Example We cannot approximate from above via simple functions when $\mu(A)=\infty$ and when f is not bounded.

<u>Case 1 (when $\mu(A)=\infty$)</u>: Let $A=[1,\infty)$ and $f(x)=\frac{1}{x^2}$. If φ is a simple function and $\varphi \geq f$, then if $a=\inf_{x\in A}\varphi(x)$ then $a>0$ and $\varphi^{-1}([a,\infty)) \supseteq f^{-1}([a,\infty)) = [1,\infty)$. So

$$\begin{aligned}\int_{[1,\infty)} f\,d\lambda &= \lim_{n\to\infty}\int_{[1,\infty)} f\chi_{[1,n]}\,d\lambda \\ &= \lim_{n\to\infty}\int_{[1,n]} f\,d\lambda \\ &= \lim_{n\to\infty}\int_1^n f\,d\lambda \\ &= 1\end{aligned}$$

<u>Case 2 (f is not bounded)</u>: Let $A=(0,1]$ and $f(x)=\frac{1}{\sqrt{x}}$. If φ is a simple function and $\varphi\geq f$, then φ does not exist as range of simple functions is a finite number of points in $[0,\infty)$ and range of f is $[1,\infty)$. Note

$$\begin{aligned}\int_{(0,1]} f\,d\lambda &= \lim_{n\to\infty}\int_{(0,1]} f\chi_{[\frac{1}{n},1]}\,d\lambda \\ &= \lim_{n\to\infty}\int_{[\frac{1}{n},1]} f\,d\lambda \\ &= \lim_{n\to\infty}\int_{\frac{1}{n}}^1 \frac{1}{\sqrt{x}}\,dx \\ &= 2.\end{aligned}$$

Fatou's Lemma

Theorem: Let $(X,\mathcal{A},\mu)$ be a measure space and for all $n\in\mathbb{N}$, let $f_n: X\to[0,\infty]$ be measurable. Then for all $A\in\mathcal{A}$,

$$\int_A \liminf_{n\to\infty} f_n\,d\mu \leq \liminf_{n\to\infty}\int_A f_n\,d\mu.$$

Proof: For each $n \in \mathbb{N}$, let $g_n = \inf_{k \geq n} f_k$. Then g_n is measurable and $g_n \leq f_n$ is measurable and $g_n \leq f_k$ for all $k \geq n$, so

$$\int_A g_n \, d\mu \leq \int_A f_k \, d\mu \quad \text{for all } k \geq n$$

$$\Rightarrow \int_A g_n \, d\mu \leq \liminf_{n \to \infty} \int_A f_n \, d\mu \quad \text{for all } n \in \mathbb{N}$$

Since $(g_n)_{n=1}^{\infty}$ converge to $\liminf_{n\to\infty} f_n$ pointwise and $g_n \leq g_{n+1}$ for all $n \in \mathbb{N}$, so by MCT

$$\int_A \liminf_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_A g_n \, d\mu$$
$$\leq \liminf_{n\to\infty} \int_A f_n \, d\mu.$$

Remark: The '$\leq$' in the Theorem can be strict. Indeed, let $\mu = \lambda$ and $f_n = \frac{1}{n}\chi_{[0,n]}$, then $\liminf_{n\to\infty} f_n = 0$ but then

$$\int_{\mathbb{R}} f_n \, d\mu = 1 \quad \text{for each } n \in \mathbb{N}$$

The Dominated Convergence Theorem

Theorem: Let $(X, \mathcal{A}, \mu)$ be a measure space and let $g : X \to [0, \infty]$ be integrable. For all $n \in \mathbb{N}$, let $f_n : X \to \mathbb{K}$ be measurable such that $|f_n| \leq g$ for all $n \in \mathbb{N}$ a.e. If $f : X \to \mathbb{K}$ is measurable and the pointwise limit of $(f_n)_{n=1}^{\infty}$ (a.e.) then

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu$$

for all $A \in \mathcal{A}$.

Remark: We cannot remove g from the DCT by the same example we used $<$ in Fatou's Lemma.

Proof: Note that $|f_n| \leq g$ implies f_n are integrable for all $n \in \mathbb{N}$. Moreover, because $f_n \to f$ pointwise almost everywhere we have $|f| \leq g$, so f is integrable. So

$\int_A f \, d\mu$ and $\int_A f_n \, d\mu$ exists.

Note that $|f - f_n|$ is measurable and

$$|f - f_n| \leq |f| + |f_n| \leq 2|g| \quad \text{a.e.}$$

so $|f - f_n|$ is integrable. Note that

$$2g - |f - f_n| \geq 0 \implies 2g - |f - f_n| \to 2g \text{ pointwise}$$

By Fatou's Lemma,

$$\begin{aligned}
\int_A 2g \, d\mu &= \int_A \liminf_{n\to\infty} 2g - |f_n - f| \, d\mu \\
&\leq \liminf_{n\to\infty} \int_A 2g - |f_n - f| \, d\mu \\
&= \liminf_{n\to\infty} \int_A 2g \, d\mu - \int_A |f_n - f| \, d\mu \\
&= \int_A 2g \, d\mu - \limsup_{n\to\infty} \int_A |f_n - f| \, d\mu
\end{aligned}$$

Thus, $\limsup_{n\to\infty} \int_A |f_n - f| \, d\mu = 0$. Moreover,

$$0 \leq \liminf_{n\to\infty} \int_A |f_n - f| \, d\mu \leq \limsup_{n\to\infty} \int_A |f_n - f| \, d\mu \leq 0$$

so $\lim_{n\to\infty} \int_A |f_n - f| \, d\mu = 0$. Therefore,

$$\begin{aligned}
\left| \int_A f_n \, d\mu - \int_A f \, d\mu \right| &= \left| \int_A f_n - f \, d\mu \right| \\
&\leq \int_A |f_n - f| \, d\mu.
\end{aligned}$$

We are done.

Remark: Note in the proof of DCT, we proved the stronger result that

$$\int_A |f_n - f| \, d\mu \to 0$$

(This is useful)

Corollary: Let $f: X \to \mathbb{K}$ be integrable and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$. are pairwise disjoint, then

$$\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu$$

Moreover the sum converges absolutely.

Proof: Note

$$\sum_{n=1}^{\infty} \left| \int_A f \, d\mu \right| \leq \sum_{n=1}^{\infty} \int_{A_n} |f| \, d\mu$$
$$= \int_{\bigcup_{n=1}^{\infty} A_n} |f| \, d\mu < \infty$$

So the sum converges absolutely. For $N \in \mathbb{N}$, let

$g_N = \sum_{n=1}^{N} f \chi_{A_n}$ Then $g_N \to f \chi_{\bigcup_{n=1}^{\infty} A_n}$ pointwise and

$|g_N| \leq \sum_{n=1}^{N} |f| \chi_{A_n} \leq |f| \chi_{\bigcup_{n=1}^{\infty} A_n}$

integrable because f is integrable. The DCT implies

$$\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \int_X f \chi_{\bigcup_{n=1}^{\infty} A_n} \, d\mu = \lim_{N \to \infty} \int_X g_N \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu .$$

L_p-Spaces

Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f: X \to \mathbb{K}$ be measurable. For $1 \leq p < \infty$, we say f is **p-integrable** if

$$\int_X |f|^p \, d\mu < \infty$$

Example: Let μ be the counting measure on $\mathbb{N}$. Then any $f: \mathbb{N} \to \mathbb{K}$ is measurable and

$$\int_{\mathbb{N}} |f|^p \, d\mu = \sum_{n=1}^{\infty} |f(n)|^p$$

The p-integrable functions form $\ell_p(\mathbb{N})$.

Notation: We use $\mathcal{L}_p(X, \mu)$ to denote the p-integrable functions on $(X, \mathcal{A}, \mu)$ for $p \in [1, \infty)$

Proposition: $\mathcal{L}_p(X, \mu)$ is a vector space.

Proof: Let $f, g \in \mathcal{L}_p(X, \mu)$ and $\alpha \in \mathbb{K}$.

- $\int_X |\alpha f|^p \, d\mu = |\alpha|^p \int_X |f|^p \, d\mu < \infty \implies \alpha f \in \mathcal{L}_p(X, \mu)$
- Note that $$\begin{aligned} |f+g|^p &\leq (|f| + |g|)^p \\ &\leq (2 \max\{|f|, |g|\})^p \\ &\leq 2^p \max\{|f|^p, |g|^p\} \\ &\leq 2^p |f|^p + 2^p |g|^p \end{aligned}$$

Remark: We want to define a norm on $\mathcal{L}_p(X, \mu)$. We want to define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

We check the triangle inequality holds. Note

$$\|\alpha f\|_p = \left(\int_X |\alpha f|^p \, d\mu\right)^{\frac{1}{p}} = |\alpha| \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} = |\alpha| \|f\|_p.$$

Note that $\|f\|_p \in [0, \infty)$. Also, if $f = 0$, so $\|f\|_p = 0$. Otherwise, if $\|f\|_p = 0$, then $f = 0$ a.e. Thus, we need not have a norm as zero a.e. not imply 0 everywhere.

Remark: Let $\mathcal{M}(X, \mathbb{K}) = \{ f : X \to \mathbb{K} : f \text{ measurable} \}$. Let $W = \{ f \in \mathcal{M}(X, \mathbb{K}) : f = 0 \text{ a.e} \}$. Then W is a subspace so we form a quotient space $\mathcal{M}(X, \mathbb{K}) / W$. We denote $[f]_W = f + W$.

Note that $[f]_W = 0$ if and only if $f \in W$ if and only if $f = 0$ a.e. Note $[f]_W = [g]_W$ if and only if $f = g$ a.e. Also, $[f]_W = [g]_W$ then $f \in \mathcal{L}_p(X, \mu)$ if and only if $g \in \mathcal{L}_p(X, \mu)$ with $\|f\|_p = \|g\|_p$.

Definition: The L_p-Space of X with respect to μ is

$$L_p(X, \mu) = \{ [f]_W : f \in \mathcal{L}_p(X, \mu), [f]_W = f + W \}$$

This is a vector space. Then the p-norm on $L_p(X, \mu)$ is defined by

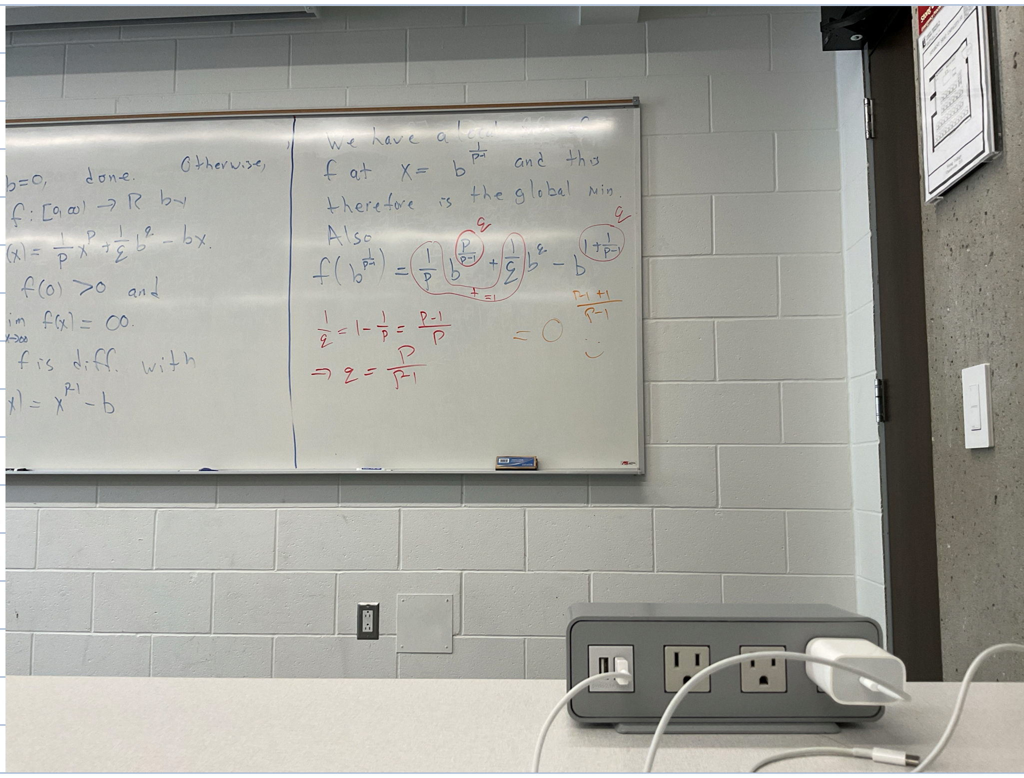
$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$$

Remark: To check the p-norm is a norm on $L_p(X, \mu)$ we need the triangle inequality.

Theorem: (Young's Inequality) If $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \geq 0$, then

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q.$$

Proof: Check Functional Analysis notes.



Theorem: (Hölder's Inequality) If $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then if $f \in \mathcal{L}_p(X, \mu)$, $g \in \mathcal{L}_q(X, \mu)$, then $fg \in \mathcal{L}_1$ and $\|fg\|_1 \le \|f\|_p \|g\|_q$

Proof: Let $\alpha = \|f\|_p$, $\beta = \|g\|_q$. If $\alpha = 0$ or $\beta = 0$, obvious.

Otherwise

$$\begin{aligned}
\int_X |fg| \, d\mu &= \alpha\beta \int_X \frac{|f|}{\alpha} \cdot \frac{|g|}{\beta} \, d\mu \\
&\le \alpha\beta \int_X \frac{1}{p}\left(\frac{|f|}{\alpha}\right)^p + \frac{1}{q}\left(\frac{|g|}{\beta}\right)^q d\mu \\
&= \frac{\alpha\beta}{p\alpha^p} \int_X |f|^p + \frac{\alpha\beta}{q\beta^q} \int_X |g|^q \, d\mu \\
&= \frac{\alpha\beta}{p\alpha^p} \alpha^p + \frac{\alpha\beta}{q\beta^q} \beta^q \\
&= \alpha\beta
\end{aligned}$$