Definition: A function $\varphi : X \to [0,\infty)$ is said to be **simple** if there exists $n \in \mathbb{N}$, $\{A_i\}_{i=1}^n \subseteq \mathcal{A}$ nonempty pairwise disjoint, and with union X and $(a_i)_{i=1}^n \subseteq [0,\infty)$ distinct such that

$$\varphi = \sum_{i=1}^n a_i \chi_{A_i}$$

Egrof's Theorem

Theorem: Let μ be a finite measure, let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions, Assume that $f : X \to K$ is measurable and $f_n \to f$ pointwise. Then for any $\delta > 0$ there exists $B \in \mathcal{A}$ such that

(i) $\mu(B) < \delta$

(ii) $f_n \to f$ uniformly on B^c.

Proof: Let $\delta > 0$. For all $m, k \in \mathbb{K}$, let

$$B_{m,k} = \bigcup_{n=m}^{\infty} \{x \in X : |f_n(x) - f(x)| > \tfrac{1}{k}\}.$$

where $B_{m,k}$ is measurable and $B_{m+1,k} \subseteq B_{m,k}$ for all $m \in \mathbb{N}$, and $\bigcap_{m=1}^\infty B_{m,k} = \emptyset$. By the Monotone Convergence Theorem, as μ is finite,

$$\lim_{m\to\infty} \mu(B_{m,k}) = 0.$$

Thus there exists $m_k \in \mathbb{N}$ such that $\mu(B_{m_k,m}) < \frac{\delta}{2^n}$

Let $B = \bigcup_{k=1}^\infty B_{m_k,m} \in \mathcal{A}$, so by subadditivity,

$$\mu(B) = \mu\left(\bigcup_{k=1}^\infty B_{m_k,m}\right) \leq \sum_{k=1}^\infty \mu(B_{m_k,m}) < \sum_{k=1}^\infty \frac{\delta}{2^n} = \delta.$$

proving (i).

To see (ii), let $\varepsilon > 0$, and choose $K \in \mathbb{N}$ such that $\frac{1}{K} < \varepsilon$. Note that $B_{m_K, m} \subseteq B$, so $B^c \subseteq B^c_{m_K, m}$. Thus, if $n \geq m_K$ and $x \in B^c$, then $x \in B_{m_K, K}$ so

$$|f_n(x) - f(x)| < \frac{1}{K} < \varepsilon.$$

Therefore, we have uniform convergence on B^c.

Remark: We can replace pointwise convergence with a.e. pointwise convergence.

Indeed, if $A \in \mathcal{A}$ such that $\mu(A^c) = 0$, and $f_n(x) \to f(x)$ for all $x \in A$. Then $f_n \chi_A \to f \chi_A$ pointwise on X. So the proof gives us $B \in \mathcal{A}$ such that $\mu(B) < \delta$ and $f_n \chi_A \to f \chi_A$ uniformly on B^c. Let $D = B \cup A^c \in \mathcal{A}$. Then

$$\mu(D) \leq \mu(B) + \mu(A^c) < \delta + 0 = \delta$$

Moreover, on $D^c = B^c \cap A$, $f_n \chi_A = f_n$ and $f \chi_A = f$, so $f_n \to f$ uniformly on D^c.

Remark: We cannot remove the condition that "μ is finite". Indeed, let $\mu = \lambda$ on $\mathbb{R}$ and $f_n = \chi_{[n,\infty)}$ $\forall n \in \mathbb{N}$. Then $f_n \to 0$ pointwise.

We claim that these fail the conclusions of Egrof's Theorem. Assume for a contradiction they do not fail. Then there exists $B \in \mathcal{M}(\mathbb{R})$ such that $\lambda(B) < 1$ and $f_n \to f$ uniformly on B^c. With $\varepsilon = 1$, there exists $N \in \mathbb{N}$

such that

$$|f_N(x)| = |f_N(x) - 0| < \varepsilon = 1 \quad \forall x \in B^c.$$

Thus, $x \notin [N, \infty)$, so $x \in (-\infty, N)$, for all $x \in B^c$ and so $B^c \subseteq (-\infty, N)$. Therefore, $[N, \infty) \subseteq B$, i.e. $\lambda(B) = \infty$.

Littlewood's First Principle

Definition: Let (X, d) be a metric space and let μ be a measure on a σ-algebra $\mathcal{A}$ of X containing the Borel sets.

(i) We say μ is outer regular if

$$\mu(A) = \inf\{\mu(U) : U \text{ is open and } A \subseteq U\}.$$

(ii) We say μ is inner regular if

$$\mu(A) = \sup\{\mu(K) : K \text{ is compact and } K \subseteq A\}.$$

(iii) We say μ is regular if μ is both outer and inner regular.

Example: (Assignment 1) λ is regular.

Theorem: Let μ be an outer regular measure on $\mathbb{R}$ and let $A \in \mathcal{A}$ be a measurable set with $\mu(A) < \infty$. Then for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and $I_1, \ldots, I_n$ pairwise disjoint open intervals such that $U = \bigcup_{i=1}^{n} I_i$ then

$$\mu(U \triangle A) < \varepsilon,$$

(where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference).

Proof: Let $\varepsilon > 0$ be arbitrary. By outer regularity, there exists U open such that $A \subseteq U$ and

$$\mu(U) < \lambda(A) + \frac{\varepsilon}{2}.$$

Because $\mu(U) < \infty$ and $\mu(A) < \infty$, then

$$\mu(U \setminus A) < \frac{\varepsilon}{2}.$$

Since U is open, then U is the union of a collection $\{I_n\}_{n=1}^{\infty}$ of pairwise disjoint open intervals.

Let $V_n = \bigcup_{i=1}^{n} I_i \in \mathcal{A}$. Then $V_n \subseteq V_{n+1}$ for all $n \in \mathbb{N}$, so by the Monotone Convergence Theorem, there exists $N \in \mathbb{N}$ such that

$$\mu(U) < \mu(V_N) + \frac{\varepsilon}{2}, \quad \text{where } U = \bigcup_{n=1}^{\infty} V_n.$$

Note that

$$\begin{aligned} \mu(V_N \setminus A) &= \mu(V_n) - \mu(A) \\ &\leq \mu(U) - \mu(A) \\ &< \frac{\varepsilon}{2} \end{aligned}$$

also
$$\begin{aligned} \mu(A \setminus V_N) &\leq \mu(U \setminus V_N) \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(V_N \triangle A) &= \mu((V_N \setminus A) + (A \setminus V_N)) \\ &\leq \mu(V_N \setminus A) + \mu(A \setminus V_N) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Lusin's Theorem

Theorem: Let μ be a regular measure on $\mathbb{R}$ such that $\mu([a,b]) < \infty$ for some $a < b \in \mathbb{R}$. Let $f : [a,b] \to \mathbb{K}$ be measurable. Then

(i) for all $\varepsilon > 0$, there exists $F \subseteq \mathbb{R}$ closed such that $\mu([a,b] \setminus F) < \varepsilon$ and $f|_F$ is continuous.

(ii) there exists $g : [a,b] \to \mathbb{K}$ continuous such that $g = f$ on F, $\mu(\{x : g(x) \neq f(x)\}) < \varepsilon$ and

$$\sup_{x \in [a,b]} |g(x)| \leq \sup_{x \in [a,b]} |f(x)|$$

Note: (ii) immediately follows from Tietz Extension Theorem

Theorem: (Tietz) If $F \subseteq \mathbb{R}$ is closed and $h : F \to \mathbb{K}$ is continuous, there exists $g : \mathbb{R} \to \mathbb{K}$ continuous, $g|_F = h$ and

$$\sup_{x \in \mathbb{R}} |g(x)| \leq \sup_{x \in F} |h(x)|$$

Proof: Because F is closed, $F^c = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}_{n=1}^{\infty}$ are pairwise disjoint open intervals. Write $I_n = (a_n, b_n)$ with $a_n < b_n$ $a_n \in \mathbb{R} \cup \{\infty\}$, $b_n \in \mathbb{R} \cup \{-\infty\}$. Then

$$g(x) = \begin{cases} h(x) & \text{if } x \in F \\ h(b_n) & \text{if } a_n = -\infty \\ h(a_n) & \text{if } b_n = \infty \\ \dfrac{x - a_n}{b_n - a_n}(h(b_n) - h(a_n)) + h(a_n) & \text{if } a_n \neq \infty, b_n \neq \infty. \end{cases}$$

Lemma: Lusin's Theorem holds for simple functions.

Proof: Assume $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $\{A_i\}_{i=1}^{n}$ are pairwise disjoint with union $[a,b]$ and $a_i \in [0,\infty]$ for all i.

Fix $\varepsilon > 0$. By inner regularity, there exists closed $\{F_i\}_{i=1}^{n}$ such that $F_i \subseteq A_i$ $\forall i$ and

$$\mu(A_i) < \mu(F_i) + \frac{\varepsilon}{n}$$

Let $F = \bigcup_{i=1}^{n} F_i$. Then F is closed, so

$$\begin{aligned}\mu([a,b] \setminus F) &= \mu\left(\left(\bigcup_{i=1}^{n} A_i\right) \setminus F\right) \\ &= \mu\left(\bigcup_{i=1}^{n} (A_i \setminus F_i)\right) \\ &\leq \sum_{n=1}^{\infty} \mu(A_i \setminus F_i) \\ &< \varepsilon.\end{aligned}$$

Now we show $f|_F$ is continuous. Let $(x_m)_{m=1}^{\infty}$ be a sequence in F such that $x_n \to x$ for some $x \in F$. Then $\exists\, k_x$ such that $x \in F_{k_x}$. Since $\{A_i\}_{i=1}^{n}$ are pairwise disjoint and $F_i \subseteq A_i$ $\forall i$, k_x is unique, so there exists $N \in \mathbb{N}$ such that $x_m \in F_{x_m}$ $\forall\, m \geq N$. Since otherwise, there exists an infinite number of x_m are in F_k, so $x \in F_k$, as F is closed. Hence, $x_m \in F_{k_x}$ $\forall m > N$, so

$f(x_n) = a_{k_x} = f(x)$. Thus, f is continuous at x.

Proof of Lusin: Let $f : [a,b] \to \mathbb{C}$ measurable. Considering the positive and negative portions of the real and imaginary parts of f, by the fact nonnegative measurable functions are pointwise limits of simple functions and Lusin's Theorem holds for simple functions, there exists a sequence $(g_n)_{n=1}^{\infty}$ of measurable functions and $\{F_n\}_{n=1}^{\infty}$ closed such that

(i) $g_n \to f$ pointwise

(ii) $g_n|_{F_n}$ is continuous

(iii) $\mu([a,b] \setminus F_n) < \frac{\varepsilon}{2^{n+1}}$

By Egrof, there exists $B \underset{\subseteq [a,b]}{\in} \mathcal{A}$ such that $\mu(B) < \frac{\varepsilon}{4}$ and $g_n \to f$ uniformly on B^c.

By outer regularity, there exists $\mathcal{U}$ open such that $B \subseteq \mathcal{U}$ and $\mu(\mathcal{U}) < \frac{\varepsilon}{2}$. Let $F' = [a,b] \setminus \mathcal{U}$. Then $\mu([a,b] \setminus F') = \mu(\mathcal{U}) < \frac{\varepsilon}{2}$