Theorem: Let X be a vector space and let A and B be Hamel bases of X. Then $\#A = \#B$.

Def: Let X be a vector space. We define the dimension of X as $\dim(X) = \#A$ where A is any Hamel basis of X.

- If $\dim(X) = n$ for some $n \in \mathbb{N}_0$, then we call X finite dimension. Otherwise, X is infinite dimensional.
- If X is finite dimensional or $\dim(X) = \#\mathbb{N}$, then we say X has a countable dimension. Otherwise, X has uncountable dimension.

Examples:

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathcal{P}_n[0,1]) = n+1$
- $\dim(c_{00}(\mathbb{N}))$ has infinite countable dimension.

Comment: We will show later that $C[0,1]$ and $\ell^p(\mathbb{N})$ are of uncountable dimension.

Proposition: Let X be a finite dimensional vector space and Y is a subspace of X with $Y \subsetneq X$, then $\dim(Y) < \dim(X)$.

Proof: Let A be a Hamel basis of Y. Then there exists a Hamel basis B of X with $A \subset B$. Because $\langle A \rangle = Y \subsetneq X = \langle B \rangle$ then $A \subsetneq B$. Because B is finite, any subset of it has strictly small cardinality, so $\dim(Y) = \#A < \#B = \dim(X)$.

Linear Operators

Def: Let X and Y be vector spaces. A mapping $T: X \to Y$ is called a linear operator if

- For $x, y \in X$, then $T(x+y) = T(x) + T(y)$
- For $x \in X$ and $\alpha \in \mathbb{R}$, $T(\alpha x) = \alpha T(x)$.

Examples:

- Let X be a vector space, then $id: X \to X$ given by $id(x) = x$ is a linear operator.
- Let X, Y be vector spaces, then the zero operator $T: X \to Y$ such that for $x \in X$, $T(x) = 0_Y$ is a linear operator.
- If $n, m \in \mathbb{N}$, then any $n \times m$ matrix A, defines a linear operator $T_A: \mathbb{R}^m \to \mathbb{R}^n$. In fact any linear $T: \mathbb{R}^m \to \mathbb{R}^n$ is of the form $T = T_A$ for some A.
- Expectation operator $\mathbb{E}: C[0,1] \to \mathbb{R}$ with $\mathbb{E}(f) = \int_0^1 f(t)\,dt$
- For $t_0 \in [0,1]$, define the dirac functional $\delta_{t_0}: C[0,1] \to \mathbb{R}$ with $\delta_{t_0}(f) = f(t_0)$.
- The differential operator $D: C^1[0,1] \to C[0,1]$ with $Df = f'$.
- The Volterra operator: $V: C[0,1] \to C^1[0,1]$ defined as
$$(Vf)(t) = \int_0^t f(x)\,dx.$$
Then $(Vf)' = f$, in particular, $Vf \in C^1[0,1]$

Remark: The composition of linear operators is linear.

Proposition: Let X and Y be vector spaces and $T: X \to Y$, then

(1) For every $W \subset X$, $T(W) = \{T(w) : w \in W\}$ is a subspace of Y.

(2) For every $Z \subset Y$, $T^{-1}(Z) = \{x \in X : T(x) \in Z\}$ is a subspace of X.

Proof: Let $x, y \in T^{-1}(Z)$. Then we show $x + y \in T^{-1}(Z)$. Indeed, $T(x+y) = T(x) + T(y) \in Z$ and also for $x \in T^{-1}(Z)$ and $\alpha \in \mathbb{R}$, $T(\alpha x) = \alpha T(x) \in Z$.

Definition: Let X and Y be vector spaces and $T: X \to Y$ is a linear operator. Define

(1) $Ker(T) = T^{-1}(\{0_Y\}) = \{x \in X : T(x) = 0_Y\}$ (Kernel of T)

(2) $T(X) = \{T(x) : x \in X\}$ (the range of T).
These are subspaces of X and Y, respectively.

Proposition: Let X and Y be vector spaces and $T: X \to Y$ be a linear operator. T is one-to-one if and only if $\ker(T) = \{0_X\}$.

Proof: If T is one-to-one, and if $x \in \ker(T)$, then
$T(x) = 0_Y = T(0_X)$, so $x = 0_X$, and $\ker(T) = \{0_X\}$.
If $\ker(T) = \{0_X\}$ and $x, y \in X$, such that $T(x) = T(y)$, then $x - y = 0_X$ so $x = y$. T is one-to-one.

Examples:

- $\mathbb{E} : C[0,1] \to \mathbb{R}$ is not one-to-one. Because if $f(x) = \sin(2\pi x)$ f is nonzero and $\mathbb{E}(f) = 0$.
- $V : C[0,1] \to C'[0,1]$ is one-to-one. Let $f \in \ker(V)$, i.e. f is the zero function, then $DV(f) = f$ is the zero function.
  Because $\ker(V) = \{0\}$, V is one-to-one.

Proposition: If $T: X \to Y$ is linear and $A \subset X$, then $T(\langle A \rangle) = \langle T(A) \rangle$. (exercise).

Proposition: Let X, Y be vector spaces and $T: X \to Y$ is linear.
(1) If $A \subset X$ is linearly independent. T is one-to-one, then $T(A)$ is linearly independent.
(2) If $B \subset T(X)$ that is linearly independent and for all $b \in B$ $a_b \in X$ such that $T(a_b) = b$, then $\{a_b : b \in B\}$ is linearly independent.

Proof (2): Take $(a_{b_i})_{i=1}^{n}$ without repetitions and $(\alpha_i)_{i=1}^{n} \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \alpha_i a_{b_i} = 0_X \implies T\left(\sum_{i=1}^{n} \alpha_i a_{b_i}\right) = T(0_X) = 0_Y$$

$$\implies \sum_{i=1}^{n} \alpha_i T(a_{b_i}) = 0_Y \implies \sum_{i=0}^{n} \alpha_i b_i = 0_Y$$

Because $\{b_i\}_{i=1}^n$ has no repetitions and B is linearly independent so $\alpha_i = 0 \cdot \forall i$.

<u>Theorem</u>: Let X and Y be vector spaces and B be a Hamel basis of X, and $T, S : X \to Y$ such that $\forall b \in B$, $T(b) = S(b)$, then $T = S$.

<u>Proof</u>: If $x \in X$ then for $(b_i)_{i=1}^n \in B$ and $(\alpha_i)_{i=1}^n \in \mathbb{R}$,

$$x = \sum_{i=1}^n \alpha_i b_i \implies T(x) = \sum_{i=1}^n \alpha_i T(b_i) = \sum_{i=1}^n \alpha_i S(b_i) = S(x)$$

<u>Theorem</u>: Let X and Y be vector spaces and B be a Hamel basis of X and $\phi : B \to Y$ is a function. Then there exists a unique operator $T_\phi : X \to Y$ such that for all $b \in B$,
$T_\phi(b) = \phi(b)$

<u>Proof</u>: We define T_ϕ as follows:
- $T_\phi(0_X) = 0_Y$.
- If $x \in X \setminus \{0_X\}$, take F be a unique nonempty subset of B and the unique nonzero scalars $(\alpha_b)_{b\in F}$ such that $x = \sum_{b\in F} \alpha_b \cdot b$.

Put $T_\phi(x) = \sum_{b\in F} \alpha_b \phi(b)$. Then T_ϕ is linear and for all $b \in B$, $T_\phi(b) = \phi(b)$. Uniqueness follows from previous theorem.

(Do Exercise 1.3.15).

<u>Linear Projections</u>

<u>Def</u>: Let X be a vector space. A linear operator $P : X \to X$ is called a linear projection if $P^2 = P$.

<u>Proposition</u>: Let X be a vector space. If $P : X \to X$ is a linear projection.
(1) $P(X)$ and $\ker(P)$ form a linear decomposition of X
(2) If Y and Z are linear decomposition of X, then there exists

a unique linear projection $P: X \to X$ such that $Y = P(X)$ and $Z = \ker(P)$

<u>Proof (1):</u> If $x \in P(X) \cap \ker(P)$. we show $x = 0_X$. Because $x \in \ker(P)$, $P(x) = 0_X$ and because $x \in P(X)$, there exists a $y \in X$ such that $x = P(y)$. Then $P(x) = P(P(y)) = P(y) = x$ so $P(x) = x = 0_X$.

$P(X) + \ker(P) = X$, so let $x \in X$. Then $x = P(x) + (x - P(x))$ because $P(x - P(x)) = P(x) - P(P(x)) = P(x) - P(x) = 0_X$ so $x - P(x) \in \ker(P)$,

<u>Proof (2):</u> Because Y and Z are a linear decomposition of X then $\forall x \in X$, there exists unique $y \in Y$, $z \in Z$ such that $x = y + z$. Define $y = P(x)$, then P is linear and $P^2 = P$.

<u>Linear Operators and Quotients</u>

<u>Prop:</u> Let X be a vector space and Y be a subspace of X. Then $Q: X \to X/Y$ defined by $Q(x) = [x]_Y$ is a linear operator such that $Q(X) = X/Y$ and $\ker(Q) = Y$.

<u>Proof:</u> Linearity follows from the definition of addition and scalar multiplication in X/Y. $Q(X) = X/Y$ is obvious. We show $\ker(Q) = Y$. If $y \in Y$, then $[y]_Y = [0_X]_Y = Y$ Therefore $Q(y) = [y] = Y = 0_{X/Y}$. Take $x \in \ker(Q)$. then $[x] = Q(x) = 0_{X/Y} = [0_X]_Y \Rightarrow x - 0_X \in Y \Rightarrow x \in Y$.

<u>Theorem:</u> Let X and Y be vector spaces and let $T: X \to Y$ be a linear operator. Define $Z = \ker(T)$ and $Q: X \to X/Z$, then there exists a unique linear operator $\overline{T}: X/Z \to Y$ such that for all $x \in X$, $\overline{T}Q(x) = T(x)$. Furthermore, $\overline{T}$ is one-to-one.

(Sounds familiar...).

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "Q"'] & Y \\ X/Z \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

<u>Proof:</u> For $x, y \in X$, $T(x) = T(y)$ if and only if $x - y \in \ker(T) = Z$

and so we have $Q(x) = Q(y)$. Indeed, if $x, y \in X$ are such that $T(x) = T(y) \Rightarrow T(x-y) = 0_Y \Rightarrow x-y \in \ker(T)$
If $x-y \in \ker(T) \Rightarrow T(x-y) = 0_Y \Rightarrow T(x) - T(y) = 0_Y \Rightarrow T(x) = T(y)$
We define $\bar{T} : X/Z \to Y$ as follows: For $[x]_Z = Q(x) \in X/Z$, we define $\bar{T}([x]_Z) = T(x)$. This is well defined because if $[x]_Y = [y]_Y$, then by the above, $T(x) = T(y)$. Then, it is obvious that for $x \in X$, $\bar{T}Q(x) = \bar{T}([x]_Z) = T(x)$. Linearity follows from the linearity of T and the definition of addition and scalar multiplication on X/Z.

Algebraic Isomorphisms

Def: A linear operator $T : X \to Y$ between vector spaces X and Y is called an algebraic isomorphism if it is a linear bijection. If such a T exists between X and Y we say that X and Y are algebraically isomorphic.

Remark: The inverse of an algebraic isomorphism is an algebraic isomorphism. The composition of algebraic isomorphisms is an algebraic isomorphism.

Theorem: Let X and Y be vector spaces. Then $X \simeq Y$ (algebraic isomorphic) if and only if $\dim(X) = \dim(Y)$.

Stopped before Section 3.6, but is a very short one.

Exercises: 1.3.3, 1.3.4, 1.3.9, 1.3.12, 1.3.15, 1.3.19, 1.3.27