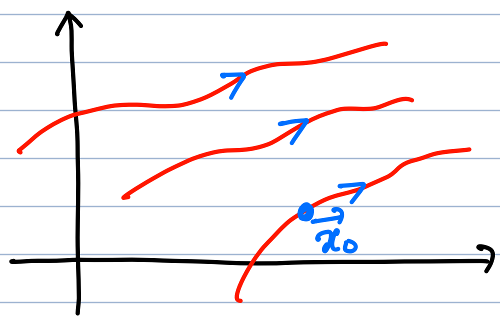
Flows Defined By Differential Equations (2.5 - Perko).

Case 1: Linear system $\begin{cases} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$ (1) has a solution $\vec{x}(t) = e^{At}\vec{x}_0$.

The set of mapping $e^{At}: \mathbb{R}^n \to \mathbb{R}^n$ is called the flow of the linear system, and denote it $\phi_t = e^{At}$.

The flow $\phi_t = e^{At}$ describes a motion of a point $\vec{x}_0 \in \mathbb{R}^n$ along the trajectory of (1)

Given a differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$ with solution $\vec{x}(t) = e^{At}\vec{x}_0$,



Lemma: For $\vec{x} \in \mathbb{R}^n$, $\phi_t(\vec{x}) = \phi(t, \vec{x})$ for $t \in \mathbb{R}$, the flow ϕ_t for the linear system (1) satisfies the following properties:

(i) $\phi_0(\vec{x}_0) = \vec{x}_0 \quad \forall \vec{x}_0 \in \mathbb{R}^n$

(ii) $(\phi_s \circ \phi_t)(\vec{x}) = \phi_{s+t}(\vec{x}) \quad \forall s, t \in \mathbb{R}, \forall \vec{x} \in \mathbb{R}$.

Moreover,

(iii) $(\phi_{-t} \circ \phi_t)(\vec{x}) = \phi_0(\vec{x}) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}, t \in \mathbb{R}$. That is, if ϕ_t is a flow, then it is invertible and $\phi_t^{-1} = \phi_{-t}$.

Proof: (i) Since ϕ_t is a flow on $\mathbb{R}^n$, then $\forall \vec{x} \in \mathbb{R}^n$, $\phi_t(\vec{x}) = \phi(t, \vec{x})$ $\forall t \in \mathbb{R}$. In particular for linear systems, $\phi_t(\vec{x}) = e^{At}\vec{x}$. Taking $t=0$, $\phi_0(\vec{x}_0) = e^{A0}\vec{x}_0 = e^{I}\vec{x}_0 = I\vec{x}_0 = \vec{x}_0 \quad \forall \vec{x}_0 \in \mathbb{R}^n$.

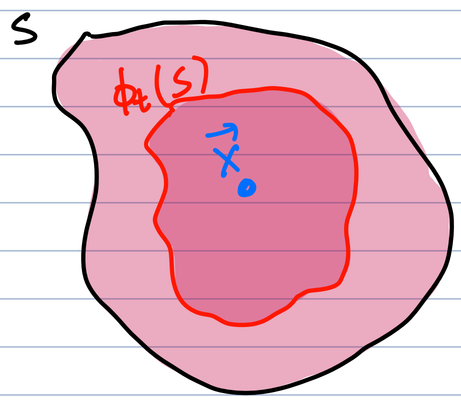
(ii) $\forall \vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$, $(\phi_s \circ \phi_t)(\vec{x}) = \phi_s(\phi_t(\vec{x})) = \phi_s(e^{At}\vec{x})$

$= e^{As} e^{At}\vec{x} = e^{A(s+t)}\vec{x} = \phi_{s+t}(\vec{x})$.

(iii) By (i) and (ii), taking $s = -t$. Then $\forall \vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$

$(\phi_{-t} \circ \phi_t)(\vec{x}) \overset{(ii)}{=} \phi_{-t+t}(\vec{x}) = \phi_0(\vec{x}) \overset{(i)}{=} \vec{x}$

Def: A set $S \subset \mathbb{R}^n$ is said to be invariant under the flow ϕ_t of the linear system (1) if $\phi_t(S) \subset S$, or $e^{At} S \subset S$. i.e if $\vec{x} \in e^{At} S \Rightarrow \vec{x} \in S$.



Remark: Recall that E_s, E_c, E_u are the stable, central, and unstable subspaces of eigenvectors. Then E_s, E_c, E_u are invariant sets for $S \subset \mathbb{R}^n$ under the flow of the linear system (1).

Case 2: Flows defined by nonlinear systems

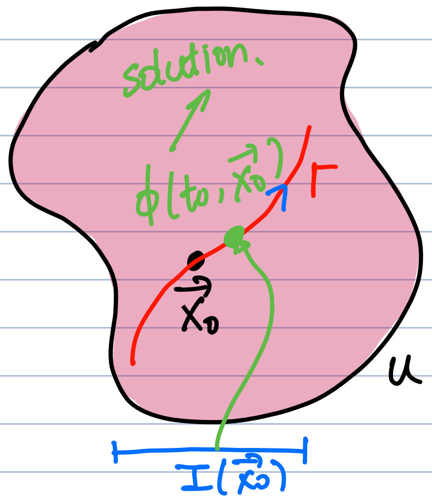
Given a nonlinear system $\begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) & (2) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$. Let $U \subset \mathbb{R}^n$ be an open subset and assume that $\vec{f} \in C^1(U)$.

Def: For $\vec{x}_0 \in U$, let $\vec{x}(t) = \phi(t, \vec{x})$ be a solution of (2) and the maximal interval of existence $I(\vec{x}_0)$. Then $\forall t \in I(\vec{x}_0)$, the collection of mappings $\phi_t : U \to U$ defined by $\phi_t(\vec{x}_0) = \phi(t, \vec{x}_0)$ is called a flow of the system $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$ or the flow defined by the nonlinear system.

Note: A vector field is a function $\vec{F}: U \to \mathbb{R}^n$ that defines a vector $\vec{v} = \vec{F}(\vec{x})$ at each point $\vec{x} \in U$ of the phase space U.

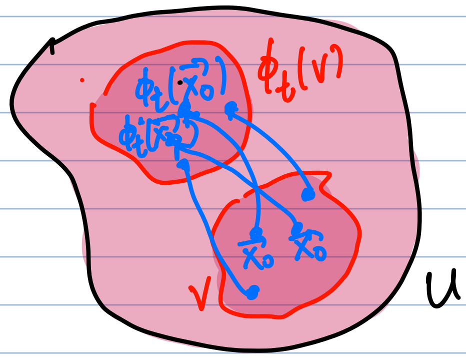
$\Rightarrow$ vector field associated to the flow: $\vec{v} = f(\vec{x}) = \left.\frac{d\phi_t(\vec{x})}{dt}\right|_{t=0}$.

Lemma: ϕ_t is a flow iff it is a solution of the nonlinear system (2).



Let $\vec{x}_0 \in \mathbb{R}^n$ be a fixed point in U. Then the solution curve that passes through $(t_0, \vec{x}_0)$ is a trajectory Γ.

Let $\vec{x}_0 \in \mathbb{R}^n$ be a varying point in $V \subset U$. Then the set of mappings that pass through $\vec{x}_0$ as it varies in V, is a flow.



Example: Let the system of differential equations $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$ describes motion of a fluid. Then

(i) The trajectory Γ of the system describing the motion of a single particle in the fluid.

(ii) Flow, ϕ_t of the system differential equation describes the motion of the entire fluid.

Remark: Let $U \subset \mathbb{R}^n$ be an open subset, $\vec{f} \in C^1(U)$. Then the set $\Omega = \{(t, \vec{x_0}) \in \mathbb{R} \times U : t \in I(\vec{x_0})\}$ is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $\phi : \Omega \to \mathbb{R}^n$. $(t, \vec{x_0}) \mapsto \phi(t, \vec{x_0}) = \phi_t(\vec{x_0})$.

Note: If $\vec{f} \in C^k(U)$, then $\phi \in C^k(\Omega)$. That is, if f is a complex valued function that is holomorphic, then f is also ϕ is holomorphic. (analytic)

Example: Find the set Ω defined above for the system in 1D: $\frac{dx}{dt} = \frac{1}{x}$ for $x(0) = x_0$.

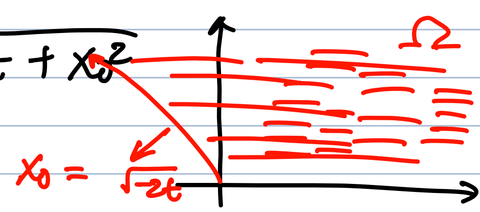
Recall $\Omega = \{(t, x_0) : \mathbb{R} \times U : t \in I(x_0)\}$. Then $f(x) = \frac{1}{x} \in C^1(U)$ whenever $x > 0$. $\Rightarrow x_0 > 0$; or if $x < 0 \Rightarrow x_0 < 0$. WLOG, assume $f(x) = \frac{1}{x} \in C^1(U)$ for $x, x_0 > 0$. $\Rightarrow$ the initial value problem has a unique solution on $I(x_0)$.

$$x\,dx = dt \Rightarrow \int x\,dx = \int dt \Rightarrow \frac{1}{2}x^2 = t + C$$

$$\Rightarrow x^2 = 2t + C \Rightarrow x \overset{\geq 0}{=} \sqrt{2t + C}.$$ When $x(0) = x_0$, so

$$x_0 = \sqrt{C} \Rightarrow C = x_0^2 \Rightarrow x(t) = \sqrt{2t + x_0^2}$$

$$\Rightarrow x(t) = \phi(t, x_0) = \sqrt{2t + x_0^2}$$



$$\sqrt{2t + x_0^2} > 0 \Rightarrow 2t + x_0^2 > 0 \Rightarrow 2t > -x_0^2 \Rightarrow t > -\frac{1}{2}x_0^2$$

$$\Rightarrow I(x_0) = \left(-\frac{1}{2}x_0^2, \infty\right) \Rightarrow \text{Maximal Interval of Existence.}$$

$$\Omega = \{(t, x_0) : t \in I(x_0),\ x_0 \in U\}.$$

The flow for non linear system (2) also satisfies the properties

(i) $\phi_0(\vec{x_0}) = \vec{x_0} \quad \forall \vec{x_0} \in U$.

(ii) $(\phi_s \circ \phi_t)(\vec{x_0}) = \phi_{s+t}(\vec{x_0}) \quad \forall t \in I(\vec{x_0}),\ s \in I(\phi_t(\vec{x_0}))$.

(iii) Moreover if $s = -t$, by (i) and (ii),

$$\phi_{-t}(\phi_t(\vec{x})) = \phi_0(\vec{x}) = \vec{x} \quad \forall \vec{x} \in B(\vec{x_0}, \varepsilon)$$

$$\phi_t(\phi_{-t}(\vec{y})) = \phi_0(\vec{y}) = \vec{y} \quad \forall \vec{y} \in \phi_t(U).$$