

Question 1

(i) Let I be an ideal of $R \times S$. Let $(r, s) \in I$.

We show that $(r, 0_S), (0_R, s) \in I$. Indeed, because R, S are rings with identity, $1_R \in R$, $1_S \in S$, so $(1_R, 0_S), (0_R, 1_S) \in R \times S$, and because I is an ideal, we have

$$(r, 0_S) = (1_R, 0_S)(r, s) \in I$$

$$(0_R, s) = (0_R, 1_S)(r, s) \in I.$$

So $(r, 0_S), (0_R, s) \in I$. We now claim there exists an ideal I_1 of R and an ideal I_2 of S such that $I = I_1 \times I_2$. To see this, take

$$I_1 = \{r \in R : (r, 0_S) \in I\} \text{ and } I_2 = \{s \in S : (0_R, s) \in I\}.$$

We claim that I_1, I_2 are ideals of R, S , respectively (we show that I_1 is an ideal of R , the other will be nearly identical). Indeed,

- $I_1 \neq \emptyset$ because $0_R \in I_1$, i.e. $0_R \in R$ and $(0_R, 0_S) \in I$.
- Let $r_1, r_2 \in I_1$. We want to show that $r_1 - r_2 \in I_1$.

Indeed, we have $(r_1, 0_S), (r_2, 0_S) \in I$ and because I is an ideal,

$$(r_1 - r_2, 0_S) = (r_1, 0_S) - (r_2, 0_S) \in I,$$

$$\text{so } r_1 - r_2 \in I_1.$$

- let $a \in R$ and $r \in I_1$. Then we want to show

that are $\in I_1$. Indeed, we have $(r, 0_S) \in I$ and $(a, 0_S) \in R \times S$, so because I is an ideal, we get

$$(ar, 0_S) = (a, 0_S)(r, 0_S) \in I$$

so $ar \in I_1$.

Therefore, we have shown that I_1 is an ideal of R . Now, we claim that $I = I_1 \times I_2$. To see this, we will show by double inclusion.

Let $(r, s) \in I$. Then from the above argument we have $(r, 0_S), (0_R, s) \in I$, so we have $r \in I_1, s \in I_2$, and so $(r, s) \in I_1 \times I_2$.

On the other hand, let $(r, s) \in I_1 \times I_2$. Then because I is an ideal,

$$(r, s) = (r, 0_S) + (0_R, s) \in I$$

which shows that $(r, s) \in I$.

Therefore, the claim that for every ideal I of $R \times S$, there is an ideal I_1 of R and an ideal I_2 of S such that $I = I_1 \times I_2$, holds.

This would then imply that $|I| = |I_1 \times I_2|$ so there exists a bijection from I to $I_1 \times I_2$ as required.

(ii) Let P be a prime ideal of $R \times S$. From (i), there exists an ideal P_1 of R and an ideal P_2 of S such that $P = P_1 \times P_2$. Now, because P is a prime ideal of $R \times S$, precisely one of P_1 or P_2 must be proper. Indeed, if both ideals are proper, then $(1_R, 0_S), (0_R, 1_S) \notin P$, however

$$(0_R, 0_S) = (1_R, 0_S)(0_R, 1_S) \in P$$

which is absurd because P is prime. This would imply that either $P = R \times P_2$ if P_2 is a proper ideal of S , or $P = P_1 \times S$ if P_1 is a proper ideal of R .

Now let $\mathcal{A} = \{P : P \text{ is a prime ideal of } R \times S\}$, $\mathcal{B} = \{P : P \text{ is an ideal of } R\}$, and $\mathcal{C} = \{P : P \text{ is a prime ideal of } S\}$. Let $f : \mathcal{B} \sqcup \mathcal{C} \rightarrow \mathcal{A}$ be the map defined by

$$f(P) = \begin{cases} R \times P & \text{if } P \subseteq S \\ P \times S & \text{if } P \subseteq R \end{cases}$$

We claim that $f : \mathcal{B} \sqcup \mathcal{C} \rightarrow \mathcal{A}$ is a bijection.

Indeed, let P be such that $f(P) = \{0\}$. Then because $\{0\}$ is a prime ideal, there are ideals I_1 of R and I_2 of S such that $\{0\} = I_1 \times I_2$. But because $\{0\}$ is prime, either $\{0\} = R \times \{0\}$ if

$\{0\} \subseteq S$ or $\{0\} = \{0\} \times S$ if $\{0\} \subset R$. In

particular, we yield $P = \{0\}$, so $\ker(f)$ is trivial, so f is injective.

To show onto, fix a prime ideal $Q \in \mathcal{A}$.

Then there are ideals I_1 of R and I_2 of S such that $Q = I_1 \times I_2$. But either $Q = R \times I_2$ if $I_2 \subseteq S$ or $Q = I_1 \times S$ if $I_1 \subseteq R$. In particular, exactly one of I_1, I_2 are proper, so choose $P = I_1$ if $I_1 \subseteq R$ and choose $P = I_2$ if $I_2 \subseteq S$.

Therefore, f is onto, and hence f is a bijection.

Moreover, we have shown that $|\mathcal{A}| = |\mathcal{B} \cup \mathcal{C}|$.

Question 2

Let $M_{p,q} = \mathbb{Z}[\frac{1}{p}] \otimes F_q$, where F_q is a field with q elements. Then since F_q is a finite field, we have $F_q \cong \mathbb{Z}_q$. Thus,

$$M_{p,q} \cong \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q.$$

We consider the following cases:

Case 1: If $p = q$ then we have

$$M_{p,p} \cong \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_p.$$

Fix $\frac{a}{p^e} \in \mathbb{Z}[\frac{1}{p}]$ and $n \in \mathbb{Z}_p$. Observe that

$$\frac{a}{p^e} \otimes n = \frac{a}{p^e} \cdot \frac{p}{p} \otimes n = \frac{a}{p^{e+1}} \otimes np = \frac{a}{p^{e+1}} \otimes 0 = 0$$

Therefore, as a, e, n were arbitrary, we get

$$\mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_p = \{0\} \Rightarrow M_{p,p} \cong \{0\}.$$

Case 2: If $p \neq q$, then we have

$$M_{p,q} \cong \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q.$$

Fix $\frac{a}{p^e} \in \mathbb{Z}[\frac{1}{p}]$, $n \in \mathbb{Z}_q$. Because p, q are relatively prime, there exists $s \in \mathbb{Z}$ such that

$$n = p^e s. \text{ Now}$$

$$\frac{a}{p^e} \otimes n = \frac{a}{p^e} \otimes p^e s = \frac{a}{p^e} p^e \otimes s = 1 \otimes as$$

From here, we claim that

$$\mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q \cong \mathbb{Z}_q$$

We can take $\phi: \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q \rightarrow \mathbb{Z}_q$ to be the map $\phi(a \otimes s) = as$.

- One-to-One: Let $a \otimes s \in \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q$ be such that $as = 0$. Because \mathbb{Z}_q is a field, it is also an integral domain so either $a=0$ or $s=0$. So $0 \otimes s = a \otimes 0 = 0 \otimes 0 = 0$, so $a \otimes s = 0$, showing injectivity.

- Onto: Fix $s \in \mathbb{Z}_q$. Take $1 \otimes s \in \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q$ so that $\phi(1 \otimes s) = s$, showing onto.
- Homomorphism: Let $a \otimes s, b \otimes t \in \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q$, $\lambda \in \mathbb{Z}$.

Then

$$\begin{aligned}
 \phi((a \otimes s) + \lambda(b \otimes t)) &= \phi(1 \otimes as + 1 \otimes \lambda bt) \\
 &= \phi(1 \otimes (as + \lambda bt)) \\
 &= as + \lambda bt \\
 &= \phi(a \otimes s) + \lambda \phi(b \otimes t)
 \end{aligned}$$

Therefore, we have shown that

$$\mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q \simeq \mathbb{Z}_q$$

Moreover, $M_{p,q} \simeq \mathbb{Z}_q$

Question 3

Let $M = \mathbb{Q}$, $N = \mathbb{Z}_2^{\mathbb{N}}$ (sequences in \mathbb{Z}_2). Fix $\frac{a}{b} \in \mathbb{Q}$ and $(x_n)_{n=1}^{\infty} \in \mathbb{Z}_2^{\mathbb{N}}$. Then observe that

$$\begin{aligned}\frac{a}{b} \otimes (x_n)_{n=1}^{\infty} &= \frac{a}{b} \cdot \frac{2}{2} \otimes (x_n)_{n=1}^{\infty} = \frac{a}{2b} \otimes 2(x_n)_{n=1}^{\infty}, \\ &= \frac{a}{2b} \otimes (2x_n)_{n=1}^{\infty} \\ &= \frac{a}{2b} \otimes 0 \\ &= 0\end{aligned}$$

Therefore, $\mathbb{Q} \otimes \mathbb{Z}_2^{\mathbb{N}} = \{0\}$.

Question 4

Note: Some elements have been borrowed from "Multivariable Mathematics" by Theodore, Shifrin., Chapter 8, Sections 1 and 2. Will denote [TS]

(i) Let $x, y \in A$. Then

$$\begin{aligned}(x+y) \wedge (x+y) &= x \wedge x + x \wedge y + y \wedge x + y \wedge y \\ 0 &= x \wedge y + y \wedge x \\ x \wedge y &= -y \wedge x.\end{aligned}$$

(ii) Assume $\{e_1, e_2\}$ is a basis for A . To see that $\{e_1 \wedge e_2\}$ is a basis for $\Lambda^2 A$, we show the following:

- Linear Independence: Assume there is an $r \in R$ such that $re_1 \wedge e_2 = 0$. If $r \neq 0_R$, then $re_1 - e_2 = 0$ implies that $e_2 = re_1$, so $\{e_1, e_2\}$ are linearly dependent, which is absurd. Hence, $r = 0_R$.
- Spanning Set: We show $\langle \{e_1 \wedge e_2\} \rangle = \Lambda^2 A$. Indeed, if $x \in \langle \{e_1 \wedge e_2\} \rangle$, there exists $r \in R$ such that $x = re_1 \wedge e_2$ but $e_1, e_2 \in A$, so $re_1 \wedge e_2 \in \Lambda^2 A$, so $x \in \Lambda^2 A$. On the other hand, if $x \wedge y \in \Lambda^2 A$, because $\{e_1, e_2\}$ is a basis for A , there are $r_1, r_2, s_1, s_2 \in R$ such that $x = r_1 e_1 + r_2 e_2, y = s_1 e_1 + s_2 e_2$. Now, by (i) we have

$$\begin{aligned}
x \wedge y &= (r_1 e_1 + r_2 e_2) \wedge (s_1 e_1 + s_2 e_2) \\
&= r_1 s_2 e_1 \wedge e_2 + r_2 s_1 e_2 \wedge e_1 \\
&= r_1 s_2 e_1 \wedge e_2 - r_2 s_1 e_1 \wedge e_2 \\
&= (r_1 s_2 - r_2 s_1) e_1 \wedge e_2
\end{aligned}$$

so $x \wedge y \in \langle \{e_1 \wedge e_2\} \rangle$, showing that $\langle \{e_1 \wedge e_2\} \rangle = \Lambda^2 A$.

Therefore $\{e_1 \wedge e_2\}$ is a basis for $\Lambda^2 A$.

Let $\phi: A \oplus A \rightarrow \Lambda^2 A$ be $\phi(x, y) = x \wedge y$. Then note that $x \wedge y = 0$ if and only if $x-y=0$ if and only if $x=y$. So $\ker(\phi) = \{(x, x) \in A \oplus A\}$.

(iii) We use induction on k to show that

$$a_{\sigma(1)} \wedge a_{\sigma(2)} \wedge \cdots \wedge a_{\sigma(k)} = \text{sgn}(\sigma) a_1 \wedge \cdots \wedge a_k$$

For $k=1$, $a_{\sigma(1)} = a_1 = \text{sgn}(\sigma) a_1$. Now assume

$$a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k)} = \text{sgn}(\sigma) a_1 \wedge \cdots \wedge a_k.$$

Then for $k+1$, we have

$$\begin{aligned}
a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k+1)} &= \text{sgn}(\sigma) a_1 \wedge \cdots \wedge a_k \wedge a_{\sigma(k+1)} \\
&= \text{sgn}(\sigma) a_1 \wedge \cdots \wedge a_{k+1}
\end{aligned}$$

so we have $a_{\sigma(1)} \wedge \cdots \wedge a_k = \text{sgn}(\sigma) a_1 \wedge \cdots \wedge a_k$

for all $k \in \mathbb{N}$.

(iv) We apply Theorem 5.6 to that there exists a homomorphism $\Lambda^m A \otimes \Lambda^n A \rightarrow \Lambda^{m+n} A$. Let us denote

$\phi: \Lambda^m A \times \Lambda^n A \rightarrow \Lambda^{m+n} A$ by

$$\phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i\right) = \bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n b_i.$$

We show that ϕ is a middle linear map. Indeed,

- If $\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i \in \Lambda^m A, \bigwedge_{i=1}^n c_i \in \Lambda^n A$, then

$$\begin{aligned}\phi\left(\bigwedge_{i=1}^m a_i + \bigwedge_{i=1}^n b_i, \bigwedge_{i=1}^n c_i\right) &= \left(\bigwedge_{i=1}^m a_i + \bigwedge_{i=1}^n b_i\right) \wedge \bigwedge_{i=1}^n c_i \\ &= \left(\bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n c_i\right) + \left(\bigwedge_{i=1}^n b_i \wedge \bigwedge_{i=1}^n c_i\right) \\ &= \phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n c_i\right) + \phi\left(\bigwedge_{i=1}^n b_i, \bigwedge_{i=1}^n c_i\right)\end{aligned}$$

- If $\bigwedge_{i=1}^m a_i \in \Lambda^m A, \bigwedge_{i=1}^n b_i, \bigwedge_{i=1}^n c_i \in \Lambda^n A$, then

$$\begin{aligned}\phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i + \bigwedge_{i=1}^n c_i\right) &= \bigwedge_{i=1}^m a_i \wedge \left(\bigwedge_{i=1}^n b_i + \bigwedge_{i=1}^n c_i\right) \\ &= \left(\bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n b_i\right) + \left(\bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n c_i\right) \\ &= \phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i\right) + \phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n c_i\right)\end{aligned}$$

- If $\bigwedge_{i=1}^m a_i \in \Lambda^m A, \bigwedge_{i=1}^n b_i \in \Lambda^n A, r \in R$

$$\begin{aligned}\phi\left(r \bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i\right) &= r \bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n b_i \\ &= \bigwedge_{i=1}^m a_i \wedge r \bigwedge_{i=1}^n b_i \\ &= \phi\left(\bigwedge_{i=1}^m a_i, r \bigwedge_{i=1}^n b_i\right).\end{aligned}$$

Therefore, we have shown that ϕ is a middle linear map, so

by Theorem 5.6, there exists a homomorphism

$\psi: \Lambda^m A \otimes \Lambda^n A \rightarrow \Lambda^{m+n} A$ given by

$$\psi\left(\bigwedge_{i=1}^m a_i \otimes \bigwedge_{i=1}^n b_i\right) = \bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n b_i$$

We now claim that ψ is onto, but not one-

to-one. Indeed, ψ is not one-to-one because by (ii)

the kernel is not trivial, so extending to $\Lambda^m A \otimes \Lambda^n A \rightarrow \Lambda^{m+n} A$

would also yield kernel is nontrivial. To see it is onto,

fix $\bigwedge_{i=1}^{m+n} a_i \in \Lambda^{m+n} A$, take $\bigwedge_{i=1}^m a_i \otimes \bigwedge_{i=1}^n a_{m+i} \in \Lambda^m A \otimes \Lambda^n A$

which yields $\psi(\bigwedge_{i=1}^m a_i \otimes \bigwedge_{i=1}^n a_{m+i}) = \bigwedge_{i=1}^{m+n} a_i$. Thus,

ψ is onto.

(v) Let $\{v_1, \dots, v_d\}$ be a basis of A . Then the number of ways of choosing k elements from v_1, \dots, v_d is

$$d(d-1)(d-2)\cdots(d-k+1) = \frac{d!}{(d-k)!}$$

By (iii), the number of increasing k -tuples is then

$$\frac{d!}{k!(d-k)!} = \binom{d}{k}$$

Hence, $\dim(\Lambda^k A) = \binom{d}{k}$.

(vi) (a) let f, g be smooth functions. Then

$$\begin{aligned} d(fg) &= \sum_{i=1}^n \frac{\partial(fg)}{\partial x_i} dx_i = \sum_{i=1}^n g \frac{\partial f}{\partial x_i} dx_i + f \frac{\partial g}{\partial x_i} dx_i \\ &= g \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i + f + \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i \\ &= g df + fdg. \end{aligned}$$

(b) For simplicity of notation, denote $I = (i_1, \dots, i_n)$

$J = (j_1, \dots, j_m)$ which are increasing n -tuple and m -tuples, respectively, denote

$$dx_I = dx_{(i_1, \dots, i_n)} = dx_{i_1} \wedge \cdots \wedge dx_{i_n}. \quad [\text{TS}]$$

Then if $\omega = f dx_I$, $\psi = g dx_J$ for some smooth functions f, g . Then

$$\omega \wedge \psi = f dx_I \wedge g dx_J = fg dx_I \wedge dx_J.$$

Now, by (a), and (v)

$$\begin{aligned}
d(\omega \wedge \psi) &= d(fg dx_I \wedge dx_J) \\
&= d(fg) \wedge dx_I \wedge dx_J \\
&= (g df + f dg) \wedge dx_I \wedge dx_J \\
&= (g df \wedge dx_I \wedge dx_J) + (f dg \wedge dx_I \wedge dx_J) \\
&= (df \wedge dx_I) \wedge g dx_J + (dg \wedge f dx_I \wedge dx_J) \xrightarrow{n\text{-tuple.}} \\
&\quad \text{requires } n \text{ switches} \\
&= d\omega \wedge \psi + (-1)^n (fdx_I) \wedge (dg \wedge dx_J) \\
&= d\omega \wedge \psi + (-1)^n \omega \wedge d\psi.
\end{aligned}$$

For the general case, let $I = (i_1, \dots, i_n)$ be an increasing n -tuple, $J = (j_1, \dots, j_m)$ be an m -tuple, let

$$\omega = \sum_{i_1 < \dots < i_n} f_I dx_I, \quad \psi = \sum_{j_1 < \dots < j_m} g_J dx_J \quad [\text{TS}]$$

Then by a similar approach,

$$\begin{aligned}
\omega \wedge \psi &= \sum_{i_1 < \dots < i_n} f_I dx_I \wedge \sum_{j_1 < \dots < j_m} g_J dx_J \\
&= \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} f_I g_J dx_I \wedge dx_J.
\end{aligned}$$

Now we have

$$\begin{aligned}
d(\omega \wedge \psi) &= d\left(\sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} f_I g_J dx_I \wedge dx_J\right) \\
&= \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} d(f_I g_J dx_I \wedge dx_J) \\
&= \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} d(f_I g_J) \wedge dx_I \wedge dx_J \\
&= \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} (g_J df_I + f_I dg_J) \wedge dx_I \wedge dx_J \\
&= \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} [g_J df_I \wedge dx_I \wedge dx_J + f_I dg_J \wedge dx_I \wedge dx_J] \\
&= \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} g_J df_I \wedge dx_I \wedge dx_J + \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} f_I dg_J \wedge dx_I \wedge dx_J \\
&= \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} (df_I \wedge dx_I) \wedge g_J dx_J + \sum_{i_1 < \dots < i_n} \sum_{j_1 < \dots < j_m} dg_J \wedge f_I dx_I \wedge dx_J \xrightarrow{n \text{ switches}}
\end{aligned}$$

$$= \sum_{i_1 < \dots < i_n} (df_{I_1} \wedge dx_{I_1}) \wedge \sum_{j_1 < \dots < j_m} g_J dx_{J_1} + (-1)^n \sum_{i_1 < \dots < i_n} f_{I_1} dx_{I_1} \wedge \sum_{j_1 < \dots < j_m} dg_{J_1} dx_{J_1}$$

$$= d\omega \wedge \psi + (-1)^n \omega \wedge d\psi$$

as desired.