Theorem: Let $p, q \in (1,\infty)$ be conjugate exponents. Then $\ell_p^*(\mathbb{N})$ is isometrically isomorphic to $\ell_q(\mathbb{N})$. More precisely, the map $\Phi : \ell_q(\mathbb{N}) \to \ell_p^*(\mathbb{N})$ such that for $x \in \ell_q(\mathbb{N})$, $y \in \ell_p(\mathbb{N})$ $(\Phi x)(y) = \sum_{n=1}^{\infty} x_n y_n$ is an onto linear isometry.

Comment: Every $x \in \ell_q(\mathbb{N})$ can be "naturally" identified with a bounded linear functional

$$x \equiv \Phi x : \ell_p(\mathbb{N}) \to \mathbb{R}$$

and all bounded linear functionals $f : \ell_p(\mathbb{N}) \to \mathbb{R}$ are of the above form.

E.g. For $p = q = 2$, take $x = (\frac{1}{n})_{n=1}^{\infty} \in \ell_2(\mathbb{N})$ with $\|x\|_{\ell^2} = \frac{\pi}{\sqrt{6}}$. This defines $\phi x : \ell_2(\mathbb{N}) \to \mathbb{R}$ with $(\phi x)(y) = \sum_{n=1}^{\infty} \frac{1}{n} y_i$ and $\|\phi x\| = \frac{\pi}{\sqrt{6}}$.

Proof: We have to show that $\Phi : \ell_q(\mathbb{N}) \to \ell_p^*(\mathbb{N})$ is a linear isometry. It remains to prove that it is onto.

Fix $f \in \ell_p^*(\mathbb{N})$. We seek an $x \in \ell_q(\mathbb{N})$ such that $f = \Phi x$.

Define $x := (f(e_i))_{n=1}^{\infty}$ where $(e_n)_{n=1}^{\infty}$ is the standard basis of $c_{00}(\mathbb{N})$. We need to show that $x \in \ell^q(\mathbb{N})$ and $f = \Phi x$. Observe that for $y \in c_{00}(\mathbb{N})$ $y = \sum_{i=1}^{n} y_i e_i$. Then $f(y) = \sum_{i=1}^{n} y_i f(e_i) = \sum_{i=1}^{n} x_i y_i$.

Fix $N \in \mathbb{N}$. We will show $\sum_{n=1}^{N} |x_n|^q \leq \|f\|^q$. Consider $(x_n)_{n=1}^{N} \in \mathbb{R}^N$ and $(f_n)_{n=1}^{N} \in \mathbb{R}^N$. By a known result,

there exists $(y_n)_{n=1}^N \in \mathbb{R}^N$ such that
$\sum_{n=1}^N y_n x_n = \left(\sum_{i=1}^N |x_i|^q\right)^{\frac{1}{q}}$ and $\left(\sum_{i=1}^N |y_i|^p\right)^{\frac{1}{p}} = 1$.

Let $z = \sum_{i=1}^N y_i e_i \in \ell^p(\mathbb{N})$. Observe that

$$f(z) = f\left(\sum_{i=1}^N y_i e_i\right) = \sum_{i=1}^N y_i f(e_i) = \sum_{i=1}^N x_i y_i = \left(\sum_{i=1}^N |x_i|^q\right)^{\frac{1}{q}}$$

$$\Rightarrow \left(\sum_{i=1}^N |x_i|^q\right)^{\frac{1}{q}} \leq |f(z)| \leq \|f\| \cdot \|z\|_p = \|f\| \left(\sum_{i=1}^N |y_i|^p\right)^{\frac{1}{p}}$$
$$= \|f\|$$

$$\Rightarrow \sum_{i=1}^N |x_i|^q \leq \|f\|^q \implies x \in \ell^q(\mathbb{N}).$$

Because $x \in \ell^q(\mathbb{N}) \Rightarrow \Phi x \in \ell_p^*(\mathbb{N})$. It remains to prove that $\Phi x = f$. Note for $j \in \mathbb{N}$,

$$(\Phi x)(e^{(j)}) = \sum_{n=1}^\infty x_n e_n^{(j)} = \sum_{n=1}^\infty f(e_n) e_n^{(j)} = f(e^{(j)})$$

Bec. $(e_n)_{n=1}^\infty$ is a Hamel basis of $c_{00}(\mathbb{N})$, $f|_{c_{00}(\mathbb{N})} = \Phi(x)|_{c_{00}(\mathbb{N})}$. Bec. $c_{00}(\mathbb{N})$ is dense in $c_{00}(\mathbb{N})$ and $f, \Phi x$ are continuous, they are equal.

Theorem: $c_0^*(\mathbb{N}) \equiv \ell^1(\mathbb{N})$ and $(\ell^1(\mathbb{N})^* \equiv \ell^\infty(\mathbb{N}))$

More precisely.

$\Psi : \ell^1(\mathbb{N}) \to (c_0(\mathbb{N}))^*$ (resp. $\Psi : \ell^\infty(\mathbb{N}) \to (\ell^1(\mathbb{N}))^*$)

such that for

$x \in \ell^1(\mathbb{N}), y \in c_0(\mathbb{N})$ (resp. $x \in \ell^\infty(\mathbb{N}), y \in \ell^1(\mathbb{N})$)

$$(\Psi x)(y) = \sum_{n=1}^\infty x_n y_n$$

is an onto linear isometry

Remark: For $1<p,q<\infty$ conjugate exponents.

$(\ell^p(\mathbb{N}))^* \equiv \ell^q(\mathbb{N})$ $\quad$ $(\ell^q(\mathbb{N}))^* \equiv \ell^p(\mathbb{N})$

$(c_0(\mathbb{N}))^* \equiv \ell^1(\mathbb{N})$ $\quad$ $(\ell^1(\mathbb{N}))^* \equiv \ell^\infty(\mathbb{N})$

We will see later $(\ell^\infty(\mathbb{N}))^* \not\equiv \ell^1(\mathbb{N})$

Hilbert Spaces

Def: Let X be a vector space. A function $\langle \cdot,\cdot \rangle : X\times X \to \mathbb{R}$ is called an inner product if it satisfies the following:

(i) $\forall x\in X$ such that $x\neq 0$, $\langle x,x\rangle > 0$

(ii) $\forall x,y\in X$, $\langle x,y\rangle = \langle y,x\rangle$

(iii) $\forall x,y,z\in X$, $\lambda\in\mathbb{R}$, $\langle x,\lambda y+z\rangle = \lambda\langle x,y\rangle + \langle x,z\rangle$

Remark: If X is an inner product space, then

- $\forall x,y,z\in X$, $\lambda\in\mathbb{R}$, $\langle \lambda x+y,z\rangle = \lambda\langle x,z\rangle + \langle y,z\rangle$
- $\forall x\in X$, $\langle 0_X,x\rangle = \langle x,0_X\rangle = \langle x, 0\cdot 0_X\rangle = 0$.

In particular, for $x\in X$, $\langle x,x\rangle \geq 0$ and is zero if and only if $x=0$.

Examples:

(i) In $\mathbb{R}^n$, the standard inner product is the dot product: For $x,y\in\mathbb{R}^n$,

$$\langle x,y\rangle = x\cdot y = \sum_{i=1}^n x_i y_i$$

(ii) In $\ell_2(\mathbb{N})$, the standard inner product is the dot product: For $x,y\in \ell^2(\mathbb{N})$

$$\langle x, y\rangle = x \cdot y = \sum_{n=1}^{\infty} x_n y_n.$$

(iii) In $C([0,1])$, the usual inner product is as follows

For $f, g \in C([0,1])$,

$$\langle f, g\rangle = \int_0^1 f(x) g(x)\, dx$$

Proposition: (Cauchy-Schwarz) Let X be an inner product space and let $x, y \in X$. Then

$$\langle x, y\rangle^2 \leq \langle x, x\rangle \langle y, y\rangle.$$

Proof: Consider for $\lambda \in \mathbb{R}$, $p(\lambda) = \langle x - \lambda y, x - \lambda y\rangle \geq 0.$

$$= \langle x, x - \lambda y\rangle - \lambda \langle y, x - \lambda y\rangle$$

$$= \langle x, x\rangle - \lambda \langle x, y\rangle - \lambda \langle y, x\rangle +$$

$$= \lambda^2 \langle y, y\rangle - 2\lambda \langle x, y\rangle + \langle x, x\rangle \geq 0$$

This is a quadratic function in λ and it is nonnegative.

Therefore, the discriminant is non positive.

$$\Delta = b^2 - 4ac \leq 0 \Rightarrow (-2\langle x, y\rangle)^2 - 4\langle y, y\rangle\langle x, x\rangle \leq 0$$

$$\Rightarrow 4\langle x, y\rangle^2 \leq 4\langle x, x\rangle\langle y, y\rangle \Rightarrow \langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

Def: Let X be an inner product space. We define

$\|\cdot\| : X \to [0, \infty)$ given by

$\|x\| = \sqrt{\langle x, x\rangle}$ and call it the inner product norm.

Remark: If X is an inner product space

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

Remark: If X is an inner product space and $x, y \in X$.

We have

- $\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$
- $\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$.

Proposition: Let X be an inner product space. Then the inner product norm is a norm.

Proof:

- $\|x\| \geq 0$ by def and $\|x\| = 0 \Leftrightarrow \|x\|^2 = 0 \Leftrightarrow \langle x,x\rangle = 0$ $\Leftrightarrow x = 0_X$.
- $\|\lambda x\|^2 = \langle \lambda x, \lambda x\rangle = \lambda\langle x, \lambda x\rangle = \lambda^2 \langle x,x\rangle = \lambda^2 \|x\|^2$
- For $x,y \in X$,

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2\end{aligned}$$

Examples:

(i) In $\mathbb{R}^n$, we have for $x \in \mathbb{R}^n$

$$\|x\|_2^2 = \langle x,x\rangle = \sum_{i=1}^{n} |x_i|^2$$

(ii) In $\ell^2(\mathbb{N})$, we have for $x \in \ell^2(\mathbb{N})$

$$\|x\|_{\ell^2}^2 = \langle x,x\rangle = \sum_{n=1}^{\infty} |x_i|^2$$

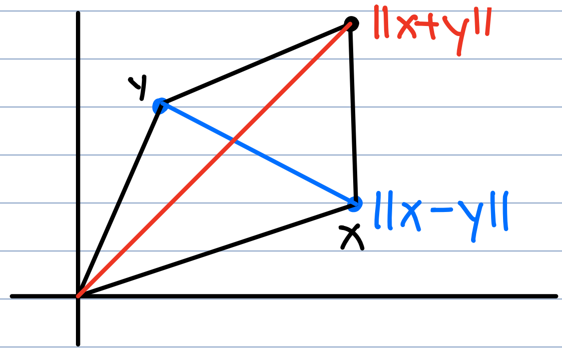
(iii) In $C([0,1])$, we have for $f \in C([0,1])$

$$\|f\|_{L^2}^2 = \int_0^1 |f(x)|^2\, dx.$$

Geometric Properties of Inner Products

Parallelogram Law: Let X be an inner product space and $x, y \in X$. Then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\| + 2\|y\|.$$



Proof:

$\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 + 2\langle x, y\rangle$

$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 2\langle x, y\rangle$

and add them.

Remark: Let X be an inner product space and for nonzero $x, y \in X$

$\dfrac{\langle x, y\rangle}{\|x\|\cdot\|y\|} \in [-1, 1]$. We can find $\theta \in [0, \pi]$ such that $\cos(\theta) = \dfrac{\langle x, y\rangle}{\|x\|\cdot\|y\|}$.

Definition: let X be an inner product space and for $x, y \in X$, we say x, y are orthogonal if $\langle x, y\rangle = 0$. Write $x \perp y$.

Pythagorean Theorem: Let X be an inner product space and $x, y \in X$. The following are equivalent

(a) $x \perp y$

(b) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Exercise: If X be a normed space such that for $x, y \in X$, $\|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 + 2\|y\|^2$ then $\exists$ an inner product on X inducing the norm. (Hint: $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$)

Exercise: Take $X = \ell^p(\mathbb{N})$ and $1 \le p \le \infty$, $p \neq 2$. Show $\|\cdot\|_p$ fails the parallelogram law.

Proposition: Let X be an inner product space. Then $\langle \cdot, \cdot \rangle$ is continuous. That is, if $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in X such that $x_n \to x$, $y_n \to y$. then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

Def: A **Hilbert space** $\mathcal{H}$ is a complete inner product space.

Examples: $(\mathbb{R}^n, \|\cdot\|_2)$, $(\ell^2(\mathbb{N}), \|\cdot\|_2)$ are Hilbert spaces. but $(C([0,1]), \|\cdot\|)$ is not.