## 4.5 The Fundamental Theorem of Calculus

Lemma: Let $f : [a,b] \to \mathbb{R}$ be Lebesgue integrable. Let $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_{[a,x]} f \, d\lambda$$

If $F$ is non-decreasing, then $f \geq 0$ almost everywhere.

Corollary: If $F$ is constant, then $f = 0$ almost everywhere.

Proof: Use Lemma for $f$ and $-f$.

Proof of Lemma: Assume for a contradiction that $f \not\geq 0$ almost everywhere. Then

$$X = \{ x \in [a,b] : f(x) < 0 \} \in \mathcal{M}(\mathbb{R})$$

and $\lambda(X) > 0$. By inner regularity, there exists $K \subseteq X \subseteq [a,b]$ compact such that $\lambda(K) > 0$.

Let $V = (a,b) \setminus K$ which is open. Then

$$\begin{aligned} F(b) - F(a) &= \int_{[a,b]} f \, d\lambda \\ &= \int_{K \cup V} f \, d\lambda \\ &= \underbrace{\int_K f \, d\lambda}_{<0} + \int_V f \, d\lambda \\ &< \int_V f \, d\lambda \end{aligned}$$

Since $V$ is open, $V = \bigcup_{n=1}^{\infty} (a_n, b_n)$ where $(a_n, b_n) \cap (a_m, b_m) = \phi$ if $n \neq m$. Thus

$$\begin{aligned} \int_V f \, d\lambda &= \int_{\bigcup_{n=1}^{\infty} (a_n, b_n)} f \, d\lambda \\ &= \sum_{n=1}^{\infty} \int_{(a_n, b_n)} f \, d\lambda \\ &= \sum_{n=1}^{\infty} \left( F(b_n) - F(a_n) \right) \end{aligned}$$

$$\leq F(b) - F(a)$$

as $F$ is nondecreasing, which is absurd. $\square$

<u>Theorem:</u> (Fundamental Theorem of Calculus, I). Let $f \in L_1[a,b]$ be real-valued and let $F : [a,b] \to \mathbb{R}$ be defined by

$$F(x) = \int_{[a,x]} f \, d\lambda$$

Then $F' = f$ almost everywhere.

<u>Proof:</u> Because $F$ is absolutely continuous, then $F'$ exists a.e. and is Lebesgue measurable.

<u>Case 1: $f$ is bounded</u>

Let $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. For each $n \in \mathbb{N}$, let $F_n : [a,b] \to \mathbb{R}$ by

$$F_n(x) = \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}} = n \int_{[x, x+\frac{1}{n}]} f \, d\lambda .$$

Extend $f$ to $\mathbb{R}$ by $f(x) = f(b)$ for all $x \geq b$ and also $f(x) = f(a)$ for all $x \leq a$. Note $f$ is still absolutely continuous.

Note $F_n$ is measurable for all $n \in \mathbb{N}$ and $F_n \to F'$ pointwise almost everywhere as $n \to \infty$. Since $f$ is bounded

$$|F_n(x)| = n \left| \int_{[x, x+\frac{1}{n}]} f \, d\lambda \right| \leq n \int_{[x, x+\frac{1}{n}]} |f| \, d\lambda$$

$$\leq n \int_{[x, x+\frac{1}{n}]} M \, d\lambda \leq M.$$

So $|F_n| \leq M$, so $F'$ and $F$ are integrable. By the Dominated Convergence, we have for all $c \in [a,b]$ that

$$\int_{[a,c]} F' \, d\lambda = \lim_{n\to\infty} \int_{[a,c]} F_n \, d\lambda$$

$$= \lim_{n\to\infty} n \int_{[a,c]} F(x+\tfrac{1}{n}) - F(x) \, d\lambda$$

$$= \lim_{n\to\infty} n \int_{[c, c+\frac{1}{n}]} F \, d\lambda - n \int_{[a, a+\frac{1}{n}]} F \, d\lambda$$

Claim: $\lim_{n\to\infty} \int_{[c,c+\frac{1}{n}]} F \, d\lambda = F(c)$.

Indeed, because is continuous, for all $\varepsilon>0$, there exists $N \in \mathbb{N}$ such that if $x \in [c, c+\frac{1}{n}]$ for all $n \geq N$ then $|F(x) - F(c)| < \varepsilon$. Hence, for all $n \geq N$

$$\begin{aligned}
\left| F(c) - \int_{[c,c+\frac{1}{n}]} F \, d\lambda \right| &= \left| n \int_{[c,c+\frac{1}{n}]} F(c) \, d\lambda - n \int_{[c,c+\frac{1}{n}]} F \, d\lambda \right| \\
&= n \left| \int_{[c,c+\frac{1}{n}]} F(c) - F(x) \, d\lambda(x) \right| \\
&\leq n \int_{[c,c+\frac{1}{n}]} |F(c) - F(x)| \, d\lambda(x) \\
&< n\left(\frac{1}{n} \cdot \varepsilon\right) \\
&= \varepsilon
\end{aligned}$$

Similarly, we have $\lim_{n\to\infty} \int_{[a,a+\frac{1}{n}]} F \, d\lambda = F(a)$. Now,

$$\begin{aligned}
\int_{[a,c]} F' \, d\lambda &= F(c) - F(a) \\
&= \int_{[a,c]} f \, d\lambda
\end{aligned}$$

Since $F'$ and $F$ are integrable, we have

$$\int_{[a,c]} F' - f \, d\lambda = 0 \qquad \text{for all } c \in [a,b]$$

By the corollary, $F' - f = 0 \Rightarrow F' = f$ almost everywhere.

Case 2: If $f \geq 0$ almost everywhere

For all $n \in \mathbb{N}$, let $f_n = \min\{f, n\}$. Then $f_n : [a,b] \to [0,n]$ and is measurable being the infimum of measurable functions. Moreover, $f_n \to f$ pointwise and $0 \leq f_n \leq f$ so $f_n \in L_1[a,b]$.

For all $n \in \mathbb{N}$ and $x \in [a,b]$, let

$$F_n(x) = \int_{[a,x]} f_n \, d\lambda \qquad \text{and} \qquad G_n(x) = \int_{[a,x]} f_n - f \, d\lambda.$$

Since $f$ and $f_n$ are integrable, $F_n, G_n$ are well-defined $F = F_n + G_n$, and $F_n$ and $G_n$ are absolutely continuous, differentiable a.e with $F' = F_n' + G_n'$ a.e.

By Case 1, $F_n' = f_n$. Since $f - f_n \geq 0$ $\forall n \in \mathbb{N}$, $G_n$ is non decreasing so $G_n' \geq 0$ a.e, and $G_n'$ is integrable, so is $F'$, so by Lebesgue Differentiation Theorem, $F' \geq f_n$ a.e. so $F' \geq f$ a.e., so

$$0 \leq \int_{[a,x]} F' - f \, d\lambda = \int_{[a,x]} F' \, d\lambda - \int_{[a,x]} f \, d\lambda$$

$$= \int_{[a,x]} F' \, d\lambda - F(x) \leq F(x) - F(a) - F(x) = 0$$

$$\text{so } \int_{[a,x]} F' - f \, d\lambda = 0, \quad \text{so } F' = f \text{ a.e.}$$

<u>Case 3</u>: If $f : [a,b] \to \mathbb{R}$.

Write $f = f_+ - f_-$ with $f_+, f_-$ nonnegative and integrable $\square$

<u>Theorem</u>: (Fundamental Theorem of Calculus, II) If $F : [a,b] \to \mathbb{R}$ is absolutely continuous, then

$$F(x) = F(a) + \int_{[a,x]} F' \, d\lambda$$

for all $x \in [a,b]$.

<u>Proof</u>: Because $F$ is absolutely continuous, $F'$ exists a.e. and $F' \in L_1[a,b]$. Define $G : [a,b] \to \mathbb{R}$ by

$$G(x) = \int_{[a,x]} F' \, d\lambda$$

Then $G$ is absolutely continuous, by I, $G' = F$, so $G' - F$ is absolutely continuous, and $(G-F)' = 0$ a.e. so $G - F$ is constant, so

$$G(x) - F(x) = G(a) - F(a) = -F(a)$$

$\square$

# Signed Measures

**Definition:** Let $(X, \mathcal{A})$ be a measurable space. A **signed measure** is a function $\nu : \mathcal{A} \to \overline{\mathbb{R}}$ such that

① $\nu(\phi) = 0$

② $\mathrm{Range}(\nu) \subseteq (-\infty, \infty]$ or $\mathrm{Range}(\nu) \subseteq [-\infty, \infty)$.

③ If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ are pairwise disjoint, then

(a) $\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) = \pm\infty$.

(b) $\left|\nu\left(\bigcup_{n=1}^{\infty} A_n\right)\right| < \infty$, then $\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) < \infty$.

**Example:**

① Measures are signed measures

② If $\mu_1, \ldots, \mu_n$ are finite measures and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ then $\nu = \sum_{k=1}^{n} \alpha_k \mu_k$ is a signed measure.

③ Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in L_1(X)$ be real-valued. Then $\nu : \mathcal{A} \to \mathbb{R}$

$$\nu(A) = \int_A f \, d\mu$$

is a signed measure.

**Remark:** $\nu(A) = \int_A f \, d\mu$ can be written as a linear combination of finite measures. Let

$$\mu_{\pm}(A) = \int_A f_{\pm} \, d\mu.$$

Then $\nu = \mu_+ - \mu_-$