Question 1. Let μ be a measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x])$$

for all $x \in \mathbb{R}$. The function F is called the *cumulative distribution function* of μ.

a) Show that if μ is finite, then F is non-decreasing, right continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$.

b) Show that if μ is finite and if μ has no atoms (that is, $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$), then F is continuous.

Solution.
a) First note that F is well-defined since $(-\infty, x] \in \mathfrak{B}(\mathbb{R})$ for all $x \in \mathbb{R}$ so $F(x) = \mu((-\infty, x])$ makes sense. To see that F is non-decreasing, notice if $x, y \in \mathbb{R}$ with $x < y$ then $(-\infty, x] \subseteq (-\infty, y]$ so by the monotonicity of measures we have that

$$F(x) = \mu((-\infty, x]) \leq \mu((-\infty, y]) = F(y).$$

Hence F is non-decreasing.

To see that F is right continuous, let $x \in \mathbb{R}$ and let $(x_n)_{n\geq 1}$ be an arbitrary non-increasing sequence in $\mathbb{R}$ such that $x \leq x_n$ for all $n \in \mathbb{N}$ and $x = \lim_{n\to\infty} x_n$. For each $n \in \mathbb{N}$ let $A_n = (-\infty, x_n]$. Clearly $\{A_n\}_{n=1}^\infty \subseteq \mathfrak{B}(\mathbb{R})$ is such that $A_n \supseteq A_{n-1}$ and $\bigcap_{n=1}^\infty A_n = (-\infty, x]$. Therefore, as μ is finite, we obtain by the Monotone Convergence Theorem for Measures that

$$F(x) = \mu((-\infty, x]) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} F(x_n).$$

Therefore, since $(x_n)_{n\geq 1}$ was arbitrary, we obtain that F is right continuous.

To see that $\lim_{x\to-\infty} F(x) = 0$, let $(x_n)_{n\geq 1}$ be an arbitrary non-increasing sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = -\infty$. For each $n \in \mathbb{N}$ let $A_n = (-\infty, x_n]$. Clearly $\{A_n\}_{n=1}^\infty \subseteq \mathfrak{B}(\mathbb{R})$ is such that $A_n \supseteq A_{n-1}$ and $\bigcap_{n=1}^\infty A_n = \emptyset$. Therefore, since μ is finite, we obtain by the Monotone Convergence Theorem for Measures that

$$0 = \mu(\emptyset) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} F(x_n).$$

Therefore, since $(x_n)_{n\geq 1}$ was arbitrary, we obtain that $\lim_{x\to-\infty} F(x) = 0$.

Finally, to see that $\lim_{x\to\infty} F(x) = 0$, let $(x_n)_{n\geq 1}$ be an arbitrary non-decreasing sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = \infty$. For each $n \in \mathbb{N}$ let $A_n = (-\infty, x_n]$. Clearly $\{A_n\}_{n=1}^\infty \subseteq \mathfrak{B}(\mathbb{R})$ is such that $A_n \subseteq A_{n+1}$ and $\bigcup_{n=1}^\infty A_n = \mathbb{R}$. Therefore by the Monotone Convergence Theorem for Measures we obtain that

$$\mu(\mathbb{R}) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} F(x_n).$$

Therefore, as $(x_n)_{n\geq 1}$ was arbitrary, we obtain that $\lim_{x\to\infty} F(x) = \mu(\mathbb{R})$.

b) To see that F is continuous, it suffices by part (a) to show that F is left continuous. To see that F is left continuous, let $(x_n)_{n\geq 1}$ be an arbitrary non-decreasing sequence in $\mathbb{R}$ such that $x_n \leq x$ for all $n \in \mathbb{N}$ and $x = \lim_{n\to\infty} x_n$. For each $n \in \mathbb{N}$ let $A_n = (-\infty, x_n]$. Clearly $\{A_n\}_{n=1}^\infty \subseteq \mathfrak{B}(\mathbb{R})$ is such that $A_n \subseteq A_{n+1}$ and $\bigcup_{n=1}^\infty A_n = (-\infty, x)$. Therefore, since μ contains no atoms, we obtain by the Monotone Convergence Theorem for Measures that

$$F(x) = \mu((-\infty, x]) = \mu(\{x\}) + \mu((-\infty, x)) = \mu((-\infty, x)) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} F(x_n).$$

Therefore, since $(x_n)_{n\geq 1}$ was arbitrary, we obtain that F is left continuous and thus continuous.

Question 2. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, let

$$\mathcal{F} = \{\emptyset, \mathbb{N}\} \cup \{\{2k-1, 2k\} \mid k \in \mathbb{N}\}$$

and define $\ell : \mathcal{F} \to [0, \infty]$ by $\ell(\emptyset) = 0$, $\ell(\{2k-1, 2k\}) = 1$ for all $k \in \mathbb{N}$, and $\ell(\mathbb{N}) = \infty$. If μ_ℓ^* denotes the outer measure associated to ℓ, describe the σ-algebra $\mathcal{A}$ of all μ_ℓ^*-measurable sets. Justify your answer.

Solution. In order to determine the μ_ℓ^*-measurable sets, we first need to describe μ_ℓ^*. We claim that if $A \subseteq \mathbb{N}$ and

$$G_A = \{k \in \mathbb{N} \mid G \cap \{2k-1, 2k\} \neq \emptyset\}$$

then

$$\mu_\ell^*(A) = |G_A|.$$

To see this, note $\mu_\ell^*(\emptyset) = 0$ as μ_ℓ^* will always be an outer measure. Otherwise assume $A \subseteq \mathbb{N}$ is non-empty. Notice that $\{\{2k-1, 2k\}\}_{k \in G_A}$ is a collection of subsets of $\mathcal{F}$ such that $A \subseteq \bigcup_{k \in G_A} \{2k-1, 2k\}$. Therefore

$$\mu_\ell^*(A) \leq \sum_{k \in G_A} \ell(\{2k-1, 2k\}) = |G_A|.$$

To see the reverse inequality, let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}$ be such that $A \subseteq \bigcup_{n=1}^\infty A_n$. If $A_n = \mathbb{N}$ for some $n \in \mathbb{N}$, then

$$|G_A| \leq \infty = \sum_{n=1}^\infty \ell(A_n).$$

Otherwise, if $A_n \neq \mathbb{N}$ for all $n \in \mathbb{N}$, for each $k \in G_A$ we must have that there exists an $n_k \in \mathbb{N}$ such that $A_{n_k} = \{2k-1, 2k\}$ as either $2k-1 \in A$ or $2k \in A$. Hence

$$\sum_{n=1}^\infty \ell(A_n) \geq \sum_{k \in G_A} \ell(A_{n_k}) = |G_A|.$$

Therefore, since $\{A_n\}_{n=1}^\infty$ was arbitrary, $\mu_\ell^*(A) = |G_A|$ as claimed.

Now we claim that the μ_ℓ^*-measurable sets are

$$\mathcal{A} = \left\{ \bigcup_{k \in I} \{2k-1, 2k\} \,\middle|\, I \subseteq \mathbb{N} \right\}.$$

To begin, recall for a set $A \subseteq \mathbb{N}$ to be μ_ℓ^*-measurable, it must be the case that for all $B \subseteq \mathbb{N}$ that

$$\mu_\ell^*(B) = \mu_\ell^*(B \cap A) + \mu_\ell^*(B \cap A^c).$$

If $A \in \mathcal{P}(\mathbb{N}) \setminus \mathcal{A}$, then there exists a $k \in \mathbb{N}$ such that either $2k-1 \in A$ and $2k \notin A$, or $2k-1 \notin A$ and $2k \in A$. Therefore, if we let $B = \{2k-1, 2k\}$ then

$$\mu_\ell^*(B) = 1 \qquad \text{where as} \qquad \mu_\ell^*(B \cap A) + \mu_\ell^*(B \cap A^c) = 1 + 1 = 2$$

as $B \cap A$ and $B \cap A^c$ will each have at exactly one of $2k-1$ and $2k$. Hence, as $1 \neq 2$, we obtain that A is not μ_ℓ^*-measurable.

To complete the claim, let $A \in \mathcal{A}$ be arbitrary. Hence $A = \bigcup_{k \in I} \{2k-1, 2k\}$ for some $I \subseteq \mathbb{N}$. To see that A is μ_ℓ^*-measurable, let $B \subseteq \mathbb{N}$ be arbitrary. Hence

$$\mu_\ell^*(B) = |G_B|.$$

However, if $k \in G_B$, then by the definition of A either $\{2k-1, 2k\} \subseteq A$ and $\{2k-1, 2k\} \cap A^c = \emptyset$, or $\{2k-1, 2k\} \cap A = \emptyset$ and $\{2k-1, 2k\} \subseteq A^c$. Thus $G_B = G_{B \cap A} \cup G_{B \cap A^c}$ and $G_{B \cap A} \cap G_{B \cap A^c} = \emptyset$. Thus

$$\mu_\ell^*(B \cap A) + \mu_\ell^*(B \cap A^c) = |G_{B \cap A}| + |G_{B \cap A^c}| = |G_B| = \mu_\ell^*(B).$$

Therefore, since B was arbitrary, A is μ_ℓ^*-measurable. Thus the claim is complete so $\mathcal{A}$ is precisely the set of μ_ℓ^*-measurable sets.

Question 3. Let $A \subseteq \mathbb{R}$. Prove the following are equivalent:

1. $A \in \mathcal{M}(\mathbb{R})$.
2. For all $\epsilon > 0$ there exists an open subset $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $\lambda^*(U \setminus A) < \epsilon$.
3. For all $\epsilon > 0$ there exists a closed subset $F \subseteq \mathbb{R}$ such that $F \subseteq A$ and $\lambda^*(A \setminus F) < \epsilon$.
4. There exists a G_δ set $G \subseteq \mathbb{R}$ such that $A \subseteq G$ and $\lambda^*(G \setminus A) = 0$.
5. There exists an F_σ set $F \subseteq \mathbb{R}$ such that $F \subseteq A$ and $\lambda^*(A \setminus F) = 0$.

(Recall a set is G_δ if it is the countable intersection of open sets and a set is F_σ if it is the countable union of closed sets).

Solution. We will show that a), b), and d) are equivalent whereas the equivalence of a), c), and e) will follow by taking complements.

Fix $A \subseteq \mathbb{R}$ and assume that d) holds. Notice if $G \subseteq \mathbb{R}$ is a G_δ-set such that $A \subseteq G$ and $\lambda^*(G \setminus A) = 0$, we obtain that $G \setminus A \in \mathcal{M}(\mathbb{R})$ since the Lebesgue measure is complete. Furthermore, since G is G_δ, we obtain that G is Borel and thus $G \in \mathcal{M}(\mathbb{R})$. Therefore, since

$$A = (G \setminus A)^c \cap G$$

and since $\mathcal{M}(\mathbb{R})$ is closed under complements and intersections, we obtain that $A \in \mathcal{M}(\mathbb{R})$. Thus d) implies a).

Next, assume that a) holds so that $A \in \mathcal{M}(\mathbb{R})$. For each $n \in \mathbb{Z}$, let

$$A_n = A \cap [n, n+1].$$

By Proposition **??** for each $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ there exists an open set $U_{n,k}$ such that $A_n \subseteq U_{n,k}$ and

$$0 \leq \lambda(U_{n,k}) \leq \lambda(A_n) + \frac{1}{k2^{-|n|}}.$$

Hence, since $0 \leq \lambda(A_n) \leq \lambda([n, n+1]) < \infty$ by the monotonicity of measures, we obtain that

$$\lambda(U_{n,k} \setminus A_n) \leq \frac{1}{k2^{-|n|}}.$$

For each $k \in \mathbb{N}$ let

$$U_k = \bigcup_{n \in \mathbb{Z}} U_{n,k}.$$

Clearly U_k is an open set being the countable union of open sets. Furthermore, since $U_k, A \in \mathcal{M}(\mathbb{R})$, we obtain by subadditivity and monotonicity of measures that

$$\begin{aligned}
\lambda(U_k \setminus A) &= \lambda\left(\bigcup_{n \in \mathbb{Z}} (U_{n,k} \setminus A)\right) \\
&\leq \sum_{n \in \mathbb{Z}} \lambda(U_{n,k} \setminus A) \\
&\leq \sum_{n \in \mathbb{Z}} \lambda(U_{n,k} \setminus A_n) \\
&\leq \sum_{n \in \mathbb{Z}} \frac{1}{k2^{-|n|}} \\
&= \frac{3}{k}.
\end{aligned}$$

Hence b) follows.

To see that b) implies d), note that b) implies for each $k \in \mathbb{N}$ there exists an open set U_k such that $A \subseteq U_k$ and $\lambda(U_k \setminus A) \leq \frac{3}{k}$. Let

$$G = \bigcap_{k=1}^{\infty} U_k.$$

Then G is a G_δ set being the countable intersection of open sets. Thus G is Borel so $G \in \mathcal{M}(\mathbb{R})$. Furthermore, notice for all $k \in \mathbb{N}$ that

$$0 \leq \lambda(G \setminus A) \leq \lambda(U_k \setminus A) \leq \frac{3}{k}$$

by the monotonicity of measures. Hence, since $\lim_{k\to\infty} \frac{3}{k} = 0$, we obtain

$$\lambda^*(G \setminus A) = \lambda(G \setminus A) = 0$$

as desired.

Question 4. Let $A \in \mathcal{M}(\mathbb{R})$. Prove that

a) $\lambda(A) = \inf\{\lambda(U) \mid U \subseteq \mathbb{R} \text{ is an open set such that } A \subseteq U\}$.

b) $\lambda(A) = \sup\{\lambda(K) \mid K \subseteq \mathbb{R} \text{ is a compact set such that } K \subseteq A\}$.

Solution.
a) Let $A \in \mathcal{M}(\mathbb{R})$. Clearly if $U \subseteq \mathbb{R}$ is an open subset such that $A \subseteq U$ then $\lambda(A) \leq \lambda(U)$ by the monotonicity of measures and thus

$$\lambda(A) \leq \inf\{\lambda(U) \mid U \subseteq \mathbb{R} \text{ is an open set such that } A \subseteq U\}.$$

To see the other inequality let $\epsilon > 0$. Since $A \in \mathcal{M}(\mathbb{R})$, we know that $\lambda(A) = \lambda^*(A)$. Hence there exists a countable collection $\{I_n\}_{n=1}^\infty$ of open intervals such that $A \subseteq \bigcup_{n=1}^\infty I_n$ and

$$\sum_{n=1}^\infty \ell(I_n) \leq \lambda^*(A) + \epsilon.$$

Therefore, if $U = \bigcup_{n=1}^\infty I_n$, then U is an open subset of $\mathbb{R}$ such that $A \subseteq U$ and

$$\lambda(U) \leq \sum_{n=1}^\infty \ell(I_n) \leq \lambda^*(A) + \epsilon.$$

Hence

$$\inf\{\lambda(U) \mid U \subseteq \mathbb{R} \text{ is an open set such that } A \subseteq U\} \leq \lambda(A) + \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, we obtain the desire inequality.

b) First note that the difficulty in using a) to directly prove this result is that we have no control of measure of the complement of a set with infinite measure. Thus fix $A \in \mathcal{M}(\mathbb{R})$. Clearly if $K \subseteq \mathbb{R}$ is a compact such that $K \subseteq A$ then $\lambda(K) \leq \lambda(A)$ by the monotonicity of measures and thus

$$\lambda(A) \geq \sup\{\lambda(K) \mid K \subseteq \mathbb{R} \text{ is a compact set such that } K \subseteq A\}.$$

For the other direction, for each $n \in \mathbb{N}$ let

$$A_n = A \cap [-n, n].$$

Clearly $A_n \in \mathcal{M}(\mathbb{R})$ and

$$\lambda(A_n) \leq \lambda([-n, n]) \leq 2n < \infty$$

by the monotonicity of measures. Furthermore, since $A = \bigcup_{n=1}^\infty A_n$ and $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, we obtain by the Monotone Convergence Theorem (Theorem **??**) that

$$\lambda(A) = \lim_{n\to\infty} \lambda(A_n).$$

For each $n \in \mathbb{N}$, let $B = A_n^c \cap [-n, n]$. Clearly $\lambda(B_n) \leq \lambda([-n, n]) \leq 2n < \infty$ by the monotonicity of measures. By part a) there exists an open subset $U_n \subseteq \mathbb{R}$ such that $B_n \subseteq U_n$ and

$$\lambda(U_n) \leq \lambda(B_n) + \frac{1}{2^n}.$$

Hence, since $\lambda(B_n) < \infty$ so $\lambda(U_n) < \infty$, we obtain that $U_n \cap [-n, n] \in \mathcal{M}(\mathbb{R})$ and

$$0 \leq \lambda(U_n \cap [-n, n]) - \lambda(B_n) \leq \lambda(U_n) - \lambda(B_n) \leq \frac{1}{2^n}.$$

For each $n \in \mathbb{N}$, let $K_n = U_n^c \cap [-n, n]$. Clearly K_n is closed being the intersection of two closed sets and is bounded by n. Hence K_n is compact and $K_n \in \mathcal{M}(\mathbb{R})$. Moreover, since $B_n \subseteq U_n$, we have $K_n = U_n^c \cap [-n, n] \subseteq B_n^c \cap [-n, n] = A_n$. Since

$$[-n, n] = K_n \cup (U_n \cap [-n, n]) \qquad \text{and} \qquad [-n, n] = A_n \cup B_n$$

are disjoint unions of measurable sets, we obtain that

$$\lambda(K_n) + \lambda(U_n \cap [-n, n]) = 2n = \lambda(A_n) + \lambda(B_n)$$

so

$$\lambda(A_n) \leq \lambda(K_n) + \lambda(U_n \cap [-n, n]) - \lambda(B_n) \leq \lambda(K_n) + \frac{1}{2^n}.$$

Therefore, since

$$\lambda(A) = \lim_{n\to\infty} \lambda(A_n) \leq \liminf_{n\to\infty} \lambda(K_n) + \frac{1}{2^n} = \liminf_{n\to\infty} \lambda(K_n),$$

we have that

$$\lambda(A) \leq \sup\{\lambda(K) \mid K \subseteq \mathbb{R} \text{ is a compact set such that } K \subseteq A\}$$

as desired.

Question 5. Let $(X, \mathcal{A}, \mu)$ be a measure space and let

$$\overline{\mathcal{A}} = \{E \subseteq X \mid \text{there exists } A, B \in \mathcal{A} \text{ such that } A \subseteq E \subseteq B \text{ and } \mu(B \setminus A) = 0\}.$$

Define $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty]$ by $\overline{\mu}(E) = \mu(A)$ where $E \in \overline{\mathcal{A}}$ and $A, B \in \mathcal{A}$ are such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$.

a) Show that $\mathcal{A} \subseteq \overline{\mathcal{A}}$, $\overline{\mu}$ is well-defined, and $\overline{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

b) Show that $\overline{\mathcal{A}}$ is a σ-algebra, $\overline{\mu}$ is a measure on $(X, \overline{\mathcal{A}})$, and that $\overline{\mu}$ is complete

Solution.
a) First, to see that $\mathcal{A} \subseteq \mathcal{A}^*$, let $E \in \mathcal{A}$ be arbitrary. By letting $A = B = E$ we see that $A, B \in \mathcal{A}$, $A \subseteq E \subseteq B$, and $\mu(B \setminus A) = \mu(\emptyset) = 0$ so $E \in \mathcal{A}^*$ by definition. Hence, since $E \in \mathcal{A}^*$ was arbitrary, we obtain that $\mathcal{A} \subseteq \mathcal{A}^*$. Furthermore, this shows us that if μ^* is well-defined, then $\mu^*(E) = \mu(E)$ for all $E \in \mathcal{A}$ as desired.

To see that μ^* is well-defined, it suffices to show that if $E \in \mathcal{A}^*$ is such that there exists $A_1, A_2, B_1, B_2 \in \mathcal{A}$ with

$$A_k \subseteq E \subseteq B_k \qquad \text{and} \qquad \mu(B_k \setminus A_k) = 0$$

for all $k \in \{1, 2\}$, then $\mu(A_1) = \mu(A_2)$. To see this, let E, A_1, A_2, B_1, B_2 have the properties listed above. Hence for each $k \in \{1, 2\}$ we have that B_k is the disjoint union of $B_k \cap A_k = A_k$ and $B_k \setminus A_k$. Therefore, as each of these sets are elements of $\mathcal{A}$, we obtain that

$$\mu(B_k) = \mu(B_k \setminus A_k) + \mu(B_k \cap A_k) = 0 + \mu(A_k) = \mu(A_k).$$

Hence $\mu(A_1) = \mu(B_1)$ and $\mu(A_2) = \mu(B_2)$. However, since $A_1 \subseteq E \subseteq B_2$ and $A_2 \subseteq E \subseteq B_1$, we obtain by the monotonicity of measures that

$$\mu(A_1) \leq \mu(B_2) \qquad \text{and} \qquad \mu(A_2) \leq \mu(B_1).$$

These inequalities combined with the previous equalities implies that $\mu(A_1) = \mu(A_2)$ as desired. Hence μ^* is well-defined.

b) First we will demonstrate that $\mathcal{A}^*$ is a σ-algebra. First, since $\mathcal{A} \subseteq \mathcal{A}^*$, we clearly have $\emptyset, X \in \mathcal{A} \subseteq \mathcal{A}^*$. Next, let $E \in \mathcal{A}$ be arbitrary. To see that $E^c \in \mathcal{A}^*$, recall since $E \in \mathcal{A}$ there exists $A, B \in \mathcal{A}$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. Clearly $A^c, B^c \in \mathcal{A}$ as $\mathcal{A}$ is a σ-algebra, clearly $B^c \subseteq E^c \subseteq A^c$, and since

$$A^c \setminus B^c = \{x \in X \mid x \in A^c, x \notin B^c\} = \{x \in X \mid x \notin A, x \in B\} = B \setminus A,$$

we see that $\mu(A^c \setminus B^c) = \mu(B \setminus A) = 0$. Hence $E^c \in \mathcal{A}$.

Finally, assume $\{E_n\}_{n=1}^\infty \subseteq \mathcal{A}^*$. Thus for each $n \in \mathbb{N}$ there exists $A_n, B_n \in \mathcal{A}$ such that $A_n \subseteq E_n \subseteq B_n$ and $\mu(B_n \setminus A_n) = 0$. Clearly if

$$A = \bigcup_{n=1}^\infty A_n, \quad B = \bigcup_{n=1}^\infty B_n, \quad \text{and} \quad E = \bigcup_{n=1}^\infty E_n,$$

then $A, B \in \mathcal{A}$ since $\mathcal{A}$ is a σ-algebra, clearly $A \subseteq E \subseteq B$, and

$$\begin{aligned}
\mu(B \setminus A) &= \mu\left(\bigcup_{n=1}^\infty (B_n \setminus A)\right) \\
&\leq \sum_{n=1}^\infty \mu(B_n \setminus A) \\
&\leq \sum_{n=1}^\infty \mu(B_n \setminus A_n) \quad \text{monotonicity} \\
&= \sum_{n=1}^\infty 0 = 0.
\end{aligned}$$

Hence $E \in \mathcal{A}^*$ by definition. Therefore $\mathcal{A}^*$ is a σ-algebra.

To see that μ^* is a measure, first notice that $\emptyset \in \mathcal{A}^*$ and if $A = B = \emptyset \in \mathcal{A}$, then $A \subseteq \emptyset \subseteq B$ and $\mu(B \setminus A) = \mu(\emptyset) = 0$. Hence, by the definition of μ^*, we obtain that

$$\mu^*(\emptyset) = \mu(A) = 0$$

as desired.

Next assume that $\{E_n\}_{n=1}^\infty \subseteq \mathcal{A}^*$ are pairwise disjoint sets. By the definition of $\mathcal{A}^*$, for each $n \in \mathbb{N}$ there exists $A_n, B_n \in \mathcal{A}$ such that $A_n \subseteq E_n \subseteq B_n$ and $\mu(B_n \setminus A_n) = 0$ and thus $\mu^*(E_n) = \mu(A_n)$. Clearly if

$$A = \bigcup_{n=1}^\infty A_n, \quad B = \bigcup_{n=1}^\infty B_n, \quad \text{and} \quad E = \bigcup_{n=1}^\infty E_n,$$

then $A, B \in \mathcal{A}$ since $\mathcal{A}$ is a σ-algebra, clearly $A \subseteq E \subseteq B$, and, using the same computation as above, $\mu(B \setminus A) = 0$. Hence $\mu^*(E) = \mu(A)$ by the definition of μ^*. However, since $\{E_n\}_{n=1}^\infty$ are pairwise disjoint and $A_n \subseteq E_n$ for all $n \in \mathbb{N}$, we see that $\{A_n\}_{n=1}^\infty$ are pairwise disjoint. Therefore, since μ is a measure, we obtain that

$$\mu^*(E) = \mu(A) = \sum_{n=1}^\infty \mu(A_n) = \sum_{n=1}^\infty \mu^*(E_n)$$

as desired. Hence μ^* is a measure.

Finally, to see that μ^* is complete, assume $E \in \mathcal{A}^*$ and $F \subseteq X$ are such that $F \subseteq E$ and $\mu^*(E) = 0$. To see that $F \in \mathcal{A}^*$ thereby completing the proof, notice that since $E \in \mathcal{A}^*$ there exists $A, B \in \mathcal{A}$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. By the proof demonstrated in part a), notice that $\mu(A) = \mu(B)$ which implies that $\mu(B) = \mu^*(E) = 0$. Therefore, we clearly have that $\emptyset, B \in \mathcal{A}$, that $\emptyset \subseteq F \subseteq B$, and $\mu(B \setminus \emptyset) = \mu(B) = 0$. Hence $F \in \mathcal{A}^*$ by definition thereby completing the problem.

Question 6. Prove that if $A \subseteq \mathbb{R}$ is such that $\lambda^*(A) > 0$, then there exists a subset $B \subseteq A$ such that B is not Lebesgue measurable.
(Hint: Reduce to the case that A is bounded and use the same technique from class to construct a non-measurable subset.)

Solution. Let $A \subseteq \mathbb{R}$ be such that $\lambda^*(A) > 0$. For each $n \in \mathbb{Z}$ let $A_n = A \cap [n, n+1)$. Therefore, since $A = \bigcup_{n=1}^{\infty} A_n$, we obtain by the subadditivity of the Lebesgue outer measure that

$$0 < \lambda^*(A) \leq \sum_{n=1}^{\infty} \lambda^*(A_n).$$

Hence there exists an $N \in \mathbb{N}$ such that $\lambda^*(A_N) > 0$.

We claim there exists a subset $B \subseteq A_N$ such that B is not Lebesgue measurable. To see this, note since the notion of Lebesgue measurability is invariant under translation, we may assume that $N = 0$.

Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Clearly every equivalence class under $\sim$ has an element in $[0, 1)$ and by the Axiom of Choice there exists a subset B of A_0 that contains precisely one element from each equivalence class with a representative from A_0. We claim that B is not Lebesgue measurable. To see this, suppose for the sake of a contradiction that B is Lebesgue measurable.

Since $\mathbb{Q}$ is countable, we may enumerate $\mathbb{Q} \cap [0, 1)$ as

$$\mathbb{Q} \cap [0, 1) = \{r_n \mid n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, let

$$B_n = \{x \in [0, 1) \mid x \in r_n + B \text{ or } x + 1 \in r_n + B\}$$

(that is, B_n is $r_n + B$ modulo 1). Since $B_n \subseteq A_n$ where $\{A_n\}_{n=1}^{\infty}$ are as in the example from class, we see that $\{B_n\}_{n=1}^{\infty}$ is a collection of pairwise disjoint subsets of $[0, 1)$.

Moreover, we claim that

$$A_0 \subseteq \bigcup_{n=1}^{\infty} B_n.$$

To see this, note if $x \in A_0$ then there exists a unique $y \in B$ such that $x \sim y$. Thus $x - y \in \mathbb{Q} \cap (-1, 1)$. If $x - y \in \mathbb{Q} \cap [0, 1)$ then $x - y = r_n$ for some n and thus $x = r_n + y \in B_n$. Otherwise if $x - y \in \mathbb{Q} \cap (-1, 0)$ then $(x + 1) - y \in (0, 1)$. Thus $(x + 1) - y = r_n$ for some n and thus $x = r_n + y - 1 \in B_n$. Thus the claim is complete.

For each $n \in \mathbb{N}$, let

$$\begin{aligned} B_{n,1} &= (r_n + B) \cap [0, 1) \\ B_{n,2} &= -1 + ((r_n + B) \cap [1, 2)). \end{aligned}$$

Clearly $B_n = B_{n,1} \cup B_{n,2}$ since $r_n + B \subseteq [0, 2)$ for all n.

We claim that $B_{n,1} \cap B_{n,2} = \emptyset$. To see this, suppose for the sake of a contradiction that $b \in B_{n,1} \cap B_{n,2}$. By definition there exists $x, y \in B$ such that $r_n + x \in [0, 1)$, $r_n + y \in [1, 2)$, and $b = r_n + x = -1 + r_n + y$. Clearly $r_n + x \in [0, 1)$ and $r_n + y \in [1, 2)$ imply that $x \neq y$ whereas we have $x - y = -1 \in \mathbb{Q}$ so $x \sim y$. Therefore, since B contains exactly one element from each equivalence class, we have obtained a contradiction. Hence $B_{n,1} \cap B_{n,2} = \emptyset$.

To obtain our contradiction, note that

$$
\begin{aligned}
0 &< \lambda(A_0)) \\
&\leq \lambda\left(\bigcup_{n=1}^{\infty} B_n\right) && \text{monotonicity} \\
&= \sum_{n=1}^{\infty} \lambda(B_n) && \{A_n\}_{n=1}^{\infty} \text{ are disjoint} \\
&= \sum_{n=1}^{\infty} \lambda(B_{n,1} \cup B_{n,2}) \\
&= \sum_{n=1}^{\infty} \lambda(B_{n,1}) + \lambda(B_{n,2}) && B_{n,1} \text{ and } B_{n,2} \text{ are disjoint} \\
&= \sum_{n=1}^{\infty} \lambda((r_n + B) \cap [0,1)) + \lambda\left(((r_n + B) \cap [1,2)\right) \\
&= \sum_{n=1}^{\infty} \lambda((r_n + B) \cap [0,2)) \\
&= \sum_{n=1}^{\infty} \lambda(r_n + B) && r_n + B \subseteq [0,2) \\
&= \sum_{n=1}^{\infty} \lambda(B).
\end{aligned}
$$

This yields our contradiction since $\lambda(B) \in [0,\infty]$ yet no number in $[0,\infty]$ when summed an infinite number of times produces a number in $(0,\infty)$. Hence we have obtained our contradiction so B is not Lebesgue measurable.