MATH 6280: Measure Theory

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Preface

These are the first edition of these lecture notes for MATH 6280 (Measure Theory). Consequently, there may be several typographical errors, missing exposition on necessary background, and more advanced topics for which there will not be time in class to cover. Future iterations of these notes will hopefully be fairly self-contained provided one has the necessary background. If you come across any typos, errors, omissions, or unclear expositions, please feel free to contact me so that I may continually improve these notes.

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Chapter 1

Measure Spaces

As per its title, this course is dedicated to the study of the theory of measures. So what sort of course would this be if we did not define the main objective of study in the first chapter? After defining the notion of a measure, we will examine several examples and properties of measures that immediately follow from definitions. We will then turn to constructing measures from various notions of length and extending measure-like functions to actual measures. This will lead us to looking at measures with important analytical properties including the Lebesgue-Stieltjes measures and metric outer measures. By using metric outer measures, we will obtain the notion of Hausdorff dimension for subsets of metric spaces.

1.1 Measure Spaces

Definition 1.1.1

Let X be a nonempty set. A σ-algebra on X is a subset $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

(i) $\emptyset, X \in \mathcal{A}$, i.e. we measure the empty event and the full event.

(ii) If $A \in \mathcal{A}$, then $A^c = X \setminus A \in \mathcal{A}$, i.e. we measure the complement of an event.

(iii) If $(A_n)_{n=1}^{\infty}$ is a countable collection of events in $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a *measurable space*, and the elements of $\mathcal{A}$ are called *measurable sets*.

Remark 1.1.2

One may ask why we only ask for countable unions of measurable sets to be measurable. One answer for this comes with the definition of a measure in that we want to have additivity over disjoint unions and adding over an uncountable set only works if only a countable number of elements are nonzero. Another reason is that restricting to countable collections is quite powerful as we will see in this course.

Remark 1.1.3

One may also ask why we have not required that the intersection of countable collection of measurable sets is measurable. The reason is that countable intersections come for free. Indeed, if $(X, \mathcal{A})$ is a measurable space and $(A_n)_{n=1}^{\infty}$ is a countable collection of events in $\mathcal{A}$, then

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{A}$$

as complements and countable unions of elements of $\mathcal{A}$ are elements of $\mathcal{A}$. Furthermore, by using $\emptyset$ in unions and X in intersections, clearly, a finite union or intersection of elements of $\mathcal{A}$ is an element of $\mathcal{A}$.

Example 1.1.4

Let X be a nonempty set.

(α) $(X, \mathcal{P}(X))$ is a measurable space and $(X, \{\emptyset, X\})$ is a measurable space.

(β) Let

$$\mathcal{A} = \{A \subseteq X : A \text{ is countable or } A^c \text{ is countable}\}$$

Then $(X, \mathcal{A})$ is a measurable space.

Moreover, if one has a collection of σ-algebras on a set X, there are ways of constructing new σ-algebras. In particular, it is elementary to verify the following using set properties and Definition 1.1.1.

Lemma 1.1.5

Let X be a nonempty set and let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of σ-algebras of X over index set I. Then

$$\bigcap_{i \in I} \mathcal{A}_i$$

is a σ-algebra of X.

Remark 1.1.6

Using Lemma 1.1.5, we can construct the smallest σ-algebra containing a collection of subsets. Indeed, let X be a nonempty set and let $A \subseteq \mathcal{P}(X)$. Define

$$I = \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ such that } A \subseteq \mathcal{A}\}$$

Clearly, $\mathcal{P}(X) \in I$, so I is nonempty. Hence, Lemma 1.1.5 implies that

$$\sigma(A) = \bigcap_{\mathcal{A} \in I} \mathcal{A}$$

is a σ-algebra. Since clearly $A \subseteq \sigma(A)$ by construction, $\sigma(A)$ is the smallest σ-algebra of X contains A. As such, $\sigma(A)$ is called the *σ-algebra generated by A.*

Definition 1.1.7

Let (X, d) be a metric space. The σ-algebra generated by the open subsets of X is called the *Borel σ-algebra* and is denoted $\mathcal{B}(X)$. In particular, $\mathcal{B}(X)$ is also the σ-algebra generated by the closed subsets of X as open and closed sets are complements of each other and as σ-algebras are closed under complements. Elements of $\mathcal{B}(X)$ are called *Borel sets.*

Remark 1.1.8

In terms of the Borel subsets of $\mathbb{R}$, the sets

$$\begin{aligned} &\{(a, b) : a < b \in \mathbb{R}\} \\ &\{(a, b] : a < b \in \mathbb{R}\} \\ &\{[a, b) : a < b \in \mathbb{R}\} \\ &\{[a, b] : a < b \in \mathbb{R}\} \\ &\{(-\infty, b) : b \in \mathbb{R}\} \\ &\{(-\infty, b] : b \in \mathbb{R}\} \\ &\{(a, \infty) : a \in \mathbb{R}\} \\ &\{[a, \infty) : a \in \mathbb{R}\} \end{aligned}$$

all can be shown to generate $\mathcal{B}(\mathbb{R})$ via unions, intersections, and complements. To show this, we verify the following

(i) $\mathcal{B}(\mathbb{R})$ contains each of these sets.

(ii) Any σ-algebra containing one of these sets contains all open intervals and thus all open sets by the fact that every open set is a countable union of open intervals.

We note it is possible to show that $|\mathcal{B}(\mathbb{R})| = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$.

Using σ-algebras, we may now define the central object of study in this course.

Definition 1.1.9

Let $(X, \mathcal{A})$ be a measurable space. A (countably additive, positive) *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $(A_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

where the sum is infinite if one of the elements is ∞ or if the sum diverges.

The triple $(X, \mathcal{A}, \mu)$ is a *measure space* and given an element $A \in \mathcal{A}$, $\mu(A)$ is called the *μ-measure* of A.

Remark 1.1.10

Notice if $(X, \mathcal{A}, \mu)$ is a measure space and $A_1, ..., A_n$ are pairwise disjoint subsets of $\mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

by using the properties of a measure with $A_i = \emptyset$ for all $i > n$.

Before we get too deep into the study of properties of measures, let us examine some common measures which are easy to define.

Example 1.1.11

(α) Let X be a nonempty set and let $x \in X$. The *point-mass measure* at x is the measure δ_x on $(X, \mathcal{P}(X))$ defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(β) Let X be a nonempty set. The *counting measure on X* is the measure μ on $(X, \mathcal{P}(X))$ defined by

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

Example 1.1.12

A function $\mu : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ of elements of $[0,\infty]$ such that

$$\mu(A) = \sum_{n\in A} a_n$$

for all $A \subseteq \mathbb{N}$.
To see this, note that if μ has the described form, then μ is a measure. Conversely, suppose μ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. As for each $n \in \mathbb{N}$, the set $\{n\}$ is measurable, for each $n \in \mathbb{N}$, and we may define

$$a_n = \mu(\{n\}) \in [0,\infty]$$

We claim that

$$\mu(A) = \sum_{n\in A} a_n$$

for all $A \subseteq \mathbb{N}$. Indeed, let $A \subseteq \mathbb{N}$. Then as A is countable and

$$A = \bigcup_{n\in A} \{n\}$$

we obtain by the properties of measure that

$$\mu(A) = \mu\left(\sum_{n\in A} \{n\}\right) = \sum_{n\in A} \mu(\{n\}) = \sum_{n\in A} a_n$$

as desired.

Note that measure in Example 1.1.12 can be constructed using Example 1.1.11 (α) and the following technique (which will be of use to us later).

Example 1.1.13

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $(A_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ and let $(a_n)_{n=1}^{\infty} \in [0,\infty]$. Define $\nu : \mathcal{A} \to [0,\infty]$ by

$$\nu(A) = \sum_{n=1}^{\infty} a_n \mu(A_n \cap A)$$

for all $A \in \mathcal{A}$ where the sum equates to ∞ if the sum diverges or one of the terms is ∞ and

$$a \times \infty = \begin{cases} 0 & \text{if } a = 0 \\ \infty & \text{if } a > 0 \end{cases}$$

Then ν is a measure on $(X, \mathcal{A})$. To see this, we clearly note that $\nu(\emptyset) = 0$. Furthermore, if $(B_m)_{m=1}^{\infty} \subseteq \mathcal{A}$ are pairwise disjoint, then $\{A_n \cap B_m\}_{m=1}^{\infty}$ are pairwise disjoint for all $n \in \mathbb{N}$, and thus, as μ is a measure,

$$\begin{aligned}
\nu\left(\bigcup_{m=1}^{\infty} B_m\right) &= \sum_{n=1}^{\infty} a_n \mu\left(A_n \cap \left(\bigcup_{m=1}^{\infty} B_m\right)\right) \\
&= \sum_{n=1}^{\infty} a_n \mu\left(\bigcup_{m=1}^{\infty} (A_n \cap B_m)\right) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \mu(A_n \cap B_m) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \mu(A_n \cap B_m) \\
&= \sum_{m=1}^{\infty} \nu(B_m)
\end{aligned}$$

Hence, ν is a measure as desired.

Although we may define more measures, we turn our attention to properties of measures immediately implied that Definition 1.1.9 and set manipulations. We begin with the following.

Proposition 1.1.14

If $E, F \in \mathcal{A}$ such that $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

Proof.

Indeed, if $E \subseteq F$, we have $F \setminus E \in \mathcal{A}$ is disjoint from E and $F = E \cup (F \setminus E)$, and so

$$\mu(E) \leq \mu(E) + \mu(F \setminus E) = \mu(E \cup (F \setminus E)) = \mu(F)$$

■

In particular, if $\mathbb{A}$ is ordered by inclusion, then μ is monotone with respect to this inclusion. This implies if $\mu(X) < \infty$, then $\mu(A) < \infty$ for all $A \in \mathcal{A}$. Moreover, notice in the above computation that if $\mu(E) < \infty$, then we may subtract $\mu(E)$ from both sides in order to obtain that

$$\mu(F \setminus E) = \mu(F) - \mu(E)$$

Note that the above formula need not hold if $\mu(E) = \infty$ as do not know how to define $\infty - \infty$.

Proposition 1.1.15

If $A, B \in \mathcal{A}$ and $\mu(A \cap B) < \infty$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

Proof.

This is easy; we have

$$\begin{aligned}\mu(A \cup B) &= \mu(A \cup (B \setminus (B \cap A))) \\ &= \mu(A) + \mu(B \setminus (B \cap A)) \\ &= \mu(A) + \mu(B) - \mu(A \cap B)\end{aligned}$$

which is a formula that may appear familiar in the context of probability. ∎

Of course, if a measure is going to represent the probability of an event occurring, we must dictate the probability of all possible events is one. As such, when discussing probability, we use the following terminology.

Definition 1.1.16

Let $(X, \mathcal{A}, \mu)$ be a measure space. It is said that $(X, \mathcal{A}, \mu)$ is a *probability space* and μ is a *probability space* if $\mu(X) = 1$. In this case, X is called the *sample space*, elements of $\mathcal{A}$ are called *events*, and given $A \in \mathcal{A}$, $P(A) = \mu(A)$ denotes the probability that the event A occurs.

Remark 1.1.17

It is not difficult to see that a probability space is the correct notion in order to study probability theory. Indeed, the probability of the entire space is one and whenever A and B are disjoint sets, which is the notion of independent events, then the probability of $A \cup B$ is the sum of the probability of A and the probability of B. Furthermore, Proposition 1.1.14 is precisely the formula for the probability of $A \cup B$ when A and B are not disjoint; that is, the formula for the probability of the union of two not necessarily independent events. Of course, when studying probability, one may only have finite additivity instead of countable additivity. As will be seen in later sections, it is not difficult to extend finitely additive measures to countably additive measures, which is far more desirable in our analytic realm.

Of course, requiring the measure of the entire space to be one is a specific property of a measure we may wish to study. The following generalizations of probability measures are vital for the course.

Definition 1.1.18

A measure μ on a measurable space $(X, \mathcal{A})$ is said to be

(i) *finite* if $\mu(X) < \infty$ (and thus, $\mu(A) < \infty$ for all $A \in \mathcal{A}$ by monotonicity).

(ii) *σ-finite* if there exists a collection $(A_n)_{n=1}^\infty \subseteq \mathcal{A}$ such that $X = \bigcup_{n=1}^\infty A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

In most cases, if one can prove a property for any finite measure, one can extend the result to all σ-finite measures using analytic techniques. This is often done using the following additional partition decompositions of a σ-finite measure space.

Remark 1.1.19

Assume μ is a σ-finite measure on $(X, \mathcal{A})$. Thus, there exists a collection $(A_n)_{n=1}^{\infty}$ of $\mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathcal{N}$. Let $B_1 = C_1 = A_1$, and for each $n \geq 2$, let

$$B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} B_k \right) \quad C_n = \bigcup_{k=1}^{n} A_k$$

Then $(B_n)_{n=1}^{\infty}$ are pairwise disjoint elements of $\mathcal{A}$ are such that $X = \bigcup_{n=1}^{\infty} B_n$ and $\mu(B_n) \leq \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Similarly, $(C_n)_{n=1}^{\infty}$ are elements of $\mathcal{A}$ are such that $X = \bigcup_{n=1}^{\infty} C_n$, $C_n \subseteq C_{n+1}$ for all $n \in \mathbb{N}$, and $\mu(C_n) < \infty$ for all $n \in \mathbb{N}$. The reason $\mu(C_n) < \infty$ can be seen via the following result as $\sum_{i=1}^{n} \mu(A_i) < \infty$.

Proposition 1.1.20: Subadditivity of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^{\infty}$ be in $\mathcal{A}$. Then

$$\mu\left(\sum_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Proof.

Let $E_1 = A_1$, and for each $n \in \mathbb{N}_{\geq 2}$, denote

$$E_n = A_n \setminus \left(\bigcup_{i=1}^{n} A_i \right)$$

Since $(A_n)_{n=1}^{\infty}$ are in $\mathcal{A}$, by the properties of σ-algebra we have that $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Furthermore, it is clear that $E_n \cap E_m = \emptyset$ if $n \neq m$, $E_n \subseteq A_n$ for all $n \in \mathbb{N}$, and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$$

Hence, by the definition and monotonicity of measures (Proposition 1.1.14), we obtain that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) = \mu\left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

as desired. ■

As seen above, being able to replace our measurable sets with disjoint measurable is a very useful technique. In particular, the same idea is helpful in proving the following theorem.

Theorem 1.1.21: Monotone Convergence Theorem for Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^{\infty}$ be a collection of $\mathcal{A}$. Then the following hold.

(i) If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(ii) If $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$ with $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

Proof.

To see that the first assertion holds, let $A_0 = \emptyset$ for conventional simplicity, and for each $n \in \mathbb{N}$, define

$$B_n = A_n \setminus A_{n-1}$$

then $(B_n)_{n=1}^{\infty}$ is a collection of pairwise disjoint elements of $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ and $\bigcup_{i=1}^{n} B_i = A_n$ for all $n \in \mathbb{N}$. Hence,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n)$$

proving (i).

To see that the second assertion holds, let $B_n = A_1 \setminus A_n$ for all $n \in \mathbb{N}$, then $(B_n)_{n=1}^{\infty}$ is a collection of elements of $\mathcal{A}$ with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Thus, since

$$\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)$$

from (i),

$$\mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n)$$

Since $\mu(A_1) < \infty$, then using the part after Proposition 1.1.14, $\mu(A_1 \setminus E) = \mu(A_1) - \mu(E)$ for all $E \in \mathcal{A}$ with $E \subseteq A_1$. Hence,

$$\begin{aligned}\mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) &= \mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right)\\ &= \lim_{n\to\infty} (A_1 \setminus A_n)\\ &= \lim_{n\to\infty} (\mu(A_1) - \mu(A_n))\end{aligned}$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$

By subtracting $\mu(A_1) < \infty$ from both sides, we obtain the desired equality, proving (ii). ■

Exercise 1.1.22: Assignment 1, Question 1

Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x])$$

for all $x \in \mathbb{R}$. The function F is called the *cumulative distribution function* of μ.

(i) Show that if μ is finite, then F is non-decreasing, right continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$.

(ii) Show that if μ is finite and μ has no atoms (that is, $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$), then F is continuous.

Solution.

(i) Assume that μ is finite, i.e. $\mu(\mathbb{R}) < \infty$ and for all $I \in \mathcal{B}(\mathbb{R})$, $\mu(I) < \infty$, where I is an interval on $\mathbb{R}$.

- To see that F is nondecreasing, let $x < y \in \mathbb{R}$ be arbitrary. Then $(-\infty, x] \subseteq (-\infty, y]$, and thus, by the monotonicity of measures, this implies that $\mu((-\infty, x]) \leq \mu((-\infty, y])$ and thus, $F(x) \leq F(y)$ as desired.

- To see that F is right-continuous, let $(x_n)_{n=1}^{\infty}$ be a decreasing sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$ be such that $x_n \to x$. For each $n \in \mathbb{N}$, denote $I_n = (-\infty, x_n]$. Then since $(x_n)_{n=1}^{\infty}$ is a decreasing sequence, we have $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, which then implies that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, and furthermore, since μ is finite, $\mu(I_1) < \infty$. Put $I = (-\infty, x]$, and we claim that $\bigcap_{n=1}^{\infty} I_n = I$. Given the claim, by the Monotone Convergence Theorem for Measures (Theorem 1.1.21), we have

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu(I_n) = \mu\left(\bigcap_{n=1}^{\infty} I_n\right) = \mu(I) = F(x)$$

  So we have shown that F is right-continuous. To prove the claim, note that $y \in \bigcap_{n=1}^{\infty} I_n$ if and only if for every $n \in \mathbb{N}$, $y \leq x_n$ if and only if as $n \to \infty$, $y \leq x$ if and only if $y \in I$.

- To see that $\lim_{x\to-\infty} F(x) = 0$, let $(x_n)_{n=1}^{\infty}$ be a decreasing sequence in $\mathbb{R}$ such that $x_n \to -\infty$. Denoting $I_n = (-\infty, x_n]$ for each $n \in \mathbb{N}$ again, we claim that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

  Given the claim and using the Monotone Convergence Theorem for Measures (Theorem

1.1.21) this would imply that

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \mu((-\infty, x]) = \mu\left(\bigcap_{n=1}^{\infty}(-\infty, x_n]\right) = \mu(\emptyset) = 0$$

Therefore, $\lim_{x\to-\infty} F(x) = 0$. Now to prove the claim, assume for a contradiction that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Then there exists an element $y \in \bigcap_{n=1}^{\infty} I_n$. Then for every $n \in \mathbb{N}$, we have that $y \in I_n$, and thus, for every $n \in \mathbb{N}$, $y \leq x_n$. But since $x_n \to -\infty$, there exists an $N \in \mathbb{N}$ such that $x_N < y$ for all $n \geq N$, implying that $y \notin (-\infty, x_N]$, which is absurd. Hence, we must have $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

- To see that $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$, let $(x_n)_{n=1}^{\infty}$ be a increasing sequence in $\mathbb{R}$ such that $x_n \to \infty$. Denoting $I_n = (-\infty, x_n]$ for each $n \in \mathbb{N}$, note that since $(x_n)_{n=1}^{\infty}$ is an increasing sequence, we have $x_{n+1} \geq x_n$ for every $n \in \mathbb{N}$, which then implies that $I_n \subseteq I_{n+1}$. Now, we claim that
$$\bigcup_{n=1}^{\infty} I_n = \mathbb{R} = (-\infty, \infty)$$
Given the claim, by the Monotone Convergence Theorem for Measures (Theorem 1.1.21)
$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \mu(I_n) = \mu\left(\bigcup_{n=1}^{\infty} I_n\right) = \mu(\mathbb{R})$$
and therefore, $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$. To prove the claim, we will show by double inclusion. First, let $y \in \bigcup_{n=1}^{\infty} I_n$. Then there exists an $n_0 \in \mathbb{N}$ such that $y \leq x_{n_0}$. In particular, since $(x_n)_{n=1}^{\infty}$ is an increasing sequence, for each $n \in \mathbb{N}$, we have $x_n < \infty$, so $y < \infty$, and thus, $(-\infty, y] \subseteq (-\infty, \infty) = \mathbb{R}$, and so $y \in \mathbb{R}$. For the other inclusion, let $y \in \mathbb{R}$ be arbitrary. Then for any $\varepsilon > 0$, there exists an $M > 0$ such that $y < M - \varepsilon$. By assumption, since $x_n \to \infty$, then there exists an $N \in \mathbb{N}$ such that $y < M - \varepsilon \leq x_n$, for all $n \geq N$, so $y \leq x_N$, implying that $(-\infty, y] \subseteq I_N$, and thus, $y \in \bigcup_{n=1}^{\infty} I_n$. Therefore, we must have $\bigcup_{n=1}^{\infty} I_n = \mathbb{R}$.

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1.2 The Carathéodory Method

Based on the above notations, it is very natural to ask whether there exists a measure λ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ that emulates the length of a set. In particular, we desire a measure to have some very natural properties, such as

(i) If I is an interval, then $\lambda(I)$ is the length of I.

(ii) If $A \in \mathcal{P}(\mathbb{R})$, $x \in \mathbb{R}$, and $x + A = \{x + a : a \in A\}$, then $\lambda(x + A) = \lambda(A)$; that is, λ is translation invariant.

However, it turns out that no such measure exists. This can be seen via the following example.

Example 1.2.1

Assume for a contradiction that λ is a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ with the above two properties. Define a relation "$\sim$" on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. It is easy to see that "$\sim$" is an equivalence relation on $\mathbb{R}$.
We claim that every element of $\mathbb{R}$ is $\sim$-equivalent to some element in $[0,1)$. Indeed, if $x \in \mathbb{R}$, then x is the sum of its integer part $\lfloor x \rfloor$ and its fractional part $\{x\}$. Since $x - \{x\} = \lfloor x \rfloor \in \mathbb{Q}$, we obtain that $x \sim \{x\}$. Therefore, since $\{x\} \in [0,1)$, x is $\sim$-equivalent to some element in $[0,1)$.
Consequently, every equivalence class under $\sim$ has an element in $[0,1)$. Let $A \subseteq [0,1)$ be a set that contains precisely one element from each equivalence class of $\sim$. Note the existence of A follows from the Axiom of Choice.
Since $\mathbb{Q}$ is countable, we enumerate $\mathbb{Q} \cap [0,1)$ as

$$\mathbb{Q} \cap [0,1) = \{r_n : n \in \mathbb{N}\}$$

For each $n \in \mathbb{N}$, denote

$$A_n = \{x \in [0,1) : x \in r_n + A \text{ or } x + 1 \in r_n + A\}$$

that is, A_n is $r_n + A$ mod 1.
We claim that $(A_n)_{n=1}^{\infty}$ are disjoint with union $[0,1)$. To see this, note that if $x \in [0,1)$, then there exists a unique $y \in A \subseteq [0,1)$ such that $x \sim y$. Thus, $x - y \in \mathbb{Q} \cap (-1,1)$. If $x - y \in \mathbb{Q} \cap [0,1)$, then $x - y = r_n$ for some $n \in \mathbb{N}$, and thus, $x = r_n + y \in A_n$. Otherwise, if $x - y \in \mathbb{Q} \cap (-1,0)$, then $(x+1) - y \in (0,1)$. Thus, $(x+1) - y = r_n$ for some $n \in \mathbb{N}$, and thus, $x = r_n + y - 1 \in A$. Hence,

$$[0,1) = \bigcup_{n=1}^{\infty} A_n$$

To see that $(A_n)_{n=1}^{\infty}$ are pairwise disjoint, suppose $x \in A_n \cap A_m$ for some $n, m \in \mathbb{N}$. By definition, there exists $y, z \in A$ and $k, l \in \{0,1\}$ such that $x + k = r_n + y$ and $x + l = r_m + z$. Therefore, $y - z = r_n - r_m + k - l \in \mathbb{Q}$ so $y \sim z$. Hence, $y = z$ as A contains exactly one element from each equivalence class of $\sim$. Thus, $0 = r_n - r_m + k - l$. Since $k - l \in \{-1,0,1\}$ and $r_n - r_m \in (-1,1)$, $0 = r_n - r_m + k - l$ can only occur when $n = m$, in which case $n = m$. Thus, $(A_n)_{n=1}^{\infty}$ is a collection of pairwise disjoint sets whose union is $[0,1)$.
For each $n \in \mathbb{N}$, let $B_{n,1} = (r_n + A) \cap [0,1)$ and $B_{n,2} = -1 + ((r_n + A) \cap [1,2))$. Clearly, $A_n = B_{n,1} \cup B_{n,2}$ since $r_n + A \subseteq [0,2)$ for all $n \in \mathbb{N}$.
We claim that $B_{n,1} \cap B_{n,2} = \emptyset$. To see this, assume for a contradiction that $b \in B_{n,1} \cap B_{n,2}$. By definition, there exists $x, y \in A$ such that $r_n + x \in [0,1)$, $r_n + y \in [1,2)$, and $b = r_n + x = -1 + r_n + y$. Clearly, $r_n + x \in [0,1)$ and $r_n + y \in [1,2)$ imply that $x \neq y$, whereas we have $x - y = -1 \in \mathbb{Q}$, so $x \sim y$. Therefore, as A contains exactly one element from each equivalence class, we have obtained a contradiction. Hence, $B_{n,1} \cap B_{n,2} = \emptyset$. To our

contradiction, note that

$$\begin{aligned} 1 = \lambda([0,1)) = \lambda\left(\bigcup_{n=1}^\infty A_n\right) &= \sum_{n=1}^\infty \lambda(A_n) = \sum_{n=1}^\infty \lambda(B_{n,1} \cup B_{n,2}) = \sum_{n=1}^\infty [\lambda(B_{n,1}) + \lambda(B_{n,2})] \\ &= \sum_{n=1}^\infty \lambda((r_n + A) \cap [0,1)) + \lambda(((r_n + A) \cap [1,2)) = \sum_{n=1}^\infty \lambda((r_n + A) \cap [0,2)) \\ &= \sum_{n=1}^\infty \lambda(r_n + A) = \sum_{n=1}^\infty \lambda(A) \end{aligned}$$

This yields our contradiction since $\lambda(A) \in [0,\infty]$, yet no number in $[0,\infty]$ when summed an infinite number of times produces 1. Thus, we have obtained a contradiction to the existence of such a λ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$.

The above example illustrates that $\mathcal{P}(\mathbb{R})$ is too large; that is, there are too many sets in $\mathcal{P}(\mathbb{R})$ to define such a measure in a consistent way. The set A in Example 1.2.1 is one of these sets.

To solve this problem, our answer is to reduce the number of sets we consider measurable. Of course, if we would like to do analysis, we need the open sets to be measurable and thus, we require all Borel sets to be measurable. However, the problem still remains, "How do we construct our measure and determine which sets are measurable?"

To answer the problem, we will invoke a technique called Carathéodory's method. The idea of this method is, given a set X, to define a function on the power set of X, that is almost a measure, but has weaker properties. We will then define sets that behave 'nicely' and show these nice sets form a σ-algebra. Finally, we will demonstrate that restricting the function to these nice sets does indeed produce a measure space that hopefully contains some nice measurable sets.

To begin, we define the 'function' that behaves almost like a measure.

Definition 1.2.2

Let X be a nonempty set. A function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ is said to be an *outer measure* if

(i) $\mu^*(\emptyset) = 0$.

(ii) For all $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, $\mu^*(A) \leq \mu^*(B)$.

(iii) If $(A_n)_{n=1}^\infty$ is a collection in $\mathcal{P}(X)$, then $\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu^*(A_n)$.

Notice that every measure is an outer measure by the results of Section 1.1 whereas an outer measure need not be a measure as it is not necessary that equality occur in the third property of Definition 1.2.2 when the collection $(A_n)_{n=1}^\infty$ are pairwise disjoint. Of course, it is a priori possible that every outer measure is automatically a measure. For an example to show this is not the case, we will need to construct some outer measures. The most natural way to do so is the following which attempts to assign certain sets a specific value.

Definition 1.2.3

Let X be a nonempty set, let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X such that $\emptyset, X \in \mathcal{F}$, and let $\ell : \mathcal{F} \to [0, \infty]$ be any function such that $\ell(\emptyset) = 0$. The *outer measure associated to* ℓ is the function $\mu_\ell^* : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\mu_\ell^*(A) = \inf\left\{\sum_{n=1}^{\infty} \ell(A_n) : (A_n)_{n=1}^{\infty} \text{ is a collection of } \mathcal{F} \text{ such that } A \subseteq \bigcup_{n=1}^{\infty} A_n\right\}$$

for all $A \subseteq X$, where $\inf\{\infty\} = \infty$.

Of course, we need to prove that the outer measure associated to ℓ is actually an outer measure.

Proposition 1.2.4

Let X be a nonempty set, $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X such that $\emptyset, X \in \mathcal{F}$, and $\ell : \mathcal{F} \to [0, \infty]$ be any function such that $\ell(\emptyset) = 0$.

(i) The outer measure associated to ℓ is an outer measure μ_ℓ^* such that $\mu_\ell^*(A) \leq \ell(A)$ for all $A \subseteq X$.

(ii) If $\nu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure such that $\nu^*(A) \leq \ell(A)$ for all $A \subseteq x$, then $\nu^*(A) \leq \mu_\ell^*(A)$ for all $A \subseteq X$. Hence, μ_ℓ^* is the largest outer measure bounded above by ℓ.

Proof.

First notice that since $X \in \mathcal{F}$, that the set whose infimum defines $\mu_\ell^*(A)$ is nonempty for all $A \subseteq X$. This fact will be used throughout the proof.

Clearly, $\mu_\ell^* : \mathcal{P}(X) \to [0, \infty]$. Furthermore, if $\emptyset \in \mathcal{F}$ and $\ell(\emptyset) = 0$, we clearly see that $\mu_\ell^*(\emptyset) = 0$ as $\{\emptyset\}_{n=1}^{\infty}$ is a cover of $\emptyset$. Moreover, if $A \subseteq X \subseteq X$, it is easy to see that $\mu_\ell^*(A) \leq \mu_\ell^*(B)$ since the infimum in the definition of $\mu_\ell^*(A)$ is taken over a larger collection of sets than the infimum in the definition of $\mu_\ell^*(B)$.

Finally, to check the subadditivity of μ_ℓ^*, let $(A_n)_{n=1}^{\infty}$ be a collection of members of $\mathcal{P}(X)$ and let $A = \bigcup_{n=1}^{\infty} A_n$. Let $\varepsilon > 0$ be arbitrary. By definition, for every $n \in \mathbb{N}$, there exists a collection $(A_{n,k})_{k=1}^{\infty}$ of $\mathcal{F}$ such that $A_n \subseteq \bigcup_{k=1}^{\infty} A_{n,k}$ and

$$\sum_{k=1}^{\infty} \ell(A_{n,k}) \leq \mu_\ell^*(A_n) + \frac{\varepsilon}{2^n}$$

It is easy to note that $(A_{n,k})_{k=1}^{\infty}$ is countable subset of $\mathcal{F}$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{n,k}$$

Thus,

$$\mu_\ell^*(A) \leq \sum_{n=1}^\infty \sum_{k=1}^\infty \ell(A_{n,k}) \leq \sum_{n=1}^\infty \mu_\ell^*(A_n) + \frac{\varepsilon}{2^n} = \varepsilon + \sum_{n=1}^\infty \mu_\ell^*(A_n)$$

Therefore, since $\varepsilon > 0$ was arbitrary, we obtain that

$$\mu_\ell^*(A) \leq \sum_{n=1}^\infty \mu_\ell^*(A_n)$$

Hence, μ_ℓ^* is an outer measure.

Now to prove the second assertion, if $\nu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure such that $\nu^*(A) \leq \ell(A)$ for all $A \subseteq X$, then for each $A \subseteq X$ and each collection $(A_n)_{n=1}^\infty$ of $\mathcal{F}$ such that $A \subseteq \bigcup_{n=1}^\infty A_n$, we have

$$\nu^*(A) \leq \nu^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \nu^*(A_n) \leq \sum_{n=1}^\infty \ell(A_n)$$

by properties of an outer measure and the assumptions on ν^*. Therefore, since $\mu_\ell^*(A)$ is the infimum of $\sum_{n=1}^\infty \ell(A_n)$ over all collections $(A_n)_{n=1}^\infty$ of $\mathcal{F}$ such that $A \subseteq \bigcup_{n=1}^\infty A_n$, we obtain $\nu^*(A) \leq \mu_\ell^*(A)$ for all $A \subseteq X$. ■

The outer measure one uses on $\mathbb{R}$ to define length is the following.

Definition 1.2.5

Let $I \subseteq \mathbb{R}$ be an interval, let $\ell(I)$ denote the length of I. The *Lebesgue outer measure*, denoted by λ^* is the outer measure associated to ℓ restricted to the open intervals. In particular, $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is defined by

$$\lambda^*(A) = \inf \left\{ \sum_{n=1}^\infty \ell(I_n) : (I_n)_{n=1}^\infty \text{ are open intervals such that } A \subseteq \bigcup_{n=1}^\infty I_n \right\}$$

for all $A \subseteq \mathbb{R}$.

Clearly, we can extend the Lebesgue outer measure on $\mathbb{R}$ to measures on $\mathbb{R}^n$ to measure areas and volumes.

Definition 1.2.6

For $n \in \mathbb{N}$, let

$$\mathcal{F} = \left\{ \prod_{i=1}^n (a_i, b_i) \subseteq \mathcal{R}^n : a_i < b_i \in \mathbb{R} \cup \{\pm\infty\} \right\}$$

and define $\ell : \mathcal{F} \to [0, \infty]$ by

$$\ell\left(\prod_{i=1}^n (a_i, b_i)\right) = \prod_{i=1}^n (b_i - a_i)$$

where the product is zero if $a_i = b_i$ for some i and otherwise, if $b_i = \infty$ or $a = -\infty$ for some i, then the product is infinite. The n-dimensional Lebesgue outer measure, denoted by λ_n^* is the outer measure on $\mathbb{R}^n$ associated to ℓ.

With the above notion of outer measures, we desire to construct measures from outer measures. To do so, we need to define a σ-algebra of sets for which the restriction of our outer measure produces a measure. These sets are described as follows.

Definition 1.2.7

Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X. A subset $A \subseteq X$ is said to be μ^**-measurable*, or *outer measurable* if for every $B \in \mathcal{P}(X)$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

Remark 1.2.8

The reason we are interested in outer measurable sets is that if $A \subseteq X$ has the property that

$$\mu^*(B) \neq \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

for some $B \in \mathcal{P}(X)$, it is likely we do not want to consider A to be measurable as it causes μ^* to fail to be additive on specific disjoint sets if B was also measurable.

Remark 1.2.9

Notice by the properties of an outer measure that if $A, B \in \mathcal{P}(X)$, then

$$\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

Thus, to show that A is measurable, it suffices to show that

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

for all $B \in \mathcal{P}(X)$, Furthermore, clearly, it suffices to restrict our attention to B such that $\mu^*(B) < \infty$.

The Carathéodory Method of constructing a measure is as follows: construct an outer measure μ^* and apply the following to get a σ-algebra $\mathcal{A}$ such that $\mu^*|_{\mathcal{A}}$ is a measure.

Theorem 1.2.10

Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X. The set $\mathcal{A}$ of all outer measurable sets is a σ-algebra. Furthermore, $\mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.

Proof.

To see that $\mathcal{A}$ is a σ-algebra, first notice that for all $B \in \mathcal{P}(X)$, that

$$\mu^*(B) = \mu^*(B) + 0 = \mu^*(B \cap \emptyset^c) + \mu^*(B \cap \emptyset)$$

Hence, $\emptyset \in \mathcal{A}$. Furthermore clearly if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ due to the symmetry in the definition of an outer measurable set. Hence, $\mathcal{A}$ is closed under compliments and $X \in \mathcal{A}$.

In order to demonstrate that $\mathcal{A}$ is closed under countable unions, let us verify that $\mathcal{A}$ is closed under finite unions. To verify that $\mathcal{A}$ is closed under finite unions, it suffices to verify that if $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$ as we can then apply recursion to take arbitrary finite unions of element of $\mathcal{A}$. Thus, let $A_1, A_2 \in \mathcal{A}$ be arbitrary. To see that $A_1 \cup A_2 \in \mathcal{A}$, let $B \subseteq X$ be arbitrary. Since A_1 is outer measurable, we know that

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c)$$

and since A_2 is outer measurable, we know that

$$\mu^*(B \cap A_1^c) = \mu^*((B \cap A_1^c) \cap A_2) + \mu^*((B \cap A_1^c) \cap A_2^c)$$

Hence,

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c)$$

However, since

$$B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap (A_2 \cap A_1^c))$$

it follows from subadditivity that

$$\begin{aligned} \mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\ &\geq \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap A_1^c \cap A_2^c) \\ &= \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap (A_1 \cup A_2)^c) \end{aligned}$$

Therefore, since $B \subseteq X$ was arbitrary, we obtain that $A_1 \cup A_2 \in \mathcal{A}$, so $\mathcal{A}$ is closed under finite unions.

Since $\mathcal{A}$ is also closed under complements, we also obtain that $\mathcal{A}$ is closed under finite intersections using a similar argument to Remark 1.1.3.

To see that $\mathcal{A}$ is closed under countable unions, let $(A_n)_{n=1}^\infty$ be a collection in $\mathcal{A}$. Let $E_1 = A_1$ and for $n \geq 1$, let

$$E_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i \right) = A_n \cap \left(\bigcup_{i=1}^{n-1} A_i \right)^c$$

Clearly, $(E_n)_{n=1}^\infty$ are pairwise disjoint such that $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty A_n$. Furthermore, $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ by the above argument.

To see that $E = \bigcup_{n=1}^\infty E_n$ is a member of $\mathcal{A}$, let $B \subseteq X$ be arbitrary. For $n \in \mathbb{N}$, let $F_n = \bigcup_{i=1}^n E_n$ which is an element of $\mathcal{A}$ since $\mathcal{A}$ is closed under finite unions. Therefore, since F_n is outer measurable, since $F_n \subseteq E$ so $E^c \subseteq F_n^c$, and since μ^* is monotone, we obtain that

$$\mu^*(B) = \mu^*(B \cap F_n) + \mu^*(B \cap F_n^c) \geq \mu^*(B \cap F_n) + \mu^*(B \cap E^c)$$

for all $n \in \mathbb{N}$.

Since $(F_n)_{n=1}^{\infty}$ are an increasing sequence of sets with union E, we would like to take the limit of the right side of the above inequality to obtain that $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$ thereby obtaining that E is outer measurable. However, since we do not know the Monotone Convergence Theorem works for outer measures, we will need another approach to take the limit.

Notice that $F_n = F_{n-1} \cup E_n$ and $F_{n-1} \cap E_n = \emptyset$ by construction, so since $E_n \in \mathcal{A}$

$$\begin{aligned}\mu^*(B \cap F_n) &= \mu^*((B \cap F) \cap E_n) + \mu^*((B \cap F_n) \cap E_n^c) \\ &= \mu^*(B \cap E_n) + \mu^*(B \cap F_{n-1})\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, recursion implies that

$$\mu^*(B \cap F_n) = \sum_{i=1}^{n} \mu^*(B \cap F_i)$$

for all $n \in \mathbb{N}$. Thus,

$$\mu^*(B) \geq \mu^*(B \cap E^c) + \sum_{i=1}^{n} \mu^*(B \cap E_i)$$

for all $n \in \mathbb{N}$. By taking the supremum on the right side of the above expression yields

$$\mu^*(B) \geq \mu^*(B \cap E^c) + \sum_{i=1}^{\infty} \mu^*(B \cap E_i)$$

Therefore, subadditivity implies that

$$\begin{aligned}\mu^*(B) &\geq \mu^*(B \cap E^c) + \mu^*\left(\bigcup_{n=1}^{\infty}(B \cap E_n)\right) \\ &= \mu^*(B \cap E^c) + \mu^*\left(B \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) \\ &= \mu^*(B \cap E^c) + \mu^*(B \cap E)\end{aligned}$$

Therefore, as $B \subseteq X$ was arbitrary, we obtain that $E \in \mathcal{A}$. Hence, $\mathcal{A}$ is a σ-algebra.

Finally, to check that $\mu^*|_{\mathcal{A}}$ is a measure, first notice that $\mu^*(\emptyset) = 0$ by construction. To check the other property, let $(E_n)_{n=1}^{\infty}$ be a collection of disjoint elements of $\mathcal{A}$ and let $E = \bigcup_{n=1}^{\infty} E_n$. Using the above computation with E in place of B, we have

$$\mu^*(E) \geq \mu^*(E \cap E^c) + \sum_{n=1}^{\infty} \mu^*(E \cap E_n) = 0 + \sum_{n=1}^{\infty} \mu^*(E_n) = \sum_{n=1}^{\infty} \mu^*(E_n)$$

However, since subadditivity of outer measures implies

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$$

we obtain that

$$\mu^*(E) = \sum_{n=1}^{\infty} \mu^*(E_n)$$

Hence, $\mu^*|_{\mathcal{A}}$ is a measure as desired. ■

Let λ^* be the Lebesgue outer measure from Definition 1.2.5. By Theorem 1.2.10, the collection $\mathcal{M}(\mathbb{R})$ of λ^*-measurable sets is a σ-algebra and $\lambda^*|_{\mathcal{M}(\mathbb{R})}$ is a measure. Since these objects will be the focus for the remainder of these notes, we make the following definition.

Definition 1.2.11

(i) The *Lebesgue measure* on $\mathbb{R}$ is the measure $\lambda = \lambda^*|_{\mathcal{M}(\mathbb{R})}$. The elements of $\mathcal{M}(\mathbb{R})$ are called the *Lebesgue measurable sets*.

(ii) The *n-dimensional Lebesgue measure* on $\mathbb{R}^n$ is the measure λ_n obtained from restricting λ_n^* to the λ_n^*-measurable subsets of $\mathbb{R}^n$.

One by-product of the Carathéodory Method is that the measures constructed have a specific additional property that we now describe.

Definition 1.2.12

A measure space $(X, \mathcal{A}, \mu)$ is said to be *complete* if whenever $A \in \mathcal{A}$ and $B \in \mathcal{P}(X)$ are such that $B \subseteq A$ and $\mu(A) = 0$, then $B \in \mathcal{A}$.

Proposition 1.2.13

Let X be a nonempty set, let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X, and let $\mathcal{A}$ be the σ-algebra of all outer measurable sets. If $A \in \mathcal{P}(X)$ and $\mu^*(A) = 0$, then $A \in \mathcal{A}$. Hence, $(X, \mathcal{A}, \mu^*|_{\mathcal{A}})$ is complete by the monotonicity of μ^*.

Proof.

Assume that $A \in \mathcal{P}(X)$ is such that $\mu^*(A) = 0$. To see that $A \in \mathcal{A}$, let $B \in \mathcal{P}(X)$ be arbitrary. Then

$$0 \leq \mu^*(B \cap A) \leq \mu^*(A) = 0$$

by monotonicity. Furthermore,

$$\mu^*(B) \geq \mu^*(B \cap A^c) = \mu^*(B \cap A^c) + \mu^*(B \cap A)$$

Therefore, as $B \in \mathcal{P}(X)$ was arbitrary, $A \in \mathcal{A}$.

Finally, to see that $(X, \mathcal{A}, \mu^*|_{\mathcal{A}})$ is complete, let $A \in \mathcal{A}$ and $B \in \mathcal{P}(X)$ such that $B \subseteq A$ and $\mu^*(A) = 0$. Monotonicity implies that $\mu^*(B) = 0$. Thus, the first part of this proof implies that $B \in \mathcal{A}$, as desired. ■

Remark 1.2.14

By Proposition 1.2.13, λ is a complete measure.

One may think the Carathéodory Method may not be that useful as it can only construct measures that are complete and thereby might be limited. However, this is not the case as it is always possible to 'complete' a measure rather simply.

Exercise 1.2.15

Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that there exists a complete measure space $(X, \overline{\mathcal{A}}, \overline{\mu})$ such that $\mathcal{A} \subseteq \overline{\mathcal{A}}$ and $\overline{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Exercise 1.2.16: Assignment 1, Question 2

Let $X = \mathbb{N}$, let

$$\mathcal{F} = \{\emptyset, \mathbb{N}\} \cup \{\{2k-1, 2k\} : k \in \mathbb{N}\}$$

and define $\ell : \mathcal{F} \to [0, \infty]$ by $\ell(\emptyset) = 0$, $\ell(\{2k-1, 2k\}) = 1$ for all $k \in \mathbb{N}$, and $\ell(\mathbb{N}) = \infty$. If μ_ℓ^* denotes the outer measure associated to ℓ, describe the σ-algebra $\mathcal{A}$ of all μ_ℓ^*-measurable sets. Justify your answer.

1.3 Extending Measures

1.4 Properties of the Lebesuge Measure

1.5 Metric Outer Measures

1.6 Hausdorff Measures

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