Section 5.4: Finite Signed Measures

Def: Let ν be a signed measure on $(X, \mathcal{A})$. The total variation measure is $|\nu| : \mathcal{A} \to [0, \infty]$ where $|\nu| = \nu_+(A) + \nu_-(A)$ for all $A \in \mathcal{A}$.

Example: If $\nu(A) = \int_A f \, d\mu$ where $f \in L_1$ is real-valued, then $\nu_\pm(A) = \int_A f_\pm \, d\mu$, so

$$|\nu|(A) = \int_A f_+ \, d\mu + \int_A -f_- \, d\mu$$

Proposition: Let ν be a signed measure on $(X, \mathcal{A})$. Let $\mathcal{P}$ be the collection of all partitions $\{A_n\}_{n=1}^\infty$ of X into pairwise disjoint measure sets. Then

$$|\nu|(A) = \sup_{\{A_n\} \in \mathcal{P}} \sum_{n=1}^\infty |\nu(A \cap A_n)|$$

Proof: By the Hahn-Decomposition Theorem, there exists $P, N \in \mathcal{A}$ such that $X = P \cup N$ and $\emptyset = P \cap N$, so we have

$\nu_+(A) = \nu(A \cap P)$ and $\nu_-(A) = \nu(A \cap N)$

So the sup is at least $|\nu(A \cap P)| + |\nu(A \cap N)| = \nu_+(A) + \nu_-(A)$
$= |\nu|(A)$.

On the other hand, for any partition $\{A_n\}_{n=1}^\infty$ of disjoint measurable sets,

$$\begin{aligned}
\sum_{n=1}^\infty |\nu(A \cap A_n)| &= \sum_{n=1}^\infty |\nu_+(A \cap A_n) - \nu_-(A \cap A_n)| \\
&\le \sum_{n=1}^\infty \nu_+(A \cap A_n) + \sum_{n=1}^\infty \nu_-(A \cap A_n) \\
&= \nu_+\left(A \cap \left(\bigcup_{n=1}^\infty A_n\right)\right) + \nu_-\left(A \cap \left(\bigcup_{n=1}^\infty A_n\right)\right) \\
&= \nu_+(A) + \nu_-(A) \\
&= |\nu|(A).
\end{aligned}$$

__Lemma:__ Let ν be a signed measure on $(X, \mathcal{A})$. Then $|\nu(A)| \leq |\nu|(A)$ for all $A \in \mathcal{A}$.

__Def:__ A signed measure ν on $\mathcal{A}$ is said to be finite if $\nu(A) \neq \pm\infty$ for all $A \in \mathcal{A}$.

__Proposition:__ Let ν be a signed measure on $(X, \mathcal{A})$. TFAE

1. ν is finite.
2. ν_+ and ν_- are finite.
3. $|\nu|$ is finite.
4. $\nu(A) \neq \pm\infty$ $\forall A \in \mathcal{A}$.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] \arrow[dr, shift left] & & 4 \arrow[ll] \\ & 3 \arrow[ul, shift left] & \end{tikzcd}$$

__Proof:__ (2) $\Leftrightarrow$ (3) as $|\nu| = \nu_+ + \nu_-$, so $|\nu|(X) < \infty$ if and only if $\nu_+(X), \nu_-(X) < \infty$.

(2) $\Rightarrow$ (1) Trivial because if ν_+ and ν_- are finite, then $\nu_+(A), \nu_-(A) \in [0, \infty]$ for all $A \in \mathcal{A}$, so

$$\nu(A) = \nu_+(A) - \nu_-(A) \in (-\infty, \infty)$$

(1) $\Rightarrow$ (4) Take $A = X$ in the definition.

(4) $\Rightarrow$ (2): Suppose ν_+ is not finite or ν_- is not finite.

- If $\nu_\pm(X) = \infty$ or $\nu_\mp(X) = \infty$, then $\nu(X) = \pm\infty$, absurd.
- If $\nu_+(X) = \infty = \nu_-(X)$. Let P and N be as in the HDT.

Then $\nu(P) = \nu_+(P) - \nu_-(P) = \nu_+(P) + \nu_+(N) = \nu_+(X) = \infty$

However

$$\nu(N) = \nu_+(N) - \nu_-(N) = 0 - \nu_-(N) = -\nu_-(N) = -\nu_-(P) - \nu_-(N)$$
$$= -\nu_-(X) = -\infty.$$

absurd because both cannot occur for a signed measure.

Theorem: Let $(X, \mathcal{A})$ be a measurable space and let

$\text{Meas}(X, \mathcal{A}) = \{ \nu : \nu \text{ is a finite signed measure on } (X, \mathcal{A}) \}$.

Then $\text{Meas}(X, \mathcal{A})$ is a vector space over $\mathbb{R}$. Define $\| \cdot \|_M : \text{Meas}(X, \mathcal{A}) \to [0, \infty)$ by $\| \nu \|_M = |\nu|(X)$. Then $(\text{Meas}(X, \mathcal{A}), \| \cdot \|_M)$ is a Banach space.

Section 5.5: The Radon-Nikodym Theorem

Definition: Let $(X, \mathcal{A})$ be a measurable space and let μ be a measure on $(X, \mathcal{A})$, and let ν be a signed measure on $(X, \mathcal{A})$. Then ν is said to be absolutely continuous with respect to μ, denoted by $\nu \ll \mu$, if whenever $A \in \mathcal{A}$ and $\mu(A) = 0$, then $\nu(A) = 0$.

Example: If μ is a measure, f is real-valued $\mathcal{L}_1$-function, and

$\nu(A) = \int_A f \, d\mu$.

Then $\nu \ll \mu$.

Remark: If $\nu \ll \mu$ and $\mu(A) = 0$, then A is null for ν, so if $B \in \mathcal{A}$ and $B \subseteq A$, then $\mu(B) = 0$, so $\nu(B) = 0$.

Lemma: Let μ be a measure and ν be a signed measure. Then $\nu \ll \mu$ if and only if $\nu_+ \ll \mu$ and $\nu_- \ll \mu$.

Proof: Let $A \in \mathcal{A}$ be such that $\mu(A) = 0$. If $\nu_+(A) = 0$ and $\nu_-(A) = 0$, then trivially, $\nu(A) = 0$. Conversely, if $\nu \ll \mu$, then A is null, so $\nu(P \cap A) = 0$ and $\nu(N \cap A) = 0$, where P and N are from the HDT. Thus, $\nu_+(A) = \nu(P \cap A) = 0$ and $\nu_-(A) = -\nu(A \cap N) = 0$.

Theorem: (Radon-Nikodym) Let μ and ν be measures on $(X, \mathcal{A})$.

If μ is σ-finite and $\nu \ll \mu$, then there exists $f: X \to [0,\infty]$ such that f is measurable and

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$. Moreover if $g: X \to [0,\infty]$ is measurable and $\int_A g \, d\mu = \nu(A)$ for all $A \in \mathcal{A}$, then $f = g$ μ-a.e.

<u>Remark</u>: Uniqueness follows from Chapter 2 and 3 for σ-finite measures.

<u>Remark</u>: If $\nu(A) = \int_A f \, d\mu$ for all $A \in \mathcal{A}$ and $B \in \mathcal{A}$ such that $a \le f \le b$ on B for some $a, b \in \mathbb{R}$, then

$$a\mu(B) \le \nu(B) \le b\mu(B) \quad (*)$$

We will try to construct sets that satisfy $(*)$ and then try to construct f from these sets.

<u>Lemma</u>: Let $Q \subseteq \mathbb{R}$ be a countable set. For each $q \in Q$, let $A_q \in \mathcal{A}$ such that if $q_1, q_2 \in Q$ and $q_1 < q_2$, then $A_{q_1} \subseteq A_{q_2}$. Then there exists $f: X \to [-\infty,\infty]$ such that f is measurable, $f(x) \ge q$ for all $x \in A_q^c$ and $f(x) \le q$ for all $x \in A_q$.

<u>Proof</u>: Let $f: X \to [-\infty,\infty]$ by

$$f(x) = \inf\{q \in Q : x \in A_q\}.$$

Thus, if $x \in A_q$, $f(x) \le q$ by definition. Moreover, if $x \in A_q^c$, then $x \notin A_q$, so $x \notin A_{q'}$ for all $q' < q$ with $q' \in Q$, so $f(x) \ge q$.

Note that for all $\alpha \in \mathbb{R}$ $\{x \in X : f(x) < \alpha\} = \bigcup_{\substack{q \in Q \\ q < \alpha}} A_q \in \mathcal{A}$

so f is measurable

<u>Lemma:</u> (The Main RN-Theorem Lemma) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $Q \subseteq \mathbb{R}$ be a countable set, and for all $q \in Q$, let $A_q \in \mathcal{A}$. If $q_1, q_2 \in Q$ such that $q_1 < q_2$ we have $\mu(A_{q_1} \setminus A_{q_2}) = 0$, then there exists a measurable function $f: X \to \overline{\mathbb{R}}$ such that $f \leq q$ μ-a.e. on A_q and $f \geq q$ μ-a.e. on A_q^c.

<u>Proof:</u> Let $Z = \bigcup_{\substack{q_1, q_2 \in Q \\ q_1 < q_2}} (A_{q_1} \setminus A_{q_2})$. Since Q is countable and $A_q \in \mathcal{A}$ for all $q \in Q$, then $Z \in \mathcal{A}$.

For each $q \in Q$, let $B_q = A_q \cup Z$. Note if $q_1 < q_2$ in Q, then

$$
\begin{aligned}
B_{q_1} = A_{q_1} \cup Z &= ((A_{q_1} \cap A_{q_2}) \cup (A_{q_1} \setminus A_{q_2})) \cup Z \\
&= (A_{q_1} \cap A_{q_2}) \cup Z \\
&\subseteq A_{q_2} \cap Z \\
&= B_{q_2}.
\end{aligned}
$$

By the previous lemma, there exists $f: X \to \overline{\mathbb{R}}$ such that $f(x) \leq q$ for all $x \in B_q$ and $f(x) \geq q$ for all $x \in B_q^c$ for all $q \in Q$.

If $x \in A_q$, then $x \in B_q$, so $f(x) \leq q$. Note $\mu(Z) = 0$ by subadditivity, so if $x \in A_q^c \cap Z^c = B_q^c$, so $f(x) \geq q$. Since $\mu(Z) = 0$, we are done.

<u>Proof of RN-Theorem:</u> Consider the following cases:

① If μ is finite. For each $q \in \mathbb{Q}$, consider the measure $\nu - q\mu$. Since μ is finite, $\nu - q\mu$ is a signed measure. By the HDT, there exists $P_q, N_q \in \mathcal{A}$ such that $P_q \cap N_q = \emptyset$, $P_q \cup N_q = X$, P_q (resp. N_q) is a positive (resp. negative) set for $\nu - q\mu$.

Without loss of generality, let $P_0 = X$, $N_0 = \phi$.

<u>Claim:</u> If $q_1, q_2 \in \mathbb{Q}$ with $q_1 < q_2$, $\mu(N_{q_1} \setminus N_{q_2}) = 0$,

Indeed note $N_{q_1} \setminus N_{q_2} \subseteq N_{q_1}$ which is a negative set for $\nu - q_1\mu$, so $(\nu - q_1\mu)(N_{q_1} \setminus N_{q_2}) \leq 0$, so

$$0 \leq \nu(N_{q_1} \setminus N_{q_2}) \leq q_1 \mu(N_{q_1} \setminus N_{q_2}) < \infty$$

Also, note that $N_{q_1} \setminus N_{q_2} \subseteq P_{q_2}$ which is a positive set for $\nu - q_2\mu$. Thus, $(\nu - q_2\mu)(N_{q_1} \setminus N_{q_2}) \geq 0$, so $\nu(N_{q_1} \setminus N_{q_2}) \geq q_2 \mu(N_{q_1} \setminus N_{q_2})$. By combining these inequalities, we have

$$\begin{aligned} q_2 \mu(N_{q_1} \setminus N_{q_2}) &\leq \nu(N_{q_1} \setminus N_{q_2}) \\ &\leq q_1 \mu(N_{q_1} \setminus N_{q_2}) < \infty, \end{aligned}$$

so $(q_2 - q_1)\mu(N_{q_1} \setminus N_{q_2}) \leq 0$, so $\mu(N_{q_1} \setminus N_{q_2}) = 0$.

By the lemma, there exists $f : X \to \overline{\mathbb{R}}$ such that $f(x) \leq q$ μ-a.e for $x \in N_q$ and $f(x) \geq q$ μ-a.e. for $x \in N_q^c$ so $f(x) \leq 0$ μ-a.e for $x \in N_0$ and $f(x) \geq 0$ μ-a.e. for $x \in P_0$, so $f : X \to [0, \infty]$.