Projective Modules and Injective Modules

Definition: A module P over R is projective if and only if for every diagram of R-modules

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed] & \\ A \arrow[r, "g"'] & B \arrow[r] & 0 \end{tikzcd}$$

with the bottom row exact, there exists a morphism $h: P \to A$ with $gh = f$.

Exercise: If R is a ring with identity, and P is a unitary R-module, it suffices to check the above A, B are unitary.

Theorem: Every free module over a ring with identity is projective.

Proof: Let F be a free module over R and assume we have a diagram

$$\begin{tikzcd} & F \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

with the bottom row exact and A, B are unitary. Since F is free on X, for example, $i: X \to F$ and since g is onto, $\forall x \in X \ \exists a_x \in A$, $g(a_x) = f(i(x))$.

We can define our homomorphism by

$$h\left(\sum c_x i(x)\right) = \sum c_x a_x \in A,$$

so $h: F \to A$. We check that $gh = f$. Indeed,

$$gh\left(\sum c_x i(x)\right) = g\left(\sum c_x a_x\right) = \sum c_x g(a_x)$$
$$= \sum c_x f(i(x)) = f\left(\sum c_x i(x)\right)$$

Therefore $gh = f$.

So every R-module is the homomorphic image of a projective module.

Theorem: Let R be a ring and P be an R-module. TFAE:

(a) P is projective.

(b) Every short exact sequence $0 \to A \to B \to P \to 0$ splits (so $B \simeq A \oplus P$)

(c). There exists a free module F and another module K so that $F \simeq K \oplus P$.

Example: $\mathbb{Z}_6 \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_2$, so $\mathbb{Z}_2$ is a projective module over $\mathbb{Z}_6$.

Proof:

(a) $\Rightarrow$ (b) Assume P is projective, and consider a short exact sequence $0 \to A \to B \to P \to 0$. The diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "id"] & \\ B \arrow[r, "g"] & P \arrow[r] & 0 \end{tikzcd}$$

By definition, $\exists\, h: P \to B$ such that $gh = id$. This is what it means for a sequence to split.

(b) $\Rightarrow$ (c) Assume (b). Then $\exists g : F \to P$ and let $K = \ker(g)$. Then

$$0 \to K \to F \xrightarrow{g} P \to 0$$

so by (ii), $F \simeq K \oplus P$.

(c) $\Rightarrow$ (a) Assume $F \simeq K \oplus P$. Let $\pi : F \to P$ be the projection, and $i : P \to F$ be the inclusion map. Suppose we have

$$\begin{tikzcd}
& F \arrow[d, "\pi"] \arrow[ddl, dashed, "h_1"'] \\
& P \arrow[u, bend right, "i"'] \arrow[d, "f"] \arrow[dl, dashed, "h"'] \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

with the bottom row exact. Because F is projective, there exists some $h_1 : F \to A$ such that $h_1 g = f\pi$.

Let $h = h_1 i$. Then $gh = gh_1 i = f\pi i = f$.

Proposition: A direct sum $\sum_{i \in I} P_i$ is projective if and only if $\forall i \in I$, P_i is projective.

Partial Proof: (see notes for full proof)

In one direction, if $\sum_{i \in I} P_i$ is projective

$$\begin{tikzcd}
& \sum_{i \in I} P_i \arrow[d, "\pi"] \arrow[ddl, "h"'] \\
& P_j \arrow[u, bend right, "i"'] \arrow[d, "f"] \\
A \arrow[r, "g"'] & B \arrow[r] & 0
\end{tikzcd}$$

same proof $\Rightarrow P_j$ is projective, so each P_j is projective.

Conversely

$$\begin{tikzcd} & P_j \arrow[ddl, "h_j"'] \arrow[d, shift right, "i_j"'] & \\ & \sum_{i\in I} P_i \arrow[u, shift right, "\pi_j"'] \arrow[d, "f"] & \\ A \arrow[r, "g"'] & B \arrow[r] & 0 \end{tikzcd}$$

$gh_j = f i_j$. "Stitch together" the $h_j : P_j \to A$ to $h : \sum_{i\in I} P_i \to A$ such that $gh = f$.

Injective Modules

Definition: An R-module J is injective if and only if given a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[dl, "h"] \\ & J & \end{tikzcd}$$

with the top row exact, there exists a morphism $h : B \to J$ with $f = hg$.

Proposition: Let R be a ring with identity and J be an R-module. TFAE:

(a) J is injective.

(b) Every short exact sequence $0 \to J \to B \to C \to 0$ splits, so $B \simeq J \oplus C$.

(c) J is a direct summand of every module of which it is a submodule.

Proposition: Every unitary module over a ring with identity can be embedded into an injective module.

Lemma: Let R be a ring with identity. A unitary R-module J is injective if and only if for every ideal L in R and an R-module homomorphism $L \to J$, there is an extension $R \to J$.

Sketch Proof: One direction follows by definition. Indeed,

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "id"] \arrow[d, "f"'] & R \arrow[dl, "h"] \\ & J & \end{tikzcd}$$

So if J is injective, morphisms on ideals extend.

For the other direction, assume J has the property. Consider

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \\ & J & \end{tikzcd}$$

with the top row exact. Let S be the set of all R-module morphisms $h: C \to J$ with $Im(g) \subseteq C \subseteq B$ and $hg = f$. First, note $S \neq \emptyset$. We can take $fg^{-1}: Im(g) \to J$. Partially Order S by extension $h_1 \leq h_2$ if and only if h_2 is an extension of h_1, i.e. $h_2|_{dom(h_1)} = h_1$, and (domain(h_1) $\subseteq$ domain(h_2)). In $(S, \leq)$ chains of upper bounds. By Zorn's Lemma, there exists a maximal element $h: H \to J$. If $H = B$, we are done. Otherwise, let $b \in B \setminus H$ and let $L = \{r \in R : rb \in H\} \leq R$. L is an ideal of R.

We have a function $r \mapsto h(rb)$, $L \to J$ and by assumption, this extends to $k: R \to J$ with $k(r) = h(rb)$ for $r \in L$. Let $c = k(1_R)$. Define an R-module homomorphism $H + Rb \to J$ by $a + rb \mapsto h(a) + rc$. (Check this is well-defined and showing it is an R-module homomorphism).
But this gives a proper extension of h to $B \supseteq H + rb \supsetneq H$. which is a contradiction. Hence, $H = B$.

Example: $\mathbb{Z}$-modules $\Leftrightarrow$ abelian groups.

Proposition: An abelian group G is divisible if and only if it is an injective $\mathbb{Z}$-module.

Note: G is divisible if and only if $\forall g \in G$, $n \in \mathbb{Z}$, there is an $h \in G$ such that $nh = g$. (e.g. $(\mathbb{Q}, +)$ is, but $(\mathbb{Z}, +)$ is not)

Proof: If G is an injective $\mathbb{Z}$-module, let $g \in G$ and $n \neq 0$ be an integer. Define $f: n\mathbb{Z} \to G$ by $f(nm) = mg$ (which is a group homomorphism). Because G is injective f extends to $\hat{f}: \mathbb{Z} \to G$. Now take $h = \hat{f}(1)$. We have

$nh = n\hat{f}(1) = f(n) = g.$

Conversely, if G is divisible, let $f: I \to G$ be a $\mathbb{Z}$-module homomorphism with $I \leq \mathbb{Z}$ an ideal. If $I \neq \{0\}$ then $I = n\mathbb{Z}$ for some $n \neq 0$ integer. Then $\tilde{f}: n\mathbb{Z} \to G$ Extend f to $\mathbb{Z}$ by $\tilde{f}(x) = \frac{1}{n} f(nx)$. (Check $\hat{f}$ is a $\mathbb{Z}$-module homomorphism). If $I = \{0\}$, put $f(x) = 0$ for $x \in \mathbb{Z}$. So every $\mathbb{Z}$-module homomorphism from an ideal of $\mathbb{Z}$ to G extends to all of $\mathbb{Z}$, so G is an injective $\mathbb{Z}$-module.

Modules Over PIDs

Theorem: Let F be a free module over a PID, and let $G \subseteq F$ be a submodule. Then G is free and $\operatorname{rank}(G)$ is at most $\operatorname{rank}(F)$.

Proof: Let R be a PID, let F be a free module over R, and $G \subseteq F$ a submodule. We apply induction on $n = \operatorname{rank}(F)$

Base Case: For $n=1$, we have $F \simeq Rx \simeq R$ and this isomorphism identifies G with a submodule of R. By assumption, this is a principal ideal. If G maps to yR then $G = Ryx$.

- If $y=0$, $\operatorname{rank}(G)=0$.
- If $y \neq 0$, $\operatorname{rank}(G) = 1$.

Inductive Step: Assume the statement holds for rank less than n and assume F is a free module of rank n. Then F has a basis $\{x_1, \dots, x_n\}$ so the set

$$F' = Fx_1 + \cdots + Fx_n \subseteq F.$$

Here, F' is a free R-module of rank $n-1$, and $G' = G \cap F'$ is a submodule of F', so its a free R-module of rank at most $n-1$.

Now consider

$$G/G' \simeq G/(G \cap F') \simeq (G+F')/F'$$

which holds by the 2nd Isomorphism Theorem, and $G, F' \subseteq F$, So

$$G/G' \cong (G+F')/F' \subseteq F/F' \cong R.$$

Thus, G/G' is a free R-module of rank at most 1.

It follows that G/G' is projective, so

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G/G' \longrightarrow 0$$

splits and $G \cong G' \oplus G/G'$, thus, G is the direct sum of a free module of rank at most $n-1$ and a free module of rank at most 1, so G is a free module of rank at most n.

Corollary: Over a PID, TFAE:

(a) A unitary module is free.

(b) A unitary module is projective.

Proof: (a) $\Rightarrow$ (b) easy

(b) $\Rightarrow$ (a) Assume that P is a projective unitary module over a PID. Then $F \cong K \oplus P$ for some free module F and some module K. So P is isomorphic to a submodule of a free module, so it is free.

Notation: Let A be a module over an integral domain R, and let $a \in A$. Define $A^{a} = \{r \in R : ra = 0\}$, which is called the annihilator of a.

Theorem: Let R be an integral domain, A be an R-module.

(i) A^{a} is an ideal of R

(ii) $A_t = \{a \in A : A^0 \neq \langle 0 \rangle\}$ is a submodule of A

(iii) For each $a \in A$, there exists an isomorphism

$$R/A^0 \simeq Ra.$$

Proof: exercise.

Note A_t is called the torsion submodule of A.

Definition: A is **torsion-free** if $A_t = \{0\}$.

Example:

(α) Consider $\mathbb{Z}_n$ as a $\mathbb{Z}$-module. Then $(\mathbb{Z}_n)_t = \mathbb{Z}_n$

(β) $\mathbb{Q}$ as a $\mathbb{Z}$-module is torsion-free because if $x \in \mathbb{Q}$ and $nx = 0$ for $n \neq 0$ integer, $x = 0$, so $\mathbb{Q}_t = \{0\}$.

Theorem: A finitely generated torsion-free module over a PID is free.

Theorem: If A is a finitely generated module over a PID, then $A \simeq A_t \oplus F$, where F is a free R-module of finite rank.

Proof: Consider A/A_t, which is torsion-free, because if $r \in R$ $r \neq 0$,

$$0 + A_t = r(x + A_t) = rx + A_t, \quad \text{so} \quad rx \in A_t$$

Hence, there is $s \in R$ nonzero such that $s(rx) = 0$, but $(sr)x = 0 \Rightarrow x \in A_t$. So A/A_t is torsion-free and finitely generated, so $A/A_t = F$ is a free module of finite rank. Now

$$0 \longrightarrow A_t \longrightarrow A \longrightarrow A/A_t \longrightarrow 0$$

is exact and A/A_t is projective, so

$$A \simeq A_t \oplus A/A_t.$$

Theorem: A finitely generated, torsion-free module over a PID is a free module of finite rank.

Proof: Let A be a finitely generated torsion-free module over a PID R. Let $X \subseteq A$ be a finite set of generators for A over R. Think about sets $S = \{x_1, \dots, x_n\} \subseteq X$ with the property that $r_1 x_1 + \cdots + r_n x_n = 0 \Rightarrow r_i = 0 \ \forall i$. There are some subsets with this property because if $x \in X$, then $rx = 0$ if and only if $r = 0$ (torsion-free). Let S be a maximal linearly independent set. The submodule $F \subseteq A$ generated by S is a free module of rank $|S|$.

Claim: There exists $r \in R$ nonzero such that $rX \subseteq F$.

Given the claim, this implies that $rA \subseteq F$. But $rA \simeq A$ as A is torsion-free, so A is a free module of rank at most $\operatorname{rank}(F)$.

To prove the claim, let $y \in X \setminus S$. By the maximality there are $r_1, \dots, r_{n+1} \in R$ not all zero with

$$r_1 x_1 + \cdots + r_n x_n + r_{n+1} y = 0$$

If $r_{n+1} = 0$, this contradicts the linear independence of S, so $r_{n+1} \neq 0$. Thus,

$$r_{n+1} y = -r_1 x_1 - \cdots - r_n x_n \in F.$$

We shown for $y \in X$, there exists $r_y \neq 0 \in R$ such that $r_y y \in F$. If $r = \prod_{y \in X} r_y \in R$, then $r \neq 0$ and $rX \subseteq F$.