Lecture 13

Recall in the previous lecture, the flow for a nonlinear system $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$ given $\vec{x}(0) = \vec{x}_0$ satisfies the following properties:

(i) $\phi_0(\vec{x}_0) = \vec{x}_0$ for all $\vec{x}_0 \in U$

(ii) $(\phi_s \circ \phi_t)(\vec{x}_0) = \phi_{s+t}(\vec{x}_0)$ for all $t \in I(\vec{x}_0)$ and $s \in I(\phi_t(\vec{x}_0))$.

(iii) Moreover, if $s = -t$, then

$$\phi_{-t}(\phi_t(\vec{x})) = \phi_0(\vec{x}) = \vec{x}$$

for all $\vec{x} \in B(\vec{x}_0, \varepsilon)$ and

$$\phi_t(\phi_{-t}(\vec{y})) = \phi_0(\vec{y}) = \vec{y}$$

for all $\vec{y} \in \phi_t(U)$.

Furthermore, also note that because ϕ_t is a bijection, then $\phi_t^{-1}(\vec{x}) = \phi_{-t}(\vec{x})$ for all $\vec{x} \in U$.

Example 1. Show that the mapping $\psi_t : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, where $I(\vec{x}_0) = \mathbb{R}$ and $U = \mathbb{R}^2$, given by

$$\psi_t(\vec{x}) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \vec{x}$$

for $t \in \mathbb{R}$, and $\vec{x} \in \mathbb{R}^2$, defines a flow for the initial value problem

$$\frac{dx_1}{dt} = f_1(\vec{x}) \quad \frac{dx_2}{dt} = f_2(\vec{x}) \quad \vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

It is given that $\psi_t(\vec{x}_0)$ solves the differential equation given above. We show that this mapping satisfies the properties (i)-(iii) above. Indeed, when $t = 0$,

$$\psi_0(\vec{x}_0) = \begin{bmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{bmatrix} \vec{x}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}_0 = \vec{x}_0$$

so the first condition is satisfied.

Next, for $t \in I(\vec{x}_0)$ and $s \in I(\psi(\vec{x}_0))$,

$$\begin{aligned}
(\psi_s \circ \psi_t)(\vec{x}_0) &= \psi_s(\psi_t(\vec{x}_0)) \\
&= \psi_s\left(\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \vec{x}_0\right) \\
&= \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \vec{x}_0 \\
&= \begin{bmatrix} \cos(s)\sin(t) - \sin(s)\sin(t) & -\cos(s)\sin(t) - \sin(s)\cos(t) \\ \sin(s)\cos(t) + \cos(s)\sin(t) & -\sin(s)\sin(t) + \cos(s)\cos(t) \end{bmatrix} \vec{x}_0 \\
&= \begin{bmatrix} \cos(s+t) & -\sin(s+t) \\ \sin(s+t) & \cos(s+t) \end{bmatrix} \vec{x}_0
\end{aligned}$$

so the second condition is satisfied.

To show (iii), we simply need to use (ii) and replace $s = -t$, to see that $\psi_{-t}(\psi_t(\vec{x}_0)) = \vec{x}_0$.

Definition 1 (Invariant Set)**.** A set S is said to be an invariant set under the flow $\phi_t(x)$ on

$$\Omega = \{(t, x) : t \in \mathbb{R}, x \in U \subset \mathbb{R}^n\}$$

whenever $\phi_t(S) \subset S$. That is, for every $\vec{x} \in \phi_t(S)$, then $\vec{x} \in S$.

Geometrically, any solution of the system $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$ must remain inside S for all $t \in \mathbb{R}$.

- S is positively invariant if $\phi_t(S) \subset S$ for all $t \geq 0$
- S is negatively invariant if $\phi_t(S) \subset S$ for all $t \leq 0$

Example 2 (*)**.** Show that the set

$$S = \left\{x \in \mathbb{R}^2 : x_2 = -\frac{1}{3}x_1^2\right\}$$

is an invariant set under the flow of the dynamical system $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$, where $\vec{x} \in \mathbb{R}^2$, where

$$\vec{f}(\vec{x}) = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$$

We show that $\phi_t(S) \subset S$, i.e. if we take any $\vec{x} \in \phi_t(S)$, then $\vec{x} \in S$ as well. But we need to know what the flow is. In order to find this, we need to solve the differential equation above. Note that

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$$

so

$$\frac{dx_1}{dt} = -x_1 \quad \frac{dx_2}{dt} = x_2 + x_1^2$$

The first differential equation is easy to solve. Indeed,

$$\frac{dx_1}{dt} = -x_1 \Rightarrow \frac{1}{x_1}dx_1 = -dt \Rightarrow \int \frac{1}{x_1}dx_1 = -\int dt \Rightarrow \ln|x_1| = -t + C_1 \Rightarrow x_1(t) = \beta_1 e^{-t} \tag{1}$$

For the second differential equation, we will take (1) so that

$$\frac{dx_2}{dt} = x_2 + (\beta_1 e^{-t})^2 \Rightarrow \frac{dx_2}{dt} - x_2 = \beta_1^2 e^{-2t}$$

This is a linear equation with $p(t) = -1$, so the integrating factor is $e^{\int p(t)dt} = e^{-t}$ and so, with $q(t) = \beta_1 e^{-2t}$

$$\begin{aligned} x_2(t) &= e^{-\int p(t)dt}\left(\int q(t)e^{\int p(t)dt}dt + \beta_2\right) \\ &= e^t\left(\int \beta_1^2 e^{-2t}e^{-t}dt + \beta_2\right) \\ &= e^t\left(\int \beta_1^2 e^{-3t}dt + \beta_2\right) \\ &= e^t\left(-\frac{\beta_1^2}{3}e^{-3t} + \beta_2\right) \\ &= -\frac{\beta_1^2}{3}e^{-2t} + \beta_2 e^t \end{aligned}$$

Therefore, the solution (flow) of the system of equations is

$$\phi_t(\vec{x}_0) = \begin{bmatrix} \beta_1 e^{-t} \\ -\frac{\beta_1^2}{3}e^{-2t} + \beta_2 e^t + \beta_1^2 e^{-2t} \end{bmatrix}$$

Now we need to show that $\phi_t(S) \subset S$. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in S$ such that $\phi_t(\vec{v}) \in \phi_t(S)$. We need to show that $\phi_t(\vec{v}) \in S$. Using the fact that $v_2 = -\frac{1}{3}v_1^2$,

$$\phi_t(\vec{v}) = \begin{bmatrix} v_1 e^{-t} \\ -\frac{v_1^2}{3}e^{-2t} - \frac{1}{3}v_1^2 e^t + v_1^2 e^{-2t} \end{bmatrix} = \begin{bmatrix} v_1 e^{-t} \\ \frac{v_1^2}{3}(2e^{-2t} - e^t) \end{bmatrix} = \begin{bmatrix} v_1 e^{-t} \\ -\frac{1}{3}(v_1^2 e^t - 2v_1^2 e^{-2t}) \end{bmatrix}$$

Something is not right here........need to double check this part.