Recall: If $f:(X,d)\to(Y,\rho)$ is a function and $x_0\in X$, then **f is continuous at x_0** if $\forall\varepsilon>0\ \exists\delta>0$ s.t. $\forall x\in X$ with $d(x,x_0)<\delta$, then $\rho(f(x),f(x_0))<\varepsilon$.

Prop: Let X and Y be normed spaces and let $T:X\to Y$ be a linear operator. The following are equivalent:

(i) T is continuous

(ii) T is continuous at 0_X

(iii) T is continuous at some $x_0\in X$

(iv) T is Lipschitz, i.e. there exists a $M\geq 0$ such that for all $x,y\in X$, $\|T(x)-T(y)\|_Y\leq\|x-y\|_X$

(v) There exists an $M\geq 0$ such that for all $x\in X$ $\|T(x)\|_Y\leq M\|x\|_X$.

(vi) $\sup_{x\in B_X}\|T(x)\|_Y<\infty$.

Henceforth, a continuous linear operator will be called a **bounded linear operator**.

Proof: (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (vi) are easy.

We will show (iii) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (iv).

(iii) $\Rightarrow$ (v): Let $x_0\in X$ such that T is continuous at x_0.

Bec. T is continuous, for $\varepsilon=1$, there exists a $\delta>0$ such that for all $x\in B[x_0,\delta]$, then $\|T(x)-T(y)\|_Y\leq 1$.

We will show $\forall x\in X$, $\|T(x)\|_Y\leq\frac{1}{\delta}\|x\|_X$. Fix $x\in X$.

If $x=0_X$, then $\|T(x)\|_Y\leq\frac{1}{\delta}\|x\|_X$ holds. If $x\neq 0_X$, then take $y = x_0 + \frac{\delta}{\|x\|_X} \cdot x$ and note $\|x_0 - y\|_X < \delta$ which implies

$$1 \geq \|T(x_0) - T(y)\|_Y = \left\| T(x_0) - T\left(x_0 + \frac{\delta}{\|x\|_X} x\right)\right\|_Y$$
$$= \frac{\delta}{\|x\|_X} \cdot \|T(x)\|_Y .$$

(vi) $\Rightarrow$ (iv): Assume $\sup_{x \in \mathbb{B}_X} \|T(x)\|_Y = M < \infty$. We will show that $\forall x \in X$, $\|T(x)\|_Y \leq M \|x\|_X$. If $x = 0_X$, result holds.

Otherwise, if $x \neq 0_X$, $\|x\|_X^{-1} \in \mathbb{B}_X$ and so

$$\|T(\|x\|_X^{-1} \cdot x)\|_Y \leq M \Rightarrow \frac{\|T(x)\|_Y}{\|x\|_X} \leq M.$$

Def: Let X and Y be a normed space and $T: X \to Y$ be a linear operator. The operator norm of T is defined as $\|T\|_{\mathbb{B}_X} = \sup_{x \in \mathbb{B}_X} \|T(x)\|$

Notation: We denote $\mathcal{L}(X,Y) = \{T : X \to Y : T \text{ is bounded}\}$ and is contained in $L(X,Y)$.

Prop: Let X and Y be normed spaces. Then $\mathcal{L}(X,Y)$ is a vector space and $\|\cdot\|$ is a norm on it.

Proof: For $T, S \in \mathcal{L}(X,Y)$ we will show that $\|T+S\| \leq \|T\| + \|S\|$, and will show $T+S \in \mathcal{L}(X,Y)$.

It suffices to show $\|T+S\| \leq \|T\| + \|S\|$. Fix $x \in \mathbb{B}_X$.

then

$$\|(T+S)(x)\| = \|T(x) + S(x)\| \leq \|T(x)\| + \|S(x)\| \leq \|T\| + \|S\|$$

Taking sup over all $x \in \mathbb{B}_X$, $\|T+S\| \leq \|T\| + \|S\|$.

For $T \in \mathcal{L}(X,Y)$ and $\lambda \in \mathbb{R}$, we will show $\lambda T \in \mathcal{L}(X,Y)$ and we will show $\|\lambda T\| = |\lambda| \cdot \|T\|$.

$\|\lambda T\| = \sup_{x\in \mathbb{B}_X} \|\lambda T(x)\| = \sup_{x\in \mathbb{B}_X} |\lambda|\cdot\|T(x)\| = |\lambda| \sup_{x\in \mathbb{B}_X} \|T(x)\| = |\lambda|\|T\|$

Prop:

(i) If X and Y are normed space and $T\in \mathcal{L}(X,Y)$, then $\forall x\in X$, $\|T(x)\| \le \|T\|\cdot\|x\|_X$

(ii) If X, Y and Z are normed space, $T\in\mathcal{L}(X,Y)$ and $S\in\mathcal{L}(Y,Z)$, then $ST\in\mathcal{L}(X,Z)$ and $\|ST\|\le\|S\|\cdot\|T\|$.

Proof of (ii): Fix $x\in\mathbb{B}_X$. Then

$$\|(ST)(x)\| = \|S(T(x))\| \le \|S\|\cdot\|T(x)\| \le \|S\|\cdot\|T\|.$$

Taking sup over all $x\in\mathbb{B}_X$, $\|ST\|\le\|S\|\cdot\|T\|$.

Prop: Let X and Y be normed spaces and $T\in L(X,Y)$.

Then $\|T\| = \sup_{x\in\mathbb{B}_X^\circ}\|T(x)\|$ (if $\dim(X)>0$)

$= \sup_{x\in S_X}\|T(x)\|$

$= \inf\{M>0 : \forall x\in X, \|T(x)\|\le M\|x\|\}$.

Theorem: Let X be a normed space and let Y be a Banach space, then $(\mathcal{L}(X,Y), \|\cdot\|)$ is a Banach space.

Proof: Let $(T_n)_{n=1}^\infty \in \mathcal{L}(X,Y)$ such that $\sum_{n=1}^\infty \|T_n\| < \infty$.

We will prove $\exists T\in\mathcal{L}(X,Y)$ such that $\sum_{n=1}^\infty \|T_n\| = T$, i.e. $\lim_{N\to\infty} \|\sum_{n=1}^N T_n - T\| = 0$. Fix $x\in X$. Then $\sum_{n=1}^\infty \|T_n(x)\| \le \sum_{n=1}^\infty \|T_n\|\cdot\|x\| < \infty$. Bec. $(T_n(x))_{n=1}^\infty$ is absolutely summable in the Banach space Y, there exists a $T(x)\in Y$ such that $\sum_{n=1}^\infty T_n(x) = T(x)$. In particular, T is linear.

Now we show T is bounded. Fix $x \in \mathbb{B}_X$. Then

$$\|T(x)\| = \left\|\sum_{n=1}^{\infty} T_n(x)\right\| \le \sum_{n=1}^{\infty} \|T_n(x)\| \le \left(\sum_{n=1}^{\infty} \|T_n\|\right)\|x\|$$

$\Rightarrow$ T is bounded and $\|T\| \le \sum_{n=1}^{\infty} \|T_n\|$.

Lastly, we show $\lim_{N\to\infty} \left\|\sum_{n=1}^{N} T_n - T\right\| = 0$. Fix $x \in X$ and $N \in \mathbb{N}$. Then

$$\left\|\left(\sum_{n=1}^{N} T_n - T\right)(x)\right\| = \left\|\sum_{n=1}^{N} T_n(x) - \sum_{n=1}^{\infty} T_n(x)\right\| = \left\|\sum_{n=N+1}^{\infty} T_n(x)\right\|$$

$$\le \sum_{n=N+1}^{\infty} \|T_n(x)\| \le \left(\sum_{n=N+1}^{\infty} \|T_n\|\right)\|x\| \le \sum_{n=N+1}^{\infty} \|T_n\|$$

$$\Rightarrow \left\|\sum_{n=1}^{N} T_n - T\right\| \le \sum_{n=N+1}^{\infty} \|T_n\| \to 0 \quad \text{as } N \to 0.$$

Specifically, $\mathcal{L}(X, \mathbb{R})$ is always a Banach space.

Theorem: Let X and Y be normed spaces and Z be a
↳ Banach space.
dense subspace of X. If $T: Z \to Y$ is a bounded linear operator, then there exists a unique bounded linear operator $\bar{T}: X \to Y$ such that $\bar{T}|_Z = T$. Furthermore, $\|\bar{T}\| = \|T\|$.

Proof:

Claim 1: $\forall x \in X$ and $(z_n)_{n=1}^{\infty}$ in Z s.t. $(z_n) \to x$, then $\lim_{n\to\infty} T(z_n)$ exists in Y.

Bec. $(z_n) \to x$, (z_n) is Cauchy so

$$\limsup_{n\to\infty}\left(\limsup_{m\to\infty} \|T(z_n) - T(z_m)\|\right)$$

$$\le \limsup_{n\to\infty}\left(\limsup_{m\to\infty} \|T\| \cdot \|z_n - z_m\|\right)$$

$$= \|T\| \limsup_{n\to\infty}\left(\limsup_{m\to\infty} \|z_n - z_m\|\right) = 0$$

Bec. Y is a Banach space, $\lim(T(z_n))$ exists.

Claim 2: $\forall x \in X$, if $(z_n), (z_n')$ are in Z such that $\lim(z_n) = \lim(z_n') = x$, then $\lim(T(z_n)) = \lim(T(z_n'))$

Indeed, $\lim \|T(z_n) - T(z_n')\| \leq \|T\| \cdot \lim \|z_n - z_n'\| = 0$.

For $x \in X$, let $\bar{T}(x)$ be the unique $y \in Y$ such that $\forall (z_n) \in Z$ with $\lim(z_n) = x$, we have $\lim(T(z_n)) = \bar{T}(x)$

- $\bar{T}$ is linear
- $\bar{T}|_Z = T$. If $z \in Z$, take $z_n = z$ $\forall n \in \mathbb{N}$. Then $\bar{T}(z) = \lim(T(z_n)) = T(z)$.
- $\|\bar{T}\| = \|T\|$. Show $\|\bar{T}\| \geq \|\bar{T}|_Z\| = \|T\|$. Finally, if $x \in \mathbb{B}_X$ then $(z_n) \in Z$ with $\lim(z_n) = x$.

$\|\bar{T}(x)\| = \lim \|T(z_n)\| \leq \|T\| \lim \|z_n\| \leq \|T\|$,

Isomorphisms of a Normed Space.

Def: Let X and Y be normed spaces.

(i) A linear operator $T: X \to Y$ such that for all $x \in X$, $\|T(x)\| = \|x\|$ is called a linear isometry

(ii) A linear isometry $T: X \to Y$ that is onto, is called an isometrical isomorphism.

If an isometrical isomorphism $T: X \to Y$ exists, we say that X and Y are isometrically isometric and write $X \equiv Y$.

Remark: Let X, Y be normed spaces

(i) If $T: X \to Y$ is a linear isometry, then $\|T\| = 1$.

(ii) If $T: X \to Y$ is a linear isometry, then $\forall x, y \in X$

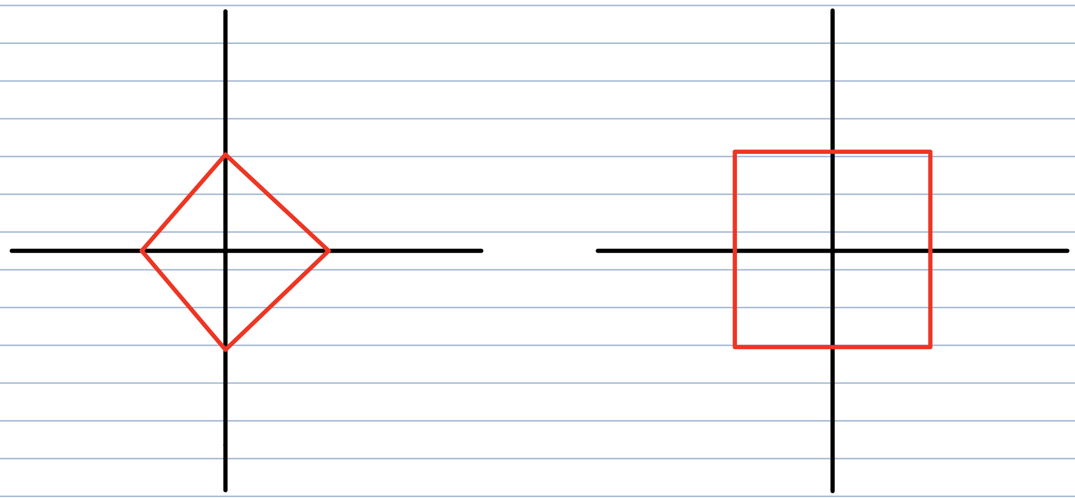
$$\|T(x) - T(y)\| = \|T(x-y)\| = \|x-y\|$$

In particular, if $x \neq y$, $T(x) \neq T(y) \Rightarrow T$ is one-to-one.

(iii) If $T: X \to Y$ is a surjective linear isometry, then $T^{-1}: Y \to X$ is a surjective linear isometry.

(iv) Compositions of linear isometries is a linear isometry..

Example: $(\mathbb{R}^2, \|\cdot\|_1) \equiv (\mathbb{R}^2, \|\cdot\|_\infty)$



Take $T: (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$ with

$$T(e_1) = e_1 + e_2, \quad T(e_2) = -e_1 + e_2$$

Def: Let X and Y be normed spaces and $T: X \to Y$ is called an **isomorphism** if

(i) T is a bijection

(ii) T is bounded

(iii) T^{-1} is bounded

If $\exists$ an isomorphism, we say X and Y are isomorphic we write $X \simeq Y$.

Remark: Let $T: X \to Y$ be an isomorphism. Then $T^{-1}: Y \to X$ is an isomorphism, and compositions of isomorphisms are isomorphisms.

Example: $\forall n \in \mathbb{N}$, $id: (\mathbb{R}^n, \|\cdot\|_1) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ is an isomorphism.

Remark: Isomorphisms $\Rightarrow$ equivalence relation.