Lemma: $(X,\mathcal{A})$ ms, $f,g:(X,\mathcal{A})\to\mathbb{K}$ measurable. Then $h:(X,\mathcal{A})\to\mathbb{K}^2$ defined by

$$h(x)=(f(x),g(x))\quad\forall x\in X$$ is measurable from $\mathcal{A}$ to $\mathcal{B}(\mathbb{K}^2)$.

Proof: We can show that $\mathcal{B}(\mathbb{K}^2)$ is countably generated by sets of the form $I_1\times I_2$ where I_1, I_2 are open in $\mathbb{K}$. Then

$$h^{-1}(I_1\times I_2)=f^{-1}(I_1)\cap g^{-1}(I_2)$$

thus, by assumption, f,g are measurable, so is h.

Corollary: If $(X,\mathcal{A})$ ms, $f,g: X\to\mathbb{K}$ measurable. Then

(i) cf is measurable $\forall c\in\mathbb{K}$

(ii) $f+g$ is measurable.

(iii) fg is measurable.

(iv) $|f|$ is measurable.

(v) If $f\neq 0$, $\frac{1}{f}$ is measurable.

(vi) $\bar{f}$ is measurable.

Proof: (i) follows from (iii). (ii) and (iii) follow by the lemma, as h is measurable and "$+$" $:\mathbb{K}\times\mathbb{K}\to\mathbb{K}$ and "$\cdot$" $:\mathbb{K}\times\mathbb{K}\to\mathbb{K}$ are continuous with

$$f+g=\text{"}+\text{"}\circ h \quad\text{and}\quad fg=\text{"}\cdot\text{"}\circ h \text{ are measurable.}$$

(iv) $-$ (vi) follow by composing with appropriate continuous functions.

Remark: If $f: X \to \mathbb{C}$ we can define $Re(f), Im(f): X \to \mathbb{R}$ by

(i) $Re(f)(x) = \dfrac{f(x) + \overline{f(x)}}{2}$

(ii) $Im(f)(x) = \dfrac{f(x) - \overline{f(x)}}{2i}$

Then $f = Re(f) + i\,Im(f)$. Note f is measurable if and only if $Re(f)$ and $Im(f)$ are measurable.

Remark: If $f: X \to \mathbb{R}$, we define $f_+, f_- : X \to [0, \infty]$ by

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

We call f_+ and f_- the positive and negative parts of f. Note that $f = f_+ + f_-$, where

$f_+ = \dfrac{f + |f|}{2}$ and $f_- = \dfrac{-f + |f|}{2}$. Thus, f is measurable if and only if f_+, f_- are measurable.

Definition: Let $(X, \mathcal{A})$ ms. An extended real-valued function $f: X \to [-\infty, \infty]$ is said to be measurable if $f^{-1}(\{\infty\}), f^{-1}(\{-\infty\}) \in \mathcal{A}$ and $f^{-1}(A) \in \mathcal{A}$ $\forall A \in \mathcal{B}(\mathbb{R})$

Remark: As $f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}((n, \infty])$ and als $f^{-1}(\{-\infty\}) = \left(\bigcup_{n=1}^{\infty} f^{-1}((-n, \infty])\right)^c$. Thus, f is measurable if and only if $\{x : f(x) > a\} \in \mathcal{A}$ $\forall a \in \mathbb{R}$.

Theorem: Let $(X, \mathcal{A})$ ms, let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions from $X \to [-\infty, \infty]$. Then

(i) $\sup_{n\in\mathbb{N}} f_n$ is measurable.

(ii) $\inf_{n\in\mathbb{N}} f_n$ is measurable.

(iii) $\limsup_{n\to\infty} f_n$ is measurable.

(iv) $\liminf_{n\to\infty} f_n$ is measurable.

Thus, if $f: X \to [-\infty, \infty]$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x\in X$. Then f is measurable.

Proof: (i) and (ii) Note that

$$\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} f_n^{-1}((a,\infty)) \in \mathcal{A}$$

$$\left(\inf_{n\in\mathbb{N}} f_n\right)^{-1}([a,\infty)) = \bigcap_{n=1}^{\infty} f_n^{-1}([a,\infty)) \in \mathcal{A}$$

(iii) and (iv) Note that

$$\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \underbrace{\sup_{n\geq k} f_k}_{\in\mathcal{A}} \qquad \liminf_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} \underbrace{\inf_{n\geq k} f_k}_{\in\mathcal{A}}$$

Furthermore, if f exists, $(f_n) \to f$ is measurable.

Corollary: If $f_n : X \to \mathbb{C}$ are measurable and $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x\in X$, then f is measurable.

Proof: Note $\mathrm{Re}(f_n)(x) \to \mathrm{Re}(f)(x)$ and $\mathrm{Im}(f_n)(x) \to \mathrm{Im}(f)(x)$ $\forall x \in X$, so $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable.

Definition: Let $(X, \mathcal{A}, \mu)$ be a ms and $P: X \to \{T, F\}$ a property at each point in X. We say that P holds μ-almost everywhere, or a.e. if there exists $A\in\mathcal{A}$, $P(x) = T$ $\forall x\in A$, and $\mu(A^c) = 0$.

Remark: Note if $B = \{x : P(x) = F\}$, need note be measurable, so $\mu(B)$ may not be defined.

Example: Let $f, g : X \to \mathbb{K}$. We say $f = g$ a.e. if $\exists A \in \mathcal{A}$ s.t. $f(x) = g(x)$ $\forall x \in A$ and $\mu(A^c) = 0$.

Note $\mu(\{x : f(x) \neq g(x)\})$ need not make sense, unless, f, g are measurable. Hence, if f, g meas. $f = g$ a.e if and only if

$$\mu(\{x : f(x) - g(x) \neq 0\}) = 0$$

Example: $\mathbb{1}_{\mathbb{Q}} = 0$ a.e.

Example: Let $(X, \mathcal{A}, \mu)$ complete ms. If $f, g : X \to \mathbb{K}$ are s.t. f is measurable and $f = g$ a.e, then g is measurable.

Proof: Because $f = g$ a.e, $\exists A \in \mathcal{A}$ s.t. $f(x) = g(x)$ $\forall x \in A$ and $\mu(A^c) = 0$. Then $\forall B \in \mathcal{B}(\mathbb{K})$,

$$\begin{aligned} g^{-1}(B) &= (g^{-1}(B) \cap A) \cup (g^{-1}(B) \cap A^c) \\ &= \underbrace{(f^{-1}(B) \cap A)}_{\in \mathcal{A}} \cup (g^{-1}(B) \cap A^c) \end{aligned}$$

Bec. $g^{-1}(B) \cap A^c \subseteq A^c$, $\mu(A^c) = 0$, μ is complete, so $g^{-1}(B) \cap A^c \in \mathcal{A}$, so $g^{-1}(B) \in \mathcal{A}$, and g is measurable.

Theorem: Let $(X, \mathcal{A}, \mu)$ complete ms, $f_n : X \to \mathbb{K}$ measurable $\forall n \in \mathbb{N}$. If $f = \lim_{n\to\infty} f_n$ pointwise a.e. then f is measurable.

Proof: Because $f = \lim_{n\to\infty} f_n$ pointwise a.e., there exists $A \in \mathcal{A}$ s.t. $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x \in A$ and $\mu(A^c) = 0$. Then $f = f\mathbb{1}_A$ a.e. and $f\mathbb{1}_A = \lim_{n\to\infty} f_n \mathbb{1}_A$ pointwise. Since $A \in \mathcal{A}$, $f_n \mathbb{1}_A$ is measurable, so $f\mathbb{1}_A$ is measurable, so f is measurable.

Simple Functions

Definition: A function $\varphi : X \to [0, \infty)$ is said to be **simple** if there exists $n \in \mathbb{N}$, $\{A_i\}_{i=1}^{n} \subseteq \mathcal{A}$ pairwise disjoint, $X = \bigcup_{i=1}^{n} A_i$, distinct $(a_i)_{i=1}^{n} \subseteq [0, \infty)$ such that

$$\varphi = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$$

Remark:

(i) Simple Functions are measurable.

(ii) Let $g : X \to [0, \infty)$ be m-able with finite range. Thus, $g(X) = \{a_1, \dots, a_k\}$ with $a_i \neq a_j$ if $i \neq j$. Let $A_k = g^{-1}(\{a_k\}) \in \mathcal{A}$. Moreover, $A_i \neq A_j$ if $i \neq j$ and $X = \bigcup_{i=1}^{n} A_i$, since $g = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$,

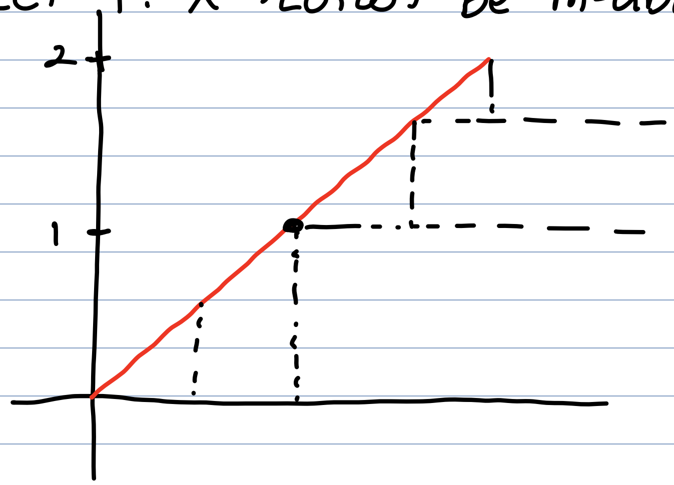
So the set of simple functions is exactly the set of non negative m-able functions with finite range. Thus, sum of 2 simple functions is simple, and non negative scalar multiple of simple function is simple.

Note, if $\{A_i\}_{i=1}^{n} \subseteq \mathcal{A}$ and $(a_i)_{i=1}^{n} \in [0,\infty)$, and $g = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$ then g is m-able. Thus, the way of writing g in the description is called the canonical decomposition of g.

Theorem: A function $f: X \to [0,\infty]$ is m-able iff there exists $(e_i)_{i=1}^{\infty}$ of m-able functions s.t. $e_n \le e_{n+1}$ $\forall n \in \mathbb{N}$ and $f = \lim_{n\to\infty} e_n$ pointwise.

Proof: Pointwise limit simple functions is m-able.

Let $f: X \to [0,\infty]$ be m-able.



For $n \in \mathbb{N}$, let

$K = \{1, 2, \dots, n2^n\}$

Let $A_{n,k} = f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right)$

$B_n = \left(\bigcup_{k=1}^{n2^n} A_{n,k}\right)^c$

Let $\varphi_n = n\mathbb{1}_{B_n} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbb{1}_{A_{n,k}}$. Then φ_n is simple.

and $\varphi_n \le \varphi_{n+1}$ by construction: $A_{n,k}$ becomes two $A_{n+1,k'}$ with $\frac{k'-1}{2^{n+1}} \ge \frac{k-1}{2^n}$

If $f(x) = \infty$, $x \in B_n$ $\forall n \in \mathbb{N}$. Otherwise, if $x \in A_{n,k}$

$\forall n,k$, so $\varphi_n(x) = n \to \infty = f(x)$. If $f(x) < \infty$, there is $N \in \mathbb{N}$ s.t. $f(x) < N$. Then $\forall n \geq N$, $x \in A_{n,k}$ for unique k, so

$x \in f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right)$ so

$|f_n(x) - \varphi_n(x)| \leq \frac{1}{2^n} \quad \forall n \geq N.$