MATH 6280: Measure Theory

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Preface

These are the first edition of these lecture notes for MATH 6280 (Measure Theory). Consequently, there may be several typographical errors, missing exposition on necessary background, and more advanced topics for which there will not be time in class to cover. Future iterations of these notes will hopefully be fairly self-contained provided one has the necessary background. If you come across any typos, errors, omissions, or unclear expositions, please feel free to contact me so that I may continually improve these notes.

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Chapter 1

MEASURE SPACES

As per its title, this course is dedicated to the study of the theory of measures. So what sort of course would this be if we did not define the main objective of study in the first chapter? After defining the notion of a measure, we will examine several examples and properties of measures that immediately follow from definitions. We will then turn to constructing measures from various notions of length and extending measure-like functions to actual measures. This will lead us to looking at measures with important analytical properties including the Lebesgue-Stieltjes measures and metric outer measures. By using metric outer measures, we will obtain the notion of Hausdorff dimension for subsets of metric spaces.

1.1 Measure Spaces

Definition 1.1.1

Let X be a nonempty set. A σ-algebra on X is a subset $\mathcal{A} \subset \mathcal{P}(X)$ such that

  (i) $\emptyset, X \in \mathcal{A}$, i.e. we measure the empty event and the full event.

  (ii) If $A \in \mathcal{A}$, then $A^c = X \setminus A \in \mathcal{A}$, i.e. we measure the complement of an event.

  (iii) If $(A_n)_{n=1}^\infty$ is a countable collection of events in $\mathcal{A}$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a *measurable space*, and the elements of $\mathcal{A}$ are called *measurable sets*.

Remark 1.1.2

One may ask why we only ask for countable unions of measurable sets to be measurable. One answer for this comes with the definition of a measure in that we want to have additivity over disjoint unions and adding over an uncountable set only works if only a countable number of elements are nonzero. Another reason is that restricting to countable collections is quite powerful as we will see in this course.

Remark 1.1.3

One may also ask why we have not required that the intersection of countable collection of measurable sets is measurable. The reason is that countable intersections come for free. Indeed, if $(X, \mathcal{A})$ is a measurable space and $(A_n)_{n=1}^{\infty}$ is a countable collection of events in $\mathcal{A}$, then

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c \in \mathcal{A}$$

as complements and countable unions of elements of $\mathcal{A}$ are elements of $\mathcal{A}$. Furthermore, by using $\emptyset$ in unions and X in intersections, clearly, a finite union or intersection of elements of $\mathcal{A}$ is an element of $\mathcal{A}$.

Example 1.1.4

Let X be a nonempty set.

(α) $(X, \mathcal{P}(X))$ is a measurable space and $(X, \{\emptyset, X\})$ is a measurable space.

(β) Let

$$\mathcal{A} = \{A \subset X : A \text{ is countable or } A^c \text{ is countable}\}$$

Then $(X, \mathcal{A})$ is a measurable space.

Moreover, if one has a collection of σ-algebras on a set X, there are ways of constructing new σ-algebras. In particular, it is elementary to verify the following using set properties and Definition 1.1.1.

Lemma 1.1.5

Let X be a nonempty set and let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of σ-algebras of X over index set I. Then

$$\bigcap_{i \in I} \mathcal{A}_i$$

is a σ-algebra of X.

Remark 1.1.6

Using Lemma 1.1.5, we can construct the smallest σ-algebra containing a collection of subsets. Indeed, let X be a nonempty set and let $A \subset \mathcal{P}(X)$. Define

$$I = \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ such that } A \subset \mathcal{A}\}$$

Clearly, $\mathcal{P}(X) \in I$, so I is nonempty. Hence, Lemma 1.1.5 implies that

$$\sigma(A) = \bigcap_{\mathcal{A} \in I} \mathcal{A}$$

is a σ-algebra. Since clearly $A \subset \sigma(A)$ by construction, $\sigma(A)$ is the smallest σ-algebra of X contains A. As such, $\sigma(A)$ is called the *σ-algebra generated by A*.

1.1. Measure Spaces

Definition 1.1.7

Let (X, d) be a metric space. The σ-algebra generated by the open subsets of X is called the *Borel σ-algebra* and is denoted $\mathcal{B}(X)$. In particular, $\mathcal{B}(X)$ is also the σ-algebra generated by the closed subsets of X as open and closed sets are complements of each other and as σ-algebras are closed under complements. Elements of $\mathcal{B}(X)$ are called *Borel sets*.

Remark 1.1.8

In terms of the Borel subsets of $\mathbb{R}$, the sets

$$\{(a, b) : a < b \in \mathbb{R}\}$$
$$\{(a, b] : a < b \in \mathbb{R}\}$$
$$\{[a, b) : a < b \in \mathbb{R}\}$$
$$\{[a, b] : a < b \in \mathbb{R}\}$$
$$\{(-\infty, b) : b \in \mathbb{R}\}$$
$$\{(-\infty, b] : b \in \mathbb{R}\}$$
$$\{(a, \infty) : a \in \mathbb{R}\}$$
$$\{[a, \infty) : a \in \mathbb{R}\}$$

all can be shown to generate $\mathcal{B}(\mathbb{R})$ via unions, intersections, and complements. To show this, we verify the following

(i) $\mathcal{B}(\mathbb{R})$ contains each of these sets.

(ii) Any σ-algebra containing one of these sets contains all open intervals and thus all open sets by the fact that every open set is a countable union of open intervals.

We note it is possible to show that $|\mathcal{B}(\mathbb{R})| = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$.

Using σ-algebras, we may now define the central object of study in this course.

Definition 1.1.9

Let $(X, \mathcal{A})$ be a measurable space. A (countably additive, positive) *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

where the sum is infinite if one of the elements is ∞ or if the sum diverges.

The triple $(X, \mathcal{A}, \mu)$ is a *measure space* and given an element $A \in \mathcal{A}$, $\mu(A)$ is called the *μ-measure* of A.

Remark 1.1.10

Notice if $(X, \mathcal{A}, \mu)$ is a measure space and $A_1, ..., A_n$ are pairwise disjoint subsets of $\mathcal{A}$, then

$$\mu \left(\bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mu(A_i)$$

by using the properties of a measure with $A_i = \emptyset$ for all $i > n$.

Before we get too deep into the study of properties of measures, let us examine some common measures which are easy to define.

Example 1.1.11

(α) Let X be a nonempty set and let $x \in X$. The *point-mass measure* at x is the measure δ_x on $(X, \mathcal{P}(X))$ defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(β) Let X be a nonempty set. The *counting measure on X* is the measure μ on $(X, \mathcal{P}(X))$ defined by

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

Example 1.1.12

A function $\mu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ of elements of $[0, \infty]$ such that

$$\mu(A) = \sum_{n \in A} a_n$$

for all $A \subset \mathbb{N}$.
To see this, note that if μ has the described form, then μ is a measure. Conversely, suppose μ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. As for each $n \in \mathbb{N}$, the set $\{n\}$ is measurable, for each $n \in \mathbb{N}$, and we may define

$$a_n = \mu(\{n\}) \in [0, \infty]$$

We claim that

$$\mu(A) = \sum_{n \in A} a_n$$

for all $A \subset \mathbb{N}$. Indeed, let $A \subset \mathbb{N}$. Then as A is countable and

$$A = \bigcup_{n \in A} \{n\}$$

we obtain by the properties of measure that

$$\mu(A) = \mu\left(\sum_{n \in A} \{n\}\right) = \sum_{n \in A} \mu(\{n\}) = \sum_{n \in A} a_n$$

as desired.

Note that measure in Example 1.1.12 can be constructed using Example 1.1.11 (α) and the following technique (which will be of use to us later).

Example 1.1.13

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $(A_n)_{n=1}^\infty \subset \mathcal{A}$ and let $(a_n)_{n=1}^\infty \in [0, \infty]$. Define $\nu : \mathcal{A} \to [0, \infty]$ by

$$\nu(A) = \sum_{n=1}^\infty a_n \mu(A_n \cap A)$$

for all $A \in \mathcal{A}$ where the sum equates to ∞ if the sum diverges or one of the terms is ∞ and

$$a \times \infty = \begin{cases} 0 & \text{if } a = 0 \\ \infty & \text{if } a > 0 \end{cases}$$

Then ν is a measure on $(X, \mathcal{A})$. To see this, we clearly note that $\nu(\emptyset) = 0$. Furthermore, if $(B_m)_{m=1}^\infty \subset \mathcal{A}$ are pairwise disjoint, then $\{A_n \cap B_m\}_{m=1}^\infty$ are pairwise disjoint for all $n \in \mathbb{N}$, and thus, as μ is a measure,

$$\begin{aligned}
\nu\left(\bigcup_{m=1}^\infty B_m\right) &= \sum_{n=1}^\infty a_n \mu\left(A_n \cap \left(\bigcup_{m=1}^\infty B_m\right)\right) \\
&= \sum_{n=1}^\infty a_n \mu\left(\bigcup_{m=1}^\infty (A_n \cap B_m)\right) \\
&= \sum_{n=1}^\infty \sum_{m=1}^\infty a_n \mu(A_n \cap B_m) \\
&= \sum_{n=1}^\infty \sum_{m=1}^\infty a_n \mu(A_n \cap B_m) \\
&= \sum_{m=1}^\infty \nu(B_m)
\end{aligned}$$

Hence, ν is a measure as desired.

Although we may define more measures, we turn our attention to properties of measures immediately implied that Definition 1.1.9 and set manipulations. We begin with the following.

Proposition 1.1.14

If $E, F \in \mathcal{A}$ such that $E \subset F$, then $\mu(E) \leq \mu(F)$.

Proof.

Indeed, if $E \subset F$, we have $F \setminus E \in \mathcal{A}$ is disjoint from E and $F = E \cup (F \setminus E)$, and so

$$\mu(E) \leq \mu(E) + \mu(F \setminus E) = \mu(E \cup (F \setminus E)) = \mu(F)$$

∎

In particular, if $\mathbb{A}$ is ordered by inclusion, then μ is monotone with respect to this inclusion. This implies if $\mu(X) < \infty$, then $\mu(A) < \infty$ for all $A \in \mathcal{A}$. Moreover, notice in the above computation that if $\mu(E) < \infty$, then we may subtract $\mu(E)$ from both sides in order to obtain that

$$\mu(F \setminus E) = \mu(F) - \mu(E)$$

Note that the above formula need not hold if $\mu(E) = \infty$ as do not know how to define $\infty - \infty$.

Proposition 1.1.15

If $A, B \in \mathcal{A}$ and $\mu(A \cap B) < \infty$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

Proof.

This is easy; we have

$$\begin{aligned}
\mu(A \cup B) &= \mu(A \cup (B \setminus (B \cap A))) \\
&= \mu(A) + \mu(B \setminus (B \cap A)) \\
&= \mu(A) + \mu(B) - \mu(A \cap B)
\end{aligned}$$

which is a formula that may appear familiar in the context of probability. ∎

Of course, if a measure is going to represent the probability of an event occurring, we must dictate the probability of all possible events is one. As such, when discussing probability, we use the following terminology.

Definition 1.1.16

Let $(X, \mathcal{A}, \mu)$ be a measure space. It is said that $(X, \mathcal{A}, \mu)$ is a *probability space* and μ is a *probability space* if $\mu(X) = 1$. In this case, X is called the *sample space*, elements of $\mathcal{A}$ are called *events*, and given $A \in \mathcal{A}$, $P(A) = \mu(A)$ denotes the probability that the event A occurs.

Remark 1.1.17

It is not difficult to see that a probability space is the correct notion in order to study probability theory. Indeed, the probability of the entire space is one and whenever A and B

are disjoint sets, which is the notion of independent events, then the probability of $A \cup B$ is the sum of the probability of A and the probability of B. Furthermore, Proposition 1.1.14 is precisely the formula for the probability of $A \cup B$ when A and B are not disjoint; that is, the formula for the probability of the union of two not necessarily independent events. Of course, when studying probability, one may only have finite additivity instead of countable additivity. As will be seen in later sections, it is not difficult to extend finitely additive measures to countably additive measures, which is far more desirable in our analytic realm.

Of course, requiring the measure of the entire space to be one is a specific property of a measure we may wish to study. The following generalizations of probability measures are vital for the course.

Definition 1.1.18

A measure μ on a measurable space $(X, \mathcal{A})$ is said to be

(i) *finite* if $\mu(X) < \infty$ (and thus, $\mu(A) < \infty$ for all $A \in \mathcal{A}$ by monotonicity).

(ii) *σ-finite* if there exists a collection $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

In most cases, if one can prove a property for any finite measure, one can extend the result to all σ-finite measures using analytic techniques. This is often done using the following additional partition decompositions of a σ-finite measure space.

Remark 1.1.19

Assume μ is a σ-finite measure on $(X, \mathcal{A})$. Thus, there exists a collection $(A_n)_{n=1}^{\infty}$ of $\mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathcal{N}$. Let $B_1 = C_1 = A_1$, and for each $n \geq 2$, let

$$B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} B_k \right) \quad C_n = \bigcup_{k=1}^{n} A_k$$

Then $(B_n)_{n=1}^{\infty}$ are pairwise disjoint elements of $\mathcal{A}$ are such that $X = \bigcup_{n=1}^{\infty} B_n$ and $\mu(B_n) \leq \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Similarly, $(C_n)_{n=1}^{\infty}$ are elements of $\mathcal{A}$ are such that $X = \bigcup_{n=1}^{\infty} C_n$, $C_n \subset C_{n+1}$ for all $n \in \mathbb{N}$, and $\mu(C_n) < \infty$ for all $n \in \mathbb{N}$. The reason $\mu(C_n) < \infty$ can be seen via the following result as $\sum_{i=1}^{n} \mu(A_i) < \infty$.

Proposition 1.1.20: Subadditivity of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^{\infty}$ be in $\mathcal{A}$. Then

$$\mu \left(\sum_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Proof.

Let $E_1 = A_1$, and for each $n \in \mathbb{N}_{\geq 2}$, denote

$$E_n = A_n \setminus \left(\bigcup_{i=1}^{n} A_i \right)$$

Since $(A_n)_{n=1}^{\infty}$ are in $\mathcal{A}$, by the properties of σ-algebra we have that $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Furthermore, it is clear that $E_n \cap E_m = \emptyset$ if $n \neq m$, $E_n \subset A_n$ for all $n \in \mathbb{N}$, and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$$

Hence, by the definition and monotonicity of measures (Proposition 1.1.14), we obtain that

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

as desired. ■

As seen above, being able to replace our measurable sets with disjoint measurable is a very useful technique. In particular, the same idea is helpful in proving the following theorem.

Theorem 1.1.21: Monotone Convergence Theorem for Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^{\infty}$ be a collection of $\mathcal{A}$. Then the following hold.

(i) If $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mu(A_n)$$

(ii) If $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ with $\mu(A_1) < \infty$, then

$$\mu \left(\bigcap_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mu(A_n)$$

Proof.

To see that the first assertion holds, let $A_0 = \emptyset$ for conventional simplicity, and for each $n \in \mathbb{N}$, define

$$B_n = A_n \setminus A_{n-1}$$

then $(B_n)_{n=1}^{\infty}$ is a collection of pairwise disjoint elements of $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ and $\bigcup_{i=1}^{n} B_i = A_n$ for all $n \in \mathbb{N}$. Hence,

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \mu \left(\bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n \to \infty} \mu \left(\bigcup_{i=1}^{n} B_i \right) = \lim_{n \to \infty} \mu(A_n)$$

proving (i).

To see that the second assertion holds, let $B_n = A_1 \setminus A_n$ for all $n \in \mathbb{N}$, then $(B_n)_{n=1}^{\infty}$ is a collection of elements of $\mathcal{A}$ with $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$. Thus, since

$$\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n \right)$$

from (i),

$$\mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n \right) \right) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \mu(A_1 \setminus A_n)$$

Since $\mu(A_1) < \infty$, then using the part after Proposition 1.1.14, $\mu(A_1 \setminus E) = \mu(A_1) - \mu(E)$ for all $E \in \mathcal{A}$ with $E \subset A_1$. Hence,

$$\begin{aligned} \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n \right) &= \mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n \right) \right) \\ &= \lim_{n \to \infty} (A_1 \setminus A_n) \\ &= \lim_{n \to \infty} (\mu(A_1) - \mu(A_n)) \\ &= \mu(A_1) - \lim_{n \to \infty} \mu(A_n) \end{aligned}$$

By subtracting $\mu(A_1) < \infty$ from both sides, we obtain the desired equality, proving (ii). ∎

Exercise 1.1.22: Assignment 1, Question 1

Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x])$$

for all $x \in \mathbb{R}$. The function F is called the *cumulative distribution function* of μ.

(i) Show that if μ is finite, then F is non-decreasing, right continuous, $\lim_{x \to -\infty} F(x) = 0$, and $\lim_{x \to \infty} F(x) = \mu(\mathbb{R})$.

(ii) Show that if μ is finite and μ has no atoms (that is, $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$), then F is continuous.

1.2   The Carathéodory Method

Based on the above notations, it is very natural to ask whether there exists a measure λ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ that emulates the length of a set. In particular, we desire a measure to have some very natural properties, such as

(i) If I is an interval, then $\lambda(I)$ is the length of I.

(ii) If $A \in \mathcal{P}(\mathbb{R})$, $x \in \mathbb{R}$, and $x + A = \{x + a : a \in A\}$, then $\lambda(x + A) = \lambda(A)$; that is, λ is translation invariant.

However, it turns out that no such measure exists. This can be seen via the following example.

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