Question 1 (Chebyshev's Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f : X \to [0, \infty)$ be measurable, and let $\alpha > 0$. Prove that if $g : [0, \infty) \to (0, \infty)$ is non-decreasing, then

$$\mu(\{x \in X \mid f(x) \geq \alpha\}) \leq \frac{1}{g(\alpha)} \int_X (g \circ f) \, d\mu.$$

Solution. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f : X \to [0, \infty)$ be measurable, let $\alpha > 0$, and let $g : [0, \infty) \to (0, \infty)$ be non-decreasing. Therefore $g^{-1}((a, \infty))$ is an interval for every $a \in \mathbb{R}$. Hence g is a Borel function and thus $g \circ f$ is measurable. (Note this was necessary to verify in order to ensure $\int_X (g \circ f) \, d\mu$ is well-defined).

Consider the set $A = \{x \in X \mid f(x) \geq \alpha\}$, which is measurable since f is measurable. Since g is non-decreasing, $g(\alpha) \leq g(f(x))$ for all $x \in A$ so

$$0 \leq g(\alpha)\chi_A \leq (g \circ f)\chi_A \leq g \circ f$$

Hence

$$g(\alpha)\mu(A) = \int_A g(\alpha) \, d\mu = \int_X g(\alpha)\chi_A \, d\mu \leq \int_X (g \circ f) \, d\mu.$$

Therefore, since $g(\alpha) > 0$, dividing by $g(\alpha)$ yields the result.

Question 2 (Convergence in Mean)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \in L_1(X, \mu)$, and let $(f_n)_{n\geq 1} \subseteq L_1(X, \mu)$. Recall it is said that $(f_n)_{n\geq 1}$ converges to f in $L_1(X, \mu)$ if $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. In the case that μ is a probability measure, this is also known as *convergence in mean*.

a) Prove that if $(f_n)_{n\geq 1}$ converges to f in $L_1(X, \mu)$, then $(f_n)_{n\geq 1}$ converges to f in measure.

b) Give an example of a finite measure μ, an $f \in L_1(X, \mu)$, and $(f_n)_{n\geq 1} \subseteq L_1(X, \mu)$ such that $(f_n)_{n\geq 1}$ converges to f in measure, $(f_n)_{n\geq 1}$ converges to f pointwise, but $(f_n)_{n\geq 1}$ does converges not to f in $L_1(X, \mu)$. Prove your example satisfies the desired conditions.

c) Give an example of a finite measure μ, an $f \in L_1(X, \mu)$, and $(f_n)_{n\geq 1} \subseteq L_1(X, \mu)$ such that $(f_n)_{n\geq 1}$ converges to f in $L_1(X, \mu)$ but $(f_n)_{n\geq 1}$ does not converges to f pointwise almost everywhere. Prove your example satisfies the desired conditions.

Solution.
a) Assume $(f_n)_{n\geq 1}$ converges to f in $L_1(X, \mu)$. To see that $(f_n)_{n\geq 1}$ converges to f in measure, let $\epsilon > 0$. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}.$$

Since f_n and f are measurable, A_n is measurable for all $n \in \mathbb{N}$. Moreover, since $|f_n(x) - f(x)| \geq \epsilon$ for all $x \in A_n$, we obtain that

$$0 \leq \mu(A_n) \leq \frac{1}{\epsilon} \int_{A_n} |f_n - f| \, d\mu \leq \frac{1}{\epsilon} \int_X |f_n - f| \, d\mu.$$

Therefore, since $\lim_{n\to\infty} \|f - f_n\|_1 = 0$, we obtain that $\lim_{n\to\infty} \mu(A_n) = 0$. Hence, since $\epsilon > 0$ was arbitrary, $(f_n)_{n\geq 1}$ converges to f in measure.

b) Let $X = [0, 1]$ and let μ be the Lebesgue measure restricted to $[0, 1]$. For each $n \in \mathbb{N}$ let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } x \geq \frac{2}{n} \end{cases}.$$

Then f_n is continuous with

$$\int_{[0,1]} |f_n| \, d\lambda = 1$$

for all $n \in \mathbb{N}$. Hence $f_n \in L_1(X, \mu)$.

Clearly $(f_n)_{n\geq 1}$ converges pointwise to $f = 0$. Hence Assignment 2 Question 2 implies that $(f_n)_{n\geq 1}$ converges to 0 in measure. However, since

$$\|f_n - 0\|_1 = \|f_n\|_1 = 1$$

for all $n \in \mathbb{N}$, we see that $\lim_{n\to\infty} \|f_n - 0\|_1 \neq 0$. Hence $(f_n)_{n\geq 1}$ does converges to f in $L_1(X, \mu)$.

c) Let $X = [0, 1]$ and let μ be the Lebesgue measure restricted to $[0, 1]$. Let $(I_n)_{n\geq 1}$ be the sequence of intervals

$$I_1 = \left[0, \frac{1}{2}\right] \qquad I_2 = \left[\frac{1}{2}, 1\right] \qquad I_3 = \left[0, \frac{1}{4}\right] \qquad I_4 = \left[\frac{1}{4}, \frac{1}{2}\right]$$
$$I_5 = \left[\frac{1}{2}, \frac{3}{4}\right] \qquad I_6 = \left[\frac{3}{4}, 1\right] \qquad I_7 = \left[0, \frac{1}{8}\right] \qquad I_i = \left[\frac{1}{8}, \frac{1}{4}\right]$$

and so on. In particular, if $\sum_{k=1}^{N-1} 2^k \leq n < \sum_{k=1}^{N} 2^k$, then

$$I_n = \left[\frac{n - \sum_{k=1}^{N-1} 2^k}{2^N}, \frac{1 + n - \sum_{k=1}^{N-1} 2^k}{2^N}\right].$$

For each $n \in \mathbb{N}$, let $f_n = \chi_{I_n}$. We claim that $(f_n)_{n\geq 1}$ converges to 0 in $L_1(X, \mu)$ but $(f_n)_{n\geq 1}$ does not converges to 0 pointwise almost everywhere. To see that $(f_n)_{n\geq 1}$ converges to 0 in $L_1(X, \mu)$, notice if $\sum_{k=1}^{N-1} 2^k \leq n < \sum_{k=1}^{N} 2^k$ then

$$\int_{[0,1]} f_n \, d\lambda = \lambda(I_n) = \frac{1}{2^N}.$$

Hence, it easily follows that $\lim_{n\to\infty} \|f_n - 0\|_1 = 0$ so $(f_n)_{n\geq 1}$ converges to 0 in $L_1(X, \mu)$.

To see that $(f_n)_{n\geq 1}$ does not converges to 0 pointwise almost everywhere, note for each $x \in [0, 1]$ that there are an infinite number of natural numbers n such that $f_n(x) = 1$. Hence $(f_n)_{n\geq 1}$ does not converges to 0 pointwise anywhere.

Question 3. Let $(X, \mathcal{A})$ be a measurable space and let $\mu, \nu : \mathcal{A} \to [0, \infty]$ be measures. Define $\mu + \nu : \mathcal{A} \to [0, \infty]$ by $(\mu+\nu)(A) = \mu(A) + \nu(A)$ for all $A \in \mathcal{A}$. It is elementary to verify that $\mu + \nu$ is a measure.

Prove that if $f : X \to \mathbb{C}$ is measurable with respect to $(X, \mathcal{A})$, then f is integrable with respect to $\mu + \nu$ if and only if f is integrable with respect to μ and with respect to ν. Moreover, when f is integrable, prove that

$$\int_X f \, d(\mu + \nu) = \int_X f \, d\mu + \int_X f \, d\nu.$$

Solution. First, notice for all $A \in \mathcal{A}$ that

$$\int_X \chi_A \, d(\mu + \nu) = (\mu + \nu)(A) = \mu(A) + \nu(A) = \int_X \chi_A \, d\mu + \int_X \chi_A \, d\nu.$$

Therefore, since integrals are linear, we obtain that

$$\int_X \varphi \, d(\mu + \nu) = \int_X \varphi \, d\mu + \int_X \varphi \, d\nu$$

for all simple functions $\varphi : X \to [0, \infty)$.

Let $g : X \to [0, \infty]$ be a measurable function. Then there exists an increasing sequence $(\varphi_n)_{n \geq 1}$ of simple functions that converge to g pointwise. Therefore, by applying the above integral relation for simple functions plus the Monotone Convergence Theorem three times, we obtain that

$$\int_X g \, d(\mu + \nu) = \int_X g \, d\mu + \int_X g \, d\nu.$$

Let $f : X \to \mathbb{C}$ be a measurable function. Since $|f| : X \to [0, \infty)$ is a measurable function, the above shows us that

$$\int_X |f| \, d(\mu + \nu) = \int_X |f| \, d\mu + \int_X |f| \, d\nu.$$

Hence f is integrable with respect to $\mu + \nu$ if and only if f is integrable with respect to μ and with respect to ν. Moreover, when f is integrable, f is a linear combination of four non-negative measurable functions that are all integrable with respect to all the measures under consideration by the results from class. Therefore, the above paragraph and the linearity of the integral implies that

$$\int_X f \, d(\mu + \nu) = \int_X f \, d\mu + \int_X f \, d\nu.$$

as desired.

Question 4. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function. Recall the function $\nu : \mathcal{A} \to [0, \infty]$ defined by

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$ is a measure.

a) Prove for all measurable functions $g : X \to [0, \infty]$ that

$$\int_X g \, d\nu = \int_X fg \, d\mu.$$

b) Prove that if f is integrable, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{A}$ and $\mu(A) < \delta$ then $\nu(A) < \epsilon$.

Solution.
a) First notice for all $A \in \mathcal{A}$ that

$$\int_X \chi_A \, d\nu = \nu(A) = \int_A f \, d\mu = \int_X \chi_A f \, d\mu.$$

Next, assume $\varphi : X \to [0, \infty)$ is a simple function on $(X, \mathcal{A}, \mu)$. Then $\varphi = \sum_{k=1}^n a_k \chi_{A_k}$ where $\{a_k\}_{k=1}^n \subseteq [0, \infty)$ and $\{A_k\}_{k=1}^n \subseteq \mathcal{A}$. Hence φ is ν-measurable and, due to the linearity of the integral,

$$\int_X \varphi \, d\nu = \sum_{k=1}^n a_k \int_X \chi_{A_k} \, d\nu = \sum_{k=1}^n a_k \int_X \chi_{A_k} f \, d\mu = \int_X \varphi f \, d\mu.$$

Finally, to see the result, assume $g : X \to [0, \infty]$ is an arbitrary μ-measurable function. Hence g is ν-measurable since ν is defined on $\mathcal{A}$. By a result from class, for each $n \in \mathbb{N}$ there exists a simple function $\varphi_n : X \to [0, \infty)$ such that $\varphi_n \leq \varphi_{n+1} \leq g$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varphi_n(x) = g(x)$ for all $x \in X$. Consequently, it is not difficult to see that $(\varphi_n f)_{n\geq 1}$ is an increasing sequence of non-negative measurable functions that converges pointwise to fg. Therefore, by applying the above and the Monotone Convergence Theorem twice, we obtain that

$$\int_X g \, d\nu = \lim_{n\to\infty} \int_X \varphi_n \, d\nu = \lim_{n\to\infty} \int_X \varphi_n f \, d\mu = \int_X fg \, d\mu.$$

Therefore, since g was arbitrary, the result follows.

b) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be an integrable function. To prove the result, we will reduce to f being a simple function.

Let $\epsilon > 0$ be arbitrary. Due to the definition of the integral of f and the fact that $\int_X f \, d\mu < \infty$, there exists a simple function $\varphi : X \to [0, \infty)$ such that $\varphi \leq f$ and

$$\int_X f \, d\mu \leq \int_X \varphi \, d\mu + \frac{\epsilon}{2}.$$

Since $0 \leq \varphi \leq f$ and f is integrable, we obtain that φ is integrable with $f - \varphi \geq 0$. Hence for all $A \in \mathcal{A}$ we obtain that

$$\int_A f \, d\mu - \int_A \varphi \, d\mu = \int_A (f - \varphi) \, d\mu \leq \int_X (f - \varphi) \, d\mu \leq \frac{\epsilon}{2}.$$

Hence

$$\int_A f \, d\mu \leq \int_A \varphi \, d\mu + \frac{\epsilon}{2}.$$

for all $A \in \mathcal{A}$.

Since φ is a simple function, we can write $\varphi = \sum_{k=1}^n a_k \chi_{A_k}$ where $n \in \mathbb{N}$, $\{a_k\}_{k=1}^n \subseteq [0, \infty)$, and $\{A_k\}_{k=1}^n$ are pairwise disjoint measurable sets. Let

$$M = \max(\{a_k\}_{k=1}^n) < \infty$$

and let $\delta = \frac{\epsilon}{2M+1}$. Then $\delta > 0$ and if $A \in \mathcal{A}$ is such that $\mu(A) < \delta$, then

$$\begin{aligned}
\int_A f \, d\mu &\leq \frac{\epsilon}{2} + \int_A \varphi \, d\mu \\
&= \frac{\epsilon}{2} + \sum_{k=1}^{n} a_k \mu(A \cap A_k) \\
&\leq \frac{\epsilon}{2} + M \sum_{k=1}^{n} \mu(A \cap A_k) \\
&\leq \frac{\epsilon}{2} + M\mu\left(\bigcup_{k=1}^{n} A \cap A_k\right) && \{A \cap A_k\}_{k=1}^{n} \text{ are pairwise disjoint} \\
&\leq \frac{\epsilon}{2} + M\delta \\
&= \frac{\epsilon}{2} + M\frac{\epsilon}{2M+1} < \epsilon.
\end{aligned}$$

Hence, since $\epsilon > 0$ was arbitrary, the result follows.

Question 5. Let $f : \mathbb{R} \to \mathbb{C}$ be a Lebesgue integrable function and let $g : \mathbb{R} \to \mathbb{C}$ be an essentially bounded Lebesgue integrable function.

a) For each $y \in \mathbb{R}$, let $f_y : \mathbb{R} \to \mathbb{C}$ be the Lebesgue integrable function where $f_y(x) = f(x-y)$ for all $x \in \mathbb{R}$. Prove for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|y| < \delta$ then $\|f - f_y\|_1 < \epsilon$.

b) Let $f * g : \mathbb{R} \to \mathbb{C}$ be defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\, d\lambda(y).$$

Prove that $f * g$ is well-defined and uniformly continuous on $\mathbb{R}$.

Solution.
a) Let $\epsilon > 0$ be arbitrary. Since f is Lebesgue integrable, a result from class implies there exists a continuous function $g : \mathbb{R} \to \mathbb{C}$ and a compact set K such that $g(x) = 0$ for all $x \notin K$ and

$$\|f - g\|_1 < \frac{\epsilon}{3}.$$

Hence

$$\|f_y - g_y\|_1 = \|f - g\|_1 < \frac{\epsilon}{3}$$

for all $y \in \mathbb{R}$ by the translation invariance of the Lebesgue integral.

Since K is compact, K is bounded. Hence there exists an $M > 0$ such that $K \subseteq [-M, M]$. Moreover, since $g(x) = 0$ for all $x \notin K$, it is elementary to verify that g is uniformly continuous on $\mathbb{R}$. Hence there exists a $\delta > 0$ such that if $|y| < \delta$ then

$$|g(x) - g_y(x)| = |g(x) - g(x-y)| < \frac{\epsilon}{3(4M+1)}$$

for all $x \in \mathbb{R}$. Therefore, if $|y| < \delta$ then

$$\begin{aligned}
\|f - f_y\|_1 &\leq \|f - g\|_1 + \|g - g_y\|_1 + \|g_y - f_y\|_1 \\
&\leq \frac{\epsilon}{3} + \int_{\mathbb{R}} |g(x) - g(x-y)|\, d\lambda(x) + \frac{\epsilon}{3} \\
&= \frac{2\epsilon}{3} + \int_{[-M,M]\cup(y+[-M,M])} |g(x) - g(x-y)|\, d\lambda(x) \\
&\leq \frac{2\epsilon}{3} + \int_{[-M,M]\cup(y+[-M,M])} \frac{\epsilon}{3(4M+1)}\, d\lambda \\
&\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3(4M+1)}\lambda([-M,M]\cup(y+[-M,M])) \\
&\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3(4M+1)}(4M) < \epsilon.
\end{aligned}$$

Hence the proof is complete.

b) To see that $f * g$ is well-defined, note since g is essentially bounded that there exists an $M > 0$ such that $|g(x)| \leq M$ almost everywhere. Hence $g \in \mathcal{L}_\infty(\mathbb{R}, \lambda)$ so Hölder's Inequality along with the translation and inversion invariance of the Lebesgue integral implies that $y \mapsto f(x-y)g(y)$ is Lebesgue integrable for all $x \in \mathbb{R}$.

To see that $f * g$ is uniformly continuous, let $\epsilon > 0$. By part a) there exists a $\delta > 0$ such that if $|t| < \delta$ then

$$\|f - f_y\|_1 < \frac{\epsilon}{M}.$$

Therefore, if $x, x_0 \in \mathbb{R}$ are such that $|x - x_0| < \delta$, then

$$\begin{aligned}
&|(f * g)(x) - (f * g)(x_0)| \\
&= \left| \int_{\mathbb{R}} (f(x-y) - f(x_0 - y))g(y)\, d\lambda(y) \right| \\
&\leq \int_{\mathbb{R}} |f(x-y) - f(x_0 - y)||g(y)|\, d\lambda(y) \\
&\leq \int_{\mathbb{R}} |f(x-y) - f(x_0 - y)| M\, d\lambda(y) \\
&= M \int_{\mathbb{R}} |f(x+y) - f(x_0 + y)|\, d\lambda(y) \\
&= M \int_{\mathbb{R}} |f(y) - f((x_0 - x) + y)|\, d\lambda(y) \\
&\leq M \frac{\epsilon}{M} = \epsilon.
\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $f * g$ is uniformly continuous.

Question 6. Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) (The Generalized Dominated Convergence Theorem)
Let g and $(g_n)_{n\geq 1}$ be non-negative integrable functions such that $\lim_{n\to\infty} g_n(x) = g(x)$ for almost every $x \in X$. Let f and $(f_n)_{n\geq 1}$ be measurable functions such that $|f_n| \leq g_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$. Prove that if

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu \qquad \text{then} \qquad \lim_{n\to\infty} \|f - f_n\|_1 = 0.$$

b) Let $p \in [1, \infty)$ and let f and $(f_n)_{n\geq 1}$ be elements of $L_p(X, \mu)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$. Prove that

$$\lim_{n\to\infty} \|f - f_n\|_p = 0 \qquad \text{if and only if} \qquad \lim_{n\to\infty} \|f_n\|_p = \|f\|_p .$$

Solution.
a) Since $|f_n| \leq g_n$ and g_n is integrable for all $n \in \mathbb{N}$, f_n is integrable for all $n \in \mathbb{N}$. Furthermore, since $\lim_{n\to\infty} g_n(x) = g(x)$ for almost every $x \in X$, $|f_n| \leq g_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$, we obtain that $|f| \leq g$ almost everywhere. Hence f is integrable and there exists an $A \in \mathcal{A}$ such that $|f(x)| \leq g(x)$ for all $x \in A$ and $\mu(A^c) = 0$. By multiplying all functions by χ_A, we can assume that $|f(x)| \leq g(x)$ everywhere.

Consider $g + g_n - |f - f_n|$. Then

$$g + g_n - |f - f_n| \geq g + g_n - |f| - |f_n| \geq 0$$

and

$$\lim_{n\to\infty} g(x) + g_n(x) - |f(x) - f_n(x)| = 2g(x)$$

by assumptions. Hence we obtain by Fatou's Lemma that

$$\begin{aligned}
\int_X 2g \, d\mu &= \int_X \liminf_{n\to\infty} g + g_n - |f - f_n| \, d\mu \\
&\leq \liminf_{n\to\infty} \int_X g + g_n - |f - f_n| \, d\mu \\
&= \liminf_{n\to\infty} \int_X g \, d\mu + \int_X g_n \, d\mu - \int_X |f - f_n| \, d\mu \\
&= \int_X g \, d\mu + \int_X g \, d\mu + \liminf_{n\to\infty} - \int_X |f - f_n| \, d\mu \\
&= \int_X 2g \, d\mu - \limsup_{n\to\infty} \int_X |f - f_n| \, d\mu
\end{aligned}$$

Therefore, since $\int_X g \, d\mu \in \mathbb{R}$, we obtain that

$$\limsup_{n\to\infty} \int_X |f - f_n| \, d\mu = 0.$$

Hence $\lim_{n\to\infty} \|f - f_n\|_1 = 0$ as desired.

b) Let f and $(f_n)_{n\geq 1}$ be elements of $L_p(X, \mu)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$. Assume $\lim_{n\to\infty} \|f - f_n\|_p = 0$. By the triangle inequality

$$\left| \|f\|_p - \|f\|_p \right| \leq \|f - f_n\|_p$$

for all $n \in \mathbb{N}$ and thus $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Conversely, assume $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. Notice that

$$(a+b)^p \leq (2\max\{a,b\})^p = 2^p \max\{a^p, b^p\} \leq 2^p(a^p + b^p)$$

for all $a, b \geq 0$. Let $g = 2(2^p)|f|^p$ and $g_n = 2^p(|f_n|^p + |f|^p)$ for all $n \in \mathbb{N}$. Then g and $(g_n)_{n\geq 1}$ are non-negative integrable functions such that $\lim_{n\to\infty} g_n(x) = g(x)$ for almost every $x \in X$ and

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \lim_{n\to\infty} \|g_n\|_1 = \|g\|_1 = \int_X g \, d\mu.$$

Therefore, since

$$|f_n - f|^p \leq (|f_n| + |f|)^p \leq g_n$$

for all $n \in \mathbb{N}$ and as $\lim_{n\to\infty} |f_n(x) - f(x)|^p = 0$ for almost every x, we obtain by part a) that

$$\lim_{n\to\infty} \|f_n - f\|_p = \lim_{n\to\infty} \int_X |f_n - f|^p \, d\mu = 0$$

as desired.

Question 7. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \leq q < p < r \leq \infty$.

a) Prove that if μ is finite and $f \in \mathcal{L}_p(X, \mu)$, then $f \in \mathcal{L}_q(X, \mu)$. (Hint: consider $s = \frac{p}{q} \in (1, \infty)$ and its conjugate.)

b) Prove that if $f : X \to \mathbb{C}$ is measurable, then $\|f\|_p \leq \max\{\|f\|_q, \|f\|_r\}$. (Hint: when $r \neq \infty$ and, $f \in \mathcal{L}_r(X, \mu) \cap \mathcal{L}_q(X, \mu)$, show that $|f|^{\frac{q(r-p)}{r-q}} \in \mathcal{L}_{\frac{r-q}{r-p}}(X, \mu)$ and $|f|^{\frac{r(p-q)}{r-q}} \in \mathcal{L}_{\frac{r-q}{p-q}}(X, \mu)$.)

Solution.
a) Let $f \in \mathcal{L}_p(X, \mu)$ be arbitrary. Consider $r = \frac{p}{q} > 1$ and notice the conjugate index of r is $w = \frac{r}{r-1} = \frac{p}{p-q}$. To see that $f \in \mathcal{L}_q(X, \mu)$, first notice since $f \in \mathcal{L}_p(X, \mu)$ that

$$\int_X (|f|^q)^r \, d\mu = \int_X |f|^p \, d\mu < \infty.$$

Therefore $|f|^q \in \mathcal{L}_r(X, \mu)$. Since μ is finite, the constant function 1 is in $\mathcal{L}_w(X, \mu)$. Hence Hölder's inequality implies that

$$|f|^q = |f|^q 1 \in \mathcal{L}_1(X, \mu)$$

Hence $f \in \mathcal{L}_q(X, \mu)$ as desired.

b) Let $M = \max(\|f\|_q, \|f\|_r)$. First notice if $f \notin \mathcal{L}_q(X, \mu)$ or $f \notin \mathcal{L}_r(X, \mu)$, then $M = \infty$ and the result holds trivially. Therefore, we may assume that $f \in \mathcal{L}_q(X, \mu)$ and $f \in \mathcal{L}_r(X, \mu)$.

First let us assume that $r = \infty$. Then we easily see that

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left(\int_X |f|^q |f|^{p-q} \, d\mu \right)^{\frac{1}{p}} \\
&\leq \left(\int_X |f|^q \, \|f\|_\infty^{p-q} \, d\mu \right)^{\frac{1}{p}} \qquad \text{as } p > q \\
&= \left(\int_X |f|^q \, d\mu \right)^{\frac{1}{p}} \|f\|_\infty^{\frac{p-q}{p}} \\
&= \|f\|_q^{\frac{q}{p}} \, \|f\|_\infty^{1-\frac{q}{p}} \\
&\leq M^{\frac{q}{p}} M^{1-\frac{q}{p}} = M.
\end{aligned}$$

Hence the result follows.

Otherwise, assume that $r < \infty$. Let

$$s = \frac{r-q}{r-p}.$$

Clearly $1 < s < \infty$ by assumption. Select $s' \in (1, \infty)$ so that $\frac{1}{s} + \frac{1}{s'} = 1$. Therefore

$$s' = \frac{s}{s-1} = \frac{\frac{r-q}{r-p}}{\frac{r-q}{r-p} - 1} = \frac{r-q}{p-q}.$$

Next consider

$$k = \frac{q}{s} = \frac{q(r-p)}{(r-q)}.$$

Since $k > 0$, we see that $|x|^k$ is well-defined for all $x \in \mathbb{C}$. Furthermore, as

$$p - k = p - \frac{q(r-p)}{r-q} = \frac{(pr - pq) - (qr - qp)}{r-q} = \frac{pr - qr}{r-q} = \frac{r(p-q)}{r-q} > 0$$

we see that $|x|^{p-k}$ is well-defined for all $x \in \mathbb{C}$.

We claim that $|f|^k \in \mathcal{L}_s(X,\mu)$ and $|f|^{p-k} \in \mathcal{L}_{s'}(X,\mu)$. To see this, we notice that

$$\int_X \left(|f|^k\right)^s \, d\mu = \int_X \left(|f|^{\frac{q}{s}}\right)^s \, d\mu = \int_X |f|^q \, d\mu < \infty.$$

and

$$\int_X \left(|f|^{p-k}\right)^{s'} \, d\mu = \int_X \left(|f|^{\frac{r(p-q)}{r-q}}\right)^{\frac{r-q}{p-q}} \, d\mu = \int_X |f|^r \, d\mu < \infty.$$

Therefore, by Hölder's inequality applied to $|f|^k \in \mathcal{L}_s(X,\mu)$ and $|f|^{p-k} \in \mathcal{L}_{s'}(X,\mu)$ (which makes sense as $\frac{1}{s} + \frac{1}{s'} = 1$), we obtain that

$$\int_X |f|^k |f|^{p-k} \, d\mu \le \left(\int_X \left(|f|^k\right)^s\right)^{\frac{1}{s}} \left(\int_X \left(|f|^{p-k}\right)^{s'} \, d\mu\right)^{\frac{1}{s'}}.$$

Hence

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \\
&= \left(\int_X |f|^k |f|^{p-k} \, d\mu\right)^{\frac{1}{p}} \\
&\le \left(\int_X \left(|f|^k\right)^s \, d\mu\right)^{\frac{1}{sp}} \left(\int_X \left(|f|^{p-k}\right)^{s'} \, d\mu\right)^{\frac{1}{s'p}} \\
&= \left(\int_X |f|^q \, d\mu\right)^{\frac{1}{sp}} \left(\int_X |f|^r \, d\mu\right)^{\frac{1}{s'p}} \\
&\le (M^q)^{\frac{1}{sp}} (M^r)^{\frac{1}{s'p}} \\
&= M^{\frac{q}{sp} + \frac{r}{s'p}} \\
&= M^{\frac{q(r-p)}{p(r-q)} + \frac{r(p-q)}{p(r-q)}} \\
&= M^{\frac{(qr-qp)+(rp-rq)}{p(r-q)}} = M^{\frac{rp-qp}{p(r-q)}} = M.
\end{aligned}$$

Thus the result follows.