Question 1

(i) Let $z \in X$ s.t., $\forall x \in X$, $z + x = z$.

Then, for $x = 0_X$, $z + 0_X = 0_X$

But 0_X is an additive identity, and thus,

$z + 0_X = z$. We conclude $z = 0_X$.

(ii) Fix $x \in X$ and let $z \in X$ s.t. $x + z = 0_X$.

Then,

$$z = 0_X + z = (-x + x) + z =$$
$$-x + (x + z) = -x + 0_X = -x.$$

(iii) $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$

Therefore,

$$0_X = -(0 \cdot x) + 0 \cdot x = -(0 \cdot x) + (0 \cdot x + 0 \cdot x)$$
$$= (-(0 \cdot x) + 0 \cdot x) + 0 \cdot x = 0_X + 0 \cdot x = 0 \cdot x.$$

(iv) We will show $(-1) \cdot x + x = 0_X$. By (ii), this yields $(-1) \cdot x = -x$.

Indeed, $(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = (-1+1) \cdot x$

$= 0 \cdot x = 0_X$ (by (iii)).

(v) $\lambda \cdot 0_X = \lambda \cdot (0_X + 0_X) = \lambda 0_X + \lambda \cdot 0_X$.

Therefore, $0_X = -(\lambda \cdot 0_X) + \lambda 0_X =$

$-(\lambda \cdot 0_X) + (\lambda \cdot 0_X + \lambda \cdot 0_X) = (-(\lambda \cdot 0_X) + \lambda \cdot 0_X) + \lambda \cdot 0_X$

$$= 0_X + \lambda \cdot 0_X = \lambda \cdot 0_X .$$

(vi) Assume that $\lambda \neq \mu$. Then,

$$x = 1 \cdot x = \frac{1}{\lambda-\mu}\left[(\lambda-\mu)\cdot x\right] = \frac{1}{\lambda-\mu}\left[\lambda x + (-\mu)x\right]$$

$$= \frac{1}{\lambda-\mu}\left[\lambda x + (-1)\cdot(\mu x)\right] =$$

$$= \frac{1}{\lambda-\mu}\left[\lambda x + (-1)\cdot(\lambda x)\right] = \frac{1}{\lambda-\mu}\left[\lambda x + (-(\lambda x))\right]$$

$$= \frac{1}{\lambda-\mu} 0_X = 0_X \quad \text{(by (v))}.$$

Question 2

Consider the function $\phi : \{e_1, e_2\} \to \mathbb{R}^2$ given by $\phi(e_1) = e_1$ & $\phi(e_2) = 2e_1$. Because $\{e_1, e_2\}$ is a Hamel basis of $\mathbb{R}^2$, there exists a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Te_1 = e_1$ & $Te_2 = 2e_1$. That is, Te_1, Te_2 are different non-zero vectors. But they are linearly dependent because $Te_1 + \left(\frac{-1}{2}\right) Te_2 = 0_{\mathbb{R}^2}$.

Question 3

We will prove by induction on $n = 0, 1, \ldots$ that $\{p_0, p_1, \ldots, p_n\}$ is linearly independent. That is,

for every $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ s.t. $\sum_{i=0}^{n} \lambda_i p_i = 0$ (the zero function) we have $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$.

For $n=0$, this is obvious, because $p_0 = 1$ (the constant function 1). So if $\lambda_0 p_0 = 0$, then $\lambda_0 = 0$.

Let $n \geq 0$ s.t. the conclusion holds. Let $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ s.t. $\sum_{i=0}^{n} \lambda_i p_i = 0$, i.e., for $t \in [0,1]$,

$$\lambda_0 + \sum_{i=1}^{n} \lambda_i t^i = 0.$$

Differentiating, $\sum_{i=1}^{n} \lambda_i \, i \cdot t^{i-1} = 0$, & therefore,

$\sum_{i=0}^{n-1} \lambda_{i+1} (i+1) p_i = 0$. By the inductive hypothesis,

$\lambda_1 = \lambda_2 \cdot 2 = \cdots = \lambda_n \cdot n = 0$, & therefore,

$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. We deduce $\lambda_0 p_0 = 0$, and thus $\lambda_0 = 0$ as well.

The induction is complete. Because $B = \bigcup_{n=0}^{\infty} \{p_0, \ldots, p_n\}$ is the union of a chain of linearly independent sets, it is linearly independent.

Question 4

Let $\mathcal{B} = \{ B \subset A : B$ is linearly independent$\}$.

(Note that $\emptyset \in \mathcal{B}$)

With inclusion, this is a partially ordered set.
By a known lemma, if $(B_i)_{i \in I}$ is a chain in $\mathcal{B}$, $B = \bigcup_{i \in I} B_i$ is linearly independent, and thus an upper bound of the chain. By Zorn's lemma, there is a maximal element B of $\mathcal{B}$.
We claim that $\langle B \rangle = \langle A \rangle$.
We first show $A \subset \langle B \rangle$. If this is false, $\exists\ a_0 \in A \setminus \langle B \rangle$. But then, by a known lemma, $B_0 = B \cup \{a_0\}$ is a linearly independent subset of A. But $B \subsetneq B_0$, which contradicts the maximality of B.

By a known property of linear spans, $A \subset \langle B \rangle$ yields $\langle A \rangle \subset \langle B \rangle$. But because $B \subset A$, $\langle B \rangle \subset \langle A \rangle$.

Question 5

We first show $Y \cap Z = \{0_X\}$, so let $x \in Y \cap Z$.

Assume $x \neq 0$.

Because $\langle A \rangle = Y$, there are (pairwise different) $x_1, \dots, x_n \in A$ and (non-zero) $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ s.t. $x = \sum_{i=1}^{n} \lambda_i x_i$.

Because $\langle B \setminus A \rangle = Z$, there are (pairwise different) $y_1, \dots, y_m \in B \setminus A$ and (non-zero) $\mu_1, \dots, \mu_m \in \mathbb{R}$ s.t. $x = \sum_{j=1}^{m} \mu_j y_j$. Then,

But then, $0_X = \sum_{i=1}^{n} \lambda_i x_i + \sum_{j=1}^{m} (-\mu_j) y_j$.

Because $\mathcal{B}$ is linearly independent and $x_1, \dots, x_n, y_1, \dots, y_m$ are pairwise different, $\lambda_1 = \dots = \lambda_n = -\mu_1 = \dots = -\mu_m = 0$. This is absurd, so $x = 0$.

Next, we show $Y + Z = X$. Let $x \in X$.

Because $0_X = 0_X + 0_X \in Y + Z$, we need to treat the case $x \neq 0$. Because $\mathcal{B}$ is a Hamel basis, there is a non-empty set $F \subset \mathcal{B}$ and scalars $(\lambda_b)_{b \in F}$ such that $x = \sum_{b \in F} \lambda_b b$.

Put $y = \sum_{b \in F \cap A} \lambda_b b \in Y$ & $z = \sum_{b \in F \cap B} \lambda_b b \in Z$

$x = y + z$.

Question 6

Let C be a Hamel basis of the vector space $R(T)$, and for each $y \in C$, let $x_y \in X$ s.t. $Tx_y = y$. By a known lemma, $A = \{x_y : y \in C\}$ is a linearly independent set.

Put $Z = \langle A \rangle$.

- $T|_Z$ is an injection

  It suffices to show $\ker(T|_Z) = \ker(T) \cap Z = \{0_X\}$.
  Indeed, let $x \in Z$ such that $Tx = 0_Y$.
  If $x \neq 0_X$, there is a non-empty finite $F = \{x_{y_1}, \dots, x_{y_n}\} \subset A$ (with $y_1, \dots, y_n$ different) and non-zero scalars $\lambda_1, \dots, \lambda_n$ such that $x = \sum_{i=1}^{n} \lambda_i x_{y_i}$. Then, $0_X = Tx = \sum_{i=1}^{n} \lambda_i y_i$.
  By the linear independence of C, $0 = \lambda_1 = \dots = \lambda_n$, which is absurd.

- $\ker(T)$, Z form a linear decomposition of X.

  We show $\ker(T) \cap Z = \{0_X\}$. Indeed, if $x \in \ker(T) \cap Z$ then $Tx = 0_X$. Because $x \in Z$ & $\ker(T|_Z) = \{0_X\}$, $x = 0_X$.

- $\ker(T) + Z = X$. Let $x \in X$. Because $x \in R(T) = \langle C \rangle$, there are $y_1, \dots, y_n \in C$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

such that $y = \sum_{i=1}^{n} \lambda_i y_i$. Put $z = \sum_{i=1}^{n} \lambda_i x_{y_i} \in Z$ and $w = x - z$. Clearly, $x = z + w$, so we need to show $w \in \ker(T)$. Indeed,

$$Tw = Tx - Tz = \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \lambda_i Tx_{y_i}$$
$$= \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \lambda_i y_i = 0_Y .$$