Recall: Let X and Y be normed spaces and let $T: X \to Y$ be a linear operator.

(i) If $\forall x \in X$, $\|T(x)\| = \|x\|$ is called a **linear isometry**.

(ii) If T satisfies (i) and is onto, we call T an **isometric isomorphism.** If such $T: X \to Y$ exists, we say **X and Y are isometrically isomorphism** and write **$X \equiv Y$.**

(iii) If T is a bounded algebraic isomorphism with $T^{-1}: Y \to X$ also bounded, then we call T an **isomorphism**. If such T exists, we say **X and Y are isomorphic**, and write **$X \simeq Y$.**

Example: $(\mathbb{R}^2, \|\cdot\|_1) \equiv (\mathbb{R}^2, \|\cdot\|_\infty)$, but $\forall n > 2$, $(\mathbb{R}^n, \|\cdot\|_1) \simeq (\mathbb{R}^n, \|\cdot\|_\infty)$, but are not isometrically isomorphic.

Proposition: Let X, Y be normed spaces and $T: X \to Y$ be a linear operator. TFAE.

(i) T is an isomorphism.

(ii) T is onto and $\exists A, B > 0$ such that $\forall x \in X$

$$\frac{1}{B}\|x\| \le \|T(x)\| \le A\|x\|.$$

Proof: (i) $\Rightarrow$ (ii) Take $A = \|T\|$ and $B = \|T^{-1}\|$. Then T is onto and for $x \in X$,

$$\|T(x)\| \le \|T\| \cdot \|x\| \le A\|x\|$$

For $x \in X$,

$$\|x\| = \|T^{-1}T(x)\| \le \|T^{-1}\| \cdot \|T(x)\| \le B\|T(x)\|.$$

(ii) $\Rightarrow$ (i): Bec. B exists, T is 1-1. If $x \in \ker(T)$, then $0 = \|T(x)\| \ge \frac{1}{B}\|x\| \Rightarrow x = 0$. Bec. $\ker(T) = \{0\}$, T is 1-1 so T is an algebraically isomorphic. so T^{-1} is well-defined and $\forall y \in Y$, $\|y\| = \|TT^{-1}(y)\| \ge B^{-1}\|T^{-1}\|$. $\Rightarrow \|T^{-1}\| \le B$.

Prop: Let X and Y be isomorphic normed spaces. The following hold:

(i) X is separable if and only if Y is.

(ii) X is a Banach space if and only if Y is.

Def: Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. We call $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent and denote $\|\cdot\|_1 \sim \|\cdot\|_2$, if

$$id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2) \text{ is an isomorphism.}$$

Example: In $\mathbb{R}^n$, $\|\cdot\|_p \sim \|\cdot\|_q$ for $1 \le p, q \le \infty$.

Def: Let X and Y be normed spaces. A linear operator $T : X \to Y$ is called an into isomorphism if seen as an operator $T : X \to T(X)$ is an isomorphism.

Do Exercises 2.3.21, 2.3.22, 2.3.23, 2.3.24, 2.3.25, 2.3.26, 2.3.27.

Quotients of Normed Spaces

Recall, if X is a vector space and Y is a subspace of X, we define $\sim_Y$ as $x \sim_Y z \Leftrightarrow x - z \in Y$, and $X/Y = \{[x]_Y : x \in X\}$ is a vector space. Then $Q : X \to X/Y$ given by $Q(x) = [x]_Y$ is an onto linear map with $\ker(Q) = Y$.

Def: Let X be a normed space and Y is a subspace of X. For $[x]_Y \in X/Y$, define the operator norm of $[x]_Y$ as $\|[x]_Y\| = \inf\{\|y\| : y \in [x]_Y\}$.

Remark:
$$\begin{aligned}\|[x]_Y\| &= \inf\{\|z\| : z \in [x]_Y\} \\ &= \inf\{\|x+y\| : y \in Y\} \\ &= \inf\{\|x-y\| : y \in Y\} \\ &= \operatorname{dist}(x, Y).\end{aligned}$$

Lemma: Let X be a normed space and $Y \subset X$. Then $\forall x \in X$ and $\lambda \in \mathbb{R}$ $\operatorname{dist}(\lambda x, Y) = |\lambda| \operatorname{dist}(x, Y)$.

Proof: If $\lambda = 0$, then $\lambda x \in$ and so

$$0 = \operatorname{dist}(\lambda x, Y) = |\lambda| \operatorname{dist}(x, Y).$$

If $\lambda \neq 0$, then

$$\|\lambda x - y\| = |\lambda| \, \|x - \tfrac{y}{\lambda}\| \geq |\lambda| \operatorname{dist}(x, Y)$$

$$\Rightarrow \operatorname{dist}(\lambda x, Y) \geq |\lambda| \operatorname{dist}(x, Y)$$

Take $x' = \lambda x$, $\lambda' = 1/\lambda$. Then

$$\operatorname{dist}(\lambda' x', Y) \geq |\lambda'| \operatorname{dist}(x', Y) \Rightarrow \operatorname{dist}(x, Y) \geq \frac{1}{|\lambda|} \operatorname{dist}(\lambda x, Y)$$

Theorem: Let X be a normed space and Y is a closed subspace of X.

(i) The quotient norm is a norm on X/Y.

(ii) If X is a Banach space, so is X/Y.

Proof:

(i) • Positive Definite: Let $[x] \in X/Y$. If $\|[x]\| = 0$

$\Rightarrow \text{dist}(x, Y) = 0 \overset{\text{closed}}{\Rightarrow} x = 0 \Rightarrow [x] = [0]$

• Let $[x]_Y \in X/Y$, $\lambda \in \mathbb{R}$. Then

$$\|\lambda[x]\| = \|[\lambda x]\| = \text{dist}(\lambda x, Y) = |\lambda| \text{dist}(x, Y)$$
$$= |\lambda| \cdot \|[x]\|.$$

• For $[x], [z] \in X/Y$

$$\|[x+z]\| = \text{dist}(x+z, Y)$$
$$= \inf\{\|x+z-y\| : y \in Y\}$$
$$\leq \inf\{\|x+z-(y+y')\| : y, y' \in Y\}$$
$$\leq \inf\{\|x-y\| + \|z-y'\| : y, y' \in Y\}$$
$$= \text{dist}(x, Y) + \text{dist}(z, Y)$$
$$= \|[x]\| + \|[z]\|.$$

(ii) Assume X is a Banach space. Fix $([x_n]_Y)_{n=1}^{\infty}$ in X/Y such that $\sum_{n=1}^{\infty} \|[x_n]_Y\| < \infty$. We seek $[x]_Y \in X/Y$ such that $\sum_{n=1}^{\infty} [x_n]_Y = [x]_Y$.

For $n \in \mathbb{N}$, pick $y_n \in Y$ such that $\|x_n - y_n\| < \|[x_n]_Y\| + \frac{1}{2^n}$

Then $\sum_{n=1}^{\infty} \|x_n - y_n\| < \sum_{n=1}^{\infty} \|[x_n]_Y\| + 1 < \infty$. X is a Banach space, $\exists x \in X$ s.t. $\sum_{n=1}^{\infty}(x_n - y_n) = x$. Then

$$\left\|\sum_{n=1}^{N}[x_n]_Y - [x]_Y\right\| = \left\|\left[\sum_{n=1}^{N} x_n - x\right]_Y\right\|$$

$$= \operatorname{dist}\left(\sum_{n=1}^{N} x_n - x, Y\right) \leq \left\|\sum_{n=1}^{N} x_n - x - \left(\sum_{n=1}^{N} y_n\right)\right\|$$

$$= \left\|\sum_{n=1}^{N}(x_n - y_n) - x\right\| \to 0$$

Def: Let X and Y be normed spaces. A bounded linear operator $T: X \to Y$ is called open if $\forall U \subset X$ open $T(U)$ is an open subset of Y

Lemma: X, Y normed space, $T: X \to Y$ bounded lin. operator. TFAE:

(i) T is open,

(ii) $\exists A \subset X$ bounded s.t. $T^{\circ}(A) \neq \emptyset$

Proof: (i) $\Rightarrow$ (ii) $\quad A = \mathbb{B}_X^{\circ}$

(ii) $\Rightarrow$ (i) : $A \subset X$ bounded with $T^{\circ}(A) \neq \emptyset$. Let $R > \sup_{x \in A}\|x\|$ and $x_0 \in A$ and $\delta > 0$ such that $B(T(x_0), \delta) \subset T(A)$.

Fix $U \subset X$ open. Show $T(U)$ is open. Fix $x \in U$. Seek a $\eta > 0$ such that $B(T(x), \eta) \subset T(U)$.

Bec. U open $\Rightarrow \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset U$. Then put $\eta = \frac{\varepsilon \cdot \delta}{2R}$. Show $B(T(x), \eta) \subset T(U)$.

Take $w \in B(T(x), \eta)$ and put $y = T(x_0) + \frac{2R}{\varepsilon}(T(x) - w)$ in $B(T(x_0), \delta) \subset T(A)$. $\Rightarrow \exists x' \in A$ s.t. $y = T(x')$

$\Rightarrow w = T\left(x - \frac{\varepsilon}{2R}(x' - x_0)\right) \Rightarrow x - \frac{\varepsilon}{2R}(x' - x_0) \in B(x, \varepsilon) \subset U$

$\Rightarrow w \in T(U)$.

Prop: Let X be a normed space and Y closed subspace of X.

(i) $\|Q\| = 1$ unless $Y = X$

(ii) $Q(\mathbb{B}^\circ_X) = \mathbb{B}^\circ_{X/Y}$. In particular, Q is open.

Proof: (ii) Fix $x \in \mathbb{B}^\circ_X$, i.e. $\|x\| < 1$. Then

$\|Q(x)\| = \|[x]\| = \text{dist}(x, Y) \leq \|x - 0_X\| < 1$.

$\Rightarrow Q(\mathbb{B}^\circ_X) \subset \mathbb{B}^\circ_{X/Y}$.

Fix $[x]_Y \in \mathbb{B}^\circ_{X/Y} \Rightarrow \|[x]\| < 1$. $\exists y \in Y$ s.t. $\|x - y\| < 1$

$\Rightarrow x - y \in \mathbb{B}^\circ_X \Rightarrow [x] = Q(x) = Q(x) - Q(y) = Q(x-y)$

$\Rightarrow [x] \in Q(\mathbb{B}^\circ_X)$.

Prop: X, Y normed space and $T: X \to Y$ linear operator. Then $\bar{T}: X/\ker(T) \to Y$ is bounded and $\|T\| = \|\bar{T}\|$.

Proof: $\ker(T)$ is closed and so $X/\ker(T)$ is a normed space. Fix $[x] \in X/Y$ such that $\|[x]\| \leq 1$. Then $\|\bar{T}([x]_Y)\| = \|\bar{T}Q(x)\| = \|T(x)\| \leq \|T\|\|x\|$. Taking inf over all $x' \in [x]_Y$

$\|\bar{T}([x])\| = \|\bar{T}([x'])\| \leq \|T\| \cdot \|x'\|$.

$\|\bar{T}([x])\| \leq \|T\| \cdot \|[x]\| \Rightarrow \bar{T}$ is bounded and $\|\bar{T}\| \leq \|T\|$.

Sums of Normed Spaces

Def: For X, Y normed spaces and $1 \leq p \leq \infty$, we define

$\|\cdot\|_p$ on $X \times Y$ by

$$\|(x,y)\|_p = \left(\|x\|^p + \|y\|^p\right)^{1/p}$$ and for $\|\cdot\|_\infty$

$$\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$$

We denote $(X \times Y, \|\cdot\|_p)$ as $(X \oplus Y)_p$

Exercise: Let X, Y be normed spaces, $1 \leq p \leq \infty$.

- $\|\cdot\|_p$ is a norm on $X \times Y$.
- The map $i_1 : X \to (X \oplus Y)_p$ given by $i_1(x) = (x, 0)$ is an into linear isometry
- The map $q_1 : (X \oplus Y)_p \to X$ given by $q_1(x,y) = x$ is an onto bounded linear operator with $\|q_1\| = 1$

Def: Let $\{X_i\}_{i \in I}$ be normed spaces indexed over I.

(i) For $1 \leq p \leq \infty$

$$\left(\bigoplus_{i \in I} X_i\right)_p = \left\{x \in \prod_{i \in I} X_i : \sum_{i \in I} \|x_i\|^p < \infty\right\}$$ with

$$\|x\|_p = \left(\sum_{i \in I} \|x_i\|^p\right)^{1/p}$$

(ii) $\left(\bigoplus_{i \in I} X_i\right)_\infty$

(iii) $\left(\bigoplus_{i \in I} X_i\right)_0 = \left\{x \in \prod_{i \in I} X_i : \forall \varepsilon > 0 \ \|x_i\| > \varepsilon \ \forall i \text{ but finitely}\right\}$.