Recall: Let $(X, \mathcal{A}, \mu)$ be a measure space. If $f \in L_1(X)$ is real-valued, then

$$\nu(A) = \int_A f \, d\mu$$

is a signed measure. In particular,

$$\nu(A) = \int_A f_+ \, d\mu - \int_A f_- \, d\mu$$

where $\int_A f_+ \, d\mu$ and $\int_A f_- \, d\mu$ are also measures.

Section 5.2: The Hahn-Decomposition Theorem

Remark: With $f \in L_1(X)$ as above, let

$$P = \{x \in X : f(x) \geq 0\} \quad \text{and} \quad N = \{x \in X : f(x) < 0\}$$

Then $P, N \in \mathcal{A}$, $P \cap N = \emptyset$ and $P \cup N = X$. Note that if $A \subseteq P$ and $B \subseteq N$, then

$$\nu(A) = \int_A f \, d\mu \geq 0 \qquad \nu(B) = \int_B f \, d\mu \leq 0.$$

Definition: Let ν be a signed measure on a measurable space $(X, \mathcal{A})$. A set $P \in \mathcal{A}$ is said to be positive for ν if for all $A \in \mathcal{A}$ such that $A \subseteq P$, $\nu(A) \geq 0$. Similarly, a set $N \in \mathcal{A}$ is said to be negative for ν if for all $A \in \mathcal{A}$ such that $A \subseteq N$, $\nu(A) \leq 0$. Finally, a set $C \in \mathcal{A}$ is said to be null for ν if for all $A \in \mathcal{A}$ such that $A \subseteq C$, we have $\nu(A) = 0$.

Lemma: Let $\{P_n\}_{n=1}^{\infty}$ be a collection of positive sets for a signed measure for ν. Then $\bigcup_{n=1}^{\infty} P_n$ is positive for ν.

Proof: Clearly, $\bigcup_{n=1}^{\infty} P_n$ is measurable. To see it is positive for ν, let $A \subseteq \bigcup_{n=1}^{\infty} P_n$. Let $A_1 = A \cap P_1$ and for $n \geq 2$, let $A_n = (A \cap P_n) \setminus \left(\bigcup_{i=1}^{n-1} P_i\right)$, so $\{A_n\}_{n=1}^{\infty}$ are pairwise disjoint and measurable such that $\bigcup_{n=1}^{\infty} A_n = A$. Because $A_n \subseteq P_n$ $\forall n \in \mathbb{N}$ and P_n is positive for all $n \in \mathbb{N}$ $\nu(A_n) \geq 0$, so by properties of signed measures,

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A_n) \geq 0.$$

<u>lemma:</u> let ν be a signed measure on $(X, \mathcal{A})$. Assume there is $A \in \mathcal{A}$ such that $\nu(A) > 0$. Then there is a $P \in \mathcal{A}$ where $P \subseteq A$ is positive and $\nu(P) > 0$.

<u>Proof:</u> If A is positive. We are done. Otherwise, there exists $A_1 \in \mathcal{A}$ such that $A_1 \subseteq A$ and $\nu(A_1) < 0$. let $m_1 \in \mathbb{N}$ be the least natural number such that there exists $B \subseteq A$ measurable such that $\nu(B) < -\frac{1}{m_1}$. Choose $B_1 \subseteq A$ measurable such that $\nu(B_1) < -\frac{1}{m_1}$. Consider $A \setminus B_1$. If $A \setminus B_1$ is positive we are done. Otherwise, there is $A_2 \subseteq A \setminus B_1$ and $\nu(A_2) < 0$. let $m_2 \in \mathbb{N}$ be such that $m_1 < m_2$ and the least natural number such that $B \subseteq A \setminus B_1$ such that $\nu(B) < -\frac{1}{m_2}$.
Choose $B_2 \subseteq A$ such that $\nu(B_2) < -\frac{1}{m_2}$. Proceeding inductively, either we find a positive set P with $\nu(P) > 0$ (if we use a later computation), or there exists $\{B_n\}_{n=1}^{\infty}$ of $\mathcal{A}$ and $(m_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ such that

$$B_n \subseteq A \setminus \bigcup_{k=1}^{n-1} B_k \quad \text{and} \quad \nu(B_n) < -\frac{1}{m_n}.$$

let $P = A \setminus \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$. Note that

$$A = P \cup \bigcup_{n=1}^{\infty} B_n. \quad \text{so}$$

$$\nu(A) = \nu(P) + \nu\left(\bigcup_{n=1}^{\infty} B_n\right)$$

Because $\{B_n\}_{n=1}^{\infty}$ are pairwise disjoint with $\nu(B_n) < 0$, we have

$$\nu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \nu(B_n) \in [-\infty, 0), \text{ but cannot be } -\infty$$

So $\lim_{n\to\infty} \nu(B_n) = 0$. So as $\lim_{n\to\infty} m_n = \infty$, we have

$$0 > -\frac{1}{m_n} > \nu(B_n) \longrightarrow 0, \text{ so } \nu(P) > 0.$$

We claim that P is positive. If not, then there is a $B \in \mathcal{A}$ with $B \subseteq P$ and $\nu(B) < 0$. Choose $m \in \mathbb{N}$ to be the least natural number such that $\nu(B) < -\frac{1}{m}$. Because $\lim_{n\to\infty} m_n = \infty$, there exists $N \in \mathbb{N}$ such that $m_n > m$ for all $n \geq N$ so then

$$\nu(B) < -\frac{1}{m_{N-1}} \quad \text{but}$$

$B \subseteq P \subseteq A \setminus \bigcup_{n=1}^{\infty} B_n$ so $B \subseteq A \setminus \bigcup_{n=1}^{N-1} B_n$ contradicting the choice of m_N.

<u>Theorem:</u> Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists $P, N \in \mathcal{A}$ such that $X = P \cup N$ and $P \cap N = \phi$ and P is positive and N is negative for ν.

<u>Proof:</u> If $\nu : \mathcal{A} \to (-\infty, \infty]$, replace ν with $-\nu$ as then $-\nu : \mathcal{A} \to [-\infty, \infty)$ and it is not hard to check

P is positive for $\nu \Leftrightarrow P$ is negative for $-\nu$

N is negative for $\nu \Leftrightarrow N$ is positive for ν.

Without loss of generality, assume $\nu : \mathcal{A} \to [-\infty, \infty)$.

Let $\alpha = \sup\{\nu(P) : P \text{ is positive for } \nu\} \in [0, \infty)$.

Note that $\alpha \neq -\infty$ because ϕ is positive. Choose positive sets P_n such that

$$P = \bigcup_{n=1}^{\infty} P_n \in \mathcal{A}$$

and P is positive for ν.

Note $P_n \subseteq P$ for all $n \in \mathbb{N}$ and $P \setminus P_n \subseteq P$, so

$$\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \geq \nu(P_n)$$

as $P \setminus P_n \subseteq P$ and P is positive implies $\nu(P) \geq \lim_{n\to\infty} \nu(P_n) = \alpha$

so $\nu(P) \leq \alpha$ as P is positive and by the supremum, so

$\nu(P) = \alpha$.

On the other hand, let $N = X \setminus P$. We claim that N is negative for ν. If not, then there exists $A \subseteq N \in \mathcal{A}$ s.t. $\nu(A) > 0$. By the second lemma there is a P_0 such that $P_0 \subseteq A \subseteq N$, P_0 is positive and $\nu(P_0) > 0$.

So $P \cup P_0$ is positive such that

$$\nu(P \cup P_0) = \nu(P) + \nu(P_0) = \alpha + \nu(P_0) > \alpha$$

contradicting the definition of α.

Section 5.3: The Jordan-Decomposition Theorem

Theorem: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists $\nu_+, \nu_- : \mathcal{A} \to [0, \infty]$ such that $\nu = \nu_+ - \nu_-$.

Proof: By the HDT, there are $P, N \in \mathcal{A}$ s.t. (P pos., N neg)

- $X = P \cup N$ and $\emptyset = P \cap N$

Define ν_+, ν_- by

$$\nu_+(A) = \nu(A \cap P)$$
$$\nu_-(A) = -\nu(A \cap N)$$

- $\nu_\pm(\emptyset) = 0$
- countable additivity ✓
- $\nu = \nu_+ - \nu_-$

Definition: Two signed measures ν_1, ν_2 on $(X, \mathcal{A})$ are said to be mutually singular (or orthogonal) denoted by $\nu_1 \perp \nu_2$ if there are $A_1, A_2 \in \mathcal{A}$ s.t. A_1 is null for ν_1

and A_2 is null for ν_2, $X = A_1 \cup A_2$, $\emptyset = A_1 \cap A_2$.

Remark: If μ is a measure, then $A \in \mathcal{A}$ is null for μ if and only if $\mu(A) = 0$, so if μ_1, μ_2 are measures, so $\mu_1 \perp \mu_2$ if and only if there are A_1, A_2 such that $\mu(A_1) = 0$ and $\mu(A_2) = 0$.

Example: For $x \in \mathbb{R}$, $\delta_x \perp \lambda$. Because $\mathbb{R} = \{x\} \cup (\mathbb{R} \setminus \{x\})$
$\lambda(\{x\}) = 0$ and $\delta_x(\mathbb{R} \setminus \{x\}) = 0$.

Example: ν_+ and ν_- in JDT are mutually singular.

Theorem (Jordan Decomposition Theorem): If ν is a signed measure, there are unique $\nu_+, \nu_- : \mathcal{A} \to [0, \infty]$ such that $\nu = \nu_+ - \nu_-$ and $\nu_+ \perp \nu_-$.

Proof: Existence : look above.

Uniqueness: If $\nu_1, \nu_2 : \mathcal{A} \to [0, \infty]$ such that $\nu = \nu_1 - \nu_2$ and $\nu_1 \perp \nu_2$.

Claim: $\nu_1 = \nu_+$, $\nu_2 = \nu_-$

let $P, N, B, C \in \mathcal{A}$ such that

$$X = P \cup N = B \cup C$$
$$\emptyset = P \cap N = B \cap C$$

$\nu_+(N) = \nu_-(P) = \nu_1(C) = \nu_2(B) = 0$.

Fix $A \in \mathcal{A}$

Case 1: $A \subseteq P \cap B$

Note $A \subseteq P$, so $\nu_-(A) = 0$
$A \subseteq B$, so $\nu_2(A) = 0$, $\Big\}$ $\nu_- = \nu_2$

and $\nu_+(A) = \nu_+(A) - \nu_-(A) = \nu(A) = \nu_1(A) - \nu_2(A) = \nu_1(A)$.

Case 2: $A \subseteq P \cap C$

$\nu_-(A) = \nu_1(A) = 0$ and

$$\left.\begin{aligned} \nu_+(A) &= \nu_+(A) - \nu_-(A) \\ &= \nu(A) \\ &= \nu_1(A) - \nu_2(A) \\ &= -\nu_2(A) \end{aligned}\right\} \quad \nu_+ = \nu_2 = 0.$$

③ $A \subseteq N \cap B$ — sim to ②

④ $A \subseteq N \cap C$ — sim to ①

⑤ $A = (A \cap P \cap B) \cup (A \cap P \cap C) \cup (A \cap N \cap B) \cup (A \cap N \cap C)$

Definition: For ν signed measure, ν_+ and ν_- are positive and negative parts of ν.