Proof of RN-Theorem: Note if $A \in \mathcal{A}$, and $A \subseteq P_q \cap N_r$ for some $q, r \in \mathbb{Q}$ with $q < r$, then $A \subseteq P_q$ so $(\nu - q\mu)(A) \geq 0$, so $\nu(A) \geq q\mu(A)$. On the other hand, if $A \subseteq N_q$, so we have $(\nu - r\mu)(A) \leq 0$, so $\nu(A) \leq r\mu(A)$. Moreover, $f(x) \leq r$ μ-almost everywhere for all $x \in A$, so

$$\int_A f \, d\mu \leq r\mu(A)$$

Hence $\int_A f \, d\mu$, $\nu(A) \in [q\mu(A), r\mu(A)]$.

Now to see that $\nu(A) = \int_A f \, d\mu$ for all $A \in \mathcal{A}$, let $A \in \mathcal{A}$, fix $m \in \mathbb{N}$. For all $n \in \mathbb{N}_0$, let

$$A_{m,n} = A \cap \left(N_{\frac{n+1}{m}} \setminus \left(\bigcup_{k=0}^{n} N_{\frac{k}{m}} \right) \right) \text{ and let}$$

$$A_{m,\infty} = A \setminus \bigcup_{n=0}^{\infty} A_{m,n} \in \mathcal{A}, \text{ and note that } A = \left(\bigcup_{n=0}^{\infty} A_{m,n} \right) \cup A_{m,\infty}.$$

Note that $A_{m,\infty} \subseteq N^c_{\frac{n+1}{m}} = P_{\frac{n+1}{m}}$ for all $n \in \mathbb{N}$. If $\mu(A_{m,\infty}) > 0$, then as $A_{m,\infty} \subseteq P_{\frac{n+1}{m}}$ for all $n \in \mathbb{N}$, we have $\nu(A_{m,\infty}) \geq \frac{n+1}{m}\mu(A_{m,\infty})$ so taking $n \to \infty$, $\nu(A_{m,\infty}) = 0$, so $\nu(A) = \infty$ as $A_{m,\infty} \subseteq A$.

Note $f(x) \geq \frac{n+1}{m}$ for μ-almost everywhere $x \in P_{\frac{n+1}{m}}$, so as $A_{m,\infty} \subseteq P_{\frac{n+1}{m}}$ for all $n \in \mathbb{N}$, we have

$$\int_{A_{m,\infty}} f \, d\mu \geq \frac{n+1}{m} \mu(A_{m,\infty}) \to \infty,$$

so $\int_A f \, d\mu \geq \int_{A_{m,\infty}} f \, d\mu = \infty$, so we have the result. Hence we may assume $\mu(A_{m,\infty}) = 0$, so $\int_{A_{m,\infty}} f \, d\mu = 0$ and thus $\nu(A_{m,\infty}) = 0$ as $\nu \ll \mu$.

Since $A_{m,n} \subseteq N_{\frac{n+1}{m}} \cap P_{\frac{n}{m}}$ the above computation yields $\frac{n}{m}\mu(A_{m,n}) \leq \nu(A_{m,n}) \leq \frac{n+1}{m}\mu(A_{m,n})$ and also $\frac{n}{m}\mu(A_{m,n}) \leq \int_{A_{m,n}} f \, d\mu \leq \frac{n+1}{m}\mu(A_{m,n})$, so either we have $\nu(A) = \int_A f \, d\mu = \infty$ or otherwise the following computation works. Now,

$$\left| \nu(A) - \int_A f \, d\mu \right| = \left| \cancel{\nu(A_{m,\infty})}^{\,0} + \sum_{n=0}^{\infty} \nu(A_{m,n}) - \cancel{\int_{A_{m,\infty}} f \, d\mu}^{\,0} - \sum_{n=0}^{\infty} \int_{A_{m,n}} f \right|$$
$$\leq \sum_{n=0}^{\infty} \left| \nu(A_{m,n}) - \sum_{n=0}^{\infty} \int_{A_{m,n}} f \, d\mu \right|$$
$$\leq \sum_{n=0}^{\infty} \frac{1}{m} \mu(A_{m,n})$$
$$\leq \frac{1}{m} \mu(A)$$
$$\leq \frac{1}{n} \underbrace{\mu(X)}_{<\infty}$$

:)

<u>Case 2: When μ is σ-finite</u>

There exists $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $\{X_n\}_{n=1}^{\infty}$ pairwise disjoint, and $X = \bigcup_{n=1}^{\infty} X_n$, $\mu(X_n) < \infty$ $\forall n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mu_n, \nu_n : \mathcal{A} \to [0, \infty]$ by $\mu_n(A) = \mu(A \cap X_n)$ and $\nu_n(A) = \nu(A \cap A_n)$ for all $A \in \mathcal{A}$. Note that $\mu_n(X) = \mu(X_n) < \infty$ and if $A \in \mathcal{A}$ and $\mu_n(A) = 0$ then $\mu(A \cap A_n) = 0$, so by absolute continuity $\nu(A \cap X_n) = 0$, so $\nu_n(A) = 0$, so $\nu_n \ll \mu$. Case 1 implies there exists $f_n : X \to [0, \infty]$ measurable such that $\nu_n(A) = \int_A f_n \, d\mu$ $\forall A \in \mathcal{A}$. Let $f : X \to [0, \infty]$ be defined by $f = \sum_{n=1}^{\infty} f_n \chi_{X_n}$. Then for all $A \in \mathcal{A}$, we have

$$\nu(A) = \nu\left(A \cap \left(\bigcup_{n=1}^{\infty} X_n\right)\right)$$
$$= \sum_{n=1}^{\infty} \nu(A \cap X_n)$$
$$= \sum_{n=1}^{\infty} \nu_n(A)$$
$$= \sum_{n=1}^{\infty} \int_A f_n \, d\mu$$
$$= \sum_{n=1}^{\infty} \int_{A \cap X_n} f_n \, d\mu$$
$$= \sum_{n=1}^{\infty} \int_A f \chi_{X_n} \, d\mu$$
$$= \int_A f \, d\mu$$

∎

Corollary: Let μ be a σ-finite measure and let ν be a finite signed measure on $(X, \mathcal{A})$. Then there exists a unique $f \in L_1(X, \mu)$ such that $\nu(A) = \int_A f \, d\mu$ for all $A \in \mathcal{A}$.

Proof: Because $\nu \ll \mu$, then we have $\nu_\pm \ll \mu$, so there are $f_\pm : X \to [0, \infty]$ measurable such that

$\nu_\pm(A) = \int_A f_\pm \, d\mu$. Let $f = f_+ - f_-$, so f is measurable.

Because ν is finite, $\nu_\pm$ are finite, so

$\int_X |f_\pm| \, d\mu = \int_X f_\pm \, d\mu = \nu_\pm(X) < \infty$, so $f_\pm$ are integrable, so f is integrable.

Suppose $g \in L_1(X, \mu)$ such that $\nu(A) = \int_A g \, d\mu$ for all $A \in \mathcal{A}$. Then $\int_A g_+ \, d\mu - \int_A g_- \, d\mu = \nu(A) = \int_A f_+ \, d\mu - \int_A f_- \, d\mu$ so

$\int_A g_+ \, d\mu + \int_A f_- \, d\mu = \int_A f_+ \, d\mu + \int_A g_- \, d\mu$, so by RN,

$g_+ + f_- = f_+ + g_-$ a.e, so $g = g_+ - g_- = f_+ - f_- = f$ a.e.

Example: RN fails when μ is not σ-finite.

Let μ be the counting measure on $\mathbb{R}$, and let $\nu = \lambda$ both restricted to $\mathcal{B}(\mathbb{R})$. Then if $A \in \mathcal{B}(\mathbb{R})$ and $\mu(A) = 0$, so $A = \emptyset$, so $\lambda(A) = 0$. Because $\mathbb{R}$ is uncountable, μ is not σ-finite.

If $f : \mathbb{R} \to [0, \infty]$ was Borel and $\lambda(A) = \int_A f \, d\mu$ for all $A \in \mathcal{B}(\mathbb{R})$, note for all $x \in \mathbb{R}$,

$0 = \lambda(\{x\}) = \int_{\{x\}} f \, d\mu = f(x)$, so $f \equiv 0$, and so

$\lambda(A) = \int_A 0 \, d\mu = 0$ for all $A \in \mathcal{B}(\mathbb{R})$

Definition: Let μ be a σ-finite measure on $(X, \mathcal{A})$, and let ν

be either a measure or a finite signed measure such that $\nu \ll \mu$. The unique $f: X \to \mathbb{R}$ which is measurable and either nonnegative when ν is a measure and an L_1-function when ν is a signed measure produced by the RN Theorem is called the Radon-Nikodym derivative of ν by μ, denoted by $\frac{d\nu}{d\mu}$

Exercise: If μ is a σ-finite and ν is a measure such that then $\int_A f \, d\nu = \int_A f \frac{d\nu}{d\mu} \, d\mu$.

Example: (Lebesgue-Stieljes Measure) Let $F: \mathbb{R} \to \mathbb{R}$ be non-decreasing and right continuous. Let λ_F be the Lebesgue Stieljes measure. Then $\lambda_F \ll \lambda$ if and only if F is absolutely continuous and moreover, $\frac{d\lambda_F}{d\lambda} = F'$, so

$$\int_A f \, d\mu = \int_A f F' \, d\mu$$

for all $A \in \mathcal{B}(\mathbb{R})$. Indeed, if $\lambda_F \ll \lambda$, then we have that

$$\lambda_F(A) = \int_A \frac{d\lambda_F}{d\lambda} \, d\lambda \text{ for all } A \in \mathcal{B}(\mathbb{R}) \text{ with } \frac{d\lambda_F}{d\lambda} : \mathbb{R} \to [0, \infty]$$

measurable. Note for all $a < b$, $a, b \in \mathbb{R}$, that

$$F(b) - F(a) = \lambda((a, b]) = \int_a^b \frac{d\lambda_F}{d\lambda} \, d\lambda, \text{ so } \frac{d\lambda_F}{d\lambda} \text{ is}$$

integrable on any closed interval as $F: \mathbb{R} \to \mathbb{R}$. Hence FTC II implies F is absolutely continuous and $F' = \frac{d\lambda_F}{d\lambda}$.

On the other hand, if F is absolutely continuous, F' exists a.e and is integrable. Since F is non-decreasing, $F' \geq 0$ a.e. Define $\nu: \mathcal{B}(\mathbb{R}) \to [0, \infty]$ by $\nu(A) = \int_A F' \, d\lambda$. Clearly ν is a measure such that $\nu \ll \lambda$ and so $\frac{d\nu}{d\lambda} = F'$. So we need to show $\nu = \lambda_F$.

Note as λ_F is σ-finite, we can use the fact that λ_F is unique measure that extends a premeasure. Note

$$\nu([a,b]) = \int_{[a,b]} F' d\lambda = F(b) - F(a) = \lambda_F([a,b])$$

Since the algebra $\mathcal{F}$ of all finite unions of such sets, we are done.

Section 5.6: Lebesgue Decomposition Theorem

Theorem: Let μ and ν be σ-finite measures on $(X, \mathcal{A})$. Then there exists a unique pair of measures ν_a and ν_s on $(X, \mathcal{A})$ such that $\nu = \nu_a + \nu_s$, $\nu_a \ll \mu$, and $\nu_s \perp \mu$.

Proof: Note $\mu \ll \mu + \nu$ and moreover $\mu + \nu$ is σ-finite. Indeed, let $\{X_n\}_{n=1}^{\infty}, \{Y_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} Y_n$, $\mu(X_n) < \infty$, $\mu(Y_m) < \infty$. Let $Z_{n,m} = X_n \cap Y_m$ $\forall n, m \in \mathbb{N}$, so $\{Z_{n,m}\}_{n,m=1}^{\infty} \subseteq \mathcal{A}$ is a countable collection with $(\mu+\nu)(Z_{n,m}) < \infty$, so $\mu + \nu$ is σ-finite.

Hence RN implies there is a unique $f: X \to [0, \infty]$ measurable such that

$$\mu(A) = \int_A f \, d(\mu+\nu) \quad \text{for all } A \in \mathcal{A}$$

Let $P = \{x \in X : f(x) > 0\}$ and $N = \{x \in X : f(x) = 0\}$, note $X = P \cup N$ and $\emptyset = P \cap N$. Let $\nu_a, \nu_s : X \to [0, \infty]$ be defined by $\nu_a(A) = \nu(A \cap P)$ and $\nu_s(A) = \nu(A \cap N)$ $\forall A \in \mathcal{A}$. Then ν_a, ν_s are measures with $\nu = \nu_a + \nu_s$, $\boxed{\nu_s \perp \mu}$

$\mu(N) = \int_N f \, d(\mu+\nu) = 0$, and $\nu_s(P) = \nu(P \cap N) = \nu(\emptyset) = 0$

$\boxed{\nu_a \ll \mu}$ Let $A \in \mathcal{A}$, $\mu(A) = 0$, then $0 = \int_A f \, d(\mu+\nu)$ so $(\mu+\nu)(A \cap P) = 0 \Rightarrow \nu(A \cap P) = 0 \Rightarrow \nu_a(A) = 0$.