Math 6461 Lecture 6

Recall: If $1 \le p \le \infty$, its conjugate exponent is the unique $1 \le q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

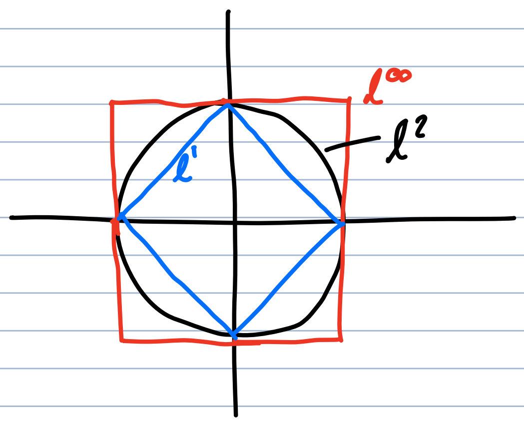
Theorem: Let $n \in \mathbb{N}$ and $1 < p, q < \infty$. The following hold:

(i) $\forall x, y \in \mathbb{R}^n$, $\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$

(Hölder's inequality)

(ii) $\forall x \in \mathbb{R}^n \ \exists y \in \mathbb{R}^n$ such that $\|y\|_p = 1$ and $\sum_{i=1}^{n} x_i y_i = \|x\|_p$.

(iii) $\forall x, y \in \mathbb{R}^n$. $\|x+y\|_p \le \|x\|_p + \|y\|_p$ (Minkowski's Inequality)



ℓ^1 : unit sphere for $\|\cdot\|_{\ell^1}$

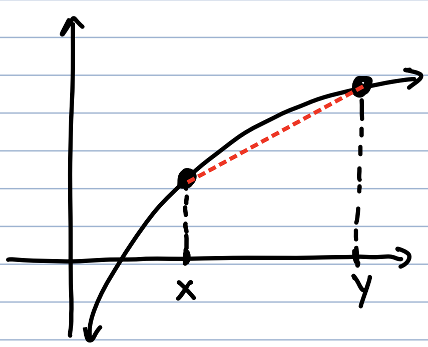
ℓ^2 : unit sphere for $\|\cdot\|_{\ell^2}$

ℓ^∞ : unit sphere for $\|\cdot\|_{\ell^\infty}$

Young's Inequality: For $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ then for all $x, y \ge 0$, then $xy \le \frac{x^p}{p} + \frac{y^q}{q}$

Proof: Recall $\ln : (0, \infty) \to \mathbb{R}$ is monotonically increasing and concave. Then for all $x, y \in (0, \infty)$ and $\lambda \in [0,1]$

$$\ln(\lambda x + (1-\lambda) y) \ge \lambda \ln(x) + (1-\lambda)\ln(y).$$



Taking $\lambda = \frac{1}{p}$ and $x \mapsto x^p$, $y \mapsto y^p$

Then $\ln\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \ge \frac{1}{p}\ln(x^p) + \frac{1}{q}\ln(y^q)$

$= \ln(x) + \ln(y)$

$= \ln(xy) \Rightarrow xy \le \frac{x^p}{p} + \frac{y^p}{q}$

Proof of Hölder: Let $x=(x_i)_{i=1}^n,\ y=(y_i)_{i=1}^n \in \mathbb{R}^n$. Put $A=\|x\|_p,\ B=\|y\|_q$, we will show that $\sum_{i=1}^n |x_i y_i| \le AB$.

If either $A=0$ or $B=0$, then this is trivial. So assume $A, B>0$. By Young's Inequality, for $1\le i\le n$

$$\frac{|x_i|}{A}\cdot\frac{|y_i|}{B} \le \frac{|x_i|^p}{A^p p} + \frac{|y_i|^q}{B^q q}$$

$$\Rightarrow \frac{1}{AB}\sum_{i=1}^n |x_i y_i| \le \frac{1}{p}\left(\frac{\sum_{i=1}^n |x_i|^p}{A^p}\right)^{\frac{1}{p}} + \frac{1}{q}\left(\frac{\sum_{i=1}^n |y_i|^q}{B^q}\right)^{\frac{1}{q}} = \frac{1}{p}+\frac{1}{q}=1$$

Proof of (ii): If $x=0$, take any y with $\|y\|_q=1$. Assume $x\neq 0$ and put $A=\|x\|_p$ and for $1\le i\le n$, $y_i=A^{1-p}\operatorname{sgn}(x_i)|x_i|^{p-1}$

Let $y=(y_i)_{i=1}^n \Rightarrow \sum_{i=1}^n x_i y_i = \cdots = \|x\|_p$. Also,

$$\|y\|_q^q = \sum_{i=1}^n |y_i|^q = A^{q-qp}\sum_{i=1}^n \underbrace{|\operatorname{sgn}(x_i)|^q}_{=1}\cdot|x_i|^{qp-q}$$

$$\Rightarrow \frac{1}{p}+\frac{1}{q}=1 \Rightarrow p+q=pq$$

$$\Rightarrow A^{-p}\sum_{i=1}^n |x_i|^p = A^{-p}A^p = 1.$$

Proof of (iii): Fix $x, y\in\mathbb{R}^n$, we show $\|x+y\|_p \le \|x\|_p+\|y\|_p$.

Take $z\in\mathbb{R}^n$ with $\|z\|_q=1$. Then

$$\|x+y\|_p = \sum_{i=1}^n z_i(x_i+y_i) = \sum_{i=1}^n z_i x_i + \sum_{i=1}^n z_i y_i$$

$$\le \sum_{i=1}^n |z_i x_i| + \sum_{i=1}^n |z_i y_i|$$

$$\le \|z\|_q\|x\|_p + \|z\|_q\|y\|_p = \|x\|_p+\|y\|_p.$$

Exercise: For $0<p<1$, define on $\ell^p(\mathbb{N})$ by $\|x\|_p=\left(\sum_{n=1}^\infty |x_n|^p\right)^{\frac{1}{p}}$

Show that this is not a norm.

Separable Normed Spaces

Def: A normed space X is said to be separable if there exists a countable $D \subset X$ such that $\overline{D} = X$, i.e. $\forall x \in X \ \forall \varepsilon > 0 \ \exists y \in D$ s.t. $\|x - y\| < \varepsilon$.

Prop: If X is a normed space of countable dimension then it is separable.

Proof: Let B be a Hamel basis of X. Assume that B is infinite countable, i.e. $B = \{x_1, x_2, \dots\}$. Let D be all finite linear combinations of B with rational coefficients. Then for $x \in X$, because $\langle B \rangle = X$ $\exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $x = \sum_{i=1}^{n} \lambda_i x_i$. Fix $\varepsilon > 0$ and for $1 \le i \le n$, take $q_i \in \mathbb{Q}$ such that $|q_i - x_i| < \frac{\varepsilon}{n\|x\|}$. Then $y = \sum_{i=1}^{n} q_i x_i \in D$ and

$$\|x - y\| = \left\| \sum_{i=1}^{n} (\lambda_i - q_i) x \right\| \le \sum_{i=1}^{n} |\lambda_i - q_i| \cdot \|x\| < n \cdot \frac{\varepsilon}{n} = \varepsilon.$$

Prop: Let X be a normed space that has a dense subspace of countable dimension. Then X is separable.

Proof: Assume Y is a dense subspace of countable dimension. Then $\exists D \subset Y$ countable that is dense in Y. We show $\overline{D} = X$. Let $x \in X$ and $\varepsilon > 0$. Bec. Y dense, $\exists y \in Y$ s.t. $\|x - y\| < \frac{\varepsilon}{2}$. Bec. D is dense in Y, $\exists d \in D$ s.t. $\|y - d\| < \frac{\varepsilon}{2}$ $\Rightarrow \|x - d\| < \varepsilon$.

Theorem: Let $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ for $1 \le p < \infty$. Then $c_{00}(\mathbb{N})$ is dense in X and X is separable.

Proof: Let $X = \ell^p(\mathbb{N})$ for $1 \le p < \infty$. Fix $x \in \ell^p(\mathbb{N})$ and $\varepsilon > 0$. Because $\sum_{i=1}^{\infty} |x_i|^p < \infty$, $\exists\, n_0 \in \mathbb{N}$ s.t. $\sum_{n=n_0+1}^{\infty} |x_i|^p < \varepsilon^p$. Put $y = (x_1, \dots, x_{n_0}, 0, \dots)$. Then $\|x-y\|_p^p = \sum_{n=n_0+1}^{\infty} |x_i|^p < \varepsilon^p$

$\Rightarrow \|x-y\|_p < \varepsilon$.

Theorem: $\ell^\infty(\mathbb{N})$ is non-separable.

Proof: $\forall A \subset \mathbb{N}$ define $\chi_A \subset \ell^\infty(\mathbb{N})$ as follows:

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A. \end{cases}$$

Claim: If $A \neq B \subset \mathbb{N}$ then $\|\chi_A - \chi_B\|_\infty = 1$.

Note $(\chi_A)_A$ is uncountable and 1-separated. Assume $\exists\, D$ countable in $\ell^\infty(\mathbb{N})$ that is dense. Then $\forall A \subset \mathbb{N}$ $\exists\, d_A \in D$ s.t. $\|\chi_A - d_A\|_\infty < \frac{1}{2}$ $\Rightarrow$ $\forall A \neq B$ $\|d_A - d_B\| > 0$

$\Rightarrow$ D is uncountable.

Theorem: For $C([0,1])$ with $\|\cdot\|_{L^\infty}$ or $\|\cdot\|_{L^p}$ for $1 \le p < \infty$ is a separable normed space.

Sketch of Proof: The space of all piecewise linear continuous functions on $[0,1]$ with rational change points is of countable dimension are dense.

Comment: $c_{00}(\mathbb{N})$ and $\mathcal{P}([0,1])$ are separable with any norm.

Banach Spaces

Recall: Let X be a normed space.

(i) A sequence $(x_n)_{n=1}^{\infty}$ is called **Cauchy** if
$\limsup_{n\to\infty}\left(\limsup_{m\to\infty}\|x_n-x_m\|\right)=0.$

(ii) A normed space X is called a **Banach space** if every Cauchy sequence in X is convergent.

Prop: Let X be a Banach space and Y be a subspace of X. TFAE: (Lemma 4.3)

(i) Y is a Banach space

(ii) Y is a closed subspace of X.

Notation: Let X be a normed space.

(i) A sequence $(x_n)_{n=1}^{\infty}$ in X is **summable** if $\exists x_0\in X$ such that $x_0=\lim_{N\to\infty}\sum_{n=1}^{N}x_n=\sum_{n=1}^{\infty}x_n$

(ii) A sequence $(x_n)_{n=1}^{\infty}$ in X is called **absolutely summable** if $\sum_{n=1}^{\infty}\|x_n\|<\infty$.

Prop: Let X be a normed space. TFAE:

(i) X is a Banach space.

(ii) Every absolutely summable sequence $(x_n)_{n=1}^{\infty}$ in X

Sketch (i) ⇒ (ii): Let $(x_n)_{n=1}^{\infty}$ s.t. $\sum_{n=1}^{\infty}\|x_n\|<\infty$. Then if $y_N=\sum_{n=1}^{N}x_n \Rightarrow (y_N)$ is Cauchy. For $m<N$
$\|y_N-y_m\|=\left\|\sum_{n=m+1}^{N}x_i\right\|\leq\sum_{n=m+1}^{N}\|x_i\|.$

(ii) ⇒ (i): Let $(x_n)_{n=1}^{\infty}$ in X be Cauchy. Recursively construct $n_1 < n_2 < \cdots$ s.t. $\forall i < j$, $\|x_{n_i} - x_{n_j}\| < \frac{1}{2}^i$. Put

$y_i = x_{n_i} - x_{n_{i+1}}$ and note

$\sum_{n=1}^{\infty} \|y_n\| \leq \sum_{i=1}^{\infty} \frac{1}{2}^i$. By (ii) $\exists y_0 \in X$ s.t.

$$y_0 = \sum_{i=1}^{\infty} y_i = \lim_{N\to\infty} \left(\sum_{i=1}^{N} (x_{n_i} - x_{n_{i+1}}) \right)$$
$$= \lim_{N\to\infty} (x_{n_1} - x_{n_{N+1}})$$

$\Rightarrow \lim_{N\to\infty} x_{n_N} = x_{n_1} - y_0 \Rightarrow$ A Cauchy seq with a convergent subsequence is Convergent.

Theorem: The following are Banach spaces:

(i) $\mathbb{R}^n$ with $\|\cdot\|_p$ for $1 \leq p \leq \infty$ — 4.2

(ii) $\ell^p(\mathbb{N})$, $1 \leq p \leq \infty$ — Theorem 4.7

(iii) $c_0(\mathbb{N})$ — Ex. 4.4

Theorem: $C([0,1])$ with $\|\cdot\|_\infty$ is a Banach space.

Theorem: $c_{00}(\mathbb{N})$ with $\|\cdot\|_\infty$ is not a Banach space.

Theorem: $C([0,1])$ with $\|\cdot\|_p$ for $1 \leq p < \infty$ is <u>not</u> a Banach space.

Exercise: 2.1.25, 2.1.26, 2.1.27, 2.1.35, 2.2.8, 2.2.10