Measure Theory

TR 10:00 – 11:30 SC 211

Grading Scheme

30% Assignments
20% Midterm Exam (March 6, 2025)
50% Final Exam (Same as Comprehensive)

Material Covered

- Measure Spaces
- Measurable Functions
- Integration over Measurable Spaces
- Differentiation and Integration
- Signed Measures
- Product Measures and Fubini's Theorem
- Riesz Representation Theorem.

Measure Spaces

Definition: Let X be a nonempty set. A σ-algebra on X is a set $\mathcal{A} \subset \mathcal{P}(X)$ s.t.

(i) $\emptyset, X \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$

(iii) If $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

The pair $(X, \mathcal{A})$ is called a measurable space and elements of $\mathcal{A}$ are called measurable sets.

Remark:

- By taking $A_n = \emptyset$ for all $n > N$, we see if $A_1, \dots, A_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in A$.
- σ-algebras are closed under countable intersections.

Indeed, let $(A_n)_{n=1}^{\infty} \in \mathcal{A}$, then

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c$$

where each $A_n^c \in \mathcal{A}$, so $\bigcup_{n=1}^{\infty} A_n^c \in \mathcal{A}$, and thus $\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{A}$.

Examples: Let X be a nonempty set.

(α) $(X, \mathcal{P}(X))$

(β) $(X, \{\emptyset, X\})$

(γ) If $\mathcal{A} = \{A \subset X : A \text{ is countable or } A^c \text{ is countable}\}$, $(X, \mathcal{A})$ is a measurable space.

Lemma: Let X be a nonempty set. Let $(\mathcal{A}_i)_{i\in I}$ be a collection of σ-algebras on X. Then $\bigcap_{i\in I} \mathcal{A}_i$ is a σ-algebra.

Corollary: If $A \subset \mathcal{P}(X)$, there exists a smallest σ-algebra containing A. This set is called the σ-algebra generated by A, and is denoted by $\sigma(A)$.

Proof: Let $I = \{\mathcal{A} : \mathcal{A}$ is a σ-algebra and $A \in \mathcal{A}\}$
Note that $I \neq \emptyset$ bec. $\mathcal{P}(X) \in I$. Then

$$\sigma(A) = \bigcap_{\mathcal{A}\in I} \mathcal{A}$$

is a σ-algebra that contains A and is smaller than every σ-algebra containing A.

Definition: Let (X,d) be a metric space. The **Borel σ-algebra**, denoted by $\mathcal{B}(X)$ is the σ-algebra generated by the open sets.

e.g. On $\mathbb{R}$

- $\{(a,b) : a < b \in \mathbb{R}\}$
- $\{(a,\infty) : a \in \mathbb{R}\}$
- $\{(-\infty,b) : b \in \mathbb{R}\}$
- $\{[a,b] : a < b \in \mathbb{R}\}$

$\vdots$

Note: $|\mathcal{B}(\mathbb{R})| = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$

Definition: A measure on a measurable space is a function $\mu : \mathcal{A} \to [0,\infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $(A_n)_{n=1}^{\infty}$ are pairwise disjoint, then

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \quad \text{(countable additivity).}$$

The triple $(X, \mathcal{A}, \mu)$ is called a measure space and for $A \in \mathcal{A}$, the value $\mu(A)$ is called the μ-measure of A.

Remark: If $A_n = \emptyset \ \forall n > N$, we have that

$$\mu\left(\bigsqcup_{i=1}^{n} A_n\right) = \sum_{i=1}^{n} \mu(A_n) \quad \text{whenever } A_1, \dots, A_n \in \mathcal{A}$$

are pairwise disjoint.

Examples:

(α) If $x \in X$, $\delta_x : \mathcal{P}(X) \to [0,\infty]$ defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

We call δ_x the point-mass measure at x.

(β) Define $\mu : \mathcal{P}(X) \to [0,\infty]$ by

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

Then μ is a measure called the counting measure.

Example: Let $(a_n)_{n=1}^{\infty}$ be a sequence in $[0,\infty]$ and let $\mu: \mathcal{P}(\mathbb{N}) \to [0,\infty]$ by

$$\mu(A) = \sum_{n\in A} a_n$$

is a measure. Indeed,

- $\mu(\emptyset) = \sum_{n\in\emptyset} a_n = 0$

Conversely, if $\mu: \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is a measure, we claim that μ has the above form. For $n \in \mathbb{N}$, let $a_n = \mu(\{a\})$. Then $\forall A \in \mathcal{P}(\mathbb{N})$, we know

$\{\{n\} : n \in A\}$ is countable, pairwise disjoint, with union A, so

$$\mu(A) = \sum_{n\in A} \mu(\{n\}) = \sum_{n\in A} a_n.$$

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space, let $(A_n)_{n=1}^{\infty}$ be a collection in $\mathcal{A}$ and let $(a_n)_{n=1}^{\infty}$ be a sequence in $[0,\infty]$. Define $\nu: \mathcal{A} \to [0,\infty]$ by

$$\nu(A) = \sum_{n=1}^{\infty} a_n \mu(A \cap A_n)$$

Then ν is a measure.

Proof: Indeed,

- $\nu(\emptyset) = \sum_{n=1}^{\infty} a_n \mu(\emptyset \cap A_n) = \sum_{n=1}^{\infty} a_n \cdot 0 = 0$
- If $(B_m)_{m=1}^{\infty}$ is a collection in $\mathcal{A}$, pairwise disjoint, then

$$\nu\left(\bigsqcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} a_n \mu\left(\left(\bigsqcup_{m=1}^{\infty} B_m\right) \cap A_n\right)$$

$$= \sum_{n=1}^{\infty} a_n \, \mu\left(\bigsqcup_{m=1}^{\infty} (B_m \cap A_n) \right)$$
$$= \sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} \mu(B_m \cap A_n)$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \, \mu(B_m \cap A_n)$$
$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n \, \mu(B_m \cap A_n) \qquad \text{(Fubini's Theorem)}$$
$$= \sum_{m=1}^{\infty} \nu(B_m)$$

Remark: Let $(X, \mathcal{A}, \mu)$ be a measure space and let $E, F \in \mathcal{A}$ such that $E \subset F$. Then

$F \setminus E = F \cap E^c \in \mathcal{A}$ that is disjoint from E.

Then

$$\mu(F) = \mu((F \setminus E) \sqcup E) = \mu(F \setminus E) + \mu(E) \geq \mu(E).$$

Thus, measures are monotone.

In particular, if $\mu(F) < \infty$, then $\mu(E) < \infty$.

Moreover, if $\mu(E) < \infty$, then

$$\mu(F \setminus E) = \mu(F) - \mu(E).$$

Remark: If $A, B \in \mathcal{A}$ such that $\mu(A \cap B) < \infty$, Then

$$\begin{aligned} \mu(A \cup B) &= \mu(A \sqcup (B \setminus A)) \\ &= \mu(A) + \mu(B \setminus A) \\ &= \mu(A) + \mu(B \setminus (B \cap A)) \\ &= \mu(A) + \mu(B) - \mu(A \cap B) \end{aligned}$$

Def: A probability space is a measure space $(X, \mathcal{A}, \mu)$ where $\mu(X) = 1$. We call such a measure a probability measure.

In this context, X is called the sample space and any $A \in \mathcal{A}$ are the events, and $\mu(A)$ is called the probability of A.

Definition: A measure space $(X, \mathcal{A}, \mu)$ is called

(i) finite if $\mu(X) < \infty$ (by monotonicity, $\forall A \in \mathcal{A}, \mu(A) < \infty$)

(ii) σ-finite if there exists $(A_n)_{n=1}^{\infty}$ of $\mathcal{A}$ such that $\mu(A_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} A_n$.

Remark: We can prove the assumptions on the sets in a σ-finite measure space.

e.g. Let $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i \right)$

Then $(B_n)_{n=1}^{\infty}$ is in $\mathcal{A}$, $B_n \subset A_n$, and by monotonicity, $\mu(B_n) \leq \mu(A_n)$, $X = \bigcup_{n=1}^{\infty} B_n$, and $B_m \cap B_n = \emptyset$ if $(B_n)_{n=1}^{\infty}$ are pairwise disjoint.

(Disjointification — not a real English word, but for math, it is).

Alternatively, let $C_n = \bigcup_{i=1}^{n} A_i$. Then $(C_n)_{n=1}^{\infty}$ is measurable, $X = \bigcup_{n=1}^{\infty} C_n$, and $\mu(C_n) < \infty$ by the following theorem:

Theorem: (Subadditivity) If $(A_n)_{n=1}^{\infty}$ are measurable sets,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} A_n.$$