Question 1 (Leibniz Integral Rule)**.** Let $E \in \mathcal{M}(\mathbb{R})$ and let $f : E \times [c,d] \to \mathbb{R}$ be such that

(I) for each $t \in [c,d]$, the function $g_t : E \to \mathbb{R}$ defined by $g(x) = f(x,t)$ is Lebesgue integrable,

(II) for almost every $x \in E$, the function $h_x : (c,d) \to \mathbb{R}$ defined by $h_x(t) = f(x,t)$ is differentiable on (c,d), and

(III) there exists a Lebesgue integrable function $\theta : E \to \mathbb{R}$ such that $|h'_x(t)| \leq \theta(x)$ for all $t \in (c,d)$ and almost every $x \in E$.

Then
$$\frac{d}{dt}\int_E f(x,t)\,d\lambda(x) = \int_E \frac{\partial f}{\partial t}(x,t)\,d\lambda(x)$$
for all $t \in (c,d)$.

Solution. To begin, let $I : (c,d) \to \mathbb{R}$ be defined by
$$I(t) = \int_E f(x,t)\,d\lambda(x)$$
for all $t \in (c,d)$. Therefore I is differentiable on (c,d) with
$$I'(t_0) = \lim_{h\to 0}\frac{I(t_0+h) - I(t_0)}{h} = \lim_{h\to 0}\int_E \frac{f(x,t_0+h) - f(x,t_0)}{h}\,dt$$
for all $t_0 \in (c,d)$ provided the limit exists.

Fix $t_0 \in (c,d)$. Suppose for the sake of a contradiction that the above limit does not exist or does not equal
$$\int_E \frac{\partial f}{\partial t}(x,t_0)\,d\lambda(x).$$
Hence, there exists a sequence $(h_n)_{n\geq 1}$ of non-zero real numbers such that $\lim_{n\to\infty} h_n = 0$ and
$$\lim_{n\to\infty}\int_E \frac{f(x,t_0+h_n) - f(x,t_0)}{h_n}\,dt$$
either does not exist or does not equal $\int_E \frac{\partial f}{\partial t}(x,t_0)\,d\lambda(x)$. For each $n \in \mathbb{N}$, let $g_n : E \to \mathbb{R}$ be defined by
$$g_n(x) = \frac{f(x,t_0+h_n) - f(x,t_0)}{h_n}$$
for all $x \in E$. Note that g_n is a Lebesgue integrable function by (I).

By (II) and the Mean Value Theorem, for almost every $x \in E$ for every $n \in \mathbb{N}$ there exists a $t_{x,n} \in (c,d)$ such that
$$|g_n(x)| = \left|\frac{f(x,t_0+h_n) - f(x,t_0)}{h_n}\right| = \left|h'_x(t_{x,n})\right|.$$
Therefore (III) implies that
$$|g_n(x)| \leq \theta(x)$$
for almost every $x \in E$ for all $n \in \mathbb{N}$. Therefore, since
$$\lim_{n\to\infty} g_n(x) = \frac{\partial f}{\partial t}(x,t_0)$$

for almost every $x \in E$ and since θ is Lebesgue integrable, we obtain by the Dominated Convergence Theorem that

$$\begin{aligned} \lim_{n\to\infty} \int_E \frac{f(x, t_0 + h_n) - f(x, t_0)}{h_n}\, dt &= \lim_{n\to\infty} \int_E g_n(x)\, dt \\ &= \int_E \lim_{n\to\infty} g_n(x)\, dt \\ &= \int_E \frac{\partial f}{\partial t}(x, t_0)\, dt. \end{aligned}$$

Hence we have a contradiction so the result follows.

Question 2. Recall that a function $f : [a, b] \to \mathbb{R}$ is said to be *Lipschitz* if there exists a constant K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [a, b]$.

Prove that an absolutely continuous function $f : [a, b] \to \mathbb{R}$ is Lipschitz if and only if $|f'| \in L_\infty([a, b], \lambda)$.

Solution. Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function. Assume f is Lipschitz. Hence there exists a constant K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [a, b]$. Since f is absolutely continuous, f is differentiable on $[a, b]$. Moreover, by the Lipschitz condition, for all $x \in (a, b)$

$$|f'(x)| = \lim_{h\to 0} \frac{|f(x + h) - f(x)|}{|h|} \leq \limsup_{h\to 0} \frac{K|(x + h) - x|}{|h|} = \limsup_{h\to 0} K = K.$$

Hence $f' \in L_\infty([a, b], \lambda)$ with $\|f'\|_\infty \leq K$.

Conversely, assume $|f'| \in L_\infty([a, b], \lambda)$. Since f is absolutely continuous, the second Fundamental Theorem of Calculus implies for all $x \in [a, b]$ that

$$f(x) = f(a) + \int_{[a,x]} f' \, d\lambda.$$

Thus for all $x, y \in [a, b]$ with $y \leq x$ we have that

$$\begin{aligned}
|f(x) - f(y)| &= \left| \int_{[a,x]} f' \, d\lambda - \int_{[a,y]} f' \, d\lambda \right| \\
&= \left| \int_{[y,x]} f' \, d\lambda \right| \\
&\leq \int_{[y,x]} |f'| \, d\lambda \\
&\leq \int_{[y,x]} \|f'\|_\infty \, d\lambda \\
&= \|f'\|_\infty \, |x - y|.
\end{aligned}$$

Hence f is Lipschitz with constant $\|f'\|_\infty$.

Question 3. Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing, absolutely continuous function.

a) Prove that if G is a G_δ-subset of (a, b), then $f(G)$ is Lebesgue measurable and $\lambda(f(G)) = \int_G f' \, d\lambda$.

b) Prove that if $A \subseteq [a, b]$ is Lebesgue measurable with $\lambda(A) = 0$ then $\lambda(f(A)) = 0$.

c) Let $c = f(a)$ and $d = f(b)$. Prove that if $g : [c, d] \to [0, \infty]$ is Borel, then

$$\int_{[c,d]} g \, d\lambda = \int_{[a,b]} (g \circ f) f' \, d\lambda.$$

Solution.
a) Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing, absolutely continuous function. Hence f' exists, $f' \geq 0$, and the second Fundamental Theorem of Calculus implies that

$$f(x) = f(a) + \int_{[a,x]} f' \, d\lambda$$

for all $x \in [a, b]$.

Consider $(c, d) \subseteq (a, b)$. Since f is a strictly increasing continuous function, the Intermediate Value Theorem implies that

$$f((c, d)) = (f(c), f(d)).$$

Hence $f((c, d))$ is Lebesgue measurable and

$$\lambda(f((c, d))) = f(d) - f(c) = \int_{[a,d]} f' \, d\lambda - \int_{[a,c]} f' \, d\lambda = \int_{(c,d)} f' \, d\lambda$$

(as the Lebesgue integral over a singleton is zero). Hence the result holds for open intervals.

Next, assume $U \subseteq (a, b)$ is an open set. Hence U is a countable union of pairwise disjoint open intervals, say $\{(c_k, d_k)\}_{k=1}^\infty$, contained in (a, b). Since f is a strictly increasing function, f is injective on (a, b) so $\{f((c_k, d_k))\}_{k=1}^\infty$ is a countable collection of disjoint sets with union $f(U)$. Hence $f(U)$ is Lebesgue measurable by the previous paragraph. Therefore, since $\nu : \mathcal{M}(\mathbb{R}) \to [0, \infty)$ defined by $\nu(A) = \int_A f' \, d\lambda$ for all $A \in \mathcal{M}(\mathbb{R})$ is a measure (as $f' \geq 0$), we obtain that

$$\begin{aligned}
\lambda(f(U)) &= \lambda\left(\bigcup_{k=1}^\infty f((c_k, d_k))\right) \\
&= \sum_{k=1}^\infty \lambda((f(c_k, d_k))) \\
&= \sum_{k=1}^\infty \int_{(c_k,d_k)} f' \, d\lambda \\
&= \int_{\bigcup_{k=1}^\infty (c_k,d_k)} f' \, d\lambda \\
&= \int_U f' \, d\lambda.
\end{aligned}$$

Finally, let $G \subseteq (a, b)$ be an arbitrary G_δ set. Hence there exists a countable collection $\{U_k\}_{k=1}^\infty$ of open subsets of (a, b) such that $G = \bigcap_{k=1}^\infty U_k$. Hence

$$f(G) = \bigcap_{k=1}^\infty f(U_k)$$

since f is injective. Hence $f(G)$ is Lebesgue measurable by the previous paragraph. Since $\lambda(f((a, b))) \leq \lambda([f(a), f(b)]) < \infty$ as f is a strictly increasing continuous function, and since

$$\nu([a, b]) = \int_{[a,b]} f' \, d\lambda = f(b) - f(a) < \infty$$

by the Second Fundamental Theorem of Calculus, we obtain by the Monotone Convergence Theorem for measures that

$$\begin{aligned}\lambda(f(G)) &= \lambda\left(\bigcap_{k=1}^{\infty} f(U_k)\right) \\ &= \lim_{n\to\infty} \lambda\left(\bigcap_{k=1}^{n} U_k\right) \\ &= \lim_{n\to\infty} \int_{\bigcap_{k=1}^{n} U_k} f'\,d\lambda \\ &= \int_G f'\,d\lambda\end{aligned}$$

since $\bigcap_{k=1}^{n} U_k$ is an open set for all $n \in \mathbb{N}$. Hence the result is complete.

b) Let $A \subseteq [a, b]$ be a Lebesgue measurable subset such that $\lambda(A) = 0$. Since the Lebesgue measure of any singleton it zero, we can assume that $A \subseteq (a, b)$. By Assignment 1, Question 3 there exists a G_δ-subset G of $\mathbb{R}$ such that $A \subseteq G$ and $\lambda(G \setminus A) = 0$. Therefore

$$\lambda(G) = \lambda(G \setminus A) + \lambda(A) = 0.$$

Hence $\lambda(f(G)) = 0$ by part a). Therefore, since $f(A) \subseteq f(G)$, we obtain that $\lambda^*(f(A)) = 0$ by monotonicity.

c) First, we claim that if $A \subseteq [a, b]$ is Lebesgue measurable, then $f(A)$ is Lebesgue measurable and $\lambda(f(A)) = \int_A f'\,d\lambda$. To see this, let $A \subseteq [a, b]$ be an arbitrary Lebesgue measurable set. By Assignment 1, Question 3 there exists a G_δ-subset G of $\mathbb{R}$ such that $A \subseteq G$ and $\lambda(G \setminus A) = 0$. Therefore

$$\lambda(G) = \lambda(G \setminus A) + \lambda(A) = \lambda(A),$$

$f(G \setminus A)$ is Lebesgue measurable by part b) and the fact that λ is complete, and $\lambda(f(G \setminus A)) = 0$ by part b). Since

$$f(G) = f(A) \cup f(G \setminus A) \qquad \text{and} \qquad f(A) \cap f(G \setminus A) = \emptyset$$

as f is injective, we see $f(A) = f(G) \setminus f(G \setminus A)$ is Lebesgue measurable by part a), and

$$\begin{aligned}\lambda(f(A)) &= \lambda(A) + \lambda(f(G \setminus A)) \\ &= \lambda(f(G)) \\ &= \int_G f'\,d\lambda \\ &= \int_A f'\,d\lambda + \int_{G\setminus A} f'\,d\lambda \\ &= \int_A f'\,d\lambda\end{aligned}$$

by part a) since $\lambda(G \setminus A) = 0$ implies $\int_{G\setminus A} f'\,d\lambda = 0$. Hence the claim is complete.

For the main result, let $c = f(a)$ and $d = f(b)$ and let $g : [c, d] \to [0, \infty]$ is Borel. First assume that $g = \chi_A$ for some Borel set $A \subseteq [c, d]$. Hence

$$\int_{[a,b]} (g \circ f) f'\,d\lambda = \int_{[a,b]} (\chi_A \circ f) f'\,d\lambda = \int_{\{x \,\mid\, f(x)\in A\}} f'\,d\lambda = \lambda(f(\{x \mid f(x) \in A\})) = \lambda(A) = \int_{[c,d]} g\,d\lambda$$

as desired (note we needed A Borel here so that $\{x \mid f(x) \in A\}$ is Lebesgue measurable). Therefore, by the linearity of the integral and by the linearity of $g \circ f$ in g, the above implies that

$$\int_{[c,d]} g\,d\lambda = \int_{[a,b]} (g \circ f) f'\,d\lambda$$

for all simple Borel functions g.

To see the final result, let $g : [c,d] \to [0,\infty]$ be Borel. Hence there exists a sequence $(\varphi_n)_{n\geq 1}$ of simple Borel functions such that $\varphi_n \leq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $(\varphi_n)_{n\geq 1}$ converges pointwise to g. Hence $(\varphi_n \circ f)$ is an increasing sequence of Borel functions that converges to $g \circ f$ pointwise. Therefore, by two applications of the Monotone Convergence Theorem and the above case, we obtain that

$$\int_{[c,d]} g \, d\lambda = \lim_{n\to\infty} \int_{[c,d]} \varphi_n \, d\lambda = \lim_{n\to\infty} \int_{[a,b]} (\varphi_n \circ f) f' \, d\lambda = \int_{[a,b]} (g \circ f) f' \, d\lambda$$

as desired.

Question 4. A monotone function $f : [a, b] \to \mathbb{R}$ is said to be *singular* if $f' = 0$ λ-almost everywhere.

a) Prove that any non-decreasing function on $[a, b]$ is the sum of an absolutely continuous non-decreasing function and a singular non-decreasing function.

b) Let $f : [a, b] \to \mathbb{R}$ be a non-decreasing singular function. Prove that f has the following property: (S) For all $\epsilon, \delta > 0$ there exists

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$$

such that

$$\sum_{k=1}^{n} |b_k - a_k| < \delta \qquad \text{and} \qquad \sum_{k=1}^{n} |f(b_k) - f(a_k)| > f(b) - f(a) - \epsilon.$$

c) Let $f : [a, b] \to \mathbb{R}$ be a non-decreasing function with property (S) from part b). Use part a) to prove that f is singular.

d) Let $(f_n)_{n \geq 1}$ be a sequence of non-decreasing singular functions on $[a, b]$ such that the function f defined for all $x \in [a, b]$ by

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is finite everywhere. Prove that f is singular.

e) Show that there exists a strictly increasing, singular, continuous function on $[0, 1]$.

Solution.
a) Let $f : [a, b] \to \mathbb{R}$ be a non-decreasing function. Hence f is differentiable λ-almost everywhere with f' Lebesgue measurable by the Lebesgue Differentiation Theorem. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_{[a,x]} f' \, d\lambda$$

for all $x \in [a, b]$. Hence F is absolutely continuous with derivative f' λ-almost everywhere by the first Fundamental Theorem of Calculus. Moreover, since f is non-decreasing, $f' \geq 0$ λ-almost everywhere and thus F is non-decreasing.

Define $g : [a, b] \to \mathbb{R}$ by $g = f - F$ so that $f = F + g$. However, since g is the difference of two λ-almost everywhere differentiable functions, g is differentiable λ-almost everywhere with $g' = f' - F' = 0$ λ-almost everywhere. Hence g is singular. Finally, to see that g is non-decreasing, let $x, y \in [a, b]$ with $x < y$. Then, by the Lebesgue Differentiation Theorem,

$$g(y) - g(x) = f(y) - f(x) - (F(y) - F(x)) = f(y) - f(x) - \int_{[x,y]} f' \, d\lambda \geq 0.$$

Hence g is non-decreasing thereby completing the proof.

b) Let $f : [a, b] \to \mathbb{R}$ be a non-decreasing singular function. Hence f' Lebesgue measurable with $f' = 0$ λ-almost everywhere. Let

$$X = \{x \in (a, b) \mid f'(x) \text{ exists and } f'(x) = 0\}.$$

Since f' is Lebesgue measurable with $f' = 0$ λ-almost everywhere, we know that X is measurable and $\lambda(X) = b - a$.

To see that f has property (S), let $\epsilon, \delta > 0$ be arbitrary. Using the definition of X together with the fact that f is non-decreasing, for each $y \in X$ and $\delta_0 > 0$ there exists a $0 < h < \delta_0$ such that $(y, y + h) \subseteq (a, b)$ and $f(y + h) - f(y) < \frac{\epsilon}{b-a} h$. Since the collection of such intervals forms a Vitali covering of X, the Vitali Covering Lemma implies there exists an $n \in \mathbb{N}$ and a collection of open intervals $\{I_k\}_{k=1}^{n}$ such that if $I_k = (b_k, a_{k+1}) \subseteq (a, b)$ for all $k \in \{1, \ldots, n\}$, $a_{k+1} \leq b_{k+1}$ for all k (by reordering the intervals if necessary),

$$f(a_{k+1}) - f(b_k) < \frac{\epsilon}{b - a}(a_{k+1} - b_k),$$

for all k, and

$$\lambda\left(X \setminus \bigcup_{k=1}^{n}(b_k, a_{k+1})\right) < \delta.$$

By setting $a_1 = a$ and $b_{n+1} = b$, we obtain that

$$a = a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_{n+1} < b_{n+1} = b$$

since $b_1 > a$ and $a_{n+1} < b$. Moreover

$$\sum_{k=1}^{n+1} |b_k - a_k| = \lambda\left([a,b] \setminus \bigcup_{k=1}^{n}(b_k, a_{k+1})\right) = \lambda\left(X \setminus \bigcup_{k=1}^{n}(b_k, a_{k+1})\right) < \delta.$$

since $\lambda([a,b] \setminus X) = 0$. Finally, since f is non-decreasing, we see that

$$\begin{aligned}
\sum_{k=1}^{n+1} |f(b_k) - f(a_k)| &= \sum_{k=1}^{n+1} f(b_k) - f(a_k) \\
&= f(b) - f(a) - \sum_{k=1}^{n} f(a_{k+1}) - f(b_k) \\
&> f(b) - f(a) - \sum_{k=1}^{n} \frac{\epsilon}{b-a}(a_{k+1} - b_k) \\
&> f(b) - f(a) - \epsilon
\end{aligned}$$

as $\sum_{k=1}^{n} a_{k+1} - b_k < b - a$. Hence the proof is complete.

c) Let $f : [a,b] \to \mathbb{R}$ be a non-decreasing function with property (S). To see that f is singular, let $F, g : [a,b] \to \mathbb{R}$ be as in the solution to part a). Therefore, since $f = F + g$ and g is singular, it suffices to show that $F = 0$.

Let $\epsilon > 0$ be arbitrary. Let $\delta > 0$ be as in the definition of absolute continuity for F.

Since f has property (S), there exists

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$$

such that

$$\sum_{k=1}^{n} |b_k - a_k| < \delta \qquad \text{and} \qquad \sum_{k=1}^{n} |f(b_k) - f(a_k)| > f(b) - f(a) - \epsilon.$$

Hence, by our choice of δ, we obtain that

$$\begin{aligned}
&(F(b) - F(a)) + (g(b) - g(a)) - \epsilon \\
&= f(b) - f(a) - \epsilon \\
&< \sum_{k=1}^{n} |f(b_k) - f(a_k)| \\
&= \sum_{k=1}^{n} |(F(b_k) - F(a_k)) + (g(b_k) - g(a_k))| \\
&= \sum_{k=1}^{n} (F(b_k) - F(a_k)) + (g(b_k) - g(a_k)) && \text{since } F \text{ and } g \text{ are non-decreasing} \\
&\leq \epsilon + \sum_{k=1}^{n} g(b_k) - g(a_k) && \text{by the choice of } \delta \\
&\leq \epsilon + g(b) - g(a) && \text{since } g \text{ is non-decreasing.}
\end{aligned}$$

Hence $F(b) - F(a) < 2\epsilon$. Therefore, since $\epsilon > 0$ was arbitrary, we obtain that $F(b) = F(a)$. Hence, since F is non-decreasing and $F(a) = 0$, $F = 0$ as desired.

d) Let $(f_n)_{n\geq 1}$ be a sequence of non-decreasing singular functions on $[a, b]$ such that the function $f : [a, b] \to \mathbb{R}$ defined for all $x \in [a, b]$ by

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is finite everywhere. To show that f is singular, note f is clearly non-decreasing so it suffices to show that f satisfies property (S) by part c).

To see that f has property (S), let $\epsilon, \delta > 0$ be arbitrary. Since $f(a)$ and $f(b)$ are finite, there exists an $N \in \mathbb{N}$ such that

$$\left| f(a) - \sum_{n=1}^{N} f_n(a) \right| < \frac{\epsilon}{3} \qquad \text{and} \qquad \left| f(b) - \sum_{n=1}^{N} f_n(b) \right| < \frac{\epsilon}{3}.$$

Consider $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = \sum_{n=1}^{N} f_n(x)$$

for all $x \in [a, b]$. Since each f_k is non-decreasing, g is non-decreasing. Moreover, since $f'_n = 0$ almost everywhere, we have by taking a finite sum of limits that $g' = 0$ almost everywhere. Hence g is singular so there exists

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m \leq b$$

such that

$$\sum_{n=1}^{N} \sum_{k=1}^{m} f_n(b_k) - f_n(a_k) = \sum_{k=1}^{m} g(b_k) - g(a_k) = \sum_{k=1}^{m} |g(b_k) - g(a_k)| > g(b) - g(a) - \frac{\epsilon}{3} = \sum_{n=1}^{N} f_n(b) - f_n(a) - \frac{\epsilon}{3}.$$

Therefore, since each f_n is non-negative and thus f is non-negative, we obtain that

$$\begin{aligned}
\sum_{k=1}^{n} |f(b_k) - f(a_k)| &= \sum_{k=1}^{n} \sum_{m=1}^{\infty} f_m(b_k) - f_m(a_k) \\
&\geq \sum_{k=1}^{n} \sum_{m=1}^{N} f_m(b_k) - f_m(a_k) && \text{since } f_m(b_k) - f_m(a_k) \geq 0 \text{ for all } m \\
&> \left(\sum_{m=1}^{N} f_m(b) - f_m(a) \right) - \frac{\epsilon}{3} \\
&> \left(\sum_{m=1}^{N} f_m(b) \right) - \left(\sum_{m=1}^{N} f_m(a) \right) - \frac{\epsilon}{3} \\
&> \left(f(b) - \frac{\epsilon}{3} \right) - \left(f(a) + \frac{\epsilon}{3} \right) - \frac{\epsilon}{3} \\
&= f(b) - f(a) - \epsilon.
\end{aligned}$$

Therefore, since $\epsilon, \delta > 0$ were arbitrary, f has property (S) and thus is singular.

e) Let $f : [0, 1] \to [0, 1]$ denote the Cantor ternary function. Define $g : \mathbb{R} \to [0, 1]$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, 1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Clearly g is a non-decreasing continuous function on $\mathbb{R}$ that is differentiable λ-almost everywhere on $\mathbb{R}$ with $g' = 0$ λ-almost everywhere.

Let $\{r_n\}_{n\geq 1}$ be an enumeration of the countable set $\mathbb{Q}\cap[0,1]$ and define $G:[0,1]\to[0,1]$ by

$$G(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}g(x-r_n).$$

Since clearly this sum converges uniformly on $[0,1]$, we see that G is a continuous function on $[0,1]$. Moreover, since g is singular and translations of singular functions is singular, we obtain via part d) that G is singular. Finally, if $x,y\in[0,1]$ are such that $x<y$, then there exists an $r_k\in\mathbb{Q}\cap[0,1]$ such that $x<r_k<y$. Therefore

$$0=\frac{1}{2^k}g(x-r_k)<\frac{1}{2^k}g(y-r_k)$$

so, since g is non-decreasing, we obtain that $G(x)<G(y)$. Hence G is a continuous, strictly increasing singular function on $[0,1]$.