**MATH 6280** **Assignment 1**

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**Question 1.** Let $\mu$ be a measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x])$$

for all $x \in \mathbb{R}$. The function $F$ is called the *cumulative distribution function* of $\mu$.

a) Show that if $\mu$ is finite, then $F$ is non-decreasing, right continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$.

b) Show that if $\mu$ is finite and if $\mu$ has no atoms (that is, $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$), then $F$ is continuous.

**Question 2.** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, let

$$\mathcal{F} = \{\emptyset, \mathbb{N}\} \cup \{\{2k-1, 2k\} \mid k \in \mathbb{N}\}$$

and define $\ell : \mathcal{F} \to [0, \infty]$ by $\ell(\emptyset) = 0$, $\ell(\{2k-1, 2k\}) = 1$ for all $k \in \mathbb{N}$, and $\ell(\mathbb{N}) = \infty$. If $\mu_\ell^*$ denotes the outer measure associated to $\ell$, describe the $\sigma$-algebra $\mathcal{A}$ of all $\mu_\ell^*$-measurable sets. Justify your answer.

**Question 3.** Let $A \subseteq \mathbb{R}$. Prove the following are equivalent:

1. $A \in \mathcal{M}(\mathbb{R})$.
2. For all $\epsilon > 0$ there exists an open subset $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $\lambda^*(U \setminus A) < \epsilon$.
3. For all $\epsilon > 0$ there exists a closed subset $F \subseteq \mathbb{R}$ such that $F \subseteq A$ and $\lambda^*(A \setminus F) < \epsilon$.
4. There exists a $G_\delta$ set $G \subseteq \mathbb{R}$ such that $A \subseteq G$ and $\lambda^*(G \setminus A) = 0$.
5. There exists an $F_\sigma$ set $F \subseteq \mathbb{R}$ such that $F \subseteq A$ and $\lambda^*(A \setminus F) = 0$.

(Recall a set is $G_\delta$ if it is the countable intersection of open sets and a set is $F_\sigma$ if it is the countable union of closed sets).

**Question 4.** Let $A \in \mathcal{M}(\mathbb{R})$. Prove that

a) $\lambda(A) = \inf\{\lambda(U) \mid U \subseteq \mathbb{R} \text{ is an open set such that } A \subseteq U\}$.

b) $\lambda(A) = \sup\{\lambda(K) \mid K \subseteq \mathbb{R} \text{ is a compact set such that } K \subseteq A\}$.

**Question 5.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let

$$\overline{\mathcal{A}} = \{E \subseteq X \mid \text{there exists } A, B \in \mathcal{A} \text{ such that } A \subseteq E \subseteq B \text{ and } \mu(B \setminus A) = 0\}.$$

Define $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty]$ by $\overline{\mu}(E) = \mu(A)$ where $E \in \overline{\mathcal{A}}$ and $A, B \in \mathcal{A}$ are such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$.

a) Show that $\mathcal{A} \subseteq \overline{\mathcal{A}}$, $\overline{\mu}$ is well-defined, and $\overline{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

b) Show that $\overline{\mathcal{A}}$ is a $\sigma$-algebra, $\overline{\mu}$ is a measure on $(X, \overline{\mathcal{A}})$, and that $\overline{\mu}$ is complete

**Question 6.** Prove that if $A \subseteq \mathbb{R}$ is such that $\lambda^*(A) > 0$, then there exists a subset $B \subseteq A$ such that $B$ is not Lebesgue measurable.
(Hint: Reduce to the case that $A$ is bounded and use the same technique from class to construct a non-measurable subset.)