Question 1 I will only show that $\|\cdot\|_p$ fails the triangle inequality. Let $x = e_1 = (1,0)$ and $y = e_2 = (0,1)$.

Then, $\|x\|_p = 1$ & $\|y\|_p = 1$, $p<1$

$$\|x+y\|_p = (1^p + 1^p)^{1/p} = 2^{1/p} > 2 = \|x\|_p + \|y\|_p,$$

because $1/p > 1$.

Question 2

(i) Let $x, y \in \bar{A}$ & $\lambda \in [0,1]$. We will show $z = \lambda x + (1-\lambda) y \in \bar{A}$.

Fix sequences $(x_n)_n$, $(y_n)_n$ in A such that

$\lim_n x_n = x$ and $\lim_n y_n = y$. By the convexity of A,

$\forall n \in \mathbb{N}$, $z_n = \lambda x_n + (1-\lambda) y_n \in A$, and

$$\|z_n - z\| = \|\lambda (x_n - x) + (1-\lambda)(y_n - y)\|$$
$$\leq |\lambda| \cdot \|x_n - x\| + |1-\lambda| \, \|y_n - y\| \to 0.$$

Therefore, $z \in \bar{A}$.

(ii) Let $x, y \in A^\circ$ & $\lambda \in [0,1]$. We will show $z = \lambda x + (1-\lambda) y \in A^\circ$.

Because $x, y \in A^\circ$, $\exists \, \varepsilon_1, \varepsilon_2 > 0$ s.t. $B(x, \varepsilon_1) \subset A$ & $B(y, \varepsilon_2) \subset A$. Put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. We will show $B(z, \varepsilon) \subset A$. Let $z' \in B(z, \varepsilon)$, i.e., $\|z - z'\| < \varepsilon$. Put $x' = x + (z' - z)$, $y' = y + (z' - z)$.

Note $x' \in B(x, \varepsilon) \subset B(x, \varepsilon_1) \subset A$ &

$y' \in B(y, \varepsilon) \subset B(y, \varepsilon_2) \subset A$.

By the convexity of A,

$$A \ni \lambda x' + (1-\lambda) y' = \lambda x + (1-\lambda) y + (\lambda + 1 - \lambda)(z' - z)$$
$$= z + z' - z = z'.$$

Therefore, $B(z, \varepsilon) \subset A$.

Question 3:

We define $T: X^{\#} \to \mathbb{R}^B$ as follows: for $f \in X^{\#}$, $Tf = (f(b))_{b \in B}$. This is linear, but I will not show this here, because it is standard. We need to check that it is a bijection.

- T is 1-1. Let $f, g \in X^{\#}$ s.t. $Tf = Tg$, i.e., $\forall\, b \in B$, $f(b) = g(b)$. Because f & g coincide on a Hamel basis, by a known proposition, $f = g$.
- T is onto. Let $a = (a(b))_{b \in B} \in \mathbb{R}^B$. By a known theorem, there exists a linear operator $f: X \to \mathbb{R}$ such that, $\forall b \in B$, $f(b) = a(b)$. But then, $f \in X^{\#}$ & $Tf = a$.

Question 4:

(i) Consider the quotient map $Q: X \to X/Z$, which is a linear surjection with $\ker(Q) = Z$. We will show $Q|_Y : Y \to X/Z$ is an algebraic isomorphism. By a known theorem, this will yield $\dim(X/Z) = \dim(Y) = n$, i.e., Z is of codimension n.

• $Q|_Y$ is 1-1. By a known characterization of injectivity for linear operators, it suffices to show that $\ker(Q|_Y)$ is trivial. Indeed,

$$\ker(Q|_Y) = \{y \in Y : Qy = 0_{X/Z}\} = \ker(Q) \cap Y = Z \cap Y = \{0_X\}.$$

• $Q|_Y$ is onto. Let $[x]_Z \in X/Z$. Because Y, Z for a linear decomposition of X, $\exists y \in Y$ & $z \in Z$ s.t. $x = y + z$. Therefore,

$[x]_Z = Qx = Qy + Qz = Q|_Y y$, because $\ker(Q) = Z$.

Question 4:

(ii) Fix $i \in \{1, \dots, n\}$. Then, $i \in \{1, \dots, n\} \setminus \{i\}$, & by a known statement, $\bigcap_{\substack{j=1 \\ j \neq i}}^{n} \ker(f_j) \not\subset \ker(f_i)$,

i.e., $\exists\ y_i \in \left(\bigcap_{\substack{j=1 \\ j \neq i}}^{n} \ker(f_j) \right) \setminus \ker(f_i)$.

In other words, $f_i(y_i) \neq 0$ &, for $j \neq i$, $f_j(y_i) = 0$.

Put $x_i = (f_i(y_i))^{-1} y_i$, & note $f_j(x_i) = \delta_{ji}$.

Let us show that $(x_i)_{i=1}^{n}$ is linearly independent,

so fix $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ s.t. $\sum_{i=1}^{n} \lambda_i x_i = 0_X$.

Then,

$$0 = \sum_{j=1}^{n} \lambda_j f_j \left(\sum_{i=1}^{n} \lambda_i x_i \right) = \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_j \lambda_i \delta_{ji} = \sum_{i=1}^{n} \lambda_i^2 .$$

Therefore, $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$.

Question 4:

(iii) Define $Y = \langle \{x_1, \dots, x_n\} \rangle$, which is n-dimensional. We will show that Y, Z form a linear decomposition of X. By (ii), this will yield the desired conclusion.

- $Y \cap Z = \{0_X\}$. Let $x \in Y \cap Z$. Because $x \in Y$, $\exists\, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ s.t. $x = \sum_{i=1}^{n} \lambda_i x_i$. But, for $1 \leq j \leq n$ $x \in \ker(f_j)$, so $0 = f_j(x) = \lambda_j$. So $x = 0_X$.
- $Y + Z = X$. Let $x \in X$. Put $y = \sum_{i=1}^{n} f_i(x) x_i \in Y$ & $z = x - y$. Obviously, $x = y + z$, and we will show $z \in Z$. Indeed, for $1 \leq j \leq n$,

$$f_j(z) = f_j(x) - f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x)$$
$$= 0.$$

Therefore, $z \in \bigcap_{j=1}^{n} \ker(f_j) = Z$.