Question 1. Extend Lusin's Theorem to measurable functions on $\mathbb{R}$. That is, show that if $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable then for all $\epsilon > 0$ there exists a closed subset $F \subseteq \mathbb{R}$ such that $\lambda(F^c) < \epsilon$ and $f|_F$ is continuous.

Solution. For each $n \in \mathbb{Z}$, let $A_n = [n, n+1]$. Then $\bigcup_{n\in\mathbb{Z}} A_n = \mathbb{R}$. We will apply Lusin's Theorem to each A_n and stitch together the results.

Let $\epsilon > 0$. Without loss of generality, $\epsilon < 1$. Since restricting the Lebesgue measure to A_n satisfies the hypotheses of Lusin's Theorem, for each $n \in \mathbb{Z}$ there exists a closed subset $F_n \subseteq [n, n+1]$ such that $f|_{F_n}$ is continuous and

$$\lambda(A_n \setminus F_n) < \frac{\epsilon}{2^{3+|n|}}.$$

For each $n \in \mathbb{Z}$, let

$$I_n = \left[n + \frac{\epsilon}{2^{4+|n|}}, n + 1 - \frac{\epsilon}{2^{4+|n|}}\right] \quad \text{and} \quad F'_n = F_n \cap I_n.$$

Then F'_n is a closed subset of F_n such that $f|_{F'_n}$ is continuous and

$$\lambda(A_n \setminus F'_n) = \lambda((A_n \setminus F_n) \cup (A_n \setminus I_n)) < \frac{\epsilon}{2^{3+|n|}} + \frac{\epsilon}{2^{3+|n|}} = \frac{\epsilon}{2^{2+|n|}}.$$

Let $F = \bigcup_{n\in\mathbb{Z}} F'_n$. Since $F'_{n\,n\in\mathbb{Z}}$ are pairwise disjoint subsets contained in the pairwise disjoint closed intervals subsets $\{I_n\}_{n\in\mathbb{Z}}$ which have positive separation and since $f|_{F'_n}$ is continuous for all n, it is elementary to show that F is closed and $f|_F$ is continuous (i.e. any sequence that is in F must eventually completely lie in I_{n_0} for some n_0 and thus has distance at least $\frac{\epsilon}{2^{4+|n_0|}}$ from any other I_n). Finally, as

$$\lambda(F^c) = \lambda\left(\bigcup_{n\in\mathbb{Z}} A_n \setminus F'_n\right) \leq \sum_{n\in\mathbb{Z}} \lambda(A_n \setminus F'_n) = \sum_{n\in\mathbb{Z}} \frac{\epsilon}{2^{2+|n|}} < \epsilon,$$

the result follows.

Question 2 (Convergence in Measure). Let $(X, \mathcal{A}, \mu)$ be a measure space and let f and $(f_n)_{n\geq 1}$ be measurable functions. It is said that $(f_n)_{n\geq 1}$ *converge in measure* (also know as *convergence in probability* when μ is a probability measure) to f if for all $\epsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

For this entire question, assume f, g, $(f_n)_{n\geq 1}$, and $(g_n)_{n\geq 1}$ are measurable.

a) Prove that if $\mu(X) < \infty$ and $(f_n)_{n\geq 1}$ converges to f pointwise almost everywhere, then $(f_n)_{n\geq 1}$ converges in measure to f.

b) Give an example of a measure μ and a sequence $(f_n)_{n\geq 1}$ that converges to f pointwise almost everywhere, yet $(f_n)_{n\geq 1}$ does not converges in measure to f. Prove your example satisfies the desired conditions.

c) Give an example of a μ such that $\mu(X) < \infty$ and a sequence $(f_n)_{n\geq 1}$ that converges in measure to f, yet $(f_n)_{n\geq 1}$ does not converges pointwise almost everywhere to f. Prove your example satisfies the desired conditions.

d) Prove that if $(f_n)_{n\geq 1}$ converges in measure to f, then there exists a subsequence $(f_{k_n})_{n\geq 1}$ that converges pointwise almost everywhere to f.

e) Prove that $(f_n)_{n\geq 1}$ converges in measure to f if and only if every subsequence of $(f_n)_{n\geq 1}$ has a further subsequence that converges in measure to f.

f) Prove that if $\mu(X) < \infty$, if $(f_n)_{n\geq 1}$ converges in measure to f, and if $(g_n)_{n\geq 1}$ converges in measure to g, then $(\alpha f_n + g_n)_{n\geq 1}$ converges in measure to $\alpha f + g$ for all $\alpha \in \mathbb{C}$ and $(f_n g_n)_{n\geq 1}$ converges in measure to fg.

g) Give an example of a measure μ and sequences $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ that converge in measure to f and g respectively, yet $(f_n g_n)_{n\geq 1}$ does not converge in measure to fg. Prove your example satisfies the desired conditions.

Solution.
a) Assume $\mu(X) < \infty$ and that $(f_n)_{n\geq 1}$ converges almost everywhere to f. To see that $(f_n)_{n\geq 1}$ converges in measure to f, let $\epsilon > 0$ be arbitrary. Furthermore, let $\delta > 0$ be arbitrary. Since $(f_n)_{n\geq 1}$ converges almost everywhere to f and since $\mu(X) < \infty$, Egoroff's Theorem implies there exists a $B \in \mathcal{A}$ such that $\mu(B) < \delta$ and $(f_n)_{n\geq 1}$ converges uniformly to f on B^c. Hence, since $\epsilon > 0$ and $(f_n)_{n\geq 1}$ converges uniformly to f on B^c, there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$ and $x \in B^c$. Thus, for all $n \geq N$ we have that

$$\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\} \subseteq B.$$

Therefore

$$\limsup_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu(B) < \delta.$$

Therefore, since $\delta > 0$ was arbitrary, we obtain that

$$0 \leq \liminf_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq \limsup_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq 0.$$

Hence $\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$. Therefore, since $\epsilon > 0$ was arbitrary, $(f_n)_{n\geq 1}$ converges in measure to f.

b) Let $\mu = \lambda$ viewed as a measure on $\mathbb{R}$. For each $n \in \mathbb{N}$, let $f_n = \chi_{[n,\infty)}$. It is elementary to see that $(f_n)_{n\geq 1}$ converges pointwise to 0. However, since $|f_n(x) - 0| \geq 1$ for all $x \geq n$, we see that

$$\lambda(\{x \in \mathbb{R} \mid |f_n(x) - f(x)| \geq 1\}) = \infty$$

for all $n \in \mathbb{N}$. Hence $(f_n)_{n\geq 1}$ does not converges in measure to f.

c) Let $X = [0,1]$ and let $\mu = \lambda$ restricted to the Borel sets on $[0,1]$. Let $(I_n)_{n\geq 1}$ be the sequence of intervals

$$I_1 = \left[0, \frac{1}{2}\right] \qquad I_2 = \left[\frac{1}{2}, 1\right] \qquad I_3 = \left[0, \frac{1}{4}\right] \qquad I_4 = \left[\frac{1}{4}, \frac{1}{2}\right]$$
$$I_5 = \left[\frac{1}{2}, \frac{3}{4}\right] \qquad I_6 = \left[\frac{3}{4}, 1\right] \qquad I_7 = \left[0, \frac{1}{8}\right] \qquad I_i = \left[\frac{1}{8}, \frac{1}{4}\right]$$

and so on. In particular, if $\sum_{k=1}^{N-1} 2^k \leq n < \sum_{k=1}^{N} 2^k$, then

$$I_n = \left[\frac{n - \sum_{k=1}^{N-1} 2^k}{2^N}, \frac{1 + n - \sum_{k=1}^{N-1} 2^k}{2^N}\right].$$

For each $n \in \mathbb{N}$, let $f_n = \chi_{I_n}$. We claim that $(f_n)_{n\geq 1}$ converges to 0 in measure yet $(f_n)_{n\geq 1}$ does not converges to 0 pointwise almost everywhere. To see that $(f_n)_{n\geq 1}$ converges to 0 in measure, let $\epsilon > 0$. Notice if $\sum_{k=1}^{N-1} 2^k \leq n < \sum_{k=1}^{N} 2^k$ then

$$\lambda(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = \lambda(\{x \in X \mid |\chi_{I_n}(x) - 0| \geq \epsilon\}) = \lambda(I_n) = \frac{1}{2^N}.$$

Hence, it easily follows that $\lim_{n\to\infty} \lambda(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$ so $(f_n)_{n\geq 0}$ converges to 0.

To see that $(f_n)_{n\geq 0}$ does not converges to 0 pointwise almost everywhere, note for each $x \in [0,1]$ that there are an infinite number of natural numbers n such that $f_n(x) = 1$. Hence $(f_n)_{n\geq 0}$ does not converges to 0 pointwise anywhere.

d) Let f and $(f_n)_{n\geq 1}$ be measurable functions such that $(f_n)_{n\geq 1}$ converges in measure to f. Hence for all $\epsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

Therefore for each $n \in \mathbb{N}$ there exists an $M_n \in \mathbb{N}$ such that if $k \geq M_n$ then

$$\mu\left(\left\{x \in X \,\middle|\, |f_k(x) - f(x)| \geq \frac{1}{2^n}\right\}\right) < \frac{1}{2^n}.$$

Clearly we can choose an increasing sequence $(k_n)_{n\geq 1}$ of natural numbers such that $k_n \geq M_n$ for all $n \in \mathbb{N}$ (i.e. choose $k_1 = M_1$ and $k_{n+1} = 1 + \max\{k_n, M_{n+1}\}$ for all $n \geq 2$).

We claim that $(f_{k_n})_{n\geq 1}$ converges to f almost everywhere. To see this, for each $n \in \mathbb{N}$ define

$$A_n = \left\{x \in X \,\middle|\, |f_{k_n}(x) - f(x)| \geq \frac{1}{2^n}\right\},$$

which is clearly measurable as f_{k_n} and f are measurable. By the above we see that $\mu(A_n) \leq \frac{1}{2^n}$. For each $m \in \mathbb{N}$ let

$$B_m = \bigcup_{m=n}^{\infty} A_m$$

and let $A = \bigcap_{n=1}^{\infty} B_m$. Since $\mathcal{A}$ is a σ-algebra, clearly $B_m \in \mathcal{A}$ for all $m \in \mathbb{N}$ so $A \in \mathcal{A}$. Furthermore, since $B_{m+1} \subseteq B_m$ for all $m \in \mathbb{N}$ and since

$$\mu(B_m) \leq \sum_{n=m}^{\infty} \mu(A_n) < \sum_{n=m}^{\infty} \frac{1}{2^n} = \frac{1}{2^{m-1}} < \infty,$$

we obtain by the Monotone Convergence Theorem for measures that

$$0 \leq \mu(A) = \lim_{m\to\infty} \mu(B_m) \leq \limsup_{m\to\infty} \frac{1}{2^{m-1}} = 0.$$

Since $\mu(A) = 0$, to complete the proof it suffices to show that $(f_{k_n})_{n\geq 1}$ converges to f on A^c. To see this, let $x \in A^c$ be arbitrary. By the definition of A there exists an $N \in \mathbb{N}$ such that $x \notin B_N$. Therefore $x \notin A_n$ for all $n \geq N$, which implies that

$$|f_{k_n}(x) - f(x)| \leq \frac{1}{2^n}$$

for all $n \geq N$ and thus $\lim_{n\to\infty} f_{k_n}(x) = f(x)$. Therefore, since $x \in A^c$ was arbitrary, we obtain that $(f_{k_n})_{n\geq 1}$ converges to f on A^c. Hence $(f_{k_n})_{n\geq 1}$ that converges almost everywhere to f.

e) First, assume that $(f_n)_{n\geq 1}$ converges in measure to f. Clearly the definition of convergence in measure implies that every subsequence of $(f_n)_{n\geq 1}$ converges in measure to f. Consequently, every subsequence of every subsequence of $(f_n)_{n\geq 1}$ converges in measure to f.

To see the converse, for the sake of a contradiction that $(f_n)_{n\geq 1}$ does not converge in measure to f. Hence there exists an $\epsilon > 0$ such that

$$(\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}))_{n\geq 1}$$

does not converge to 0. Hence, by elementary properties of the real numbers, there exists a $\delta > 0$ and a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ such that

$$\mu(\{x \in X \mid |f_{n_k}(x) - f(x)| \geq \epsilon\}) \geq \delta$$

for all $k \geq 1$. Consequently, if $(f_{n_{k_j}})_{j\geq 1}$ is any subsequence of $(f_{n_k})_{k\geq 1}$, then

$$\mu(\{x \in X \mid |f_{n_{k_j}}(x) - f(x)| \geq \epsilon\}) \geq \delta$$

for all $j \geq 1$ so

$$\lim_{j\to\infty} \mu(\{x \in X \mid |f_{n_{k_j}}(x) - f(x)| \geq \epsilon\} \neq 0$$

and thus $(f_{n_{k_j}})_{j\geq 1}$ does not converge in measure to f. Hence, $(f_{n_k})_{k\geq 1}$ is a subsequence of $(f_n)_{n\geq 1}$ such that no subsequence of $(f_{n_k})_{k\geq 1}$ converges in measure to f. Thus the proof is complete.

f) First we will demonstrate that $(\alpha f_n + g)_{n\geq 1}$ converges in measure to $(\alpha f + g)$. Clearly if $\alpha = 0$ the result is trivial. Hence we may assume that $\alpha \neq 0$. Notice for a fixed $\epsilon > 0$ that if $x \in X$ is such that

$$\epsilon \leq |(\alpha f_n(x) + g_n(x) - (\alpha f(x) + g(x))| \leq |\alpha||f_n(x) - f(x)| + |g_n(x) - g(x)|,$$

then it must be the case that either $|\alpha||f_n(x) - f(x)| \geq \frac{\epsilon}{2}$ or $|g_n(x) - g(x)| \geq \frac{\epsilon}{2}$. Thus

$$\begin{aligned}
&\{x \in X \mid |(\alpha f_n(x) + g_n(x) - (\alpha f(x) + g(x))| \geq \epsilon\} \\
&\subseteq \left\{x \in X \,\middle|\, |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\right\} \cup \left\{x \in X \,\middle|\, |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}.
\end{aligned}$$

Therefore, due to the subadditivity of measures and the facts that $(f_n)_{n\geq 1}$ converges in measure to f and $(g_n)_{n\geq 1}$ converges in measure to g, we have that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \mu(\{x \in X \mid |(\alpha f_n(x) + g_n(x) - (\alpha f(x) + g(x))| \geq \epsilon\}) \\
&\leq \limsup_{n\to\infty} \mu\left(\left\{x \in X \,\middle|\, |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\right\}\right) + \mu\left(\left\{x \in X \,\middle|\, |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}\right) \\
&= 0 + 0 = 0.
\end{aligned}$$

Hence $\lim_{n\to\infty} \mu(\{x \in X \mid |(\alpha f_n(x) + g_n(x) - (\alpha f(x) + g(x))| \geq \epsilon\}) = 0$. Therefore, since $\epsilon > 0$ was arbitrary, $(\alpha f_n + g)_{n\geq 1}$ converges in measure to $(\alpha f + g)$.

To see that $(f_n g_n)_{n\geq 1}$ converges in measure to fg, we will invoke parts a), d), and e). To being, let $(f_{n_k} g_{n_k})_{k\geq 1}$ be an arbitrary subsequence of $(f_n g_n)_{n\geq 1}$. Since $(f_{n_k})_{k\geq 1}$ is a subsequence of $(f_n)_{n\geq 1}$ and thus converges in measure to f, part d) of this problem implies there exists a subsequence $(f_{n_{k_j}})_{j\geq 1}$ that converges to f pointwise. Since $(g_{n_{k_j}})_{j\geq 1}$ is a subsequence of $(g_n)_{n\geq 1}$ and thus converges in measure to g, part d) of this problem implies there exists a subsequence $(g_{n_{k_{j_l}}})_{l\geq 1}$ that converges to g pointwise. However since $(f_{n_{k_j}})_{j\geq 1}$ converges to f pointwise, so too does $(f_{n_{k_{j_l}}})_{l\geq 1}$. Hence $(f_{n_{k_{j_l}}} g_{n_{k_{j_l}}})_{l\geq 1}$ converges to fg pointwise. Thus part a) of this problem implies that the subsequence $(f_{n_{k_{j_l}}} g_{n_{k_{j_l}}})_{l\geq 1}$ of $(f_{n_k} g_{n_k})_{k\geq 1}$ converges in measure to fg as μ is finite. Therefore, since $(f_{n_k} g_{n_k})_{k\geq 1}$ was arbitrary, $(f_n g_n)_{n\geq 1}$ converges in measure to fg by part e) of this problem.

g) Let $\mu = \lambda$. For each $n \in \mathbb{N}$, let $f_n = \frac{1}{n}\chi_{(0,n)}$ and let $g_n(x) = x$ for all $x \in \mathbb{R}$. Clearly if $g(x) = x$ for all $x \in \mathbb{R}$, then $(g_n)_{n\geq 1}$ converges to g in measure. We claim that $(f_n)_{n\geq 1}$ converges to 0 in measure. Indeed, if $\epsilon > 0$ and $N \in \mathbb{N}$ is such that $\frac{1}{N} < \epsilon$, then for all $n \geq N$

$$\lambda(\{x \in \mathbb{R} \mid |f_n(x) - f(x)| \geq \epsilon\}) = \lambda\left(\left\{x \in (0,n) \,\middle|\, \frac{1}{n}\chi_{(0,n)}(x) \geq \epsilon\right\}\right) = 0.$$

Hence $(f_n)_{n\geq 1}$ converges to 0 in measure.

We claim that $(f_n g_n)_{n\geq 1}$ does not converge to $fg = 0$ in measure. To see this, notice that

$$(f_n g_n)(x) = \frac{x}{n}\chi_{(0,n)}.$$

Therefore, when $\epsilon = \frac{1}{2}$, we see that

$$\lambda(\{x \in \mathbb{R} \mid |(f_n g_n)(x) - 0| \geq \epsilon\}) = \lambda\left(\left[\frac{n}{2}, n\right]\right) = \frac{n}{2}$$

and thus

$$\lim_{n\to\infty} \lambda(\{x \in \mathbb{R} \mid |(f_n g_n)(x) - 0| \geq \epsilon\}) = \infty.$$

Hence $(f_n g_n)_{n\geq 1}$ does not converge to $fg = 0$ in measure.

Question 3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Borel* if $f^{-1}(U)$ is a Borel set for all open subsets $U \subseteq \mathbb{R}$.

a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is Borel.

b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, bijective, and has a continuous inverse (i.e. f is a homeomorphism), then $\{f(B) \mid B \in \mathfrak{B}(\mathbb{R})\} = \mathfrak{B}(\mathbb{R})$ (that is, f induces a bijective on the Borel sets).

c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ λ-almost everywhere.

Solution.
a) Recall that the Borel subsets of $\mathbb{R}$, denoted $\mathfrak{B}(\mathbb{R})$, is a σ-algebra and note that the notion that a function $f : \mathbb{R} \to \mathbb{R}$ being Borel is equivalent to the notion that f is measurable with respect to the Borel σ-algebra. Therefore, it is elementary to see that every continuous function is Borel.

Assume $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Hence f is continuous. For each $n \in \mathbb{N}$ consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}}$$

for all $n \in \mathbb{N}$. Clearly f_n is continuous for all $n \in \mathbb{N}$. Moreover, since f is differentiable, we see that f' is the pointwise limit of $(f_n)_{n\geq 1}$. Therefore, since each f_n is Borel (being continuous) and the pointwise limit of measurable functions is measurable, we obtain that f' is Borel.

b) Let

$$\mathcal{A} = \{A \subseteq \mathbb{R} \mid g^{-1}(A) \in \mathfrak{B}(\mathbb{R})\}.$$

By the proof from class, we see that $\mathcal{A}$ is a σ-algebra on $\mathbb{R}$ that contains the open subsets of $\mathbb{R}$. Hence $\mathfrak{B}(\mathbb{R}) \subseteq \mathcal{A}$. Thus

$$\begin{aligned} g^{-1}(\mathfrak{B}(\mathbb{R})) &\subseteq g^{-1}(\mathcal{A}) \\ &= \{g^{-1}(A) \mid A \subseteq \mathbb{R}, g^{-1}(A) \in \mathfrak{B}(\mathbb{R})\} \\ &\subseteq \mathfrak{B}(\mathbb{R}) \end{aligned}$$

so $\mathfrak{B}(\mathbb{R}) \subseteq g(\mathfrak{B}(\mathbb{R}))$. Similarly

$$\mathcal{A}' = \{A \subseteq \mathbb{R} \mid g(A) = (g^{-1})^{-1}(A) \in \mathfrak{B}(\mathbb{R})\}$$

is a σ-algebra on $\mathbb{R}$ such that $\mathfrak{B}(\mathbb{R}) \subseteq \mathcal{A}'$. Thus

$$\begin{aligned} g(\mathfrak{B}(\mathbb{R})) &\subseteq g(\mathcal{A}') \\ &= \{g(A) \subseteq \mathbb{R} \mid A \subseteq \mathbb{R}, g(A) \text{ is Borel in } \mathbb{R}\} \\ &\subseteq \mathfrak{B}(\mathbb{R}). \end{aligned}$$

Hence g is a bijection between the Borel subsets of $\mathbb{R}$.

c) First we claim that if $\varphi : \mathbb{R} \to [0, \infty)$ is a simple function with respect to the Lebesgue measure, then there exists a Borel function $\psi : \mathbb{R} \to [0, \infty)$ such that $\varphi = \psi$ λ-almost everywhere. To see this, let $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$ be the canonical representation of φ. By Assignment 1, Question 3, for each $k \in \{1, \ldots, n\}$ there exists a Borel set B_k such that $B_k \subseteq A_k$ and $\lambda(A_k \setminus B_k) = 0$. Let $\psi = \sum_{k=1}^{n} a_k \chi_{B_k}$, which is clearly a Borel function. Furthermore, notice that

$$\{x \in \mathbb{R} \mid \psi(x) \neq \varphi\} \subseteq \bigcup_{k=1}^{n} A_k \setminus B_k$$

due to the fact that $\mathbb{R} = \bigcup_{k=1}^{n} A_k$ by the canonical representation of a simple function. Hence

$$0 \leq \lambda(\{x \in \mathbb{R} \mid \psi(x) \neq \varphi\}) \leq \sum_{k=1}^{n} \lambda(A_k \setminus B_k) = 0$$

so $\varphi = \psi$ λ-almost everywhere as desired.

To proceed, let $f : \mathbb{R} \to [0, \infty)$ be Lebesgue measurable. Thus there exists an increasing sequence of simple functions $(\varphi_n)_{n\geq 1}$ that converges to f pointwise. By the first part of this proof, there exists a sequence $(\psi_n)_{n\geq 1}$ of Borel functions such that $\varphi_n = \psi_n$ λ-almost everywhere. Therefore, since the countable union of measure zero sets has measure zero, we obtain that $(\psi_n)_{n\geq 1}$ is a sequence of Borel functions that converge to f λ-almost everywhere. However, to obtain a Borel function, we must verify convergence everwhere.

Since $(\psi_n)_{n\geq 1}$ converge to f λ-almost everywhere, there exists a Lebesgue measurable set E such that $\lambda(E) = 0$ and $\lim_{n\to\infty} \psi_n(x) = f(x)$ for all $x \in E^c$. By Assignment 1, Question 3 there exists a Borel set G such that $E \subseteq G$ and $\lambda(G \setminus E) = 0$. Hence $\lambda(G) = 0$ and $\lim_{n\to\infty} \psi(x) = f(x)$ for all $x \in G^c$.

Consider the sequence $(\psi_n \chi_{G^c})$. Since the product of measurable functions is measurable, the product of Borel functions is Borel. Therefore, since G and thus G^c is Borel, $\psi_n \chi_{G^c}$ is Borel for all n. Furthermore since $(\psi_n \chi_{G^c})_{n\geq 1}$ converges to $f\chi_{G^c}$ pointwise by construction and since the pointwise limit of Borel functions is Borel, $f\chi_{G^c}$ is Borel. Therefore, since $\lambda(G) = 0$, we obtain that $f = f\chi_{G^c}$ λ-almost everywhere. Hence the proof is complete for non-negative Lebesgue measurable functions.

Finally let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary Lebesgue measurable. Therefore $f_+, f_- : \mathbb{R} \to [0, \infty)$ are Lebesgue measurable. Hence there exists Borel functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that $f_+ = g_1$ λ-almost everywhere and $f_- = g_2$ λ-almost everywhere. Since $g = g_1 - g_2$ is a Borel function (since the sum of two measurable functions using any σ-algebra on the domain is measurable) and $f = g$ λ-almost everywhere, the proof is complete.

Question 4 (The Cantor Ternary Function)**.** Given a sequence $\vec{a} = (a_n)_{n\geq 1}$ of elements of $\{0,1,2\}$, define

$$K_{\vec{a}} = \begin{cases} N & \text{if } a_N = 1 \text{ and } a_k \neq 1 \text{ for all } k < N \\ \infty & \text{otherwise} \end{cases}$$

and define a sequence $\vec{b}_{\vec{a}} = (b_n)_{n\geq 1}$ of elements of $\{0,1\}$ by

$$b_n = \begin{cases} \frac{a_n}{2} & \text{if } n \leq K_{\vec{a}} \\ 1 & \text{if } n = K_{\vec{a}} \\ 0 & \text{otherwise} \end{cases}.$$

The *Cantor ternary function* is the function $f : [0,1] \to [0,1]$ defined as follow: if $x \in [0,1]$, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for a sequence $\vec{a} = (a_n)_{n\geq 1}$ of elements of $\{0,1,2\}$, and $\vec{b}_{\vec{a}} = (b_n)_{n\geq 1}$ is the sequence of elements of $\{0,1\}$ as defined above, then

$$f(x) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}.$$

(i.e. Write a ternary expansion of x. If N is the first index where a 1 occurs, replace each $\frac{0}{3^n}$ with $n < N$ with $\frac{0}{2^n}$, replace each $\frac{2}{3^n}$ with $n < N$ with $\frac{1}{2^n}$, replace $\frac{1}{3^N}$ with $\frac{1}{2^N}$, and change all terms of index greater than N to zero). As ternary expansions are not unique, one must verify that f is well-defined. Convince yourself that f is well-defined (you do not need to hand-in a proof).

a) Prove that f is a continuous, non-decreasing function on $[0,1]$ such that f is constant on each interval in $\mathcal{C}^c$ and $f(\mathcal{C}) = [0,1]$.

b) Define $\psi : [0,1] \to [0,2]$ by $\psi(x) = x + f(x)$ for all $x \in [0,1]$. Prove that ψ is a strictly increasing continuous function such that $\psi(\mathcal{C})$ is Lebesgue measurable with $\lambda(\psi(\mathcal{C})) > 0$ and that there exists a subset $B \subseteq \mathcal{C}$ such that $\psi(B)$ is not Lebesgue measurable.

c) Prove that there exists a subset $B \subseteq \mathcal{C}$ such that B is not Borel. (Therefore B is a Lebesgue measurable subset that is not Borel.)

d) Prove there exists Lebesgue measurable functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $g \circ h$ is not Lebesgue measurable.

Solution. a) Since f is well-defined we know that the ternary expansion we choose for each point in $[0,1]$ does not affect the value of f. Therefore, for each point in $[0,1]$ with two ternary expansions, select one to use throughout.

To see that f is constant on $\mathcal{C}^c$, notice by the definition of $\mathcal{C}$ (Definition **??**) that

$$\mathcal{C}^c = \bigcup_{n\geq 0} \bigcup_{a_1,\ldots,a_n \in \{0,2\}} I_{n;a_1,\ldots,a_n}$$

where

$$I_{n;a_1,\ldots,a_n} = \left\{ x = \sum_{k=1}^{\infty} \frac{a'_k}{3^{-k}} \,\middle|\, \begin{array}{l} a'_k \in \{0,1,2\}, a'_{n+1} = 1, \text{ and} \\ a'_k = a_k \text{ for all } k \in \{1,\ldots,n\} \end{array} \right\}.$$

Therefore, by the definition of f we see that

$$f(x) = \sum_{k=1}^{n} \frac{\frac{1}{2}a_n}{2^n} + \frac{1}{2^{n+1}}$$

for all $x \in I_{n;a_1,\ldots,a_n}$. Hence f is constant on each interval in $\mathcal{C}^c$.

To see that f is non-decreasing, let $x, y \in [0,1]$ be such that $x < y$ and write the ternary expansions of x and y as

$$x = \sum_{k=1}^{\infty} \frac{a_k(x)}{3^k} \qquad \text{and} \qquad y = \sum_{k=1}^{\infty} \frac{a_k(y)}{3^k}.$$

Since $x \neq y$, due to our assumed uniqueness of the ternary expansions there exists a $q \in \mathbb{N}$ such that $a_q(x) \neq a_q(y)$ and $a_k(x) = a_k(y)$ for all $k < q$. We claim that $a_q(x) < a_q(y)$. Indeed if $a_q(x) > a_q(y)$ then, since $a_k(x), a_k(y) \in \{0, 1, 2\}$ for all $k \in \mathbb{N}$, we see that

$$\begin{aligned} y - x &= \sum_{k=1}^{\infty} \frac{a_k(y)}{3^k} - \sum_{k=1}^{\infty} \frac{a_k(x)}{3^k} \\ &= \frac{a_q(y) - a_q(x)}{3^q} + \sum_{k=q+1}^{\infty} \frac{a_k(y) - a_k(x)}{3^k} \\ &\leq \frac{-1}{3^q} + \sum_{k=q+1}^{\infty} \frac{a_k(y) - a_k(x)}{3^k} \\ &\leq \frac{-1}{3^q} + \sum_{k=q+1}^{\infty} \frac{2}{3^k} \\ &= 0, \end{aligned}$$

which is a contradiction. Hence $a_q(x) < a_q(y)$.

Using the index q we can show that $f(x) \leq f(y)$. To do this we divide the proof into three cases:

(1) There exists an $k < q$ such that $a_k(x) = a_k(y) = 1$.

(2) Case (1) does not occur and $a_q(x) = 0$ (and thus $a_q(y) \in \{1, 2\}$).

(3) Case (1) does not occur and $a_q(x) = 1$ (and thus $a_q(y) = 2$).

To begin, in all cases write

$$f(x) = \sum_{k=1}^{\infty} \frac{b_k(x)}{2^k} \qquad \text{and} \qquad f(y) = \sum_{k=1}^{\infty} \frac{b_k(y)}{2^k}$$

where the sequences $(b_k(x))_{k\geq 1}$ and $(b_k(y))_{k\geq 1}$ are determined from the sequences $(a_k(x))_{k\geq 1}$ and $(a_k(y))_{k\geq 1}$ via the construction of the Cantor ternary function.

In case (1), note that $(b_k(x))_{k\geq 1} = (b_k(y))_{k\geq 1}$ by definition. Hence $f(x) = f(y)$ as desired.

In case (2), note that $b_k(x) = b_k(y)$ for all $k < q$, that $b_q(x) = 0$, and that $b_q(y) = 1$. Therefore, since $b_k(x), b_k(y) \in \{0, 1\}$ for all $k \in \mathbb{N}$, we see that

$$\begin{aligned} f(y) - f(x) &= \sum_{k=1}^{\infty} \frac{b_k(y)}{2^k} - \sum_{k=1}^{\infty} \frac{b_k(x)}{2^k} \\ &= \frac{1}{2^q} + \sum_{k=q+1}^{\infty} \frac{b_k(y) - b_k(x)}{2^k} \\ &\geq \frac{1}{2^q} + \sum_{k=q+1}^{\infty} \frac{-1}{2^k} \\ &= 0. \end{aligned}$$

Hence $f(x) \leq f(y)$ in case (2).

Finally, in case (3), note that $b_k(x) = b_k(y)$ for all $k < q$, that $b_q(x) = 1$, that $b_k(x) = 0$ for all $k > q$, and that $b_q(y) = 1$. Therefore, since $b_k(x), b_k(y) \in \{0, 1\}$ for all $k \in \mathbb{N}$, we see that

$$\begin{aligned} f(y) - f(x) &= \sum_{k=1}^{\infty} \frac{b_k(y)}{2^k} - \sum_{k=1}^{\infty} \frac{b_k(x)}{2^k} \\ &= \sum_{k=q+1}^{\infty} \frac{b_k(y) - b_k(x)}{2^k} \\ &= \sum_{k=q+1}^{\infty} \frac{b_k(y)}{2^k} \\ &\geq 0. \end{aligned}$$

Hence $f(x) \leq f(y)$ in case (3). Therefore, by combining all of the cases, we obtain that f is non-decreasing and thus monotone.

To see that f is continuous, first notice that f is continuous at each point in $\mathcal{C}^c$ since f is constant on each open interval of $\mathcal{C}^c$. Thus it remains to demonstrate that f is continuous at each point in $\mathcal{C}$. To see this, fix $x \in \mathcal{C}$ and let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$. By Definition **??** there exists $a_1, \ldots, a_n \in \{0, 2\}$ such that

$$x \in \left[\sum_{k=1}^{n} \frac{a_k}{3^k}, \frac{1}{3^n} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right].$$

Consider the open interval $I = (y, z)$ where

$$y = -\frac{1}{3^n} + \sum_{k=1}^{n} \frac{a_k}{3^k} \qquad \text{and} \qquad z = \frac{2}{3^n} + \sum_{k=1}^{n} \frac{a_k}{3^k}$$

Clearly $x \in I$. We divide the discussion into two cases based on the value of a_n.

Assume $a_n = 0$. Let m be the greatest natural number such that $a_k = 0$ for all $k \geq m$ yet $a_{m-1} \neq 0$ (so $a_{m-1} = 2$). Then

$$f(y) = f\left(\sum_{k=1}^{m-2} \frac{a_k}{3^k} + \frac{1}{3^{m-1}} + \sum_{k=m}^{n-1} \frac{2}{3^m} + \frac{1}{3^n} + \sum_{k=n+1}^{\infty} \frac{2}{3^n}\right) = \sum_{k=1}^{m-2} \frac{\frac{a_k}{2}}{2^k} + \frac{1}{2^{m-1}}$$

whereas

$$f(z) = \sum_{k=1}^{n-1} \frac{\frac{a_k}{2}}{2^k} + \frac{1}{2^n} = \sum_{k=1}^{m-1} \frac{\frac{a_k}{2}}{2^k} + \frac{1}{2^n} = f(y) + \frac{1}{2^n}$$

(since $a_k = 0$ for all $k \geq m$). Therefore, since f is non-decreasing, we see for all $q \in I$ that

$$f(y) \leq f(q) \leq f(z) = f(y) + \frac{1}{2^n}.$$

Hence $|f(x) - f(q)| < \frac{1}{2^n} < \epsilon$ for all $q \in I$ so f is continuous at x.

Otherwise $a_n = 2$. Let m be the greatest natural number such that $a_k = 2$ for all $k \geq m$ yet $a_{m-1} \neq 2$ (so $a_{m-1} = 0$). Then

$$f(z) = f\left(\sum_{k=1}^{m-2} \frac{a_k}{3^k} + \frac{1}{3^{m-1}} + \sum_{k=m}^{n-1} \frac{0}{3^m} + \frac{1}{3^n}\right) = \sum_{k=1}^{m-2} \frac{\frac{a_k}{2}}{2^k} + \frac{1}{2^{m-1}}$$

whereas

$$f(y) = \sum_{k=1}^{n} \frac{\frac{a_k}{2}}{2^k} = \sum_{k=1}^{m-2} \frac{\frac{a_k}{2}}{2^k} + \sum_{k=m}^{n} \frac{1}{2^k} = f(z) - \frac{1}{2^{m-1}} + \sum_{k=m}^{n} \frac{1}{2^k} = f(z) - \frac{1}{2^n}.$$

Therefore, since f is non-decreasing, we see for all $q \in I$ that

$$f(y) \leq f(q) \leq f(z) = f(y) + \frac{1}{2^n}.$$

Hence $|f(x) - f(q)| < \frac{1}{2^n} < \epsilon$ for all $q \in I$ so f is continuous at x. Hence f is continuous on $[0, 1]$.

Finally, clearly $f(0) = 0$ and $f(1) = 1$. Therefore, since f is non-decreasing, the Intermediate Value Theorem immediately implies that $f(\mathcal{C}) = [0, 1]$.

b) To see that ψ is a strictly increasing continuous function, first note that ψ is continuous being the sum of continuous functions. To see that ψ is strictly increasing, note if $x_1, x_2 \in [0, 1]$ are such that $x_1 < x_2$, then, since f is non-decreasing,

$$\psi(x_1) = x_1 + f(x_1) < x_2 + f(x_1) \leq x_2 + f(x_2) = \psi(x_2).$$

Hence ψ is strictly increasing.

Since ψ is a continuous function and since $\mathcal{C}$ is compact, $\psi(\mathcal{C})$ is a compact set. Therefore, since compact sets are Lebesgue measurable, $\psi(\mathcal{C})$ is Lebesgue measurable. To see that $\lambda(\psi(\mathcal{C})) > 0$, first notice since ψ is a strictly increasing continuous function that if $[a,b] \subseteq [0,1]$ then $\psi([a,b]) = [\psi(a), \psi(b)]$. Therefore, if $(a,b) \subseteq \mathcal{C}^c$ then $f(a) = f(b)$ since f is continuous and constant on each interval of $\mathcal{C}^c$ by construction. Thus

$$\lambda^*(\psi(a,b)) \leq \lambda^*(\psi([a,b])) = \lambda([\psi(a), \psi(b)]) = \psi(b) - \psi(a) = b - a.$$

Moreover, since ψ is strictly increasing (and thus injective), notice $[0,2] = \psi(\mathcal{C}) \cup \psi(\mathcal{C}^c)$ and $\psi(\mathcal{C}) \cap \psi(\mathcal{C}^c) = \emptyset$. Hence, since $\mathcal{C}^c$ is a disjoint union of intervals whose sum of lengths is one, the above computation shows that

$$\lambda^*(\psi(\mathcal{C}^c)) \leq 1$$

so $\lambda(\psi(\mathcal{C})) \geq 1 > 0$.

To construct $B \subseteq \mathcal{C}$ such that $\psi(B)$ is not Lebesgue measurable, note that since $\lambda(\psi(\mathcal{C})) > 0$, there exists a subset $D \subseteq \psi(C)$ such that D is not Lebesgue measurable by Assignment 1, Question 6. Therefore, if $B = \psi^{-1}(D) \subseteq \mathcal{C}$ then $\psi(B) = D$ is not Lebesgue measurable.

c) Let $B \subseteq \mathcal{C}$ be as in part b). We claim that B is not Borel. To see this note since ψ is a continuous, strictly increasing function, ψ^{-1} is a continuous and strictly increasing function. Hence ψ^{-1} is a Borel function. Therefore, if B is a Borel set, then $\psi^{-1}(B)$ is a Borel set which contradicts the fact that $\psi^{-1}(B)$ is not Lebesgue measurable. Hence B is not a Borel set.

d) Let ψ be as in part b). Since $\psi : [0,1] \to [0,2]$ is continuous and strictly increasing, $\psi^{-1} : [0,2] \to [0,1]$ exists, is strictly increasing, and continuous. Thus $h = \psi^{-1}$ is Lebesgue measurable.

Let $g = \chi_B$ where B is as in part b). Since $B \subseteq \mathcal{C}$, since $\lambda(\mathcal{C}) = 0$, and since the Lebesgue measure is complete, B is Lebesgue measurable and thus g is Lebesgue measurable. However, notice that

$$(g \circ h)^{-1}(\{1\}) = h^{-1}(g^{-1}(\{1\})) = h^{-1}(B) = \psi(B)$$

which is not Lebesgue measurable by part b). Hence $g, h : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable functions such that $g \circ h$ is not Lebesgue measurable.