 2.6. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subset \mathcal{T}$. The following are equivalent:*

(1) *$\mathcal{B}$ is a basis for $\mathcal{T}$.*
(2) *For every $x \in X$, $U \in \mathcal{T}$ with $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U$.*

PROOF. (1) $\Rightarrow$ (2): Assume that $\mathcal{B}$ is a basis and take $x \in X$ and $U \in \mathcal{T}$ containing $x \in U$. By assumption, there exists $(V_i)_{i \in I}$ in $\mathcal{B}$ such that $U = \bigcup_{i \in I} V_i$. Since $x \in U$, there exists $i_0 \in I$ such that $x \in V_{i_0}$. Then because $V_{i_0} \subset U$, it is the member of $\mathcal{B}$ we are looking for.

(2) $\Rightarrow$ (1): Take $\emptyset \neq U \in \mathcal{T}$. By assumption, for each $x \in U$, there exists a $V_x \in \mathcal{B}$ such that $x \in V_x$ and, and $V_x \subset U$. Then $U = \bigcup_{x \in U} V_x$. $\square$

EXAMPLE 2.7. In any metric space (X, d),

$$\mathcal{B} = \{B(x, q) : x \in X, q \in \mathbb{Q} \cap (0, \infty)\}$$

This is a basis for $\mathcal{T}_d$. (Verify)

EXAMPLE 2.8. In $\mathbb{R}$ with the usual topology,

$$\mathcal{B} = \{(q, r) : q < r \in \mathbb{Q}\}$$

is a basis. (Verify)

PROPOSITION 2.9. *Let X be a set with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The following are equivalent*

(1) $\mathcal{T}_1 \subset \mathcal{T}_2$ *($\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ or $\mathcal{T}_2$ is finer than $\mathcal{T}_1$)*
(2) *For every $V \in \mathcal{B}_1$ and $x \in V$, there exists a $U \in \mathcal{B}_2$ with $x \in U$ and $U \subset V$*

EXERCISE 2.10. Let X be a nonempty set and d_1, d_2 be metrics on X such that for all $x, y \in X$, $d_1(x,y) \leq d_2(x,y)$. Then $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$. For example, on $C[0,1] = \{\text{all } f : [0,1] \to \mathbb{R}\}$ continuous take $d_\infty(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$ and $d_1(f,g) = \int_0^1 |f(t) - g(t)|dt$. Examine whether $\mathcal{T}_d$ and $\mathcal{T}_\infty$ compare and if yes, how.

EXAMPLE 2.11. On $\mathbb{R}^n$ define $d_\infty(x,y)$ for $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ by taking

$$d_\infty = \max_{1\leq i\leq n} |x_i - y_i|$$

Then $\mathcal{B} = \{(a_1, , b_1) \times \cdots \times (a_n, b_n) : a_1 < b_1 \in \mathbb{R}, a_2 < b_2 \in \mathbb{R}^2, ...a_n < b_n \in \mathbb{R}\}$ is a basis for d_∞.

Also if $\mathcal{T}$ denotes the Euclidean topology on $\mathbb{R}^n$ and show $\mathcal{T}_{d_\infty} = \mathcal{T}$

DEFINITION 2.12. Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{F} \subset \mathcal{T}$ is called a *subbasis* for $\mathcal{T}$ if the collection

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^n V_i : n \in \mathbb{N}, V_1, ..., V_n \in \mathcal{F} \right\}$$

is a basis for $\mathcal{T}$.

EXAMPLE 2.13. In $\mathbb{R}^n$, denote $\mathcal{F}$ to be

$$\mathcal{F} = \bigcup_{i=1}^n \{\mathbb{R} \times \cdots \times (a,b) \times \mathbb{R} \times \mathbb{R} : a < b \in \mathbb{R}\}$$

Then $\mathcal{B}(\mathcal{F})$ contains the family $\mathcal{B}$ from the previous example and thus $\mathcal{F}$ is called a subbasis of $\mathbb{R}^n$

Note that anything that is labelled in blue are stuff that weren't mentioned during the lecture or requires clarification.

THEOREM 2.14. *Let X be a nonempty set and let $B \subset \mathcal{P}(X)$ be such that*

(1) *If $x \in X$, then there exists a $B \in \mathcal{B}$ such that $x \in B$*
(2) *If $x \in X$ and $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$ then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$, $B_3 \subset B_1$ and $B_3 \subset B_2$.*

Let $\mathcal{T}_\mathcal{B}$ be the set of all subsets U of X such that for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Then $\mathcal{T}_\mathcal{B}$ is a topology on X such that $\mathcal{B} \subset \mathcal{T}$.

Proof. To see that $\mathcal{T}_{\mathcal{B}}$ is a topology, we must verify the three properties of a topology. Clearly, $\emptyset \in \mathcal{T}_{\mathcal{B}}$. To see that $X \in \mathcal{T}_{\mathcal{B}}$, recall that for each $x \in X$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x$. As $B_x \subset X$ by definition, we obtain that $X \in \mathcal{T}_{\mathcal{B}}$ by the definition of $\mathcal{T}_{\mathcal{B}}$.

Next, suppose that $(U_i)_{i\in I}$ is a collection of elements of $\mathcal{T}_{\mathcal{B}}$. To see that $\bigcup_{i\in I} U_i \in \mathcal{T}_{\mathcal{B}}$, let $x \in \bigcup_{i\in I} U_i$ be arbitrary. Then there exists a $i_0 \in I$ such that $x \in U_{i_0}$. Since $U_{i_0} \in \mathcal{T}_{\mathcal{B}}$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U_{i_0}$. Hence, $B \subset U \subset \bigcup_{i\in I} U_i$. As $x \in \bigcup_{i\in I} U_i$ was arbitrary, we obtain that $\bigcup_{i\in I} U_i \in \mathcal{T}_{\mathcal{B}}$ by definition.

Finally, suppose $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$. To see that $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{B}}$, let $x \in U_1 \cap U_2$ be arbitrary. Hence, $x \in U_1$ and $x \in U_2$ so as $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$, there exists $B_k \in \mathcal{B}$ (either $k = 1$ or 2) such that $x \in B_k$ and $B_k \subset U_k$. By applying (2) for another time, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset B_k$. Hence, $B \in \mathcal{B}$, $x \in B$, and $B \subset B_k \subset U_k$ so that $B \subset U_1 \cap U_2$. Therefore, as $x \in X$ was arbitrary, $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{B}}$ as desired.

Finally, the fact that $\mathcal{B} \subset \mathcal{T}_{\mathcal{B}}$ follows from the definition of $\mathcal{T}_{\mathcal{B}}$. □

Definition 2.15. Let X be a nonempty set. A *basis for a topology on* X is a collection of subsets $\mathcal{B} \subset \mathcal{P}(X)$ such that

(1) If $x \in X$, then there exists a $B \in \mathcal{B}$ such that $x \in B$
(2) If $x \in X$ and $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1$ and $B_3 \subset B_2$.

Example 2.16. Let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$. We claim that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. To see this, it suffices to verify the two defining properties of being a basis from Definition 2.15. To begin, notice if $x \in \mathbb{R}$, $x \in [x, x + 1) \in \mathcal{B}$. Hence, the first property is satisfied. To see the second property, let $[a_1, b_1), [a_2, b_2) \in \mathcal{B}$ and $x \in \mathbb{R}$ such that $x \in [a_1, b_1) \cap [a_2, b_2)$ be arbitrary. Let $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$ and let $B = [a, b)$. Since $x \in [a_1, b_1) \cap [a_2, b_2)$, we see that $a \leq x < b$, so $B \in \mathcal{B}$ and $x \in B$. Furthermore by construction, $B \subset [a_1, b_1) \cap [a_2, b_2)$. Hence, since $[a_1, b_1), [a_2, b_2) \in \mathcal{B}$ and $x \in \mathbb{R}$ were arbitrary, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

Theorem 2.17. *Let X be a nonempty set and let $\mathcal{B}$ be a basis for a topology on X. Then*

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{B\in\mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\}$$

Proof. Nice snce $\mathcal{T}_{\mathcal{B}}$ is a topology and since $\mathcal{B} \subset \mathcal{T}_{\mathcal{B}}$, we know that for all $\mathcal{B}_0 \subset \mathcal{B}$ that $\bigcup_{B\in\mathcal{B}_0} B$ is a union of elements of $\mathcal{T}_{\mathcal{B}}$ and thus in $\mathcal{T}_{\mathcal{B}}$. Hence,

$$\left\{ \bigcup_{B\in\mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\} \subset \mathcal{T}_{\mathcal{B}}$$

To show the other inclusion, let $U \in \mathcal{T}_{\mathcal{B}}$ be arbitrary. By the definition of $\mathcal{T}_{\mathcal{B}}$, for each $x \in U$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. Hence, we see that

$$U = \bigcup_{x \in U} B_x$$

Therefore, as $U \in \mathcal{T}_{\mathcal{B}}$ was arbitrary, we obtain that

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{B \in \mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\}$$

as desired. □

DEFINITION 2.18. Let $(X, \mathcal{T})$ be a topological space. A set $\mathcal{B} \subset \mathcal{P}(X)$ is said to be a *basis for* $(X, \mathcal{T})$ if $\mathcal{B}$ is a basis for a topology on X and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$.

REMARK 2.19. Of course, $\mathcal{B} \subset \mathcal{T}_{\mathcal{B}}$ for any basis of a topology $\mathcal{B}$, for a set $\mathcal{B} \subset \mathcal{P}(X)$ to be a basis for a topology $\mathcal{T}$, it must be the case that $\mathcal{B} \subset \mathcal{T}$. Furthermore, by Theorem 2.14 and Theorem 2.17, we see that if $\mathcal{B}$ is a basis for $(X, \mathcal{T})$, then

(1) A set $U \subset X$ is open if and only if for every $x \in U$ there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$
(2) The open sets in $(X, \mathcal{T})$ are exactly the union of elements of $\mathcal{B}$.

EXAMPLE 2.20. Let X be a nonempty set and let $\mathcal{T}$ be the discrete topology on X. Then

$$\mathcal{B} = \{\{x\} : x \in X\}$$

is a basis for $(X, \mathcal{T})$. Indeed, $\mathcal{B} \subset \mathcal{T}$ as $\mathcal{T}$ is the discrete topology. Next, clearly the first property from Definition 2.15 holds and the second property also holds since the only was $x \in X$ and $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$ is if $B_1 = B_2 = \{x\} \in \mathcal{B}$. Hence, $\mathcal{B}$ is a basis for $(X, \mathcal{T})$.

EXAMPLE 2.21. Let (X, d) be a metric space. Then the set $\mathcal{B}$ of all open balls forms a basis for $(X, \mathcal{T}_d)$. Indeed clearly, $\mathcal{B} \subset \mathcal{T}$ and if $x \in X$ then $B(x, 1) \in \mathcal{T}_d$ so the first property of Definition 2.15 is satisfied. To see the second property of Definition 2.15 is satisfied, let $x \in X$ and $B_1, B_2 \in \mathcal{B}$ be arbitrary such that $x \in B_1 \cap B_2$. Then there exists points $x_1, x_2 \in X$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $B_1 = B(x_1, \varepsilon_1)$ and $B_2 = B(x_2, \varepsilon_2)$. Thus, as $x \in B_1 \cap B_2$, we see that

$$d(x, x_1) < \varepsilon_1 \quad d(x, x_2) < \varepsilon_2$$

Let

$$\varepsilon = \min\{\varepsilon_1 - d(x, x_1), \varepsilon_2 - d(x, x_2)\}$$

Then $\varepsilon > 0$. We see that $B(x, \varepsilon) \subset B(x_1, \varepsilon_1)$ and $B(x, \varepsilon) \subset B(x_2, \varepsilon_2)$. Hence, as $x \in X$ and $B_1, B_2 \in \mathcal{B}$ were arbitrary, the second property of Definition 2.15 is satisfied, so $\mathcal{B}$ is a basis for $(X, \mathcal{T}_d)$.

EXAMPLE 2.22. Let (X, d) be a metric space and let $\varepsilon > 0$. The set $\mathcal{B}$ of all open balls with radius at most ε forms a basis for $(X, \mathcal{T}_d)$. Indeed the proof is identical to that of Example 2.21 with the additional restraint that all radii involved are at most ε.

PROPOSITION 2.23. *Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{B} \subset \mathcal{T}$ has the property that for all $U \in \mathcal{T}$ and for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Then $\mathcal{B}$ is a basis for $(X, \mathcal{T})$.*

PROOF. To see that $\mathcal{B}$ is a basis for a topology on X, we will simply verify the two properties of Definition 2.15. Let $x \in X$ be arbitrary. Then as $X \in \mathcal{T}$, the assumptions of the proposition imply there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset X$. Hence, as $x \in X$ was arbitrary, the first assumption of Definition 2.15 is satisfied.

To see the second property of Definition 2.15 holds, let $x \in X$ and $B_1, B_2 \in \mathcal{B}$ be such that $x \in B_1 \cap B_2$ be arbitrary. As $\mathcal{B} \subset \mathcal{T}$, we see that $B_1, B_2 \in \mathcal{T}$ and thus $B_1 \cap B_2 \in \mathcal{T}$. Therefore, the assumptions of the proposition there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. Hence, as $x \in X$ and $B_1, B_2 \in \mathcal{B}$ were arbitrary, the second property of Definition 2.15 are satisfied. Thus, $\mathcal{B}$ is a basis for a topology on X.

To see that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$, we first note that $\mathcal{B} \subset \mathcal{T}$ and as $\mathcal{T}$ is closed under unions, Theorem 2.17 imply that

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{B \in \mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\} \subset \mathcal{T}$$

Conversely, if $U \in \mathcal{T}$, then the assumptions of the proposition imply that $U \in \mathcal{T}_{\mathcal{B}}$ by Definition 2.15. Hnece $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ as desired. $\square$

COROLLARY 2.24. *Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{B} \subset \mathcal{T}$ has the property that for every $U \in \mathcal{T}$ there exists a subset $\mathcal{B}_0 \subset \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{B}_0} B$. Then $\mathcal{B}$ is a basis for $(X, \mathcal{T})$.*

PROOF. To prove this result, we will verify that the assumption of Proposition 2.23 holds. To see this, let $U \in \mathcal{T}$ and $x \in U$ be arbitrary. Then by the assumptions of this corollary, there exists a subset $\mathcal{B}_0 \subset \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{B}_0} B$. Hence as $x \in U$, there exists a $B_x \in \mathcal{B}_0$ such that $x \in B_x$ and $B_x \subset \bigcup_{B \in \mathcal{B}_0} B = U$. Therefore, as $U \in \mathcal{T}$ and $x \in U$ were arbitrary, the assumption of Proposition 2.23 holds. Hence, the result follows. $\square$

DEFINITION 2.25. Let $(X, \mathcal{T})$ be a topological space. A *subbasis* for $(X, \mathcal{T})$ is a ocllection of subsets $\mathcal{B}_{\mathcal{S}} \subset \mathcal{T}$ such that the set of all finite intersections of elements of $\mathcal{B}_{\mathcal{S}}$ is a basis for $(X, \mathcal{T})$. For a set $\mathcal{B}_{\mathcal{S}}$ to be a subbasis on some topology $\mathcal{T}$ on X, it is necessary that

$$X = \bigcup_{S \in \mathcal{B}_{\mathcal{S}}} S$$

THEOREM 2.26. *Let X be a nonempty set and let $\mathcal{B}_{\mathcal{S}} \subset \mathcal{P}(X)$ be such that*

$$X = \bigcup_{S \in \mathcal{B}_{\mathcal{S}}} S$$

Let $\mathcal{B} \subset \mathcal{P}(X)$ be the set of all finite intersections of elements of $\mathcal{B}_{\mathcal{S}}$. Then $\mathcal{B}$ is a basis for a topology on X for which $\mathcal{B}_{\mathcal{S}}$ is a subbasis.

PROOF. To see that $\mathcal{B}$ is a basis for a topology on X, we only need to check the two conditions from Definition 2.15. For the first, let $x \in X$ be arbitrary. Since $X = \bigcup_{S \in \mathcal{B}_{\mathcal{S}}} S$, there exists an $S_x \in \mathcal{B}_{\mathcal{S}}$ such that $x \in S_x$. As $\mathcal{B}_{\mathcal{S}} \subset \mathcal{B}$, the first property of being a basis holds for $\mathcal{B}$.

For the second property, let $x \in X$ and $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$ be arbitrary. Since $B_1, B_2 \in \mathcal{B}$, B_1 and B_2 are the finite intersections of elements of $\mathcal{B}_{\mathcal{S}}$. Hence, $B_1 \cap B_2$ is a finite intersection of elements of $\mathcal{B}_{\mathcal{S}}$ so that $B_1 \cap B_2 \in \mathcal{B}$. Therefore, as $x \in X$ and $B_1, B_2 \in \mathcal{B}$ were arbitrary, the second property of being a basis holds for a topology on X. The fact that $\mathcal{B}_{\mathcal{S}}$ is a subbasis is a subbasis for $\mathcal{T}_{\mathcal{B}}$ is then trivial. $\square$

3. Creating Topologies

THEOREM 3.1. *Let X be a nonempty set, $\mathcal{W}$ be a collection of susets of X, then define*

$$\mathcal{T} = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members} \right\} \cup \{X, \emptyset\}$$

The following are equivalent:

(1) *$\mathcal{T}$ is a topology on X and $\mathcal{W} \cup \{X\}$ is a basis for $\mathcal{T}$*
(2) *For every $A, B \in \mathcal{W}$, and $x \in A \cap B$ there exists a $C \in \mathcal{W}$ with $x \in C$ and $C \subset A \cap B$. In particular, if $\mathcal{F}$ is any collection of subsets of X then the collection*

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} A_i : n \in \mathbb{N}, A_1, A_2, ..., A_n \in \mathcal{F} \right\} \cup \{X\}$$

is a basis for the topology

$$\mathcal{T}(\mathcal{F}) = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members of } \mathcal{B}(\mathcal{F}) \right\} \cup \{X, \emptyset\}$$

called the topology generated by $\mathcal{F}$.

PROOF. (2) $\Rightarrow$ (1): We take $\mathcal{F} \subset \mathcal{P}(X)$ and show that for any $A, B \in \mathcal{B}(\mathcal{F})$, $A \cap B \in \mathcal{B}(\mathcal{F})$. This easily implies that $\mathcal{B}(\mathcal{F})$ satisfies assumption (2) and is a basis for $\mathcal{T}(\mathcal{F})$. As $A \in \mathcal{B}(\mathcal{F})$, there are $A_1, A_2, ..., A_n \in \mathcal{F}$ such that $A = \bigcap_{i=1}^{n} A_i$ and similarly, there are $B_1, B_2, ..., B_m$ with $B = \bigcap_{j=1}^{m} B_j$. Therefore, $A \cap B = (\bigcap_{i=1}^{n} A_i) \cap \left(\bigcap_{j=1}^{m} B_j\right)$, which is in $\mathcal{B}(\mathcal{F})$ as finite intersection of members of $\mathcal{F}$. $\square$

We will prove (1) $\Rightarrow$ (2) next time.

Week 3

September 18-22

LECTURE 4

September 18, 2023

Recall if $(X, \mathcal{T})$ be a topological space. Let $\mathcal{F} \subset \mathcal{T}$ is called a *subbasis* if

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} V_i : n \in \mathbb{N}, V_1, V_2, ..., V_n \in \mathcal{F} \right\}$$

is a basis for $\mathcal{T}$.

Example 0.1. In $\mathbb{R}$ with the usual topology

$$\mathcal{F} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$$

is a subbasis for $\mathcal{T}$. [Listen back to the lecture later...]

1. Creating Topologies

Theorem 1.1. *Let X be a nonempty set, $\mathcal{W}$ be a collection of subsets of X, then define*

$$\mathcal{T} = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members} \right\} \cup \{X, \emptyset\}$$

The following are equivalent:

1. *$\mathcal{T}$ is a topology on X and $\mathcal{W} \cup \{X\}$ is a basis for $\mathcal{T}$*
2. *For every $A, B \in \mathcal{W}$, and $x \in A \cap B$ there exists a $C \in \mathcal{W}$ with $x \in C$ and $C \subset A \cap B$. In particular, if $\mathcal{F}$ is any collection of subsets of X then the collection*

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} A_i : n \in \mathbb{N}, A_1, A_2, ..., A_n \in \mathcal{F} \right\} \cup \{X\}$$

is a basis for the topology

$$\mathcal{T}(\mathcal{F}) = \left\{ \bigcup_{i \in I}^{n} V_i : (V_i)_{i \in I} \text{ is an arbitrary collection of members of } \mathcal{B}(\mathcal{F}) \right\} \cup \{X, \emptyset\}$$

called the topology generated by $\mathcal{F}$.

Proof. We will prove (1) $\Rightarrow$ (2): Take $A, B \in \mathcal{W}$ and $x \in A \cap B$. Since $\mathcal{T}$ is a topology and $\mathcal{W} \subset \mathcal{T}$, we have that $A, B \in \mathcal{T}$, $A \cap B \in \mathcal{T}$ as well. Since $\mathcal{W} \cup \{X\}$ is a basis, there exists some $C \in \mathcal{W} \cup \{X\}$ such that $x \in C$ and $C \subset A \cap B$. We claim that $C \in \mathcal{W}$.

Next we will prove (2) $\Rightarrow$ (1): We first show that $\emptyset, X \in \mathcal{T}$, which is obvious by definition. Let $(U_i)_{i\in I}$ be an arbitrary collection of members of $\mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$. If $U, V \in \mathcal{T}$, we need to show that $U \cap V \in \mathcal{T}$. If $U = \emptyset$ or $V = \emptyset$, then $U \cap V = \emptyset$, and if $U = X$ or $V = X$, trivial as well. Without loss of generality, there exists collections $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ of members of $\mathcal{W}$ such that $U = \bigcup_{i\in I} U_i$ and $V = \bigcup_{j\in J} V_j$. Take $x \in U \cap V$. We will find $O_x \in \mathcal{W}$ such that $x \in O_x$ and $O_x \subset U \cap V$. If we find this, we will find that $U \cap V = \bigcup_{x\in U\cap V} O_x \in \mathcal{T}$. Since $x \in U$, then there exists an $i_x \in I$ such that $x \in U_{i_x} \in \mathcal{W}$, and since $x \in V$, then there exists an $j_x \in J$ such that $x \in V_{j_x} \in \mathcal{W}$. Thus, since $x \in U_{i_x} \cap V_{j_x}$. By (2), there exists an $O_x \in \mathcal{W}$ such that $x \in O_x \in \mathcal{W}$ and $O_x \subset U_{i_x} \cap V_{j_x} \subset U \cap V$. $\square$

REMARK 1.2. Let X be a nonempty set and $\mathcal{F} \subset \mathcal{P}(X)$ such that $\bigcup \mathcal{F} = X$. Then

$$\mathcal{B}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} U_i : n \in \mathbb{N}, U_1, U_2, ..., U_n \in \mathcal{F} \right\}$$

is a basis for the topology

$$\mathcal{T}(\mathcal{F}) = \left\{ \bigcup_{i\in I} U_i : (U_i)_{i\in I} \text{ is a collection of members of } \mathcal{B}(\mathcal{F}) \right\} \cup \{\emptyset\}$$

Recall that if $\mathcal{F} \subset \mathcal{P}(X)$, then $\mathcal{T}(\mathcal{F})$ is called the topology generated by $\mathcal{F}$. If $\bigcup \mathcal{F} = X$, then $\mathcal{F}$ is a subbasis for a topology.

EXAMPLE 1.3. Let $\mathcal{B}$ be the following collection of subsets of $\mathbb{R}$:

$$\mathcal{B} = \{[a, b) : a < b \in \mathbb{R}\}$$

The topology generated by $\mathcal{B}$ is called the *lower limit topology of* $\mathbb{R}$, denoted by $\mathcal{T}_\ell$. Sometimes $(\mathbb{R}, \mathcal{T}_\ell)$ is called the Sorgenfrey Line.

EXERCISE 1.4.
(1) Show that $(\mathbb{R}, \mathcal{T}_\ell)$ is Hausdorff.
(2) Show that $\mathcal{B} = \{[a, b) : a < b \in \mathbb{R}\}$ is a basis for $\mathcal{T}_\ell$.
(3) If $\mathcal{T}$ is the usual topology on $\mathbb{R}$, then $\mathcal{T} \subset \mathcal{T}_\ell$.

REMARK 1.5. If X is a nonempty set and $\mathcal{F}_1 \subset \mathcal{F}_2$, then $\mathcal{T}(\mathcal{F})_1 \subset \mathcal{T}(\mathcal{F})_2$.

PROPOSITION 1.6. *Let X be a nonempty set and let $\mathcal{F} \subset \mathcal{P}(X)$. Then*

$$\mathcal{T}(\mathcal{F}) = \bigcap \{\mathcal{T} : \mathcal{T} \textit{ is a topology of } X, \mathcal{F} \subset \mathcal{T}\}$$

Thus, $\mathcal{T}(\mathcal{F})$ is the smallest topology on X containing $\mathcal{F}$.

PROOF. Denote

$$\mathscr{T} = \{\mathcal{T} : \mathcal{T} \text{ is a topology on } X \text{ and } \mathcal{F} \subset \mathcal{T}\} \neq \emptyset$$

since $\mathcal{T} = \mathcal{P}(X)$ is a topology and $\mathcal{F} \subset \mathcal{T}$, and so $\mathcal{T} \in \mathscr{T}$. Next, denote

$$\mathscr{S} = \bigcap \mathscr{T} = \{A \subset X : \text{for all } \mathcal{T} \in \mathscr{T},\, A \in \mathcal{T}\}$$

Then $\mathscr{S}$ is a topology and $\mathcal{F} \subset \mathscr{S}$. We show that $\mathscr{S} \subset \mathcal{T}(\mathcal{F})$ which is true because $\mathcal{T}(\mathcal{F}) \in \mathscr{T}$.

Secondly, we show that $\mathcal{T}(\mathcal{F}) \subset \mathscr{S}$. Since $\mathcal{F} \cup X$ is a subbasis of $\mathcal{T}(\mathcal{F})$, we get that every member of $\mathcal{T}(\mathcal{F})$ is produced by taking finite intersections and then arbitrary unions of members of $\mathcal{F} \subset \mathscr{S}$, thus it is also a member of $\mathscr{S}$. □

2. Elementary Concepts of Topology

DEFINITION 2.1. Let $(X, \mathcal{T})$ be a topological space.

(1) A subset U of X is called open if $U \in \mathcal{T}$.
(2) A subset F of X is called *closed* if its complement is open, i.e. $X \setminus F$ is open.

EXAMPLE 2.2. In $\mathbb{R}$ with the usual topology,

(1) $(0, 1)$ is an open set.
(2) $[0, 1]$ is an closed set because $\mathbb{R} \setminus [0, 1] = (-\infty, 0) \cup (1, \infty)$ which is open.
(3) The set $\mathbb{R}$ is open and closed, or, a *clopen* set.
(4) The set $\mathbb{Q}$ is neither open or closed.

EXERCISE 2.3. In $(\mathbb{R}, \mathcal{T}_\ell)$, for $a, b \in \mathbb{R}$ with $a < b$, then $[a, b)$ is a clopen set.

PROPOSITION 2.4. *Let $(X, \mathcal{T})$ be a topological space. The following hold:*

(1) *The sets $\emptyset$ and X are closed.*
(2) *If F and G are closed, then $F \cap G$ are closed.*
(3) *If $(F_i)_{i \in I}$ is an arbitrary collection of closed subsets of X, then $\bigcap_{i \in I} F_i$ is closed.*

PROOF. This is an immediate application of definition of a topology and De Morgan's Law. □

PROPOSITION 2.5. *If $(X, \mathcal{T})$ is Hausdorff, then for every $x_0 \in X$, $\{x_0\}$ is closed.*

PROOF. We show that $X \setminus \{x_0\}$ is open. Take $x \in X \setminus \{x_0\}$, i.e. $x \neq x_0$. By Hausdorff, there exists a U, V open and disjoint such that $x \in U$ and $x_0 \in V$. In particular, $x \in U \subset X \setminus \{x_0\}$, therefore, $X \setminus \{x_0\}$ is open. □

DEFINITION 2.6. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. An $x_0 \in X$ is called a

(1) isolated point of A if $x_0 \in A$ and there exists a $U \in \mathcal{T}$ containing x_0 such that $U \cap A = \{x_0\}$.
(2) limit point of A if every $U \in \mathcal{T}$ containing x_0 intersects A, i.e. $U \cap A \neq \emptyset$.
(3) cluster point of A if it is a limit point of $A \setminus \{x_0\}$.

EXAMPLE 2.7. In $\mathbb{R}$, for $A = (0,1] \cup \{2\}$, 2 is an isolated point and a limit point of A, but not a cluster point of A and any $x \in [0,1]$ is a limit point and cluster point of A, but not an isolated point of A. Note that $x = 0$ is a limit and cluster point of A, despite not being in A.

PROPOSITION 2.8. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, $A \subset X$ and $x \in X$.*

(1) *x is an isolated point of A if and only if $a \in A$ and there exists $U \in \mathcal{T}$ containing x such that $U \cap A$ is finite.*
(2) *x is a cluster point of A if and only if every for every $U \in \mathcal{T}$ containing x, $U \cap A$ is infinite.*

PROOF. If x is an isolated point of A, then $x \in A$ and there exists $U \in \mathcal{T}$ containing x such that $U \cap A = \{x\}$ which is finite. If on the other hand, $x \in A$, and there exists $U \in \mathcal{T}$ containing x such that $U \cap A = \{x_1, x_2, ..., x_n\}$, where $x_1, x_2, ..., x_n$ are finitely many points. As $x \in U \cap A$ we may assume without loss in generality, that $x_1 = x$. Because $\mathcal{T}$ is Hausdorff, there are pairs of disjoint open sets U_i and V_i with $2 \leq i \leq n$ such that $x_1 \in U_i$ and $x_i \in V_i$. Define $W = U \cap (\bigcap_{i=2}^{n} U_i)$, which contains x_1, but none of the $x_2, ..., x_n$ and is a subset of U. Thus, $x \in W \cap A \subset (U \cap A) \setminus \{x_2, ..., x_n\} = \{x\}$, i.e. $W \cap A = \{x\}$. $\square$