Theorem: (Hölder's Inequality) If $f \in \mathcal{L}_p(X,\mu)$ and $g \in \mathcal{L}_q(X,\mu)$ with $p, q \in (1,\infty)$ conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in \mathcal{L}_1(X,\mu)$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$

Corollary: If $\mu(X) < \infty$ and $f \in \mathcal{L}_p(X,\mu)$, then $f \in \mathcal{L}_1(X,\mu)$ and

$$\|f\|_1 \leq \mu(X)^{\frac{1}{q}} \|f\|_p.$$

Theorem: (Minkowski's Inequality) For $1 \leq p < \infty$ and $f, g \in \mathcal{L}_p(X,\mu)$ then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof: Note for $p=1$, we have

$$\int_X |f+g| \, d\mu \leq \int_X |f| + |g| \, d\mu = \int_X |f| \, d\mu + \int_X |g| \, d\mu.$$

Otherwise, for $p > 1$, we have

$$\begin{aligned}
\int_X |f+g|^p \, d\mu &= \int_X |f+g| |f+g|^{p-1} \, d\mu \\
&\leq \int_X (|f| + |g|) |f+g|^{p-1} \, d\mu \\
&= \int_X |f| |f+g|^{p-1} \, d\mu + \int_X |g| |f+g|^{p-1} \, d\mu \\
&\leq \|f\|_p \|(f+g)^{p-1}\|_q + \|g\|_p \|(f+g)^{p-1}\|_q
\end{aligned}$$

If $\|(f+g)^{p-1}\|_q = 0$, we are done. Otherwise we have

$$\left(\int_X |f+g|^p \, d\mu \right)^{1-\frac{1}{p}} \leq \|f\|_p + \|g\|_p.$$

Corollary: $(L_p(X,\mu), \|\cdot\|_p)$ is a normed space.

Theorem: (Riesz-Markov Theorem): For $1 \leq p < \infty$, $(L_p(X,\mu), \|\cdot\|_p)$ is complete.

Proof: Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L_p(X,\mu)$ and let $(f_{n_k})_{k=1}^{\infty}$ be a subsequence such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{2^k}$$

Note that $(f_n)_{n=1}^{\infty}$ converges if and only if $(f_{n_k})_{k=1}^{\infty}$ converges because $(f_n)_{n=1}^{\infty}$ is Cauchy.

For each $x \in X$, let $g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ so $g: X \to [0, \infty]$ and g is measurable. By Fatou's Lemma

$$\left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}} \le \liminf_{N\to\infty} \left(\int_X |f_{n_1}| + \sum_{k=1}^{N} |f_{n_{k+1}} - f_{n_k}| \, d\mu\right)^{\frac{1}{p}}$$
$$\le \liminf_{N\to\infty} \left(\|f_{n_1}\|_p + \sum_{k=1}^{N} \|f_{n_{k+1}} - f_{n_k}\| \, d\mu\right)^{\frac{1}{p}}$$
$$\le \|f_{n_1}\|_p + 1$$

so $g \in \mathcal{L}_p(X, \mu)$, thus, $\mu(\{x : g(x) = \infty\}) = 0$. So with $A = \{x : g(x) = \infty\} \in \mathcal{A}$ we can multiply $(f_n)_{n=1}^{\infty}$ by χ_{A^c} to assume $g(x) < \infty$ for all $x \in X$.

Since $L_p(X, \mu)$ is the equivalence class under a.e. equivalence and $\mu(A) = 0$. Thus,

$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$ defines a function $f: X \to \mathbb{K}$, since $\mathbb{K}$ is complete. Note f is a pointwise limit of measurable functions, thus, measurable.

Moreover,

$$f(x) = \lim_{N\to\infty} f_{n_1}(x) + \sum_{k=1}^{N} (f_{n_{k+1}} - f_{n_k}(x))$$
$$= \lim_{N\to\infty} f_{n_{N+1}}(x)$$

Now we need to show $\lim_{k\to\infty} \|f_{n_{k+1}} - f_{n_k}\|_p = 0$. Indeed,

$$|f_{n_{k+1}}(x) - f_{n_k}(x)|^p \le (|f_{n_{k+1}}(x)| + |f_n(x)|)^p$$
$$\le 2^p |g(x)|^p$$

for all $x \in X$ by DCT. So $g \in \mathcal{L}_1(X, \mu)$ and we have

$\lim_{k\to\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| = 0$, so DCT implies

$$\lim_{k\to\infty} \int_X |f_{n_{k+1}}(x) - f_{n_k}(x)|^p \, d\mu = 0$$

Corollary: $L_2(X,\mu)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_X f\bar{g} \, d\mu.$$

Corollary: For $1 \le p < \infty$, if $(f_n)_{n=1}^{\infty}$ is a sequence in $L_p(X,\mu)$ that converges to f with respect to $\|\cdot\|_p$, then there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ that converges pointwise to f.

Proof: Riesz-Fischer implies there exists a $h \in L_p(X,\mu)$ and a subsequence $(f_{n_k})_{k=1}^{\infty}$ that converges to h in $L_p(X,\mu)$ pointwise a.e. By uniqueness, $f = h$.

Definition: We say a measurable function $f: X \to \mathbb{K}$ is **essentially bounded** if there exists $M > 0$ such that $\mu(\{x : |f(x)| > M\}) = 0$. We denote all essentially bounded functions by $\mathcal{L}_\infty(X,\mu)$ and

$$L_\infty(X,\mu) = \{[f] : f \in \mathcal{L}_\infty(X,\mu)\}$$

Call this the **L_∞ space of (X,μ)**.

Definition: We define the **$\|\cdot\|_\infty$-norm on $L_\infty(X,\mu)$** by

$$\|f\|_\infty = \inf\{M > 0 : \mu(\{x \in X : |f(x)| > M\}) = 0\}$$

Theorem: $L_\infty(X,\mu)$ is a vector space! $\|\cdot\|_\infty$ is a norm on $L_\infty(X,\mu)$.

Proof: Easy to note that $\|\cdot\|_\infty$ is well-defined, $\|\cdot\|_\infty \in [0,\infty)$, $\|0\|_\infty = 0$. If $\|f\|_\infty = 0$, then $f = 0$ because there exists $(a_n)_{n=1}^{\infty}$ in $(0,\infty)$ such that $a_n \to 0$ and

$\mu(\{x : |f(x)| > a_n\}) = 0$, so

$\{x \in X : |f(x)| > 0\} = \bigcup_{n=1}^{\infty} \{x : |f(x)| > a_n\}$ so

$\mu(\{x : |f(x)| > 0\}) = 0$ by subadditivity.

Now let $\alpha \in \mathbb{R}$ and $f \in \mathcal{L}_\infty(X,\mu)$, we show $\alpha f \in \mathcal{L}_\infty(X,\mu)$

if $\alpha = 0$, we are done. Otherwise

$\{x \in X : |\alpha f(x)| > M\} = \{x \in X : |f(x)| > \frac{M}{|\alpha|}\}$

so $\alpha f \in \mathcal{L}_\infty(X,\mu)$ and $\|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$ by computation with inf.

Now let $f, g \in \mathcal{L}_\infty(X,\mu)$. If $M_1, M_2 > 0$, then

$\{x \in X : |f(x) + g(x)| > M_1 + M_2\} \subseteq \{x : |f(x)| + |g(x)| > M_1 + M_2\}$

$\subseteq \{x : |f(x)| > M_1\} \cup \{x : |g(x)| > M_2\}$

Thus if $M_1, M_2 > 0$ are such that

$\mu(\{x : |f(x)| > M_1\}) = 0$ and $\mu(\{x : |g(x)| > M_2\}) = 0$

so $\mu(\{x : |f(x) + g(x)| > M_1 + M_2\}) = 0$. Thus, $f + g \in \mathcal{L}_\infty(X,\mu)$

and $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

Remark: If $f \in \mathcal{L}_\infty(X,\mu)$, then

$\mu(\{x : |f(x)| \geq \|f\|_\infty\}) = 0$. Thus, in $L_\infty(X,\mu)$, we can assume $|f(x)| \leq \|f\|_\infty$ a.e.

Indeed. if $n \in \mathbb{N}$, then by the definition of inf, there exists $M \in (\|f\|_\infty, \|f\|_\infty + \frac{1}{n})$ such that

$\mu(\{x : |f(x)| > M\}) = 0$ but then note

$\{x : |f(x)| > \|f\|_\infty + \frac{1}{n}\} \subseteq \{x : |f(x)| > M\}$, so

$\mu(\{x : |f(x)| > \|f\|_\infty + \frac{1}{n}\}) = 0$. Because

$$\{x : |f(x)| > \|f\|_\infty\} = \bigcup_{n=1}^{\infty} \{x : |f(x)| \geq \|f\|_\infty + \tfrac{1}{n}\}$$

and by subadditivity, we are done.

Remark: If $f \in C[a,b]$, the Extreme Value Theorem implies there exists $M > 0$ such that $[a,b] = \{x : |f(x)| \leq M\}$. Thus $\lambda(\{x : |f(x)| > M\}) = 0$, so $f \in \mathcal{L}_\infty([a,b], \lambda)$.

We claim $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$. Indeed, we have shown "$\leq$" above. On the other hand, if $M < \sup_{x \in [a,b]} |f(x)|$, let $x_0 \in [a,b]$ and $\varepsilon = \frac{|f(x_0)| - M}{2} > 0$. Because f is continuous, there exists $\delta > 0$ such that if $x \in [a,b]$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$ so

$$|f(x)| > |f(x_0)| - \varepsilon > M, \text{ thus } \|f\|_\infty \geq M.$$

Theorem: (Riesz-Fischer) $(L_\infty(X,\mu), \|\cdot\|_\infty)$ is complete. $(*)$

Proof: Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L_\infty(X,\mu)$. WLOG, $|f_n(x)| \leq \|f_n\|_\infty$ $\forall x \in X$, $\forall n \in \mathbb{N}$, and so

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \text{ for all } x \in X,\ n, m \in \mathbb{N}.$$

Now, $(f_n(x))_{n=1}^{\infty}$ is Cauchy on $\mathbb{K}$ for all $x \in X$, which is complete, so there exists $f : X \to \mathbb{K}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Note $\forall x \in X$, $n \in \mathbb{N}$,

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)|$$
$$\leq \liminf_{m\to\infty} \|f_n - f_m\|_\infty$$

With $n=1$,

$$|f_1(x) - f(x)| \leq \liminf_{m\to\infty} \|f_1 - f_m\|_\infty$$

$$\leq \liminf_{m\to\infty} (\|f_1\|_\infty + \|f_m\|_\infty)$$ Cauchy sequence are bounded

$$\leq \|f_1\|_\infty + M \quad \text{for some } M > 0$$

thus $f - f_1 \in \mathcal{L}_\infty(X,\mu)$, so because $\mathcal{L}_\infty(X,\mu)$ is a vector space and $f_1 \in \mathcal{L}_\infty(X,\mu)$, $f \in \mathcal{L}_\infty(X,\mu)$, so

$$\|f_1 - f\|_\infty \leq \liminf_{m\to\infty} \|f_n - f_m\|_\infty \to 0 \quad \text{because } (f_n)_{n=1}^{\infty} \text{ Cauchy}$$

Corollary: (Hölder's Inequality) If $f \in \mathcal{L}_1(X,\mu)$ and $g \in \mathcal{L}_\infty(X,\mu)$, then $fg \in \mathcal{L}_1(X,\mu)$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

Proof: $\int_X |fg|\, d\mu \leq \int_X |f| \, \|g\|_\infty \, d\mu = \|f\|_1 \|g\|_\infty$.

Corollary: If $\mu(X) < \infty$, $g \in \mathcal{L}_\infty(X,\mu)$, $p \in [1,\infty)$, then $g \in \mathcal{L}_p(X,\mu)$ and $\|g\|_p \leq \mu(X)^{\frac{1}{p}} \|g\|_\infty$

Proof: $\left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X \|g\|_\infty^p \, d\mu\right)^{\frac{1}{p}} = \left(\|g\|_\infty^p \, \mu(X)\right)^{\frac{1}{p}} =$

Theorem: Let $1 \leq p < \infty$. If

$$\mathcal{F} = \langle \{ \varphi : \varphi \text{ is simple such that } \exists A \in \mathcal{A}, \mu(A) < \infty, \ \varphi|_{A^c} = 0 \} \rangle$$

Then $[\mathcal{F}] = L_p(X,\mu)$

Theorem: Let $C_c(\mathbb{R},\mathbb{K})$ denote the $\mathbb{K}$-valued compact support i.e. $\mathrm{supp}(f) = \overline{\{x \in \mathbb{R} : |f(x)| > 0\}}$ is compact. Then $\overline{C_c(\mathbb{R},\mathbb{K})} = L_p(\mathbb{R},\lambda)$, $1 \leq p < \infty$.