Theorem: (Subadditivity) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^{\infty}$ be a collection in $\mathcal{A}$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n).$$

Proof: Let $E_1 = A_1 \in \mathcal{A}$ and for $n \geq 2$, let $E_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$

Then $E_n \subset A_n$ $\forall n \in \mathbb{N}$, and $(E_n)_{n=1}^{\infty}$ are pairwise disjoint, and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n$. Now,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= \sum_{n=1}^{\infty} \mu(E_n) \\
&\le \sum_{n=1}^{\infty} \mu(A_n) \quad \text{by monotonicity.}
\end{aligned}$$

Theorem: (Monotone Convergence Theorem) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{A}$

(i) If $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(ii) If $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

Proof: Let $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$.

It is easy to show $\forall n \in \mathbb{N}$, $B_n \in \mathcal{A}$. Then $(B_n)_{n=1}^{\infty}$ are pairwise disjoint, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ and $A_n = \bigcup_{i=1}^{n} B_i$.

Thus,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} B_n\right) \\
&= \lim_{N\to\infty} \mu(A_N)
\end{aligned}$$

(ii) $\forall n \in \mathbb{N}$, let $C_n = A_1 \setminus A_n \in \mathcal{A}$. Then $C_n \subset C_{n+1}$ for all $n \in \mathbb{N}$ and

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (A_1 \cap A^c) = A_1 \cap \left(\bigcup_{n=1}^{\infty} A_n^c \right)$$
$$= A_1 \cap \left(\bigcap_{n=1}^{\infty} A_n \right)^c$$
$$= A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n \right)$$

From (i) and because $\mu(A_1) < \infty$,

$$\mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right) = \lim_{n\to\infty} \mu(A_1 \setminus A_n)$$
$$\cancel{\mu(A_1)} - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \cancel{\mu(A_1)} - \mu(A_n)$$

$$\Rightarrow \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

The Carathéodory Method

We desire to have a measure of length $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ such that

(i) $\lambda(I) :=$ length of I for all intervals I.

(ii) If $A \subset \mathbb{R}$, $x \in \mathbb{R}$, $x + A = \{x + a : a \in A\}$, then $\lambda(x+A) = \lambda(A)$.

Theorem: There is no such measure.

Proof: Assume for a contradiction that there is such a measure λ. Define an equivalence relation "$\sim$" on $\mathbb{R}$ by $x \sim y$ iff $x - y \in \mathbb{Q}$.

Note if $x \in \mathbb{R}$, then x is equivalent to some element in $[0,1)$. Let $A \subset [0,1)$ be the set that consists of exactly one element from each equivalence class.

Write $\mathbb{Q} \cap [0,1) = \{r_n\}_{n=1}^{\infty}$. $\forall n \in \mathbb{N}$, let $A_n = \{x \in [0,1): x \in r_n + A$ or $x+1 \in r_n + A\}$. Note that $r_n + A \subset [0,2)$. In other words, A_n is "$r_n + A$ mod 1".

Claim: (*) $\bigcup_{n=1}^{\infty} A_n = [0,1)$.

Note if $x \in [0,1)$, then there exists $y \in A$ such that $x \sim y$. So $x - y \in \mathbb{Q} \cap (-1,1)$.

- If $x - y \in \mathbb{Q} \cap [0,1)$, then $x - y = r_n$ for some $n \in \mathbb{N}$ so $x = r_n + y \in A_n$.
- Otherwise, if $x - y \in \mathbb{Q} \cap (-1,0)$, then $(x+1) - y \in \mathbb{Q} \cap (0,1)$ so $(x+1) - y = r_n$ for some $n \in \mathbb{N}$, so $x+1 = r_n + y \in A$, so $x \in A_n$.

Claim: (**) $A_n \cap A_m = \phi$ if $n \neq m$.

If $x \in A_n \cap A_m$, then $x = r_n + a + k$ where $a \in A$ and $k \in \{0,1\}$ and $x = r_m + b + \ell$, where $b \in A$, $\ell \in \{0,1\}$ so $a - b = (r_m - r_n) + (\ell - k) \in \mathbb{Q}$ so $a \sim b$ and $a, b \in A$, so $a = b$. In particular, $\underbrace{r_n - r_m}_{n=m} = \underbrace{\ell - k}_{\in\{-1,0,1\}}$, so $r_n = r_m$ and $\ell = k$, so $n = m$.

Claim: (***) $\lambda(A_n) = \lambda(A)$ for all $n \in \mathbb{N}$.

Let $B_{n,1} = (r_n + A) \cap [0,1)$ and $B_{n,2} = (-1 + r_n + A) \cap [0,1)$. Then $A_n = B_{n,1} \cup B_{n,2}$. We claim $B_{n,1} \cap B_{n,2} = \phi$. If not, there are $a, b \in A$ such that

$$r_n + a = -1 + r_n + b \Rightarrow a - b = -1 \in \mathbb{Q}, \text{ so } a \neq b$$

and $a \sim b$, $a \neq b$ and $a = b$, which is absurd, so $B_{n,1} \cap B_{n,2} = \phi$. Moreover,

$$\lambda(A_n) = \lambda(B_{n,1} \cup B_{n,2}) = \lambda(B_{n,1}) + \lambda(B_{n,2})$$

$$= \lambda((r_n + A) \cap [0,1)) + \lambda((-1 + r_n + A) \cap [0,1)).$$

$$= \lambda((r_n + A) \cap [0,1)) + \lambda((r_n + A) \cap [1,2))$$

$$= \lambda((r_n + A) \cap [0,2)).$$

$$= \lambda((r_n + A))$$

$$= \lambda(A).$$

Finally, note that

$$1 = \lambda([0,1)) = \lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} \lambda(A)$$

but $\lambda(A) \in [0, \infty]$, which is absurd.

The problem is $\mathcal{P}(\mathbb{R})$ is too large to define λ. Our goal is to find a way to define λ on a suitably large σ-algebra. We will use the Carathéodory Method.

Definition: An outer measure on X is a map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

(i) $\mu^*(\phi) = 0$.

(ii) If $A \subset B \subset X$, then $\mu^*(A) \leq \mu^*(B)$.

(iii) If $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{P}(X)$. Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Definition: Let X be nonempty and let $\mathcal{F} \subset \mathcal{P}(X)$ such that $\emptyset, X \in \mathcal{F}$, and let $\ell : \mathcal{F} \to [0,\infty]$ be such that $\ell(\emptyset) = 0$. The outer measure associated to ℓ, denote μ_ℓ^* defined by

$$\mu_\ell^* = \inf \left\{ \sum_{n=1}^{\infty} \ell(B_n) : (B_n)_{n=1}^{\infty} \text{ is a collection in } \mathcal{F} \text{ and } A \subset \bigcup_{n=1}^{\infty} B_n \right\}$$

Theorem: μ_ℓ^* is an outer measure such that $\mu_\ell^*(B) \leq \ell(B)$ for all $B \in \mathcal{F}$.

More over, if ν^* is an outer measure such that $\nu^*(B) \leq \ell(B)$ for all $B \in \mathcal{F}$, then $\nu^*(A) \leq \mu_\ell^*(A)$ for all $A \subset X$.

Proof: Note if $B \in \mathcal{F}$ then

$B = B \cup \left(\bigcup_{n=2}^{\infty} \emptyset \right)$ so $\mu_\ell^*(B) \leq \ell(B) + \ell(\emptyset) + \cdots = \ell(B)$.

Note also $\mu_\ell^* : \mathcal{F} \to [0,\infty]$ as the set we take the infimum is nonempty and each element of the set is nonnegative, so

$$0 \leq \mu_\ell^*(\emptyset) \leq \ell(\emptyset) = 0 \implies \mu_\ell^*(\emptyset) = 0$$

Now let $A_1 \subset A_2$. Then $\mu_\ell^*(A_1) \leq \mu_\ell^*(A_2)$ as any cover of A_2 also covers A_1.

To check subadditivity, let $(A_n)_{n=1}^{\infty}$ be a collection in $\mathcal{P}(X)$, and let $\varepsilon > 0$. For each $n \in \mathbb{N}$, let $\{B_{n,m}\}_{m=1}^{\infty}$ be a collection of $\mathcal{F}$ such that

$A_n \subset \bigcup_{m=1}^{\infty} B_{n,m}$ and $\sum_{m=1}^{\infty} \ell(B_{n,m}) \le \mu_\ell^*(A_n) + \frac{\varepsilon}{2^n}$.

Then $\{B_{n,m}\}_{n,m\in\mathbb{N}}$ is a countable collection in $\mathcal{F}$ whose union contains $\bigcup_{n=1}^{\infty} A_n$ so

$$\mu_\ell^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \ell(B_{n,m}) \le \sum_{n=1}^{\infty}\left(\mu_\ell^*(A_n) + \frac{\varepsilon}{2^n}\right) = \varepsilon + \sum_{n=1}^{\infty} \mu_\ell^*(A_n)$$

Thus, as $\varepsilon > 0$ was arbitrary,

$$\mu_\ell^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu_\ell^*(A_n).$$

Now checking the "moreover" part, if ν^* is an outer measure. Let $A \subset X$. If $(B_n)_{n=1}^{\infty}$ is a collection of $\mathcal{F}$ such that $A \subset \bigcup_{n=1}^{\infty} B_n$ then $\nu^*(A) \le \nu^*\left(\bigcup_{n=1}^{\infty} B_n\right)$
$\le \sum_{n=1}^{\infty} \nu^*(B_n) \le \sum_{n=1}^{\infty} \ell(B_n)$

so by taking infimum, $\nu^*(A) \le \mu_\ell^*(A)$.

Example:

(α) Let $\mathcal{F} = \{$all open intervals of $\mathbb{R}\}$ and let ℓ be the usual length. Then the outer measure is called the Lebesgue Outer Measure and is denoted by λ^*. Thus, if $A \subset \mathbb{R}$

$$\lambda^*(A) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : I_n \text{ intervals}, A \subset \bigcup_{n=1}^{\infty} I_n\right\}$$

(β) Let $\mathcal{F} = \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i \in \mathbb{R} \cup \{-\infty\}, b_i \in \mathbb{R} \cup \{\infty\}, a_i < b_i\}$

Define $\ell : \mathcal{F} \to [0, \infty]$ by

$$l\left(\prod_{i=1}^{n}(a_i, b_i)\right) = \prod_{i=1}^{n}(b_i - a_i)$$

Then the outer measure μ_l^* is called the n-dim Lebesgue Outer Measure, λ_n^*.