MATH 4271: Dynamical Systems

Joe Tran

January 12, 2024

Contents

1 Linear Systems of ODEs **5**
1.1 Uncoupled Linear Systems 5
1.2 Diagonalization 8

Chapter 1

Linear Systems of ODEs

This chapter presents a study of linear systems of ordinary differential equations of the form

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

where $\vec{x} \in \mathbb{R}^n$, A is an $n \times n$ matrix, and

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

It is shown that the solution of the linear system above together with the initial condition $\vec{x}(0) = \vec{x}_0$, is given by

$$\vec{x}(t) = \vec{x}_0 e^{At}$$

where e^{At} is an $n \times n$ matrix defined by its Taylor series. A good portion of this chapter is concerned with the computation of the matrix e^{At} in terms of the eigenvalues and eigenvectors of the square matrix A.

1.1 Uncoupled Linear Systems

The method of separation of variables can be used to solve the first-order linear differential equation

$$\frac{dx}{dt} = ax$$

where the general solution is given as

$$x(t) = ce^{at}$$

where $c = x(0)$, the value of the function $x(t)$ at time $t = 0$.

Now consider the uncoupled linear system

$$\frac{dx_1}{dt} = -x_1$$
$$\frac{dx_2}{dt} = 2x_2$$

This system can be written in matrix form as

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

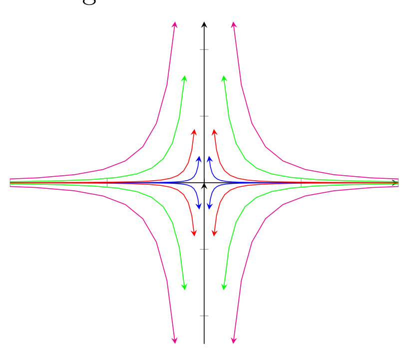
where $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Note that in this case, A is a diagonal matrix, i.e. $A = \operatorname{diag}(-1, 2)$, and in general, whenever A is a diagonal matrix, the system reduces to an uncoupled linear system. The general solution of the above uncoupled linear system can be found by the method of separation of variables. It is given by

$$x_1(t) = c_1 e^{-t}$$
$$x_2(t) = c_2 e^{2t}$$

or equivalently,

$$\vec{x}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \vec{c}$$

where $\vec{c} = \vec{x}(0)$. The solution defines a motion along these curves. That is, each $\vec{c} \in \mathbb{R}^2$ moves to a point $\vec{x}(t) \in \mathbb{R}^2$ after time t. These motions can be described geometrically by drawing the solution curves in the x_1x_2-plane, which we refer to as the *phase plane*, and using arrows to indicate the direction of motion along these curves with increasing time t.



For $c_1 = c_2 = 0$, $x_1(t) = 0$ and $x_2(t) = 0$ for all $t \in \mathbb{R}$, and the origin is referred to as an *equilibrium point* in this example. Note that solutions starting on the x_1-axis approach the origin as $t \to \infty$, and that solutions starting on the x_2-axis approach the origin as $t \to -\infty$.

The *phase portrait* of a system of differential equations with $\vec{x} \in \mathbb{R}^n$ is the set of all solution curves of $\frac{d\vec{x}}{dt} = A\vec{x}$ in the phase space $\mathbb{R}^n$. The above figure provides a geometric representation of the phase portrait of the uncoupled linear system as described above.

The *dynamical system* defined by the linear system $\frac{d\vec{x}}{dt} = A\vec{x}$ in this example is simply the mapping $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by the solution $\vec{x}(t, \vec{c})$ given by

$$\vec{x}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \vec{c}$$

That is,

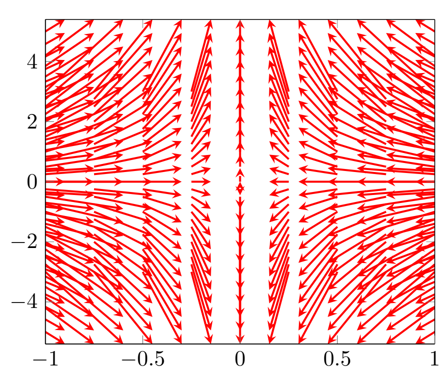
$$\phi(t, \vec{c}) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \vec{c}$$

Geometrically, the dynamical system describes the motion of the points in phase space along the solution curves defined by the system of differential equations.

The function

$$\vec{f}(\vec{x}) = A\vec{x}$$

on the right hand side of $\frac{d\vec{x}}{dt} = A\vec{x}$ defines a mapping $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ (linear in this case). This mapping (which need not be linear) defines a *vector field* on $\mathbb{R}^2$, i.e. to each point $\vec{x} \in \mathbb{R}^2$, the mapping $\vec{f}$ assigns a vector $\vec{f}(\vec{x})$. If we draw each vector $\vec{f}(\vec{x})$ with its initial point at the point $\vec{x} \in \mathbb{R}^2$, we obtain a geometric representation of the vector field



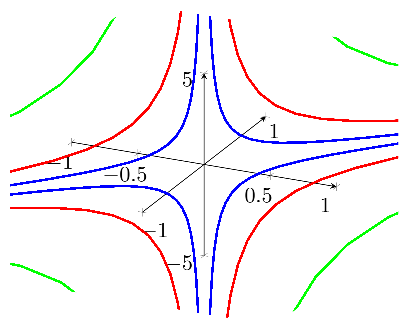
Now let us consider the following uncoupled linear system in $\mathbb{R}^3$,

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = y$$
$$\frac{dz}{dt} = -z$$

The general solution is given by

$$x(t) = c_1 e^t$$
$$y(t) = c_2 e^t$$
$$z(t) = c_3 e^{-t}$$

and the phase portrait for the system is shown below.



The xy-plane is referred to as the *unstable subspace* of the system and the z-axis is called the *stable subspace* of the system.

1.2 Diagonalization

The algebraic technique of diagonalizing a square matrix A can be used to reduce the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x} \tag{1}$$

to an uncoupled linear system. We consider the case when A has $\mathbb{K}$ distinct eigenvalues. The following theorem from linear algebra then allows us to solve the linear system.

Theorem 1.2.1. *If the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of an $n \times n$ matrix A are in $\mathbb{K}$ and are distinct, then any corresponding eigenvectors $\{\vec{v}_1, ..., \vec{v}_n\}$ forms a basis for $\mathbb{K}^n$, the matrix $P = [\vec{v}_1, \vec{v}_2, ..., \vec{v}_n]$ is invertible, and*

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$$

This theorem says that if a linear transformation $T : \mathbb{K}^n \to \mathbb{K}^n$ is represented by the $n \times n$ matrix A with respect to the standard basis $\{\vec{e}_1, ..., \vec{e}_n\}$ for $\mathbb{K}^n$, then with respect to any basis of eigenvectors $\{\vec{v}_1, ..., \vec{v}_n\}$, T is represented by the diagonal matrix of eigenvalues $\operatorname{diag}(\lambda_1, ..., \lambda_n)$.

In order to reduce the system (1) to an uncoupled linear systems using the above theorem, define the linear transformation of coordinates

$$\vec{y} = P^{-1}\vec{x}$$

where P is the invertible matrix defined in the theorem. Then

$$\vec{x} = P\vec{y} \quad \frac{d\vec{y}}{dt} = P^{-1}\frac{d\vec{x}}{dt} = P^{-1}A\vec{x} = P^{-1}AP\vec{y}$$

and, according to Theorem 1.2.1, we obtain the uncoupled linear system

$$\frac{d\vec{y}}{dt} = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)\vec{y}$$

This uncoupled linear system has the solution

$$\vec{y}(t) = \operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t})\vec{y}(0)$$

And then since $\vec{y}(0) = P^{-1}\vec{x}(0)$ and $\vec{x}(t) = P\vec{y}(t)$, it follows that (1) has the solution

$$\vec{x}(t) = PE(t)P^{-1}\vec{x}(0) \tag{2}$$

where $E(t)$ is the diagonal matrix

$$E(t) = \operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t})$$

Corollary 1.2.2. *Under the hypotheses of the above theorem, the solution of the linear system (1) is given by the function $\vec{x}(t)$ defined by (2).*

Example 1.2.3. Consider the linear system

$$\frac{dx_1}{dt} = -x_1 - 3x_2 \quad \frac{dx_2}{dt} = 2x_2$$

which can be written in the form (1) with the matrix

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$. A pair of corresponding eigenvectors is given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The matrix P and its inverse are then given by

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

It is elementary to verify that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then under the coordinate transformation $\vec{y} = P^{-1}\vec{x}$, we obtain the uncoupled linear system,

$$\frac{dy_1}{dt} = -y_1 \quad \frac{dy_2}{dt} = 2y_2$$

which has the general solution $y_1(t) = c_1e^{-t}$ and $y_2(t) = c_2e^{2t}$. The phase portrait for this system is given in Section 1.1. And by Corollary 1.2.2, the general solution to the original linear system of this example is given by

$$\vec{x}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\vec{c}$$

where $\vec{c} = \vec{x}(0)$, or equivalently,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1e^{-t} + c_2(e^{-t} - e^{2t}) \\ c_2e^{2t} \end{bmatrix} \tag{3}$$

Note that the subspaces spanned by the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of the matrix A determine the stable and unstable subspaces of the linear system (1) according to the following definition.

Definition 1.2.4. Suppose that the $n \times n$ matrix A has k negative eigenvalues $\lambda_1, ..., \lambda_k$ and $n - k$ positive eigenvalues $\lambda_{k+1}, ...\lambda_n$ and that these eigenvalues are distinct, and are in $\mathbb{K}$. Let $\{\vec{v}_1, ..., \vec{v}_n\}$ be a corresponding set of eigenvectors. Then the *stable* and *unstable subspaces* of the linear system (1), E^s and E^u are the linear subspaces spanned by $\{\vec{v}_1, ..., \vec{v}_k\}$ and $\{\vec{v}_{k+1}, ..., \vec{v}_n\}$, respectively. That is,

$$E^s = \text{span}\{\vec{v}_1, ..., \vec{v}_k\} \quad E^u = \text{span}\{\vec{v}_{k+1}, ..., \vec{v}_n\}$$