MATH 6280 **Assignment 2**

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Question 1. Extend Lusin's Theorem to measurable functions on $\mathbb{R}$. That is, show that if $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable then for all $\epsilon > 0$ there exists a closed subset $F \subseteq \mathbb{R}$ such that $\lambda(F^c) < \epsilon$ and $f|_F$ is continuous.

Question 2 (Convergence in Measure)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let f and $(f_n)_{n\geq1}$ be measurable functions. It is said that $(f_n)_{n\geq1}$ *converge in measure* (also know as *convergence in probability* when μ is a probability measure) to f if for all $\epsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

For this entire question, assume f, g, $(f_n)_{n\geq1}$, and $(g_n)_{n\geq1}$ are measurable.

a) Prove that if $\mu(X) < \infty$ and $(f_n)_{n\geq1}$ converges to f pointwise almost everywhere, then $(f_n)_{n\geq1}$ converges in measure to f.

b) Give an example of a measure μ and a sequence $(f_n)_{n\geq1}$ that converges to f pointwise almost everywhere, yet $(f_n)_{n\geq1}$ does not converges in measure to f. Prove your example satisfies the desired conditions.

c) Give an example of a μ such that $\mu(X) < \infty$ and a sequence $(f_n)_{n\geq1}$ that converges in measure to f, yet $(f_n)_{n\geq1}$ does not converges pointwise almost everywhere to f. Prove your example satisfies the desired conditions.

d) Prove that if $(f_n)_{n\geq1}$ converges in measure to f, then there exists a subsequence $(f_{k_n})_{n\geq1}$ that converges pointwise almost everywhere to f.

e) Prove that $(f_n)_{n\geq1}$ converges in measure to f if and only if every subsequence of $(f_n)_{n\geq1}$ has a further subsequence that converges in measure to f.

f) Prove that if $\mu(X) < \infty$, if $(f_n)_{n\geq1}$ converges in measure to f, and if $(g_n)_{n\geq1}$ converges in measure to g, then $(\alpha f_n + g_n)_{n\geq1}$ converges in measure to $\alpha f + g$ for all $\alpha \in \mathbb{C}$ and $(f_n g_n)_{n\geq1}$ converges in measure to fg.

g) Give an example of a measure μ and sequences $(f_n)_{n\geq1}$ and $(g_n)_{n\geq1}$ that converge in measure to f and g respectively, yet $(f_n g_n)_{n\geq1}$ does not converge in measure to fg. Prove your example satisfies the desired conditions.

Question 3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Borel* if $f^{-1}(U)$ is a Borel set for all open subsets $U \subseteq \mathbb{R}$.

a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is Borel.

b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, bijective, and has a continuous inverse (i.e. f is a homeomorphism), then $\{f(B) \mid B \in \mathfrak{B}(\mathbb{R})\} = \mathfrak{B}(\mathbb{R})$ (that is, f induces a bijective on the Borel sets).

c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ λ-almost everywhere.

Question 4 (The Cantor Ternary Function)**.** Given a sequence $\vec{a} = (a_n)_{n\geq1}$ of elements of $\{0, 1, 2\}$, define

$$K_{\vec{a}} = \begin{cases} N & \text{if } a_N = 1 \text{ and } a_k \neq 1 \text{ for all } k < N \\ \infty & \text{otherwise} \end{cases}$$

and define a sequence $\vec{b_{\vec{a}}} = (b_n)_{n\geq 1}$ of elements of $\{0,1\}$ by

$$b_n = \begin{cases} \frac{a_n}{2} & \text{if } n \leq K_{\vec{a}} \\ 1 & \text{if } n = K_{\vec{a}} \\ 0 & \text{otherwise} \end{cases}.$$

The *Cantor ternary function* is the function $f : [0,1] \to [0,1]$ defined as follow: if $x \in [0,1]$, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for a sequence $\vec{a} = (a_n)_{n\geq 1}$ of elements of $\{0,1,2\}$, and $\vec{b_{\vec{a}}} = (b_n)_{n\geq 1}$ is the sequence of elements of $\{0,1\}$ as defined above, then

$$f(x) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}.$$

(i.e. Write a ternary expansion of x. If N is the first index where a 1 occurs, replace each $\frac{0}{3^n}$ with $n < N$ with $\frac{0}{2^n}$, replace each $\frac{2}{3^n}$ with $n < N$ with $\frac{1}{2^n}$, replace $\frac{1}{3^N}$ with $\frac{1}{2^N}$, and change all terms of index greater than N to zero). As ternary expansions are not unique, one must verify that f is well-defined. Convince yourself that f is well-defined (you do not need to hand-in a proof).

a) Prove that f is a continuous, non-decreasing function on $[0,1]$ such that f is constant on each interval in $\mathcal{C}^c$ and $f(\mathcal{C}) = [0,1]$.

b) Define $\psi : [0,1] \to [0,2]$ by $\psi(x) = x + f(x)$ for all $x \in [0,1]$. Prove that ψ is a strictly increasing continuous function such that $\psi(\mathcal{C})$ is Lebesgue measurable with $\lambda(\psi(\mathcal{C})) > 0$ and that there exists a subset $B \subseteq \mathcal{C}$ such that $\psi(B)$ is not Lebesgue measurable.

c) Prove that there exists a subset $B \subseteq \mathcal{C}$ such that B is not Borel. (Therefore B is a Lebesgue measurable subset that is not Borel.)

d) Prove there exists Lebesgue measurable functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $g \circ h$ is not Lebesgue measurable.