Recall in the previous lecture we have started to generalize subcase 2 to $n \times n$. We define and generalized eigenvectors.

Definition 1. If λ is an eigenvalue of A with multiplicity $m \leq n$, then any nonzero solution $\vec{v}$ of $(A - \lambda I_n)^k \vec{v} = \vec{0}$ for $k = 1, 2, ..., m$, is called a generalized eigenvector of A.

In particular, for 2×2, if $\vec{v}_2$ is a generalized eigenvector, it can be obtained as

$$(A - \lambda I_2)\vec{v}_2 = \vec{v}_1$$

and so

$$A\vec{v}_2 = \vec{v}_1 + \lambda I_2 \vec{v}_2$$

The general solution is given by

$$\vec{x}(t) = \sum_{j=1}^{m} c_j \vec{x}_j(t)$$

where if $j = 1$, $\vec{x}_1(t) = \vec{v}_1 e^{\lambda t}$, if $j = 2$, then $\vec{v}_1 t e^{\lambda t} + \vec{v}_2 e^{\lambda t}$, and for when $j = 3$, then

$$\vec{x}_3(t) = \frac{1}{2!} t^2 \vec{x}_1(t) + t\vec{v}_2 e^{\lambda t} + \vec{v}_3 e^{\lambda t}$$

In general,

$$\vec{x}_n(t) = \frac{1}{(n-1)!} t^{n-1} \vec{x}_1(t) + \frac{1}{(n-2)!} t^{n-2} \vec{x}_2(t) + \cdots + \vec{v}_{n-1} t e^{\lambda t} + \vec{v}_n e^{\lambda t}$$

<u>Case 3:</u> If A has k pairs of complex conjugate eigenvalues, i.e. $\lambda_j = a_j + ib_j$, where $j = 1, 2, ...k$. Assume that $\vec{v}_j = \vec{\alpha}_j + i\vec{\beta}_j$ be the corresponding eigenvectors. Then the solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is

$$\vec{x}(t) = \sum_{j=1}^{k} (c_j \vec{u}_j + d_j \vec{v}_j)$$

where $\vec{u}_j = (\vec{\alpha}_j \cos(b_j t) - \vec{\beta}_j \sin(b_j t))e^{a_j t}$ and $\vec{v}_j = (\vec{\alpha}_j \cos(b_j t) + \vec{\beta}_j \sin(b_j t))e^{a_j t}$.

Example 1. Solve the system $\frac{dx_1}{dt} = 3x_1 - 4x_2$, $\frac{dx_2}{dt} = x_1 - x_2$.

Recall that to find the eigenvalues, we solve $\det(A - \lambda I_2) = 0$. Indeed,

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$. Finding the eigenvalues gives

$$\begin{aligned} \chi_A(\lambda) = \det(A - \lambda I_2) &= \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} \\ &= (3-\lambda)(-1-\lambda) + 4 \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda - 1)^2 = 0 \end{aligned}$$

We have that $\lambda = 1$. Next, we find the eigenvector corresponding to $\lambda = 1$. Indeed,

$$\begin{aligned}
\mathcal{E}_A(1) &= \ker\left\{\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix}\right\} \\
&= \ker\left\{\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\right\} \\
&= \ker\left\{\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right\} \\
&= \ker\left\{\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} 2s \\ s \end{bmatrix} : s \in \mathbb{R}\right\} \\
&= \operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}
\end{aligned}$$

Therefore, $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of the eigenspace $\mathcal{E}_A(1)$. Then using the generalized eigenvector formula, we have $\vec{v}_1$, so we find $\vec{v}_2$ by $\vec{v}_2 = (A - \lambda I_2)\vec{v}_1$. Indeed,

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and so solving the system of equations above, we obtain

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore, the general solution of the system is

$$\begin{aligned}
\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) \\
&= c_1\vec{v}_1 e^t + c_2(t\vec{x}_1(t) + \vec{v}_2 e^{\lambda t}) \\
&= c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2\left(t\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t\right)
\end{aligned}$$

Example 2. Solve the system given by

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\vec{x}$$

Then $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, and so finding the eigenvalues we have

$$\begin{aligned}
\chi_A(\lambda) &= \det(A - \lambda I_2) \\
&= \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 + 1 \\
&= (\lambda - 1)^2 - i^2 \\
&= (\lambda - 1 + i)(\lambda - 1 - i) = 0
\end{aligned}$$

So we have eigenvalues $\lambda = 1-i$ and $\lambda = 1+i$. Now we find the eigenspaces of $\lambda = 1-i$ and $\lambda = 1+i$. Indeed, for $\lambda = 1-i$,

$$\begin{aligned}
\mathcal{E}_A(1-i) &= \ker\left\{\begin{bmatrix} 1-(1-i) & 1 \\ -1 & 1-(1-i) \end{bmatrix}\right\} \\
&= \ker\left\{\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\right\} \\
&= \ker\left\{\begin{bmatrix} 1 & -i \\ -1 & i \end{bmatrix}\right\} \\
&= \ker\left\{\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} is \\ s \end{bmatrix} : s \in \mathbb{R}\right\} \\
&= \operatorname{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}
\end{aligned}$$

and similarly, one can find that $\mathcal{E}_A(1+i) = \operatorname{span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$. So we may take two eigenvectors $\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \in \mathcal{E}_A(1-i)$ and $\vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \in \mathcal{E}_A(1+i)$. Then note that

$$\begin{aligned}
\vec{v}_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{\alpha}_1 + i\vec{\alpha}_2 \\
\vec{v}_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \vec{\alpha}_1 - i\vec{\alpha}_2
\end{aligned}$$

and so

$$\begin{aligned}
\vec{u}_1 &= e^{at}\left(\vec{\alpha}_1\cos(bt) - \vec{\alpha}_2\sin(bt)\right) = e^t\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t)\right) \\
\vec{u}_2 &= e^{at}\left(\vec{\alpha}_1\cos(bt) + \vec{\alpha}_2\sin(bt)\right) = e^t\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t)\right)
\end{aligned}$$

Therefore,

$$\vec{x}(t) = c_1\vec{u}_1 + d_1\vec{u}_2$$

Definition 2. Any set $\{\vec{x}_1(t), ..., \vec{x}_n(t)\}$ of solution (1) is said to be a Fundamental set of solution of (1) if

1. The set is linearly independent
2. For any $\vec{x}(t) \in \mathbb{R}^n$ of (1) can be written in the form

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}(t) + \cdots + c_n\vec{x}_n(t)$$

The matrix

$$\phi(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \vdots & \vec{x}_n(t) \end{bmatrix}$$

is called the Fundamental Matrix of (1), which is an $n \times n$ matrix.

Remark 1. Note that $\phi(t)$ is an invertible matrix, since $\vec{x}_1(t), ..., \vec{x}_n(t)$ are linearly independent vectors.

Let $\vec{x}_1(t)$, $\vec{x}_2(t)$,..., $\vec{x}_n(t)$ is a linearly independent solution. The general solution of (1) is

$$\begin{aligned}
\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t) \\
&= \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\
&= \phi(t)\vec{c}
\end{aligned}$$

where $\vec{c}$ is an arbitrary constant vector. In particular, suppose we are given an initial condition that $\vec{x}(0) = \vec{x}_0$. Then $\vec{x}_0 = \phi(0)\vec{c}$, and so $\phi^{-1}(0)\vec{x}_0 = \vec{c}$. Thus, the solution of the initial value problem is

$$\vec{x}(t) = \phi(t)\phi^{-1}(0)\vec{x}_0 \tag{*}$$