MATH 4271 Dynamical Systems **Problem Set 1**
PRACTICE JOE TRAN

Question 1. Find the general solution of the following linear systems

(a) $\frac{dx}{dt} = x$, $\frac{dy}{dt} = y$
(b) $\frac{dx}{dt} = x$, $\frac{dy}{dt} = 2y$
(c) $\frac{dx}{dt} = x$, $\frac{dy}{dt} = 3y$
(d) $\frac{dx}{dt} = -y$, $\frac{dy}{dt} = x$
(e) $\frac{dx}{dt} = -x + y$, $\frac{dy}{dt} = -y$

Question 2. Find the general solution of the following linear systems.

(a) $\frac{dx}{dt} = x$, $\frac{dy}{dt} = y$, $\frac{dz}{dt} = z$
(b) $\frac{dx}{dt} = -x$, $\frac{dy}{dt} = -y$, $\frac{dz}{dt} = z$
(c) $\frac{dx}{dt} = -y$, $\frac{dy}{dt} = x$, $\frac{dz}{dt} = -z$

Question 3. Find the general solution of the linear system

$$\frac{dx}{dt} = x \quad \frac{dy}{dt} = ay$$

where a is a constant.

Question 4. Find the general solution of the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x} \tag{1}$$

when A is the $n \times n$ diagonal matrix $A = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. What condition on the eigenvalues $\lambda_1, ..., \lambda_n$ will guarantee that $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$ for all solutions $\vec{x}(t)$ of (1)?

Question 5. What is the relationship between the vector fields defined by

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

and

$$\frac{d\vec{x}}{dt} = kA\vec{x}$$

where k is a nonzero constant?

Question 6. (a) If $\vec{u}(t)$ and $\vec{v}(t)$ are solutions of the linear system (1), prove that for any constants a and b, $\vec{w}(t) = a\vec{u}(t) + b\vec{v}(t)$ is a solution.

(b) For

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

find solutions $\vec{u}(t)$ and $\vec{v}(t)$ of $\frac{d\vec{x}}{dt} = A\vec{x}$ such that every solution is a linear combination of $\vec{u}(t)$ and $\vec{v}(t)$.

Question 7. Find the eigenvalues and eigenvectors of the matrix A, and show that $D = P^{-1}AP$ is a diagonal matrix. Solve the linear system $\frac{d\vec{y}}{dt} = D\vec{y}$ and then solve $\frac{d\vec{x}}{dt} = A\vec{x}$.

(a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

Question 8. Find the eigenvalues and eigenvectors for the matrix A, solve the linear system $\dfrac{d\vec{x}}{dt} = A\vec{x}$, determine the stable and unstable subspaces for the linear system

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \vec{x}$$

Question 9. Write the following linear differential equations with constant coefficients in the form of the linear system (1), then solve.

(a) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - 2x = 0$

(b) $\dfrac{d^2x}{dt^2} + x = 0$

(c) $\dfrac{d^3x}{dt^2} - 2\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + 2x = 0$

Question 10. Solve the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \vec{x}(0) = \vec{x}_0$$

(a) with A given by Question 7(a) above and $\vec{x}_0 = (1, 2)^T$

(b) with A given by Question 8 and $\vec{x}_0 = (1, 2, 3)^T$.

Question 11. Let A be an $n \times n$ matrix with real and distinct eigenvalues. Find conditions on the eigenvalues that are necessary and sufficient for $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$, where $\vec{x}(t)$ is any solution of $\dfrac{d\vec{x}}{dt} = A\vec{x}$.

Question 12. Let A be an $n \times n$ matrix with real and distinct eigenvalues. Let $\phi(t, \vec{x}_0)$ be the solution of the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \vec{x}(0) = \vec{x}_0$$

Show that for each fixed $t \in \mathbb{R}$,

$$\lim_{\vec{y}_0 \to \vec{x}_0} = \phi(t, \vec{y}_0) = \phi(t, \vec{x}_0)$$