MATH 6280 **Assignment 3**

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Question 1 (Chebyshev's Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f : X \to [0, \infty)$ be measurable, and let $\alpha > 0$. Prove that if $g : [0, \infty) \to (0, \infty)$ is non-decreasing, then

$$\mu(\{x \in X \mid f(x) \geq \alpha\}) \leq \frac{1}{g(\alpha)} \int_X (g \circ f)\, d\mu.$$

Question 2 (Convergence in Mean)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \in L_1(X, \mu)$, and let $(f_n)_{n\geq 1} \subseteq L_1(X, \mu)$. Recall it is said that $(f_n)_{n\geq 1}$ converges to f in $L_1(X, \mu)$ if $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. In the case that μ is a probability measure, this is also known as *convergence in mean*.

a) Prove that if $(f_n)_{n\geq 1}$ converges to f in $L_1(X, \mu)$, then $(f_n)_{n\geq 1}$ converges to f in measure.

b) Give an example of a finite measure μ, an $f \in L_1(X, \mu)$, and $(f_n)_{n\geq 1} \subseteq L_1(X, \mu)$ such that $(f_n)_{n\geq 1}$ converges to f in measure, $(f_n)_{n\geq 1}$ converges to f pointwise, but $(f_n)_{n\geq 1}$ does converges to f in $L_1(X, \mu)$. Prove your example satisfies the desired conditions.

c) Give an example of a finite measure μ, an $f \in L_1(X, \mu)$, and $(f_n)_{n\geq 1} \subseteq L_1(X, \mu)$ such that $(f_n)_{n\geq 1}$ converges to f in $L_1(X, \mu)$ but $(f_n)_{n\geq 1}$ does converges to f pointwise almost everywhere. Prove your example satisfies the desired conditions.

Question 3. Let $(X, \mathcal{A})$ be a measurable space and let $\mu, \nu : \mathcal{A} \to [0, \infty]$ be measures. Define $\mu + \nu : \mathcal{A} \to [0, \infty]$ by $(\mu + \nu)(A) = \mu(A) + \nu(A)$ for all $A \in \mathcal{A}$. It is elementary to verify that $\mu + \nu$ is a measure.

Prove that if $f : X \to \mathbb{C}$ is measurable with respect to $(X, \mathcal{A})$, then f is integrable with respect to $\mu + \nu$ if and only if f is integrable with respect to μ and with respect to ν. Moreover, when f is integrable, prove that

$$\int_X f\, d(\mu + \nu) = \int_X f\, d\mu + \int_X f\, d\nu.$$

Question 4. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function. Recall the function $\nu : \mathcal{A} \to [0, \infty]$ defined by

$$\nu(A) = \int_A f\, d\mu$$

for all $A \in \mathcal{A}$ is a measure.

a) Prove for all measurable functions $g : X \to [0, \infty]$ that

$$\int_X g\, d\nu = \int_X fg\, d\mu.$$

b) Prove that if f is integrable, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{A}$ and $\mu(A) < \delta$ then $\nu(A) < \epsilon$.

Question 5. Let $f : \mathbb{R} \to \mathbb{C}$ be a Lebesgue integrable function and let $g : \mathbb{R} \to \mathbb{C}$ be an essentially bounded Lebesgue integrable function.

a) For each $y \in \mathbb{R}$, let $f_y : \mathbb{R} \to \mathbb{C}$ be the Lebesgue integrable function where $f_y(x) = f(x - y)$ for all $x \in \mathbb{R}$. Prove for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|y| < \delta$ then $\|f - f_y\|_1 < \epsilon$.

b) Let $f * g : \mathbb{R} \to \mathbb{C}$ be defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y)\, d\lambda(y).$$

Prove that $f * g$ is well-defined and uniformly continuous on $\mathbb{R}$.

Question 6. Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) (The Generalized Dominated Convergence Theorem)
Let g and $(g_n)_{n\geq 1}$ be non-negative integrable functions such that $\lim_{n\to\infty} g_n(x) = g(x)$ for almost every $x \in X$. Let f and $(f_n)_{n\geq 1}$ be measurable functions such that $|f_n| \leq g_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$. Prove that if

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu \qquad \text{then} \qquad \lim_{n\to\infty} \|f - f_n\|_1 = 0.$$

b) Let $p \in [1, \infty)$ and let f and $(f_n)_{n\geq 1}$ be elements of $L_p(X, \mu)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$. Prove that

$$\lim_{n\to\infty} \|f - f_n\|_p = 0 \qquad \text{if and only if} \qquad \lim_{n\to\infty} \|f_n\|_p = \|f\|_p \,.$$

Question 7. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \leq q < p < r \leq \infty$.

a) Prove that if μ is finite and $f \in \mathcal{L}_p(X, \mu)$, then $f \in \mathcal{L}_q(X, \mu)$. (Hint: consider $s = \frac{p}{q} \in (1, \infty)$ and its conjugate.)

b) Prove that if $f : X \to \mathbb{C}$ is measurable, then $\|f\|_p \leq \max\{\|f\|_q, \|f\|_r\}$. (Hint: when $r \neq \infty$ and, $f \in \mathcal{L}_r(X, \mu) \cap \mathcal{L}_q(X, \mu)$, show that $|f|^{\frac{q(r-p)}{r-q}} \in \mathcal{L}_{\frac{r-q}{r-p}}(X, \mu)$ and $|f|^{\frac{r(p-q)}{r-q}} \in \mathcal{L}_{\frac{r-q}{p-q}}(X, \mu)$.)