Recall: If $\mathcal{F} \subseteq X$ is such that $\emptyset, X \in \mathcal{F}$ and $\ell : \mathcal{F} \to [0,\infty]$ is such that $\ell(\emptyset) = 0$, then

$$\mu^*_\ell(A) = \inf\left\{ \sum_{n=1}^{\infty} \ell(B_n) : (B_n)_{n=1}^{\infty} \subseteq \mathcal{F} \text{ and } A \subseteq \bigcup_{n=1}^{\infty} B_n \right\}.$$

for all $A \subseteq X$ defines an outer measure on X.

Definition: Let μ^* be an outer measure on X. A set $A \subset X$ is said to be μ^*-measurable if for all $B \subset X$, we have

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

Remark: Because μ^* is an outer measure and is subadditive, then

$$\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

For the other inequality, we have by the following.

Theorem: Let μ^* be an outer measure on X and let $\mathcal{A}$ be the set of all μ^*-measurable sets. Then

(i) $\mathcal{A}$ is a σ-algebra

(ii) $\mu^*|_{\mathcal{A}}$ is a measure.

Proof: Note $\emptyset, X \in \mathcal{A}$, because for all $B \subset X$,

$$\mu^*(B) = \mu^*(B \cap \emptyset) + \mu^*(B \cap \emptyset^c)$$

Clearly, $\mathcal{A}$ is closed under complements. Let us first try to show that $\mathcal{A}$ is closed under finite unions. Fix $A_1, A_2 \in \mathcal{A}$, and fix $B \subseteq X$. Because $A_1 \in \mathcal{A}$,

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c). \quad (*)$$

and since $A_2 \in \mathcal{A}$ and $B \cap A_1^c \subset X$, then we know

$$\mu^*(B \cap A_1^c) = \mu^*((B \cap A_1^c) \cap A_2) + \mu^*((B \cap A_1^c) \cap A_2^c). \quad (**)$$

Put $(**)$ into $(*)$, we have

$$\begin{aligned}\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap (A_1 \cup A_2)^c) \\ &\geq \mu^*((B \cap A_1) \cup (B \cap A_1^c \cap A_2)) + \mu^*(B \cap (A_1 \cup A_2)^c) \\ &= \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap (A_1 \cup A_2)^c)\end{aligned}$$

Hence, $A_1 \cup A_2 \in \mathcal{A}$, so $\mathcal{A}$ is closed under finite unions. Because $\mathcal{A}$ is closed under complements $\mathcal{A}$ is also closed under finite intersections.

Let $(A_n)_{n=1}^{\infty}$ be a collection in $\mathcal{A}$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Let $E_1 = A_1$ and $E_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i \right)$ for $n \geq 2$, which are in $\mathcal{A}$. Then $(E_n)_{n=1}^{\infty}$ are pairwise disjoint and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n = A$.

For all $n \in \mathbb{N}$, let $F_n = \bigcup_{i=1}^{n} E_n \in \mathcal{A}$. Note that $F_n \subset A$ so $A^c \subset F_n^c$. Hence, if $B \subset X$, then

$$\begin{aligned}\mu^*(B) &= \mu^*(B \cap F_n) + \mu^*(B \cap F_n^c). \\ &\geq \mu^*(B \cap F_n) + \mu^*(B \cap A^c)\end{aligned}$$

We want to show $\lim_{n \to \infty} \mu^*(B \cap F_n) = \mu^*(B \cap A)$. Indeed observe $F_n = E_n \cup F_{n-1}$ and $E_n \cap F_{n-1} = \emptyset$ for all $n \in \mathbb{N}$

So because $E_n \in \mathcal{A}$

$$\mu^*(B \cap F_n) = \mu^*(B \cap F_n \cap E_n) + \mu^*(B \cap F_n \cap E_n^c)$$

$$= \mu^*(B \cap E_n) + \mu^*(B \cap F_{n-1})$$

By repetition,

$$\mu^*(B \cap F_n) \geq \sum_{i=1}^{n} \mu^*(B \cap E_i)$$
$$\geq \sum_{i=1}^{n} \mu^*(B \cap E_i) + \mu^*(B \cap A^c)$$

By taking $n \to \infty$, we obtain

$$\mu^*(B) \geq \sum_{n=1}^{\infty} \mu^*(B \cap E_n) + \mu^*(B \cap A^c) \quad ***$$
$$\geq \mu^*\left(\bigcup_{n=1}^{\infty} (B \cap E_n)\right) + \mu^*(B \cap A^c)$$
$$= \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

Thus, $A \in \mathcal{A}$, proving (i)

To prove (ii), we check $\mu^*|_{\mathcal{A}}(\emptyset) = 0$, which is easy. Let $(E_n)_{n=1}^{\infty}$ be a collection in $\mathcal{A}$ be pairwise disjoint and $A = \bigcup_{n=1}^{\infty} E_n$. By *** with $B = A$,

$$\mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(E_k).$$

Finally, by subadditivity

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(E_k)$$

Thus, $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(E_k)$ proving (ii)

Definition: The λ^*-measurable sets, denoted by $\mathcal{M}(\mathbb{R})$ are called the Lebesgue measurable sets. The measure $\lambda = \lambda^*|_{\mathcal{M}(\mathbb{R})}$ is called the Lebesgue measure.

Definition: For $n \in \mathbb{N}$, the n-dimensional Lebesgue measure denoted λ_n is the restriction of λ_n^* to the λ_n^*-outer measurable sets.

Definition: A measure space $(X, \mathcal{A}, \mu)$ is said to be **Complete** if

(i) $A \in \mathcal{A}$

(ii) $B \subset A$ and $\mu(A) = 0$

Then $B \in \mathcal{A}$.

Corollary: If μ^* is an outer measure and $B \subset X$ is such that $\mu^*(B) = 0$, then B are outer measurable

Thus, μ^* is complete on the σ-algebra $\mathcal{A}$ of μ^*-measurable sets.

Proof: Let $C \subset X$. Then $C \cap B \subset B$ and so

$$0 \leq \mu^*(C \cap B) \leq \mu^*(B) = 0$$

so $\mu^*(C \cap B) = 0$. Thus,

$$\mu^*(C) \geq \mu^*(C \cap B) + 0 = \mu^*(C \cap B^c) + \mu^*(C \cap B)$$

Therefore, $B \in \mathcal{A}$

Extending Measures

Def: For a nonempty set X, an **algebra** on X is a set $\mathcal{A} \subset \mathcal{P}(X)$ such that

(i) $\emptyset, X \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

(iii) If $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$.

Remark: Algebras are closed under finite unions and intersection.

Example: Let $\mathcal{F} = \{(a,b] : a < b \in \mathbb{R} \cup \{-\infty\}\}$

$$\cup \{(a,\infty) : a \in \mathbb{R} \cup \{-\infty\}\}$$

Let $\mathcal{A}$ be all finite unions of all elements of $\mathcal{F}$. Then $\mathcal{A}$ is an algebra because the complement of $\mathcal{F}$ is a finite union of elements of $\mathcal{F}$. But $\mathcal{A}$ is not a σ-algebra. Note $(2n, 2n+1] \in \mathcal{A}$ for all $n \in \mathbb{N}$, but $\bigcup_{n=1}^{\infty} (2n, 2n+1] \notin \mathcal{A}$. So $\mathcal{A}$ is not a σ-algebra.

Def: Let $\mathcal{A}$ be an algebra on X. A **pre-measure** μ on $\mathcal{A}$ is a map $\mu : \mathcal{A} \to [0,\infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $(A_n)_{n=1}^{\infty}$ are a collection in $\mathcal{A}$ pairwise disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Remark: We can repeat proofs to show all premeasures

(i) $\mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu(A_n)$

$\mu^*(B) = \inf\{\sum_{n=1}^{\infty} \mu(A_n) \dots\}$

(ii) For all $A, B \in \mathcal{A}$ and $A \subset B$, $\mu(A) \leq \mu(B)$.

Theorem: (Carathéodory-Hahn Extension Theorem)

Let X be a nonempty set, $\mathcal{A}$ be an algebra on X, $\mu : \mathcal{A} \to [0,\infty]$ be a premeasure. Let μ^* be the outer measure associated to μ. Let $\mathcal{A}^*$ be the outer measurable sets.

(i) $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$

(ii) $\mathcal{A} \subset \mathcal{A}^*$

(iii) If μ is σ-finite in the sense there is $(X_n)_{n=1}^{\infty}$ is a collection in $\mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$, then if $\nu : \mathcal{A}^* \to [0,\infty]$ is a measure such that $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$, then

$$\nu|_{\mathcal{A}^*} = \mu^*|_{\mathcal{A}^*}.$$

Proof: We know $\mu^*(A) \leq \mu(A)$ by the infimum.

For the other inequality, let $(A_n)_{n=1}^{\infty}$ be a collection of $\mathcal{A}$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$. Let $B_1 = A_1$ and $B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ for $n \geq 2$, which are in $\mathcal{A}$. Then $(B_n)_{n=1}^{\infty}$ are pairwise disjoint and $A \subset \bigcup_{n=1}^{\infty} B_n$. Note $A_n \supset B_n \cap A$, so

$$\sum_{n=1}^{\infty} \mu(A_n) \geq \sum_{n=1}^{\infty} \mu(B_n \cap A) = \mu\left(\bigcup_{n=1}^{\infty} (B_n \cap A)\right) = \mu(A)$$

Since $B_n \cap A \in \mathcal{A}$ for all $n \in \mathbb{N}$, $\{B_n \cap A\}_{n=1}^{\infty}$ are pairwise disjoint, $\bigcup_{n=1}^{\infty} B_n \cap A = A$, so the second property of premeasures applies.