Theorem: Let $(X, \mathcal{A}, \mu)$ be a measure space, $A \in \mathcal{A}$, and $f, g : X \to [0, \infty]$ be measurable.

(i) If $c \geq 0$, $\int_A cf \, d\mu = c \int_A f \, d\mu$

(ii) If $B \in \mathcal{A}$ and $B \subseteq A$, $\int_B f \, d\mu \leq \int_A f \, d\mu$.

(iii) $\int_X \chi_A f \, d\mu = \int_A f \, d\mu$

(iv) If $f \chi_A \leq g \chi_A$, $\int_A f \, d\mu \leq \int_A g \, d\mu$

(v) $\int_A f \, d\mu = 0$ if and only if $\mu(\{x : f(x) > 0\} \cap A) = 0$.

(vi) If $\mu(A) = 0$, then $\int_A f \, d\mu = 0$.

Proof: (iii) Note that $\int_A f \, d\mu = \sup \left\{ \int_A \varphi \, d\mu : \varphi \text{ simple}, \varphi \leq f \right\}$

$= \sup \left\{ \int_X \varphi \chi_A \, d\mu : \varphi \text{ simple}, \varphi \leq f \right\}$

$= \sup \left\{ \int_X \psi \, d\mu : \psi \text{ simple and } \psi \leq f \chi_A \right\}$

$= \int_X f \chi_A \, d\mu$

(v) Assume $\int_A f \, d\mu = 0$. For each $n \in \mathbb{N}$, let

$B_n = \{x : f(x) > \frac{1}{n}\} \cap A \in \mathcal{A}$.

Then $\frac{1}{n} \chi_{B_n} \leq f$ is a simple function, so

$\frac{1}{n} \mu(B_n) = \int_A \frac{1}{n} \chi_{B_n} \, d\mu \leq \int_A f \, d\mu = 0 \Rightarrow \mu(B_n) = 0$.

Because $\bigcup_{n=1}^{\infty} B_n = \{x : f(x) > 0\} \cap A$, by monotone convergence theorem, we get $\mu(\{x : f(x) > 0\} \cap A) = 0$.

Conversely, if $\mu(\{x : f(x) > 0\} \cap A) = 0$. Assume φ is simple and $\varphi \leq f \chi_A$. Write $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ with $A_i \in \mathcal{A}$ and $a_i \in (0, \infty)$. Then $A_i \subseteq \{x : f(x) > 0\} \cap A$

So $\mu(A_i) = 0$, i.e. $\int_X \varphi \, d\mu = 0$, so $\int_X f \chi_A \, d\mu = 0$

$\Rightarrow \int_A f \, d\mu = 0.$

Question: Is $\int_A f + g \, d\mu = \int_A f \, d\mu + \int_A g \, d\mu$?

If $\varphi \le f$ and $\psi \le g$ are simple functions, because

$$\int_A \varphi + \psi \, d\mu = \int_A \varphi \, d\mu + \int_A \psi \, d\mu \text{, we have}$$

$$\int_A f \, d\mu + \int_A g \, d\mu \le \int_A f + g \, d\mu.$$

Monotone Convergence Theorem for Integrals

Theorem: Let $f : X \to [0, \infty]$ be measurable and for each $n \in \mathbb{N}$, let $f_n : X \to [0, \infty]$ be measurable. If $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise. Then

$$\lim_{n \to \infty} \int_A f_n \, d\mu = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$.

Proof: By multiplying by χ_A, we can assume $A = X$.

Because $f_n \to f$ pointwise and $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$

so $\int_X f_n \, d\mu \le \int_X f \, d\mu$ for all $n \in \mathbb{N}$, so

$$\limsup_{n \to \infty} \int_X f_n \, d\mu \le \int_X f \, d\mu.$$

Now we show $\int_X f \, d\mu \le \liminf_{n \to \infty} \int_X f_n \, d\mu$. It suffices to show if φ is simple and $\varphi \le f$, then

$$\alpha \int_X \varphi \, d\mu \le \liminf_{n \to \infty} \int_X f_n \, d\mu. \text{ for all } \alpha \in (0,1).$$

Note that $f - \alpha\varphi \ge 0$, so for each $n \in \mathbb{N}$, let

$$A_n = \{ x : f_n(x) - \alpha \varphi(x) \ge 0 \} \in \mathcal{A}.$$

Then $\alpha \int_{A_n} \varphi \, d\mu = \int_{A_n} \alpha \varphi \, d\mu \le \int_{A_n} f_n \, d\mu \le \int_X f_n \, d\mu$

$$= \liminf_{n \to \infty} \int_X f_n \, d\mu$$

We want to replace A_n with X. Note $A_n \subseteq A_{n+1}$ $\forall n \in \mathbb{N}$.

We claim that $\bigcup_{n=1}^{\infty} A_n = X$. Indeed, let $x \in X$. Then we have the following cases

- If $f(x) = 0$, then $0 \leq \varphi(x) \leq f(x) = 0$, $\varphi(x) = 0$, so $\alpha \varphi(x) = 0 \leq f_n(x)$, so $x \in A_n$ for all $n \in \mathbb{N}$.
- If $f(x) \neq 0$, then $f(x) - \alpha \varphi(x) > 0$. Then as $f_n \nearrow f$ pointwise, there is a $N \in \mathbb{N}$ such that $x \in A_n$ $\forall n \geq N$

so $\bigcup_{n=1}^{\infty} A_n = X$.

Because $\nu(A) = \int_A \varphi \, d\mu$ is a measure, the Monotone Convergence Theorem for Measures gives

$$\lim_{n \to \infty} \int_{A_n} \varphi \, d\mu = \int_X \varphi \, d\mu$$

Hence,

$$\alpha \int_X \varphi \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.$$

Corollary: Let $f, g : X \to [0, \infty]$ be measurable.

(i) $\int_X f + g \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$

(ii) If $f = g$ almost everywhere, $\int_X f \, d\mu = \int_X g \, d\mu$.

Proof: (i) By earlier, there exists a sequence of simple functions $(\varphi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ that converge pointwise to f and g respectively. So $(\varphi_n + \psi_n)_{n=1}^{\infty}$ is an increasing sequence of simple functions that converge to $f + g$, so

$$\int_X f \, d\mu + \int_X g \, d\mu = \lim_{n \to \infty} \left(\int_X \varphi_n \, d\mu + \int_X \psi_n \, d\mu \right)$$

$$= \lim_{n\to\infty} \left(\int_X \varphi_n + \psi_n \, d\mu \right)$$
$$= \int_X f + g \, d\mu.$$

(ii) Let $B = \{x : f(x) \neq g(x)\} \in \mathcal{A}$. Then $\mu(B) = 0$ as $f = g$ almost everywhere. Then

$$\int_X f \, d\mu = \int_X f\chi_B + f\chi_{B^c} \, d\mu$$
$$= \int_X f\chi_B \, d\mu + \int_X f\chi_{B^c} \, d\mu$$
$$= \int_B f \, d\mu + \int_{B^c} f \, d\mu$$
$$= 0 + \int_{B^c} g \, d\mu$$
$$= \text{(reverse the steps)}$$
$$= \int_X g \, d\mu$$

Remark: In the statement of the Monotone Convergence Theorem, $f_n \le f_{n+1}$ almost everywhere and $f_n \to f$ pointwise almost everywhere is enough as these and integrals are preserved under almost everywhere equivalence.

Corollary: Let $f, g : X \to [0, \infty]$ be measurable such that $\int_A f \, d\mu = \int_A g \, d\mu$ for all $A \in \mathcal{A}$, then if μ is σ-finite then $f = g$ almost everywhere.

Proof: Let $B = \{x : f(x) > g(x)\}$. We want to show $\mu(B) = 0$.

Case 1: If $\int_B g \, d\mu < \infty$

Note

$$\int_B f \, d\mu = \int_B g + (f-g) \, d\mu = \int_B g + \int_B (f-g)\chi_B \, d\mu$$
$$= \int_B f \, d\mu + \int (f-g) \, d\mu < \infty$$

Since $\int_B f\,d\mu = \int_B g\,d\mu$, we have $\int_B (f-g)\chi_B\,d\mu = 0$

so $\mu(B) = \mu(B \cap \{x : (f(x)-g(x))\chi_B(x) > 0\}) = 0$.

Case 2: Since μ is σ-finite, there exists $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$

such that $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$ and $X_n \subseteq X_{n+1}$

for all $n \in \mathbb{N}$. For $n,m \in \mathbb{N}$, let

$Y_{n,m} = \{x \in X_n : g(x) \le m\}$. Then if $B_{n,m} = Y_{n,m} \cap B$,

we have $\int_{B_{n,m}} g\,d\mu < \infty$. By repeating the proof

we get $\mu(B_{n,m}) = 0$, so by MCT

$$\mu(X_n \cap B) = \lim_{n\to\infty} \mu(B_{n,m}) = 0$$

Corollary: For all $n \in \mathbb{N}$, let $f_n : X \to [0,\infty]$ be measurable

and let $f : X \to [0,\infty]$ be defined by

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{almost everywhere}$$

If f is measurable, then $\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu$.

Proof: For all $N \in \mathbb{N}$, let $g_N = \sum_{n=1}^{N} f_n$. Then g_N is measurable

$g_N \le g_{N+1}$ $\forall N \in \mathbb{N}$ and $g_N \nearrow f$ a.e. so by MCT

$$\begin{aligned}\int_X f\,d\mu &= \lim_{N\to\infty} \int_X g_N\,d\mu \\ &= \lim_{N\to\infty} \int_X \sum_{n=1}^{N} f_n\,d\mu \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n\,d\mu \\ &= \sum_{n=1}^{\infty} \int_X f_n\,d\mu\end{aligned}$$

Corollary: Let $f : X \to [0,\infty]$ be measurable, let

$\nu : \mathcal{A} \to [0,\infty]$ defined by

$$\nu(A) = \int_A f\,d\mu \quad \text{for all } A \in \mathcal{A}$$

Then ν is a measure such that if $A \in \mathcal{A}$ and $\mu(A) = 0$, then $\nu(A) = 0$

Proof:

- $\nu(\emptyset) = \int_\emptyset f \, d\mu = 0$ as $\mu(\emptyset) = 0$.
- Let $\{A_n\}_{n=1}^\infty$ be disjoint. Then

$$\nu\left(\bigcup_{n=1}^\infty A_n\right) = \int_{\bigcup_{n=1}^\infty A_n} f \, d\mu = \int_X f \chi_{\bigcup_{n=1}^\infty A_n} \, d\mu$$
$$= \int_X \sum_{n=1}^\infty f \chi_{A_n} \, d\mu$$
$$= \sum_{n=1}^\infty \int_X f \chi_{A_n} \, d\mu$$
$$= \sum_{n=1}^\infty \int_{A_n} f \, d\mu$$
$$= \sum_{n=1}^\infty \nu(A_n).$$

Integral of a Complex Function.

If $f : X \to \mathbb{R}$ is measurable, we know how to integrate f_+ and f_-. If the integral is to be linear we want

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu$$

Definition: A measurable function $f : X \to \mathbb{K}$ is said to be **integrable** if $\int_X |f| \, d\mu < \infty$

In the case where $\mu = \lambda$, we say f is **Lebesgue Integrable**.

Remark: Since $\int_A |f| \, d\mu \le \int_X |f| \, d\mu$, if f is integrable, then $\int_A |f| \, d\mu < \infty \quad \forall A \in \mathcal{A}$.

Remark: If f is measurable, write

$$f = Re(f)_+ - Re(f)_- + i\,Im(f)_+ - i\,Im(f)_-$$

Since $|Re(f)_\pm|, |Im(f)_\pm| \le |f|$, if f is integrable,

then $\int_A Re(f)_\pm \, d\mu < \infty$, $\int_A Im(f)_\pm \, d\mu < \infty$.

Moreover, if $Re(f)_\pm$, $Im(f)_\pm$ are integrable,

$$\begin{aligned} |f| &= \sqrt{|Re(f)|^2 + |Im(f)|^2} \\ &\le |Re(f)| + |Im(f)| \\ &\le Re(f)_+ + Re(f)_- + Im(f)_+ + Im(f)_- \end{aligned}$$

so f is integrable.

Definition: If $f: X \to \mathbb{K}$ is integrable and $A \in \mathcal{A}$, the integral of f over A with respect to μ is

$$\int_A f \, d\mu = \int_A Re(f)_+ \, d\mu - \int_A Re(f)_- + i\int_A Im(f)_+ \, d\mu - i\int_A Im(f)_- \, d\mu$$

Remark:

(i) If $f: X \to [0, \infty]$, then $Re(f)_-$, $Im(f)_+$, $Im(f)_-$ are zero, agreeing with the definition of nonnegative.

(ii) Can replace A with X.