Definition: Let F/K be an extension field. Then $\alpha \in F$ is **algebraic over K** if $f(\alpha) = 0$ for some nonzero $f \in K[x]$, and α is called **transcendental over K** if α is not algebraic over K. Otherwise, F is said to be **algebraic** if every element of F is algebraic over K, and **transcendental** over K, otherwise.

Example:

(α) i is algebraic over $\mathbb{Q}$ because it is a root of x^2+1. Also, $\mathbb{Q}(i)$ is algebraic over $\mathbb{Q}$ because $a+bi$ is a root of $x^2-2ax+(a^2+b^2)=0$.

(β) If $\alpha \in K$, then α is algebraic over K $(x-\alpha)$, so K is an algebraic extension of itself.

(γ) $\mathbb{R}$ is transcendental over $\mathbb{Q}$. For example, $\pi, e, \sum_{n=1}^{\infty} 2^{-n!}$ are transcendental.

Theorem: Let F/K be an extension of fields. TFAE:

(a) $\alpha \in F$ is transcendental over K

(b) there is an isomorphism $K(\alpha) \simeq K(x)$ which is the identity on K.

Proof: (b) $\Rightarrow$ (a) exercise.

(a) $\Rightarrow$ (b) Suppose that $\alpha \in F$ is transcendental over K. Let $\phi: K[x] \to K(\alpha)$ by $\phi(f(x)) = f(\alpha)$. Then ϕ is a homomorphism and $\ker(\phi) = \{0\}$. So we can extend to $K(x)$ by

$\phi\left(\frac{f(x)}{g(x)}\right) = \frac{f(\alpha)}{g(\alpha)}$ defined because $g(\alpha) \neq 0$ for $g(x) \neq 0$. Then ϕ is an into homomorphism $K(x) \to K(\alpha)$ which is the identity on K. But ϕ is also onto.

Theorem: Let F/K be an extension and let $\alpha \in F$ be algebraic over K.

(i) $K(\alpha) = K[\alpha]$

(ii) $K(\alpha) \simeq K[x]/\langle f(x)\rangle$ where f is the least degree monic polynomial with $f(\alpha) = 0$.

(iii) $[K(\alpha) : K] = \deg(f)$ where f is the minimal polynomial.

(iv) Every element $K(\alpha)$ can be written uniquely as $c_0 + c_1\alpha + c_2\alpha + \cdots + c_{n-1}\alpha^{n-1}$ with $c_i \in K$ and $n = \deg(f)$, i.e. $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis for $K(\alpha)$ over K.

Proof: (i)-(ii) Let $\phi : K[x] \to K(\alpha)$ be $\phi(g(x)) = g(\alpha)$ be a homomorphism. Because α is algebraic over K, $\ker(\phi) \neq \{0\}$, but $K[x]$ is a principal ideal domain so $\ker(\phi) = \langle f(x)\rangle$ for some $f \in K[x]$. Replace $f(x)$ by $c^{-1}f$ to make $f(x)$, without loss of generality, monic.

Note that $Im(\phi) = K[\alpha]$, so by the First Isomorphism Theorem, $K[\alpha] \simeq K[x]/\langle f(x)\rangle$. Now, $f(x)$ is irreducible. because if $f(x) = g(x)h(x)$, then $f(\alpha) = g(\alpha)h(\alpha)$ $\Rightarrow 0 = g(\alpha)h(\alpha)$ so either $g(\alpha) = 0$ or $h(\alpha) = 0$. Thus $g(x)$ or $h(x)$ is a multiple of f, and the other is a constant (degrees). Because $f(x)$ is irreducible, then $\langle f(x)\rangle$ is maximal, so $K[x]/\langle f(x)\rangle$ is a field. Thus, $K[\alpha]$ is a field and $K \cup \{\alpha\} \subseteq K[\alpha] \subseteq K(\alpha)$, so $K[\alpha] = K(\alpha)$ because $K(\alpha)$ is the smallest field.

Now, (ii) follows immediately because $K[\alpha] = K(\alpha) \simeq K[x]/\langle f(x)\rangle$. Note now that $\phi: K[x]/\langle f(x)\rangle \to K(\alpha)$ is an isomorphism of fields which is the identity on K, so it also gives an isomorphism of vector spaces over K. Thus, $\{1, x, \ldots, x^{n-1}\}$ form a basis for $K[x]/\langle f(x)\rangle$ as a vector space (with $n = \deg(f)$) over K because the coset $g(x) + \langle f(x)\rangle$ is represented uniquely by $r(x)$ where $g(x) = q(x)f(x) + r(x)$ and $\deg(r) < \deg(f)$. Then $Im(\phi)$ is a basis for $K(\alpha)$ over K, this is $\{1, \alpha, \ldots, \alpha^{n-1}\}$, proving (iii)-(iv)

Let $\alpha \in F$ be algebraic over K, and let $f(x)$ be its minimal polynomial. If $f(\beta) = 0$ where β is in some extension), then there exists an isomorphism $\phi: K(\alpha) \to K(\beta)$ such that

(i) $\phi(a) = a$ for all $a \in K$

(ii) $\phi(\alpha) = \beta$.

Indeed, $K(\alpha) \simeq K[x]/\langle f(x) \rangle \simeq K(\beta)$.

Example:

(α) $\mathbb{Q}(\sqrt{2})$, the minimal polynomial is $f(x) = x^2 - 2$ but also $f(-\sqrt{2}) = (-\sqrt{2})^2 - 2 = 0$, so there exists an isomorphism $\phi: \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(-\sqrt{2})$ such that $\phi(a) = a$ for all $a \in \mathbb{Q}$ and $\phi(\sqrt{2}) = -\sqrt{2}$. So

$$\phi(a + b\sqrt{2}) = \phi(a) + \phi(b\sqrt{2}) = a - b\sqrt{2}.$$

(β) $\mathbb{Q}(\sqrt[3]{2})$, the minimal polynomial is $x^3 - 2$.

$$x^3 - 2 = 0 \Rightarrow x^3 = 2 \qquad x^3 = 2e^{2\pi i k} \Rightarrow x = \sqrt[3]{2}e^{\frac{2\pi i k}{3}}$$

$$x = \sqrt[3]{2}, \quad x = \sqrt[3]{2}e^{\frac{2\pi i}{3}}, \quad x = \sqrt[3]{2}e^{\frac{4\pi i}{3}}$$

Then, $\underbrace{\mathbb{Q}(\sqrt[3]{2})}_{\subset \mathbb{R}} \simeq \mathbb{Q}(\underbrace{\sqrt[3]{2}e^{\frac{2\pi i}{3}}}_{\not\subset \mathbb{R}}) \simeq \mathbb{Q}(\underbrace{\sqrt[3]{2}e^{\frac{4\pi i}{3}}}_{\not\subset \mathbb{R}})$

Theorem: If F/K is an extension and $[F:K] < \infty$ then F is algebraic and finitely generated over K.

Proof: If $[F:K] = n$, and $\{\alpha_1, \dots, \alpha_n\}$ is a basis for F as a vector space over K, then

$$F = \left\{ \sum_{i=1}^{n} c_i \alpha_i : c_i \in K \right\} \subset K(\alpha_1, \dots, \alpha_n) \subset F$$

so F is generated over K as a field by $\alpha_1, \dots, \alpha_n$.

Now let $\alpha \in F$. Since $\dim_K(F) = n$, we know that $\{1, \alpha, \dots, \alpha^{n-1}\}$ are linearly dependent over K.

So there exists $c_0, \dots, c_n \in K$ not all 0 such that

$$c_0 + c_1\alpha + \cdots + c_n\alpha^n = 0$$

But α is a root of the nonzero polynomial, so

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

Example: $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$

Then $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4$

$\sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$ so $\sqrt{2} + \sqrt{3}$ is algebraic.

$\{1, \sqrt{2} + \sqrt{3}, (\sqrt{2} + \sqrt{3})^2, (\sqrt{2} + \sqrt{3})^3\}$ are linearly dependent over $\mathbb{Q}$.