Lebesgue Differentiation Theorem

Definition: Let $f: \mathbb{R} \to \mathbb{R}$. For each $x \in \mathbb{R}$, define

- $D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$
- $D_+ f(x) = \liminf_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$
- $D^- f(x) = \limsup_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$
- $D_- f(x) = \liminf_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$

and note that $D_+ f(x) \leq D^+ f(x)$ and $D_- f(x) \leq D^- f(x)$. We say that f is differentiable at x if

$$D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \quad \forall x \in \mathbb{R}.$$

If f is differentiable at x, then the derivative of f at x, denoted by $f'(x)$.

Theorem: (Lebesgue Differentiation Theorem). If $f: [a,b] \to \mathbb{R}$ is a non-decreasing function, then f is differentiable λ-almost everywhere, f' is Lebesgue measurable, $f' \geq 0$ λ-almost everywhere, and

$$\int_{[a,b]} f' \, d\lambda \leq f(b) - f(a).$$

Proof: Extend f to a function on $\mathbb{R}$ by setting $f(x) = \begin{cases} f(a) & \text{if } x < a \\ f(b) & \text{if } x > b. \end{cases}$ Then f is non-decreasing so $f^{-1}((a,\infty))$ is always an interval and thus, f is measurable.

To see that f is differentiable a.e, we must check for all $s,t \in \{+,-\}$ that

$\{x : D^s f(x) \neq D_t f(x)\}$, $\overset{(*)}{\{x : D^+ f(x) \neq D^- f(x)\}}$, and

$\{x : D_+ f(x) \neq D_- f(x)\}$ all have measure 0, in which case these are Lebesgue measurable, and outside of these sets, $D^+ f(x) \neq \pm\infty$.

Let us check that $(*)$ has λ measure 0. The other are similar.

For each $p, q \in \mathbb{Q}$ with $p > q$, let

$E_{p,q} = \{x : D^+ f(x) > p > q > D^- f(x)\}$.

$F_{p,q} = \{x : D^- f(x) > p > q > D^+ f(x)\}$.

Then we have that

$$\{x : D^+ f(x) \neq D^- f(x)\} = \left(\bigcup_{\substack{p,q \in \mathbb{Q} \\ p > q}} E_{p,q}\right) \cup \left(\bigcup_{\substack{p,q \in \mathbb{Q} \\ p > q}} F_{p,q}\right)$$

Hence, it suffices to show that $\lambda^*(E_{p,q}) = 0$ by subadditivity and $\lambda^*(F_{p,q}) = 0$. We will check for $F_{p,q}$.

Let $p, q \in \mathbb{Q}$ with $p > q$. Let $r = \lambda^*(F_{p,q}) \leq \lambda^*([a,b])$ and let $\varepsilon > 0$. Then there exists an open $U \subseteq \mathbb{R}$ such that $E_{p,q} \subseteq U$ and

$$\lambda(U) \leq \lambda(F_{p,q}) + \varepsilon = r + \varepsilon.$$

Note if $x \in A \subseteq F_{p,q}$, then

$$\inf_{\delta > 0} \sup_{0 < h < \delta} \frac{f(x-h) - f(x)}{-h} = \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = D^- f(x) > p.$$

Thus for all $x \in A$ and $\delta > 0$ there is $h_{x,\delta} > 0$ such that $0 < h_{x,\delta} < \delta$ and $\frac{f(x)-f(x-h)}{h} > p$. As the collection of intervals of the form $[x-h_{x,\delta}, x] \subseteq \bigcup_{j=1}^{n} I_j$ form a Vitali covering of A. By the VCL $\exists\, m \in \mathbb{N}$, $y_1, \ldots, y_m \in A$ and $l_1, \ldots, l_m > 0$ such that if $J_k = (y_k - l_k, y_k)$ then $J_k \subseteq \bigcup_{j=1}^{n} I_j$, $\{J_k\}_{k=1}^{\infty}$ are pairwise disjoint,
$\lambda^*(A \setminus \bigcup_{k=1}^{m} J_k) < \varepsilon$, and $f(y_k) - f(y_k - l_k) > p\, l_k$. Let $B = A \cap (\bigcup_{k=1}^{m} J_k) \subseteq \bigcup_{j=1}^{n} I_j$. $A = B \cup (A \setminus \bigcup_{k=1}^{m} J_k)$

so

$$r - \varepsilon \leq \lambda^*(A) \leq \lambda^*(B) + \lambda^*\left(A \setminus \bigcup_{k=1}^{m} J_k\right)$$
$$\leq \lambda^*(B) + \varepsilon$$

so $\lambda^*(B) \geq r - 2\varepsilon$. Moreover,

$$\sum_{k=1}^{m} f(y_k) - f(y_k - l_k) > p \sum_{k=1}^{m} l_k$$
$$= p \sum_{k=1}^{m} \lambda(J_k)$$
$$= p\, \lambda\left(\bigcup_{k=1}^{m} J_k\right)$$
$$\geq p \lambda^*(B)$$
$$\geq p(r - 2\varepsilon).$$

However, note

$$\sum_{k : J_k \subseteq J_j} f(y_k) - f(y_k - l_k) < f(x_j + h_j) - f(x_j)$$

since the J_k are disjoint and f is nondecreasing, so

$$p(r - 2\varepsilon) \leq \sum_{k=1}^{m} f(y_k) - f(y_k - l_k)$$
$$\leq \sum_{j=1}^{n} f(x_j + h_j) - f(x_j)$$

$\leq q(r+\varepsilon)$.

so $pr \leq qr$, so $r=0$.

Hence, we have f is differentiable a.e, provided we allow $f'(x) = \pm\infty$. Note $f'(x) = -\infty$ is impossible as f is non-decreasing so $D^s f, D_s f \geq 0$, so $f' \geq 0$ a.e. if f' exists.

For all $n \in \mathbb{N}$, let $g_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$g_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{1/n} \quad (**) \quad \forall x \in \mathbb{R}.$$

Then g_n is measurable for all $n \in \mathbb{N}$ and converge pointwise almost everywhere to a function $g : \mathbb{R} \to [0,\infty]$ which is f' provided that $g(x) \neq \infty$ a.e. Thus, g is measurable. Note

$$\begin{aligned}
\int_{[a,b]} g \, d\lambda &\leq \liminf_{n\to\infty} \int_{[a,b]} g_n \, d\lambda \\
&= \liminf_{n\to\infty} n \int_{[a,b]} f(x+\tfrac{1}{n}) - f(x) \, d\lambda(x) \\
&= \liminf_{n\to\infty} n \int_{[a,b]} f_{1/n} \, d\lambda - n \int_{[a,b]} f \, d\lambda \\
&= \liminf_{n\to\infty} n \int_{[a+\frac{1}{n}, b+\frac{1}{n}]} f \, d\lambda - n \int_{[a,b]} f \, d\lambda \\
&= \liminf_{n\to\infty} n \int_{[b, b+\frac{1}{n}]} f \, d\lambda - n \int_{[a, a+\frac{1}{n}]} f \, d\lambda \\
&= \liminf_{n\to\infty} f(b) - n \int_{[a, a+\frac{1}{n}]} f \, d\lambda \\
&= f(b) - \liminf_{n\to\infty} n \int_{[a, a+\frac{1}{n}]} f \, d\lambda
\end{aligned}$$

f is int. as f is non dec.

(Note $n \int_{[a,a+\frac{1}{n}]} f \, d\lambda \geq n \int_{[a,a+\frac{1}{n}]} f(a) \, d\lambda = f(a)$.)

$$\leq f(b) - f(a).$$

Remark: If f is non increasing, then $-f$ is nondecreasing so $(-f)' = -f'$ exists a.e. and is measure, so f' exists a.e and measurable.

Corollary: If $f:[a,b] \to \mathbb{R}$ is Lebesgue measurable a.e, and differentiable a.e. then f' is measurable.

Question: Can we get "=" instead of "≤" in LDT ?

No.

Example: Let $f:[0,1] \to [0,1]$ be the Cantor Ternary function. Then f is constant on each interval in $\mathcal{C}^c$, $f'=0$ on $\mathcal{C}^c$. so $f'=0$ a.e. However,

$$\int_{[0,1]} f' \, d\lambda = 0 < 1 = f(1) - f(0).$$

Bounded Variation

Definition: A function $f:[a,b] \to \mathbb{C}$ is said to be of **bounded variation** if

$$V(f) = \sup_{x_0 < \cdots < x_n} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty$$

Remark: f is of bounded variation if and only if $Re(f)$ and $Im(f)$ are of bounded variation. Hence, we focus on real-valued functions.

Example: Let $f:[a,b] \to \mathbb{R}$ be differentiable such that there exists $M>0$ such that $|f'(x)| \le M$ for all $x \in (a,b)$. By the MVT,

$$|f(x_k) - f(x_{k-1})| \le M(x_k - x_{k-1}), \text{ so } V(f) \le M(x_k - x_{k-1}).$$

Example: Let $f:[0,1] \to [-1,1]$ be such that $f(0)=0$ and $f(x) = x\cos\left(\frac{\pi}{2x}\right)$ for all $x \in [0,1]$. We claim that f is not of bounded variation. Indeed, for $n \in \mathbb{N}$, let $x_0 = 0$, $x_k = \frac{1}{2n+2-k}$

Then

$$|f(x_k)| = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{1}{2n+2-k} & \text{if } k \text{ is even} \end{cases}$$

so $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = 2\sum_{j=1}^{n} \frac{1}{2n+2-2j} = \sum_{k=1}^{n} \frac{1}{k}$