MATH 6280
Measure Theory

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Winter 2025

Preface

These are the first edition of these lecture notes for MATH 6280 (Measure Theory). Consequently, there may be several typographical errors, missing exposition on necessary background, and more advanced topics for which there will not be time in class to cover. Future iterations of these notes will hopefully be fairly self-contained provided one has the necessary background. If you come across any typos, errors, omissions, or unclear expositions, please feel free to contact me so that I may continually improve these notes.

Contents

Chapter 1

Measure Spaces

As per its title, this course is dedicated to the study of the theory of measures. So what sort of course would this be if we did not define the main objective of study in the first chapter? After defining the notion of a measure, we will examine several examples and properties of measures that immediately follow from definitions. We will then turn to constructing measures from various notions of length and extending measure-like functions to actual measures. This will lead us to looking at measures with important analytical properties including the Lebesgue-Stieltjes measures and metric outer measures. By using metric outer measures, we will obtain the notion of Hausdorff dimension for subsets of metric spaces.

1.1 Measure Spaces

Definition 1.1.1

Let X be a nonempty set. A σ-algebra on X is a subset $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

(i) $\emptyset, X \in \mathcal{A}$, i.e. we measure the empty event and the full event.

(ii) If $A \in \mathcal{A}$, then $A^c = X \setminus A \in \mathcal{A}$, i.e. we measure the complement of an event.

(iii) If $(A_n)_{n=1}^\infty$ is a countable collection of events in $\mathcal{A}$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a *measurable space*, and the elements of $\mathcal{A}$ are called *measurable sets*.

Remark 1.1.2

One may ask why we only ask for countable unions of measurable sets to be measurable. One answer for this comes with the definition of a measure in that we want to have additivity over disjoint unions and adding over an uncountable set only works if only a countable number of elements are nonzero. Another reason is that restricting to countable collections is quite powerful as we will see in this course.

Remark 1.1.3

One may also ask why we have not required that the intersection of countable collection of measurable sets is measurable. The reason is that countable intersections come for free. Indeed, if $(X, \mathcal{A})$ is a measurable space and $(A_n)_{n=1}^{\infty}$ is a countable collection of events in $\mathcal{A}$, then

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{A}$$

as complements and countable unions of elements of $\mathcal{A}$ are elements of $\mathcal{A}$. Furthermore, by using $\emptyset$ in unions and X in intersections, clearly, a finite union or intersection of elements of $\mathcal{A}$ is an element of $\mathcal{A}$.

Example 1.1.4

Let X be a nonempty set.

(α) $(X, \mathcal{P}(X))$ is a measurable space and $(X, \{\emptyset, X\})$ is a measurable space.

(β) Let

$$\mathcal{A} = \{A \subseteq X : A \text{ is countable or } A^c \text{ is countable}\}$$

Then $(X, \mathcal{A})$ is a measurable space.

Moreover, if one has a collection of σ-algebras on a set X, there are ways of constructing new σ-algebras. In particular, it is elementary to verify the following using set properties and Definition 1.1.1.

Lemma 1.1.5

Let X be a nonempty set and let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of σ-algebras of X over index set I. Then

$$\bigcap_{i \in I} \mathcal{A}_i$$

is a σ-algebra of X.

Remark 1.1.6

Using Lemma 1.1.5, we can construct the smallest σ-algebra containing a collection of subsets. Indeed, let X be a nonempty set and let $A \subseteq \mathcal{P}(X)$. Define

$$I = \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ such that } A \subseteq \mathcal{A}\}$$

Clearly, $\mathcal{P}(X) \in I$, so I is nonempty. Hence, Lemma 1.1.5 implies that

$$\sigma(A) = \bigcap_{\mathcal{A} \in I} \mathcal{A}$$

is a σ-algebra. Since clearly $A \subseteq \sigma(A)$ by construction, $\sigma(A)$ is the smallest σ-algebra of X contains A. As such, $\sigma(A)$ is called the *σ-algebra generated by A.*

Definition 1.1.7

Let (X, d) be a metric space. The σ-algebra generated by the open subsets of X is called the *Borel σ-algebra* and is denoted $\mathcal{B}(X)$. In particular, $\mathcal{B}(X)$ is also the σ-algebra generated by the closed subsets of X as open and closed sets are complements of each other and as σ-algebras are closed under complements. Elements of $\mathcal{B}(X)$ are called *Borel sets*.

Remark 1.1.8

In terms of the Borel subsets of $\mathbb{R}$, the sets

$$\begin{aligned}
&\{(a, b) : a < b \in \mathbb{R}\}\\
&\{(a, b] : a < b \in \mathbb{R}\}\\
&\{[a, b) : a < b \in \mathbb{R}\}\\
&\{[a, b] : a < b \in \mathbb{R}\}\\
&\{(-\infty, b) : b \in \mathbb{R}\}\\
&\{(-\infty, b] : b \in \mathbb{R}\}\\
&\{(a, \infty) : a \in \mathbb{R}\}\\
&\{[a, \infty) : a \in \mathbb{R}\}
\end{aligned}$$

all can be shown to generate $\mathcal{B}(\mathbb{R})$ via unions, intersections, and complements. To show this, we verify the following

(i) $\mathcal{B}(\mathbb{R})$ contains each of these sets.

(ii) Any σ-algebra containing one of these sets contains all open intervals and thus all open sets by the fact that every open set is a countable union of open intervals.

We note it is possible to show that $|\mathcal{B}(\mathbb{R})| = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$.

Using σ-algebras, we may now define the central object of study in this course.

Definition 1.1.9

Let $(X, \mathcal{A})$ be a measurable space. A (countably additive, positive) *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $(A_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

where the sum is infinite if one of the elements is ∞ or if the sum diverges.

The triple $(X, \mathcal{A}, \mu)$ is a *measure space* and given an element $A \in \mathcal{A}$, $\mu(A)$ is called the *μ-measure* of A.

Remark 1.1.10

Notice if $(X, \mathcal{A}, \mu)$ is a measure space and $A_1, ..., A_n$ are pairwise disjoint subsets of $\mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

by using the properties of a measure with $A_i = \emptyset$ for all $i > n$.

Before we get too deep into the study of properties of measures, let us examine some common measures which are easy to define.

Example 1.1.11

(α) Let X be a nonempty set and let $x \in X$. The *point-mass measure* at x is the measure δ_x on $(X, \mathcal{P}(X))$ defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(β) Let X be a nonempty set. The *counting measure on X* is the measure μ on $(X, \mathcal{P}(X))$ defined by

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

Example 1.1.12

A function $\mu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ of elements of $[0, \infty]$ such that

$$\mu(A) = \sum_{n \in A} a_n$$

for all $A \subseteq \mathbb{N}$.

To see this, note that if μ has the described form, then μ is a measure. Conversely, suppose μ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. As for each $n \in \mathbb{N}$, the set $\{n\}$ is measurable, for each $n \in \mathbb{N}$, and we may define

$$a_n = \mu(\{n\}) \in [0, \infty]$$

We claim that

$$\mu(A) = \sum_{n \in A} a_n$$

for all $A \subseteq \mathbb{N}$. Indeed, let $A \subseteq \mathbb{N}$. Then as A is countable and

$$A = \bigcup_{n \in A} \{n\}$$

we obtain by the properties of measure that

$$\mu(A) = \mu\left(\sum_{n \in A} \{n\}\right) = \sum_{n \in A} \mu(\{n\}) = \sum_{n \in A} a_n$$

as desired.

Note that measure in Example 1.1.12 can be constructed using Example 1.1.11 (α) and the following technique (which will be of use to us later).

Example 1.1.13

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $(A_n)_{n=1}^\infty \subseteq \mathcal{A}$ and let $(a_n)_{n=1}^\infty \in [0, \infty]$. Define $\nu : \mathcal{A} \to [0, \infty]$ by

$$\nu(A) = \sum_{n=1}^\infty a_n \mu(A_n \cap A)$$

for all $A \in \mathcal{A}$ where the sum equates to ∞ if the sum diverges or one of the terms is ∞ and

$$a \times \infty = \begin{cases} 0 & \text{if } a = 0 \\ \infty & \text{if } a > 0 \end{cases}$$

Then ν is a measure on $(X, \mathcal{A})$. To see this, we clearly note that $\nu(\emptyset) = 0$. Furthermore, if $(B_m)_{m=1}^\infty \subseteq \mathcal{A}$ are pairwise disjoint, then $\{A_n \cap B_m\}_{m=1}^\infty$ are pairwise disjoint for all $n \in \mathbb{N}$, and thus, as μ is a measure,

$$\begin{aligned} \nu\left(\bigcup_{m=1}^\infty B_m\right) &= \sum_{n=1}^\infty a_n \mu\left(A_n \cap \left(\bigcup_{m=1}^\infty B_m\right)\right) \\ &= \sum_{n=1}^\infty a_n \mu\left(\bigcup_{m=1}^\infty (A_n \cap B_m)\right) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \mu(A_n \cap B_m)$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \mu(A_n \cap B_m)$$
$$= \sum_{m=1}^{\infty} \nu(B_m)$$

Hence, ν is a measure as desired.

Although we may define more measures, we turn our attention to properties of measures immediately implied that Definition 1.1.9 and set manipulations. We begin with the following.

Proposition 1.1.14

If $E, F \in \mathcal{A}$ such that $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

Proof.

Indeed, if $E \subseteq F$, we have $F \setminus E \in \mathcal{A}$ is disjoint from E and $F = E \cup (F \setminus E)$, and so

$$\mu(E) \leq \mu(E) + \mu(F \setminus E) = \mu(E \cup (F \setminus E)) = \mu(F)$$

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In particular, if $\mathcal{A}$ is ordered by inclusion, then μ is monotone with respect to this inclusion. This implies if $\mu(X) < \infty$, then $\mu(A) < \infty$ for all $A \in \mathcal{A}$. Moreover, notice in the above computation that if $\mu(E) < \infty$, then we may subtract $\mu(E)$ from both sides in order to obtain that

$$\mu(F \setminus E) = \mu(F) - \mu(E)$$

Note that the above formula need not hold if $\mu(E) = \infty$ as do not know how to define $\infty - \infty$.

Proposition 1.1.15

If $A, B \in \mathcal{A}$ and $\mu(A \cap B) < \infty$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

Proof.

This is easy; we have

$$\begin{aligned} \mu(A \cup B) &= \mu(A \cup (B \setminus (B \cap A))) \\ &= \mu(A) + \mu(B \setminus (B \cap A)) \\ &= \mu(A) + \mu(B) - \mu(A \cap B) \end{aligned}$$

which is a formula that may appear familiar in the context of probability.

Of course, if a measure is going to represent the probability of an event occurring, we must dictate the probability of all possible events is one. As such, when discussing probability, we use the following terminology.

Definition 1.1.16

Let $(X, \mathcal{A}, \mu)$ be a measure space. It is said that $(X, \mathcal{A}, \mu)$ is a *probability space* and μ is a *probability space* if $\mu(X) = 1$. In this case, X is called the *sample space*, elements of $\mathcal{A}$ are called *events*, and given $A \in \mathcal{A}$, $P(A) = \mu(A)$ denotes the probability that the event A occurs.

Remark 1.1.17

It is not difficult to see that a probability space is the correct notion in order to study probability theory. Indeed, the probability of the entire space is one and whenever A and B are disjoint sets, which is the notion of independent events, then the probability of $A \cup B$ is the sum of the probability of A and the probability of B. Furthermore, Proposition 1.1.14 is precisely the formula for the probability of $A \cup B$ when A and B are not disjoint; that is, the formula for the probability of the union of two not necessarily independent events. Of course, when studying probability, one may only have finite additivity instead of countable additivity. As will be seen in later sections, it is not difficult to extend finitely additive measures to countably additive measures, which is far more desirable in our analytic realm.

Of course, requiring the measure of the entire space to be one is a specific property of a measure we may wish to study. The following generalizations of probability measures are vital for the course.

Definition 1.1.18

A measure μ on a measurable space $(X, \mathcal{A})$ is said to be

(i) *finite* if $\mu(X) < \infty$ (and thus, $\mu(A) < \infty$ for all $A \in \mathcal{A}$ by monotonicity).

(ii) *σ-finite* if there exists a collection $(A_n)_{n=1}^\infty \subseteq \mathcal{A}$ such that $X = \bigcup_{n=1}^\infty A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

In most cases, if one can prove a property for any finite measure, one can extend the result to all σ-finite measures using analytic techniques. This is often done using the following additional partition decompositions of a σ-finite measure space.

Remark 1.1.19

Assume μ is a σ-finite measure on $(X, \mathcal{A})$. Thus, there exists a collection $(A_n)_{n=1}^\infty$ of $\mathcal{A}$ such that $X = \bigcup_{n=1}^\infty A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathcal{N}$. Let $B_1 = C_1 = A_1$, and for each $n \geq 2$, let

$$B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} B_k\right) \quad C_n = \bigcup_{k=1}^{n} A_k$$

Then $(B_n)_{n=1}^\infty$ are pairwise disjoint elements of $\mathcal{A}$ are such that $X = \bigcup_{n=1}^\infty B_n$ and $\mu(B_n) \leq \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Similarly, $(C_n)_{n=1}^\infty$ are elements of $\mathcal{A}$ are such that $X = \bigcup_{n=1}^\infty C_n$, $C_n \subseteq C_{n+1}$ for all $n \in \mathbb{N}$, and $\mu(C_n) < \infty$ for all $n \in \mathbb{N}$. The reason $\mu(C_n) < \infty$ can be seen via the following result as $\sum_{i=1}^n \mu(A_i) < \infty$.

Proposition 1.1.20: Subadditivity of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^\infty$ be in $\mathcal{A}$. Then

$$\mu\left(\sum_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n)$$

Proof.

Let $E_1 = A_1$, and for each $n \in \mathbb{N}_{\geq 2}$, denote

$$E_n = A_n \setminus \left(\bigcup_{i=1}^{n} A_i\right)$$

Since $(A_n)_{n=1}^\infty$ are in $\mathcal{A}$, by the properties of σ-algebra we have that $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Furthermore, it is clear that $E_n \cap E_m = \emptyset$ if $n \neq m$, $E_n \subseteq A_n$ for all $n \in \mathbb{N}$, and

$$\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty E_n$$

Hence, by the definition and monotonicity of measures (Proposition 1.1.14), we obtain that

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \mu\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty \mu(E_n) \leq \sum_{n=1}^\infty \mu(A_n)$$

as desired. ∎

As seen above, being able to replace our measurable sets with disjoint measurable is a very useful technique. In particular, the same idea is helpful in proving the following theorem.

Theorem 1.1.21: Monotone Convergence Theorem for Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^\infty$ be a collection of $\mathcal{A}$. Then the following hold.

(i) If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(ii) If $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$ with $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

Proof.

To see that the first assertion holds, let $A_0 = \emptyset$ for conventional simplicity, and for each $n \in \mathbb{N}$, define

$$B_n = A_n \setminus A_{n-1}$$

then $(B_n)_{n=1}^{\infty}$ is a collection of pairwise disjoint elements of $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ and $\bigcup_{i=1}^{n} B_i = A_n$ for all $n \in \mathbb{N}$. Hence,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n)$$

proving (i).

To see that the second assertion holds, let $B_n = A_1 \setminus A_n$ for all $n \in \mathbb{N}$, then $(B_n)_{n=1}^{\infty}$ is a collection of elements of $\mathcal{A}$ with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Thus, since

$$\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)$$

from (i),

$$\mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n)$$

Since $\mu(A_1) < \infty$, then using the part after Proposition 1.1.14, $\mu(A_1 \setminus E) = \mu(A_1) - \mu(E)$ for all $E \in \mathcal{A}$ with $E \subseteq A_1$. Hence,

$$\begin{aligned}
\mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) &= \mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right) \\
&= \lim_{n\to\infty} (A_1 \setminus A_n) \\
&= \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) \\
&= \mu(A_1) - \lim_{n\to\infty} \mu(A_n)
\end{aligned}$$

By subtracting $\mu(A_1) < \infty$ from both sides, we obtain the desired equality, proving (ii). ∎

Exercise 1.1.22: Assignment 1, Question 1

Let μ be a measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x])$$

for all $x \in \mathbb{R}$. The function F is called the *cumulative distribution function* of μ.

(i) Show that if μ is finite, then F is non-decreasing, right continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$.

(ii) Show that if μ is finite and μ has no atoms (that is, $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$), then F is continuous.

Solution.

(a) Assume that μ is finite.

F is non-decreasing: To see that F is non-decreasing, let $x < y \in \mathbb{R}$. Then we have $(-\infty, x] \subseteq (-\infty, y]$ so by monotonicity, we get $\mu((-\infty, x]) \leq \mu((-\infty, y])$, and thus, $F(x) \leq F(y)$, so F is non-decreasing.

F is right-continuous: To see that F is right continuous, let $(x_n)_{n=1}^\infty$ be a decreasing sequence such that $(x_n) \to x$ for some $x \in \mathbb{R}$. Then for each $n \in \mathbb{N}$, we have $x_{n+1} \leq x_n$, and so $(-\infty, x_{n+1}] \subseteq (-\infty, x_n]$ for all $n \in \mathbb{N}$. Moreover, since μ is finite, we have $\mu((-\infty, x_1]) < \infty$. We claim that $\bigcap_{n=1}^\infty (-\infty, x_n] = (-\infty, x]$. Indeed, to see this, if $y \in \bigcap_{n=1}^\infty (-\infty, x_n]$, then for every $n \in \mathbb{N}$, we have $y \in (-\infty, x_n]$, and so $y \leq x_n$. Because $x_n \to x$, we have that $y \leq x$, and so $y \in (-\infty, x]$. For the other inclusion, let $y \in (-\infty, x]$. Then $y \leq x$. But then since $x_n \to x$ for each $n \in \mathbb{N}$, we get $y \leq x_n$, so $y \in (-\infty, x_n]$ for each $n \in \mathbb{N}$, and hence, $y \in \bigcap_{n=1}^\infty (-\infty, x_n]$. Therefore, we must have $\bigcap_{n=1}^\infty (-\infty, x_n] = (-\infty, x]$. Now by the claim and the Monotone Convergence Theorem for Measures, we obtain

$$\begin{aligned}
\lim_{n\to\infty} F(x_n) &= \lim_{n\to\infty} \mu((-\infty, x_n]) \\
&= \mu\left(\bigcap_{n=1}^\infty (-\infty, x_n]\right) \\
&= \mu((-\infty, x]) \\
&= F(x)
\end{aligned}$$

Therefore, this implies that F is right continuous, as required.

$\lim_{x\to-\infty} F(x) = 0$: To see that $\lim_{x\to\infty} F(x) = 0$, let $(x_n)_{n=1}^\infty$ be a decreasing sequence such that $(x_n) \to -\infty$. Then for each $n \in \mathbb{N}$, we have $x_{n+1} \leq x_n$, and so $(-\infty, x_{n+1}] \subseteq (-\infty, x_n]$ for all $n \in \mathbb{N}$. Moreover, since μ is finite, we have $\mu((-\infty, x_1]) < \infty$. We claim that $\bigcap_{n=1}^\infty (-\infty, x_n] = \emptyset$. Assume for a contradiction that $\bigcap_{n=1}^\infty (-\infty, x_n] \neq \emptyset$. Then there exists an element $x \in \bigcap_{n=1}^\infty (-\infty, x_n]$ such that for all $n \in \mathbb{N}$, we have $x \in (-\infty, x_n]$, so $x \leq x_n$ for each $n \in \mathbb{N}$. In particular, since $(x_n)_{n=1}^\infty$ is a decreasing sequence such that $x_n \to -\infty$, there exists an $N \in \mathbb{N}$ such that $x > x_n$ for all $n \geq N$, which implies that $x \notin (-\infty, x_n]$ for all $n \geq N$, which is absurd. Hence, we must have $\bigcap_{n=1}^\infty (-\infty, x_n] = \emptyset$. With the claim and by the Monotone

Convergence Theorem for Measures, we get

$$\begin{aligned}\lim_{n\to\infty} F(x_n) &= \lim_{n\to\infty} \mu((-\infty, x_n])\\ &= \mu\left(\bigcap_{n=1}^{\infty}(-\infty, x_n]\right)\\ &= \mu(\emptyset)\\ &= 0\end{aligned}$$

Therefore, we have shown that $\lim_{x\to-\infty} F(x) = 0$.

$\underline{\lim_{x\to\infty} F(x) = \mu(\mathbb{R})}$: To see that $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$, let $(x_n)_{n=1}^{\infty}$ be an increasing sequence such that $(x_n) \to \infty$. Then since $(x_n)_{n=1}^{\infty}$ is increasing, we have $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, so $(-\infty, x_n] \subseteq (-\infty, x_{n+1}]$ for all $n \in \mathbb{N}$. We claim that $\bigcup_{n=1}^{\infty}(-\infty, x_n] = \mathbb{R}$. Indeed, to see this, if $x \in \bigcup_{n=1}^{\infty}(-\infty, x_n]$, then there exists an $N \in \mathbb{N}$ such that $x \in (-\infty, x_N]$, so $(-\infty, x] \subseteq (-\infty, x_N]$. Since $(x_n)_{n=1}^{\infty} \to \infty$ and is increasing, then $x_n < \infty$ for all $n \in \mathbb{N}$, so $x \leq x_N < \infty$ implies that $x < \infty$, so $x \in \mathbb{R}$. On the other hand, if $x \in \mathbb{R}$, then for any $\varepsilon > 0$, there exists an $M > 0$ such that $x < M - \varepsilon$. By assumption since $(x_n) \to \infty$, there exists an $N \in \mathbb{N}$ such that $x < M - \varepsilon \leq x_n$ for all $n \geq N$, so $x < x_n$, and thus, $x \in (-\infty, x_n]$ for all $n \geq N$, and moreover, $x \in \bigcup_{n=1}^{\infty}(-\infty, x_n]$. Therefore, we get $\bigcup_{n=1}^{\infty}(-\infty, x_n] = \mathbb{R}$. Now, with the claim and the Monotone Convergence Theorem for Measures, we get

$$\begin{aligned}\lim_{n\to\infty} F(x_n) &= \lim_{n\to\infty} \mu((-\infty, x_n])\\ &= \mu\left(\bigcup_{n=1}^{\infty}(-\infty, x_n]\right)\\ &= \mu(\mathbb{R})\end{aligned}$$

Therefore, we have shown that $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$.

(b) To see that F is continuous, it suffices to check that F is left-continuous. Indeed, let $(x_n)_{n=1}^{\infty}$ be an increasing sequence such that $(x_n) \to x$. Then since $(x_n)_{n=1}^{\infty}$ is increasing, we get $(-\infty, x_n] \subseteq (-\infty, x_{n+1}]$ for all $n \in \mathbb{N}$. In particular, we claim that $\bigcup_{n=1}^{\infty}(-\infty, x_n] \cup \{x\} = (-\infty, x]$. Indeed, if $y \in \bigcup_{n=1}^{\infty}(-\infty, x_n] \cup \{x\}$, then either $y \in \bigcup_{n=1}^{\infty}(-\infty, x_n]$ or $y \in \{x\}$. If $y \in \{x\}$, then we have $y = x$ so $y \in (-\infty, x]$. Otherwise, there exists an $N \in \mathbb{N}$ such that $y \leq x_n$ for all $n \geq N$. In particular, as $n \to \infty$, $y \leq x$, so $y \in (-\infty, x]$. For the inclusion, let $y \in (-\infty, x]$. If $y = x$, then $y \in \{x\}$. Otherwise, if $y \neq x$, we claim that $y \in \bigcup_{n=1}^{\infty}(-\infty, x_n]$. Assume for a contradiction that $y \notin \bigcup_{n=1}^{\infty}(-\infty, x_n]$. Then for every $n \in \mathbb{N}$, $y \notin (-\infty, x_n]$, so for every $n \in \mathbb{N}$, $y \in (x_n, \infty)$, i.e. $x_n < y$ for all $n \in \mathbb{N}$. But since $x_n \to x$, we get that $x < y$, so $y \in (x, \infty)$, i.e. $y \notin (-\infty, x]$, which is a contradiction. Hence, we have $y \in \bigcup_{n=1}^{\infty}(-\infty, x_n]$, so $\bigcup_{n=1}^{\infty}(-\infty, x_n] \cup \{x\} = (-\infty, x]$. Observe that $\bigcup_{n=1}^{\infty}(-\infty, x_n] \cap \{x\} = \emptyset$, and also, for each $n \in \mathbb{N}$,

$$\mu((-\infty, x_n] \cup \{x\}) = \mu((-\infty, x_n]) + \mu(\{x\}) = \mu((-\infty, x_n])$$

since $\mu(\{x\}) = 0$. Which also implies that

$$\mu\left(\bigcup_{n=1}^{\infty}(-\infty, x_n] \cup \{x\}\right) = \mu\left(\bigcup_{n=1}^{\infty}(-\infty, x_n]\right)$$

Therefore, by the Monotone Convergence Theorem,

$$\begin{aligned}
\lim_{n\to\infty} F(x_n) &= \lim_{n\to\infty} \mu((-\infty, x_n]) \\
&= \lim_{n\to\infty} \mu((-\infty, x_n] \cup \{x\}) \\
&= \mu\left(\bigcup_{n=1}^{\infty}(-\infty, x_n] \cup \{x\}\right) \\
&= \mu((-\infty, x]) \\
&= F(x)
\end{aligned}$$

Therefore, we have shown that F is continuous, and thus, F is continuous. ■

1.2 The Carathéodory Method

Based on the above notations, it is very natural to ask whether there exists a measure λ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ that emulates the length of a set. In particular, we desire a measure to have some very natural properties, such as

(i) If I is an interval, then $\lambda(I)$ is the length of I.

(ii) If $A \in \mathcal{P}(\mathbb{R})$, $x \in \mathbb{R}$, and $x + A = \{x + a : a \in A\}$, then $\lambda(x + A) = \lambda(A)$; that is, λ is translation invariant.

However, it turns out that no such measure exists. This can be seen via the following example.

Example 1.2.1

Assume for a contradiction that λ is a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ with the above two properties. Define a relation "$\sim$" on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. It is easy to see that "$\sim$" is an equivalence relation on $\mathbb{R}$.

We claim that every element of $\mathbb{R}$ is $\sim$-equivalent to some element in $[0, 1)$. Indeed, if $x \in \mathbb{R}$, then x is the sum of its integer part $\lfloor x \rfloor$ and its fractional part $\{x\}$. Since $x - \{x\} = \lfloor x \rfloor \in \mathbb{Q}$, we obtain that $x \sim \{x\}$. Therefore, since $\{x\} \in [0, 1)$, x is $\sim$-equivalent to some element in $[0, 1)$.

Consequently, every equivalence class under $\sim$ has an element in $[0, 1)$. Let $A \subseteq [0, 1)$ be a set that contains precisely one element from each equivalence class of $\sim$. Note the existence of A follows from the Axiom of Choice.

Since $\mathbb{Q}$ is countable, we enumerate $\mathbb{Q} \cap [0, 1)$ as

$$\mathbb{Q} \cap [0, 1) = \{r_n : n \in \mathbb{N}\}$$

For each $n \in \mathbb{N}$, denote

$$A_n = \{x \in [0,1) : x \in r_n + A \text{ or } x + 1 \in r_n + A\}$$

that is, A_n is $r_n + A$ mod 1.

We claim that $(A_n)_{n=1}^{\infty}$ are disjoint with union $[0,1)$. To see this, note that if $x \in [0,1)$, then there exists a unique $y \in A \subseteq [0,1)$ such that $x \sim y$. Thus, $x - y \in \mathbb{Q} \cap (-1,1)$. If $x - y \in \mathbb{Q} \cap [0,1)$, then $x - y = r_n$ for some $n \in \mathbb{N}$, and thus, $x = r_n + y \in A_n$. Otherwise, if $x - y \in \mathbb{Q} \cap (-1,0)$, then $(x+1) - y \in (0,1)$. Thus, $(x+1) - y = r_n$ for some $n \in \mathbb{N}$, and thus, $x = r_n + y - 1 \in A$. Hence,

$$[0,1) = \bigcup_{n=1}^{\infty} A_n$$

To see that $(A_n)_{n=1}^{\infty}$ are pairwise disjoint, suppose $x \in A_n \cap A_m$ for some $n, m \in \mathbb{N}$. By definition, there exists $y, z \in A$ and $k, l \in \{0,1\}$ such that $x + k = r_n + y$ and $x + l = r_m + z$. Therefore, $y - z = r_n - r_m + k - l \in \mathbb{Q}$ so $y \sim z$. Hence, $y = z$ as A contains exactly one element from each equivalence class of $\sim$. Thus, $0 = r_n - r_m + k - l$. Since $k - l \in \{-1,0,1\}$ and $r_n - r_m \in (-1,1)$, $0 = r_n - r_m + k - l$ can only occur when $n = m$, in which case $n = m$. Thus, $(A_n)_{n=1}^{\infty}$ is a collection of pairwise disjoint sets whose union is $[0,1)$.

For each $n \in \mathbb{N}$, let $B_{n,1} = (r_n + A) \cap [0,1)$ and $B_{n,2} = -1 + ((r_n + A) \cap [1,2))$. Clearly, $A_n = B_{n,1} \cup B_{n,2}$ since $r_n + A \subseteq [0,2)$ for all $n \in \mathbb{N}$.

We claim that $B_{n,1} \cap B_{n,2} = \emptyset$. To see this, assume for a contradiction that $b \in B_{n,1} \cap B_{n,2}$. By definition, there exists $x, y \in A$ such that $r_n + x \in [0,1)$, $r_n + y \in [1,2)$, and $b = r_n + x = -1 + r_n + y$. Clearly, $r_n + x \in [0,1)$ and $r_n + y \in [1,2)$ imply that $x \neq y$, whereas we have $x - y = -1 \in \mathbb{Q}$, so $x \sim y$. Therefore, as A contains exactly one element from each equivalence class, we have obtained a contradiction. Hence, $B_{n,1} \cap B_{n,2} = \emptyset$. To our contradiction, note that

$$\begin{aligned}
1 = \lambda([0,1)) = \lambda\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} \lambda(B_{n,1} \cup B_{n,2}) = \sum_{n=1}^{\infty} [\lambda(B_{n,1}) + \lambda(B_{n,2})] \\
&= \sum_{n=1}^{\infty} \lambda((r_n + A) \cap [0,1)) + \lambda(((r_n + A) \cap [1,2)) = \sum_{n=1}^{\infty} \lambda((r_n + A) \cap [0,2)) \\
&= \sum_{n=1}^{\infty} \lambda(r_n + A) = \sum_{n=1}^{\infty} \lambda(A)
\end{aligned}$$

This yields our contradiction since $\lambda(A) \in [0,\infty]$, yet no number in $[0,\infty]$ when summed an infinite number of times produces 1. Thus, we have obtained a contradiction to the existence of such a λ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$.

The above example illustrates that $\mathcal{P}(\mathbb{R})$ is too large; that is, there are too many sets in $\mathcal{P}(\mathbb{R})$ to define such a measure in a consistent way. The set A in Example 1.2.1 is one of these sets.

To solve this problem, our answer is to reduce the number of sets we consider measurable. Of course, if we would like to do analysis, we need the open sets to be measurable and thus, we require all Borel sets to be measurable. However, the problem still remains, "How do we construct our measure and determine which sets are measurable?"

To answer the problem, we will invoke a technique called Carathéodory's method. The idea of this method is, given a set X, to define a function on the power set of X, that is almost a measure, but has weaker properties. We will then define sets that behave 'nicely' and show these nice sets form a σ-algebra. Finally, we will demonstrate that restricting the function to these nice sets does indeed produce a measure space that hopefully contains some nice measurable sets.

To begin, we define the 'function' that behaves almost like a measure.

Definition 1.2.2

Let X be a nonempty set. A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is said to be an *outer measure* if

(i) $\mu^*(\emptyset) = 0$.

(ii) For all $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, $\mu^*(A) \leq \mu^*(B)$.

(iii) If $(A_n)_{n=1}^\infty$ is a collection in $\mathcal{P}(X)$, then $\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu^*(A_n)$.

Notice that every measure is an outer measure by the results of Section 1.1 whereas an outer measure need not be a measure as it is not necessary that equality occur in the third property of Definition 1.2.2 when the collection $(A_n)_{n=1}^\infty$ are pairwise disjoint. Of course, it is a priori possible that every outer measure is automatically a measure. For an example to show this is not the case, we will need to construct some outer measures. The most natural way to do so is the following which attempts to assign certain sets a specific value.

Definition 1.2.3

Let X be a nonempty set, let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X such that $\emptyset, X \in \mathcal{F}$, and let $\ell : \mathcal{F} \to [0, \infty]$ be any function such that $\ell(\emptyset) = 0$. The *outer measure associated to* ℓ is the function $\mu_\ell^* : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\mu_\ell^*(A) = \inf\left\{\sum_{n=1}^\infty \ell(A_n) : (A_n)_{n=1}^\infty \text{ is a collection of } \mathcal{F} \text{ such that } A \subseteq \bigcup_{n=1}^\infty A_n\right\}$$

for all $A \subseteq X$, where $\inf\{\infty\} = \infty$.

Of course, we need to prove that the outer measure associated to ℓ is actually an outer measure.

Proposition 1.2.4

Let X be a nonempty set, $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X such that $\emptyset, X \in \mathcal{F}$, and $\ell : \mathcal{F} \to [0, \infty]$ be any function such that $\ell(\emptyset) = 0$.

(i) The outer measure associated to ℓ is an outer measure μ_ℓ^* such that $\mu_\ell^*(A) \leq \ell(A)$ for all $A \subseteq X$.

(ii) If $\nu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure such that $\nu^*(A) \leq \ell(A)$ for all $A \subseteq x$, then $\nu^*(A) \leq \mu_\ell^*(A)$ for all $A \subseteq X$. Hence, μ_ℓ^* is the largest outer measure bounded above by ℓ.

Proof.

First notice that since $X \in \mathcal{F}$, that the set whose infimum defines $\mu_\ell^*(A)$ is nonempty for all $A \subseteq X$. This fact will be used throughout the proof.

Clearly, $\mu_\ell^* : \mathcal{P}(X) \to [0, \infty]$. Furthermore, if $\emptyset \in \mathcal{F}$ and $\ell(\emptyset) = 0$, we clearly see that $\mu_\ell^*(\emptyset) = 0$ as $\{\emptyset\}_{n=1}^\infty$ is a cover of $\emptyset$. Moreover, if $A \subseteq X \subseteq X$, it is easy to see that $\mu_\ell^*(A) \leq \mu_\ell^*(B)$ since the infimum in the definition of $\mu_\ell^*(A)$ is taken over a larger collection of sets than the infimum in the definition of $\mu_\ell^*(B)$.

Finally, to check the subadditivity of μ_ℓ^*, let $(A_n)_{n=1}^\infty$ be a collection of members of $\mathcal{P}(X)$ and let $A = \bigcup_{n=1}^\infty A_n$. Let $\varepsilon > 0$ be arbitrary. By definition, for every $n \in \mathbb{N}$, there exists a collection $(A_{n,k})_{k=1}^\infty$ of $\mathcal{F}$ such that $A_n \subseteq \bigcup_{k=1}^\infty A_{n,k}$ and

$$\sum_{k=1}^\infty \ell(A_{n,k}) \leq \mu_\ell^*(A_n) + \frac{\varepsilon}{2^n}$$

It is easy to note that $(A_{n,k})_{k=1}^\infty$ is countable subset of $\mathcal{F}$ such that

$$A \subseteq \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty A_{n,k}$$

Thus,

$$\mu_\ell^*(A) \leq \sum_{n=1}^\infty \sum_{k=1}^\infty \ell(A_{n,k}) \leq \sum_{n=1}^\infty \mu_\ell^*(A_n) + \frac{\varepsilon}{2^n} = \varepsilon + \sum_{n=1}^\infty \mu_\ell^*(A_n)$$

Therefore, since $\varepsilon > 0$ was arbitrary, we obtain that

$$\mu_\ell^*(A) \leq \sum_{n=1}^\infty \mu_\ell^*(A_n)$$

Hence, μ_ℓ^* is an outer measure.

Now to prove the second assertion, if $\nu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure such that $\nu^*(A) \leq \ell(A)$ for all $A \subseteq X$, then for each $A \subseteq X$ and each collection $(A_n)_{n=1}^\infty$ of $\mathcal{F}$ such that $A \subseteq \bigcup_{n=1}^\infty A_n$, we have

$$\nu^*(A) \leq \nu^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \nu^*(A_n) \leq \sum_{n=1}^\infty \ell(A_n)$$

by properties of an outer measure and the assumptions on ν^*. Therefore, since $\mu_\ell^*(A)$ is the infimum of $\sum_{n=1}^\infty \ell(A_n)$ over all collections $(A_n)_{n=1}^\infty$ of $\mathcal{F}$ such that $A \subseteq \bigcup_{n=1}^\infty A_n$, we obtain $\nu^*(A) \leq \mu_\ell^*(A)$ for all $A \subseteq X$. ∎

The outer measure one uses on $\mathbb{R}$ to define length is the following.

Definition 1.2.5

Let $I \subseteq \mathbb{R}$ be an interval, let $\ell(I)$ denote the length of I. The *Lebesgue outer measure*, denoted by λ^* is the outer measure associated to ℓ restricted to the open intervals. In particular, $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is defined by

$$\lambda^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : (I_n)_{n=1}^{\infty} \text{ are open intervals such that } A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

for all $A \subseteq \mathbb{R}$.

Clearly, we can extend the Lebesgue outer measure on $\mathbb{R}$ to measures on $\mathbb{R}^n$ to measure areas and volumes.

Definition 1.2.6

For $n \in \mathbb{N}$, let

$$\mathcal{F} = \left\{ \prod_{i=1}^{n} (a_i, b_i) \subseteq \mathcal{R}^n : a_i < b_i \in \mathbb{R} \cup \{\pm\infty\} \right\}$$

and define $\ell : \mathcal{F} \to [0, \infty]$ by

$$\ell\left(\prod_{i=1}^{n} (a_i, b_i) \right) = \prod_{i=1}^{n} (b_i - a_i)$$

where the product is zero if $a_i = b_i$ for some i and otherwise, if $b_i = \infty$ or $a = -\infty$ for some i, then the product is infinite. The n-dimensional Lebesgue outer measure, denoted by λ_n^* is the outer measure on $\mathbb{R}^n$ associated to ℓ.

With the above notion of outer measures, we desire to construct measures from outer measures. To do so, we need to define a σ-algebra of sets for which the restriction of our outer measure produces a measure. These sets are described as follows.

Definition 1.2.7

Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X. A subset $A \subseteq X$ is said to be μ^**-measurable*, or *outer measurable* if for every $B \in \mathcal{P}(X)$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

Remark 1.2.8

The reason we are interested in outer measurable sets is that if $A \subseteq X$ has the property that

$$\mu^*(B) \neq \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

for some $B \in \mathcal{P}(X)$, it is likely we do not want to consider A to be measurable as it causes μ^* to fail to be additive on specific disjoint sets if B was also measurable.

Remark 1.2.9

Notice by the properties of an outer measure that if $A, B \in \mathcal{P}(X)$, then

$$\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

Thus, to show that A is measurable, it suffices to show that

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

for all $B \in \mathcal{P}(X)$, Furthermore, clearly, it suffices to restrict our attention to B such that $\mu^*(B) < \infty$.

The Carathéodory Method of constructing a measure is as follows: construct an outer measure μ^* and apply the following to get a σ-algebra $\mathcal{A}$ such that $\mu^*|_{\mathcal{A}}$ is a measure.

Theorem 1.2.10

Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X. The set $\mathcal{A}$ of all outer measurable sets is a σ-algebra. Furthermore, $\mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.

Proof.

To see that $\mathcal{A}$ is a σ-algebra, first notice that for all $B \in \mathcal{P}(X)$, that

$$\mu^*(B) = \mu^*(B) + 0 = \mu^*(B \cap \emptyset^c) + \mu^*(B \cap \emptyset)$$

Hence, $\emptyset \in \mathcal{A}$. Furthermore clearly if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ due to the symmetry in the definition of an outer measurable set. Hence, $\mathcal{A}$ is closed under compliments and $X \in \mathcal{A}$.

In order to demonstrate that $\mathcal{A}$ is closed under countable unions, let us verify that $\mathcal{A}$ is closed under finite unions. To verify that $\mathcal{A}$ is closed under finite unions, it suffices to verify that if $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$ as we can then apply recursion to take arbitrary finite unions of element of $\mathcal{A}$. Thus, let $A_1, A_2 \in \mathcal{A}$ be arbitrary. To see that $A_1 \cup A_2 \in \mathcal{A}$, let $B \subseteq X$ be arbitrary. Since A_1 is outer measurable, we know that

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c)$$

and since A_2 is outer measurable, we know that

$$\mu^*(B \cap A_1^c) = \mu^*((B \cap A_1^c) \cap A_2) + \mu^*((B \cap A_1^c) \cap A_2^c)$$

Hence,

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c)$$

However, since

$$B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap (A_2 \cap A_1^c))$$

it follows from subadditivity that

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\
&\geq \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap A_1^c \cap A_2^c) \\
&= \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap (A_1 \cup A_2)^c)
\end{aligned}$$

Therefore, since $B \subseteq X$ was arbitrary, we obtain that $A_1 \cup A_2 \in \mathcal{A}$, so $\mathcal{A}$ is closed under finite unions.

Since $\mathcal{A}$ is also closed under complements, we also obtain that $\mathcal{A}$ is closed under finite intersections using a similar argument to Remark 1.1.3.

To see that $\mathcal{A}$ is closed under countable unions, let $(A_n)_{n=1}^\infty$ be a collection in $\mathcal{A}$. Let $E_1 = A_1$ and for $n \geq 1$, let

$$E_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i \right) = A_n \cap \left(\bigcup_{i=1}^{n-1} A_i \right)^c$$

Clearly, $(E_n)_{n=1}^\infty$ are pairwise disjoint such that $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty A_n$. Furthermore, $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ by the above argument.

To see that $E = \bigcup_{n=1}^\infty E_n$ is a member of $\mathcal{A}$, let $B \subseteq X$ be arbitrary. For $n \in \mathbb{N}$, let $F_n = \bigcup_{i=1}^n E_n$ which is an element of $\mathcal{A}$ since $\mathcal{A}$ is closed under finite unions. Therefore, since F_n is outer measurable, since $F_n \subseteq E$ so $E^c \subseteq F_n^c$, and since μ^* is monotone, we obtain that

$$\mu^*(B) = \mu^*(B \cap F_n) + \mu^*(B \cap F_n^c) \geq \mu^*(B \cap F_n) + \mu^*(B \cap E^c)$$

for all $n \in \mathbb{N}$.

Since $(F_n)_{n=1}^\infty$ are an increasing sequence of sets with union E, we would like to take the limit of the right side of the above inequality to obtain that $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$ thereby obtaining that E is outer measurable. However, since we do not know the Monotone Convergence Theorem works for outer measures, we will need another approach to take the limit.

Notice that $F_n = F_{n-1} \cup E_n$ and $F_{n-1} \cap E_n = \emptyset$ by construction, so since $E_n \in \mathcal{A}$

$$\begin{aligned}
\mu^*(B \cap F_n) &= \mu^*((B \cap F) \cap E_n) + \mu^*((B \cap F_n) \cap E_n^c) \\
&= \mu^*(B \cap E_n) + \mu^*(B \cap F_{n-1})
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, recursion implies that

$$\mu^*(B \cap F_n) = \sum_{i=1}^n \mu^*(B \cap F_i)$$

for all $n \in \mathbb{N}$. Thus,

$$\mu^*(B) \geq \mu^*(B \cap E^c) + \sum_{i=1}^n \mu^*(B \cap E_i)$$

for all $n \in \mathbb{N}$. By taking the supremum on the right side of the above expression yields

$$\mu^*(B) \geq \mu^*(B \cap E^c) + \sum_{i=1}^{\infty} \mu^*(B \cap E_i)$$

Therefore, subadditivity implies that

$$\begin{aligned}
\mu^*(B) &\geq \mu^*(B \cap E^c) + \mu^* \left(\bigcup_{n=1}^{\infty} (B \cap E_n) \right) \\
&= \mu^*(B \cap E^c) + \mu^* \left(B \cap \left(\bigcup_{n=1}^{\infty} E_n \right) \right) \\
&= \mu^*(B \cap E^c) + \mu^*(B \cap E)
\end{aligned}$$

Therefore, as $B \subseteq X$ was arbitrary, we obtain that $E \in \mathcal{A}$. Hence, $\mathcal{A}$ is a σ-algebra.

Finally, to check that $\mu^*|_{\mathcal{A}}$ is a measure, first notice that $\mu^*(\emptyset) = 0$ by construction. To check the other property, let $(E_n)_{n=1}^{\infty}$ be a collection of disjoint elements of $\mathcal{A}$ and let $E = \bigcup_{n=1}^{\infty} E_n$. Using the above computation with E in place of B, we have

$$\mu^*(E) \geq \mu^*(E \cap E^c) + \sum_{n=1}^{\infty} \mu^*(E \cap E_n) = 0 + \sum_{n=1}^{\infty} \mu^*(E_n) = \sum_{n=1}^{\infty} \mu^*(E_n)$$

However, since subadditivity of outer measures implies

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$$

we obtain that

$$\mu^*(E) = \sum_{n=1}^{\infty} \mu^*(E_n)$$

Hence, $\mu^*|_{\mathcal{A}}$ is a measure as desired. ■

Let λ^* be the Lebesgue outer measure from Definition 1.2.5. By Theorem 1.2.10, the collection $\mathcal{M}(\mathbb{R})$ of λ^*-measurable sets is a σ-algebra and $\lambda^*|_{\mathcal{M}(\mathbb{R})}$ is a measure. Since these objects will be the focus for the remainder of these notes, we make the following definition.

Definition 1.2.11

(i) The *Lebesgue measure* on $\mathbb{R}$ is the measure $\lambda = \lambda^*|_{\mathcal{M}(\mathbb{R})}$. The elements of $\mathcal{M}(\mathbb{R})$ are called the *Lebesgue measurable sets.*

(ii) The *n-dimensional Lebesgue measure* on $\mathbb{R}^n$ is the measure λ_n obtained from restricting λ_n^* to the λ_n^*-measurable subsets of $\mathbb{R}^n$.

One by-product of the Carathéodory Method is that the measures constructed have a specific additional property that we now describe.

Definition 1.2.12

A measure space $(X, \mathcal{A}, \mu)$ is said to be *complete* if whenever $A \in \mathcal{A}$ and $B \in \mathcal{P}(X)$ are such that $B \subseteq A$ and $\mu(A) = 0$, then $B \in \mathcal{A}$.

Proposition 1.2.13

Let X be a nonempty set, let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X, and let $\mathcal{A}$ be the σ-algebra of all outer measurable sets. If $A \in \mathcal{P}(X)$ and $\mu^*(A) = 0$, then $A \in \mathcal{A}$. Hence, $(X, \mathcal{A}, \mu^*|_{\mathcal{A}})$ is complete by the monotonicity of μ^*.

Proof.

Assume that $A \in \mathcal{P}(X)$ is such that $\mu^*(A) = 0$. To see that $A \in \mathcal{A}$, let $B \in \mathcal{P}(X)$ be arbitrary. Then

$$0 \leq \mu^*(B \cap A) \leq \mu^*(A) = 0$$

by monotonicity. Furthermore,

$$\mu^*(B) \geq \mu^*(B \cap A^c) = \mu^*(B \cap A^c) + \mu^*(B \cap A)$$

Therefore, as $B \in \mathcal{P}(X)$ was arbitrary, $A \in \mathcal{A}$.

Finally, to see that $(X, \mathcal{A}, \mu^*|_{\mathcal{A}})$ is complete, let $A \in \mathcal{A}$ and $B \in \mathcal{P}(X)$ such that $B \subseteq A$ and $\mu^*(A) = 0$. Monotonicity implies that $\mu^*(B) = 0$. Thus, the first part of this proof implies that $B \in \mathcal{A}$, as desired. ■

Remark 1.2.14

By Proposition 1.2.13, λ is a complete measure.

One may think the Carathéodory Method may not be that useful as it can only construct measures that are complete and thereby might be limited. However, this is not the case as it is always possible to 'complete' a measure rather simply.

Exercise 1.2.15

Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that there exists a complete measure space $(X, \overline{\mathcal{A}}, \overline{\mu})$ such that $\mathcal{A} \subseteq \overline{\mathcal{A}}$ and $\overline{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Exercise 1.2.16: Assignment 1, Question 2

Let $X = \mathbb{N}$, let

$$\mathcal{F} = \{\emptyset, \mathbb{N}\} \cup \{\{2k-1, 2k\} : k \in \mathbb{N}\}$$

and define $\ell : \mathcal{F} \to [0, \infty]$ by $\ell(\emptyset) = 0$, $\ell(\{2k-1, 2k\}) = 1$ for all $k \in \mathbb{N}$, and $\ell(\mathbb{N}) = \infty$. If μ_ℓ^* denotes the outer measure associated to ℓ, describe the σ-algebra $\mathcal{A}$ of all μ_ℓ^*-measurable sets. Justify your answer.

Solution.

By Theorem 1.2.10, we have that $\mathcal{A}$ of all μ_ℓ^*-measurable sets is a σ-algebra. We claim that

$$\mathcal{A} = \left\{ \bigcup_{n=1}^{\infty} F_n : F_n \in \mathcal{F} \right\}$$

To see this, we will first show that for any $k \in \mathbb{N}$, $\mu_\ell^*(\{k\}) = 1$. Indeed, let $(F_n)_{n=1}^\infty$ be the collection such that $F_1 = \{k, k+1\}$, and for $n \geq 2$, $F_n = \emptyset$. Then clearly, $\{k\} \subseteq \bigcup_{n=1}^\infty F_n = \{k, k+1\}$. Then

$$\mu_\ell^*(\{k\}) = \sum_{n=1}^{\infty} \ell(F_n) = \ell(F_1) + \sum_{n=2}^{\infty} \ell(F_n) = \ell(\{k, k+1\}) + \sum_{n=2}^{\infty} \ell(\emptyset) = 1$$

as claimed. Now, denote $\mathcal{F}' = \{\bigcup_{n=1}^\infty F_n : F_n \in \mathcal{F}\}$, and we show that $\mathcal{A} = \mathcal{F}'$ by double inclusion.

To see that $\mathcal{F}' \subseteq \mathcal{A}$, let $F \in \mathcal{F}'$, i.e. $F = \bigcup_{n=1}^\infty F_n$, where for each $n \in \mathbb{N}$, $F_n \in \mathcal{F}$. Then since $\mathcal{F} \subseteq \mathcal{A}$, it follows that $F_n \in \mathcal{A}$, and since $\mathcal{A}$ is a σ-algebra, $F = \bigcup_{n=1}^\infty F_n \in \mathcal{A}$ which shows that $F \in \mathcal{A}$.

To see that $\mathcal{A} \subseteq \mathcal{F}'$, let $A \in \mathcal{A}$. We want to show that $A = \bigcup_{n=1}^\infty F_n$ where for $n \in \mathbb{N}$, $F_n \in \mathcal{F}$. Indeed, assume for a contradiction that $A \neq \bigcup_{n=1}^\infty F_n$. Then there exists an $N \in A$ such that either

(i) $N = 2k - 1$ for some $k \in \mathbb{N}$ and $N + 1 = 2k \notin A$ or

(ii) $N = 2k$ for some $k \in \mathbb{N}$ and $N - 1 = 2k - 1 \notin A$

Assume that (i) happens. Let $B = \{2k-1, 2k\}$. Note that since $2k - 1 \in A$ and $2k \notin A$, then $B \cap A = \{2k-1\}$ and $B \cap A^c = \{2k\}$. Then, by definition of μ_ℓ^*-measurable sets and by the above claim

$$\begin{aligned} \mu_\ell^*(B) &= \mu_\ell^*(B \cap A) + \mu_\ell^*(B \cap A^c) \\ \mu_\ell^*(\{2k-1, 2k\}) &= \mu_\ell^*(\{2k-1\}) + \mu_\ell^*(\{2k\}) \\ 1 &= 1 + 1 \\ 1 &= 2 \end{aligned}$$

which is absurd. On the other hand, assume that (ii) happens. Let $B = \{2k-1, 2k\}$. Note that since $2k \in A$ and $2k - 1 \notin A$, then $B \cap A = \{2k\}$ and $B \cap A^c = \{2k-1\}$, and similarly,

$$\begin{aligned} \mu_\ell^*(B) &= \mu_\ell^*(B \cap A) + \mu_\ell^*(B \cap A^c) \\ \mu_\ell^*(\{2k-1, 2k\}) &= \mu_\ell^*(\{2k\}) + \mu_\ell^*(\{2k-1\}) \\ 1 &= 1 + 1 \\ 1 &= 2 \end{aligned}$$

which is absurd.

Therefore, it must be the case that $A = \bigcup_{n=1}^{\infty} F_n$, so $A \in \mathcal{F}'$. Hence, we conclude that $\mathcal{A} = \mathcal{F}'$, where $\mathcal{F}' = \{\bigcup_{n=1}^{\infty} F_n : F_n \in \mathcal{F}\}$.

Exercise 1.2.17: Assignment 1, Question 5

Let $(X, \mathcal{A}, \mu)$ be a measure space and let

$$\overline{\mathcal{A}} = \{E \subseteq X : \text{there exists } A, B \in \mathcal{A} \text{ such that } A \subseteq E \subseteq B \text{ and } \mu(B \setminus A) = 0\}$$

Define $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty]$ by $\overline{\mu}(E) = \mu(A)$, where $E \in \overline{\mathcal{A}}$, and $A, B \in \mathcal{A}$ are such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$.

(a) Show that $\mathcal{A} \subseteq \overline{\mathcal{A}}$, $\overline{\mu}$ is well-defined, and $\overline{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

(b) Show that $\overline{\mathcal{A}}$ is a σ-algebra, $\overline{\mu}$ is a measure on $(X, \overline{\mathcal{A}})$, and that $\overline{\mu}$ is complete.

Solution.

(a) To see that $\mathcal{A} \subseteq \mathcal{A}$, let $A \in \mathcal{A}$ be arbitrary, and note that $A \subseteq A \subseteq A$ and $\mu(A \setminus A) = 0$, so $A \in \overline{\mathcal{A}}$.

To see that $\overline{\mu}$ is well-defined, let $E \in \overline{\mathcal{A}}$ be arbitrary, and choose $A_1, A_2, B_1, B_2 \in \mathcal{A}$ such that $A_1 \subseteq E \subseteq B_1$, $A_2 \subseteq E \subseteq B_2$, and that $\mu(B_1 \setminus A_1) = 0$ and $\mu(B_2 \setminus A_2) = 0$. We claim that $\mu(A_1) = \mu(A_2)$. Observe that

$$\begin{aligned}
\mu(B_1) &= \mu(B_1 \cap A_1) + \mu(B_1 \setminus A_1) \\
&= \mu(A_1) + 0 \qquad (\text{since } A_1 \subseteq B_1, \text{ so } A_1 \cap B_1 = A_1, \text{ and } \mu(B_1 \setminus A_1) = 0 \text{ by assumption}) \\
&= \mu(A_1)
\end{aligned}$$

and similarly, we can conclude that $\mu(B_2) = \mu(A_2)$. Furthermore, observe that since $A_1 \subseteq E \subseteq B_1$ and $A_2 \subseteq E \subseteq B_2$, it follows that $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$. Now we consider the following cases:

<u>Case 1 (If $\mu(B_1) < \infty$ and $\mu(B_2) < \infty$):</u> Then observe that

$$\begin{aligned}
\mu(A_1) = \mu(B_1) = \mu(B_1 \cap A_2) + \mu(B_1 \setminus A_2) = \mu(A_2) + \mu(B_1 \setminus A_2) \qquad &(\text{since } A_2 \subseteq B_1) \\
\mu(A_2) = \mu(B_2) = \mu(B_2 \cap A_1) + \mu(B_2 \setminus A_1) = \mu(A_1) + \mu(B_2 \setminus A_1) \qquad &(\text{since } A_1 \subseteq B_2)
\end{aligned}$$

Then adding the two equations, we yield

$$\begin{aligned}
\mu(A_1) + \mu(A_2) &= \mu(A_2) + \mu(A_1) + \mu(B_1 \setminus A_2) + \mu(B_2 \setminus A_1) \\
0 &= \mu(B_1 \setminus A_2) + \mu(B_2 \setminus A_1)
\end{aligned}$$

and since $\mu \geq 0$, it follows that $\mu(B_1 \setminus A_2) = 0$ and $\mu(B_2 \setminus A_1) = 0$. Thus, we obtain

$$\mu(A_1) = \mu(A_2) + \mu(B_1 \setminus A_2) = \mu(A_2) + 0 = \mu(A_2)$$

Case 2 (If $\mu(B_1) = \infty$ and $\mu(B_2) = \infty$): Then $\mu(A_1) = \infty$ and $\mu(A_2) = \infty$, so $\mu(A_1) = \mu(A_2)$.

Case 3 (If $\mu(B_1) < \infty$ and $\mu(B_2) = \infty$): We claim that this case cannot happen. Indeed, if $\mu(B_2) = \infty$, then $\mu(A_2) = \infty$, but if $\mu(B_1)$ is finite, then so is $\mu(A_1)$. But then $\mu(A_1) = \mu(A_2)$ implies "finite" = "infinity", which cannot happen.

Thus, we conclude that $\mu(A_1) = \mu(A_2)$, and thus, $\overline{\mu}$ is well-defined.

Finally, to see that $\overline{\mu}|_{\mathcal{A}} = \mu$, let $A \in \mathcal{A}$ be arbitrary. Then since $\mathcal{A} \subseteq \overline{\mathcal{A}}$, we have that $A \subseteq A \subseteq A$, and $\mu(A \setminus A) = 0$, and by the definition of $\overline{\mu}$, since $A \in \overline{\mathcal{A}}$ and $\overline{\mu}$ is well-defined, we have $\overline{\mu}(A) = \mu(A)$.

(b) We first show that $\overline{\mathcal{A}}$ is a σ-algebra as follows:

- Note that $\emptyset \in \overline{\mathcal{A}}$ since we can take $A = B = \emptyset \in \mathcal{A}$ and see that $\emptyset \subseteq \emptyset \subseteq \emptyset$ and also, $\mu(\emptyset \setminus \emptyset) = 0$. Also, note that $X \in \overline{\mathcal{A}}$ since we can take $A = B = X$ in $\mathcal{A}$ and so $X \subseteq X \subseteq X$ and $\mu(X \setminus X) = 0$. Therefore, $\emptyset, X \in \overline{\mathcal{A}}$.

- Let $E \in \overline{\mathcal{A}}$ be arbitrary. Then there exists subsets $A, B \in \mathcal{A}$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. We seek subsets $C, D \in \mathcal{A}$ such that $C \subseteq E^c \subseteq D$ and $\mu(D \setminus C) = 0$, thereby showing that $E^c \in \overline{\mathcal{A}}$. Indeed, take $C = B^c$ and $D = A^c$, in which because by assumption $A \subseteq E \subseteq B$, we have $B^c \subseteq E^c \subseteq A^c$, and furthermore, note that

$$A^c \setminus B^c = A^c \cap (B^c)^c = A^c \cap B = B \setminus A$$

  and so $\mu(A^c \setminus B^c) = \mu(B \setminus A) = 0$, by assumption. Therefore, $E^c \in \overline{\mathcal{A}}$.

- Let $(E_n)_{n=1}^{\infty}$ be a countable collection of members of $\overline{\mathcal{A}}$. Then for each $n \in \mathbb{N}$, there exists $A_n, B_n \in \mathcal{A}$ such that $A_n \subseteq E_n \subseteq B_n$ and $\mu(B_n \setminus A_n) = 0$. Then note $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} B_n$, so it remains to show that $\mu\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)\right) = 0$.

  Claim: $\left(\bigcup_{n=1}^{\infty} B_n\right) \setminus \left(\bigcup_{n=1}^{\infty} A_n\right) \subseteq \bigcup_{n=1}^{\infty}(B_n \setminus A_n)$.

  To see the claim, let $x \in \left(\bigcup_{n=1}^{\infty} B_n\right) \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)$. Then there exists an $N \in \mathbb{N}$ such that $x \in B_N \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)$. But since $x \notin \bigcup_{n=1}^{\infty} A_n$, $x \notin A_N$, so $x \in B_N \setminus A_N$, and thus, $x \in \bigcup_{n=1}^{\infty}(B_n \setminus A_n)$.

  Now with the claim, we have

$$\begin{aligned}
\mu\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)\right) &\leq \mu\left(\bigcup_{n=1}^{\infty}(B_n \setminus A_n)\right) && \text{(Monotonicity)} \\
&\leq \sum_{n=1}^{\infty} \mu(B_n \setminus A_n) && \text{(Subadditivity)} \\
&= \sum_{n=1}^{\infty} 0 && \text{(Assumption)} \\
&= 0
\end{aligned}$$

  and so $\bigcup_{n=1}^{\infty} E_n \in \overline{\mathcal{A}}$.

Therefore, we have shown that $\overline{\mathcal{A}}$ is a σ-algebra.

Next, we show that $\overline{\mu}$ is a measure on $(X, \overline{\mathcal{A}})$.

- Since for all $A \in \mathcal{A}$, we have $\overline{\mu}(A) = \mu(A)$, then since $\emptyset \in \mathcal{A}$ and since μ is a measure, $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$.
- Let $(E_n)_{n=1}^{\infty}$ a collection of countable disjoint sets in $\overline{\mathcal{A}}$. Then for each $n \in \mathbb{N}$, there exists $A_n, B_n \in \mathcal{A}$ such that $A_n \subseteq E_n \subseteq B_n$ and $\overline{\mu}(E_n) = \mu(A_n)$. Then since $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} B_n$, where $\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$. Then it follows that the collection $(A_n)_{n=1}^{\infty}$ is also disjoint, and since μ is a measure
$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \overline{\mu}(E_n)$$
Therefore, $\overline{\mu}$ is countably additive.

Therefore, we have shown that $\overline{\mu}$ is a measure on $(X, \overline{\mathcal{A}})$.

Finally, to show that $\overline{\mu}$ is complete, fix $E \in \overline{\mathcal{A}}$ and let $F \in \mathcal{P}(X)$ such that $F \subseteq E$ and $\overline{\mu}(E) = 0$. Since $E \in \overline{\mathcal{A}}$, there exists $A, B \in \mathcal{A}$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. But by definition of $\overline{\mu}$, we have $\overline{\mu}(E) = \mu(A) = 0$. Furthermore, since $\mu(A) = \mu(B)$ whenever $\mu(B \setminus A) = 0$, then $\mu(B) = 0$ as well. Now we want to find $C, D \in \mathcal{A}$ such that $C \subseteq F \subseteq D$ and $\mu(D \setminus C) = 0$. In particular, take $C = \emptyset$ and $D = B$, so that $\emptyset \subseteq F \subseteq B$, which is true, because $F \subseteq E \subseteq B$, and also $\mu(B \setminus \emptyset) = \mu(B) = 0$, in which we can conclude that $F \in \overline{\mathcal{A}}$. ■

1.3 Extending Measures

Although the Carathéodory Method has enabled us to construct the Lebesgue measure and other measures, the process produces startlingly little information about the properties the measure inherits from the length function ℓ used to define the outer measure μ_ℓ^*. In particular, does the Lebesgue measure have the properties described in the beginning of Section 1.2 and is every Borel set Lebesgue measurable? Verifying the desired properties for the Lebesgue measure is not difficult to do directly (and will be demonstrated in Section 1.4). However, we will take a more indirect approach to produce further results and obtain the properties of the Lebesgue measure for free.

In this section, we will analyze how properties of specific length functions are immediately inherited by the measures produced by the Carathéodory Method. In particular, we will see how a 'finitely additive measure' can be extended uniquely to an actual measure. Consequently, the results of this section will immediately give almost all properties of the Lebesgue measure we desire in the next section.

To begin, we desire to describe functions that are similar to measures on a more general notion than a σ-algebra.

Definition 1.3.1

Let X be a nonempty set. An *algebra* on X is a subset $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

(i) $\emptyset, X \in \mathcal{A}$.

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

(iii) If $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$.

Remark 1.3.2

Notice that if $\mathcal{A}$ is an algebra, then $\mathcal{A}$ is closed under *finite* unions by iterating the third property of Definition 1.3.1. Furthermore, if $A_1, A_2 \in \mathcal{A}$, then

$$A_1 \cap A_2 = (A_1^c \cup A_2^c)^c \in \mathcal{A}$$

so $\mathcal{A}$ is closed under finite intersections.

Example 1.3.3

Clearly, every σ-algebra is an algebra. However, there are some algebras that are not necessarily σ-algebras. Consider $X = \mathbb{R}$ and the family

$$\mathcal{F} = \{(a, b] : a < b \in \mathbb{R} \cup \{-\infty\}\} \cup \{(a, \infty) : a \in \mathbb{R} \cup \{-\infty\}\}$$

and let $\mathcal{A}$ denote the collection of all sets obtained by taking all finite unions of elements of $\mathcal{F}$, including the empty set. It is easy to see that $\mathcal{A}$ is an algebra as the complement of each element of $\mathcal{F}$ is a finite union of elements of $\mathcal{F}$. However, $\mathcal{A}$ is not a σ-algebra since $\bigcup_{n=1}^{\infty}(2n, 2n+1] \notin \mathcal{A}$, yet $(2n, 2n+1] \in \mathcal{A}$ for all $n \in \mathbb{N}$.

Using algebras in place of σ-algebras, we obtain the beginnings of a measure.

Definition 1.3.4

Let X be a nonempty set and let $\mathcal{A}$ be an algebra on X. A *premeasure* on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$.

(ii) If $(A_n)_{n=1}^{\infty}$ are pairwise disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Remark 1.3.5

Notice the difference between a premeasure on an algebra and a measure on a σ-algebra stems from the fact that if $\mathcal{A}$ is an algebra and $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{A}$, then it not need be the case that $\bigcup_{n=1}^{\infty} \in \mathcal{A}$. Hence, if $\mathcal{A}$ is a σ-algebra, there is no difference between Definitions 1.1.9 and 1.3.4. It is not difficult to see that a premeasure shares several properties with measures by repeating some of the proofs in Section 1.1. Indeed, every premeasure is finitely addtivie and is monotone. To see the monotonicity of premeasures, assume that $A, B \in \mathcal{A}$ are such that $A \subseteq B$. Then $B \cap A^c \in \mathcal{A}$ as $\mathcal{A}$ is an algebra. Therefore, as A and $B \cap A^c$ are disjoint subsets, we notice for any premeasure μ on $\mathcal{A}$, that

$$\mu(B) = \mu(A \cup (B \cap A^c)) = \mu(A) + \mu(B \cap A^c) \geq \mu(A)$$

as claimed.

Instead of repeating the theory to show premeasures have properties similar to measures, we will demonstrate that every premeasure can be extended to a measure.

Lemma 1.3.6

Let X be a nonempty set, let $\mathcal{A}$ be an algebra on X, and let $\mu : \mathcal{A}\ to[0, \infty]$ be a premeasure on $\mathcal{A}$. Let μ^* be the outer measure associated to μ, that is, $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is defined by

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : (A_n)_{n=1}^{\infty} \subseteq \mathcal{A}, A \subseteq \bigcup_{n=1}^{\infty} A_n \right\}$$

for all $A \subseteq X$. Then μ^* is an outer measure on X such that $\mu^*|_{\mathcal{A}} = \mu$, i.e. $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof.

It is easy to see that μ^* is an outer measure by Proposition 1.2.4. To see that $\mu^*|_{\mathcal{A}} = \mu$, let $A \in \mathcal{A}$ be arbitrary. It is easy to note that $\mu^*(A) \leq \mu(A)$ by definition.

For the other inequality, assume that $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{A}$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$. Take $B_1 = A_1 \cap A$ and for $n \geq 2$,

$$B_n = (A \cap A_n) \setminus \left(\bigcup_{i=1}^{n-1} A_i \right)$$

Since $\mathcal{A}$ is an algebra, we see that $B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Furthermore, $(B_n)_{n=1}^{\infty}$ are disjoint subsets such that for all $n \in \mathbb{N}$, $B_n \subseteq A_n$, and

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A \cap A_n = A$$

Therefore, since μ is a premeasure, we obtain

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

where the inequality follows from the fact that μ is monotone as demonstrated in Remark 1.3.5. Therefore, $\mu^*|_{\mathcal{A}} = \mu$. ∎

Theorem 1.3.7: Carathéodory-Hahn Extension Theorem

Let X be a nonempty set, let $\mathcal{A}$ be an algebra on X, and let $\mu : \mathcal{A} \to [0, \infty]$ be a premeasure on $\mathcal{A}$. Let μ^* be the outer measure associated to μ, and let $\mathcal{A}^*$ denote the set of all μ^*-measurable sets. Recall from Theorem 1.2.10 that $\mathcal{A}^*$ is a σ-algebra on X and $\overline{\mu} = \mu^*|_{\mathcal{A}^*}$ is a measure on $\mathcal{A}^*$. The following hold.

(i) $\mathcal{A} \subseteq \mathcal{A}^*$ and $\overline{\mu}|_{\mathcal{A}} = \mu$.

(ii) If μ is a σ-finite in the sense that there exists a countable collection $(B_n)_{n=1}^{\infty}$ of $\mathcal{A}$ such that $B = \bigcup_{n=1}^{\infty} B_n$ and $\mu(B_n) < \infty$, and if $\nu : \mathcal{A}^* \to [0, \infty]$ is a measure such that $\nu|_{\mathcal{A}} = \mu$, then $\nu = \overline{\mu}$.

Proof.

(i) Recall from Lemma 1.3.6 that $\mu^*|_{\mathcal{A}} = \mu$, so it suffices to check that $\mathcal{A} \subseteq \mathcal{A}^*$. Indeed, let $A \in \mathcal{A}$ be arbitrary. To see that A is μ^*-measurable, let $B \subseteq X$ and $\varepsilon > 0$ be arbitrary. By definition of μ^*, there exists a countable collection $(A_n)_{n=1}^{\infty}$ of $\mathcal{A}$ and $B \subseteq \bigcup_{n=1}^{\infty} A_n$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(B) + \varepsilon$$

Notice that

$$B \cap A \subseteq \bigcup_{n=1}^{\infty} A_n \cap A \quad B \cap A^c \subseteq \bigcup_{n=1}^{\infty} A_n \cap A^c$$

Since $\mathcal{A}$ is an algebra, $A_n \cap A$ and $A_n \cap A^c$ are in $\mathcal{A}$ for each $n \in \mathbb{N}$. Therefore, by monotonicity

$$\mu^*(B \cap A) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n \cap A\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n \cap A) = \sum_{n=1}^{\infty} \mu(A_n \cap A)$$
$$\mu^*(B \cap A^c) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n \cap A^c\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n \cap A^c) = \sum_{n=1}^{\infty} \mu(A_n \cap A^c)$$

as μ^* is an outer measure. Therefore,

$$\begin{aligned}
\mu^*(B \cap A) + \mu^*(B \cap A^c) &\leq \sum_{n=1}^{\infty} \mu(A_n \cap A) + \sum_{n=1}^{\infty} \mu(A_n \cap A^c) \\
&\leq \sum_{n=1}^{\infty} [\mu(A_n \cap A) + \mu(A_n \cap A^c)] \\
&\leq \sum_{n=1}^{\infty} \mu(A_n) \\
&\leq \mu^*(B) + \varepsilon
\end{aligned}$$

where $\mu(A_n \cap A) + \mu(A_n \cap A^c) = \mu(A_n)$ follows from the fact that μ is a pre measure and $A_n \cap A$

and $A_n \cap A^c$ are disjoint sets. Therefore, as $\varepsilon > 0$ was arbitrary, we obtain that

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \mu^*(B)$$

Therefore, as $B \subseteq X$ was arbitrary, A is μ^*-measurable as desired.

(ii) For the uniqueness, assume there exists a countable collection $(B_n)_{n=1}^\infty$ of $\mathcal{A}$ such that $B = \bigcup_{n=1}^\infty B_n$ and $\mu(B_n) < \infty$ and $\nu : \mathcal{A}^* \to [0, \infty]$ is a measure such that $\nu|_\mathcal{A} = \mu$. Notice if $C_n = \bigcup_{i=1}^n C_i$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^\infty C_n = B$, $C_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, $C_n \subseteq C_{n+1}$ for all $n \in \mathbb{N}$, and

$$\mu(C_n) = \overline{\mu}(C_n) = \overline{\mu}\left(\bigcup_{i=1}^n C_i\right) \leq \sum_{i=1}^n \overline{\mu}(B_n) = \sum_{i=1}^n \mu(C_i) < \infty$$

To see that $\nu_\mathcal{A} = \overline{\mu}$, let $D \in \mathcal{A}$ be arbitrary. Then for every i and collection $(A_n)_{n=1}^\infty$ of $\mathcal{A}$ such that $D \cap C_i \subseteq \bigcup_{n=1}^\infty A_n$, we have by the properties of the measure

$$\nu(D \cap C_i) \leq \nu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \nu(A_n) = \sum_{n=1}^\infty \mu(A_n)$$

Hence, $\nu(D \cap C_i) \leq \mu^*(D \cap C_i) = \overline{\mu}(D \cap C_i)$ as $D \cap C_i \in \mathcal{A}^*$. By repeating the above with D^c in place of D, we obtain $\nu(D^c \cap C_i) \leq \overline{\mu}(D^c \cap C_i)$. However,

$$\begin{aligned} \mu(C_i) = \nu(C_i) &= \nu(D \cap C_i) + \nu(D^c + C_i) \\ &\leq \overline{\mu}(D \cap C_i) + \overline{\mu}(D^c \cap C_i) \\ &= \overline{\mu}(C_i) = \mu(C_i) \end{aligned}$$

Since $\mu(C_i) < \infty$, we obtain that $\nu(D \cap C_i) = \overline{\mu}(D \cap C_i)$ for all i. Therefore, as $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty C_n = B$, we obtain by the Monotone Convergence Theorem for Measures,

$$\nu(D) = \lim_{n\to\infty} \nu(D \cap C_i) = \lim_{n\to\infty} \overline{\mu}(D \cap C_n) = \overline{\mu}(D)$$

Therefore, $\nu = \overline{\mu}$, as desired. ■

Due to the Carathéodory-Hahn Extension Theorem (1.3.7), we make the following definition.

Definition 1.3.8

Let X be a nonempty set, $\mathcal{A}$ be an algebra on X, and $\mu : \mathcal{A} \to [0, \infty]$ be a premeasure. The measure $\overline{\mu}$ constructed in Theorem 1.3.7 is called the *Carathéodory extension of* μ.

Before seeing what the Carathéodory-Hahn Extension Theorem (Theorem 1.3.7) produces in regards to the Lebesgue measure, we note the following example demonstrating why the σ-finite condition is necessary in order to prove uniqueness.

Example 1.3.9

If μ is a premeasure on an algebra $\mathcal{A}$ that is not σ-finite, the Carathéodory extension of μ need not be the unique extension of μ to the set of all μ^*-measurable sets. Indeed, consider $X = \mathbb{Q} \cap (0, 1]$ and let $\mathcal{A}$ be the collection of all finite unions of sets of the form $\mathbb{Q} \cap (a, b]$. It is not difficult to see that $\mathcal{A}$ is an algebra on X.

Let $\mu : \mathcal{A} \to [0, 1]$ be defined by

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$$

Clearly, μ is a premeasure on $\mathcal{A}$. Let μ^* be the outer measure associated to μ and let $\mathcal{A}^*$ denote the σ-algebra of all μ^*-measurable sets. Clearly, we see that

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$$

For all $A \subseteq X$.

We claim that μ has multiple extensions to $\mathcal{A}^*$. First, we claim that $\mathcal{A} = \mathcal{P}(X)$. Indeed, as

$$\{q\} = \bigcap_{n=1}^{\infty} \left(q - \frac{1}{n}, q\right] \cap \mathbb{Q}$$

for all $q \in \mathbb{Q}$, we have $\{q\} \in \mathcal{A}^*$ forall $q \in X$. Therefore, as X is countable, so every subset of X is countable, and as σ-algebras are closed under countable unions, the claim follows.

Since $\mathcal{A} = \mathcal{P}(X)$, we see that the Carathéodory extension of μ is μ^*. However, since the counting measure on X is a measure that extends μ, but does not equal to μ^*, the claim is complete.

To see the full power of the Carathéodory-Hahn Extension of μ is μ^*, we note the following generalization of the Lebesgue measure.

Example 1.3.10

Recall from Example 1.3.3 that if

$$\mathcal{F} = \{((a, b] : a, b \in [-\infty, \infty), a < b\} \cup \{(a, \infty) : a \in \mathbb{R} \cup \{-\infty\}\}$$

then the st $\mathcal{A}$ consisting of all finite unions of elements of $\mathcal{F}$ (including the empty set), is an algebra. Notice that if $A, B \in \mathcal{F}$ and

$$\operatorname{dist}(A, B) = \inf\{|a - b| : a \in A, b \in B\} = 0$$

then $A \cup B \in \mathcal{F}$. Hence, it is easy to see that if $A \in \mathcal{A}$ then there exists a unique $n \in \mathbb{N}$ and a unique collection $A_1, ..., A_n \in \mathcal{F}$ such that $A = \bigcup_{i=1}^{n} A_i$ and $\operatorname{dist}(A_i, A_j) > 0$ for all $i \neq j$.

Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function such that

$$F(c) = \lim_{x \to c^+} F(x)$$

for all $c \in \mathbb{R}$ (that is, F is right continuous). Since F is non-decreasing, $\lim_{x\to\infty} F(x)$ either exists or equals ∞ and $\lim_{x\to-\infty} F(x)$ either exists or equals $-\infty$.

Define $\lambda_F : \mathcal{F} \to [0, \infty]$ by

$$\lambda_F(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ F(b) - F(a) & \text{if } A = (a, b] \\ \lim_{x\to\infty} F(x) - F(a) & \text{if } A = (a, \infty) \\ F(b) - \lim_{x\to-\infty} F(x) & \text{if } A = (-\infty, b] \\ \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x) & \text{if } A = (-\infty, \infty) \end{cases}$$

for all $a, b \in \mathbb{R}$ with $a < b$, (where $\infty + c = \infty$ for all $c \in [-\infty, \infty)$ and $d - (-\infty) = \infty$ for all $d \in (-\infty, \infty]$).

Notice that we can extend λ_F to a function $\mathcal{A}$, which will also be denoted by λ_F as follows: if $A \in \mathcal{A}$, define

$$\lambda_F(A) = \sum_{i=1}^{n} \lambda_F(A_i)$$

where $A_1, ..., A_n \in \mathcal{F}$ are the unique elements such that $A = \bigcup_{i=1}^n A_i$ and $\operatorname{dist}(A_i, A_j) > 0$ for all $i \neq j$.

We claim that λ_F is a premeasure on $\mathcal{A}$. To see this, first notice that $\lambda_F(\emptyset) = 0$ and $\lambda_F(A) \geq 0$ for all $A \in \mathcal{A}$ as F is non-decreasing.

Before we demonstrated that λ_F is countably additive, first notice that if $A, B \in \mathcal{F}$ are such that $A \cap B = \emptyset$ and $\operatorname{dist}(A, B) = 0$, then

$$\lambda_F(A \cup B) = \lambda_F(A) + \lambda_F(B)$$

trivially. Hence, it is easy to see that if $(A_i)_{i=1}^n$ are in $\mathcal{A}$ are pairwise disjoint, then

$$\lambda_F\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \lambda_F(A_i)$$

(that is, λ_F is finitely additive). Moreover, if $A, B \in \mathcal{A}$ are such that $A \subseteq B$, we see that $B \cap A^c \in \mathcal{A}$ and

$$\lambda_F(B) = \lambda_F(A \cup (B \cap A^c)) = \lambda_F(A) + \lambda_F(B \cap A^c) \geq \lambda_F(A)$$

Hence, λ_F is monotone. This further implies that λ_F is finitely subadditive. Indeed, if

$(A_i)_{i=1}^n$ is a finite collection in $\mathcal{A}$, let $B_1 = A_1$, and

$$B_j = A_j \setminus \left(\bigcup_{i=1}^{j-1} A_i \right)$$

for all $j \in \{1, 2, ..., n\}$. Then $(B_j)_{j=1}^n$ are pairwise disjoint elements of $\mathcal{A}$ such that $B_j \subseteq A_j$ for all $1 \leq j \leq n$ and $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$. Consequently,

$$\lambda_F \left(\bigcup_{i=1}^n A_i \right) = \lambda_F \left(\bigcup_{i=1}^n B_i \right) = \sum_{i=1}^n \lambda_F(B_i) \leq \sum_{i=1}^n \lambda_F(A_i)$$

Hence, λ_F is finitely subadditive.

To see that λ_F is countably additive, suppose $(A_n)_{n=1}^\infty$ are pairwise disjoint sets in $\mathcal{A}$ such that $A = \bigcup_{n=1}^\infty A_n \in \mathcal{A}$. To see that $\lambda_F(A) = \sum_{n=1}^\infty \lambda_F(A_n)$ we note that we may assume that $A \in \mathcal{F}$ and $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ as every element of $\mathcal{A}$ is a finite union of elements of $\mathcal{F}$ and λ_F is finitely additive.

Notice that for all $m \in \mathbb{N}$

$$\lambda_F(A) \geq \lambda_F \left(\bigcup_{n=1}^m A_n \right) = \sum_{n=1}^\infty \lambda_F(A_n)$$

as λ_F is monotone. Hence, by taking the limit as $m \to \infty$, we see that

$$\sum_{n=1}^\infty \lambda_F(A_n) \leq \lambda_F(A)$$

To see the reverse inequality, first suppose that $A = (a, b]$ for some $a, b \in \mathbb{R}$. Therefore, as $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, we have for each $n \in \mathbb{N}$, we have that $A_n = (a_n, b_n]$ for some $a_n, b_n \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary, and notice that for each $n \in \mathbb{N}$, there exists a $c_n > b_n$ such that

$$F(c_n) < F(b_n) + \frac{\varepsilon}{2^n}$$

as $F(b_n) = \lim_{x \to b_n^+} F(x)$. Furthermore, there exists a $\delta > 0$ such that

$$F(a + \delta) < F(a) + \varepsilon$$

Since $(a, b] = \bigcup_{n=1}^\infty (a_n, b_n]$ we see that

$$[a + \delta, b] \subseteq \bigcup_{n=1}^\infty (a_n, c_n)$$

Hence, $\{(a_n, c_n)\}_{n=1}^\infty$ is an open cover of the compact set $[a + \delta, b]$ and thus, has a finite subcover, say $\{(a_{n_k}, c_{n_k})\}_{k=1}^m$. Thus,

$$(a+\delta, b] \subseteq \bigcup_{i=1}^m (a_{n_i}, c_{n_i}]$$

Therefore, by the monotonicity and finite subadditivity of λ_F, we see that

$$\begin{aligned}
\lambda_F(A) &= F(b) - F(a) \\
&< \varepsilon + F(b) - F(a+\delta) \\
&= \varepsilon + \lambda_F((a+\delta, b]) \\
&\leq \varepsilon + \lambda_F\left(\bigcup_{i=1}^m (a_{n_i}, c_{n_i}]\right) \\
&= \varepsilon + \sum_{i=1}^m F(c_{n_i}) - F(a_{n_i}) \\
&\leq \varepsilon + \sum_{i=1}^m \frac{\varepsilon}{2^{n_i}} + F(b_{n_i}) - F(a_{n_i}) \\
&\leq 2\varepsilon + \sum_{i=1}^m \lambda_F(A_{n_i}) \\
&\leq 2\varepsilon + \sum_{n=1}^\infty \lambda_F(A_n)
\end{aligned}$$

Therefore, as $\varepsilon > 0$ was arbitrary, then the claim follows in the case that $A = (a, b]$.

To see that $\lambda_F(A) \leq \sum_{n=1}^\infty \lambda_F(A_n)$ for arbitrary $A \in \mathcal{F}$, note that $A \cap (-m, m]$ is of the form $(a, b]$ for some $a, b \in \mathbb{N}$ for all $m \in \mathbb{N}$. Therefore, the previous case along with monotonicity implies that

$$\lambda_F(A \cap (-m, m]) = \sum_{n=1}^\infty \lambda_F(A_n \cap (-m, m]) \leq \sum_{n=1}^\infty \lambda_F(A_n)$$

Therefore, as it is easily seen by the definition of λ_F, that

$$\lim_{m\to\infty} \lambda_F(A \cap (-m, m]) = \lambda_F(A)$$

the claim follows.

Hence, λ_F is a premeasure on the algebra $\mathcal{A}$. Furthermore, as

$$\lambda_F((-n, n]) = F(n) - F(-n) < \infty$$

for all $n \in \mathbb{N}$, λ_F is σ-finite. Hence, the Carathéodory-Hahn Extension Theorem implies that λ_F has a unique extension, which we will also denote λ_F, to the set of all λ_F^*-measurable sets. This Carathéodory extension is called the *Lebesgue-Stieltjes measure associated to F*.

Notice since every element of $\mathcal{F}$ is λ_F^*-measurable and since $\sigma(\mathcal{F}) = \mathfrak{B}(\mathbb{R})$, all Borel sets are Lebesgue-Stieltjes measurable. Consequently, $\lambda_F|_{\mathfrak{B}(\mathbb{R})}$ is often called the *Borel-Stieltjes measure associated to F.*

Example 1.3.11

We claim that the Lebesgue measure λ is a specific instance of a Lebesgue-Stieltjes measure. Indeed, if $F(x) = x$ for all $x \in \mathbb{R}$, we claim that $\lambda = \lambda_F$. To see this, it suffices to show that $\lambda^* = \lambda_F^*$. Recall for all $A \subseteq \mathbb{R}$, that

$$\begin{aligned}
\lambda^*(A) &= \inf\left\{\sum_{n=1}^\infty \ell(I_n) : (I_n)_{n=1}^\infty \text{ are open intervals such that } A \subseteq \bigcup_{n=1}^\infty I_n\right\} \\
\lambda_F^*(A) &= \inf\left\{\sum_{n=1}^\infty \ell(J_n) : (J_n)_{n=1}^\infty \text{ are open on the left and closed}\right. \\
&\qquad\qquad \left.\text{on the right such that } A \subseteq \bigcup_{n=1}^\infty J_n\right\}
\end{aligned}$$

where ℓ denotes the length of an interval. Clearly, $\lambda_F(A) \leq \lambda^*(A)$ for all $A \subseteq \mathbb{R}$.

For the reverse inequality, let $\varepsilon > 0$ be arbitrary. Then there exists a collection $(J_n)_{n=1}^\infty$ of left-open-right-closed intervals such that $A \subseteq \bigcup_{n=1}^\infty$ and

$$\sum_{n=1}^\infty \ell(J_n) < \lambda_F^*(A) + \varepsilon$$

Clearly, for each $n \in \mathbb{N}$, there exists an open interval I_n such that $J_n \subseteq I_n$ and $\ell(I_n) \leq \ell(J_n) + \frac{\varepsilon}{2^n}$. Therefore, $(I_n)_{n=1}^\infty$ are open intervals such that $A \subseteq \bigcup_{n=1}^\infty I_n$. Thus,

$$\lambda^*(A) \leq \sum_{n=1}^\infty \ell(I_n) \leq \sum_{n=1}^\infty \ell(J_n) + \frac{\varepsilon}{2^n} < \lambda_F^*(A) + 2\varepsilon$$

Therefore, as $\varepsilon > 0$ and $A \subseteq \mathbb{R}$ were arbitrary, we have $\lambda_F^* = \lambda^*$ as required.

1.4 Properties of the Lebesgue Measure

Since the Lebesgue measure is a specific instance of the Lebesgue-Stieltjes measure which was constructed using the Carathéodory-Hahn Extension Theorem, we immediately obtain several properties.

Corollary 1.4.1

Every Borel subset of $\mathbb{R}$ is Lebesgue measurable. In particular, if $I \subseteq \mathbb{R}$ is an interval, then $\lambda(I) = \ell(I)$.

Proof.

As the Lebesgue measure is a specific example of the Lebesgue-Stieltjes measure by Example 1.3.11, Example 1.3.10 shows all Borel subsets of $\mathbb{R}$ are Lebesgue measurable. Moreover, by the Carathéodory-Hahn Extension Theorem, we obtain that $\lambda(I) = \ell(I)$ for all intervals I of the form $(a, b]$, (a, ∞), and $(-\infty, b]$ for $a, b \in \mathbb{R}$. Therefore, sine

$$\{c\} = \bigcap_{n=1}^{\infty} \left(c - \frac{1}{n}, c\right]$$

for all $c \in \mathbb{R}$, we obtain by the Monotone Convergence Theorem

$$\lambda(\{c\}) = \lim_{n\to\infty} \lambda\left(\left(c - \frac{1}{n}, c\right]\right) = \lim_{n\to\infty} c - c + \frac{1}{n} = 0$$

Therefore, as every interval of $\mathbb{R}$ differs from an interval of the form $(a, b]$, (a, ∞) and $(-\infty, b]$ for $a, b \in \mathbb{R}$ by at most two points, the results follows by properties of measures. ■

Using the above, we easily obtain the following important property of the Lebesgue measure.

Corollary 1.4.2

The Lebesgue measure is σ-finite.

Proof.

For each $n \in \mathbb{N}$, let $X_n = [-n, n]$. Then $(X_n)_{n=1}^{\infty}$ are Borel sets (and hence, Lebesgue measurable sets) such that $X = \bigcup_{n=1}^{\infty} X_n$ and $\lambda(X_n) = 2n < \infty$. Hence, λ is σ-finite by definition. ■

Unsurprisingly, it is easy to compute the Lebesgue measure of any countable set.

Proposition 1.4.3

Let $A \subseteq \mathbb{R}$ be countable. Then $A \in \mathcal{M}(\mathbb{R})$ and $\lambda(A) = 0$.

Proof.

Let $A \subseteq \mathbb{R}$ be countable. We will first show that $\lambda^*(A) = 0$. This implies that A is Lebesgue measurable and $\lambda(A) = 0$, as λ is complete. Indeed, let $\varepsilon > 0$ be arbitrary. Since A is countable, we may write $A = (a_n)_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, let

$$I_n = \left(a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}}\right)$$

Clearly, for all $n \in \mathbb{N}$, I_n is an open interval of length $\frac{\varepsilon}{2^n}$ with $a_n \in I_n$. Hence, we obtain that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

Therefore, by definition of the Lebesgue outer measure, we obtain that

$$0 \leq \lambda^*(A) \leq \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

Therefore, as $\varepsilon > 0$ was arbitrary, we obtain $\lambda^*(A) = 0$, as desired. ∎

Definition 1.4.4

Let $P_0 = [0,1]$. Construct P_1 from P_0 by removing the open interval of the length $\frac{1}{3}$ from the middle of P_0, i.e. $P_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Then construct P_2 from P_1 by removing the pen intervals of length $\frac{1}{3^2}$ from the middle of each closed subinterval of P_1. Subsequently, having constructed P_n, construct P_{n+1} by removing the open intervals of length $\frac{1}{3^{n+1}}$ from the middle of each of the 2^n closed subintervals of P_n. Specifically, P_n is the union of the 2^n closed intervals of the form

$$\left[\sum_{i=1}^{n} \frac{a_i}{3^i}, \frac{1}{3^n} + \sum_{i=1}^{n} \frac{a_i}{3^i}\right]$$

where $a_1, ..., a_n \in \{0,2\}$. The set

$$\mathcal{C} = \bigcap_{n=1}^{\infty} P_n$$

is known as the *Cantor set*.

Remark 1.4.5

The Cantor set has many interesting properties. In particular, the Cantor set is a compact set with no interior.

Lemma 1.4.6

Let $x \in \mathbb{R}$. The following assertions are equivalent.

(a) $x \in \mathcal{C}$.

(b) There exists a sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in \{0,2\}$ for each $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i}{3^i}$, i.e. $x \in [0,1]$ and x has a ternary expansion using only 0s and 2s.

Proof.

(a) $\Rightarrow$ (b) Let $x \in \mathcal{C}$. Then $x \in P_n$ for each $n \in \mathbb{N}$, so by the recursive construction of P_n, there exists $(a_n)_{n=1}^{\infty} \in \{0,2\}^{\mathbb{N}}$ such that

$$x \in \left[\sum_{i=1}^{n} \frac{a_i}{3^i}, \frac{1}{3^n} + \sum_{i=1}^{n} \frac{a_i}{3^i}\right] \subseteq P_n$$

for all $n \in \mathbb{N}$. To see that $x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i}{3^i}$, we note that

$$\left| x - \sum_{i=1}^{n} \frac{a_i}{3^i} \right| \leq \left| \left(\frac{1}{3^n} + \sum_{i=1}^{n} \frac{a_i}{3^i} \right) - \sum_{i=1}^{n} \frac{a_i}{3^i} \right| = \frac{1}{3^n}$$

Therefore, since $\lim_{n\to\infty} \frac{1}{3^n} = 0$, we obtain that $x = \lim_{i=1}^{n} \frac{a_i}{3^i}$ as required.

(b) $\Rightarrow$ (a) Assume that $x \in \mathbb{R}$ is such that there exists a sequence $(a_n)_{n=1}^{\infty} \in \{0,2\}^{\mathbb{N}}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i}{3^i}$. For each $n \in \mathbb{N}$, let $s_n = \sum_{i=1}^{n} \frac{a_i}{3^i}$. By the description of P_n, we obtain that $s_n \in P_n$ for all $n \in \mathbb{N}$. In particular, we have $s_m \in P_n$ for $m \geq n$. Hence,

$$\begin{aligned} \sum_{i=1}^{n} \frac{a_i}{3^i} &\leq \sum_{i=1}^{m} \frac{a_i}{3^i} \\ &= \sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{m} \frac{2}{3^i} \\ &= \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{2}{3^{n+1}} \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{1 - \frac{1}{3}} \\ &= \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{3^n} \\ &\leq \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n} \end{aligned}$$

Since each P_n is a closed set, since $x = \lim_{m\to\infty} s_m$ and since $s_m \in P_n$ whenever $m \geq n$, we obtain that $x \in P_n$ for each $n \in \mathbb{N}$ by the sequential description of closed sets. Hence,

$$x \in \bigcap_{n=1}^{\infty} P_n = \mathcal{C}$$

■

Example 1.4.7

We claim that the Cantor set $\mathcal{C}$ has Lebesgue measure zero. To see this, note that $\mathcal{C}$ is a closed set, hence, $\mathcal{C}$ is Borel, and thus, Lebesgue measurable. To see that $\lambda(\mathcal{C}) = 0$, recall that

$$\mathcal{C} = \bigcap_{n=1}^{\infty} P_n$$

where $P_n \subseteq [0,1]$ is the union of 2^n closed intervals each of length $\frac{1}{3^n}$. Therefore, we obtain that for each $n \in \mathbb{N}$

$$0 \leq \lambda(\mathcal{C}) \leq \lambda(P_n) \leq \frac{2^n}{3^n}$$

and thus, $\lim_{n\to\infty} \frac{2^n}{3^n} = 0$, so $\lambda(\mathcal{C}) = 0$ as desired.

One important property of the Lebesgue measure is its invariance under translation and multiplicative under scaling.

Proposition 1.4.8

If $A \in \mathcal{M}(\mathbb{R})$, $x \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, define the sets

$$\begin{aligned} x + A &= \{x + a : a \in A\} \\ \alpha A &= \{\alpha a : a \in A\} \end{aligned}$$

Then the following hold.

(i) $x + A \in \mathcal{M}(\mathbb{R})$ *and* $\lambda(x + A) = \lambda(A)$.

(ii) $\alpha A \in \mathcal{M}(\mathbb{R})$ *and* $\lambda(\alpha A) = |\alpha|\lambda(A)$.

Proof.

To see that (i) holds, let $A \in \mathcal{M}(\mathbb{R})$ and $x \in \mathbb{R}$. Since the translation of an open interval is an open interval of the same length, it is elementary to see that if $B \subseteq \mathbb{R}$, then

$$\lambda^*(x + B) = \lambda^*(B)$$

Thus, it suffices to show that $x + A$ is measurable. Indeed, to see that $x + A$ is Lebesgue measurable, let $B \subseteq \mathbb{R}$ be arbitrary. Then see that

$$\begin{aligned} \lambda^*(B) &= \lambda^*(-x + B) \\ &= \lambda^*((-x + B) \cap A) + \lambda^*((-x + B) \cap A^c) && \text{(since } A \in \mathcal{M}(\mathbb{R})\text{)} \\ &= \lambda^*(B \cap (x + A)) + \lambda^*(B \cap (x + A^c)) \\ &= \lambda^*(B \cap (x + A)) + \lambda^*(B \cap (x + A)^c) \end{aligned}$$

Therefore, since $B \subseteq \mathbb{R}$ was arbitrary, we have $x + A \in \mathcal{M}(\mathbb{R})$ as desired.

To see that (ii) holds, let $A \in \mathcal{M}(\mathbb{R})$ and let $\alpha \in \mathbb{R} \setminus \{0\}$. Note that if I is an open interval, then αI is an open interval with length $\ell(\alpha I) = |\alpha|\ell(I)$, so it is elementary to see that if $B \subseteq \mathbb{R}$, then

$$\lambda^*(\alpha B) = |\alpha|\lambda^*(B)$$

Thus, it suffices to show that αA is Lebesgue measurable. Indeed, let $B \subseteq \mathbb{R}$ be arbitrary. Then we have

$$\begin{aligned} \lambda^*(B) &= \lambda^*(\alpha B) \\ &= \lambda^*((\alpha B) \cap A) + \lambda^*((\alpha B) \cap A^c) \\ &= \lambda^*(B \cap \alpha^{-1} A) + \lambda^*(B \cap (\alpha^{-1} A^c)) \\ &= \lambda^*(B \cap \alpha^{-1} A) + \lambda^*(B \cap (\alpha^{-1} A)^c) \end{aligned}$$

Therefore, as $B \subseteq \mathbb{R}$ was arbitrary, we have $\alpha A \in \mathcal{M}(\mathbb{R})$, as desired. ■

Remark 1.4.9

Note Corollary 1.4.1 shows that

$$\mathfrak{B}(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$$

However, we have seen that $|\mathfrak{B}(\mathbb{R})| = |\mathbb{R}|$, whereas Cantor's Theorem implies that $|\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$. Thus, it is natural to ask, what is the cardinality of $\mathcal{M}(\mathbb{R})$? After all, if not that many subsets of $\mathcal{R}$ are Lebesgue measurable, do we really have a suitably general measure?

Recall by Remark 1.4.5 that the Cantor set $\mathcal{C}$ is Lebesgue measurable with $\lambda(\mathcal{C}) = 0$. Hence, every subset of the Cantor set must be Lebesgue measurable as the Lebesgue measure is complete. Moreover, since $|\mathcal{C}| = |\mathbb{R}|$, we obtain that $|\mathcal{P}(\mathbb{C})| = |\mathcal{P}(\mathbb{R})|$. Therefore, since $\mathcal{P}(\mathcal{C}) \subseteq \mathcal{M}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$ and since $|\mathcal{P}(\mathcal{C})| = |\mathcal{P}(\mathbb{R})|$, we obtain that $|\mathcal{M}(\mathbb{R})| = |\mathcal{P}(\mathbb{R})|$. Thus, in terms of cardinality, the set of Lebesgue measurable subsets of $\mathbb{R}$ is as large as possible.

Of course, $\mathcal{M}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$ since Example 1.2.1 implies there is a set $A \subseteq [0,1)$ that is not Lebesgue measurable. Using this set, we can show that there exists $|\mathcal{P}(\mathbb{R})|$ subsets of $\mathbb{R}$ that are not Lebesgue measurable. Indeed, $A' = 2 + A \subseteq [2,3)$ is not Lebesgue measurable being the translation of a set that is not Lebesgue measurable. If $A' \cup \mathcal{C}$ was Lebesgue measurable, then since $A' \cap \mathcal{C} = \emptyset$, we would have $(A' \cup \mathcal{C}) \cap \mathcal{C}^c = A'$ being the intersection of Lebesgue measurable sets and thus being Lebesgue measurable. Since this is a contradiction, we have that $A' \cup \mathcal{C}$ is not Lebesgue measurable. Similarly if $S \subseteq \mathcal{C}$, then $A' \cup S$ is not Lebesgue measurable. Therefore, since $A' \cap \mathcal{C} = \emptyset$ and as there are $|\mathcal{P}(\mathcal{C})| = |\mathcal{P}(\mathbb{R})|$ subsets of $\mathcal{C}$, we obtain that there are $|\mathcal{P}(\mathbb{R})|$ subsets of $\mathbb{R}$ that are not Lebesgue measurable.

By modifying the proof used in Example 1.2.1, one may prove the following exercise.

Exercise 1.4.10: Assignment 1, Question 6

If $A \subseteq \mathbb{R}$ is such that $\lambda^*(A) > 0$, show that there exists a subset $B \subseteq A$ such that B is not Lebesgue measurable.

Solution.

Assume that $A \subseteq \mathbb{R}$ such that $\lambda^*(A) > 0$. For $n \in \mathbb{N}$, let $A_n = A \cap [n-1, n)$ such that $A = \bigcup_{n=1}^{\infty} A_n$. First observe that by subadditivity, we have that

$$0 < \lambda^*(A) = \lambda^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \lambda^*(A_n) < \infty$$

So by properties of a measure, there exists a $N \in \mathbb{N}$ such that $\lambda^*(A_N) > 0$ since $\lambda^*(A) > 0$ by assumption. By properties of the Lebesgue measure, we have translation invariant, so assume without loss of generality that $N = 1$.

We want to show that there exists a $B \subseteq A_1 \subseteq A$ such that B is not Lebesgue measurable. Assume for a contradiction that there is such a Lebesgue measurable set. Recall from Example 1.2.1 we have defined an equivalence relation "$\sim$" on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. We have also mentioned by the Axiom of Choice that there exists a subset $B \subseteq A_1 \subseteq [0,1)$ that contains exactly one element from each equivalence class from A_1.

Following Example 1.2.1, let us enumerate $\mathbb{Q} \cap [0,1) = \{r_n\}_{n=1}^{\infty}$, which is valid since $\mathbb{Q}$ is countable. Now define the modulo-1 set by

$$B_n = \{x \in [0,1) : x \in r_n + B \text{ or } x + 1 \in r_n + B\}$$

From Example 1.2.1, we have shown that $\{B_n\}_{n=1}^{\infty}$ are disjoint subsets of $[0,1)$.

We now claim that $\{B_n\}_{n=1}^{\infty}$ is a cover for A_1, i.e. $A_1 \subseteq \bigcup_{n=1}^{\infty} B_n$. Indeed, let $x \in A_1$. Then since B is a subset of A_1 that contains exactly one element from each equivalence class from A_1, then there exists a $y \in B$ such that $x \sim y$, so $x - y \in \mathbb{Q}$. We consider the following cases

1. Assume that $x - y \geq 0$. Since $x \in A_1$ and $y \in B$, we have $x \in [0,1)$, so $x - y < 1$, so for some $n_0 \in \mathbb{N}$, we have $x - y = r_{n_0}$, and thus, $x = r_{n_0} + y$, i.e. $x \in B_{n_0}$.

2. Assume that $x - y < 0$. Since $x \in A_1$ and $y \in B$, we have $x - y \geq -1$, so for some $n_0 \in \mathbb{N}$, we have $(x - y) + 1 = r_{n_0}$, so $x + 1 = r_{n_0} + y$, i.e. $x \in B_{n_0}$.

Since $x \in B_{n_0}$ for some $n_0 \in \mathbb{N}$, it follows that $x \in \bigcup_{n=1}^{\infty} B_n$, and thus, we conclude $A_1 \subseteq \bigcup_{n=1}^{\infty} B_n$ as desired.

Now for $n \in \mathbb{N}$, let $B_{n,1} = (r_n + B) \cap [0,1)$ and $B_{n,2} = (r_n + B) \cap [1,2)$, which as in Example 1.2.1, are disjoint. Indeed, we simply use B in place of A in Example 1.2.1 to get that the two sets are disjoint. Furthermore, we have that $B_n = B_{n,1} \cup B_{n,2}$ for all $n \in \mathbb{N}$ since $r_n + A \in [0,2)$ for all $n \in \mathbb{N}$.

Finally, observe that

$$\begin{aligned}
0 &< \lambda(A_1) && \text{(assumption)} \\
&\leq \lambda\left(\bigcup_{n=1}^{\infty} B_n\right) && \text{(since } A_1 \subseteq \textstyle\bigcup_{n=1}^{\infty} B_n\text{, so monotonicity applies)} \\
&= \sum_{n=1}^{\infty} \lambda(B_n) && \text{(since } \{B_n\}_{n=1}^{\infty} \text{ are disjoint.)} \\
&= \sum_{n=1}^{\infty} \lambda(B_{n,1} \cup B_{n,2}) && \text{(since } B_n = B_{n,1} \cup B_{n,2} \text{ for all } n \in \mathbb{N}) \\
&= \sum_{n=1}^{\infty} \lambda(B_{n,1}) + \lambda^*(B_{n,2}) && \text{(since } B_{n,1} \cap B_{n,2} = \emptyset) \\
&= \sum_{n=1}^{\infty} \lambda((r_n + B) \cap [0,1)) + \lambda((r_n + B) \cap [1,2)) \\
&= \sum_{n=1}^{\infty} \lambda(r_n + B) && \text{(translation invariant by Proposition 1.4.8)} \\
&= \sum_{n=1}^{\infty} \lambda(B) && \text{(translation invariant by Proposition 1.4.8)}
\end{aligned}$$

This is absurd since there is no number $[0, \infty]$ such that when taking an infinite sum produces a positive number.

The Lebesgue measure has many additional important properties. The most important properties are described in the following two results and are used later in these notes.

Exercise 1.4.11: Assignment 1, Question 4

Let $A \in \mathcal{M}(\mathbb{R})$. Prove the following.

(i) $\lambda(A) = \inf\{\lambda(U) : U \subseteq \mathbb{R}$ is an open set such that $A \subseteq U\}$. This property of λ is called the *outer regularity*.

(ii) $\lambda(A) = \sup\{\lambda(K) : K \subseteq \mathbb{R}$ is a compact set such that $K \subseteq A\}$. This property of λ is called the *inner regularity*.

Solution.

(i) Let $B = \{\lambda(U) : U \subseteq \mathbb{R}$ is an open set such that $A \subseteq U\}$. To see that $\lambda(A) = \inf(B)$, we will show that $\lambda(A) \leq \inf(B)$ and $\lambda(A) \geq \inf(B)$.

For the former, let U be an open subset of $\mathbb{R}$ such that $A \subseteq U$. Then by monotonicity, we obtain that

$$\lambda(A) \leq \lambda(U)$$

and so taking the infimum over all open U such that $A \subseteq U$ on the right side allows us to conclude that

$$\lambda(A) \leq \inf(B)$$

For the latter, let $\varepsilon > 0$ be arbitrary. Since $A \in \mathcal{M}(\mathbb{R})$, then we have that $\lambda = \lambda^*$. Let $\{I_n\}_{n=1}^{\infty}$ be a countable collection of open intervals such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} \lambda(I_n) \leq \lambda(A) + \varepsilon$$

Let $U = \bigcup_{n=1}^{\infty} I_n$, which is open as the countable union of open subsets. Then see that by subadditivity

$$\lambda(U) = \lambda\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} \lambda(I_n) \leq \lambda(A) + \varepsilon$$

Therefore,

$$\lambda(U) \leq \lambda(A) + \varepsilon$$

so taking the infimum over all open U such that $A \subseteq U$ on the left side yields

$$\inf(B) \leq \lambda(A) + \varepsilon$$

and therefore, as $\varepsilon > 0$ was arbitrary, we obtain that

$$\lambda(A) = \inf\{\lambda(U) : U \subseteq \mathbb{R} \text{ is an open set such that } A \subseteq U\}$$

(ii) Let $C = \{\lambda(K) : K \subseteq \mathbb{R}$ is a compact set such that $K \subseteq A\}$. To see that $\lambda(A) = \sup(C)$, we will show that $\lambda(A) \leq \sup(C)$ and $\lambda(A) \geq \sup(C)$.

For the latter, let K be a compact subset of $\mathbb{R}$ such that $K \subseteq A$. Then by monotonicity, we obtain that

$$\lambda(K) \leq \lambda(A)$$

so by taking the supremum over all compact K such that $K \subseteq A$ on the left side allows us to conclude that

$$\sup(C) \leq \lambda(A)$$

For the former, let $\varepsilon > 0$ be arbitrary. Since $A \in \mathcal{M}(\mathbb{R})$, we have $\lambda = \lambda^*$. Let $(F_n)_{n=1}^{\infty}$ be a countable collection defined as

$$F_n = A \cap [-n, n]$$

and we note that $F_n \in \mathcal{M}(\mathbb{R})$ since $A \in \mathcal{M}(\mathbb{R})$ by assumption and $\lambda(F_n) \leq \lambda([-n, n]) = 2n < \infty$. Furthermore, note that $A = \bigcup_{n=1}^{\infty} F_n$ and $F_n \subseteq F_{n+1}$ for each $n \in \mathbb{N}$, so by the Monotone Convergence Theorem for Measures, we have $\lambda(F_n) \to \lambda(A)$.

Now for each $n \in \mathbb{N}$, let $B_n = A_n^c \cap [-n, n]$, in which by monotonicity, we have

$$\lambda(B_n) \leq \lambda([-n, n]) = 2n < \infty$$

Then by using (i), there exists an open subset $U_n \subseteq \mathbb{R}$ such that $B_n \subseteq U_n$ and

$$\lambda(U_n) \leq \lambda(B_n) + \varepsilon$$

Hence, since $\lambda(B_n) < \infty$, we also have $\lambda(U_n) < \infty$, and furthermore, $B_n \subseteq U_n \cap [-n, n] \in \mathcal{M}(\mathbb{R})$ and

$$0 \leq \lambda((U_n \cap [-n, n]) \setminus B_n) = \lambda(U_n \cap [-n, n]) - \lambda(B_n) \leq \lambda(U_n) - \lambda(B_n) = \lambda(U_n \setminus B_n) \leq \varepsilon$$

Now consider $K_n = U_n^c \cap [-n, n]$, which is closed as the intersection of two closed sets and is bounded by n. In particular, by Heine-Borel, K_n is compact and so $K_n \in \mathcal{M}(\mathbb{R})$. In particular,

$$K_n = U_n^c \cap [-n, n] \subseteq B_n^c \cap [-n, n] = A_n \cap [-n, n] \subseteq A$$

Since

$$\begin{aligned}
[-n, n] &= K_n \cup (U_n \cap [-n, n]) \\
[-n, n] &= A_n \cup B_n
\end{aligned}$$

are disjoint unions of measurable sets, we obtain that

$$\lambda(K_n) + \lambda(U_n \cap [-n, n]) = 2n = \lambda(A_n) + \lambda(B_n)$$

so

$$\lambda(A_n) \leq \lambda(K_n) + \lambda(U_n \cap [-n, n]) - \lambda(B_n) \leq \lambda(K_n) + \varepsilon$$

Hence,

$$\lambda(A) = \lim_{n\to\infty} \lambda(A_n) \leq \limsup_{n\to\infty} \lambda(K_n) + \varepsilon$$

so as $K_n \subseteq A$ for all $n \in \mathbb{N}$, we have

$$\lambda(A) \leq \sup(C) + \varepsilon$$

Therefore, as $\varepsilon > 0$ was arbitrary, we obtain the other inequality as desired.

Exercise 1.4.12: Assignment 1, Question 3

Let $A \subseteq \mathbb{R}$. Prove that the following are equivalent.

(1) $A \in \mathcal{M}(\mathbb{R})$

(2) For all $\varepsilon > 0$, there exists an open subset $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $\lambda^*(U \setminus A) < \varepsilon$.

(3) For all $\varepsilon > 0$, there exists an closed subset $F \subseteq \mathbb{R}$ such that $F \subseteq A$ and $\lambda^*(A \setminus F) < \varepsilon$.

(4) There exists a G_δ set $G \subseteq \mathbb{R}$ such that $A \subseteq G$ and $\lambda^*(G \setminus A) = 0$.

(5) There exists a F_σ set $F \subseteq \mathbb{R}$ such that $F \subseteq A$ and $\lambda^*(A \setminus F) = 0$.

Note that a set is said to be G_δ if it is the countable intersection of open sets, and a set is said to be F_σ if it is the countable union of closed sets.

Solution.

It suffices to prove (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1), and we note (1) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (1) would follow by taking complements.

(1) $\Rightarrow$ (2): Assume that $A \in \mathcal{M}(\mathbb{R})$ and let $\varepsilon > 0$ be arbitrary. For each $n \in \mathbb{N}$, define $A_n = A \cap [-n, n]$. For $n \in \mathbb{N}$, let U_n be an open subset of $\mathbb{R}$ such that $A_n \subseteq U_n$, and

$$0 \leq \lambda^*(U_n) \leq \lambda^*(A_n) + \frac{\varepsilon}{2^n} \tag{1}$$

Then note that since $A_n \subseteq [-n, n]$, by monotonicity,

$$\lambda^*(A_n) \leq \lambda^*([-n, n]) = 2n < \infty$$

So then we can write (1) as

$$\lambda^*(U_n) - \lambda^*(A_n) = \lambda^*(U_n \setminus A_n) \leq \frac{\varepsilon}{2^n} \tag{2}$$

Let

$$U = \bigcup_{n=1}^{\infty} U_n$$

which is an open subset of $\mathbb{R}$ as the countable union of open subsets, and furthermore, $A_n \subseteq U$ for each $n \in \mathbb{N}$. Thus, we have $U \in \mathcal{M}(\mathbb{R})$. Therefore, since $A^c \subseteq A_n^c$, by subadditivity, and by

(2), we conclude that

$$\begin{aligned}
\lambda^*(U \setminus A) &= \lambda^*\left(\left(\bigcup_{n=1}^{\infty} U_n\right) \setminus A\right) \\
&= \lambda^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus A)\right) \\
&\leq \lambda^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus A_n)\right) && \text{(since } A^c \subseteq A_n^c\text{)} \\
&\leq \sum_{n=1}^{\infty} \lambda^*(U_n \setminus A_n) && \text{(by subadditivity)} \\
&\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} && \text{(by (2))} \\
&= \varepsilon
\end{aligned}$$

Therefore, as $\varepsilon > 0$ was arbitrary, $\lambda^*(U \setminus A) < \varepsilon$.

(2) $\Rightarrow$ (4): For each $n \in \mathbb{N}$, let U_n be an open subset of $\mathbb{R}$ such that $A \subseteq U_n$ and $\lambda^*(U_n \setminus A) < \frac{1}{2^n}$. Let $G = \bigcap_{n=1}^{\infty} U_n$ be a G_δ-set, in which G is G_δ since it is the countable intersection of open subsets. In particular, $G \in \mathfrak{B}(\mathbb{R})$, so by Corollary 1.4.1, $G \in \mathcal{M}(\mathbb{R})$. Since $G \subseteq U_n$ for each $n \in \mathbb{N}$, it follows that $G \setminus A \subseteq U_n \setminus A$ for each $n \in \mathbb{N}$, so by monotonicity,

$$0 \leq \lambda^*(G \setminus A) \leq \lambda^*(U_n \setminus A) \leq \frac{1}{2^n}$$

By taking $n \to \infty$, note that the right side of the inequality yields

$$0 \leq \lambda^*(G \setminus A) \leq 0$$

in other words, $\lambda^*(G \setminus A) = 0$.

(4) $\Rightarrow$ (1): Assume that there exists a G_δ set $G \subseteq \mathbb{R}$ such that $A \subseteq G$ and $\lambda^*(G \setminus A) = 0$. Then e note that $G \setminus A \in \mathcal{M}(\mathbb{R})$. By assumption, since G is G_δ, $G \in \mathfrak{B}(\mathbb{R})$ so by Corollary 1.4.1, $G \in \mathcal{M}(\mathbb{R})$. Observe that we can write

$$A = (G \setminus A)^c \cap G$$

since $\mathcal{M}(\mathbb{R})$ is closed under complements and intersections, it follows that $A \in \mathcal{M}(\mathbb{R})$. ■

Chapter 2

Measurable Functions

As with everything in mathematics, once one has defined the main objects one desires to study, one then defines the morphisms or functions related to ones' central object. These so-called "measurable functions" will be the focus of this chapter. After developing the basic properties of real- and complex-valued measurable functions, we will demonstrate that every measurable function can be 'approximated' by 'simple' functions. We will also demonstrate that convergence of measurable functions on the reals are 'almost everywhere continuous'. The theory of measurable functions is vital for a theory of integration as we will see in the next chapter.

2.1 Measurable Functions

To begin, we define the notion of a measurable function. Note the flavour of this definition is very similar to the definition of continuous functions between topological spaces where it is said that a function is continuous if and only if the inverse image of every open set is open.

Definition 2.1.1

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. A function $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is said to be *measurable* if $f^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{B}$. That is, the inverse image of every measurable set in Y is measurable in X.

Of course, we have a collection of trivial examples.

Example 2.1.2

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$.

(α) If f is constant, then f is measurable as either $f^{-1}(A) = X$ or $f^{-1}(A) = \emptyset$ for all $A \in \mathcal{B}$.

(β) If $\mathcal{A} = \mathcal{P}(X)$, then f is automatically measurable.

(γ) If $\mathcal{B} = \{\emptyset, Y\}$, then f is automatically measurable as $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$.

For a more robust collection of examples, we look at the following.

Definition 2.1.3

Let X be a nonempty set and let $A \subseteq X$. The *characteristic function* of A (or *indicator function*) is the function $\mathbf{1}_A : X \to \mathbb{R}$ defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for all $x \in X$.

In the sense of probability theory, the characteristic function of an event takes on the value one at a point where the event occurs and zero otherwise. Of course, for a characteristic function to make sense in probability, we would want the event to be in our probability space; that is, we would want the set to be measurable.

Example 2.1.4

Let $(X, \mathcal{A})$ be a measurable space and $A \subseteq X$. The characteristic function $\mathbf{1}_A$ is measurable as a function to $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ if and only if $A \in \mathcal{A}$. Indeed, notice for all $B \subseteq \mathbb{R}$,

$$\mathbf{1}_A^{-1}(B) = \begin{cases} \emptyset & \text{if } 0, 1 \notin B \\ A & \text{if } 0 \notin B,\, 1 \in B \\ A^c & \text{if } 0 \in B,\, 1 \notin B \\ X & \text{if } 0, 1 \in B \end{cases}$$

From this and the fact that all cases are possibly by choosing select $B \in \mathfrak{B}(\mathbb{R})$, clearly $\mathbf{1}_A$ is measurable if and only if $A, A^c \in \mathcal{A}$ if and only if $A \in \mathcal{A}$.

Of course, we will mainly be interested in functions from a measure space into either the real or complex numbers. As such, we will use $\mathbb{K}$ to denote either the real or complex numbers.

However, we have a notion of a measurable function for each σ-algebra on $\mathbb{K}$. One may think we could use the σ-algebra $\{0, \mathbb{K}\}$ to force every function to be measurable. However, this would imply the characteristic functions of nonmeasurable sets are measurable, which is undesirable if we want to construct an integral for measurable functions. Since we desire any continuous function to be measurable, the σ-algebra on $\mathbb{K}$ should contain at least every open set, and thus must contain the Borel σ-algebra. Thus, we define the following notion.

Definition 2.1.5

Let $(X, \mathcal{A})$ be a measurable space. A function $f : (X, \mathcal{A}) \to \mathbb{K}$ is said to be *measurable* if f is measurable as a function from $(X, \mathcal{A})$ to $(\mathbb{K}, \mathfrak{B}(\mathbb{K}))$. The set of all measurable functions from $(X, \mathcal{A})$ to $(\mathbb{K}, \mathfrak{B}(\mathbb{K}))$ is denoted by $\mathcal{M}(X, \mathbb{K})$.

Of course, one natural question to ask when $\mathbb{K} = \mathbb{R}$ is why we did not use the Lebesgue measurable functions. To see the reason why, we require the following peculiar function.

Definition 2.1.6: The Cantor Ternary Function

Given a sequence $\mathbf{a} = (a_n)_{n=1}^{\infty}$ of elements $\{0, 1, 2\}$, define

$$K_{\mathbf{a}} = \begin{cases} N & \text{if } a_N = 1 \text{ and } a_k \neq 1 \text{ for all } k < N \\ \infty & \text{otherwise} \end{cases}$$

and define a sequence $\mathbf{b_a} = (b_n)_{n=1}^{\infty}$ of elements $\{0, 1\}$ by

$$b_n = \begin{cases} \frac{a_n}{2} & \text{if } n \leq K_{\mathbf{a}} \\ 1 & \text{if } n = K_{\mathbf{a}} \\ 0 & \text{otherwise} \end{cases}$$

The *Cantor ternary function* is the function $f : [0, 1] \to [0, 1]$ defined as follows: if $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in [0, 1]$, for a sequence $\mathbf{a} = (a_n)_{n=1}^{\infty}$ of elements $\{0, 1, 2\}$ and $\mathbf{b_a} = (b_n)_{n=1}^{\infty}$ is the sequence of elements of $\{0, 1\}$ as defined above, then

$$f(x) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$$

That is, write a ternary expansion of x. If N is the first index where a 1 occurs, replace each $\frac{0}{3^n}$ with $n < N$ with $\frac{0}{2^n}$, replace each $\frac{2}{3^n}$ with $n < N$ with $\frac{1}{2^n}$, replace $\frac{1}{3^N}$ with $\frac{1}{2^N}$ and change all terms of index greater than N to zero.

With the following properties of the Cantor ternary function, we can now demonstrate why we do not want to use the set of Lebesgue measurable functions for the σ-algebra of the codomain of measurable functions; the inverse image under a continuous function of a Lebesgue measurable set need not need Lebesgue measurable. Consequently, if we defined a function $f : X \to \mathbb{R}$ to be Lebesgue measurable if and only if the inverse image of a Lebesgue measurable set is Lebesgue measurable there would be continuous functions that are not Lebesgue measurable.

Example 2.1.7

Let f be the Cantor ternary function and define $\psi : [0, 1] \to [0, 2]$ by $\psi(x) = x + f(x)$. Then ψ is a strictly increasing function. We claim that $\psi(\mathcal{C})$ is Lebesgue measurable such that $\lambda(\psi(\mathcal{C})) > 0$. To see this, see Exercise 2.1.8. By Exercise 1.4.11, there exists a subset $A \subseteq \mathcal{C}$ such that $B = \psi(A)$ is not Lebesgue measurable. Since ψ is a strictly increasing continuous function, the function $\phi = \psi^{-1} : [0, 2] \to [0, 1]$ is continuous. However, since A is Lebesgue measurable since $A \subseteq \mathcal{C}$ and $\lambda(\mathcal{C}) = 0$, and λ is complete, yet $\phi^{-1}(A) = \psi(A)$ is not Lebesgue measurable. Hence, there exists a continuous function on $\mathbb{R}$ such that the inverse image of a Lebesgue measurable set is not Lebesgue measurable.

Exercise 2.1.8: Assignment 2, Question 4

Prove the following properties of the Cantor Ternary function.

(i) Show that the Cantor ternary function is well-defined.

(ii) Let $\mathcal{C}$ denote the cantor set and let f be the Cantor ternary function. Show that f is a non-decreasing continuous function which is constant on each interval of $\mathcal{C}^c$. Furthermore, show that $f(\mathcal{C}) = [0, 1]$.

(iii) Define $\psi : [0, 1] \to [0, 2]$ by $\psi(x) = x + f(x)$ for all $x \in [0, 1]$. Prove that ψ is a strictly increasing continuous function such that $\psi(\mathcal{C})$ is Lebesgue measurable with $\lambda(\psi(\mathcal{C})) > 0$ and that there exists a subset $B \subseteq \mathcal{C}$ such that $\psi(B)$ is not Lebesgue measurable.

(iv) Prove that there exists a subset $B \subseteq \mathcal{C}$ such that B is not Borel.

(v) Prove there exists Lebesgue measurable functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $g \circ h$ is not Lebesgue measurable.

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