Remark: Monotone functions are of bounded variation. Linear combinations of bounded variation are also of bounded variation. If $f:[a,b]\to\mathbb{C}$ is of bounded variation and $a\le x<y\le b$, then $f|_{[x,y]}$ is of bounded variation.

Definition: If $f:[a,b]\to\mathbb{C}$ is of bounded variation, then for all $a\le x\le y\le b$ we define

$$V_f(x,y)=\sup_{x_0<\cdots<x_n}\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|$$

We call $V_f(x,y)$ the total variation of f on $[x,y]$

Theorem: (Jordan Decomposition Theorem) Let $f:[a,b]\to\mathbb{R}$ be of bounded variation. Define $V,D:[a,b]\to\mathbb{R}$ by $V(x)=V_f(a,x)$ for all $x\in[a,b]$ and $D=V-f$. Then V and D are non-dec. Hence, a $\mathbb{R}$-valued function is of bounded variation if and only if it is the difference of two non decreasing functions.

Proof: We claim for all $a\le x\le y\le b$, that $V_f(a,y)=V_f(a,x)+V_f(x,y)$. This will imply V is non-decreasing. If $\mathcal{P}$ is a partition of $[a,x]$ and $\mathcal{Q}$ is a partition of $[x,y]$, then $\mathcal{P}\cup\mathcal{Q}$ is a partition of $[a,y]$, so $V_f(a,y)\ge V_f(a,x)+V_f(x,y)$.

Conversely, if $\mathcal{P}$ is a partition of $[a,y]$, add in x to get $\mathcal{P}$ being the union of a partition on $[a,x]$ with a partition on $[x,y]$. As this does not decrease the term, we take the sup of in $V_f(a,y)$, we get the desired inequality.

Next, note that for all $a\le x\le y\le b$, we have

$$D(y) - D(x) = (V(y) - V(x)) - (f(y) - f(x))$$
$$= V_f(x,y) - (f(y) - f(x))$$
$$\geq |f(y) - f(x)| - (f(y) - f(x)) \geq 0.$$

Corollary: If $f : [a,b] \to \mathbb{C}$ is of bounded variation, then f' exists almost everywhere and $f' \in L_1([a,b], \lambda)$.

Proof: Without loss of generality, say f is $\mathbb{R}$-valued. Then $f = V - D$ where V and D are nondecreasing. By LDT, f' almost everywhere and $f' = V' - D'$.

Recall $V', D' \geq 0$ by LDT, so $|f'| \leq V' + D'$ almost everywhere, so

$$\int_{[a,b]} |f'| \, d\lambda \leq \int_{[a,b]} V' + D' \, d\lambda$$
$$\leq V(b) - V(a) + D(b) - D(a)$$
$$< \infty$$

by the LDT.

Absolutely Continuous Functions

Definition: Let $f : [a,b] \to \mathbb{C}$. We say f is **absolutely continuous on $[a,b]$** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$ is such that $\sum_{k=1}^{n} |b_k - a_k| < \delta$, then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$.

Remark: The set of absolutely continuous functions are closed under linear combinations and complex conjugates, so it suffices to consider $\mathbb{R}$-valued functions.

Example: If $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$ with $|f'|\leq M$ for some $M>0$, then for all $\varepsilon>0$, let $\delta=\frac{\varepsilon}{M}>0$. Then if $a\leq a_1<b_1\leq a_2<b_2\leq\cdots\leq a_n<b_n\leq b$ is such that $\sum_{k=1}^{n}|b_k-a_k|<\delta$, then by MVT, we know that
$|f(b_k)-f(a_k)|\leq M|b_k-a_k|$, so $\sum_{k=1}^{n}|f(b_k)-f(a_k)|\leq M\delta=\varepsilon$
so f is absolutely continuous.

Example: $f(x)=|x|$ is absolutely continuous but not differentiable on $[-1,1]$.

Theorem: Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous. Then f is continuous and f is of bounded variation.

Proof: Take $n=1$ and f is continuous. For bounded variation, let $\varepsilon=1$. Because f is continuous, choose δ for such ε in the def of absolute continuity. Let $\mathcal{P}=\{x_k\}_{k=0}^{n}$ be a partition of $[a,b]$. Let $\ell\in\mathbb{N}$ be such that

$$a+\frac{\delta}{2}\ell\leq b\leq a+\frac{\delta}{2}(\ell+1)$$

Let $\mathcal{P}'=\mathcal{P}\cup\{a+\frac{\delta}{2}m\}_{m=1}^{\ell}=\{t_k\}_{k=0}^{N}$. Then

$$\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\leq\sum_{k=1}^{N}|f(t_k)-f(t_{k-1})|$$

Note for $\mathcal{P}'\cap[a+\frac{\delta}{2}m, a+\frac{\delta}{2}(m+1)]$, for all $0\leq m\leq\ell$, we get a collection of end points of intervals that when ordered via increasing order yields intervals whose sums of lengths is at most $\frac{\delta}{2}$, so absolutely continuous implies that $\sum_{k=1}^{n}|f(t_k)-f(t_{k-1})|<1$ for each of these $\ell+1$ intervals so

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{N} |f(t_k) - f(t_{k-1})| < \ell + 1 < \infty$$

Corollary: If $f : [a,b] \to \mathbb{C}$ is absolutely continuous then f' exists almost everywhere, f' is measurable and $f' \in L_1[a,b]$.

Proof: f is of bounded variation.

Theorem: Let $f : [a,b] \to \mathbb{R}$ is absolutely continuous. If $f' = 0$ almost everywhere, then f is constant.

Corollary: The Cantor Ternary Function is not absolutely cont.

Proof: $f' = 0$ almost everywhere but f is not continuous.

Proof of Theorem: Let $c \in (a,k]$. We will show $f(c) = f(a)$.

Let $\varepsilon > 0$. Since f is absolutely continuous, choose $\delta > 0$.

Because $f' = 0$ a.e. there exists $X \subseteq \mathbb{R}$ such that $X \in \mathcal{M}(\mathbb{R})$, $f'(x) = 0$ $\forall x \in X$ and $\lambda([a,b] \setminus X) = 0$. Consider $x \in X \cap [a,c)$. Then

$$0 = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Hence for all $x \in X \cap [a,c)$ and all $\delta > 0$, there is $h < \delta_0$ such that $[x, x+h) \subseteq [a,c)$ and $|f(x+h) - f(x)| < \varepsilon h$. As the set of all such intervals form a Vitali covering of $X \cap [a,c]$, there exists $n \in \mathbb{N}$, $x_1, \dots, x_n \in X \cap [a,c]$ such that $x_k < x_{k+1}$ and $h_1, \dots, h_n > 0$ such that if $I_k = [x_k, x_k + h_k)$, then $\{I_k\}_{k=1}^{n}$ are pairwise disjoint.

$|f(x_k + h_k) - f(x_k)| < h_k \varepsilon$ and also

$\lambda^*((X \cap [a,c) \setminus (\bigcup_{k=1}^{n} I_k)) < \delta$, so $\lambda^*([a,b] \setminus X) = 0$

Let $y_k = x_k + h_k$ for $1 \le k \le n$, $y_0 = a$, $x_{n+1} = c$. Then

$a = y_0 \le x_1 < y_1 \le x_2 < y_2 \le \cdots \le x_n < y_n \le x_{n+1} = c$. Note

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| \le \sum_{k=1}^{n} h_k \varepsilon = \lambda\left(\bigcup_{k=1}^{n} I_k\right) \varepsilon \le (c-a)\varepsilon$$

Also,

$$\sum_{k=1}^{n+1} |f(x_k) - f(x_{k-1})| < \varepsilon$$ because $\sum_{k=1}^{n+1} |x_k - y_{k-1}| < \delta$ so

absolute continuity applies. Hence

$$|f(c) - f(a)| \le \sum_{k=1}^{n} |f(y_k) - f(x_k)| + \sum_{k=1}^{n+1} |f(x_k) - f(y_{k-1})|$$
$$< (c-a+1)\varepsilon.$$

Therefore as $\varepsilon > 0$ was arbitrary, $f(c) = f(a)$.

Theorem: Let $f \in L_1[a,b]$ and let $F: [a,b] \to \mathbb{C}$ be defined by

$$F(x) = \int_{[a,x]} f \, d\lambda$$

Then F is absolutely continuous.

Proof: Let $\varepsilon > 0$ and let $\nu : \mathcal{M}(\mathbb{R}) \to [0,\infty]$ by

$$\nu(A) = \int_A |f| \, d\lambda$$

Then ν is a measure. By Assignment 3, there exists a $\delta > 0$ such that if $A \in \mathcal{M}(\mathbb{R})$ and $\lambda(A) < \delta$ then $\nu(A) < \varepsilon$.

Then if $a \le a_1 < b_1 \le \cdots \le a_n < b_n \le b$, s.t. $\sum_{k=1}^{n} |b_k - a_k| < \delta$ then with $I_k = (a_k, b_k)$ we have $\lambda(\bigcup_{k=1}^{n} I_k) < \delta$, so

$\nu(\bigcup_{k=1}^{n} I_k) < \varepsilon$, so now

$$\left|\sum_{k=1}^{n} F(b_k) - F(a_k)\right| = \left|\sum_{k=1}^{n} \int_{[a,b_k]} f \, d\lambda - \int_{[a,a_k]} f \, d\lambda\right|$$

$$= \left| \sum_{k=1}^{n} \int_{[a_k, b_k]} f \, d\lambda \right| \le \sum_{k=1}^{n} \int_{[a_k, b_k]} |f| \, d\lambda = \int_{A} |f| \, d\lambda < \varepsilon.$$