MATH 6122 **Homework 2**

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Question 1. Prove that $\mathrm{Gal}(\mathbb{R}/\mathbb{Q})$ is trivial. *Hint:* Use the fact that $x \leq y \in \mathbb{R}$ if and only if there exists a z with $y - x = z^2$ to show that elements of $\mathrm{Gal}(\mathbb{R}/\mathbb{Q})$ must preserve order.

Question 2. Prove that $\mathbb{Q}(\sqrt{D})$ is Galois over $\mathbb{Q}$ for any $D \in \mathbb{Q}$.

Question 3. Let $GL_2(\mathbb{C})$ be the group of 2×2 invertible matrices with complex entries (under multiplication). Let $H = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in \mathbb{C}^* \right\}$ be the scalar matrices and let $PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/H$. Prove that $\mathbb{C}(x)$ is a Galois extension of $\mathbb{C}$, and that $\mathrm{Gal}(\mathbb{C}(x)/\mathbb{C}) \simeq PGL_2(\mathbb{C})$. *Hint:* If $\sigma \in \mathrm{Gal}(\mathbb{C}(x)/\mathbb{C})$, then $\sigma(x) \in \mathbb{C}(x)$. What is the degree of $\sigma(x)$?

Question 4. Let $\alpha^3 = 2$, $\omega^3 = 1$, but $\omega \neq 1$, and let $F = \mathbb{Q}(\alpha, \omega)$.

(a) Show that $[F : \mathbb{Q}] = 6$.

(b) Show that F is a Galois extension of $\mathbb{Q}$.

(c) Explicitly describe $\mathrm{Gal}(F/\mathbb{Q})$ and the action of this group on α and ω.

(d) Write out the lattice of subgroups of $\mathrm{Gal}(F/\mathbb{Q})$ and the corresponding lattice of intermediate fields. Which intermediate fields are Galois extensions of $\mathbb{Q}$?

Question 5. Let F be an extension of K, let L and M be intermediate fields (not necessarily contained in each other) and let LM be the composition, the field generated by $L \cup M$.

(a) Show that LM is a finite extension of K if and only if both L and M are.

(b) If LM is a finite field extension of K, show that

$$[LM : K] \leq [L : K][M : K]$$

(c) If LM is finite over K and $[L : K]$ is relatively prime to $[M : K]$, then $[LM : K] = [L : K][M : K]$.

Question 6. A complex number $\alpha \in \mathbb{C}$ is an *algebraic integer* if and only if α is a root of a monic polynomiaal with *integer* coefficients.

(a) Prove that $\alpha \in \mathbb{Q}$ is an algebraic integer if and only if $\alpha \in \mathbb{Z}$.

(b) Prove that $\alpha \in \mathbb{C}$ is an algebraic integer if and only if there is a finitely-generated subgroup $H \leq (\mathbb{C}, +)$ such that $\alpha H \leq H$. Use this to prove that the algebraic integers form a subring of $\mathbb{C}$.

Hint: If $H \leq (\mathbb{C}, +)$ is finitely generated, then it is a finite rank free $\mathbb{Z}$-module, so it has a basis $e_1, ..., e_m$. If $\alpha H \leq H$, then $x \mapsto \alpha x$ is a $\mathbb{Z}$-linear transformation from H to H, so it is represented by some matrix with entries in $\mathbb{Z}$. Show that α is an eigenvalue of this matrix.