MATH 4271 Dynamical Systems **Problem Set 2**

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Question 1. (Problem Set 1.2 Question 1, Perko) Find the eigenvalues and eigenvectors of the matrix A and show that $B = P^{-1}AP$ is a diagonal matrix. Solve the linear system $\dfrac{d\vec{y}}{dt} = B\vec{y}$ and then solve $\dfrac{d\vec{x}}{dt} = A\vec{x}$ using the corollary.

(a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

Solution. (a) We first find the eigenvalues of A as follows:

$$\begin{aligned} \chi_A(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \\ &= (\lambda - 3)^2 - 1 \\ &= (\lambda - 3 - 1)(\lambda - 3 + 1) \\ &= (\lambda - 4)(\lambda - 2) = 0 \end{aligned}$$

Therefore, our eigenvalues of $\lambda = 4$ and $\lambda = 2$. Now we find the eigenspaces. When $\lambda = 2$,

$$\begin{aligned} \mathcal{E}_A(2) &= \ker(A - 2I) \\ &= \ker \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \\ &= \ker \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Therefore, we may take $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to be the eigenvector corresponding to $\lambda = 2$. When $\lambda = 4$,

$$\begin{aligned} \mathcal{E}_A(4) &= \ker(A - 4I) \\ &= \ker \left\{ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\} \\ &= \ker \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Therefore, we may take $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to be the eigenvector corresponding to $\lambda = 4$. With our eigenvectors, the matrix P is given by

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and P^{-1} is given by

$$\begin{aligned} P^{-1} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{\det(P)} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \\ &= -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

and therefore, we can express $B = P^{-1}AP$ as

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Now we want to solve the equation $\frac{d\vec{y}}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Corresponding the coordinates, we have

$$\frac{dy_1}{dt} = 2y_1 \quad \frac{dy_2}{dt} = 4y_2$$

So solving each equation individually, we have $y_1(t) = b_1 e^{2t}$ and $y_2(t) = b_2 e^{4t}$ and so the solution of the linear system is given by

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} b_1 e^{2t} \\ b_2 e^{4t} \end{bmatrix}$$

Finally, we solve the system given by $\frac{d\vec{x}}{dt} = A\vec{x}$. Using the Corollary, we have

$$\begin{aligned} \vec{x}(t) &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & -e^{4t} \\ -e^{2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} + e^{4t} & -e^{2t} - e^{4t} \\ -e^{2t} + e^{4t} & e^{2t} + e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

as required.

(b) We first find the eigenvalues of A as follows:

$$\begin{aligned} \chi_A(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} \\ &= (\lambda - 1)^2 - 3^2 \\ &= (\lambda - 1 - 3)(\lambda - 1 + 3) \\ &= (\lambda - 4)(\lambda + 2) = 0 \end{aligned}$$

Therefore, our eigenvalues are $\lambda = 4$ and $\lambda = -2$. Now we find the eigenspaces. When $\lambda = -2$,

$$\begin{aligned}
\mathcal{E}_A(-2) &= \ker(A + 2I) \\
&= \ker \left\{ \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \right\} \\
&= \ker \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\
&= \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

So we may take $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as one of our eigenvectors. When $\lambda = 4$,

$$\begin{aligned}
\mathcal{E}_A(4) &= \ker(A - 4I) \\
&= \ker \left\{ \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \right\} \\
&= \ker \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\
&= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

So we may take $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as one of our eigenvectors. With our two eigenvectors, the matrix P is given by

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and P^{-1} is given by

$$P^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

and therefore, we can express $B = P^{-1}AP$ as

$$\begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Now we want to solve $\dfrac{d\vec{y}}{dt} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Corresponding the coordinates, we have

$$\frac{dy_1}{dt} = -2y_1 \quad \frac{dy_2}{dt} = 4y_2$$

So solving each of equation individually, we have $y_1(t) = c_1 e^{-2t}$ and $y_2(t) = c_2 e^{4t}$ and so the solution of the linear system is given by

$$\vec{y}(t) = \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{4t} \end{bmatrix}$$

Finally, we solve the system given by $\dfrac{d\vec{x}}{dt} = A\vec{x}$. Using the Corollary, we have

$$\begin{aligned}\vec{x}(t) &= \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} & -e^{4t} \\ -e^{-2t} & -e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & e^{-2t} + e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\end{aligned}$$

Question 2. (Problem Set 1.2 Question 2, Perko) Find the eigenvalues and the eigenvectors for the matrix A, solve the linear system $\dfrac{d\vec{x}}{dt} = A\vec{x}$, and determine the stable and unstable subspaces for the linear system.

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \vec{x}$$

Solution. First finding the eigenvalues of A, we have

$$\begin{aligned}\chi_A(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda)(-1-\lambda) = 0\end{aligned}$$

So we have eigenvalues $\lambda = 1$, $\lambda = 2$, and $\lambda = -1$.

Now we find the eigenspaces for each eigenvalue. For $\lambda = -1$,

$$\begin{aligned}\mathcal{E}_A(-1) &= \ker(A + I) \\ &= \ker \left\{ \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\} \\ &= \ker \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}\end{aligned}$$

So we may take $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to $\lambda = -1$ for A. For $\lambda = 1$,

$$\begin{aligned}
\mathcal{E}_A(1) &= \ker(A - I) \\
&= \ker \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \right\} \\
&= \ker \left\{ \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \\
&= \operatorname{span} \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

So we may take $\vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to $\lambda = 1$ for A. For $\lambda = 2$,

$$\begin{aligned}
\mathcal{E}_A(2) &= \ker(A - 2I) \\
&= \ker \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix} \right\} \\
&= \ker \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\} \\
&= \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}
\end{aligned}$$

So we may take $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as an eigenvector corresponding to $\lambda = 2$ for A.

Now we find the matrices P and P^{-1}. Indeed, with our eigenvectors

$$P = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and thus,

$$P^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

Therefore, the solution of the linear system is given as

$$
\begin{aligned}
\vec{x}(t) &= \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2e^{t} & 0 \\ 0 & -2e^{t} & e^{2t} \\ e^{-t} & e^{t} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} -2e^{t} & 0 & 0 \\ 2e^{t} - 2e^{2t} & -2e^{2t} & 0 \\ e^{-t} - e^{t} & 0 & -2e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\end{aligned}
$$

Question 3. (Problem Set 1.2 Question 3, Perko) Write the following linear differential equations with constant coefficients in the form of the linear system (1) and solve

$$\frac{d^3x}{dt^3} - 2\frac{d^2x}{dt^2} - \frac{dx}{dt} + 2x = 0$$

(Hint: Let $x_1 = x$ and $x_2 = \dfrac{dx_1}{dt}$, etc.)

Solution. Assume that $x(t) = e^{\alpha t}$ is a solution of the above equation. Then

$$
\begin{aligned}
e^{\alpha t}(\alpha^3 - 2\alpha^2 - \alpha + 2) &= 0 \\
e^{\alpha t}[\alpha^2(\alpha - 2) - (\alpha - 2)] &= 0 \\
e^{\alpha t}(\alpha^2 - 1)(\alpha - 2) &= 0 \\
e^{\alpha t}(\alpha + 1)(\alpha - 1)(\alpha - 2) &= 0
\end{aligned}
$$

So we have $\alpha = -1$, $\alpha = 1$ and $\alpha = 2$. Therefore, the general solution of the above equation is

$$x(t) = c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t}$$

as required.

Question 4. (Problem Set 1.2 Question 4, Perko) Using the corollary of this section, solve the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \vec{x}(0) = \vec{x}_0$$

(a) With A given by Question 1(a) and $\vec{x}_0 = (1, 2)^T$.

(b) With A given by Question 2 and $\vec{x}_0 = (1, 2, 3)^T$.

Solution. (a) We have the solution given as

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then solving the system, we obtain $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$ and so the unique solution of the system is given by

$$\vec{x}(t) = \begin{bmatrix} e^{2t} + e^{4t} & -e^{2t} - e^{4t} \\ -e^{2t} + e^{4t} & e^{2t} + e^{4t} \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ 2 \end{bmatrix}$$

(b) We have the solution given as

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Then solving the system, we obtain $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ -\frac{3}{2} \end{bmatrix}$ and so the unique solution is given by

$$\vec{x}(t) = \begin{bmatrix} -2e^t & 0 & 0 \\ 2e^t - 2e^{2t} & -2e^{2t} & 0 \\ e^{-t} - e^t & 0 & -2e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -1 \\ -\frac{3}{2} \end{bmatrix}$$

Question 5. (Problem Set 1.2 Question 5, Perko) Let A be the $n \times n$ matrix with real and distinct eigenvalues. Find conditions on the eigenvalues that are necessary and sufficient for $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$, where $\vec{x}(t)$ is any solution of $\frac{d\vec{x}}{dt} = A\vec{x}$.

Solution. As we have previously solved before, the general solution is given by

$$\vec{x}(t) = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} P^{-1}\vec{c}$$

Therefore, in order for $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$, we need that all eigenvalues $\lambda_i < 0$.