Math 6122, Winter 2025
Problem Set 1

due January 16th

The first assignment is mostly about reviewing some concepts from 6121, and maybe learning one or two new things. Look up any definitions you don't know in an algebra book (Hungerford, or similar) or Wikipedia, but do the exercises yourself!

Problem 1. Let R and S be commutative rings with identity, and let $R \times S$ be the usual product ring.

i. Prove that there is a natural bijection between ideals of $R \times S$, and pairs of an ideal of R and an ideal of S. (Hint: First show that if $(r, s) \in I \subseteq R \times S$, then $(r, 0), (0, s) \in I$.)

ii. Prove that there is a natural bijection between the prime ideals of $R \times S$ and the disjoint union of the prime ideals of R and the prime ideals of S.

Problem 2. Let $p, q \in \mathbb{Z}$ be primes, let

$$\mathbb{Z}\left[\frac{1}{p}\right] = \left\{\frac{a}{p^e} : a, e \in \mathbb{Z}, e \geq 0\right\} = \{f(1/p) : f(x) \in \mathbb{Z}[x]\}$$

(which Hungerford calls $\mathbf{Z}(p^\infty)$), and let $\mathbb{F}_q$ be the field with q elements, both viewed as $\mathbb{Z}$-modules. Describe

$$M_{p,q} := \mathbb{Z}\left[\frac{1}{p}\right] \otimes_{\mathbb{Z}} \mathbb{F}_q.$$

(There are two cases.)

Problem 3. Give an example of two infinite unitary $\mathbb{Z}$-modules M and N such that $M \otimes_{\mathbb{Z}} N$ is trivial.

Problem 4. Let R be a commutative ring with identity, and let A be a unitary R-module. Define the kth exterior power of A to be

$$\bigwedge_R^k A = (A \otimes_R A \otimes_R \cdots \otimes_R A)/M,$$

where there are k terms in the tensor product, and M is the submodule generated by all terms $a_1 \otimes \cdots \otimes a_k$ such that $a_i = a_j$ for some $i \neq j$ (we will usually drop the subscript-R, and for $k = 1$ we adopt the convention that $M = \{0\}$). We denote the image of $a_1 \otimes \cdots \otimes a_k$ in the quotient by $a_1 \wedge \cdots \wedge a_k$, and an element of this form is sometimes called a pure wedge or a blade (in analogy to a pure tensor, which is an element of the form $a_1 \otimes \cdots \otimes a_k$.) Note that various basic properties follow from the analogous properties for the tensor product, for example

$$a_1 \wedge \cdots \wedge ra_j \wedge \cdots \wedge a_k = ra_1 \wedge \cdots \wedge a_j \wedge \cdots \wedge a_k$$

and

$$a_1 \wedge \cdots \wedge (a_j + b_j) \wedge \cdots \wedge a_k = (a_1 \wedge \cdots \wedge a_j \wedge \cdots \wedge a_k) + (a_1 \wedge \cdots \wedge b_j \wedge \cdots \wedge a_k).$$

i. In $\bigwedge^2 A$, prove that $x \wedge y = -y \wedge x$. (Hint: Use the fact that $(x + y) \wedge (x + y) = 0$.)

ii. If A is a free R-module, and e_1, e_2 form a basis for A over R, show that $e_1 \wedge e_2$ is a basis for $\bigwedge^2 A$. Describe the kernel of the map $A \oplus A \to \bigwedge^2 A$ given by $(x, y) \mapsto x \wedge y$.

iii. If σ is a permutation of $\{1, ..., k\}$, show that

$$a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k)} = \mathrm{sgn}(\sigma)(a_1 \wedge \cdots \wedge a_k),$$

where sgn is the homomorphism $S_k \to \{\pm 1\}$ such that $\mathrm{sgn}(\sigma) = (-1)^n$ if σ is a product of n transpositions.

iv. Show that there is a natural homomorphism $\bigwedge^m A \otimes_R \bigwedge^n A \to \bigwedge^{m+n} A$ given by

$$(a_1 \wedge \cdots \wedge a_m) \otimes (b_1 \wedge \cdots \wedge b_n) \mapsto (a_1 \wedge \cdots \wedge a_m) \wedge (b_1 \wedge \cdots \wedge b_n) := a_1 \wedge \cdots \wedge a_m \wedge b_1 \wedge \cdots \wedge b_m.$$

Is this surjective? Injective?

v. Show that, if A is a free module of rank d over R, then $\bigwedge^k A$ is a free module of rank $\binom{d}{k}$.

vi. Let $U \subseteq \mathbb{R}^n$ be an open set, and let R be the ring of smooth functions of U (that is, functions of whom you can take partial derivatives arbitrarily many times). Let Ω^1 be the R-module of smooth 1-forms on U, which is the free module over R with the basis $dx_1, ..., dx_n$. So, a 1-form just looks like

$$\omega = f_1 dx_1 + \cdots + f_n dx_n,$$

with f_i smooth functions. We now define $\Omega^0 = R$, and $\Omega^k = \bigwedge^k \Omega^1$, noting that, since constant functions are smooth, we may view all of these R-modules as $\mathbb{R}$-modules. We also define the exterior derivative $d : \Omega^k \to \Omega^{k+1}$ by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for $k = 0$ (note that by definition the ring of smooth functions is closed under partial differentiation), and

$$d\left(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for $k \geq 1$ (extended to all of Ω^k by the rule $d(\omega + \psi) = d\omega + d\psi$... the notation was just too messy to write it that way!).

(a) Prove that for smooth functions f and g, $d(fg) = g df + f dg$.

(b) Prove that for $\omega \in \Omega^n$ and $\psi \in \Omega^m$, we have $d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^n \omega \wedge d\psi$.

(c) Prove that the exterior derivatives

$$0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \to 0$$

form a chain complex (of abelian groups, or $\mathbb{R}$-modules). Recall that a sequence of homomorphisms

$$\cdots \to A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \to \cdots$$

of abelian groups forms a chain complex if and only if $\mathrm{im}(f_i) \subseteq \ker(f_{i+1})$ for all i. (So being a chain complex is part way to being an exact sequence; Poincaré's Lemma says that, if U is contractible, then this chain complex is actually an exact sequence.)