Day 9

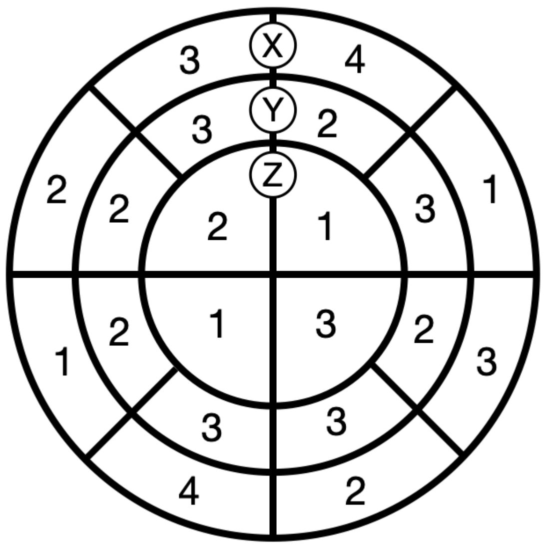
1. Exam from day 8
2. Monty Hall paradox
3. Craps!

Test 1 : MATH/EECS 4161 ::: OPEN BOOK, OPEN NOTES, CLOSED FRIENDS AND ENEMIES ::: Febraury 1, 2024

NAME: ____

Q 1	Q 2	Q 3	TOTAL
35	45	20	100

(1) (5 pts each) Answer the questions about the random variables defined by the following wheel.



a) Is X independent of Y? Why or why not?:

b) Is X dependent on Y and Z? Why or why not?:

c) Is Z dependent on X and Y? Why or why not?:

d) $P(X = 1 \text{ and } Y = 2) =$

e) $P(Z = 1 \text{ or } Y = 2) =$

f) $P(X = 1|Y = 2) =$

g) $P(Y = 2|X = 1) =$

(2) (5 pts each) Alice and Bob keep a common passbook and agree on which key to use on a given day from that book according to a schedule. Eve obtains the passbook, but does not know how they agree on the key. As far as she knows, the key is chosen with equal probability from the list of words below:

`BRAT`	`PRINT`	`RAT`	`ASH`
`JUMP`	`FRONT`	`PIT`	`TOP`
`WANT`	`STRIP`	`WET`	`LET`
`WITH`	`NORTH`	`LIP`	`HUH`

Let X represent the random variable whose value is the length of the keyword.
Let Y represent the random variable whose value is the first letter of the keyword.
Let Z represent the random variable whose value is the last letter of the keyword.

a) What is the probability that the key is a three letter word?:

b) What is the probability that the key is a three letter word or the last letter is T?:

c) What is the probability that the key is a three letter word and the last letter is T?:

d) What is the probability that first letter of the word is W given the last letter is T?:

e) What is the probability that the key is of length 3 given that the first letter is in the first half of the alphabet (begins with an A through M)?

f) Are Y and Z independent as random variables? Why or why not?

g) Are X and Y independent as random variables? Why or why not?

h) Are X and Z independent as random variables? Why or why not?

i) Is X dependent on Y and Z? Why or why not?

(3) (20 pts) We know that 2.5% of the population has AB-positive blood type. There is a breathalyzer test that can be used to determine if someone has AB-positive blood type or not. This test is good, but not 100% accurate because it has a 1% false positive rate and a 3% false negative rate (that is, someone that does not have AB-positive blood type will test positive 1% of the time; someone that does have AB-positive blood type will test negative 3% of the time).

If a random person takes this breathalyzer test and it indicates a positive test, what is the probability that the person actually has AB-positive blood type?

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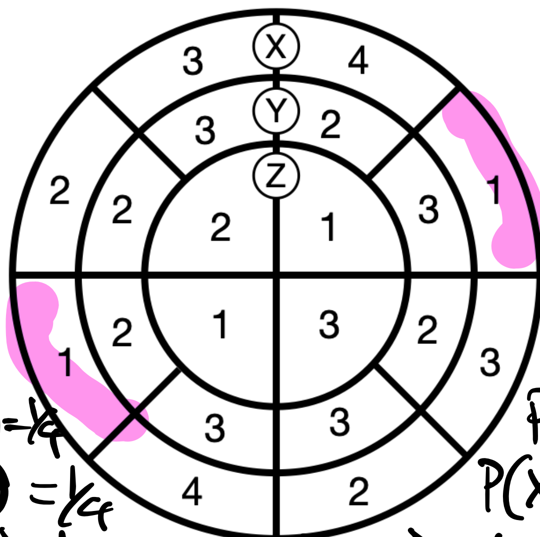
(1) (5 pts each) Answer the questions about the random variables defined by the following wheel.



$P(X=a) = P(X=a|Y=b)$ for all $a \in \{1,2,3,4\}$ and $b \in \{2,3\}$

$P(X=4) = 1/4$ $\quad P(X=4|Y=2) = P(X=4|Y=3) = 1/4$

$P(X=3) = 1/4$ $\quad P(X=3|Y=2) = P(X=3|Y=3) = 1/4$

$P(X=2) = 1/4$ $\quad P(X=2|Y=2) = 1/4$ $\quad P(X=2|Y=3) = 1/4$

Other

a) Is X independent of Y? Why or why not?:

Yes $\quad P(X=1) = 1/4$ $\quad P(X=1|Y=2) = 1/4$ $\quad P(X=1|Y=3) = 1/4$

N

b) Is X dependent on Y and Z? Why or why not?:

No. if $Y=3$ and $Z=1$ then $X = 1$ OR 4

Y

c) Is Z dependent on X and Y? Why or why not?:

Yes. $\quad (X,Y) = (1,2) \Rightarrow Z=1$ $\quad (2,2),(3,3) \Rightarrow Z=2$ $\quad (4,2),(1,3) \Rightarrow Z=1$

$(4,3) \Rightarrow Z=1$ $\quad (2,3)(3,2) \Rightarrow Z=3$

Y

d) $P(X=1 \text{ and } Y=2) =$ $1/8$

1/4

e) $P(Z=1 \text{ or } Y=2) =$ $3/4$

3/4

f) $P(X=1|Y=2) =$ $1/4$

1/2

g) $P(Y=2|X=1) =$ $1/2$

1

(2) (5 pts each) Alice and Bob keep a common passbook and agree on which key to use on a given day from that book according to a schedule. Eve obtains the passbook, but does not know how they agree on the key. As far as she knows, the key is chosen with equal probability from the list of words below:

BRAT	PRINT	RAT	ASH
JUMP	FRONT	PIT	TOP
WANT	STRIP	WET	LET
WITH	NORTH	LIP	HUH

Let X represent the random variable whose value is the length of the keyword.
Let Y represent the random variable whose value is the first letter of the keyword.
Let Z represent the random variable whose value is the last letter of the keyword.

1/2

a) What is the probability that the key is a three letter word?: 1/2

3/4

b) What is the probability that the key is a three letter word or the last letter is T?:

12/16 = 3/4

1/4

c) What is the probability that the key is a three letter word and the last letter is T?:

4/16 = 1/4

0

d) What is the probability that first letter of the word is W given the last letter is T?:

2/8 = 1/4

1/2

e) What is the probability that the key is of length 3 given that the first letter is in the first half of the alphabet (begins with an A through M)?

4/7

N

f) Are Y and Z independent as random variables? Why or why not?

No. $P(Y=W) = 3/16$ $P(Y=W \mid Z=T) = 1/4$

Y

g) Are X and Y independent as random variables? Why or why not?

No. $P(Y=P) = 2/16$ $P(Y=P \mid X=4) = 0$

N

h) Are X and Z independent as random variables? Why or why not?

Yes. $P(Z=a \mid X=b) = P(Z=a)$ $\forall a \in \{T,P,H\}$ $b \in \{3,4,5\}$

N

i) Is X dependent on Y and Z? Why or why not?

No. if $Y=W$ and $Z=T$ then $X=4$ or 3

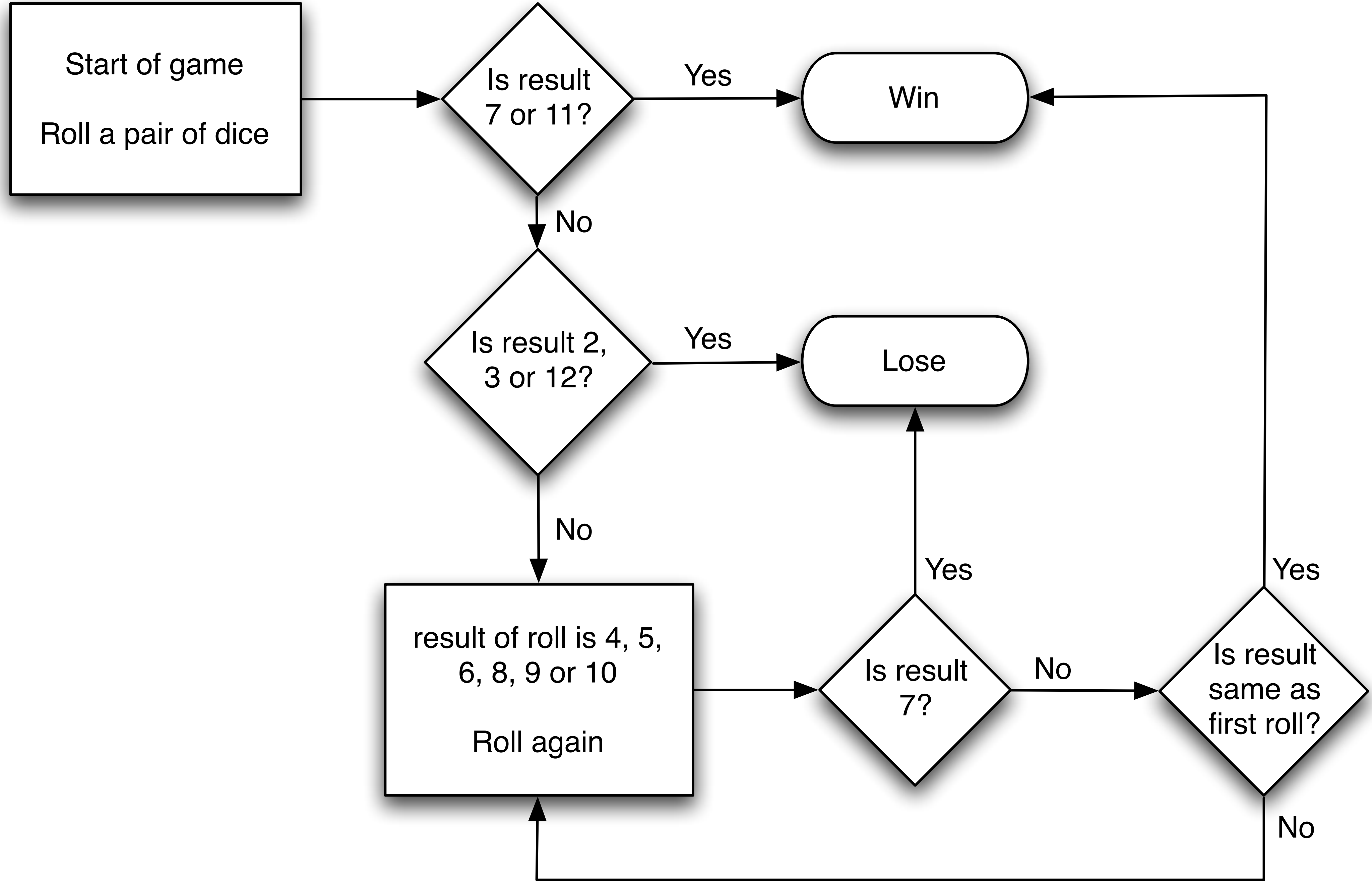
(3) (20 pts) We know that 2.5% of the population has AB-positive blood type. There is a breathalyzer test that can be used to determine if someone has AB-positive blood type or not. This test is good, but not 100% accurate because it has a 1% false positive rate and a 3% false negative rate (that is, someone that does not have AB-positive blood type will test positive 1% of the time; someone that does have AB-positive blood type will test negative 3% of the time).

If a random person takes this breathalyzer test and it indicates a positive test, what is the probability that the person actually has AB-positive blood type?

.4585

$$P(\text{have AB+ blood type} \mid \text{test positive}) = \frac{P(\text{have AB+ \& test+})}{P(\text{test+})}$$

$$= \frac{.025 \cdot .97}{.025 \cdot .97 + .975 \cdot .01} = .71323529$$



Start of game
Roll a pair of dice
Is result 7 or 11?
Yes
Win
No
Is result 2, 3 or 12?
Yes
Lose
No
result of roll is 4, 5, 6, 8, 9 or 10
Roll again
Is result 7?
Yes
No
Is result same as first roll?
Yes
No

$E(X) = 1 \cdot .49\overline{29} - 1 \cdot .5\overline{07}$

$= -.014\ldots$

Name of Bet	Description	Payoff per $1	P(Win)	P(Lose)	House Advantange
Pass Bet	2,3,12 -lose 7,11-win 4,5,6,8,9,10- this is the point shooter rolls again until either 7 or the point comes up if point is first then win if 7 is first then lose	$1	.492929	.507070	1.4%
Don't Pass Bet	2,3- win 12-roll again 7,11- lose 4,5,6,7,9,10- this is the point shooter rolls again until either 7 or the point comes up if the point is first then lose if 7 is first then win	$1			
Field Bet	The next roll is 2,3,4,9,10,11,12-win 5, 6, 7, 8-lose	$2 for 2 or 12 $1 otherwise			
Any Craps	The next roll is 2, 3, 12-win 4, 5, 6, 7, 8, 9, 10, 11- lose	$7			
Any 7	The next roll is a 7- win 2,3,4,5,6,8,9,10,11,12- lose	$4			
Big 6	If a 6 is rolled before a 7-win If a 7 is rolled before a 6-lose	$1			
Big 8	If a 8 is rolled before a 7-win If a 7 is rolled before a 8-lose	$1			
4 Hardway	If a pair of twos is rolled before a 7 or before a 1 and 3- win otherwise lose	$7			
10 Hardway	If a pair of fives is rolled before a 7 or before a 4 and a 6- win otherwise lose	$7			
6 Hardway	If a pair of threes is rolled before a 7 or before 2&4 or 1&5-win otherwise lose	$9			
8 Hardway	If a pair of fours is rolled before a 7 or before 2&6 or 3&5-win otherwise lose	$9			

PROBABILITY MEASURE: Our experiment $\mathbb{E}$ associates to each event A of F a number P[A] in the interval [0, 1] which is reflects our confidence that the outcome falls in A. We refer to P[A] as the "probability of A." Note that we should have P[Ω] = 1 and that if A and B are mutually exclusive events then

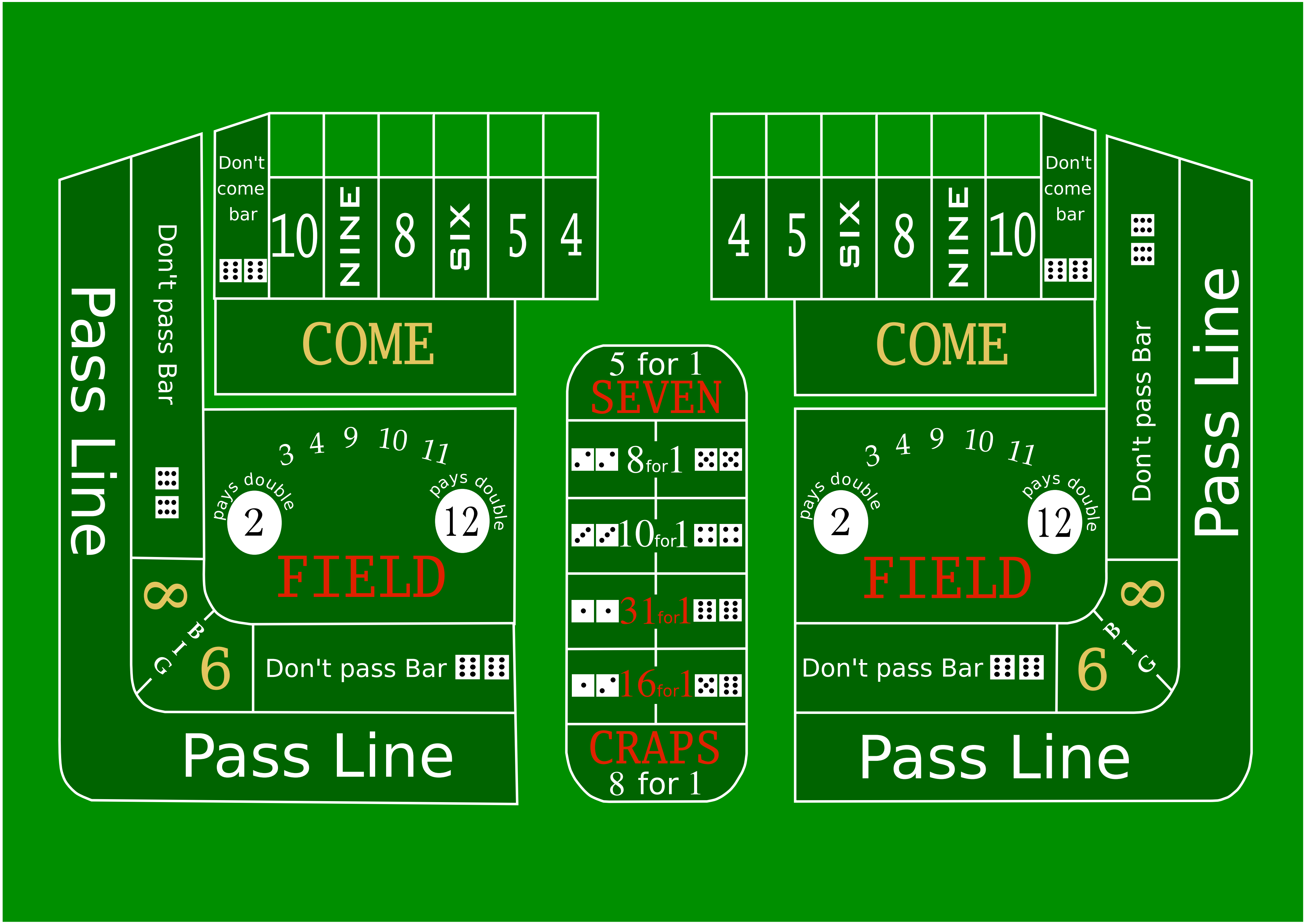
$$P[A \cup B] = P[A] + P[B]$$

A set function with these properties is usually referred to as a PROBABILITY MEASURE.

EXPECTATION OF A RANDOM VARIABLE: Any function of the outcome of our experiment can be referred to as a RANDOM VARIABLE. Mathematically, a random variable is simply a function on the sample space. If the events A_1, A_2, . . . , A_k are mutually exclusive and decompose Ω , and the random variable X takes the value x_i when A_i occurs then the expression

$$E[X] = x_1 P[A_1] + x_2 P[A_2] + \cdots + x_k P[A_k]$$

is referred to as the EXPECTATION OF X . If we repeat E a very large number of times, and average out the successive values of X we get, then we should expect the resulting average to be close to E[X].



Pass Line
Don't pass Bar
Don't come bar
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NINE
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SIX
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4
COME
3 4 9 10 11
pays double
2
12
FIELD
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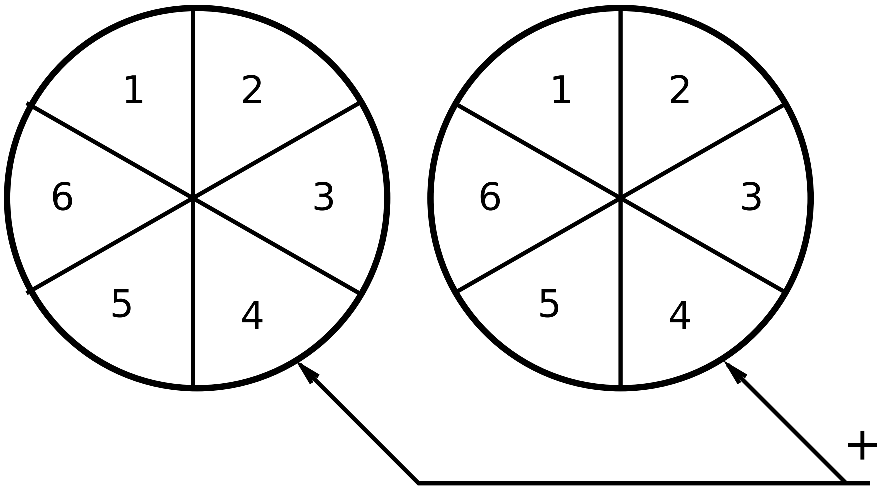
BIG
6
Don't pass Bar
Pass Line
5 for 1
SEVEN
8 for 1
10 for 1
31 for 1
16 for 1
CRAPS
8 for 1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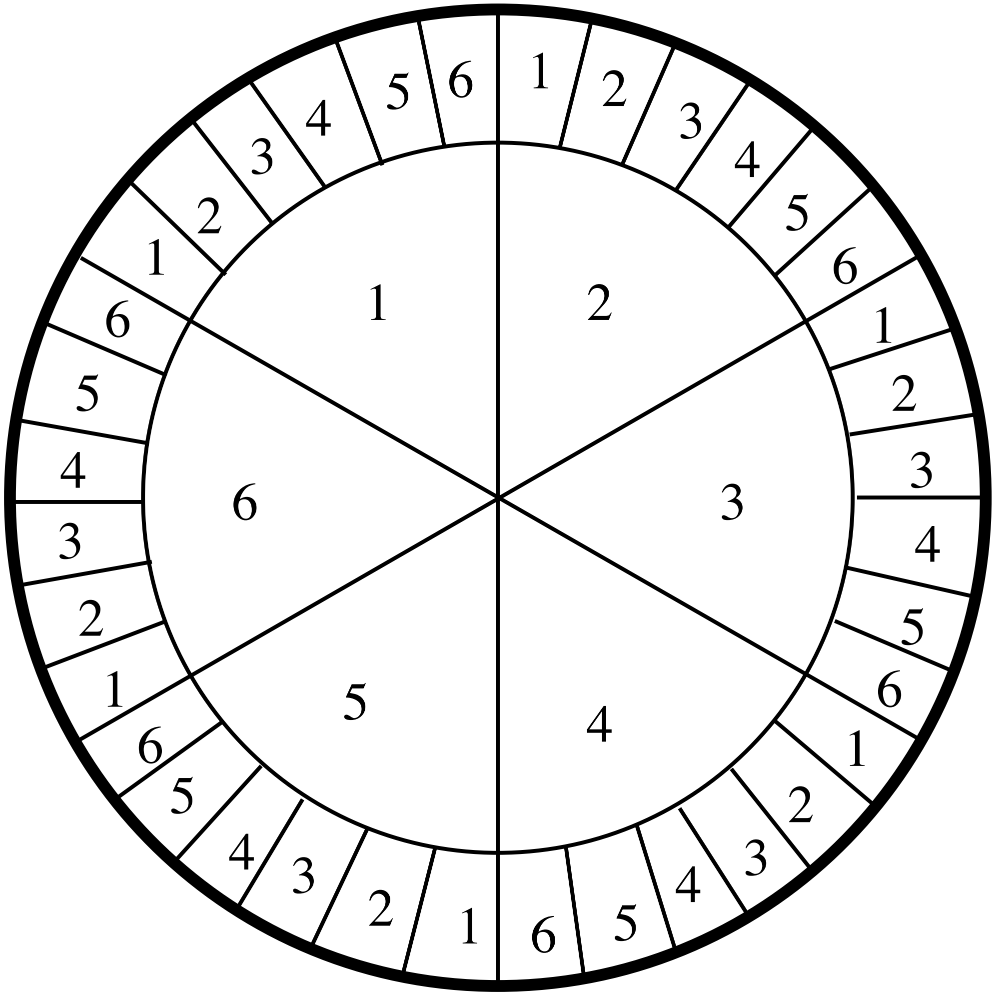
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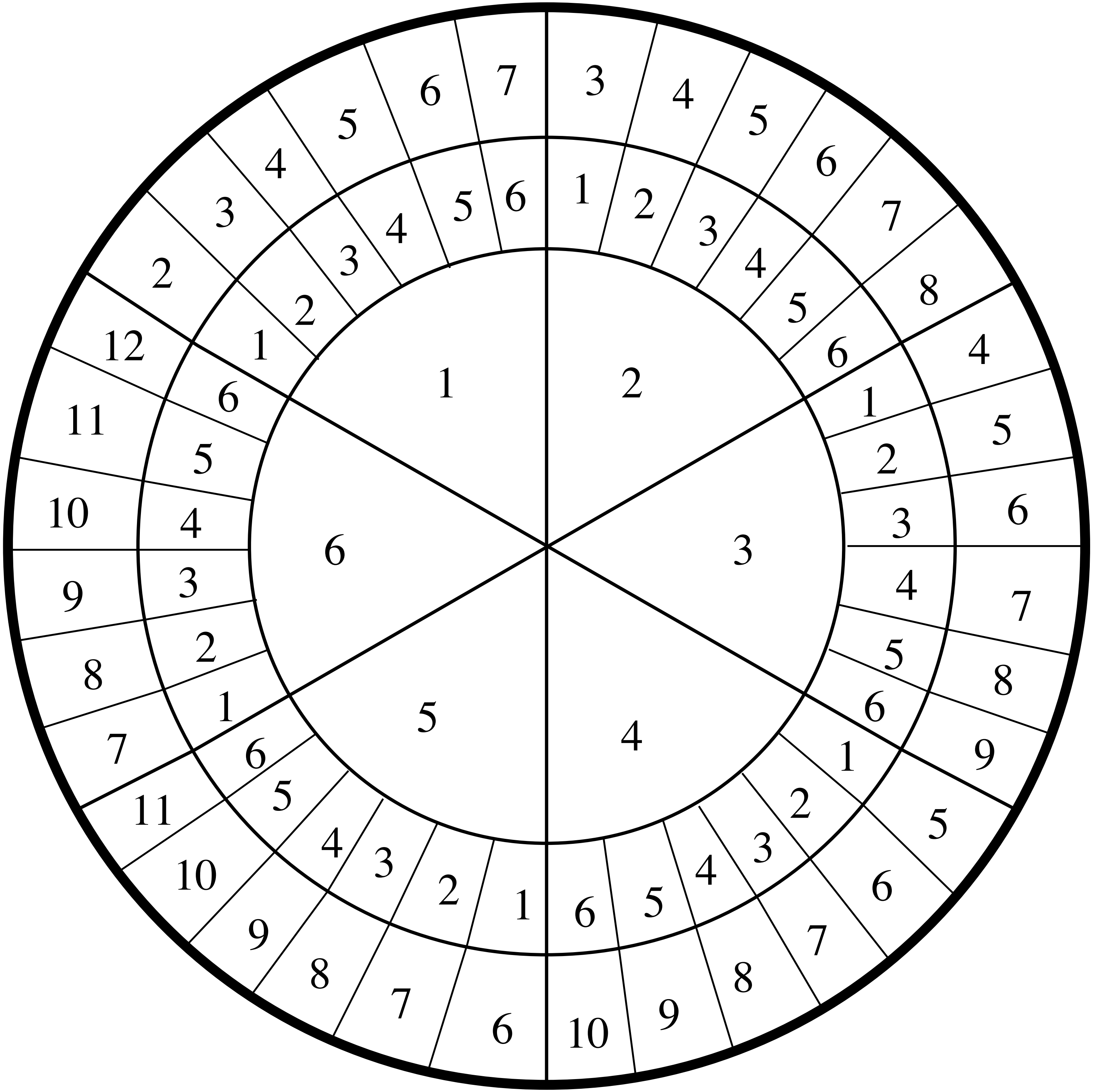

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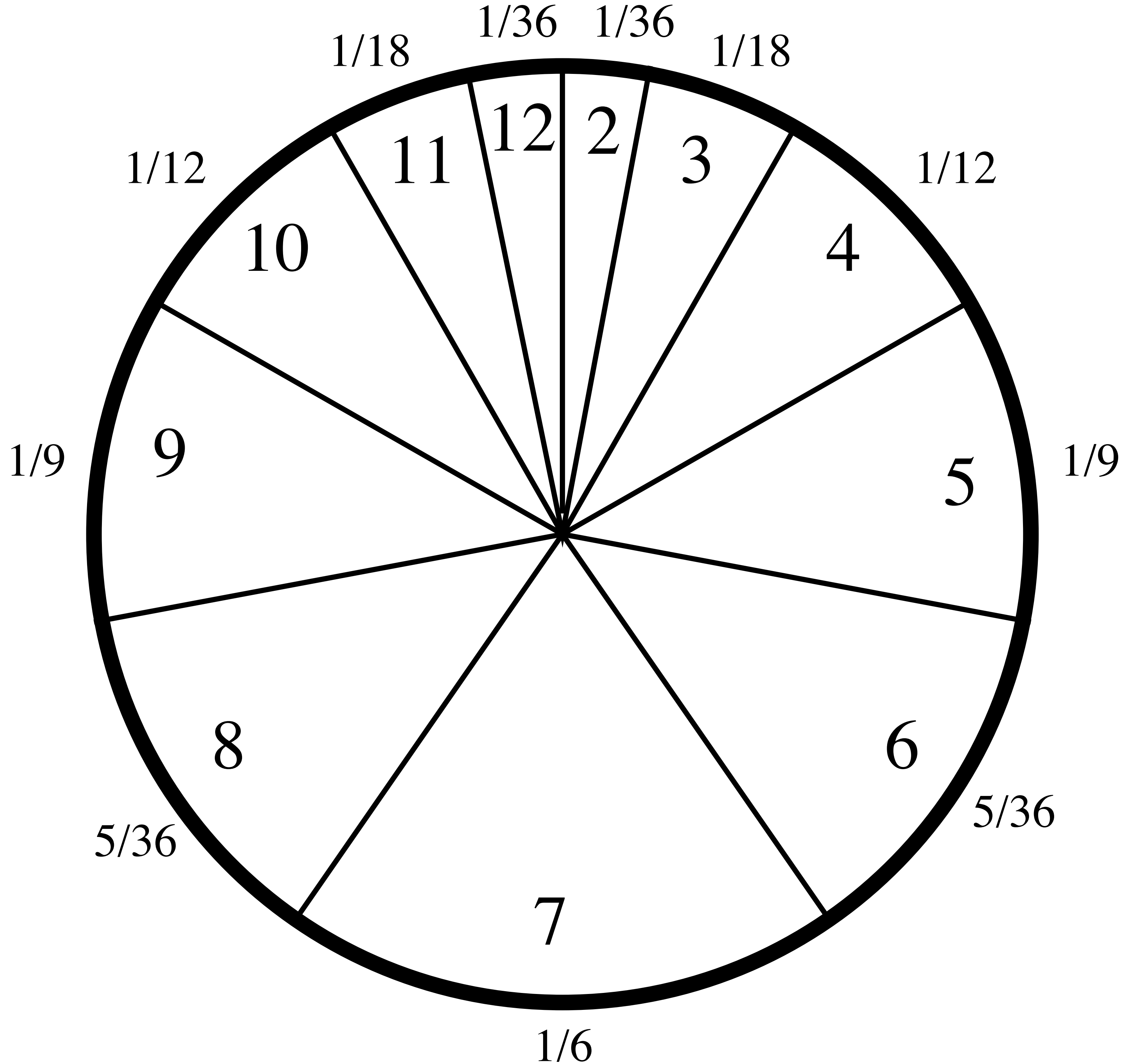

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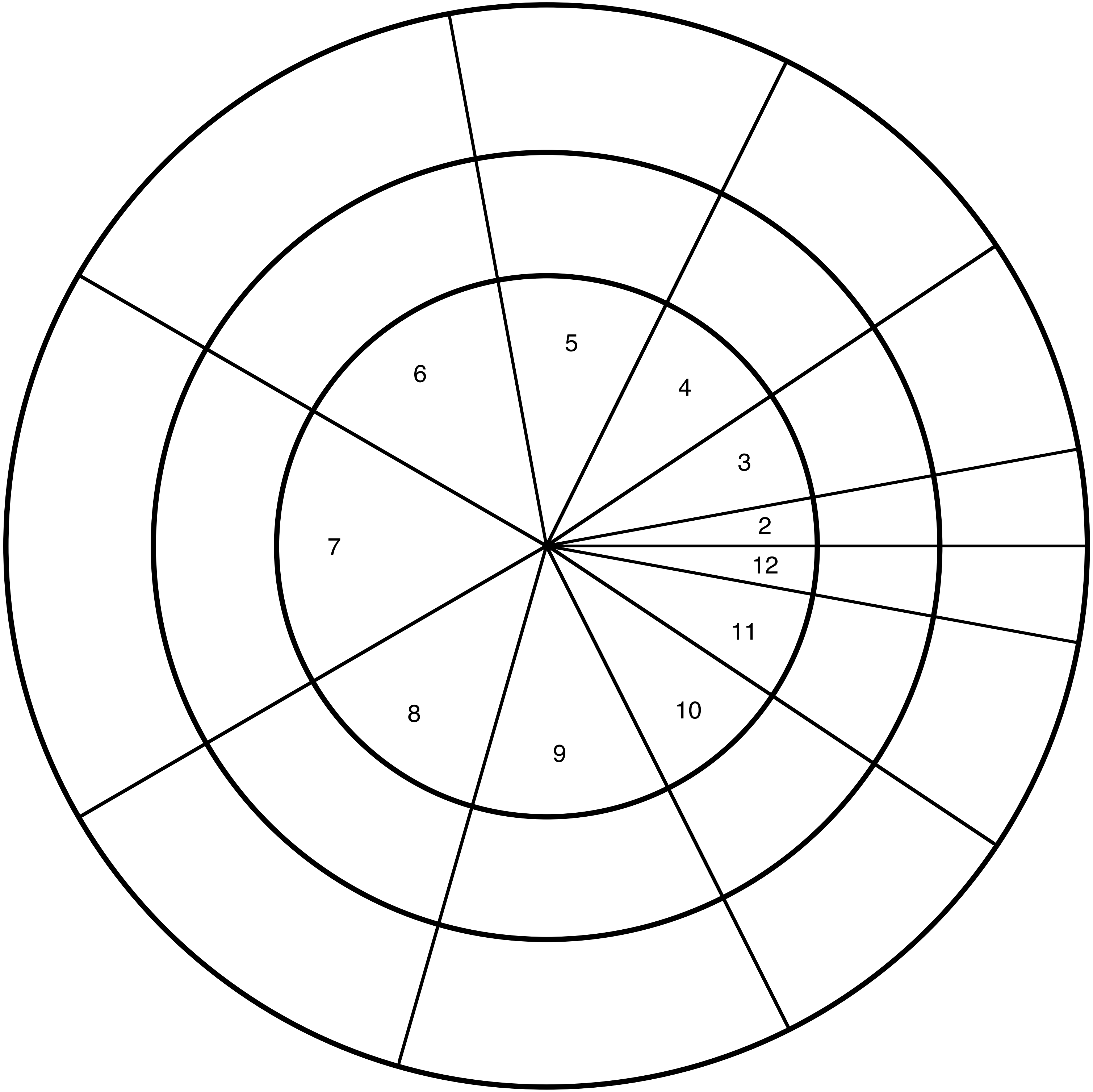

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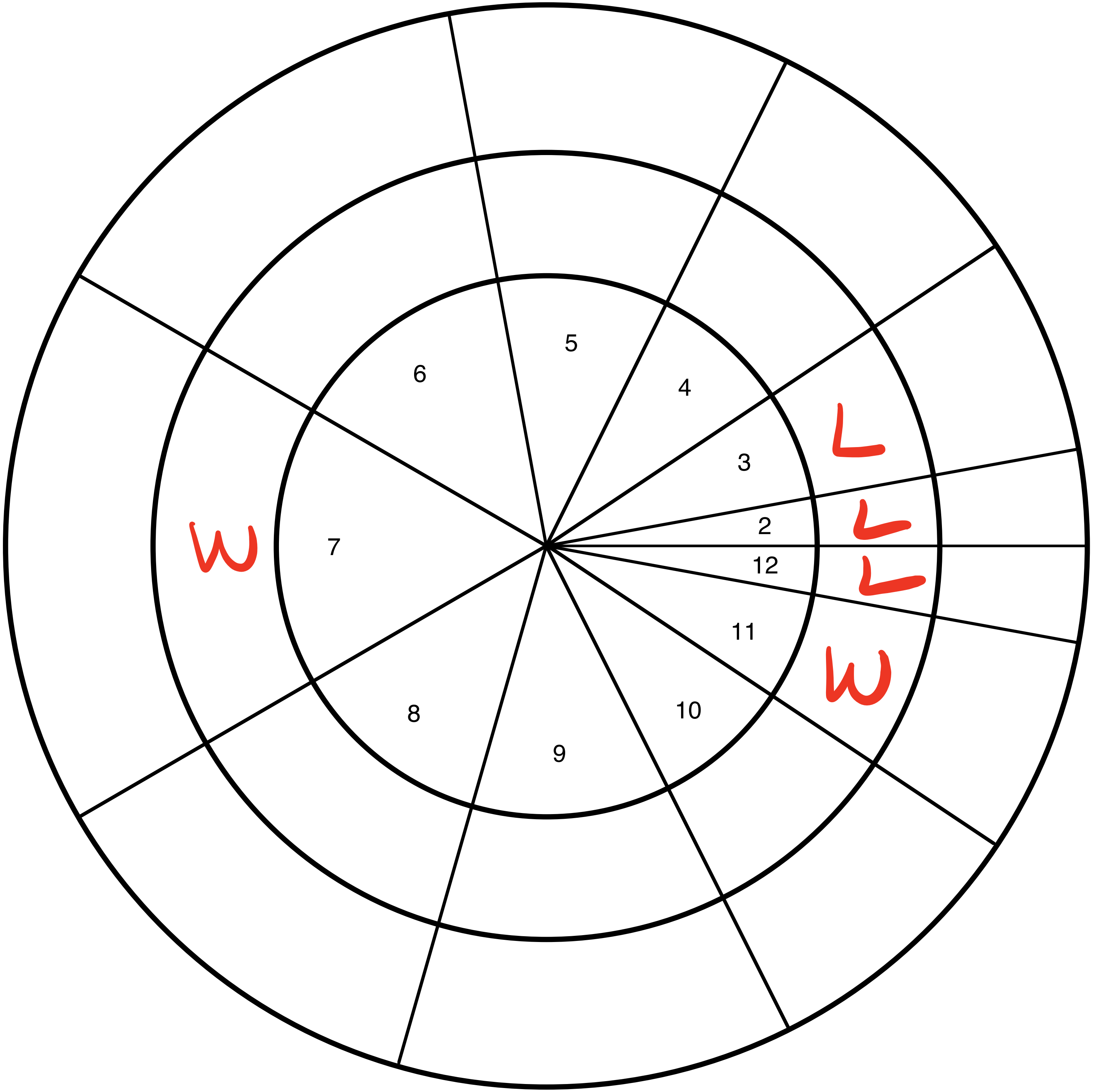
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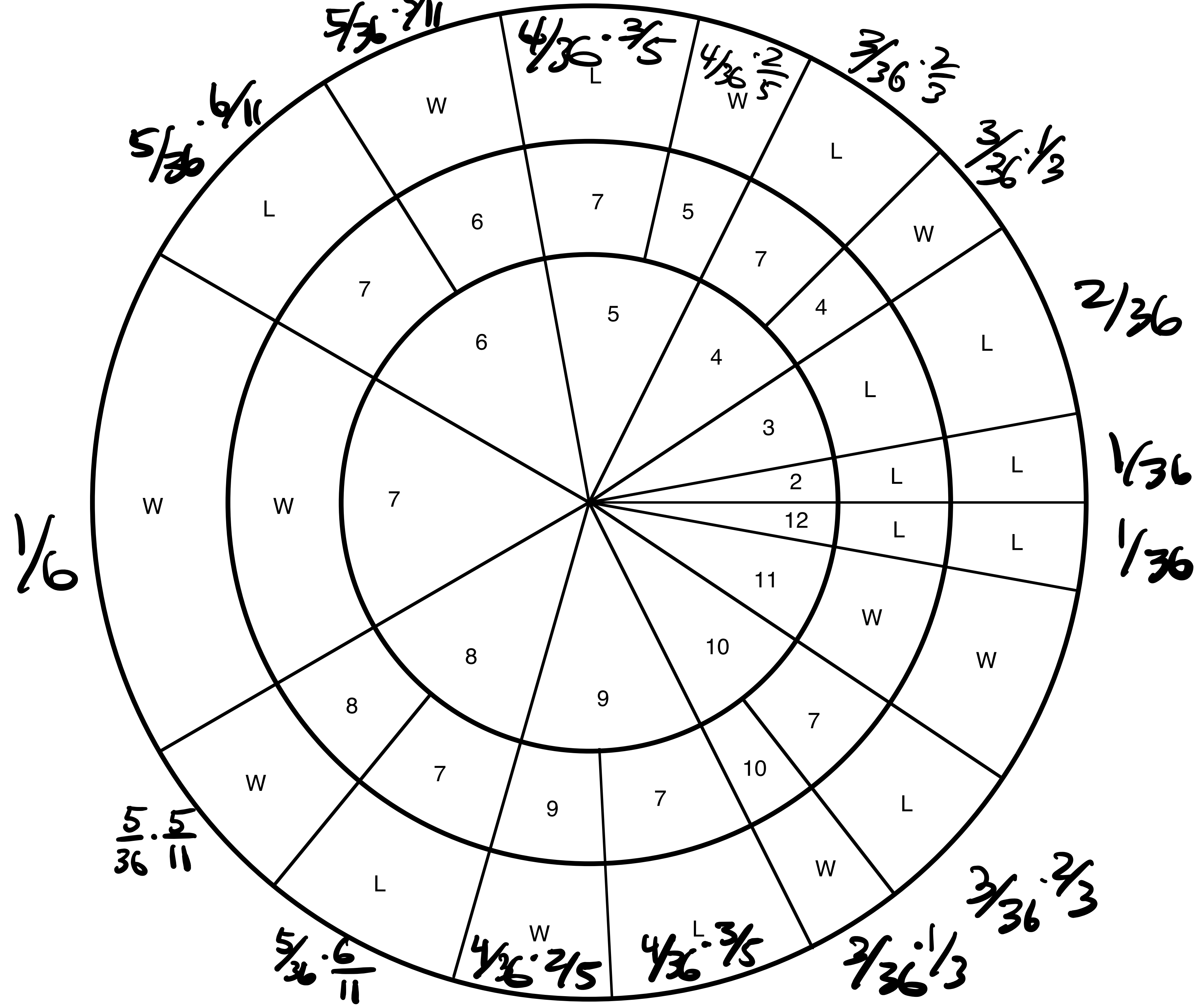

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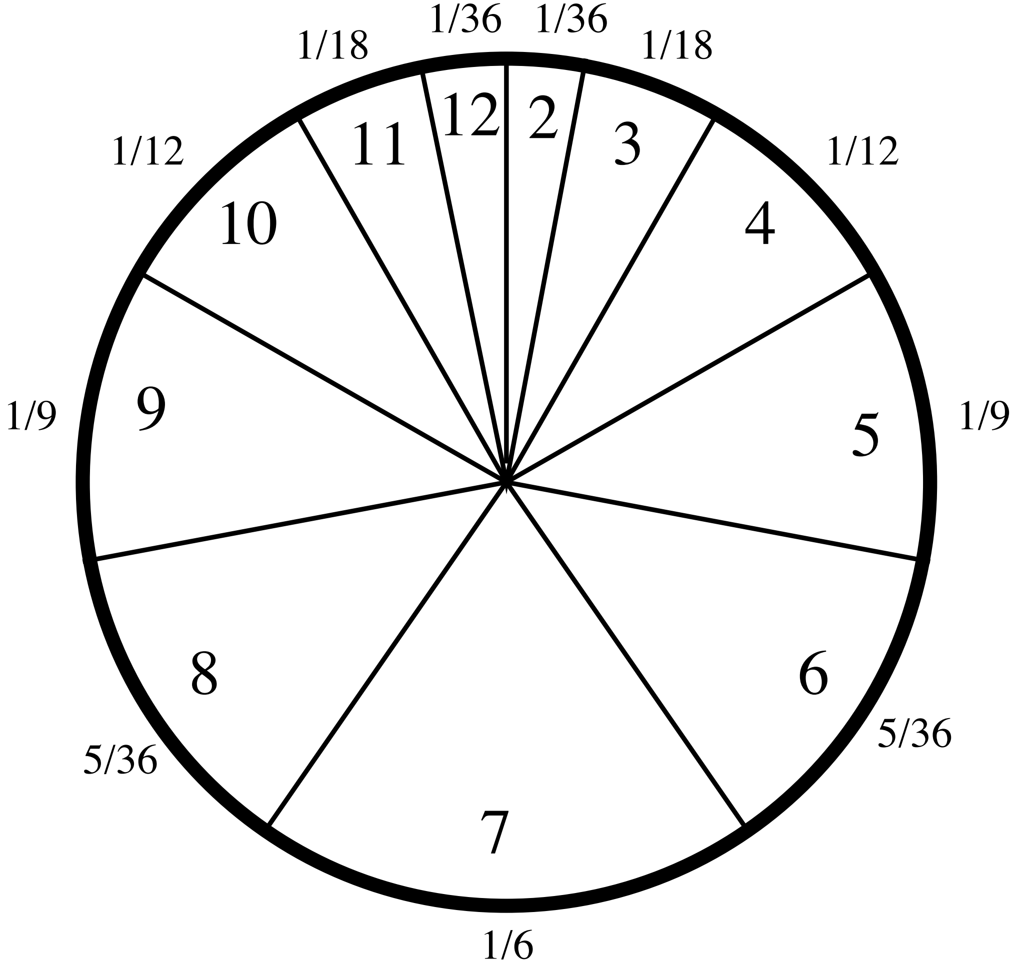

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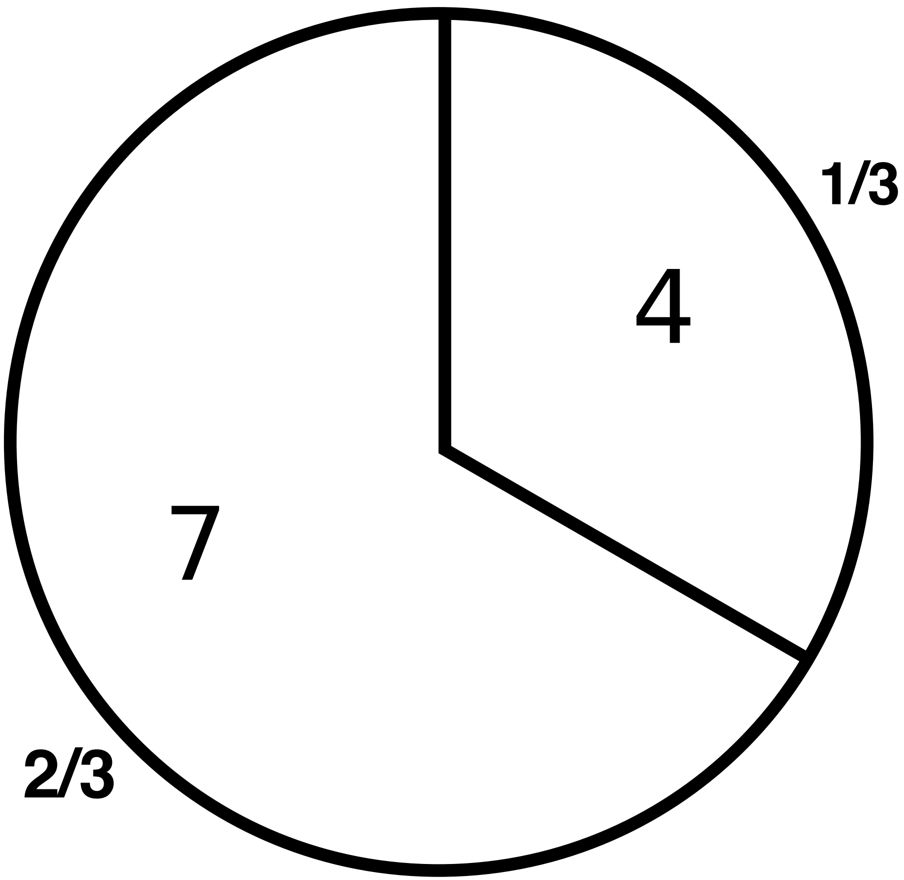

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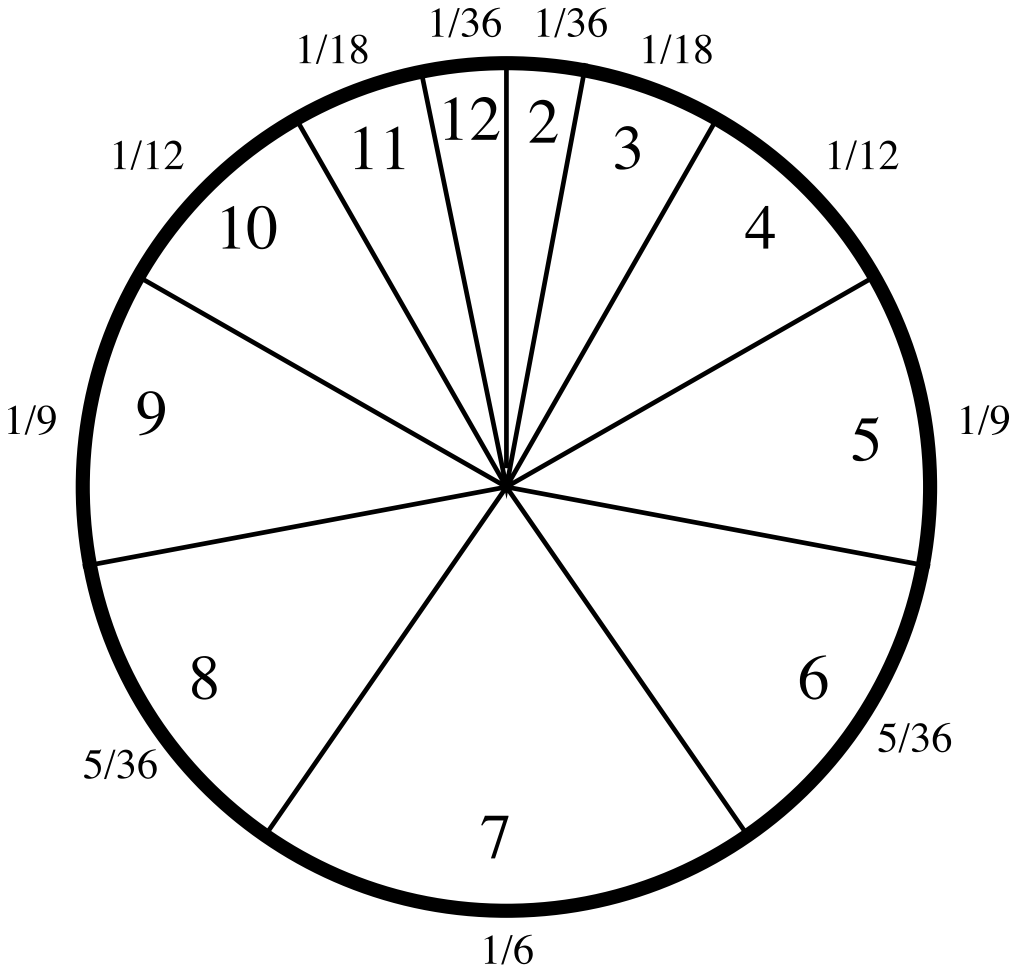
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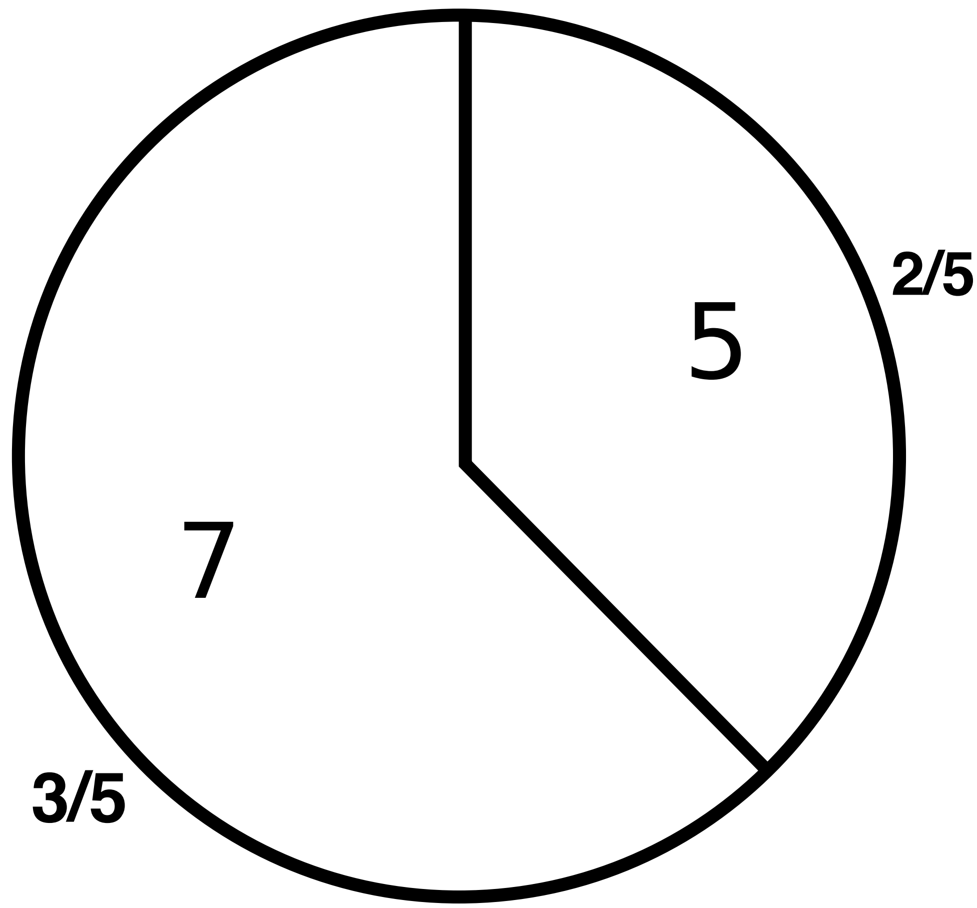
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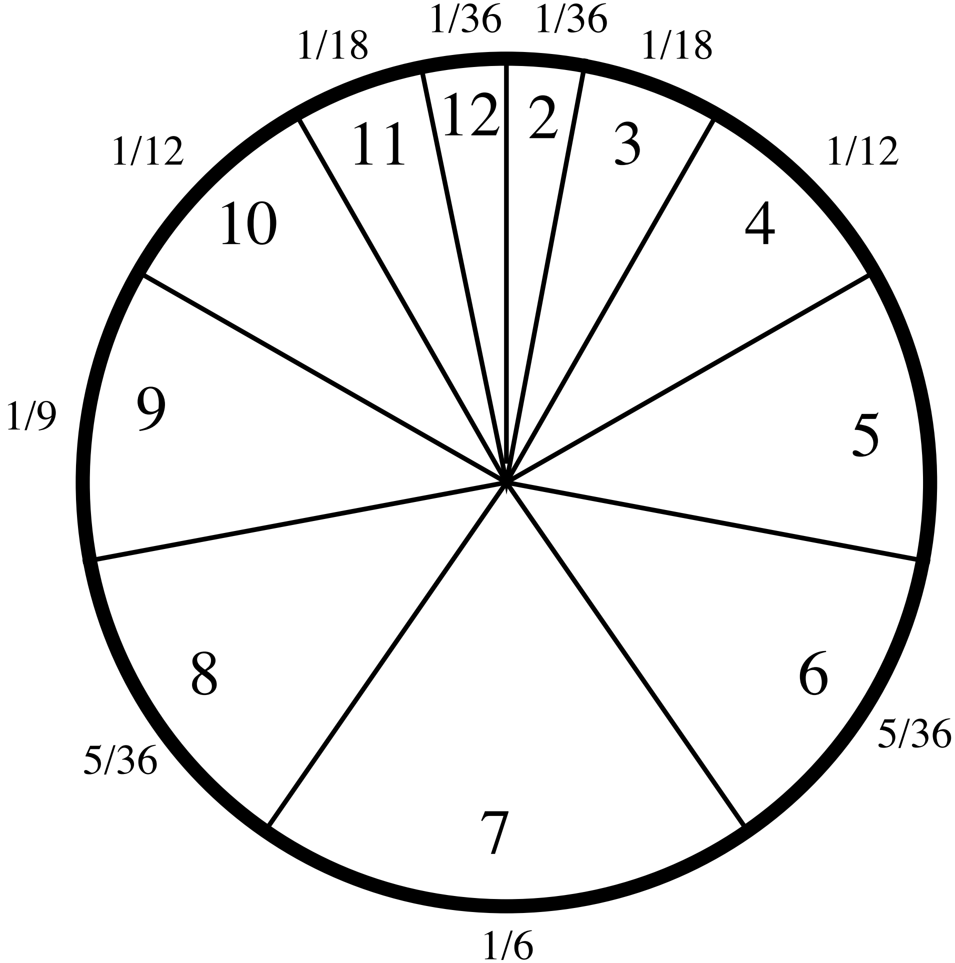
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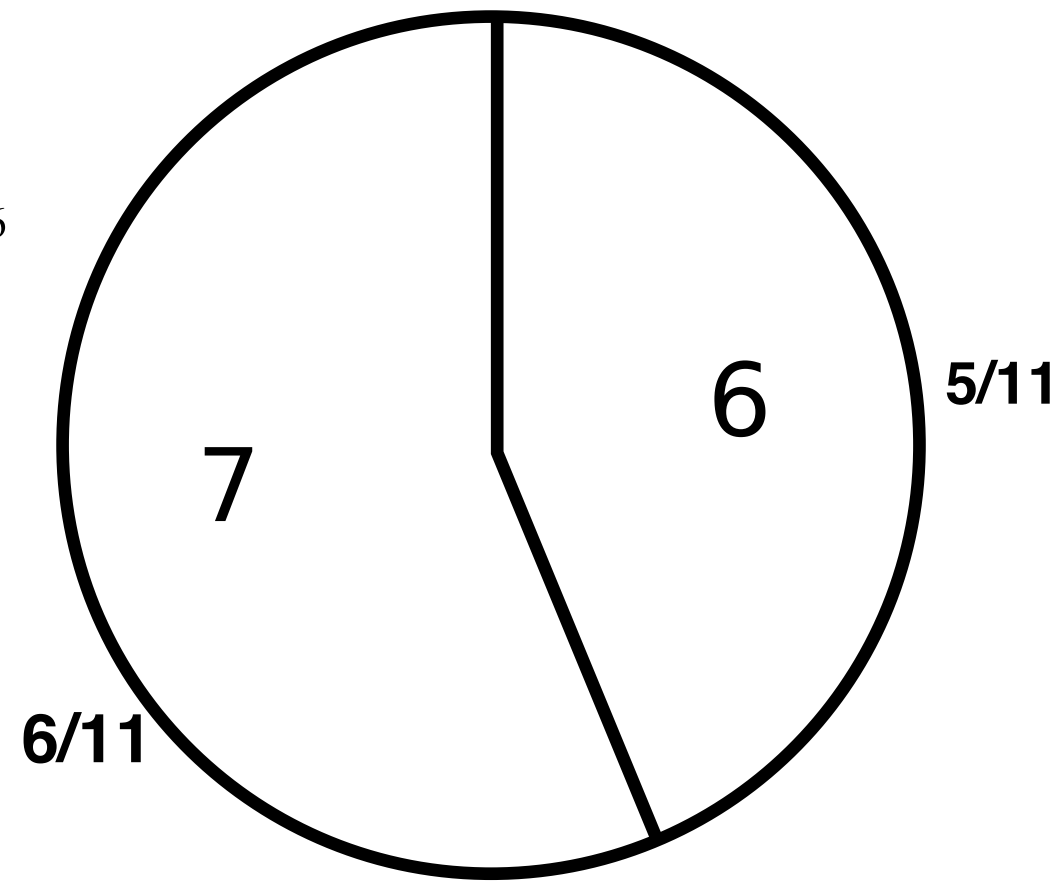
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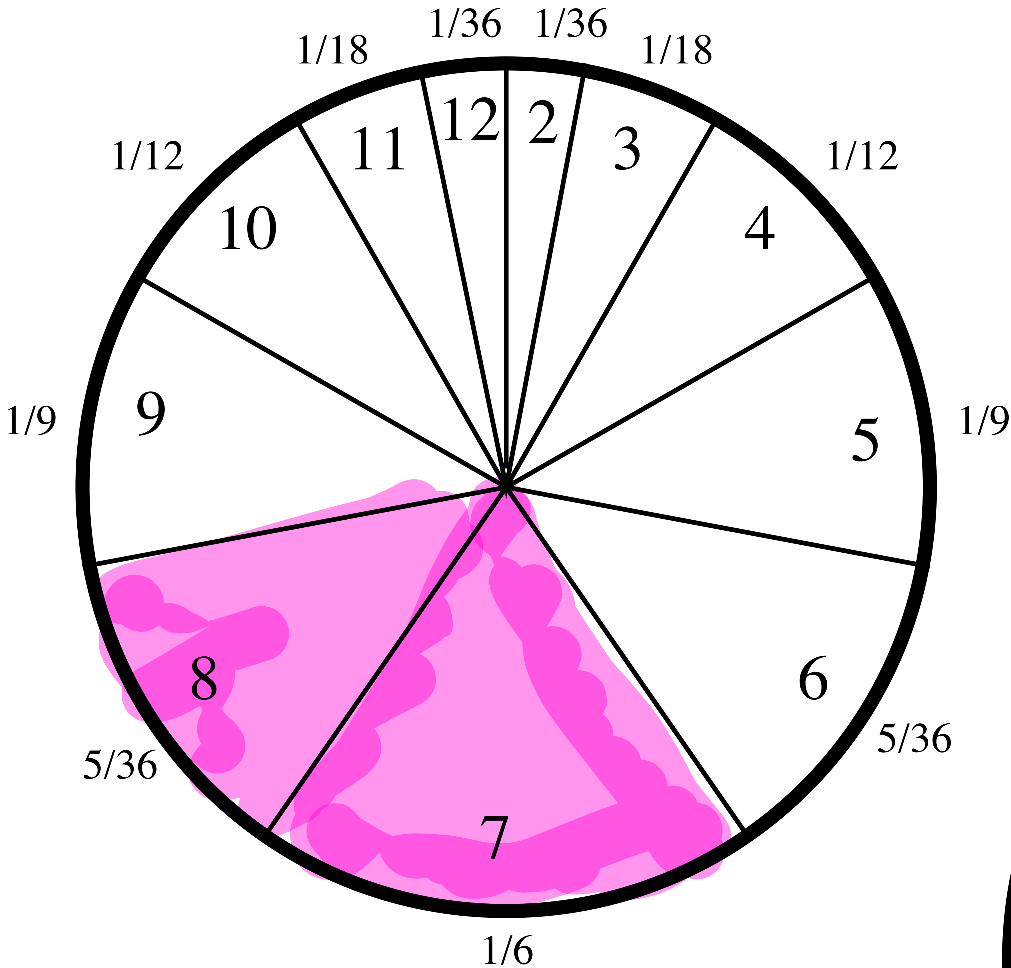
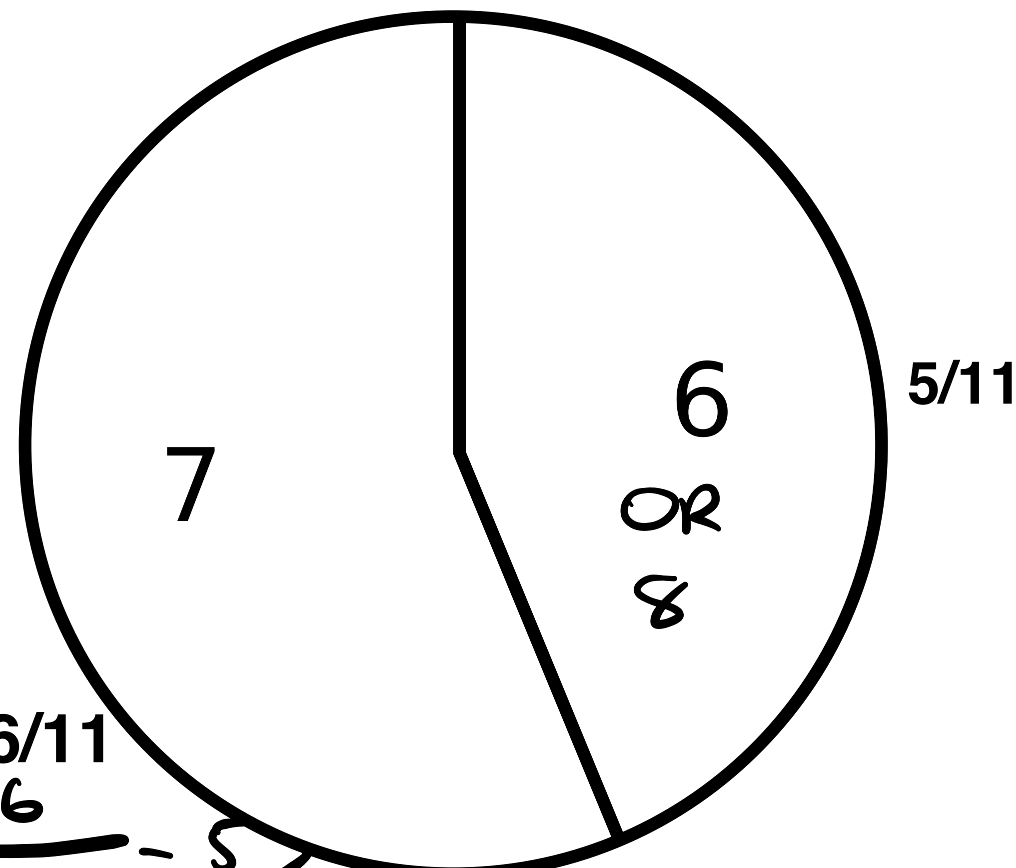
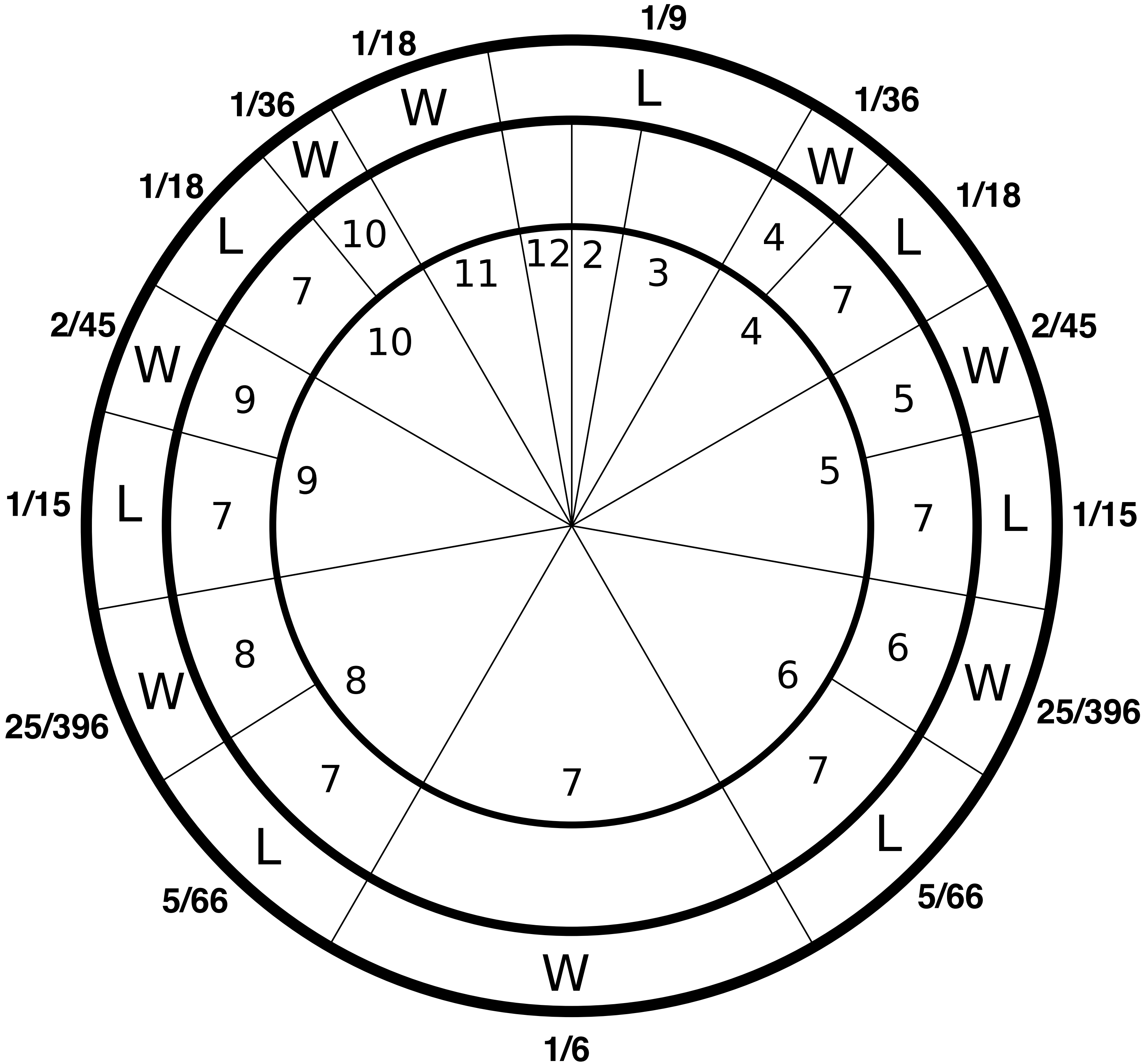




$$P(\text{next}=8 \mid \text{next}=7 \text{ OR } 8) = \frac{P(\text{next}=8 \,\&\, \text{next}=7 \text{ OR } 8)}{P(\text{next}=7 \text{ OR } 8)} = \frac{5/36}{11/36} = \frac{5}{11}$$



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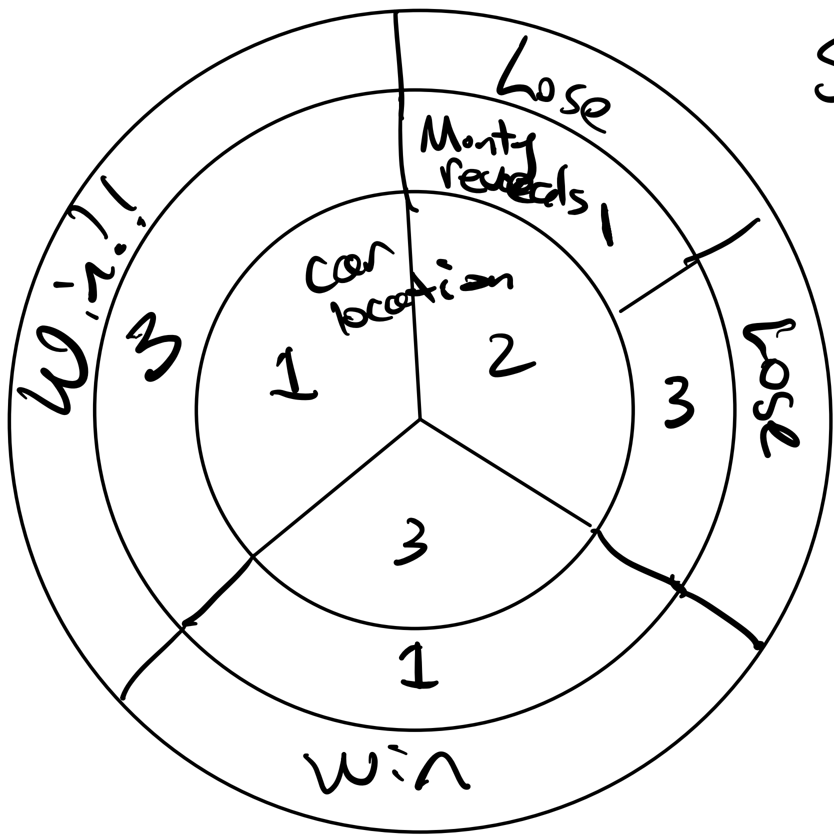
Suppose you are on a game show and you are given the choice of three doors: Behind one door is a car, behind the others, goats. You pick a door, say No. 1, and the host **who knows** what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to take the switch?

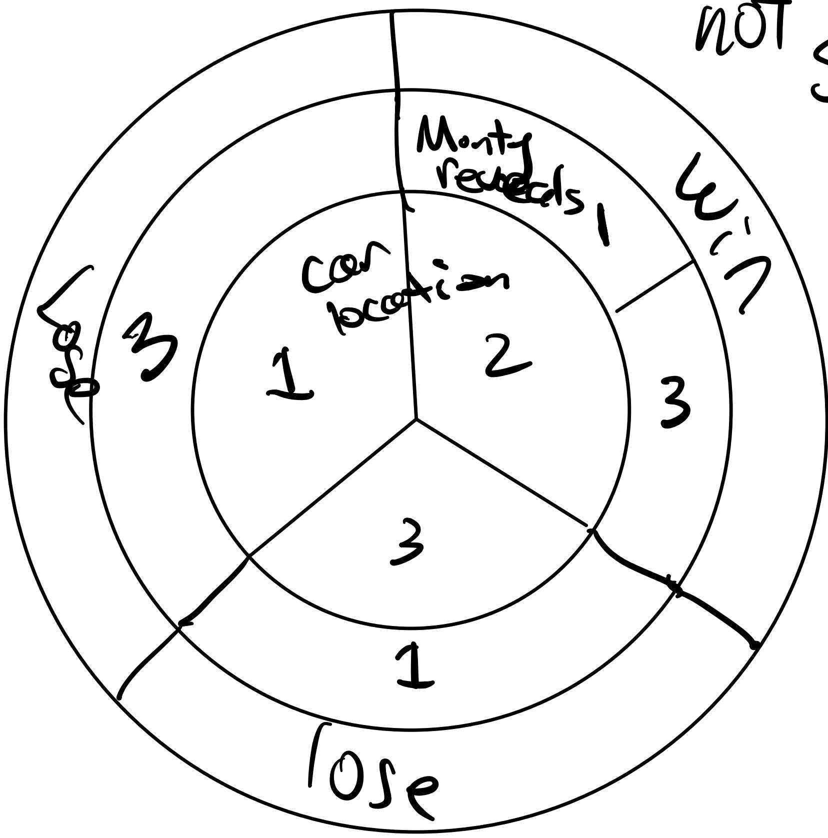
A) Yes. Switch and the probability the probability that you will win is higher.
B) No. Keep your door and you will be more likely to win.
C) Doesn't matter. Since there are two doors, only one can be the winner, it doesn't matter which one.
D) Don't know/don't care.



Switch

$P(\text{win}) = 2/3$

I choose door #2



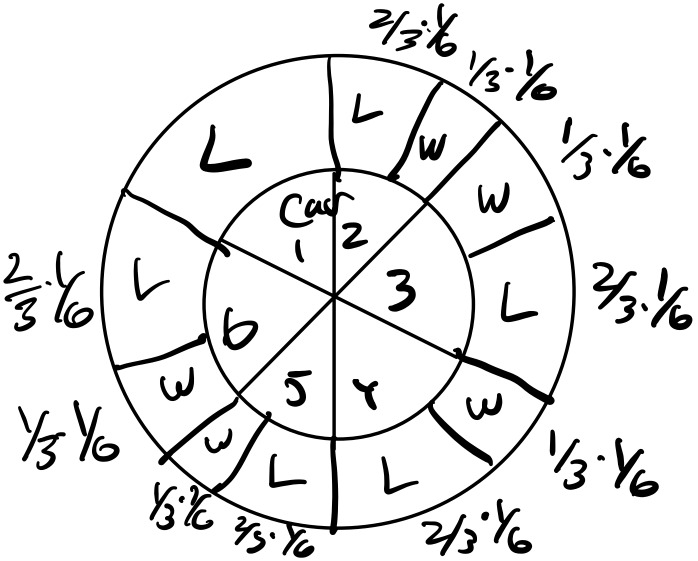
not switch

$P(win) = 1/3$

(2) Monty Hall invites you to play a game with 6 doors, behind one of the doors is a car, behind the other five are goats. The player picks a door and Monty Hall, who knows where the goats are located, opens two of the doors that were not picked to reveal where goats are located. There are 4 doors closed (one is your door and there are 3 others).

(a) What is the probability that you win if you switch your door to one of the unopened three doors?

(b) What is the probability that you win if you keep your door? $P(\text{win}) = 1/6$



$$P(\text{Win}) = 5 \cdot \frac{1}{3} \cdot \frac{1}{6}$$