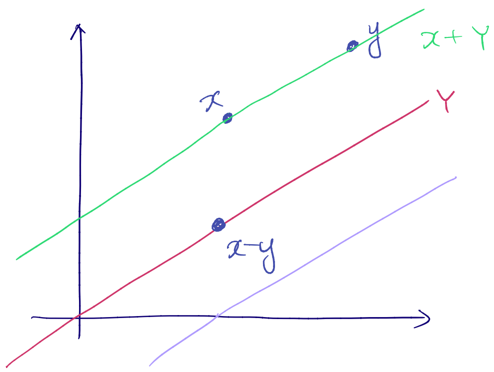
Quotients of Vector Spaces

Def: Let X be a vector space and let Y be a subspace of X. Define the relation "$\sim_Y$" on X as follows: $x \sim_Y y$ is $x - y \in Y$. Then $\sim$ is an equivalence relation on X and we denote an equivalence class $[x]_Y = \{x \in X : x - y \in Y\}$.

Remark: For $x \in X$, $[x]_Y = x + Y$



Def: Let X be a vector space and let Y be a subspace of X. Define the quotient of X over Y as the set

$$X/Y = \{[x]_Y : x \in X\}$$

Note: It is the collection of lines that are "parallel" to the line Y.

Def: Let X be a vector space and Y is a subspace of X. We define operations $\tilde{+} : X/Y \times X/Y \to X/Y$ and $\tilde{\cdot} : \mathbb{R} \times X/Y \to X/Y$ as follows:

(1) For $[x], [y] \in X/Y$, define $[x] \tilde{+} [y] = [x+y]$

(2) For $\alpha \in \mathbb{R}$, $[x] \in X/Y$, define $\alpha \tilde{\cdot} [x] = [\alpha x]$.

These are indeed well-defined. That is, $[x] = [x']$ and $[y] = [y']$, then $[x+y] = [x'+y']$. (exercise) ✓✓

The set X/Y with "addition" $\tilde{+}$ and "scalar multiplication" $\tilde{\cdot}$ is a vector space.

Notation: Henceforth, on X/Y, we denote the operations as the usual $+$ and $\cdot$.

Linear Independence

Def: Let X be a vector space.

(1) Let $A = \{x_i\}_{i=1}^{n} \subset X$, where each x_i are distinct. Then the set A is linearly independent if for every $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, then

$\sum_{i=1}^{n} \alpha_i x_i = 0$, then $\alpha_i = 0$ for every i.

(2) A subset A of X is linearly independent if every nonempty finite subset of A is linearly independent according to (1).

Remark:

- In (1), it is important to note that the x_i are nonrepeating.
- If $A \subset X$ that satisfies (1), then it also satisfys (2).
- If $A \subset X$ is linearly independent, then $0_X \notin A$.
- $\emptyset$ is linearly independent.

Notation: A non linearly independent set is a linearly dependent set.

Example:

(a) Recall $\{e_i\}_{i=1}^{n}$ is the standard basis of $\mathbb{R}^n$, where

$$e_i = (0, \dots, \underset{\substack{\uparrow \\ \text{ith coordinate}}}{1}, 0, \dots, 0)$$

Then the set is linearly independent. That is, if $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. are such that

$$\sum_{i=1}^{n} \alpha_i e_i = 0, \text{ then } \alpha_i = 0 \text{ for every } i.$$

(b) Recall $\{e_n\}_{n=1}^{\infty}$ is the standard basis of $C_{00}(\mathbb{N})$, where

$$e_n = (0, \dots, 1, 0, \dots)$$

If $A = \{e_{n_i}\}_{i=1}^{k}$ be a subset of the standard basis and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, are such that

$$\sum_{i=1}^{k} \alpha_i e_{n_i} = 0 \text{ (the zero sequence), then } \alpha_i = 0 \text{ for all } i.$$

(c) Let $X = C[0,1]$. Recall $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2, \dots,$ $p_n(x) = x^n$. Then $A = \{p_0, p_1, \dots, p_n, \dots\}$ is linearly independent.
(exercise, by differentiating) ✓✓

Proposition: Let X be a vector space and let $A \subset X$. The following are equivalent.

(1) A is linearly independent

(2) For every $x \in A$, $x \notin \operatorname{span}(A \setminus \{x\})$.

Proof: $\neg(2) \Rightarrow \neg(1)$. Assume there $x \in A$ such that $x \in \text{span}(A \setminus \{x\})$. Then there exists $y_1, \dots, y_n \in A \setminus \{x\}$ and $\alpha_1, \dots, \alpha_n$ such that $x = \sum_{i=1}^{n} \alpha_i y_i$. Then $0 = \sum_{i=1}^{n} \alpha_i y_i + (-1)x$. Then $\{y_1, \dots, y_n, x\}$ is not linearly independent as witnessed by $\alpha_1, \dots, \alpha_n, -1$, so A is not linearly independent.

$\neg(1) \Rightarrow \neg(2)$ Assume that A is not linearly independent. Then there exists $x_1, \dots, x_n \in A$ (distinct) and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ not all zero such that $0 = \sum_{i=1}^{n} \alpha_i x_i$. Assume $\lambda_n \neq 0$. Then $x_n = \sum_{i=1}^{n-1} \left(-\frac{\lambda_i}{\lambda_n}\right) x_i \in \text{span}(A \setminus \{x\})$, so we showed $\neg(2)$.

Proposition: Let X be a vector space, and A be a linearly independent subset of X. If $x_0 \in X \setminus \text{span}(A)$, then $A \cup \{x_0\}$ is linearly independent.

(Do Exercise 1.2.9 and Exercise 1.2.10).

Hamel Bases

Def: Let X be a vector space and let B be a subset of X. Then B is called a Hamel basis if B is a linearly independent (1) spanning set (2).

Examples:

- The standard basis of $\mathbb{R}^n$, $\{e_i\}_{i=1}^{n}$ is a Hamel basis of $\mathbb{R}^n$.
- The standard basis of $c_{00}(\mathbb{N})$, $\{e_n\}_{n=1}^{\infty}$ is a Hamel basis of $c_{00}(\mathbb{N})$.
- $B = \{p_0, p_1, \dots\}$, then B is a Hamel basis of $\mathcal{P}[0,1]$.
- $\{p_0, \dots, p_n\}$ is a Hamel basis of $\mathcal{P}_n[0,1]$.

Theorem: Let X be a vector space and $B \subset X \setminus \{0_X\}$. Then B is a Hamel basis if and only if for every $x \in X \setminus \{0_X\}$ there exists a unique nonempty finite $F \subset B$ and unique non zero scalars $(\lambda_b)_{b \in F}$ such that $x = \sum_{b \in F} \lambda_b b$.

Proof: $(\Rightarrow)$ Let $x \in X \setminus \{0_X\}$. By assumption, $x \in \text{span}(B)$. Then there exists $b_1, b_2, \dots, b_n \in B$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^{n} \alpha_i b_i$. By merging, we assume $b_1, \dots, b_n$ are pairwise distinct and by dropping the zero scalars, we may assume that $\alpha_i \neq 0$ for all i.

Let $F = \{b_1, \ldots, b_n\}$ and for $b \in F$, if $b = b_i$. let $\alpha_b = \alpha_i$. Then

$$x = \sum_{b \in F} \alpha_b \cdot b.$$

For uniqueness, let G be a finite nonempty subset of B. and $(\beta_b)_{b\in G}$ be nonzero scalars such that

$$x = \sum_{b \in G} \beta_b \cdot b.$$

Then $0 = \sum_{b\in F} \alpha_b b - \sum_{b \in G} \beta_b \cdot b. = \sum_{b \in F \setminus G} \alpha_b \cdot b - \sum_{b \in G \setminus F} \beta_b \cdot b + \sum_{b \in F \cap G} (\alpha_b - \beta_b) b$

Because B is linearly independent, we need the following.

- $\alpha_b = 0$ for $b \in F \setminus G$ but $\forall b \in F$, $\alpha_b \neq 0 \Rightarrow F \setminus G = \emptyset$
- $-\beta_b = 0$ for $b \in G \setminus F$, but $\forall b \in G$, $\beta_b \neq 0 \Rightarrow G \setminus F = \emptyset$
- $\alpha_b - \beta_b = 0$, $\forall b \in F \cap G = F = G \Rightarrow \alpha_b = \beta_b \ \forall b \in F = G$.

Theorem: Every vector space has a Hamel basis.

Definition 1.25 A *partial order* on a set $\mathcal{P}$ is a binary relation $\preceq$ on $\mathcal{P}$ such that for $a, b, c \in \mathcal{P}$

(i) $a \preceq a$;
(ii) $a \preceq b$ and $b \preceq a$ implies that $a = b$; and
(iii) $a \preceq b$ and $b \preceq c$ implies that $a \preceq c$.

Definition 1.26 Two elements $a, b \in \mathcal{P}$ are *comparable* if $a \preceq b$ or $b \preceq a$ (or both if $a = b$). A subset $\mathcal{C}$ of $\mathcal{P}$ is called a *chain* if any pair of elements of $\mathcal{C}$ are comparable.

An element $b \in \mathcal{P}$ is an *upper bound* for a subset $\mathcal{S}$ of $\mathcal{P}$ if $s \preceq b$ for all $s \in \mathcal{S}$. An element m of $\mathcal{P}$ is *maximal* if $m \preceq a$ for some $a \in \mathcal{P}$ implies that $a = m$.

Theorem 1.27 (Zorn's Lemma) *If $\mathcal{P}$ is a non-empty partially ordered set in which every chain has an upper bound, then $\mathcal{P}$ has at least one maximal element.*

Lemma: Let X be a vector space and $(A_i)_{i \in I}$ be a chain of linearly independent subsets of X with respect to inclusion. Then $A = \bigcup_{i \in I} A_i$ is linearly independent.

Theorem: Let X be a vector space and $A \subset X$ be a linearly independent set. Then there exists a Hamel basis B of X such that $A \subset B$.

Proof: Define $\mathcal{A} = \{B \subset X : B \text{ is linearly independent and } A \subset B\}$. Note that $A \in \mathcal{A}$. This is a partially ordered set with "$\subset$". We show the assumptions of Zorn's lemma hold.

Let $(A_i)_{i \in I}$ be a chain of $\mathcal{A}$. Then by the above lemma, $C = \bigcup_{i \in I} A_i$ is linearly independent and $A \subset C \Rightarrow A \in \mathcal{A}$, and A is an upperbound for $(A_i)_{i \in I}$. By Zorn's, $\exists$ maximal $B \in \mathcal{A}$.

Claim: $X = \operatorname{span}(B)$.

If not, $\exists\, x_0 \in X \setminus \operatorname{span}(B) \Rightarrow B_0 = B \cup \{x\}$ is linearly independent $\Rightarrow B_0 \in \mathcal{A}$ and $B_0 \neq B$ and $B \subset B_0$. Contradicts the maximality of B.

(Do Exercise 1.2.24)