Question 1. *Let A be a nonempty set. Prove that* $\dim(c_{00}(A)) = \#A$.

Question 2. *Give an example of an infinite-dimensional vector space X and a proper subspace Y of X such that* $\dim(X) = \dim(Y)$.

Question 3. *Let X be a vector space and let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of subspace of X such that each Y_n is finite-dimensional and $Y_1 \subset Y_2 \subset \cdots$. Show that $Y = \bigcup_{n=1}^{\infty} Y_n$ is a subspace of X and that Y has countable dimension.*

Question 4. *Prove that the following statements are true:*

(a) *If X is a vector space, the identity operator* $\mathrm{id} : X \to X$ *given by* $\mathrm{id}\, x = x$ *for all $x \in X$ is a linear operator.*

(b) *If X and Y are vector spaces, the zero operator $O : X \to Y$ given by $Ox = 0_Y$ for all $x \in X$ is a linear operator.*

(c) *If $m, n \in \mathbb{N}$, $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $A = [a_{i,j}]$ is an $n \times m$ matrix, then $T_A : \mathbb{R}^m \to \mathbb{R}^n$ given by*

$$T_A x = \begin{pmatrix} \sum_{j=1}^{m} a_{1,j} x_j & \sum_{j=1}^{m} a_{2,j} x_j & \cdots & \sum_{j=1}^{m} a_{n,j} x_j \end{pmatrix}$$

for $x = (x_1, ..., x_m) \in \mathbb{R}^m$ is a linear operator.

(d) *If $X = \mathcal{C}([0,1])$ and $Y = \mathbb{R}$, the* expectation operator $\mathbb{E} : \mathcal{C}([0,1]) \to \mathbb{R}$ *given by*

$$\mathbb{E} f = \int_0^1 f(x) dx$$

for all $f \in \mathcal{C}([0,1])$ is a linear operator.

(e) *If $X = \mathcal{C}([0,1])$ and $Y = \mathbb{R}$, for $x_0 \in [0,1]$, the dirac functional at x_0, $\delta_{x_0} : \mathcal{C}([0,1]) \to \mathbb{R}$ given by*

$$\delta_{x_0}(f) = f(x_0)$$

for all $f \in \mathcal{C}([0,1])$ is a linear operator.

(f) *If $X = \mathcal{C}^1([0,1])$ and $Y = \mathcal{C}([0,1])$, the differential operator $D : \mathcal{C}^1([0,1]) \to \mathcal{C}([0,1])$ given by $Df = f'$ is a linear operator.*

(g) *If $X = \mathcal{C}([0,1])$ and $Y = \mathcal{C}^1([0,1])$, the Volterra operator $V : \mathcal{C}([0,1]) \to \mathcal{C}^1([0,1])$ is a linear operator described as follows: for $f \in \mathcal{C}([0,1])$, $Vf : [0,1] \to \mathbb{R}$ is given by*

$$(Vf)(x) = \int_0^x f(t) dt$$

for $x \in [0,1]$.

Question 5. *Let X, Y and Z be vector spaces and $T : X \to Y$, $S : Y \to Z$ be linear operators. Show that their composition $ST : X \to Z$ is a linear operator.*

Question 6. *Let X and Y be vector spaces and $T : X \to Y$ be a linear operator. Let B be a Hamel basis of Y. The following statements are equivalent:*

(a) T is onto.

(b) $B \subset T[X]$

Question 7. *Let X and Y be vector spaces.*

(i) Let $T : X \to Y$ be a linear operator and $A \subset X$ such that $T|_A$ is one-to-one, and $T[A]$ is linearly independent. Prove that A is a linearly independent subset of X.

(ii) Give an example of vector spaces X and Y, a linear operator $T : X \to Y$ and a linearly dependent subset A of X such that $T[A]$ is linearly independent.

Question 8. *Let X and Y be vector spaces, and Z be a subspace of X, and $T : Z \to Y$ be a linear operator. Show that there exists a linear operator $\hat{T} : X \to Y$ such that $\hat{T}|_Z = T$.*

Question 9. *Let X be a vector space and $T : X \to X$ be a linear operator. Then, T is a linear projection if and only if $T|_{T[X]} = \mathrm{id}$.*

Question 10. *Let X be a vector space with a Hamel basis B. Prove that X is algebraically isomorphic to $c_{00}(B)$.*

