MATH 6280 **Assignment 5**

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Question 1. Let μ and ν be finite measures on a measurable space $(X, \mathcal{A})$. Prove that there exists a measurable function $f : X \to [0, 1]$ such that

$$\int_A f \, d\mu = \int_A 1 - f \, d\nu$$

for all $A \in \mathcal{A}$.

Question 2. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{A})$. Prove that if $\nu \ll \mu$ and $f : X \to \mathbb{C}$ is measurable, then f is integrable with respect to ν if and only if $f\frac{d\nu}{d\mu}$ is integrable with respect to μ. Furthermore, prove that if f is integrable with respect to ν then

$$\int_X f \, d\nu = \int_X f \frac{d\nu}{d\mu} \, d\mu.$$

Question 3. Let μ, ν, and ρ be σ-finite measures on a measurable space $(X, \mathcal{A})$ such that $\nu \ll \mu$ and $\rho \ll \nu$. Prove that $\rho \ll \mu$ and that $\frac{d\rho}{d\mu} = \frac{d\rho}{d\nu}\frac{d\nu}{d\mu}$ almost everywhere.

Question 4. Let μ_1 and μ_2 be two finite measures on a measurable space $(X, \mathcal{A})$. Consider the finite signed measure $\nu = \mu_1 - \mu_2$. Prove that if $|\nu|(X) = \mu_1(X) + \mu_2(X)$, then $\nu_+ = \mu_1$ and $\nu_- = \mu_2$.

Question 5. Let μ be a finite Borel measure on $(a, b]$ and let $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \mu((a, x])$ be the cumulative distribution function of μ.

a) Prove that if $\mu \perp \lambda$, then F is singular on $[a, b]$.

b) Prove that if F is singular, on $[a, b]$, then $\mu \perp \lambda$.

Question 6. Let $(X, \mathcal{A})$ be a measurable space. A *complex measure* is a function $\nu : \mathcal{A} \to \mathbb{C}$ such that

- $\nu(\emptyset) = 0$, and
- if $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$ are pairwise disjoint, then $\nu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \nu(A_n)$ with the sum converging absolutely.

a) Prove that if ν is a complex measure, then there exists finite measures μ_1, μ_2, μ_3, and μ_4 such that

$$\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4.$$

b) Prove that if ν is a complex measure on $(X, \mathcal{A})$ then there exists finite measure μ (denoted $|\nu|$) and a measurable function $\varphi : X \to \mathbb{C}$ such that $|\varphi(x)| = 1$ for all $x \in X$ and

$$\nu(A) = \int_A \varphi \, d\mu$$

for all $A \in \mathcal{A}$. (Hint: Consider $\sum_{k=1}^4 \mu_k$ but note this is not necessarily μ.)

c) Prove that, in part b), the measure μ is unique and that φ is unique upto sets of μ-measure zero.