**MATH 4271 Dynamical Systems** **Lecture 1**

LECTURE JOE TRAN

## 1. Course Information

**Course Weighting (TBD):**

- Assignment 1: Homework Assignment via Crowdmark (20%)
- Assignment 2: Reading Assignment and Group Presentation (25%)
- Midterm Test (25%)
- Final Exam (30%)

**Main topics in this class:**

(1) Solving linear systems
(2) What is a dynamical system: general introduction, basic concepts and definitions, classification of dynamical systems
  - Dynamical flows
  - Qualitative analysis of scalar autonomous dynamical system (1D Flows)
  - Some definitions
(3) Phase Plane Analysis of Linear Systems
  - Classification of equilibrium
  - Local stability analysis and phase plane
  - Extension to 3D linear systems
(4) Dynamical theory of linear and nonlinear autonomous ODEs: existence, uniqueness
(5) Nonlinear systems and local and global stability.
(6) Theory of Bifurcation
(7) Oscillations in nonlinear systems
(8) Applications in ecology and epidemiology
(9) Discrete systems and chaos

**Textbook:** Perko, Lawrence, *Differential equations and dynamical systems.* Vol. 7, Springer Science & Business Media, 2013.

## 2. Linear Systems of ODEs

The goal in this section is to be able to solve systems of linear ODEs. Furthermore, the goal is the study the qualitative behaviour of linear systems of ODEs.

**Definition 1.** An ordinary differential equation (or system of ODEs) is said to be an *autonomous ODE* if it does not explicitly depend on the independent variable. In other words, if the equation is written given in the form $\dfrac{dy}{dx} = (f \circ y)(x)$, that is, $x$ does not appear explicitly on the right side of the equation.

**Example 1.** Consider the one-dimensional autonomous ordinary differential equation given by

$$\frac{dy}{dx} = f(x)$$

But if $x = x(t)$, then we may write

$$\frac{dy}{dx} = f(x(t))$$

For example, we can write $\dfrac{dy}{dx} = 2y$, where $x$ is the independent variable, or we can also write $\dfrac{dx}{dt} = -3x^2 + 6$, where $t$ is the independent variable, in which both do not appear. Both of these examples are autonomous.

As another example, consider $\dfrac{dx}{dt} = -3x^2 + 6t$. Then this is not an autonomous ODE because the function depends on the independent variable.

**Definition 2.** A non-homogeneous $n \times n$ system of linear ODEs can be written as follows

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n
\end{aligned}$$

or as a vector,

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$

where $\vec{x} = (x_1, x_2, ..., x_n)^T$, $\vec{b} = (b_1, b_2, ..., b_n)^T$, and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, and $t \geq 0$.

**Definition 3.** A homogeneous $n \times n$ system of linear ODEs can be written as

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

or as a vector,

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

where $\vec{x} = (x_1, x_2, ..., x_n)^T$, and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, and $t \geq 0$.[1]

**Notation 1.** We may express the system as follows:

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \vec{x}' = A\vec{x}$$

[1] Homogeneous system only occurs whenever $\vec{b} = \vec{0}$.

**Example 2.** Consider the system given as follows:

$$\frac{dx_1}{dt} = -2x_2$$
$$\frac{dx_2}{dt} = -3x_1$$

We can write the system as

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}$$

**Example 3.** Consider the system given by

$$\frac{dx_1}{dt} = -2x_1$$
$$\frac{dx_2}{dt} = -3x_2$$

Then we can write the system as

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}$$

Such systems of this form are called *uncoupled systems*. It is elementary to find the general solutions of uncoupled systems. To solve this system, observe that

$$\frac{dx_1}{dt} = -2x_1$$
$$\frac{1}{x_1}dx_1 = -2dt$$
$$\int \frac{1}{x_1}dx_1 = -2\int dt$$
$$\ln|x_1| = -2t + c$$
$$x_1 = c_1e^{-2t}$$

and similarly, $x_2 = c_2e^{-3t}$, so the general solution of $\vec{x}$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1e^{-2t} \\ c_2e^{-3t} \end{bmatrix}$$

*Remark* 1. If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions of the homogeneous equation $\frac{d\vec{x}}{dt} = A\vec{x}$, then any linear combination

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

is also a solution (by the superposition principle).

*Remark* 2. If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent vector solutions of the ordinary differential equation, then the general solution is written as

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

In order to determine whether the vectors $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent, then we would have to either

- Find the eigenvalue solutions of the vectors
- Show that the determinant is nonzero.

## 3. Eigenvalue and Eigenvector Method

Consider the scalar ordinary differential equation of the form

$$\frac{dx}{dt} = ax$$

where $a \neq 0$. Then the general solution is $x(t) = ce^{at}$, where $c \neq 0$ (to avoid triviality).

Now consider, in general,

$$\frac{d\vec{x}}{dt} = A\vec{x} \tag{1}$$

Since (1) is linear and homogeneous, we can assume that $\vec{x}(t) = \vec{c}e^{\lambda t}$ is a solution of (1), where $\lambda \in \mathbb{K}$.[2] We will write now

$$\vec{x}(t) = \vec{v}e^{\lambda t} \tag{2}$$

Then

$$\frac{d\vec{x}}{dt} = \lambda\vec{v}e^{\lambda t}$$

So putting (2) into (1),

$$\lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t}$$

and therefore,

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

In this case, either $\vec{v} = \vec{0}$, or $A - \lambda I_n$. If $\vec{v} = \vec{0}$, we obtain a trivial solution. Otherwise, $A - \lambda I_n = \vec{0}$ and so $\det(A - \lambda I_n) = 0$.

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[2] $\mathbb{K}$ is the notation to describe either $\mathbb{R}$ or $\mathbb{C}$