# Properties of the Lebesgue Measure

**Corollary:** The Borel sets are Lebesgue Measurable and $\lambda(I) = \ell(I)$ for all intervals $I$.

**Proof:** Recall every set of the form $(a,b]$, $(a,\infty)$ and $(-\infty,b]$ are Lebesgue-Stieltjes measurable and thus Lebesgue measurable by the Carathéodory-Hahn extension theorem and $\lambda(I) = \ell(I)$ for all $I$ of this form. Since $(a,b]$, $(a,\infty)$ and $[-\infty, b)$ generate $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R})$. Note that

$$\lambda(\{c\}) = \lim_{n\to\infty} \lambda((c-\tfrac{1}{n}, c]) = 0$$

by the Monotone Convergence Theorem for Measures.

Since every other interval can be obtained by adding or removing at most 2 singletons, the claim follows.

**Corollary:** $\lambda$ is $\sigma$-finite.

**Proof:** $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n,n]$ with $[-n,n] \in \mathcal{M}(\mathbb{R})$ and $\lambda([-n,n]) = 2n < \infty$.

**Corollary:** If $A \subseteq \mathbb{R}$ is countable, then $A \in \mathcal{M}(\mathbb{R})$ and $\lambda(A) = 0$.

**Proof:** Let $A = (a_n)_{n=1}^{\infty}$. Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, let $I_n = (a_n - \frac{\varepsilon}{2^n}, a_n + \frac{\varepsilon}{2^n})$. Then $A \subseteq \bigcup_{n=1}^{\infty} I_n$, so $0 \leq \lambda^*(A) \leq \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n-1}}$. Thus, as $\varepsilon > 0$ was arbitrary, $\lambda^*(A) = 0$, so $A \in \mathcal{M}(\mathbb{R})$.

**Example:** Let $P_0 = [0,1]$. Let $P_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$ (remove middle third of $P_0$). Let $P_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$, i.e. remove the open interval in the centre of each closed interval of length $\frac{1}{3}$ of each closed set.

Repeat to get a sequence $P_n$, where $P_n$ is the union of $2^n$ disjoint closed intervals of length $\frac{1}{3^n}$. Then


$\mathcal{C} = \bigcap_{n=1}^{\infty} P_n$ is called the Cantor Set,

**Facts:**

- $\mathcal{C}$ is closed, so $\mathcal{C} \in \mathcal{M}(\mathbb{R})$
- $\mathcal{C}^\circ = \emptyset$
- If $x \in [0,1]$, $x \in \mathcal{C}$ if and only if
$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \qquad a_n \in \{0,2\} \quad \forall n \in \mathbb{N}.$$
- $f: \mathcal{C} \to [0,1]$ by $f\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$ is a bijection, so $|\mathcal{C}| = |\mathbb{R}| = |[0,1]|$
- $\lambda(\mathcal{C}) = 0$. Indeed, $\lambda(P_n) = \left(\frac{2}{3}\right)^n \to 0$, so $\lambda(\mathcal{C}) = 0$ by Monotone Convergence Theorem.
- If $A \subseteq \mathcal{C}$ then $A \in \mathcal{M}(\mathbb{R})$ because $\lambda$ is complete.

Thus, $\mathcal{P}(\mathcal{C}) \subseteq \mathcal{M}(\mathbb{R})$. Hence, $|\mathcal{M}(\mathbb{R})| = |\mathcal{P}(\mathbb{R})|$.

**Example:** The "one-element from each equivalence class set is <u>not</u> Lebesgue Measurable.

**Lemma:** If $A \in \mathcal{M}(\mathbb{R})$, then $x+A \in \mathcal{M}(\mathbb{R})$ for all $x \in \mathbb{R}$ with $\lambda(x+A) = \lambda(A)$.

**Proof:** Note that $\lambda^*(x+A) = \lambda^*(A)$ as translation of open interval covers yields open interval covers of the same length. Assume $A \in \mathcal{M}(\mathbb{R})$. Let $B \subseteq \mathbb{R}$. Then

$$\begin{aligned}\lambda^*(B) &= \lambda^*(-x+B) \\ &= \lambda^*((-x+B)\cap A) + \lambda((-x+B)\cap A^c) \\ &= \lambda^*(B\cap(x+A)) + \lambda(B\cap(x+A^c))\end{aligned}$$

Therefore, $x+A \in \mathcal{M}(\mathbb{R})$.

**Lemma:** If $A \in \mathcal{M}(\mathbb{R})$ and $\alpha \in \mathbb{R}$, then $\alpha A \in \mathcal{M}(\mathbb{R})$ and $\lambda(\alpha A) = |\alpha| \lambda(A)$.

**Remark:** The "one element from each equivalence class set" cannot be Lebesgue Measurable, as we can repeat the proof we did earlier using the first lemma to get a contradiction. Hence forth let $A$ be this set. Indeed $42+A$ is not Lebesgue Measurable. So if $B \subseteq \mathcal{C}$, then $B \cup (42+A)$ is not Lebesgue measurable since $B \in \mathcal{M}(\mathbb{R})$, so $B^c \in \mathcal{M}(\mathbb{R})$ and $B^c \cap (B \cup (42+A))^c = 42+A$. Then

$$|\mathcal{M}(\mathbb{R})^c| = |\mathcal{P}(\mathbb{R})|.$$

## Measurable Functions

**Definition:** Let $(X, \mathcal{A}_X)$, $(Y, \mathcal{A}_Y)$ be measurable functions. A function $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ is said to be **measurable** if $f^{-1}(A) \in \mathcal{A}_X$ for all $A \in \mathcal{A}_Y$.

**Examples:**

($\alpha$) If $f$ is constant, then $f$ is measurable.

($\beta$) If $\mathcal{A}_X = \mathcal{P}(X)$ then all functions are measurable.

($\gamma$) If $\mathcal{A}_Y = \{\emptyset, Y\}$, all functions are measurable.

**Example:** Let $(X, \mathcal{A})$ be a measurable space and let $A \subseteq X$. Define $\mathbb{1}_A : X \to \{0,1\}$ by

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We call $\mathbb{1}_A$ the **characteristic function** on $A$.

Using $\mathcal{P}(\{0,1\})$, $\mathbb{1}_A$ is measurable if and only if

$$\mathbb{1}_A^{-1}(B) = \begin{cases} \emptyset & \text{if } 0, 1 \notin B \\ A & \text{if } 1 \in B,\ 0 \notin B \\ A^c & \text{if } 0 \in B,\ 1 \notin B, \\ X & \text{if } 0, 1 \in B. \end{cases}$$

**Example:** Let $f : [0,1] \to [0,1]$ be the Cantor Ternary Function that is

(i) Non-decreasing

(ii) Onto

(iii) Continuous

(iv) Constant on each interval in $\mathcal{C}^c$.

(v) $f(0) = 0$, $f(1) = 1$.

If $\phi : [0,1] \to [0,2]$ is defined by $\phi(x) = x + f(x)$.

Then $\phi$ is onto, continuous, and strictly increasing.

So $\psi = \phi^{-1} : [0,2] \to [0,1]$ is continuous.

By Assignment 2, there exists $B \subseteq \mathcal{C}$ such that $\phi(B)$ is not Lebegue Measurable. So $B \in \mathcal{M}(\mathbb{R})$ but

$\psi^{-1}(B) = \phi(B) \notin \mathcal{M}(\mathbb{R})$

Thus, we do <u>not</u> want to use $\mathcal{M}(\mathbb{R})$ as the $\sigma$-algebra on the codomain for functions that are continuous functions that are <u>not</u> measurable.

**Notation:** $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**Definition:** Let $(X, \mathcal{A})$ be a measurable space. A function $f : (X, \mathcal{A}) \to \mathbb{K}$ is said to be measurable using $\mathcal{B}(\mathbb{K})$ on $\mathbb{K}$.

**Proposition:** Let $(X, \mathcal{A}_X), (Y, \mathcal{A}_Y)$ be measurable spaces. Let $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ and let $A \subseteq \mathcal{A}_Y$ be such that $\mathcal{A}_Y = \sigma(A)$. TFAE:

(a) $f$ is measurable.

(b) $f^{-1}(B) \in \mathcal{A}_X \quad \forall B \in A$.

**Proof:** (a) $\Rightarrow$ (b) easy by definition.

(b) $\Rightarrow$ (a) Assume $f^{-1}(B) \in \mathcal{A}$ for any $B \in A$. Let

$\mathcal{A} = \{C \subseteq Y : f^{-1}(C) \in \mathcal{A}_X\}$

Note $A \subseteq \mathcal{A}$ by assumption.

Claim: $\mathcal{A}$ is a $\sigma$-algebra

(i) $\emptyset, Y \in \mathcal{A}$.

(ii) Because $f^{-1}(C^c) = f^{-1}(C)^c$, $C \in \mathcal{A} \Rightarrow C^c \in \mathcal{A}$.

(iii) Because $f^{-1}(\bigcup_{n=1}^{\infty} C_n) = \bigcup_{n=1}^{\infty} f^{-1}(C_n)$, $\bigcup_{n=1}^{\infty} C_n \in \mathcal{A}$.

Thus $\sigma(A) \subseteq \mathcal{A}$, so $\mathcal{A}_Y \subseteq \mathcal{A}$, hence $f$ is measurable.

**Corollary:** $(X, \mathcal{A})$ ms. Let $f: (X, \mathcal{A}) \to \mathbb{C}$. TFAE:

(a) $f: (X, \mathcal{A}) \to \mathbb{C}$ is measurable

(b) $f^{-1}(U) \in \mathcal{A}$ $\forall$ $U$ open in $\mathbb{C}$

Let $g: (X, \mathcal{A}) \to \mathbb{R}$. TFAE

(a) $g$ is measurable.

(b) $g^{-1}(U) \in \mathcal{A}$ $\forall$ $U \subseteq \mathbb{R}$ open.

(c) $\{x : f(x) > a\} = f^{-1}((a, \infty)) \in \mathcal{A}$

**Corollary:** Let $(X, \mathcal{T})$ be a topological space (ts) and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a $\sigma$-algebra containing $\mathcal{B}(\mathbb{K})$. If $f: X \to \mathbb{K}$ is continuous, then $f$ is measurable.

**Proposition:** Let $(X, \mathcal{A})$ be a ms, let $(Y, \mathcal{T}), (Z, \mathcal{S})$ be ts, and equip $Y, Z$ with the Borel $\sigma$-algebras and $f: X \to Y$, $g: Y \to Z$ be measurable Then $gf: X \to Z$ is measurable.

**Proof:** Let $U \subseteq Z$ be open. Because $g$ is measurable $g^{-1}(U)$ is Borel. Thus, because $f$ is measurable, $f^{-1}(g^{-1}(U))$ is measurable, so $gf$ is measurable.

**Remark:** If $f, g : (\mathbb{R}, \mathcal{M}(\mathbb{R})) \to \mathbb{R}$ be Lebesgue measurable functions, $gf$ may not be Lebesgue measurable.