Theorem: (Lusin's Theorem) Let μ be a regular measure on $\mathbb{R}$ such that $\mu([a,b]) < \infty$ for some $a < b \in \mathbb{R}$. Let $f : [a,b] \to \mathbb{K}$ be measurable. Then

(i) for all $\varepsilon > 0$, there exists $F \subseteq \mathbb{R}$ closed such that $\mu([a,b] \setminus F) < \varepsilon$ and $f|_F$ is continuous.

(ii) there exists $g : [a,b] \to \mathbb{K}$ continuous such that $g = f$ on F, $\mu(\{x : g(x) \neq f(x)\}) < \varepsilon$ and

$$\sup_{x \in [a,b]} |g(x)| \leq \sup_{x \in [a,b]} |f(x)|$$

Theorem: (Tietz) If $F \subseteq \mathbb{R}$ is closed and $h : F \to \mathbb{K}$ is continuous, there exists $g : \mathbb{R} \to \mathbb{K}$ continuous, $g|_F = h$ and

$$\sup_{x \in \mathbb{R}} |g(x)| \leq \sup_{x \in F} |h(x)|$$

Lemma: Lusin's Theorem holds for simple functions.

Proof of Lusin: Let $f : [a,b] \to \mathbb{C}$ measurable. Considering the positive and negative portions of the real and imaginary parts of f, by the fact nonnegative measurable functions are pointwise limits of simple functions and Lusin's Theorem holds for simple functions, there exists a sequence $(g_n)_{n=1}^{\infty}$ of measurable functions and $\{F_n\}_{n=1}^{\infty}$ closed such that

(i) $g_n \to f$ pointwise

(ii) $g_n|_{F_n}$ is continuous

(iii) $\mu([a,b] \setminus F_n) < \frac{\varepsilon}{2^{n+1}}$

By Egrof, there exists $B \in \mathcal{A}$, $B \subseteq [a,b]$ such that $\mu(B) < \frac{\varepsilon}{4}$ and $g_n \to f$ uniformly on B^c.

By outer regularity, there exists $\mathcal{U}$ open such that $B \subseteq \mathcal{U}$ and $\mu(\mathcal{U}) < \frac{\varepsilon}{2}$. Let $F' = [a,b] \setminus \mathcal{U}$. Then $\mu([a,b] \setminus F') = \mu(\mathcal{U}) < \frac{\varepsilon}{2}$

Let $F = F' \cap \left(\bigcap_{n=1}^{\infty} F_n \right)$, which is closed and

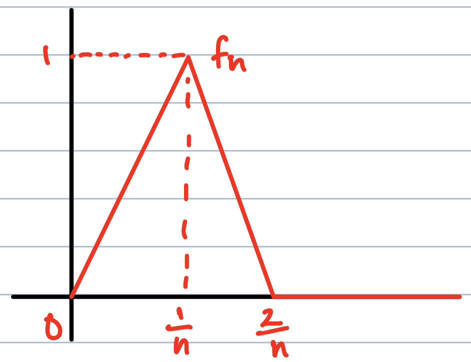
$$\begin{aligned} \mu([a,b] \setminus F) &\leq \mu([a,b] \setminus F') + \sum_{n=1}^{\infty} \mu([a,b] \setminus F_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

To see that $f|_F$ is continuous, note that $F \subseteq F'$, so $g_n \to f$ uniformly on F. Moreover, because $F \subseteq F_n$, so $g_n|_F$ is continuous, so $f|_F$ is the uniform limit of the continuous functions $(g_n|_F)_{n=1}^{\infty}$, so $f|_F$ is continuous.

Integration of Measurable Functions

Note that $\chi_{\mathbb{Q}}$ is Lebesgue measurable, but $\chi_{\mathbb{Q}}$ is not Riemann integrable! Also, note that the Riemann integral does not respect pointwise limits.

Example: Consider



Note that $f_n \to 0$ pointwise, however

$$\int_0^1 f_n(x)\,dx = 1$$

Integrals of Non-negative Measurable Functions

Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\varphi : X \to [0,\infty)$ be a simple function with representation

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

For $A \in \mathcal{A}$, we define the integral of φ against μ over A to be

$$\int_A \varphi \, d\mu = \sum_{i=1}^{n} a_i \mu(A_k \cap A)$$

Remark: We have seen $\nu(A) = \int_A \varphi \, d\mu$ for a simple function μ is a measure.

Example: Let A be a measurable set. Then

$$\int_X \chi_A \, d\mu \quad \text{is}$$

<u>Case 1:</u> If $A = X$, then χ_A is its own canonical representation so $\int_X \chi_A \, d\mu = \mu(A \cap X) = \mu(A)$

Case 2: If $A = \phi$, then $\chi_A = 0 \cdot \chi_X$ is the canonical representation

so $\int_X \chi_A \, d\mu = 0\mu(X \cap X) = 0 = \mu(A)$.

Case 3: If $A \neq \phi$ and $A \neq X$, then $\chi_A = 1\chi_A + 0\chi_{A^c}$,

$$\int_X \chi_A \, d\mu = 1\mu(A \cap A) + 0\mu(A^c \cap A) = \mu(A).$$

Remark: Suppose $g = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $\{A_i\}_{i=1}^{n}$ are pairwise disjoint, measurable, and $\bigcup_{i=1}^{n} A_i = X$ and $a_i \in [0,\infty)$.

We claim for this other (possibly not canonical) representation, the integral formula holds. Write $\varphi(X) = \{b_1, \dots, b_m\}$ with $b_i \neq b_j \ \forall i \neq j$ and $B_j = \varphi^{-1}(\{b_j\})$. Thus, $\{B_j\}_{j=1}^{m}$ are pairwise disjoint with union X. Thus,

$$\varphi = \sum_{j=1}^{m} b_j \chi_{B_j} \text{ is the canonical representation.}$$

For $1 \leq j \leq m$, let $K_j = \{1 \leq i \leq n : a_i = b_j\}$. Then $\bigcup_{j=1}^{m} K_j = \{1 \leq i \leq n : A_i \neq \phi\}$. Moreover, for $1 \leq j \leq m$,

$\bigcup_{i \in K_j} A_i = B_j$. Thus,

$$\begin{aligned}
\int_A \varphi \, d\mu &= \sum_{j=1}^{m} b_j \mu(B_j \cap A) = \sum_{j=1}^{m} b_j \mu\Big(\big(\bigcup_{i \in K_j} A_i\big) \cap A\Big) \\
&= \sum_{j=1}^{m} \sum_{i \in K_j} b_j \mu(A_i \cap A) \\
&= \sum_{j=1}^{m} \sum_{i \in K_j} a_i \mu(A_i \cap A) \\
&= \sum_{i \in K} a_i \mu(A_i \cap A) \qquad \Big(K = \bigcup_{j=1}^{m} K_j\Big) \\
&= \sum_{i=1}^{n} a_i \mu(A_i \cap A).
\end{aligned}$$

Theorem: Let $(X, \mathcal{A}, \mu)$ be a measure space, let $A \in \mathcal{A}$, and let $\varphi, \psi : X \to [0,\infty)$ be simple functions.

(i) $c\varphi$ is simple for all $c \geq 0$ and

$$\int_A c\varphi \, d\mu = c\int_A \varphi \, d\mu.$$

(ii) $\varphi + \psi$ is simple and

$$\int_A \varphi + \psi \, d\mu = \int_A \varphi \, d\mu + \int_A \psi \, d\mu.$$

(iii) If $B \in \mathcal{A}$ and $B \subseteq A$, then

$$\int_B \varphi \, d\mu \leq \int_A \varphi \, d\mu$$

(iv) $\varphi \chi_A$ is simple and

$$\int_A \varphi \, d\mu = \int_X \varphi \chi_A \, d\mu$$

(v) If $\varphi \chi_A \leq \psi \chi_A$, $\int_A \varphi \, d\mu \leq \int_A \psi \, d\mu$.

Proof:

(i) If $c = 0$, we are done. Otherwise, if $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$

then $c\varphi = \sum_{i=1}^{n} (ca_i)\chi_{A_i}$ is the canonical representation for $c\varphi$.

(ii). Let $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$, $\psi = \sum_{j=1}^{m} b_j \chi_{B_j}$ be the canonical representations. For $1 \leq i \leq n$, $1 \leq j \leq m$, let $C_{i,j} = A_i \cap B_j$. Then

$$\varphi + \psi = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i + b_j)\chi_{C_{i,j}}$$

This is not the canonical representation, but $\{C_{i,j}\}$ are pairwise disjoint with union X. By the remark

$$\begin{aligned}
\int_A \varphi + \psi \, d\mu &= \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i + b_j)\mu(C_{i,j} \cap A) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(C_{i,j} \cap A) + \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mu(C_{i,j} \cap A) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu\left(\left(\bigcup_{j=1}^{m} C_{i,j}\right) \cap A\right) + \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mu\left(\left(\bigcup_{i=1}^{n} C_{i,j}\right) \cap B\right) \\
&= \sum_{i=1}^{n} a_i \mu(A_i \cap A) + \sum_{j=1}^{m} b_j \mu(B_j \cap A)
\end{aligned}$$

$$= \int_A \varphi \, d\mu + \int_A \psi \, d\mu.$$

Corollary $\int_A \sum_{i=1}^n a_i \chi_{A_i} \, d\mu = \sum_{i=1}^n a_i \mu(A_i \cap A)$

(iii) Integration against a simple function is a measure.

(iv) Write $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$. Then $\varphi \chi_A = \sum_{i=1}^n a_i \chi_{A_i \cap A}$.

So

$$\int_X \varphi \chi_A \, d\mu = \sum_{i=1}^n a_i \mu((A_i \cap A) \cap X) = \int_A \varphi \, d\mu$$

(v) Assume $\chi_A \varphi \leq \chi_A \psi$. Then $\chi_A \psi - \chi_A \varphi \geq 0$ is measurable, has finite range, so is simple. Then

$$\int_A \psi \, d\mu = \int_X \psi \chi_A \, d\mu = \int_X \varphi \chi_A + (\psi \chi_A - \varphi \chi_A) \, d\mu$$

$$= \int_X \varphi \chi_A \, d\mu + \int_X \psi \chi_A - \varphi \chi_A \, d\mu$$

$$\geq \int_A \varphi \, d\mu.$$

Definition: Let $f : X \to [0, \infty)$ be measurable. For $A \in \mathcal{A}$, we define the integral against μ over A to be

$$\int_A f \, d\mu = \sup \left\{ \int_A \varphi \, d\mu : \varphi \text{ is simple}, \varphi \leq f \right\}.$$

In the case where $\mu = \lambda$, we call the above the Lebesgue Integral.

Remark: The above definition has a problem: We have defined $\int_A \varphi \, d\mu$ for a simple function in two ways.

We check if they coincide. Let φ be simple, let $\alpha = \int_A \varphi \, d\mu$ as a simple function and $\beta = \int_A \varphi \, d\mu$ as a nonnegative function.

By the definition of β, it is larger than α.

If ψ is simple and $\psi \leq \varphi$, then

$\int_A \psi \, d\mu \leq \alpha$ by (e).

Example: $f : X \to [0, \infty)$ is measurable

(α) If $x \in X$, then

$$\int_X f \, d\delta_x = f(x)$$

(β) If μ is the counting measure on $\mathbb{N}$,

$$\int_{\mathbb{N}} f \, d\mu = \sum_{n=1}^{\infty} f(n).$$

Theorem: Let $(X, \mathcal{A}, \mu)$ be a measure space, $A \in \mathcal{A}$, and $f, g : X \to [0, \infty]$ be measurable.

(i) If $c \geq 0$, $\int_A cf \, d\mu = c \int_A f \, d\mu$

(ii) If $B \in \mathcal{A}$ and $B \subseteq A$, $\int_B f \, d\mu \leq \int_A f \, d\mu$.

(iii) $\int_X \chi_A f \, d\mu = \int_A f \, d\mu$

(iv) If $f \chi_A \leq g \chi_A$, $\int_A f \, d\mu \leq \int_A g \, d\mu$

(v) $\int_A f \, d\mu = 0$ if and only if $\mu(\{x : f(x) > 0\} \cap A) = 0$.

(vi) If $\mu(A) = 0$, then $\int_A f \, d\mu = 0$.

Proof:

(i) If $c = 0$, we are done. Otherwise, if $c > 0$,

$\varphi \leq f$ if and only if $c\varphi \leq cf$ if and only if

$\frac{1}{c} \psi \leq f$. if and only if $\psi \leq cf$.

(ii) holds because it holds for simple functions.

(iv) If $\varphi \leq f \chi_A$, $\varphi \leq g \chi_A$ by (iii)

(vi) Follows from (v).