Theorem: Let $1 \le p < \infty$ and let

$\mathcal{F} = \langle \{ \varphi : \varphi$ is simple and $\exists A \in \mathcal{A}$ s.t. $\mu(A) < \infty$ and $\varphi|_{A^c} = 0 \} \rangle$

Then $\mathcal{F}$ is dense in $L_p(X, \mu)$.

Proof: We check that a simple function φ is in $L_p(X,\mu)$ if and only if there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$ and $\varphi|_{A^c} = 0$. Indeed, let $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$ with $a_k \in (0, \infty)$ and $A_k \in \mathcal{A}$. Let $m = \min\{a_1, \dots, a_n\}$ and $M = \max\{a_1, \dots, a_n\}$

Then

$$m^p \mu\left(\bigcup_{k=1}^{\infty} A_k\right) \le \int_X |\varphi|^p \, d\mu \le M^p \mu\left(\bigcup_{k=1}^{n} A_k\right)$$

Let $A = \bigcup_{k=1}^{n} A_k \in \mathcal{A}$. If $\varphi \in L_p(X,\mu)$, then $m^p \mu(A) \le \int_X \varphi^p d\mu < \infty$ then $\mu(A) < \infty$ and $\varphi|_{A^c} = 0$.

Otherwise, if there exists $B \in \mathcal{A}$ such that $\mu(B) < \infty$ and $\varphi|_{B^c} = 0$, then $A \subseteq B$, so $\mu(A) < \infty$. Because $L_p(X,\mu)$ is a vector space we have $\mathcal{F} \subseteq L_p(X,\mu)$

Let $f \in L_p(X,\mu)$ be such that $f \ge 0$. Then there exists a sequence $(\varphi_n)_{n=1}^{\infty}$ of simple functions such that $\varphi_n \le \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $\varphi_n \to f$ pointwise. Because $0 \le \varphi_n \le f$ for all $n \in \mathbb{N}$,

$$\int_X |\varphi_n|^p \, d\mu \le \int_X |f|^p \, d\mu < \infty$$

so $\varphi_n \in \mathcal{F}$ $\forall n \in \mathbb{N}$. Note that $|\varphi_n - f|^p \le |f|^p$ and $|f|^p \in L^1(X,\mu)$

so by DCT,

$$\int_X |f - \varphi_n|^p \, d\mu \to \int_X 0 \, d\mu = 0, \text{ so}$$

$\lim_{n\to\infty} \|f - \mu_n\|_p = 0.$

For $f \in L_p(X,\mu)$, use a linear combination via +ve and -ve of the real and imaginary parts of f.

Theorem: Let $1 \leq p < \infty$ and let

$C_c(\mathbb{R}, \mathbb{K}) = \{ f : \mathbb{R} \to \mathbb{K} : f \text{ cont. and } \operatorname{supp}(f) \text{ is compact} \}$

where $\operatorname{supp}(f) = \overline{\{ x \in \mathbb{R} : f(x) \neq 0 \}}$. Then $C_c(\mathbb{R}, \mathbb{K})$ is dense in $L_p(\mathbb{R}, \lambda)$.

Proof: We first show that $C_c(\mathbb{R}, \mathbb{K}) \subseteq L_p(\mathbb{R}, \lambda)$. Indeed, if $f \in C_c(\mathbb{R}, \mathbb{K})$, then there exists $M_1, M_2 > 0$ such that $|f| \leq M_1$ and $f|_{[-M_2, M_2]^c} = 0$. Thus,

$$\int_{\mathbb{R}} |f|^p \, d\lambda \leq M_1^p (2M_2) < \infty, \text{ so } f \in L_p(X,\mu).$$

By the previous result, it suffices to show that if $\varphi \in L_p(\mathbb{R}, \lambda)$ is simple, then for all $\varepsilon > 0$, there exists $f \in C_c(\mathbb{R}, \mathbb{K})$ such that $\|f - \varphi_n\|_p < \varepsilon$

Since $\varphi \chi_{[-n,n]} \nearrow \varphi$ pointwise, by repeating the proof, using DCT, we consider $\varphi \chi_{[-n,n]}$ in place of φ. By Lusin's theorem, there exists $g : [-n,n] \to \mathbb{R}$ such that g is continuous, $\lambda(\{x : g(x) \neq \varphi(x)\}) < \frac{\varepsilon}{3(2M^p+1)}$ $\max\{|g(x)| : x \in [-n,n]\} \leq \max\{|\varphi(x)| : x \in [-n,n]\}$. Then

$$\int_{\mathbb{R}} |f - \varphi \chi_{[-n,n]}|^p \, d\lambda$$

$$= \int_{[-n,n]} |f - \varphi \chi_{[-n,n]}|^p \, d\lambda + \int_{[-n,n]^c} |f - \varphi \chi_{[-n,n]}|^p \, d\lambda$$

$$\leq \frac{\varepsilon}{3(2M^p+1)} (2M)^p + 2M^p (2\lambda(\_\_)) < \varepsilon$$

Vitali Covering Lemma

Definition: A collection $\mathcal{I}$ of intervals of $\mathbb{R}$ that contains no singleton points is said to be a Vitali covering of a set $X \subseteq \mathbb{R}$ if for all $x \in X$ and for all $\delta > 0$, there exists $I \in \mathcal{I}$ such that $x \in I$ and $\lambda(I) < \delta$.

Examples:

- All intervals (excluding singleton) of $\mathbb{R}$ is a Vitali covering.
- All intervals of length more than one is not.

Theorem: (Vitali Covering Lemma) Let $X \subseteq \mathbb{R}$ be such that $\lambda^*(X) < \infty$. If $\mathcal{I}$ is a Vitali covering of X, then for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\{I_k\}_{k=1}^{\infty} \subseteq \mathcal{I}$ pairwise disjoint such that

$$\lambda^*(X \setminus \bigcup_{k=1}^{n} I_k) < \varepsilon.$$

Proof: By the definition of λ^*, there exists $U \subseteq \mathbb{R}$ such that U is open, $X \subseteq U$ and $\lambda(U) < \infty$.

Let $\mathcal{J} = \{J : J = \overline{I} \text{ with } I \in \mathcal{I}, J \subseteq U\}$. We claim that $\mathcal{J}$ is a Vitali covering of X. Indeed, we do not have singletons. Let $x \in X$ and $\delta > 0$. Because $x \in U$, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq U$. Choose $I \in \mathcal{I}$ such that $x \in I$ and $\lambda(I) \leq \min\{\frac{1}{2}\varepsilon, \frac{1}{2}, \delta\}$. Let $J = \overline{I}$ so $x \in J$ and $\lambda(J) = \lambda(I)$ so $J \subseteq [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}] \subseteq (x-\varepsilon, x+\varepsilon) \subseteq U$.

Hence, $\mathcal{J}$ is a Vitali covering of X. It suffices to prove the result using $\mathcal{J}$ in place of $\mathcal{I}$ as this changes the set difference $X \setminus \bigcup_{k=1}^{n} I_k$ by at most $2n$ points which has measure 0.

Let $\varepsilon > 0$. We will construct $\{J_n\}_{n=1}^{\infty} \subseteq \mathcal{J}$ recursively with specific properties.

- Choose $J_1 \in \mathcal{J}$.
- Assume we have constructed $\{J_k\}_{k=1}^{n} \in \mathcal{J}$ pairwise disjoint with properties. If $X \setminus \bigcup_{k=1}^{n} J_k = \emptyset$, we are done. Otherwise, assume $X \setminus \bigcup_{k=1}^{n} J_k \neq \emptyset$. Let

$$M_n = \sup\{\lambda(J) : J \in \mathcal{J} \text{ and } J \cap \left(\bigcup_{k=1}^{n} J_k\right) = \emptyset\}$$

Note for all $J \in \mathcal{J}$ that $J \subseteq U$, so $M_n \leq \lambda(U) < \infty$.

To see $M_n > 0$, note $X \setminus \bigcup_{k=1}^{n} J_k \neq \emptyset$ so there exists $x \in X \setminus \bigcup_{k=1}^{n} J_k$. Since $\mathcal{J}$ contains closed sets, $\bigcup_{k=1}^{n} J_k$ is closed, so $\delta = \operatorname{dist}(\{x\}, \bigcup_{k=1}^{n} J_k) > 0$. Since $\mathcal{J}$ is a Vitali covering there exists $J \in \mathcal{J}$ such that $x \in J$ and $\lambda(J) < \delta$. The above implies $J \cap \left(\bigcup_{k=1}^{n} J_k\right) = \emptyset$, so $M_n \geq \lambda(J) > 0$.

Hence, we can choose J_{n+1} such that $J_1, \ldots, J_{n+1}$ are pairwise disjoint and

$$M_n \geq \lambda(J_{n+1}) \geq \frac{1}{2} M_n > 0.$$

This process either stops yielding the result, or constructs a sequence $\{J_n\}_{n=1}^{\infty} \subseteq \mathcal{J}$ pairwise disjoint and

$M_n \geq \lambda(J_{n+1}) \geq \frac{1}{2} M_n > 0$. Note

$$\sum_{n=1}^{\infty} \lambda(J_n) = \lambda\left(\bigcup_{n=1}^{\infty} J_n\right) \leq \lambda(U) < \infty.$$

Thus, $\lim_{n\to\infty} \lambda(J_n) = 0$, so $\lim_{n\to\infty} M_n = 0$. Choose $N \in \mathbb{N}$ such that

"Start by going against the US by spelling centre correctly".

$$\sum_{n=N+1}^{\infty} \lambda(J_n) < \frac{\varepsilon}{5}.$$

For each $n \in \mathbb{N}$, let I_n be the closed interval with same centre as J_n but with 5 times the length. Then

$$\sum_{n=N+1}^{\infty} \lambda(I_n) < \varepsilon.$$

Claim: $X \setminus \bigcup_{k=1}^{N} J_n \subseteq \bigcup_{n=N+1}^{\infty} I_n$

Given the claim, we are done. To prove the claim let $x \in X \setminus \bigcup_{n=1}^{N} J_n$ be arbitrary. So $x \notin J_n$ for all $n \in \{1, 2, \dots, N\}$. As before, if $\delta = \operatorname{dist}(\{x\}, \bigcup_{n=1}^{N} J_n)$, then $\delta > 0$. Since $\lim_{n\to\infty} M_n = 0$, there exists $m \in \mathbb{N}$ such that $M_m < \delta^{\leq \lambda(J_x)}$ so for any $J_x \in \mathcal{J}$ such that $x \in J_x$ we have J_x intersects one of $J_1, \dots, J_m$. Choose the smallest $n_x \in \mathbb{N}$ such that any interval $J_x \in \mathcal{J}$ such that $x \in J_x$ intersecting one of $J_1, \dots, J_{n_x}$, so, $J_x \cap J_{n_x} \neq \emptyset$ for $k < n_x$. Note $n_x > N$. Then $M_{n_x - 1} \geq \lambda(J_x)$ and $M_{n_x-1} \leq 2\lambda(J_{n_x})$. Thus, $\lambda(J_x) \leq 2\lambda(J_{n_x})$. Hence, the distance from x to the centre of J_{n_x} is $\lambda(J_x) + \frac{1}{2}\lambda(J_{n_x}) \leq \frac{5}{2}\lambda(J_{n_x})$ so $x \in I_{n_x}$