Math 6122 Algebra II

TR 2:30 - 4:00 SC 211

Grading Scheme

- 15% Assignments
- 15% Participation
- 20% Midterm Exam (Feb. 27)
- 30% Final Exam

Material Covered

- Rings and Modules
- Fields and Galois Theory
- Structure of Fields
- Commutative Algebras
- Algebraic Geometry
- Categories.

Fields and Galois Theory

Field Extensions

Definition: A field F is an extension of a field K if and only if K is a subfield of F. Write F/K for the extension.

Example: $\mathbb{C}$ is an extension of $\mathbb{R}$. Note that $\mathbb{C} \simeq \mathbb{R}^2$ as a vector space over $\mathbb{R}$.

In general, if F/K is an extension field, F is also a K-vector space.

Definition: The degree of the extension F/K is $[F:K]$ (e.g. $[\mathbb{C}:\mathbb{R}] = 2$)

Note that $[\mathbb{R}:\mathbb{Q}] = \infty$ or $|\mathbb{R}|$.

Theorem: Let F/E and E/K be field extensions. Then

$$[F:K] = [F:E][E:K]$$ and F/K is also a field extension.

Proof: If $\{a_i\}_{i\in I}$ is a basis for E/K and $\{b_j\}_{j\in J}$ is a basis for F/E, then $\{a_i b_j\}_{i\in I, j\in J}$ is a basis for F/K

Definition: An intermediate field of F/K is a field E such that $K \leq E \leq F$.

Definition: If F is a field and $X \subset F$, then the subfield generated by X, is

$\langle X \rangle = \bigcap \{ R : R \text{ is a subfield and } X \subset R \}$

Def: If K is a subfield of F and $X \subset F$, then

- $K(X)$ is the subfield generated by $K \cup X$
- $K[X]$ is the subring generated by $K \cup X$.

Example: $\mathbb{Q}(i) = \overset{E}{\overbrace{\{ a+bi : a,b \in \mathbb{Q} \}}}$. Indeed,

$\mathbb{Q} \cup \{i\} \subset \mathbb{Q}(i)$, so $E \subset \mathbb{Q}(i)$. Also, E is also a field and $\mathbb{Q} \cup \{i\} \subset E$, so $\mathbb{Q}(i) \subset E$.

Example: $\mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} : a,b \in \mathbb{Q} \}$. In particular $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ because $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

Definition: F/K is called a simple extension if and only if $F = K(\alpha)$ for some $\alpha \in F$.

Theorem: For F/K a field extension and $\alpha \in F$, then

(i) $K[\alpha] = \{ f(\alpha) : f \in K[x] \}$ (polynomial)

(ii) $K(\alpha) = \{ f(\alpha) : f \in K(x) \text{ with } f \text{ defined at } \alpha \}$ (rational functions)