1 Exponentials of Linear Operators and Matrices

The ultimate objective in this section is to be able to write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$, where $\vec{x}(0) = \vec{x}_0$ as

$$\vec{x}(t) = \vec{x}_0 e^{At}$$

How do we compute e^{At} or for any linear operator T, e^T?

Recall that an operator T on a vector space V is a mapping that maps $\vec{v} \in V$ to $\vec{w} = T(\vec{v}) \in V$, i.e. $T : V \to V$, or $\vec{v} \mapsto T(\vec{v})$.

Definition 1. We call L to be a linear operator if

- For any $\vec{v}, \vec{w} \in V$, $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$.
- For any $\alpha \in \mathbb{K}$ and $\vec{v} \in V$, then $T(\alpha\vec{v}) = \alpha \in T(\vec{v})$.

If $V = \mathbb{R}^n$, then $T : \mathbb{R}^n \to \mathbb{R}^n$, or $\vec{x} \mapsto T(\vec{x})$.

Definition 2. The operator norm of T is defined by

$$\|T\|_\infty = \sup_{\|\vec{x}\|_2 \leq 1} |T(\vec{x})|$$

where

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

is the Euclidean norm of $\vec{x} \in \mathbb{K}^n$.

Example 1. Let $T(\vec{x}) = A\vec{x}$, and let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The $T(\vec{x})$ is given by

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

Now how do we find $\|T\|_\infty$, i.e.

$$\|T\|_\infty = \sup_{\|\vec{x}\|_2 \leq 1} |T(\vec{x})| = \sup_{\|\vec{x}\|_2 \leq 1} |A\vec{x}| = \sup_{\|\vec{x}\|_2 \leq 1} \sqrt{(2x_1 + x_2)^2 + x_2^2}$$

We use the method of Lagrange's multiplier to find the max of a function of two variables. Let

$$f(x_1, x_2) = \|T\|^2 = (2x_1 + x_2)^2 + x_2^2 = 4x_1^2 + 4x_1x_2 + x_2^2 + x_2^2$$

Definition 3. A sequence of operators $(T_k)_{k=1}^{n}$ is said to converge to an operator $T \in L(\mathbb{K}^n)$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\|T_k - T\| < \varepsilon$ for all $k \geq N$.

Notation 1. We use the notation $\mathscr{L}(\mathbb{K}^n)$ to be the linear space of linear operators on $\mathbb{K}^n$.

Proposition 1. *Let $T, S \in \mathscr{L}(\mathbb{K}^n)$ and $A, B \in \mathcal{M}_n(\mathbb{K})$. Then*

1. $\|TS\| \leq \|T\|\|S\|$ *or* $\|AB\| \leq \|A\|\|B\|$
2. $\|T^k\| \leq \|T\|^k$ *for all* $k = 0, 1, 2, \ldots.$ *or* $\|A^k\| \leq \|A\|^k$ *for all* $k = 0, 1, 2, \ldots.$

Theorem 1. *Let $T \in \mathscr{L}(\mathbb{K}^n)$ and let $t_0 > 0$. Then the series*

$$\sum_{n=0}^{\infty} \frac{T^n t^n}{n!}$$

is absolutely and uniformly convergent.

Recall that $\sum_{n=1}^{\infty} x_n$ is a convergent series if the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ is convergent, where for each $n \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} x_k$.

Recall that $\sum_{n=1}^{\infty} x_n$ is said to be absolute convergent if $\sum_{n=1}^{\infty} |x_n|$ converges.

Definition 4. The exponential of $T \in \mathscr{L}(\mathbb{K}^n)$ is defined as

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}$$

In particular, if $T : \mathbb{K}^n \to \mathbb{K}^n$, or $A = [a_{ij}]_{n\times n}$, then

$$e^{At} = \sum_{k=0}^{n} \frac{A^k t^k}{k!}$$

Recall from calculus, the Taylor series tells us for $\theta \in \mathbb{R}^n$,

$$e^\theta = \sum_{k=0}^{n} \frac{\theta^k}{k!}$$

Example 2. Let $D = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Find e^D.

To find e^D, observe that

$$\begin{aligned} e^D &= \sum_{n=0}^{\infty} \frac{D^n}{n!} \\ &= I + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + a + \frac{1}{2!}a^2 + \cdots & 0 \\ 0 & 1 + a + \frac{1}{2!}a^2 + \cdots \end{bmatrix} \\ &= \begin{bmatrix} \displaystyle\sum_{n=0}^{\infty} \frac{a^n}{n!} & 0 \\ 0 & \displaystyle\sum_{n=0}^{\infty} \frac{a^n}{n!} \end{bmatrix} = \sum_{n=0}^{\infty} \frac{a^n}{n!} I = e^a I \end{aligned}$$