MATH 4271 Dynamical System **Problem Set 1**

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Question 1. (Problem Set 1.1 Question 1, Perko) Find the general solution of the following system of linear equations:

(a) $\dfrac{dx_1}{dt} = x_1, \dfrac{dx_2}{dt} = x_2$

(c) $\dfrac{dx_2}{dt} = x_1, \dfrac{dx_2}{dt} = 3x_2$

(e) $\dfrac{dx_1}{dt} = -x_1 + x_2, \dfrac{dx_2}{dt} = -x_2$

Solution. (a) Solving each equation individually (very easy to compute), we have $x_1(t) = c_1e^t$ and $x_2(t) = c_2e^t$, and we can express it as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1e^t \\ c_2e^t \end{bmatrix}$$

as required.

(c) Solving each equation individually (very easy to compute), we have $x_1(t) = c_1e^t$ and $x_2(t) = c_2e^{3t}$, and we can express it as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1e^t \\ c_2e^{3t} \end{bmatrix}$$

as required.

(e) First, solving the second equation (very easy to compute), we have $x_2(t) = c_2e^{-t}$. Now that we have a solution for the second equation, now we substitute it to the first equation, so that we have the differential equation

$$\frac{dx_1}{dt} = -x_1 + c_2e^{-t} \Rightarrow \frac{dx_1}{dt} + x_1 = c_2e^{-t}$$

which then becomes a linear equation. So if $p(t) = 1$, then $\displaystyle\int p(t)dt = t$ and so $e^{\int p(t)dt} = e^t$ is our integrating factor. Then,

$$\begin{aligned} x_1(t) &= e^{-\int p(t)dt}\left(\int q(t)e^{\int p(t)dt}dt + c_1\right) \\ &= e^{-t}\left(\int c_2e^{-t}e^tdt + c_1\right) \\ &= e^{-t}\left(\int c_2dt + c_1\right) \\ &= e^{-t}\left(c_2t + c_1\right) \\ &= c_1e^{-t} + c_2te^{-t} \end{aligned}$$

Therefore, the general solution is

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1te^{-t} + c_2e^{-t} \\ c_2e^{-t} \end{bmatrix}$$

as required.

Question 2. (Problem Set 1.1 Question 2, Perko) Find the general solution of the following system of linear equations:

(a) $\dfrac{dx_1}{dt} = x_1, \ \dfrac{dx_2}{dt} = x_2, \ \dfrac{dx_3}{dt} = x_3$

(c) $\dfrac{dx_1}{dt} = -x_2, \ \dfrac{dx_2}{dt} = x_1, \ \dfrac{dx_3}{dt} = -x_3$

Solution. (a) Solving the ith equation (for $i = 1, 2, 3$), we have $x_i(t) = c_i e^t$ so we can express it as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_2 e^t \\ c_3 e^t \end{bmatrix}$$

as required.

(b) Solving the third equation first gives us $x_3(t) = c_3 e^{-t}$. Now we just need to solve $\dfrac{dx_1}{dt} = -x_2$ and $\dfrac{dx_2}{dt} = x_1$. Substituting the latter equation to the former we have

$$\frac{dx_1}{dt} = \frac{d}{dt}\left(\frac{dx_2}{dt}\right) = \frac{d^2 x_2}{dt^2} = -x_2$$

which then implies that $\dfrac{d^2 x_2}{dt^2} + x_2 = 0$. So solving the homogeneous linear equation, if $x_2(t) = e^{\alpha t}$ is a solution, then

$$e^{\alpha t}(\alpha^2 + 1) = 0$$

yielding $\alpha = \pm i$. So then,

$$\begin{aligned} x_1(t) &= c_1 e^{it} + c_2 e^{-it} \\ &= c_1(\cos(t) + i\sin(t)) + c_2(\cos(t) - i\sin(t)) \\ &= c_1\cos(t) + ic_1\sin(t) + c_2\cos(t) - ic_2\sin(t) \\ &= (c_1 + c_2)\cos(t) + (ic_1 - ic_2)\sin(t) \\ &= \beta_1\cos(t) + \beta_2\sin(t) \end{aligned}$$

Then finding the solution of the second equation yields,

$$\begin{aligned} x_2(t) &= -\frac{dx_1}{dt} \\ &= -(-\beta_1\sin(t) + \beta_2\cos(t)) \\ &= \beta_1\sin(t) - \beta_2\cos(t) \end{aligned}$$

Therefore, the general solution of the system of linear equations is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \beta_1\cos(t) + \beta_2\sin(t) \\ \beta_1\sin(t) - \beta_2\cos(t) \\ c_3 e^{-t} \end{bmatrix}$$

Question 3. (Problem Set 1.1 Question 4, Perko) Find the general solution of the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x} \tag{1}$$

when A is the $n \times n$ diagonal matrix $A = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. What condition on the eigenvalues $\lambda_1, ..., \lambda_n$ will guarantee that $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$ for all $\vec{x}(t)$ of (1)?

Solution. Solving the system of equations presented by (1) will yield

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

In order for $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$, we require that the entries in the diagonal matrix to be less than 0, that is, for all $1 \leq i \leq n$, $\lambda_i < 0$ will suffice to conclude that $\vec{x}(t) \to 0$ as $t \to \infty$.

Question 4. (Problem Set 1.1 Question 6)

(a) If $\vec{u}(t)$ and $\vec{v}(t)$ are solutions of the linear system (1), prove that for any constants a and b, $\vec{w}(t) = a\vec{u}(t) + b\vec{v}(t)$ is a solution.

(b) For $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, find solutions $\vec{u}(t)$ and $\vec{v}(t)$ of $\frac{d\vec{x}}{dt} = A\vec{x}$ such that every solution is a linear combination of $\vec{u}(t)$ and $\vec{v}(t)$.

Solution. (a) Assume that $\vec{u}(t)$ and $\vec{v}(t)$ are solutions of the linear system provided by (1). Then given that $\vec{w}(t) = a\vec{u}(t) + b\vec{v}(t)$, we have

$$\begin{aligned} \frac{d\vec{w}}{dt} &= \frac{d}{dt}(a\vec{u} + b\vec{v}) \\ &= a\frac{d\vec{u}}{dt} + b\frac{d\vec{v}}{dt} \\ &= aA\vec{u}(t) + bA\vec{v}(t) \\ &= A(a\vec{u}(t) + b\vec{v}(t)) \end{aligned}$$

Therefore, we have shown that $\vec{w}(t) = a\vec{u}(t) + b\vec{v}(t)$ is a linear combination that is a solution of (1).

(b) Given $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, the general solution of the linear system is given by

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ c_2 e^{-2t} \end{bmatrix}$$

Then we want to write $\vec{x}(t)$ as a linear combination. Observe that

$$\vec{x}(t) = c_1 \begin{bmatrix} e^{t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$$

is a linear combination with coefficients c_1 and c_2.