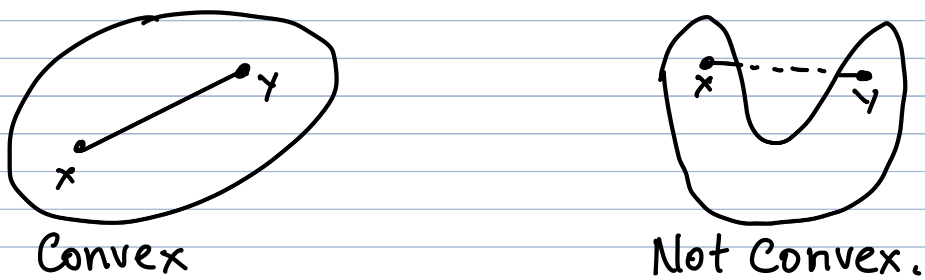
Math 6461 Lecture 5

Def: Let X be a vector space.

(1) For $x, y \in X$, we denote $[x,y] = \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ the linear segment between x and y.

(2) A subset A of X is called a convex set if for all $x, y \in A$, $[x,y] \subset A$. Equivalently, A is convex if for all $x, y \in A$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in A$.



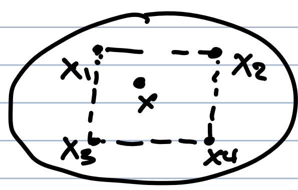
Examples:

(1) If X is a vector space. Every singleton $\{x\}$ is convex.
Every subspace is convex

(2) In $\mathbb{R}$, a subset is convex precisely

Notation: Let X be a vector space $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then $x = \sum_{i=1}^{n} \lambda_i x_i$ is called a convex combination of $x_1, \ldots, x_n$.

Proposition: Let X be a vector space and $A \subset X$ convex. Then for all $x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$, then $\sum_{i=1}^{n} \lambda_i x_i \in A$.

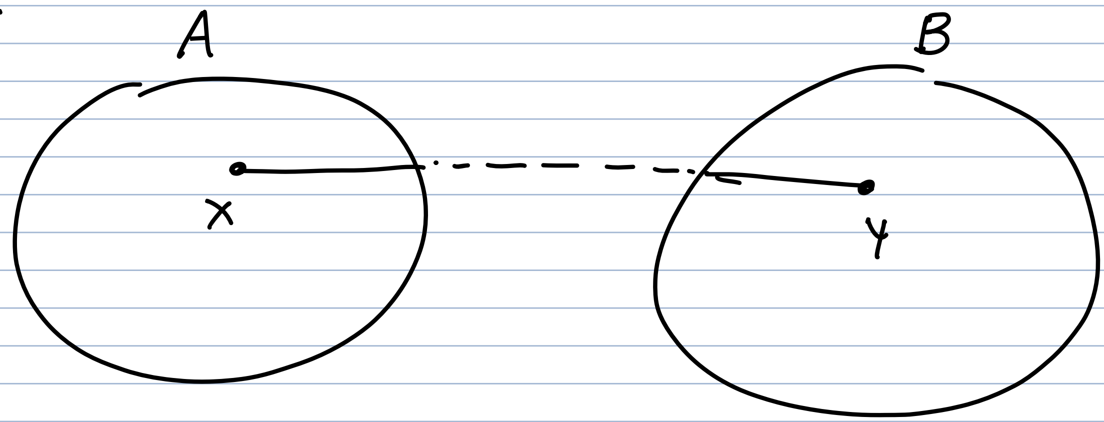


Proof: by Induction (exercise)

Proposition: Let X be a vector space and $(A_i)_{i\in I}$ are convex subsets of X, then $A = \bigcap_{i\in I} A_i$ is convex.

Proof: exercise.

Remark: The union of convex subsets may not be convex.



Remark: $\emptyset$ is convex.

Def: Let X be a vector space and $A \subset X$. The convex hull of A is the set

$$\text{conv}(A) = \bigcap \{B : B \text{ is convex and } A \subset B\}.$$

i.e. the set of all convex combinations of vectors of A.

Prop: Let X be a vector space and $A \subset X$. Then

(i) $\text{conv}(A)$ is a convex subset of X and $A \subset \text{conv}(A)$

(ii) If B is a convex subset of X and $A \subset B$, then $\text{conv}(A) \subset B$.

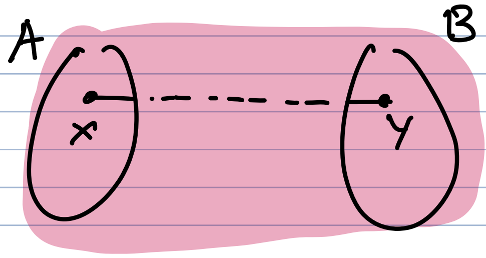
That is, $\text{conv}(A)$ is the smallest convex subset of X containing A.

Theorem: Let X be a vector space and $A \subset X$. Then

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N},\ x_1, \dots, x_n \in A,\ \lambda_1, \dots, \lambda_n \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

Prop: Let X be a vector space and $A, B \subset X$ that are nonempty convex. Then

$$\mathrm{conv}(A \cup B) = \{ \lambda x + (1-\lambda) y : x \in A, y \in B, \lambda \in [0,1] \}$$



See Robinson Ch. 3-7

Normed Spaces

Def: Let X be a vector space. A function $\|\cdot\| : X \to [0, \infty)$ is called a **norm** if the following hold:

(i) $\forall x \in X$, $\|x\| = 0$ if and only if $x = 0$,

(ii) $\forall x \in X$ and $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \cdot \|x\|$

(iii) $\forall x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$.

Prop: Let X be a normed space. Then the function $d : X \times X \to [0, \infty)$ given by $d(x,y) = \|x-y\|$ is a metric. That is,

(i) $\forall x, y \in X$, $d(x,y) = 0$ if and only if $x = y$

(ii) $\forall x, y \in X$, $d(x,y) = d(y,x)$

(iii) $\forall x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

Proof: (iii) For $x, y, z \in X$,

$$d(x,z) = \|x-z\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z).$$

This metric is called the induced metric on X.

Remark: If X is a normed space and $x, y \in X$, then
$\big| \|x\| - \|y\| \big| \leq \|x-y\|$. (exercise)

Def: If X is a normed space and $(x_n)_{n=1}^{\infty} \in X$, $x_0 \in X$, then
$\lim_{n\to\infty} x_n = x_0$ means $\lim_{n\to\infty} \|x_n - x_0\| = 0$.

Prop: Let X be a normed space,

(i) $+ : X \times X \to X$ is continuous, i.e. $\forall (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, x_0, y_0 \in X$ such that $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} y_n = y_0$. Then $\lim_{n\to\infty} (x_n + y_n) = x_0 + y_0$.

(ii) $\cdot : \mathbb{R} \times X \to X$ is continuous, i.e. $\forall (x_n)_{n=1}^{\infty}, x_0 \in X$ and $(\lambda_n)_{n=1}^{\infty}, \lambda_0 \in \mathbb{R}$ such that $(x_n) \to x_0$ and $\lambda_n \to \lambda_0$, then $(\lambda_n x_n) \to \lambda_0 x_0$.

(iii) $\|\cdot\| : X \to X$ is continuous, i.e. $\forall (x_n)_{n=1}^{\infty}, x_0 \in X$,
$\|x_n\| \to \|x_0\|$

Proof (ii):
$$
\begin{aligned}
\|\lambda_n x_n - \lambda_0 x_0\| &= \|\lambda_n x_n - \lambda_0 x_n + \lambda_0 x_n - \lambda_0 x_0\| \\
&\leq \|\lambda_n x_n - \lambda_0 x_n\| + \|\lambda_0 x_n - \lambda_0 x_0\| \\
&= \|x_n\| \, |\lambda_n - \lambda_0| + \|x_n - x_0\| \cdot |\lambda_0| \\
&\to 0.
\end{aligned}
$$

Notation: Let X be a normed space.

(i) For $x_0 \in X$ and $\varepsilon > 0$, denote
$B(x_0, \varepsilon) = \{x \in X : \|x_0 - x\| < \varepsilon\}$
called the open ball at x_0 with radius ε.

(ii) For $x_0 \in X$ and $\varepsilon > 0$, denote

$B[x_0, \varepsilon] = \{x \in X : \|x_0 - x\| \le \varepsilon\}$.

called the closed ball at x_0 with radius ε.

(iii) $\mathbb{B}_X = \{x \in X : \|x\| \le 1\}$ called the closed unit ball on X

(iv) $S_X = \{x \in X : \|x\| = 1\}$ called the unit circle on X.

Prop: $B(x, \varepsilon)$ and $\mathbb{B}_X$ are convex sets,

Prop: Let X be a normed space and Y is a subspace of X

(i) $\overline{Y} = \{$all limit points of $Y\}$ is a subspace of X

(ii) If $Y^\circ = \{$union of all open balls contained in $Y\}$ is nonempty then $Y = X$.

Proof: (i) Uses continuity of $+$ and $\cdot$ (exercise)

(ii) If $Y^\circ \neq \emptyset$ then $\exists y \in Y$ and $\varepsilon > 0$ such that $B(y, \varepsilon) \subset Y^\circ$

Fix $x \in X$. Show $x \in Y$. Put $z = y + \frac{\varepsilon}{\|x\| + 2} x$

Then $z \in B(y, \varepsilon)$ because $\|y - z\| = \frac{\varepsilon}{\|x\| + 2} \|x\| < \varepsilon$. Then

$z \in Y \Rightarrow x = \frac{\|x\| + 2}{\varepsilon}(z - y) \in Y$

Exercise: Let X be a normed space and $x_0 \in X$ and $\varepsilon > 0$

(i) $\overline{B(x_0, \varepsilon)} = B(x_0, \varepsilon)$

(ii) $(B[x_0, \varepsilon])^\circ = B(x_0, \varepsilon)$

(iii) $y_0 + B(x_0, \varepsilon) = B(x_0 + y_0, \varepsilon)$

(iv) $(\lambda x_0) \lambda\, B(x_0, \varepsilon) = B(\lambda x_0, |\lambda|, \varepsilon)$

Examples of Normed Spaces

(i) $\mathbb{R}$ with $|\cdot|$

(ii) $\mathbb{R}^n$ with $\|\cdot\|_1$ and $\|\cdot\|_\infty$, i.e. $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ and

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

(iii) $\ell^1(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with $\|x\|_{\ell^1} = \sum_{n=1}^{\infty} |x_n|$

(iv) Consider

$$C_{00}(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N} \text{ s.t } x_n = 0 \ \ \forall n \ge N\}$$

$$C_0(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : x_n \to 0 \text{ as } n \to \infty\}$$

$$\ell^\infty(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} |x_n| < \infty\}$$

with $\|x\|_{\ell^\infty} = \sup_{n\in\mathbb{N}} |x_n|$

(v) $C([0,1])$ with

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \quad \|f\|_{L^1} = \int_0^1 |f(x)|\,dx.$$

(vi) $C^1([0,1])$ then $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$.

Exercise: Show above are normed spaces.

Additional Normed Spaces

(i) $\mathbb{R}^n$ with $\|\cdot\|_p$ for $1 \le p < \infty$ where

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

(ii) $\ell^p(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ with $\|\cdot\|_{\ell^p}$ for $1 \le p < \infty$

where $\|x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$

(iii) $C([0,1])$ with

$$\|f\|_{L^p} = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \text{ for } 1 \le p < \infty.$$

Hölder and Minkowski Inequalities

Notation: Let $1 \le p \le \infty$. We denote the conjugate exponent $1 \le q \le \infty$ of p as follows:

(i) If $1 < p < \infty$ then $q = \frac{p}{p-1}$

(ii) If $p=1$, then $q=\infty$

(iii) If $p=\infty$, then $q=1$.

Remark: For $1 \le p, q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$

Theorem: Let $1<p,q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$,

(i) (Hölder) For all $x, y \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

$$\Rightarrow \|xy\|_1 \le \|x\|_p \|y\|_q$$

(ii) $\forall x \in \mathbb{R}^n \ \exists y \in \mathbb{R}^n$ s.t. $\|y\|_q = 1 \Rightarrow \sum_{i=1}^{n} x_i y_i = \|x\|_p$

(iii) (Minkowski) For all $x, y \in \mathbb{R}^n$

$$\left(\sum_{i=1}^{n} |x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

$$\Rightarrow \|x+y\|_p \le \|x\|_p + \|y\|_p.$$

Note: There are similar formulations for $\ell^p(\mathbb{N})$ and $C([0,1])$.