**Recall:** Given $\frac{d\vec{x}}{dt} = A\vec{x}$, we defined, for $n = 2$, the *uncoupled* system given by, for example,

$$\begin{cases} \frac{dx_1}{dt} = a_1 x_1 \\ \frac{dx_2}{dt} = a_2 x_2 \end{cases}$$

and we can write it as

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or write $\operatorname{diag}(a_1, a_2) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$. To solve the system of ODEs, we would have to use method of separation of variables.

On the other hand, we can also consider *coupled* system. Consider $\frac{d\vec{x}}{dt} = A\vec{x}$ and assume that $\vec{x}(t) = \vec{v}e^{\lambda t}$ for nonzero vectors $\vec{v}$. Then we obtain that

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

In this case, $\det(A - \lambda I_n) = 0$ gives the eigenvalues of the matrix $A$, which determines that we have nontrivial solutions.

**Definition 1.** A vector $\vec{v}$ is called an *eigenvalue* corresponding to the eigenvalue $\lambda \in \mathbb{K}$, if $\lambda\vec{v} = A\vec{v}$.

**Definition 2.** The equation $\chi_A(\lambda) = \det(A - \lambda I_n)$ is called the characteristic polynomial of $A$.

*Remark* 1. If $\vec{v} \neq \vec{0}$ is an eigenvector of $A$ corresponding to an eigenvalue $\lambda \in \mathbb{K}$, then $\vec{x}(t) = \vec{v}e^{\lambda t}$ is a nontrivial solution of the homogeneous equation $\frac{d\vec{x}}{dt} = A\vec{x}$.

To find the general solution, we require linearly independent vectors such that the form of the solution space (or vector space), we consider the following cases.

<u>Case 1.</u> Assume that $A$ is an $n \times n$ matrix that has real and distinct entries. Then $\lambda_j \in \mathbb{R}$ for all $j = 1, 2, ..., n$ and $j \neq k$. Then we have $n$ linearly independent eigenvectors, $\vec{v}_j$ for all $j = 1, 2, ..., n$ and $j \neq k$. Then

$$\vec{x}_j(t) = \vec{v}_j e^{\lambda_j t}$$

are solutions of the system of ODEs. Then we have linearly independent solution $\vec{x}_j(t)$ for each $j$. Therefore, the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is

$$\vec{x}(t) = \sum_{j=1}^{n} c_j \vec{v}_j e^{\lambda_j t}$$

where $c_j$ are arbitrary real constants, but not all are zero.

<u>Special Case.</u> Assume that $A$ is a $2 \times 2$ system of ordinary differential equations. Then we have $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \neq \lambda_2$, are eigenvalues of $A$. Then the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

where $c_1, c_2 \in \mathbb{R}$ but not both zero.

**Example 1.** Solve

$$\frac{dx_1}{dt} = x_1 + 2x_2$$
$$\frac{dx_2}{dt} = 2x_1 + x_2$$

To find the eigenvalues, observe that

$$\begin{aligned}\det(A - \lambda I_n) = \det\left(\begin{bmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{bmatrix}\right) &= (1-\lambda)^2 - 4\\ &= (1-\lambda-2)(1-\lambda+2)\\ &= (-\lambda-1)(-\lambda+3)\\ &= (\lambda+1)(\lambda-3) = 0\end{aligned}$$

So we have $\lambda = -1$ or $\lambda = 3$. Because our matrix is a $2 \times 2$, and we have two distinct eigenvalues, we would expect two distinct eigenvectors. We find the eigenspaces as follows: For $\lambda = -1$,

$$\mathcal{E}_A(-1) = \ker\left\{\begin{bmatrix}1-(-1) & 2\\ 2 & 1-(-1)\end{bmatrix}\right\} = \ker\left\{\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\right\} = \ker\left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\right\} = \left\{\begin{bmatrix}-1\\ 1\end{bmatrix}\right\}$$

So an eigenvector of the eigenspace is $\vec{v}_1 = \begin{bmatrix}-1\\ 1\end{bmatrix}$. Now for $\lambda = 3$,

$$\mathcal{E}_A(3) = \ker\left\{\begin{bmatrix}1-3 & 2\\ 2 & 1-3\end{bmatrix}\right\} = \ker\left\{\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix}\right\} = \ker\left\{\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\right\} = \left\{\begin{bmatrix}1\\ 1\end{bmatrix}\right\}$$

So an eigenvector of the eigenspace is $\vec{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$. Our eigenvectors are $\left\{\begin{bmatrix}-1\\ 1\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix}\right\}$, and therefore, the general solution is

$$\vec{x}(t) = c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-t} + c_2\begin{bmatrix}-1\\ 1\end{bmatrix}e^{3t} = \begin{bmatrix}c_1e^{-t} - c_2e^{3t}\\ c_1e^{-t} + c_2e^{3t}\end{bmatrix}$$

as desired.

<u>Case 2.</u> Real but repeated roots. Assume that $A$ is an $n \times n$ matrix with $\lambda_j \in \mathbb{R}$, where $j = 1, ..., n$ with $\lambda$ has algebraic multiplicity of $m$. Corresponding to the repeated eigenvalues $\lambda$, $A$ may have $n$ linearly independent eigenvectors or one or many (less than $n$) linearly independent eigenvectors.

- <u>Subcase 1.</u> Repeated root has $n$ linearly independent eigenvectors given by $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$. Then the general solution is

$$\vec{x}(t) = \sum_{j=1}^{n} c_j \vec{v}_j e^{\lambda_j t}$$

- <u>Subcase 2.</u> If $A$ has one or many (less than $n$) linearly independent eigenvectors, then

$$\vec{x}(t) = \sum_{j=1}^{n} c_j t^{j-1} e^{\lambda t}$$