MATH 6280 - Midterm

SOLUTIONS

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Instructions:

- This examination has 7 two-sided pages including this one. Please verify that this printed copy has all of the pages. There are empty pages at the end of the examination for rough and additional work space.
- The length of this examination is 75 minutes.
- All answers, mathematical statements, and proofs must be sufficiently justified, contain all essential details, and be displayed in a clean and clear fashion.
- Books, calculators, notes, electronic devices, and additional aids are not permitted.
- If you finish or are stuck on problems, take time to check your answers or add additional details.
- Good Luck!!!

Grading:

Question	Q1)	Q2)	Q3)	Q4)	Bonus	Total Points
Value	10	13	9	8	4	40

Question 1. State and provide a proof of the Monotone Convergence Theorem for Integrals of Non-Negative Measurable Functions. (10 points)

Solution. Statement: Let $(X, \mathcal{A}, \mu)$ be a measure space. For each $n \in \mathbb{N}$ let $f_n : X \to [0, \infty]$ be a measurable function such that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$. If $f : X \to [0, \infty]$ is a measurable function and the pointwise limit of $(f_n)_{n\geq 1}$, then for all $A \in \mathcal{A}$

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu.$$

Proof. First note since f is the pointwise limit of measurable functions that f is measurable. Next note we may assume that $A = X$ since multiplying by a characteristic function will preserve measurability, pointwise limits, and the value of the integral.

Since $f_n \leq f$ for all $n \in \mathbb{N}$, we have that

$$\int_X f_n \, d\mu \leq \int_X f \, d\mu$$

for all $n \in \mathbb{N}$. Hence

$$\limsup_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

Thus, to complete the proof, it suffices to show that

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

In order to facilitate some 'wiggle room', we will show that

$$\alpha \int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$$

for all $\alpha \in (0, 1)$ from which the desired inequality will follow by take the limit $\alpha \to 1$.

To obtain the desired inequality, fix $\alpha \in (0, 1)$. Let $\varphi : X \to [0, \infty)$ be an arbitrary simple function such that $\varphi \leq f$. Thus, if we can prove that

$$\alpha \int_X \varphi \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu,$$

the proof will be complete by the definition of the integral of f.

Notice $\alpha\varphi$ is a simple function such that $\alpha\varphi \leq f$. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in X \mid f_n(x) - \alpha\varphi(x) \geq 0\}.$$

Since each $f_n - \alpha\varphi$ is a measurable function, A_n is measurable for all $n \in \mathbb{N}$. Moreover we have for all $n \in \mathbb{N}$ that

$$\begin{aligned}
\alpha \int_{A_n} \varphi \, d\mu &= \int_{A_n} \alpha\varphi \, d\mu \\
&\leq \int_{A_n} f_n \, d\mu && \text{since } \alpha\varphi\chi_{A_n} \leq f_n\chi_{A_n} \\
&\leq \int_X f_n \, d\mu && \text{since } A_n \subseteq X \\
&\leq \liminf_{k\to\infty} \int_X f_k \, d\mu && \text{since } f_k \leq f_{k+1} \text{ so } \left(\textstyle\int_X f_k \, d\mu\right)_{k\geq 1} \text{ is an increasing sequence.}
\end{aligned}$$

Thus, to complete the proof, it suffices to replace A_n with X in the above inequality.

Since $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$, clearly $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$. We claim that

$$X = \bigcup_{n \geq 1} A_n.$$

To see this, let $x \in X$ be arbitrary. If $f(x) = 0$ then $f_n \leq f$ and $\varphi \leq f$ implies that $f_n(x) = 0 = \alpha\varphi(x)$ and thus $x \in A_n$ for all $n \in \mathbb{N}$. Otherwise, if $f(x) > 0$, then we notice $\varphi \leq f$ implies that $f(x) > \alpha\varphi(x)$ since $\alpha < 1$ (this is why we needed the wiggle room). Hence, since $\lim_{n\to\infty} f_n(x) = f(x)$, there exists an $N \in \mathbb{N}$ such that $f(x) \geq f_N(x) > \alpha\varphi(x)$ and thus $x \in A_N$. Hence $X = \bigcup_{n\geq 1} A_n$.

Let $\nu : X \to [0, \infty]$ be defined by

$$\nu(A) = \int_A \varphi \, d\mu$$

for all $A \in \mathcal{A}$. Since φ is a simple function, ν is a measure on $(X, \mathcal{A})$. Therefore, since $\{A_n\}_{n=1}^\infty$ is an increasing sequence of measurable sets with $X = \bigcup_{n\geq 1} A_n$, the Monotone Convergence Theorem for Measures implies that

$$\begin{aligned} \alpha \int_X \varphi \, d\mu &= \alpha\nu(X) \\ &= \alpha \lim_{n\to\infty} \nu(A_n) \\ &= \alpha \lim_{n\to\infty} \int_{A_n} \varphi \, d\mu \\ &\leq \liminf_{k\to\infty} \int_X f_k \, d\mu. \end{aligned}$$

Hence the proof is complete. □

Question 2.
a) Let X be a non-empty set, let μ^* be an outer measure on X, and let $A \subseteq X$. Define what it means for A to be *outer measurable*. (2 points)

Solution. The set $A \subseteq X$ is said to be outer measurable if for all $B \subseteq X$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

b) Let X be a non-empty set and let μ^* be an outer measure on X. Provide a proof that the set of outer measurable sets is a σ-algebra. (11 points)

Proof. To see that $\mathcal{A}$ is a σ-algebra, first notice for all $B \in \mathcal{P}(X)$ that

$$\mu^*(B) = \mu^*(B) + 0 = \mu^*(B \cap \emptyset^c) + \mu^*(B \cap \emptyset).$$

Hence $\emptyset \in \mathcal{A}$. Furthermore, clearly if $A \in \mathcal{A}$ then clearly $A^c \in \mathcal{A}$ due to the symmetry in the definition of an outer measurable set. Hence $\mathcal{A}$ is closed under compliments and $X \in \mathcal{A}$.

In order to demonstrate that $\mathcal{A}$ is closed under countable unions, let's first verify that $\mathcal{A}$ is closed under finite unions. To verify that $\mathcal{A}$ is closed under finite unions, it suffices to verify that if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cup A_2 \in A$ as we can then apply recursion to take arbitrary finite unions of element of $\mathcal{A}$. Thus let $A_1, A_2 \in \mathcal{A}$ be arbitrary. To see that $A_1 \cup A_2 \in \mathcal{A}$, let $B \subseteq X$ be arbitrary. Since A_1 is outer measurable, we know that

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c).$$

Furthermore, since A_2 is outer measurable, we know that

$$\mu^*(B \cap A_1^c) = \mu^*((B \cap A_1^c) \cap A_2) + \mu^*((B \cap A_1^c) \cap A_2^c).$$

Hence

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c).$$

However, since

$$B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap (A_2 \cap A_1^c)),$$

subadditivity implies that

$$\begin{aligned} \mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\ &\geq \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap A_1^c \cap A_2^c) \\ &= \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap (A_1 \cup A_2)^c) \end{aligned}$$

Therefore, since $B \subseteq X$ was arbitrary, we obtain that $A_1 \cup A_2 \in \mathcal{A}$. Hence $\mathcal{A}$ is closed under finite unions.

Since $\mathcal{A}$ is also closed under complements, we also obtain that $\mathcal{A}$ is closed under finite intersections.

To see that $\mathcal{A}$ is closed under countable unions, let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$ be arbitrary. Let $E_1 = A_1$ and for $n \geq 1$ let

$$E_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k \right) = A_n \cap \left(\bigcup_{k=1}^{n-1} A_l \right)^c.$$

Clearly $\{E_n\}_{n=1}^\infty$ are pairwise disjoint such that $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty A_n$. Furthermore, $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ by the above argument.

To see that $E = \bigcup_{n=1}^\infty E_n$ is an element of $\mathcal{A}$, let $B \subseteq X$ be arbitrary. For each $n \in \mathbb{N}$, let $F_n = \bigcup_{k=1}^n E_k$, which is an element of $\mathcal{A}$ since $\mathcal{A}$ is closed under finite unions. Therefore, since F_n is outer measurable, since $F_n \subseteq E$ so $E^c \subseteq F_n^c$, and since μ^* is monotone, we obtain that

$$\mu^*(B) = \mu^*(B \cap F_n) + \mu^*(B \cap F_n^c) \geq \mu^*(B \cap F_n) + \mu^*(B \cap E^c)$$

for all $n \in \mathbb{N}$.

Notice that $F_n = F_{n-1} \cup E_n$ and $F_{n-1} \cap E_n = \emptyset$ by construction. Therefore, since $E_n \in \mathcal{A}$, we obtain that

$$\begin{aligned} \mu^*(B \cap F_n) &= \mu^*((B \cap F_n) \cap E_n) + \mu^*((B \cap F_n) \cap E_n^c) \\ &= \mu^*(B \cap E_n) + \mu^*(B \cap F_{n-1}) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore recursion implies that

$$\mu^*(B \cap F_n) = \sum_{k=1}^n \mu^*(B \cap E_k)$$

for all $n \in \mathbb{N}$. Hence

$$\mu^*(B) \geq \mu^*(B \cap E^c) + \sum_{k=1}^n \mu^*(B \cap E_k)$$

for all $n \in \mathbb{N}$. By taking the supremum of the right-hand-side of the above expression, we obtain that

$$\mu^*(B) \geq \mu^*(B \cap E^c) + \sum_{k=1}^\infty \mu^*(B \cap E_k).$$

Therefore subadditivity implies that

$$\begin{aligned} \mu^*(B) &\geq \mu^*(B \cap E^c) + \mu^*\left(\bigcup_{n=1}^\infty (B \cap E_k)\right). \\ &= \mu^*(B \cap E^c) + \mu^*\left(B \cap \left(\bigcup_{k=1}^\infty E_k\right)\right) \\ &= \mu^*(B \cap E^c) + \mu^*(B \cap E). \end{aligned}$$

Therefore, as $B \subseteq X$ was arbitrary, we obtain that $E \in \mathcal{A}$ as desired. Hence $\mathcal{A}$ is a σ-algebra. $\square$

Question 3.

a) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $f : X \to (0, \infty)$ be integrable. Prove that if $0 < \alpha < \mu(X)$, then

$$\inf\left\{ \int_A f \, d\mu \,\middle|\, A \in \mathcal{A}, \mu(A) \geq \alpha \right\} > 0. \qquad \text{(6 points)}$$

Proof. There exists an increasing sequence of simple functions $(\varphi_n)_{n\geq 1}$ that converge pointwise to f. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in X \mid \varphi_n(x) = 0\}.$$

Then $\{A_n\}_{n=1}^\infty$ is a sequence of measurable sets such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Moreover, since $f(x) > 0$ for all $x \in X$ and since $(\varphi_n)_{n\geq 1}$ that converge pointwise to f, we have that $\bigcap_{n=1}^\infty A_n = \emptyset$. Therefore, since $\mu(X) < \infty$, by the Monotone Convergence Theorem for Measures, we have that

$$0 = \mu(\emptyset) = \lim_{n\to\infty} \mu(A_n).$$

Choose $N \in \mathbb{N}$ such that $\mu(A_N) < \frac{\alpha}{2}$. Since φ_N is a simple function, we can write $\varphi_N = \sum_{k=1}^m a_k \chi_{B_k}$ where $\{B_k\}_{k=1}^m$ are pairwise disjoint measurable sets and $a_k \in (0, \infty)$. Note $A_k = \left(\bigcup_{k=1}^m B_k\right)^c$ by construction.

Let $A \in \mathcal{A}$ be such that $\mu(A) \geq \alpha$. Therefore, since μ is finite,

$$\mu\left(A \cap \left(\bigcup_{k=1}^m B_k\right)\right) = \mu(A) - \mu(A \cap A_N) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

Hence, if $M = \min\{a_k \mid k \in \{1, \ldots, m\}\} > 0$, then

$$\begin{aligned}
\int_A f \, d\mu &\geq \int_A \varphi_N \, d\mu \\
&= \sum_{k=1}^m a_k \varphi(A \cap B_k) \\
&\geq \sum_{k=1}^m M\varphi(A \cap B_k) \\
&\geq M\varphi\left(A \cap \left(\bigcup_{k=1}^m B_k\right)\right) \\
&\geq M\frac{\alpha}{2}.
\end{aligned}$$

Hence

$$\inf\left\{ \int_A f \, d\mu \,\middle|\, A \in \mathcal{A}, \mu(A) \geq \alpha \right\} \geq M\frac{\alpha}{2} > 0.$$

$\square$

b) Given an example of a non-finite measure μ where the conclusions of part a) fail. (3 points)

Solution. Let $\mu = \lambda$, let $\alpha = 1$, and let $f(x) = \frac{1}{x^2}\chi_{[1,\infty)}(x)$ for all $x \in \mathbb{R}$. Then f is Lebesgue measurable with $f(x) > 0$ for all $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} f \, d\lambda = \int_1^\infty \frac{1}{x^2} = 1.$$

Hence f is integrable. However, if for each $n \in \mathbb{N}$ we let $A_n = [n, n+1]$, then $\lambda(A_n) = 1$ yet

$$\int_{A_n} f \, d\lambda = \int_n^{n+1} \frac{1}{x^2} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, since $\lim_{n\to\infty} \frac{1}{n} - \frac{1}{n+1} = 0$, we have

$$\inf\left\{ \int_A f \, d\mu \,\middle|\, A \in \mathcal{A}, \mu(A) \geq \alpha \right\} = 0.$$

Question 4. Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let $(f_n)_{n\geq 1}$ be a sequence of measurable functions on X. It is said that $(f_n)_{n\geq 1}$ is *Cauchy in measure* if for all $\epsilon, \delta > 0$, there exists an $N \in \mathbb{N}$ such that

$$\mu(\{x \mid |f_n(x) - f_m(x)| \geq \delta\}) < \epsilon$$

for all $n, m \geq N$. Prove that if $(f_n)_{n\geq 1}$ is Cauchy in measure, then there exists a measurable function $f : X \to \mathbb{C}$ and a subsequence $(f_{k_n})_{n\geq 1}$ of $(f_n)_{n\geq 1}$ such that $(f_{k_n})_{n\geq 1}$ converges to f pointwise almost everywhere. (8 points)

[Hint: Verify that if $(z_n)_{n\geq 1}$ is a complex sequence and $|z_{n+1} - z_n| < \frac{1}{2^n}$ eventually, then $(z_n)_{n\geq 1}$ is Cauchy and thus converges.]

Proof. Let $(f_n)_{n\geq 1}$ be Cauchy in measure. Hence there exists an $n_1 \in \mathbb{N}$ such that

$$\lambda\left(\left\{x \in \mathbb{R} \,\middle|\, |f_n(x) - f_m(x)| \geq \frac{1}{2}\right\}\right) < \frac{1}{2}$$

for all $n, m \geq n_1$. By iteration, there exists an increasing sequence $(n_k)_{k\geq 1}$ of natural numbers such that

$$\lambda\left(\left\{x \in \mathbb{R} \,\middle|\, |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$$

for all $n, m \geq n_k$.

We claim that the subsequence $(f_{n_k})_{k\geq 1}$ converges almost everywhere. To see this, for each $k \in \mathbb{N}$ let

$$B_k = \left\{x \in \mathbb{R} \,\middle|\, |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq \frac{1}{2^k}\right\}$$

and for each $n \in \mathbb{N}$, let

$$A_n = \bigcup_{k\geq n} B_k.$$

Note B_k and A_n are measurable for all $k, n \in \mathbb{N}$. Moreover, by assumption we know that

$$\lambda(B_k) \leq \frac{1}{2^k}$$

for all $k \in \mathbb{N}$ so

$$\lambda(A_n) \leq \sum_{k=n}^{\infty} \lambda(B_k) \leq \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}.$$

Let $B = \bigcap_{n=1}^{\infty} A_n$. Then B is measurable being the countable intersection of measurable sets. Moreover, since $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$ and since $\lambda(A_n) < \infty$ for all $n \in \mathbb{N}$, we have by the Monotone Convergence Theorem that

$$\lambda(B) = \lim_{n\to\infty} \lambda(A_n) = 0.$$

We claim that $(f_{n_k})_{k\geq 1}$ converges pointwise on B^c. To see this, let $x \in B^c$. Hence there exists an $N \in \mathbb{N}$ such that $x \notin A_N$. Therefore, by the definition of A_N, we have that

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < \frac{1}{2^k}$$

for all $k \geq N$. Therefore, for all $m > k \geq N$ we have that

$$|f_{n_m}(x) - f_{n_k}(x)| < \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

This implies that $(f_{n_k}(x))_{k\geq 1}$ is Cauchy and thus converges.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in B \\ \lim_{k\to\infty} f_{n_k}(x) & \text{if } x \in B^c \end{cases}.$$

Hence f is an almost everywhere pointwise limit of measurable functions and thus is measurable. $\square$

Bonus. Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let $(f_n)_{n\geq 1}$ be a sequence of measurable function on X that are Cauchy in measure. Prove there exists a measurable function $f : X \to \mathbb{C}$ such that $(f_n)_{n\geq 1}$ converges to f in measure. (4 points)

Proof. Using the notation from Question 4c), we claim that $(f_n)_{n\geq 1}$ converges to f in measure. To see this, let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N-1}} < \epsilon$. By the above, we have that if $x \notin A_N$ then

$$|f_{n_m}(x) - f_{n_N}(x)| < \frac{1}{2^{N-1}}.$$

Therefore, since $f(x) = \lim_{k\to\infty} f_{n_k}(x)$, we obtain that

$$|f(x) - f_{n_N}(x)| \leq \frac{1}{2^{N-1}}$$

for all $x \notin A_N$. Therefore, since for all $n \geq N$ we have

$$\begin{aligned}
&\{x \in \mathbb{R} \mid |f(x) - f_n(x)| \geq \epsilon\} \\
&\subseteq \left\{x \in \mathbb{R} \,\middle|\, |f(x) - f_n(x)| \geq \frac{1}{2^{N-1}}\right\} \\
&\subseteq \left\{x \in \mathbb{R} \,\middle|\, |f(x) - f_{n_N}(x)| \geq \frac{1}{2^{N}}\right\} \cup \left\{x \in \mathbb{R} \,\middle|\, |f_{n_N}(x) - f_n(x)| \geq \frac{1}{2^{N}}\right\} \\
&\subseteq A_N \cup \left\{x \in \mathbb{R} \,\middle|\, |f_{n_N}(x) - f_n(x)| \geq \frac{1}{2^{N}}\right\}
\end{aligned}$$

Therefore, since

$$\lambda(A_N) \leq \frac{1}{2^{N-1}} \qquad \text{and} \qquad \lambda\left(\left\{x \in \mathbb{R} \,\middle|\, |f_{n_N}(x) - f_n(x)| \geq \frac{1}{2^{N}}\right\}\right) < \frac{1}{2^{N}},$$

we obtain that

$$\lambda(\{x \in \mathbb{R} \mid |f(x) - f_n(x)| \geq \epsilon\}) < \frac{1}{2^{N-1}} + \frac{1}{2^{N}}$$

for all $n \geq N$. Hence

$$\lim_{n\to\infty} \lambda(\{x \in \mathbb{R} \mid |f(x) - f_n(x)| \geq \epsilon\}) = 0$$

for all $\epsilon > 0$. Thus $(f_n)_{n\geq 1}$ converges to f in measure.

$\square$