Definition: If $f: X \to \mathbb{K}$ is integrable and $A \in \mathcal{A}$, the **integral of f over A with respect to μ** is

$$\int_A f \, d\mu = \int_A Re(f)_+ \, d\mu - \int_A Re(f)_- + i \int_A Im(f)_+ \, d\mu - i \int_A Im(f)_- d\mu$$

Lemma: If f is integrable and g is measurable such that $f = g$ almost every where, then g is integrable and

$$\int_X f \, d\mu = \int_X g \, d\mu$$

Proof: If $f = g$ almost everywhere, then $|f| = |g|$ almoot everywhere, so

$$\int_X |g| \, d\mu = \int_X |f| \, d\mu < \infty,$$ so g is integrable. Furthermore

$Re(f)_\pm = Re(g)_\pm$ and $Im(f)_\pm = Im(g)_\pm$ almost everywhere, so

$$\int_X f \, d\mu = \int_X g \, d\mu$$

Remark: Let $f: X \to [-\infty, \infty]$ be into. If $B = \{x \in X : f(x) = \pm\infty\}$ then

$$\left.\begin{aligned} \int_B |f| \, d\mu &= \mu(B) \cdot \infty \\ \int_B |f| \, d\mu &\leq \int_X |f| \, d\mu < \infty \end{aligned}\right\} \mu(B) = 0.$$

So $f \chi_B = f$ almost everywhere.

Theorem: The set of integrable functions L_1 is a vector space and the integral is linear.

Proof: Let $f, g \in L_1$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\int_X |\alpha f + \beta g| \, d\mu \leq |\alpha| \int_X |f| \, d\mu + |\beta| \int_X |g| \, d\mu < \infty$$

Therefore, $\alpha f + \beta g \in L_1$.

Let $h_1, h_2, h_3, h_4 : X \to [0, \infty)$ be integrable and let

$h = h_1 - h_2 + ih_3 - ih_4$

Claim: $\int_X h \, d\mu = \int_X h_1 \, d\mu - \int_X h_2 \, d\mu + i\int_X h_3 \, d\mu - i\int_X h_4 \, d\mu$

Note that

$$Re(h)_+ - Re(h)_- + i Im(h)_+ - i Im(h)_- = h_1 - h_2 + ih_3 - ih_4$$

So $Re(h)_+ - Re(h)_- = h_1 - h_2$ and $Im(h)_+ - Im(h)_- = h_3 - h_4$

Note now $Re(h)_+ + h_2 = h_1 + Re(h)_-$, so now

$$\int_X Re(h)_+ + h_2 \, d\mu = \int_X h_1 + Re(h)_- \, d\mu, \quad \text{so}$$

$$\int_X Re(h)_+ \, d\mu + \int_X h_2 \, d\mu = \int_X h_1 d\mu + \int_X Re(h)_- \, d\mu$$

Because all functions are integrable.

$$\int_X Re(h)_+ \, d\mu - \int_X Re(h)_- \, d\mu = \int_X h_1 d\mu - \int_X h_2 \, d\mu$$

Similarly, the claim holds for the imaginary parts.

Let $f_1 = Re(f)_+$, $f_2 = Re(f)_-$, $f_3 = Im(f)_+$, $f_4 = Im(f)_-$,

Same for g_i, then

$$\int_X f + g \, d\mu = \int_X (f_1 + g_1) - (f_2 + g_2) + i(f_3 + g_3) - i(f_4 + g_4) \, d\mu$$

$$= \cdots = \int_X f \, d\mu + \int_X g \, d\mu$$

If $a \geq 0$

$$\int_X af \, d\mu = \int_X af_1 - af_2 + iaf_3 - iaf_4 \, d\mu$$

$$= \int_X af_1 - \int_X af_2 + ia\int_X f_3 - ia\int_X f_4 \, d\mu$$

$$= \cdots = a\int_X f \, d\mu$$

Similar for when $a < 0$.

Remark: If $f : X \to \mathbb{C}$ is integrable and $\bar{f}(x) = \overline{f(x)}$, then $\bar{f}$ is integrable and $\int_X \bar{f}\, d\mu = \overline{\int_X f\, d\mu}$ as $Re(f) = Re(\bar{f})$ and $Im(\bar{f}) = -Im(f)$.

Proposition: If $f : X \to \mathbb{K}$ is integrable, then

$$\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu.$$

Proof: Let $z \in \mathbb{C}$ s.t. $|z| = 1$ and $z\int_X f\, d\mu = \left|\int_X f\, d\mu\right|$

Then

$$\begin{aligned} 0 \leq \left|\int_X f\, d\mu\right| &= z\int_X f\, d\mu = \int_X zf\, d\mu \\ &= \int_X Re(zf)\, d\mu + \int_X Im(zf)\, d\mu \\ &= \int_X Re(zf)\, d\mu \\ &\leq \int_X |zf|\, d\mu \\ &= \int_X |f|\, d\mu. \end{aligned}$$

Proposition: Let $f : \mathbb{R} \to \mathbb{K}$ be Lebesgue integrable. For $y \in \mathbb{R}$ define $f_y : \mathbb{R} \to \mathbb{R}$ by $f_y(x) = f(x-y)$. Then f_y is integrable and $\int_{\mathbb{R}} f_y\, d\lambda = \int_{\mathbb{R}} f\, d\lambda$.

Proof: If $f = \chi_A$ and $f_y = \chi_B$ where $B = y + A$, then f_y is measurable and

$$\int_{\mathbb{R}} f_y\, d\lambda = \lambda(B) = \lambda(A) = \int_{\mathbb{R}} f\, d\lambda.$$

By linearity, it works for simple functions. Approximation by simple functions, nonnegative, so by MCT, also works for all nonnegative functions, and linearity gives it works for all integrable functions.

Proposition: Let $f: \mathbb{R} \to \mathbb{K}$ be Lebesgue measurable. Let $\check{f}, g_a: \mathbb{R} \to \mathbb{K}$ for $a \in \mathbb{R} \setminus \{0\}$ defined by $\check{f}(x) = f(-x)$ and $g_a(x) = f(ax)$. Then $\check{f}$ and g_a are integrable and

$$\int_{\mathbb{R}} \check{f}\, d\mu = \int_{\mathbb{R}} f\, d\mu \quad \text{and} \quad \int_{\mathbb{R}} g_a\, d\mu = \frac{1}{a}\int_{\mathbb{R}} f\, d\mu$$

Revisiting The Riemann Integrable.

Proposition: Let $f: [a,b] \to \mathbb{R}$. Let

$$D(f) = \{x \in [a,b] : f \text{ is discontinuous at } x\}$$

For $n \in \mathbb{N}$, let

$$D_n(f) = \{x \in [a,b] : \forall \delta > 0\ \exists y, z \in [a,b] \text{ s.t. } |x-y| < \delta,\ |x-z| < \delta, \text{ and } |f(y) - f(z)| \geq \tfrac{1}{n}\}$$

Then $D_n(f)$ are closed $\forall n$ and $D(f) = \bigcup_{n=1}^{\infty} D_n(f)$

Theorem: Let $f: [a,b] \to \mathbb{R}$. Then f is Riemann Integrable and only if f is bounded and continuous a.e.

Proof: Assume f is Riemann Integrable. Then f is bounded. It suffices to show $\forall n \in \mathbb{N}$ $\lambda(D_n(f)) = 0$ by subadditivity.

Suppose for a contradiction that $\lambda(D_q(f)) > 0$ for some $q \in \mathbb{N}$. Because f is Riemann integrable, there exists a partition P such that

$$U(f,P) - L(f,P) < \tfrac{1}{q}\lambda(D_q(f)).$$

Put $P = \{a = t_0 < \cdots < t_n = b\}$. Let $M_k = \sup_{t \in [t_{k-1}, t_k]} f(t)$ and $m_k = \inf_{t \in [t_{k-1}, t_k]} f(t)$. If $x \in [t_{k-1}, t_k] \cap D_q(f)$, then

$M_k - m_k \geq \frac{1}{q}$. Moreover, the sums of lengths of the intervals in the partition that intersect $D_q(f)$ is at least $D_\varepsilon(f)$, so

$$\frac{1}{q}\lambda(D_\varepsilon(f)) > U(f,P) - L(f,P) \geq \lambda(D_\varepsilon(f)) \cdot \frac{1}{q}.$$

Absurd.

Conversely, assume f is bounded and Riemann integrable. Then $\lambda(D(f)) = 0$. Let $\varepsilon > 0$. Let $M = \sup_{x \in [a,b]} |f(x)|$.

By definition of Lebegue measure, there exists open intervals $\{I_i\}_{i=1}^{\infty}$ such that $D(f) = \bigcup_{i=1}^{\infty} I_i$ and

$$\sum_{i=1}^{\infty} \lambda(I_i) < \frac{\varepsilon}{4(M+1)}.$$

Because $D(f)$ is compact, there exists $N \in \mathbb{N}$ such that $D(f) \subseteq \bigcup_{i=1}^{N} I_i$ and $\sum_{i=1}^{N} \lambda(I_k) < \frac{\varepsilon}{4(M+1)}$

Construct a partition using the endpoints of all I_i for $1 \leq i \leq N$ with a, b. If there is a discontinuity in $[t_{k-1}, t]$ then $M_k - m_k \leq 2M$.

We add at most $2M \sum_{i=1}^{N} \lambda(I_i) < \frac{\varepsilon}{2}$ for all parts of the partition containing a discontinuity.

If f is continuous on $[t_{k-1}, t]$, use uniform continuity to further refine the partition to get

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Corollary: If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then f is Lebesgue measurable.

Proof: We want to show $f^{-1}((a,\infty))$ is lebesgue measurable $\forall a$.

Let $D \in \mathcal{M}(\mathbb{R})$ such that $\lambda(D) = 0$ and f continuous on D^c.

Then

$$f^{-1}((a,\infty)) = (f^{-1}((a,\infty)) \cap D) \cup (f^{-1}((a,\infty)) \cap D^c)$$

Note $f^{-1}((a,\infty)) \cap D$ is measurable because of completeness and $f^{-1}((a,\infty)) \cap D^c$ is measurable because $f|_{D^c}$ is continuous so $f^{-1}((a,\infty)) \cap D^c = f|_{D^c}^{-1}((a,\infty))$ which is open in D^c, so is Lebesgue measurable.

Theorem: If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,d\lambda$$

Proof: Without loss of generality, let $f \geq 0$ and $\varepsilon > 0$. Choose a partition $P = \{a = t_0 < \cdots < t_n = b\}$ with

$$M_k = \sup_{x \in [t_{k-1}, t_k]} f(x), \quad m_k = \inf_{x \in [t_{k-1}, t_k]} f(x), \text{ so}$$

$$U(f,P) - L(f,P) < \varepsilon.$$

So $L(f,P) \leq \int_a^b f(x)\,dx \leq U(f,P) < L(f,P) + \varepsilon$.

Let $\varphi = \sum_{i=1}^n M_k \chi_{[t_{k-1}, t_k]}$ and $\psi = \sum_{i=1}^n m_k \chi_{[t_{k-1}, t_k]}$, so $\psi \leq f \leq \varphi$, thus

$$L(f,P) = \int_{[a,b]} \psi\,d\lambda \leq \int_{[a,b]} f\,d\lambda \leq \int_{[a,b]} \varphi\,d\lambda = U(f,P)$$

$$\Rightarrow \left| \int_a^b f(x)\,dx - \int_{[a,b]} f\,d\lambda \right| < \varepsilon$$

Remark: If $f : X \to [0,\infty)$ is measurable, bounded, and $\mu(A) < \infty$

$$\int_A f\,d\mu = \inf\left\{ \int_A \varphi\,d\mu : \varphi \text{ simple } \varphi \geq f \right\}.$$