Theorem: (Caratheodory - Hahn Extension Theorem)

Let $\mathcal{A}$ be an algebra on X, let μ be a premeasure on $\mathcal{A}$, let μ^* be the outer measure associated to μ, and let $\mathcal{A}^*$ be the μ^*-measure sets. Then

(i) $\mu^*(A) = \mu(A) \quad \forall A \in \mathcal{A}$

(ii) $\mathcal{A} \subset \mathcal{A}^*$

(iii) If $\nu^* : \mathcal{A}^* \to [0,\infty]$ is a measure and μ is σ-finite, then $\nu^* = \mu^*|_{\mathcal{A}^*}$.

We call $\bar{\mu} = \mu^*|_{\mathcal{A}^*}$ the Carathéodory extension of μ.

Proof: (ii) Fix $A \in \mathcal{A}$ and let $B \subset X$ and $\varepsilon > 0$. Then $\exists (A_n)_{n=1}^{\infty}$ in $\mathcal{A}$ s.t.

- $B \subset \bigcup_{n=1}^{\infty} A_n$
- $\sum_{n=1}^{\infty} \mu(A_n) \le \mu^*(B) + \varepsilon$.

So $B \cap A \subset \bigcup_{n=1}^{\infty} (A_n \cap A)$ where $\forall n \in \mathbb{N}$, $A_n \cap A \in \mathcal{A}$. So

$$\mu^*(B \cap A) \le \sum_{n=1}^{\infty} \mu(A_n \cap A).$$

Also,

$$B \cap A^c \subset \bigcup_{n=1}^{\infty} (A_n \cap A^c) \text{ where } \forall n \in \mathbb{N}, A_n \cap A^c \in \mathcal{A}.$$

$$\mu^*(B \cap A^c) \le \sum_{n=1}^{\infty} \mu(A_n \cap A^c)$$

by the definition of μ^*. Thus,

$$\begin{aligned}
\mu^*(B \cap A) + \mu^*(B \cap A^c) &\le \sum_{n=1}^{\infty} \mu(A_n \cap A) + \sum_{n=1}^{\infty} \mu(A_n \cap A^c) \\
&= \sum_{n=1}^{\infty} \mu(A_n) \\
&\le \mu^*(B) + \varepsilon. \longrightarrow \mu^*(B)
\end{aligned}$$

(iii) If μ is σ-finite, $\exists (X_n)_{n=1}^{\infty}$ of $\mathcal{A}$ s.t. $\mu(X_n) < \infty$.

$\forall n \in \mathbb{N}$, let (Note $X = \bigcup_{n=1}^{\infty} X_n$)

$$Y_n = \bigcup_{i=1}^{n} X_i$$

Then

- $Y_n \in \mathcal{A}$
- $Y_n \subset Y_{n+1}$ $\forall n \in \mathbb{N}$
- $\mu(Y_n) = \bar{\mu}(Y_n) \leq \sum_{i=1}^{n} \bar{\mu}(Y_i) = \sum_{=1}^{n} \mu(X_i) < \infty.$

If $\nu : \mathcal{A}^* \to [0,\infty]$ is a measure s.t. $\nu(A) = \mu(A)$ $\forall A \in \mathcal{A}$, we want $\nu^* = \mu^*$ on $\mathcal{A}$.

Fix $B \in \mathcal{A}^*$, let $(A_n)_{n=1}^{\infty}$ be in $\mathcal{A}$ be s.t. $B \subset \bigcup_{n=1}^{\infty} A_n$. Then

$$\begin{aligned}
\nu(B \cap Y_N) &\leq \nu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \cap Y_N\right) \\
&\leq \sum_{n=1}^{\infty} \nu(\underbrace{A_n \cap Y_N}_{\in \mathcal{A}}) \\
&= \sum_{n=1}^{\infty} \mu(A_n \cap Y_N)
\end{aligned}$$

Thus, $\nu(B \cap Y_N) \leq \mu^*(B \cap Y_N)$. By using B^c in place of B, $\nu(B^c \cap Y_N) \leq \mu^*(B^c \cap Y_N)$, so

$$\begin{aligned}
\nu(B \cap Y_N) + \nu(B^c \cap Y_N) &\leq \mu^*(B \cap Y_N) + \mu^*(B^c \cap Y_N) \\
&= \mu^*(Y_N) = \nu(Y_N)
\end{aligned}$$

$\Rightarrow \mu(Y_N) = \nu(Y_N) \leq \mu(Y_N).$

Bec. $\mu(Y_N) < \infty$,

$$\begin{aligned}
\nu(B \cap Y_N) &= \mu^*(B \cap Y_N) \quad \forall N \in \mathbb{N} \\
&= \bar{\mu}(B \cap Y_N) \quad \text{by MCT}
\end{aligned}$$

$\Rightarrow \nu(B) = \bar{\mu}(B) \quad \Rightarrow \nu = \bar{\mu} \quad \forall B \in \mathcal{A}^*$

Example: We do <u>not</u> have uniqueness when μ is not σ-finite. Let $X = \mathbb{Q}$

Let $\mathcal{A}$ be the set of all finite unions of sets of the form $\mathbb{Q} \cap (a,b]$, where $a < b \in \mathbb{R}$. Then $\mathcal{A}$ is an algebra. Define the "stupid measure" by

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases} \qquad A \in \mathcal{A}.$$

Then μ is a premeasure. Then

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases} \qquad \text{for } A \subset \mathbb{Q}$$

We claim $\mathcal{A}^* = \mathcal{P}(\mathbb{Q})$. Note $\mathcal{A} \subset \mathcal{A}^*$, so $(a,b] \in \mathcal{A}^*$ $\forall a,b \in \mathbb{Q}$, also

$$\{b\} = \bigcup_{n=1}^{\infty} (b - \tfrac{1}{n}, b] \in \mathcal{A}^* \quad \forall b \in \mathbb{Q}.$$ Therefore,

$\mathcal{P}(\mathbb{Q}) = \mathcal{A}^*$.

Thus, the counting measure is another extension of μ.

Example: $\mathcal{F} = \{(a,b] : a < b \in \mathbb{R}\} \cup \{(a,\infty) : a \in \mathbb{R}\} \cup \{\emptyset\}$ and $\mathcal{A}$ be the set of all finite unions of elements of $\mathcal{F}$. Also note $X = \mathbb{R}$.

Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. Let $\lambda_F : \mathcal{F} \to [0,\infty]$ be defined by

$$\lambda_F(A) = \begin{cases} 0 & : A = \emptyset \\ F(b) - F(a) & : A = (a,b] \\ F(b) - \lim_{x \to -\infty} F(x) & : A = (-\infty, b] \\ \lim_{x \to \infty} F(x) - F(a) & : A = (a, \infty) \\ \lim_{x \to \infty} F(x) - \lim_{x \to -\infty} F(x) & : A = \mathbb{R} \end{cases}$$

Note if $A, B \in \mathcal{F}$ and

$$\text{dist}(A,B) = \inf\{|a-b| : a \in A, b \in B\} = 0$$

then $A \cup B \in \mathcal{F}$. Thus, if $A \in \mathcal{A}$, there are unique $A_1, \dots, A_n \in \mathcal{F}$ s.t.

- $A = \bigcup_{i=1}^{n} A_n$
- $\text{dist}(A_i, A_j) > 0$ if $i \neq j$.

We extend λ_F to $\mathcal{A}$ by defining

$$\lambda_F(A) = \sum_{i=1}^{n} \lambda_F(A_i)$$

using this decomposition.

Claim: λ_F is a premeasure.

- $\lambda_F(\emptyset) = 0$

Subclaim 1: λ_F is finitely additive on $\mathcal{F}$.

Subclaim 2: λ_F is finitely additive on $\mathcal{A}$

Subclaim 3: λ_F is monotone on $\mathcal{A}$.

Subclaim 4: λ_F is finitely subadditive on $\mathcal{A}$.

Now, assume $A \in \mathcal{A}$ and $A = \bigcup_{n=1}^{\infty} A_n$, $\forall n \in \mathbb{N}, A_n \in \mathcal{A}$.

We want to show

$$\lambda_F(A) = \sum_{n=1}^{\infty} \lambda_F(A_n)$$

Considering our definitions, it suffices to assume $A \in \mathcal{F}$.

Assume $A = (a, b]$, $a < b \in \mathbb{R}$. WLOG, $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. So, $A_n = (a_n, b_n]$ for some $a_n < b_n \in \mathbb{R}$.

Let $\varepsilon > 0$. Since F is right-continuous, $\exists c_n > b_n$ s.t. $F(c_n) < F(b_n) + \frac{\varepsilon}{2^n}$. Similarly, $\exists \delta > 0$ s.t. $F(a+\delta) < F(a) + \varepsilon$. By construction,

$[a+\delta, b] \subset \bigcup_{n=1}^{\infty} (a_n, c_n]$. Since $[a+\delta, b]$ is compact, there is $N \in \mathbb{N}$ s.t.

$$[a+\delta, b] \subset \bigcup_{n=1}^{N} (a_n, c_n)$$

Now,

$$\lambda_F((a,b]) \geq \lambda_F\left(\bigcup_{n=1}^{N} (a_n, b_n]\right) \quad \text{(monotonicity)}$$
$$= \sum_{n=1}^{N} \lambda_F((a_n, b_n]) \quad \text{(finite additivity)}$$

so $\lambda_F((a,b]) \geq \sum_{n=1}^{\infty} \lambda_F((a_n, b_n])$. Also, because

$$\begin{aligned}
\lambda_F((a,b]) &= F(b) - F(a) \\
&\leq F(b) - F(a+\delta) + \varepsilon \\
&= \lambda_F((a+\delta, b]) + \varepsilon \\
&\leq \lambda_F\left(\bigcup_{n=1}^{N} (a_n, c_n]\right) + \varepsilon \\
&\leq \sum_{n=1}^{N} \lambda_F((a_n, c_n]) + \varepsilon \\
&= \sum_{n=1}^{N} \left(F(c_n) - F(a_n)\right) + \varepsilon \\
&\leq \sum_{n=1}^{N} F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} + \varepsilon \\
&= \sum_{n=1}^{N} \lambda_F((a_n, b_n]) + 2\varepsilon. \\
&\leq \sum_{n=1}^{\infty} \lambda_F((a_n, b_n]) + 2\varepsilon \longrightarrow \sum_{n=1}^{\infty} \lambda_F((a_n, b_n])
\end{aligned}$$

So we have shown for intervals of the form $(a,b]$.

Otherwise, if $A \in \mathcal{F}$ and $A \neq (a,b]$, then for all $m \in \mathbb{N}$, let $B_m = A \cap (-m, m] \in \mathcal{F}$ and of the form $(a,b]$. By taking $m \to \infty$, we have

$\lambda_F(A) = \lim_{m \to \infty} \lambda_F(B_n)$ by construction.

By "careful consideration", we get the right thing to show λ_F is a premeasure.

So since $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n]$ and

$\lambda_F((-n,n]) = F(n) - F(-n)$, λ_F is σ-finite.

Hence, λ_F extends to a measure on all λ_F^*-measurable sets called the Lebesgue-Stieljes measure associated to F, which is denoted by λ_F.

By construction, every element of $\mathcal{F}$ is λ_F^*-measurable sets contains $\mathcal{F}$ and thus contains the Borel sets $\mathcal{B}(\mathbb{R})$. The restriction $\lambda_F|_{\mathcal{B}(\mathbb{R})}$ is called the Borel-Stieljes measure associated to F.

Example: If $F(x)$, we claim $\lambda_F = \lambda$. Note that

$\lambda^*(A) = \inf \{ \sum_{n=1}^{\infty} \ell(I_n) : I_n \text{ open intervals}, A \subset \bigcup_{n=1}^{\infty} I_n \}$

$\lambda_F^*(A) = \inf \{ \sum_{n=1}^{\infty} \ell(J_n) : J_n \text{ open-closed}, A \subset \bigcup_{n=1}^{\infty} J_n \}$

Note, $\lambda(A) \geq \lambda_F^*(A)$ as open cover yields open-closed cover.

Conversely, $\forall \varepsilon > 0$, if $A \subset \bigcup_{n=1}^{\infty} J_n$, with $J_n \in \mathcal{F}$

$\exists$ open interval I_n s.t. $J_n \subset I_n$

$l(I_n) < l(J_n) + \frac{\varepsilon}{2^n}$

$\Rightarrow A \subset \bigcup_{n=1}^{\infty} I_n$

$\sum_{n=1}^{\infty} l(I_n) \leq \sum_{n=1}^{\infty} l(J_n) + \varepsilon$

$\Rightarrow \lambda^*(A) \leq \lambda_F^*(A) + \varepsilon$.