MATH 6280 **Assignment 6**

Due TBD Skoufranis

Question 1. Let $f : [0, 1] \to \mathbb{R}$ be continuous and let $G(f)$ be the graph of f; that is,

$$G(f) = \{(x, f(x)) \mid x \in [0, 1]\} \subseteq \mathbb{R}^2.$$

Prove that $\lambda_2(G(f)) = 0$.

Question 2. a) Let $B \subseteq \mathbb{R}$ be a Borel set. Prove

$$B' = \{(x, y) \in \mathbb{R}^2 \mid x - y \in B\} \in \mathcal{M}(\mathbb{R}^2).$$

b) Let $A \in \mathcal{M}(\mathbb{R})$ be such that $\lambda(A) = 0$. Prove

$$A' = \{(x, y) \in \mathbb{R}^2 \mid x - y \in A\} \in \mathcal{M}(\mathbb{R}^2)$$

and $\lambda_2(A') = 0$.

c) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function and define $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x, y) = f(x - y)$$

for all $(x, y) \in \mathbb{R}^2$. Prove h is 2-dimensional Lebesgue measurable.

Question 3. In this question, we will delve into Fourier analysis on $\mathbb{R}$. For $f, g \in L_1(\mathbb{R}, \lambda)$, define $f * g : \mathbb{R} \to \mathbb{C}$ by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y)\, d\lambda(y).$$

a) Prove that $f * g$ is a well-defined Lebesgue measurable function. Furthermore, prove that $f * g \in L_1(\mathbb{R}, \lambda)$ with $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

b) Given $h \in L_1(\mathbb{R}, \lambda)$, define $\widehat{h} : \mathbb{R} \to \mathbb{C}$ by

$$\widehat{h}(y) = \int_{\mathbb{R}} e^{-iyx} h(x)\, d\lambda(x)$$

for all $y \in \mathbb{R}$. It is an application of the Dominated Convergence Theorem to prove that $\widehat{h}$ is a well-defined, continuous, bounded function.

Prove if $f, g \in L_1(\mathbb{R}, \lambda)$, then $\widehat{f * g}(y) = \widehat{f}(y)\widehat{g}(y)$ for all $y \in \mathbb{R}$.

c) Let $G : \mathbb{R} \to [0, \infty)$ be defined by

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for all $x \in \mathbb{R}$. Note $\int_{\mathbb{R}} |G|\, d\lambda = 1$. Prove that $\widehat{G}(y) = e^{-\frac{y^2}{2}} = \sqrt{2\pi} G(y)$ for all $y \in \mathbb{R}$.

d) For each $\epsilon > 0$, let $G_\epsilon : \mathbb{R} \to [0, \infty)$ be defined by

$$G_\epsilon(x) = \frac{1}{\epsilon} G\left(\frac{x}{\epsilon}\right)$$

for all $x \in \mathbb{R}$. Using the above, it is not difficult to verify that

(I) $\widehat{G_\epsilon}(y) = e^{-\frac{\epsilon^2 y^2}{2}}$ for all $y \in \mathbb{R}$,

(II) $\int_{\mathbb{R}} G_\epsilon\, d\lambda = 1$,

(III) for all $\epsilon_0 > 0$ and $\delta > 0$ there exists an $\epsilon' > 0$ such that $|G_\epsilon(x)| < \epsilon_0$ for all $|x| \geq \delta$ and all $0 < \epsilon \leq \epsilon'$, and

(IV) $\lim_{\epsilon\to 0^+} \int_{\mathbb{R}\setminus[-\delta,\delta]} G_\epsilon \, d\lambda = 0$ for all $\delta > 0$.

Prove that if $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue integrable and continuous at a point $x_0 \in \mathbb{R}$, then

$$\lim_{\epsilon\to 0^+} (f * G_\epsilon)(x_0) = f(x_0).$$

[Hint: Deal with bounded f first.]

e) Let $f, g : \mathbb{R} \to \mathbb{C}$ be Lebesgue integrable. Prove

$$\int_{\mathbb{R}} \widehat{f} g \, d\lambda = \int_{\mathbb{R}} f \widehat{g} \, d\lambda.$$

f) Prove that if $f : \mathbb{R} \to \mathbb{C}$ is continuous and Lebesgue integrable and if $\widehat{f}$ is Lebesgue integrable, then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(y) e^{iyx} \, d\lambda(y)$$

for all $x \in \mathbb{R}$.

[Hint: Let $h_\epsilon(x) = \frac{1}{\sqrt{2\pi}} e^{ix_0 x} G(\epsilon x)$.]