MATH 6280 **Assignment 4**

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Question 1 (Leibniz Integral Rule)**.** Let $E \in \mathcal{M}(\mathbb{R})$ and let $f : E \times [c, d] \to \mathbb{R}$ be such that

(I) for each $t \in [c, d]$, the function $g_t : E \to \mathbb{R}$ defined by $g(x) = f(x, t)$ is Lebesgue integrable,

(II) for almost every $x \in E$, the function $h_x : (c, d) \to \mathbb{R}$ defined by $h_x(t) = f(x, t)$ is differentiable on (c, d), and

(III) there exists a Lebesgue integrable function $\theta : E \to \mathbb{R}$ such that $|h'_x(t)| \leq \theta(x)$ for all $t \in (c, d)$ and almost every $x \in E$.

Then

$$\frac{d}{dt} \int_E f(x, t)\, d\lambda(x) = \int_E \frac{\partial f}{\partial t}(x, t)\, d\lambda(x)$$

for all $t \in (c, d)$.

Question 2. Recall that a function $f : [a, b] \to \mathbb{R}$ is said to be *Lipschitz* if there exists a constant K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [a, b]$.

Prove that an absolutely continuous function $f : [a, b] \to \mathbb{R}$ is Lipschitz if and only if $|f'| \in L_\infty([a, b], \lambda)$.

Question 3. Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing, absolutely continuous function.

a) Prove that if G is a G_δ-subset of (a, b), then $f(G)$ is Lebesgue measurable and $\lambda(f(G)) = \int_G f'\, d\lambda$.

b) Prove that if $A \subseteq [a, b]$ is Lebesgue measurable with $\lambda(A) = 0$ then $\lambda(f(A)) = 0$.

c) Let $c = f(a)$ and $d = f(b)$. Prove that if $g : [c, d] \to [0, \infty]$ is Borel, then

$$\int_{[c,d]} g\, d\lambda = \int_{[a,b]} (g \circ f) f'\, d\lambda.$$

Question 4. A monotone function $f : [a, b] \to \mathbb{R}$ is said to be *singular* if $f' = 0$ λ-almost everywhere.

a) Prove that any non-decreasing function on $[a, b]$ is the sum of an absolutely continuous non-decreasing function and a singular non-decreasing function.

b) Let $f : [a, b] \to \mathbb{R}$ be a non-decreasing singular function. Prove that f has the following property: (S) For all $\epsilon, \delta > 0$ there exists

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$$

such that

$$\sum_{k=1}^{n} |b_k - a_k| < \delta \qquad \text{and} \qquad \sum_{k=1}^{n} |f(b_k) - f(a_k)| > f(b) - f(a) - \epsilon.$$

c) Let $f : [a, b] \to \mathbb{R}$ be a non-decreasing function with property (S) from part b). Use part a) to prove that f is singular.

d) Let $(f_n)_{n\geq 1}$ be a sequence of non-decreasing singular functions on $[a, b]$ such that the function f defined for all $x \in [a, b]$ by

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is finite everywhere. Prove that f is singular.

e) Show that there exists a strictly increasing, singular, continuous function on $[0, 1]$.