Question 1

(i) Let I be an ideal of $R \times S$. Let $(r,s) \in I$. We show that $(r, 0_S), (0_R, s) \in I$. Indeed, because R, S are rings with identity, $1_R \in R$, $1_S \in S$, so $(1_R, 0_S), (0_R, 1_S) \in R \times S$, and because I is an ideal, we have

$$(r, 0_S) = (1_R, 0_S)(r,s) \in I$$

$$(0_R, s) = (0_R, 1_S)(r,s) \in I.$$

So $(r, 0_S), (0_R, s) \in I$. We now claim there exists an ideal I_1 of R and an ideal I_2 of S such that $I = I_1 \times I_2$. To see this, take $I_1 = \{ r \in R : (r, 0_S) \in I \}$ and $I_2 = \{ s \in S : (0_R, s) \in I \}$.

We claim that I_1, I_2 are ideals of R, S, respectively (we show that I_1 is an ideal of R, the other will be nearly identical). Indeed,

- $I_1 \neq \emptyset$ because $0_R \in I_1$, i.e. $0_R \in R$ and $(0_R, 0_S) \in I$.
- Let $r_1, r_2 \in I_1$. We want to show that $r_1 - r_2 \in I_1$. Indeed, we have $(r_1, 0_S), (r_2, 0_S) \in I$ and because I is an ideal,

  $$(r_1 - r_2, 0_S) = (r_1, 0_S) - (r_2, 0_S) \in I,$$

  so $r_1 - r_2 \in I_1$.
- let $a \in R$ and $r \in I_1$. Then we want to show

that $ar \in I_1$. Indeed, we have $(r, 0_S) \in I$ and $(a, 0_S) \in R \times S$, so because I is an ideal, we get

$$(ar, 0_S) = (a, 0_S)(r, 0_S) \in I$$

so $ar \in I_1$.

Therefore, we have shown that I_1 is an ideal of R. Now, we claim that $I = I_1 \times I_2$. To see this, we will show by double inclusion.

Let $(r, s) \in I$. Then from the above argument we have $(r, 0_S), (0_R, s) \in I$, so we have $r \in I_1$, $s \in I_2$, and so $(r, s) \in I_1 \times I_2$.

On the other hand, let $(r, s) \in I_1 \times I_2$. Then because I is an ideal,

$$(r, s) = (r, 0_S) + (0_R, s) \in I$$

which shows that $(r, s) \in I$.

Therefore, the claim that for every ideal I of $R \times S$, there is an ideal I_1 of R and an ideal I_2 of S such that $I = I_1 \times I_2$, holds.

This would then imply that $|I| = |I_1 \times I_2|$ so there exists a bijection from I to $I_1 \times I_2$ as required.

(ii) Let P be a prime ideal of $R \times S$. From (i), there exists an ideal P_1 of R and an ideal P_2 of S such that $P = P_1 \times P_2$. Now, because P is a prime ideal of $R \times S$, precisely one of P_1 or P_2 must be proper. Indeed, if both ideals are proper, then $(1_R, 0_S), (0_R, 1_S) \notin P$, however

$$(0_R, 0_S) = (1_R, 0_S)(0_R, 1_S) \in P$$

which is absurd because P is prime. This would imply that either $P = R \times P_2$ if P_2 is a proper ideal of S, or $P = P_1 \times S$ if P_1 is a proper ideal of R.

Now let $\mathcal{A} = \{P : P \text{ is a prime ideal of } R \times S\}$ $\mathcal{B} = \{P : P \text{ is an ideal of } R\}$, and $\mathcal{C} = \{P : P \text{ is a prime ideal of } S\}$. Let $f : \mathcal{B} \sqcup \mathcal{C} \to \mathcal{A}$ be the map defined by

$$f(P) = \begin{cases} R \times P & \text{if } P \subseteq S \\ P \times S & \text{if } P \subseteq R. \end{cases}$$

We claim that $f : \mathcal{B} \sqcup \mathcal{C} \to \mathcal{A}$ is a bijection. Indeed, let P be such that $f(P) = \{0\}$. Then because $\{0\}$ is a prime ideal, there are ideals I_1 of R and I_2 of S such that $\{0\} = I_1 \times I_2$. But because $\{0\}$ is prime, either $\{0\} = R \times \{0\}$ if

$\{0\} \subsetneq S$ or $\{0\} = \{0\} \times S$ if $\{0\} \subsetneq R$. In particular, we yield $P = \{0\}$, so $\ker(f)$ is trivial, so f is injective.

To show onto, fix a prime ideal $Q \in \mathcal{A}$. Then there are ideals I_1 of R and I_2 of S such that $Q = I_1 \times I_2$. But either $Q = R \times I_2$ if $I_2 \subsetneq S$ or $Q = I_1 \times S$ if $I_1 \subsetneq R$. In particular, exactly one of I_1, I_2 are proper, so choose $P = I_1$ if $I_1 \subsetneq R$ and choose $P = I_2$ if $I_2 \subsetneq S$.

Therefore, f is onto, and hence f is a bijection. Moreover, we have shown that $|\mathcal{A}| = |\mathcal{B} \sqcup \mathcal{C}|$.

Question 2

Let $M_{p,q} = \mathbb{Z}[\frac{1}{p}] \otimes F_q$, where F_q is a field with q elements. Then since F_q is a finite field, we have $F_q \simeq \mathbb{Z}_q$. Thus,

$$M_{p,q} \simeq \mathbb{Z}[\tfrac{1}{p}] \otimes \mathbb{Z}_q.$$

We consider the following cases:

Case 1: If $p = q$ then we have

$$M_{p,p} \simeq \mathbb{Z}[\tfrac{1}{p}] \otimes \mathbb{Z}_p.$$

Fix $\frac{a}{p^e} \in \mathbb{Z}[\frac{1}{p}]$ and $n \in \mathbb{Z}_p$. Observe that

$$\frac{a}{p^e} \otimes n = \frac{a}{p^e} \cdot \frac{p}{p} \otimes n = \frac{a}{p^{e+1}} \otimes np = \frac{a}{p^{e+1}} \otimes 0 = 0$$

Therefore, as a, e, n were arbitrary, we get

$$\mathbb{Z}[\tfrac{1}{p}] \otimes \mathbb{Z}_p = \{0\} \Rightarrow M_{p,p} \simeq \{0\}.$$

Case 2: If $p \neq q$, then we have

$$M_{p,q} \simeq \mathbb{Z}[\tfrac{1}{p}] \otimes \mathbb{Z}_q.$$

Fix $\frac{a}{p^e} \in \mathbb{Z}[\frac{1}{p}]$, $n \in \mathbb{Z}_q$. Because p, q are relatively prime, there exists $s \in \mathbb{Z}$ such that $n = p^e s$. Now

$$\frac{a}{p^e} \otimes n = \frac{a}{p^e} \otimes p^e s = \frac{a}{p^e} p^e \otimes s = 1 \otimes as$$

From here, we claim that

$$\mathbb{Z}[\tfrac{1}{p}] \otimes \mathbb{Z}_q \simeq \mathbb{Z}_q$$

We can take $\phi : \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q \rightarrow \mathbb{Z}_q$ to be the map $\phi(a \otimes s) = as$.

- One-to-One: Let $a \otimes s \in \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q$ be such that $as = 0$. Because $\mathbb{Z}_q$ is a field, it is also an integral domain so either $a = 0$ or $s = 0$. So $0 \otimes s = a \otimes 0 = 0 \otimes 0 = 0$, so $a \otimes s = 0$, showing injectivity.
- Onto: Fix $s \in \mathbb{Z}_q$. Take $1 \otimes s \in \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q$ so that $\phi(1 \otimes s) = s$, showing onto.
- Homomorphism: Let $a \otimes s, b \otimes t \in \mathbb{Z}[\frac{1}{p}] \otimes \mathbb{Z}_q$, $\lambda \in \mathbb{Z}$. Then

$$\begin{aligned}\phi((a \otimes s) + \lambda(b \otimes t)) &= \phi(1 \otimes as + 1 \otimes \lambda bt) \\ &= \phi(1 \otimes (as + \lambda bt)) \\ &= as + \lambda bt \\ &= \phi(a \otimes s) + \lambda \phi(b \otimes t)\end{aligned}$$

Therefore, we have shown that

$$\mathbb{Z}[\tfrac{1}{p}] \otimes \mathbb{Z}_q \simeq \mathbb{Z}_q$$

Moreover, $M_{p,q} \simeq \mathbb{Z}_q$

Question 3

Let $M = \mathbb{Q}$, $N = \mathbb{Z}_2^{\mathbb{N}}$ (sequences in $\mathbb{Z}_2$). Fix $\frac{a}{b} \in \mathbb{Q}$ and $(x_n)_{n=1}^{\infty} \in \mathbb{Z}_2^{\mathbb{N}}$. Then observe that

$$\begin{aligned} \frac{a}{b} \otimes (x_n)_{n=1}^{\infty} &= \frac{a}{b} \cdot \frac{2}{2} \otimes (x_n)_{n=1}^{\infty} = \frac{a}{2b} \otimes 2(x_n)_{n=1}^{\infty} \\ &= \frac{a}{2b} \otimes (2x_n)_{n=1}^{\infty} \\ &= \frac{a}{2b} \otimes 0 \\ &= 0 \end{aligned}$$

Therefore, $\mathbb{Q} \otimes \mathbb{Z}_2^{\mathbb{N}} = \{0\}$.

Question 4

Note: Some elements have been borrowed from "Multivariable Mathematics" by Theodore, Shifrin., Chapter 8, Sections 1 and 2. Will denote [TS]

(i) Let $x, y \in A$. Then

$$(x+y)\wedge(x+y) = x\wedge x + x\wedge y + y\wedge x + y\wedge y$$
$$0 = x\wedge y + y\wedge x$$
$$x\wedge y = -y\wedge x.$$

(ii) Assume $\{e_1, e_2\}$ is a basis for A. To see that $\{e_1\wedge e_2\}$ is a basis for $\Lambda^2 A$, we show the following:

- Linear Independence: Assume there is an $r\in R$ such that $re_1\wedge e_2 = 0$. If $r \neq 0_R$, then $re_1 - e_2 = 0$ implies that $e_2 = re_1$, so $\{e_1, e_2\}$ are linearly dependent, which is absurd. Hence, $r = 0_R$.
- Spanning Set: We show $\langle\{e_1\wedge e_2\}\rangle = \Lambda^2 A$. Indeed, if $x \in \langle\{e_1\wedge e_2\}\rangle$, there exists $r\in R$ such that

  $x = re_1\wedge e_2$ but $re_1, e_2 \in A$, so $re_1\wedge e_2 \in \Lambda^2 A$, so $x\in\Lambda^2 A$. On the other hand, if $x\wedge y \in \Lambda^2 A$, because $\{e_1, e_2\}$ is a basis for A, there are $r_1, r_2, s_1, s_2 \in R$ such that $x = r_1e_1 + r_2e_2$, $y = s_1e_1 + s_2e_2$. Now, by (i) we have

$$x \wedge y = (r_1 e_1 + r_2 e_2) \wedge (s_1 e_1 + s_2 e_2)$$
$$= r_1 s_2 e_1 \wedge e_2 + r_2 s_1 e_2 \wedge e_1$$
$$= r_1 s_2 e_1 \wedge e_2 - r_2 s_1 e_1 \wedge e_2$$
$$= (r_1 s_2 - r_2 s_1) e_1 \wedge e_2$$

so $x \wedge y \in \langle \{e_1 \wedge e_2\} \rangle$, showing that $\langle \{e_1 \wedge e_2\} \rangle = \Lambda^2 A$.
Therefore $\{e_1 \wedge e_2\}$ is a basis for $\Lambda^2 A$.

Let $\phi : A \oplus A \to \Lambda^2 A$ be $\phi(x,y) = x \wedge y$. Then note that $x \wedge y = 0$ if and only if $x - y = 0$ if and only if $x = y$ so $\ker(\phi) = \{(x,x) \in A \oplus A\}$.

(iii) We use induction on k to show that

$$a_{\sigma(1)} \wedge a_{\sigma(2)} \wedge \cdots \wedge a_{\sigma(k)} = \operatorname{sgn}(\sigma) a_1 \wedge \cdots a_k$$

For $k=1$, $a_{\sigma(1)} = a_1 = \operatorname{sgn}(\sigma) a_1$. Now assume

$a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k)} = \operatorname{sgn}(\sigma) a_1 \wedge \cdots \wedge a_k$.

Then for $k+1$, we have

$$a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k+1)} = \operatorname{sgn}(\sigma)\, a_1 \wedge \cdots \wedge a_k \wedge a_{\sigma(k+1)}$$
$$= \operatorname{sgn}(\sigma)\, a_1 \wedge \cdots \wedge a_{k+1}$$

so we have $a_{\sigma(1)} \wedge \cdots \wedge a_k = \operatorname{sgn}(\sigma) a_1 \wedge \cdots \wedge a_k$ for all $k \in \mathbb{N}$.

(iv) We apply Theorem 5.6 to that there exists a homomorphism $\Lambda^m A \otimes \Lambda^n A \to \Lambda^{m+n} A$. Let us denote $\phi : \Lambda^m A \times \Lambda^n A \to \Lambda^{m+m} A$ by

$$\phi\left(\bigwedge_{i=1}^{m} a_i, \bigwedge_{i=1}^{n} b_i\right) = \bigwedge_{i=1}^{m} a_i \wedge \bigwedge_{i=1}^{n} b_i.$$

We show that ϕ is a middle linear map. Indeed,

- If $\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^m b_i \in \Lambda^m A$, $\bigwedge_{i=1}^n c_i \in \Lambda^n A$, then

$$\phi\left(\bigwedge_{i=1}^m a_i + \bigwedge_{i=1}^m b_i, \bigwedge_{i=1}^n c_i\right) = \left(\bigwedge_{i=1}^m a_i + \bigwedge_{i=1}^m b_i\right) \wedge \bigwedge_{i=1}^n c_i$$
$$= \left(\bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n c_i\right) + \left(\bigwedge_{i=1}^m b_i \wedge \bigwedge_{i=1}^n c_i\right)$$
$$= \phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n c_i\right) + \phi\left(\bigwedge_{i=1}^m b_i, \bigwedge_{i=1}^n c_i\right)$$

- If $\bigwedge_{i=1}^m a_i \in \Lambda^m A$, $\bigwedge_{i=1}^n b_i, \bigwedge_{i=1}^n c_i \in \Lambda^n A$, then

$$\phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i + \bigwedge_{i=1}^n c_i\right) = \bigwedge_{i=1}^m a_i \wedge \left(\bigwedge_{i=1}^n b_i + \bigwedge_{i=1}^n c_i\right)$$
$$= \left(\bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n b_i\right) + \left(\bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n c_i\right)$$
$$= \phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i\right) + \phi\left(\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n c_i\right)$$

- If $\bigwedge_{i=1}^m a_i \in \Lambda^m A$, $\bigwedge_{i=1}^n b_i \in \Lambda^n A$, $r \in R$

$$\phi\left(r\bigwedge_{i=1}^m a_i, \bigwedge_{i=1}^n b_i\right) = r\bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n b_i$$
$$= \bigwedge_{i=1}^m a_i \wedge r\bigwedge_{i=1}^n b_i$$
$$= \phi\left(\bigwedge_{i=1}^m a_i, r\bigwedge_{i=1}^n b_i\right).$$

Therefore, we have shown that ϕ is a middle linear map, so by Theorem 5.6, there exists a homomorphism

$\psi : \Lambda^m A \otimes \Lambda^n A \to \Lambda^{m+n} A$ given by

$$\psi\left(\bigwedge_{i=1}^m a_i \otimes \bigwedge_{i=1}^n b_i\right) = \bigwedge_{i=1}^m a_i \wedge \bigwedge_{i=1}^n b_i$$

We now claim that ψ is onto, but not one-to-one. Indeed, ψ is not one-to-one because by (ii) the kernel is not trivial, so extending to $\Lambda^m A \otimes \Lambda^n A \to \Lambda^{m+n} A$ would also yield kernel is nontrivial. To see it is onto, fix $\bigwedge_{i=1}^{m+n} a_i \in \Lambda^{m+n} A$, take $\bigwedge_{i=1}^m a_i \otimes \bigwedge_{i=1}^n a_{m+i} \in \Lambda^m A \otimes \Lambda^n A$

which yields $\psi\left(\bigwedge_{i=1}^{m} a_i \otimes \bigwedge_{i=1}^{n} a_{m+i}\right) = \bigwedge_{i=1}^{m+n} a_i$. Thus, ψ is onto.

(v) Let $\{v_1, \dots, v_d\}$ be a basis of A. Then the number of ways of choosing k elements from $v_1, \dots, v_d$ is

$$d(d-1)(d-2)\cdots(d-k+1) = \frac{d!}{(d-k)!}$$

By (iii), the number of increasing k-tuples is then

$$\frac{d!}{k!(d-k)!} = \binom{d}{k}$$

Hence, $\dim(\Lambda^k A) = \binom{d}{k}$.

(vi) (a) Let f, g be smooth functions. Then

$$\begin{aligned} d(fg) &= \sum_{i=1}^{n} \frac{\partial(fg)}{\partial x_i} dx_i = \sum_{i=1}^{n} g \frac{\partial f}{\partial x_i} dx_i + f \frac{\partial g}{\partial x_i} dx_i \\ &= g \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + f + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i \\ &= g\,df + f\,dg. \end{aligned}$$

(b) For simplicity of notation, denote $I = (i_1, \dots, i_n)$ $J = (j_1, \dots, j_m)$ which are increasing n-tuple and m-tuples, respectively, denote

$$dx_I = dx_{(i_1, \dots, i_n)} = dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$ [TS]

Then if $\omega = f\,dx_I$, $\psi = g\,dx_J$ for some smooth functions f, g. Then

$$\omega \wedge \psi = f\,dx_I \wedge g\,dx_J = fg\,dx_I \wedge dx_J.$$

Now, by (a), and (v)

$$
\begin{aligned}
d(\omega\wedge\psi) &= d(fg\, dx_I \wedge dx_J) \\
&= d(fg)\wedge dx_I \wedge dx_J \\
&= (g\,df + f\,dg)\wedge dx_I \wedge dx_J \\
&= (g\,df \wedge dx_I \wedge dx_J) + (f\,dg \wedge dx_I \wedge dx_J) \\
&= (df \wedge dx_I)\wedge g\,dx_J + (dg \wedge f\,dx_I \wedge dx_J)
\end{aligned}
$$

n-tuple.

requires n switches

$$
\begin{aligned}
&= d\omega\wedge\psi + (-1)^n (f\,dx_I)\wedge(dg\wedge dx_J) \\
&= d\omega\wedge\psi + (-1)^n\,\omega\wedge d\psi.
\end{aligned}
$$

For the general case, let $I=(i_1,\dots,i_n)$ be an increasing n-tuple, $J=(j_1,\dots,j_m)$ be an m-tuple, let

$$\omega = \sum_{i_1<\cdots<i_n} f_I\, dx_I, \quad \psi = \sum_{j_1<\cdots<j_m} g_J\, dx_J \quad \text{[TS]}$$

Then by a similar approach,

$$
\begin{aligned}
\omega\wedge\psi &= \sum_{i_1<\cdots<i_n} f_I\, dx_I \wedge \sum_{j_1<\cdots<j_m} g_J\, dx_J \\
&= \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} f_I\, g_J\, dx_I\wedge dx_J.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
d(\omega\wedge\psi) &= d\Big(\sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} f_I\, g_J\, dx_I\wedge dx_J\Big) \\
&= \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} d(f_I\, g_J\, dx_I\wedge dx_J) \\
&= \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} d(f_I\, g_J)\wedge dx_I\wedge dx_J \\
&= \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} (g_J\,df_I + f_I\,dg_J)\wedge dx_I\wedge dx_J \\
&= \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} [g_J\,df_I\wedge dx_I\wedge dx_J + f_I\,dg_J\wedge dx_I\wedge dx_J] \\
&= \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} g_J\,df_I\wedge dx_I\wedge dx_J + \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} f_I\,dg_J\wedge dx_I\wedge dx_J
\end{aligned}
$$

n switches

$$
= \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} (df_I\wedge dx_I)\wedge g_J\,dx_J + \sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} dg_J\wedge f_I\,dx_I\wedge dx_J
$$

$$= \sum_{i_1<\cdots<i_n} (df_I \wedge dx_I) \wedge \sum_{j_1<\cdots<j_m} g_J\, dx_J + (-1)^n \sum_{i_1<\cdots<i_n} f_I\, dx_I \wedge \sum_{j_1<\cdots<j_m} dg_J \wedge dx_J$$

$$= d\omega \wedge \psi + (-1)^n \omega \wedge d\psi$$

as desired.

(c) We show that $0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0$, that is, we show for all k, $\mathrm{Im}(d) \subseteq \ker(d)$. So fix a k, and let $d: \Omega^k \to \Omega^{k+1}$ denote the exterior derivative. Let $\omega \in \Omega^k$ given as $\omega = f\, dx_I$, where f is a smooth function on U, and $dx_I = dx_{(i_1,\ldots,i_k)} = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Note that

$\partial_x = \frac{\partial}{\partial x}$

$$d\omega = d(f\, dx_I) = df \wedge dx_I = \sum_{j=1}^{n} \partial_{x_j} f\, dx_j \wedge dx_I.$$

so $d\omega \in \mathrm{Im}(d)$. We show $d\omega \in \ker(d)$. Indeed,

$$d(d\omega) = d\left(\sum_{j=1}^{n} \partial_{x_j} f\, dx_j \wedge dx_I\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial x_i \partial x_j f\, dx_i \wedge dx_j \wedge dx_I. \quad (*)$$

Because $dx_i \wedge dx_j = -dx_j \wedge dx_i$, $(*)$ can be reduced to

$$= \sum_{i<j} (\partial_{x_i} \partial_{x_j} f - \partial_{x_j} \partial_{x_i} f)\, dx_i \wedge dx_j \wedge dx_I$$

By a known theorem, called Clairaut's Theorem,

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}$$

so

$$d(d\omega) = \sum_{i<j} (\partial_{x_i} \partial_{x_j} f - \partial_{x_j} \partial_{x_i} f)\, dx_i \wedge dx_j \wedge dx_I$$

$$= 0$$

so $d\omega \in \ker(d)$. Since k was arbitrary, $\mathrm{Im}(d) \subset \ker(d)$

so $0 \to \Omega^0 \to \cdots \to \Omega^n \to 0$ is a chain complex.