Fundamental Theorem of Linear Systems of ODEs.

Let $A \in M_n(\mathbb{K})$. For $\vec{x}(0) = \vec{x} \in \mathbb{K}^n$, the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$ and $\vec{x}(0) = \vec{x}_0$ (*) has a unique solution : $\vec{x}(t) = e^{At}\vec{x}_0$. (*)

Remark: Let $\Phi(t) = [\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)]$ be the fundamental matrix. Then the solution of (*) is $\vec{x}(t) = \Phi(t)\Phi^{-1}(0)\vec{x}_0$. (*) and so $e^{At} = \Phi(t)\Phi^{-1}(0)$.

Example: Let $\Phi(t) = \begin{bmatrix} e^t & e^{-2t} \\ 2e^t & 3e^{-2t} \end{bmatrix}$.

(i) Find A.

(ii) Find e^{At}.

(iii) Find the general solution given $\vec{x}_0 \in \mathbb{R}^2$.

Solution: (i) Recall that if $\Phi(t)$ is a fundamental matrix and $\vec{x}(t) = \Phi(t)$ is a solution. Then $\frac{d}{dt}(\Phi(t)) = A\Phi(t) \Rightarrow \Phi'(t) = A\Phi(t)$. Now let $t=0$. We have $A = \Phi'(0)\Phi^{-1}(0) = \begin{bmatrix} e^0 & -2e^0 \\ 2e^0 & -6e^0 \end{bmatrix}\begin{bmatrix} 3e^0 & -e^0 \\ -2e^0 & e^0 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 18 & -8 \end{bmatrix}$$

(ii) Because $e^{At} = \Phi(t)\Phi^{-1}(0)$, we obtain that

$$e^{At} = \begin{bmatrix} e^t & e^{-2t} \\ 2e^t & 3e^{-2t} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3e^t - 2e^{-2t} & -e^t + e^{-2t} \\ 6e^t - 6e^{-2t} & -2e^t + 3e^{-2t} \end{bmatrix}$$

(iii) Say $\vec{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then

$$\vec{x}(t) = e^{At}\vec{x}_0 = \begin{bmatrix} 3e^t - 2e^{-2t} & -e^t + e^{2t} \\ 6e^t - 6e^{-2t} & -2e^t + 3e^{-2t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4e^t - 3e^{-2t} \\ 8e^t - 9e^{-2t} \end{bmatrix}.$$

Computing e^{At} for $A \in M_n(\mathbb{K})$.

Lemma: Let $A \in M_n(\mathbb{K})$. Then we can write $B = P^{-1}AP$ so that $A = PBP^{-1}$ (Jordan Canonical Form) and thus $e^{At} = Pe^{Bt}P^{-1}$. The solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is $\vec{x}(t) = e^{At}\vec{x}_0 = Pe^{Bt}P^{-1}\vec{x}_0$. P is a matrix chosen so that B has simplest form.

Case 1: If $A \in \mathcal{M}_n(\mathbb{R})$ has real and distinct eigenvalues.

Then $P = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ is an invertible matrix and $B = P^{-1}AP = D$, where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then $e^{At} = Pe^{Dt}P^{-1}$ and $\vec{x}(t) = e^{At}\vec{x}_0$.

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Then

$$\chi_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = -2 + \lambda + \lambda^2 - 4 = \lambda^2 + \lambda - 6$$

$$= (\lambda+3)(\lambda-2) = 0 \implies \lambda = -3, \lambda = 2.$$

Let $\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ be an eigenvector corresponding to $\lambda = -3$. Then,

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{v}_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Let $\vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ be an eigenvector corresponding to $\lambda = 2$. Then

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then $P = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$, $P^{-1} = -\frac{1}{5}\begin{bmatrix} 1 & -1 \\ -1 & -4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}$

Therefore, $\vec{x}(t) = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix}\frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Case 2: If A has complex conjugate eigenvalues only.

Let $A \in \mathcal{M}_{2m}(\mathbb{C})$. Let $\lambda_j = a_j \pm i b_j$ and $\vec{w}_j = \vec{u}_j + i\vec{v}_j$ for $1 \le j \le m$ be the corresponding eigenvector. Then

$$\bigoplus_{i=1}^{n} A = \begin{bmatrix} A & & O \\ & A & \\ O & & \ddots\, A \end{bmatrix}$$

(i) $\mathcal{B} = \{\vec{u}_1, \vec{v}_1, \vec{u}_2, \vec{v}_2, \ldots, \vec{u}_n, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$

(ii) $P = [\vec{v}_1, \vec{u}_1, \vec{v}_2, \vec{u}_2, \ldots, \vec{v}_n, \vec{u}_n]$ is an invertible matrix.

(iii) $B = P^{-1} \bigoplus_{j=1}^{m} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix} P$, which is a $2m \times 2m$ matrix with 2×2 block matrix along the diagonal.

(iv) The solution of $\circledast$ is $\vec{x}(t) = P \bigoplus_{j=1}^{m} \begin{bmatrix} e^{a_j t}\cos(b_j t) & -e^{a_j t}\sin(b_j t) \\ e^{a_j t}\sin(b_j t) & e^{a_j t}\cos(b_j t) \end{bmatrix} P^{-1}\vec{x}_0$.

Remark: Instead of $P = [\vec{v}_1, \vec{u}_1, \dots, \vec{v}_n, \vec{u}_n]$, we can use Q, where $Q = [\vec{u}_1, \vec{v}_1, \dots, \vec{u}_n, \vec{v}_n]$ and $B = Q^{-1}AQ = \bigoplus_{j=1}^{m} \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$ Then
$\vec{x}(t) = P \bigoplus_{j=1}^{m} \begin{bmatrix} e^{a_jt}\cos(b_jt) & e^{a_jt}\sin(b_jt) \\ -e^{a_jt}\sin(b_jt) & e^{a_jt}\cos(b_jt) \end{bmatrix} P^{-1} \vec{x}_0$.

Example: Solve the system $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\chi_A(\lambda) = \begin{vmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 + 1 = (\lambda+2-i)(\lambda+2+i) = 0$$

$$\Rightarrow \lambda = -2 \pm i. \quad a = -2, \ b = 1$$

When $\lambda = -2-i$,

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & i & 0 \\ i & -1 & 0 \end{array}\right] \to \vec{w} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = P^{-1}AP = P^{-1}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} P$

$A = PBP^{-1} = P\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} P^{-1} \Rightarrow e^{At} = Pe^{-2t}\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} P^{-1}$

$$\vec{x}(t) = Pe^{-2t}\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} P^{-1}\vec{x}_0 = e^{-2t}\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= e^{-2t}\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$

Example: $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

Solution: $\chi_A(\lambda) = \begin{vmatrix} 1-\lambda & -1 & & \\ 1 & 1-\lambda & & 0 \\ & 0 & 3-\lambda & 2 \\ & & 1 & 1-\lambda \end{vmatrix}$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & & \\ & 3-\lambda & 2 \\ & 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -1 & & \\ & 3-\lambda & 2 \\ & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(1-\lambda)[(3-\lambda)(1-\lambda)-2]] - [-[(3-\lambda)(1-\lambda)-2]]$$

$$= (1-\lambda)^2[(3-\lambda)(1-\lambda)-2] + [(3-\lambda)(1-\lambda)-2]$$

$$= [(3-\lambda)(1-\lambda)-2][(1-\lambda)^2+1] = (\lambda-1+i)(\lambda-1-i)(x-2+\sqrt{3})(x-2-\sqrt{3}) = 0$$