Question 1. *Give an example of two convex subsets of a vector space with non-convex union.*

Question 2. *Let X and Y be vector spaces, $A \subset X$ and $T : X \to Y$. Show that $T[\operatorname{conv}(A)] = \operatorname{conv}(T[A])$*

Question 3. *Let X be a vector space.*

(i) Let $n \in \mathbb{N}$ and $A = \{x_1, ..., x_n\}$ be a subset of X. Show that

$$\operatorname{conv}\left(\bigcup_{i=1}^{n} A_i\right) = \left\{\sum_{i=1}^{n} \lambda_i x_i : \lambda_1, ..., \lambda_n \in \mathbb{N} \text{ with } \sum_{i=1}^{n} \lambda_i = 1\right\}$$

(ii) Let $n \in \mathbb{N}$ and $A_1, ..., A_n$ be convex subsets of X. Show that

$$\operatorname{conv}\left(\bigcup_{i=1}^{n} A_i\right) = \left\{\sum_{i=1}^{n} \lambda_i x_i : x_1 \in A_1, ..., x_n \in A, \lambda_1, ..., \lambda_n \in [0,1] \text{ with } \sum_{i=1}^{n} \lambda_i = 1\right\}$$

Question 4. *Let X be a normed space, $x_0 \in X$, and $r > 0$. Prove the following:*

(i) For $\lambda \in \mathbb{R} \setminus \{0\}$,

(a) $\lambda B(x_0, r) = B(\lambda x_0, |\lambda| r)$

(b) $\lambda B[x_0, r] = B[\lambda x_0, |\lambda| r]$

(ii) For $y_0 \in X$, $y_0 + B(x_0, r) = B(y_0, x_0 + r)$

(iii) $\overline{B(x_0, r)} = B[x_0, r]$

(iv) $B^\circ[x_0, r] = B(x_0, r)$.

Question 5. *Prove that the following statements are true:*

(a) The real line $\mathbb{R}$ with the absolute value $|\cdot|$ is a normed space.

(b) For $n \in \mathbb{N}$, the space $\mathbb{R}^n$ with the ℓ^1-norm or the ℓ^∞-norm given by

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

is a normed space.

(c) The space

$$\ell^1(\mathbb{N}) = \left\{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty\right\}$$

with the ℓ^1-norm given by

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$

is a normed space.

(d) *The following spaces:*

$$c_{00}(\mathbb{N}) = \{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \text{there exists an } N \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n \geq N\}$$
$$c_0(\mathbb{N}) = \left\{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\right\}$$
$$\ell^{\infty}(\mathbb{N}) = \left\{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} |x_n| < \infty\right\}$$

with the ℓ^{∞}-norm given by

$$\|x\|_{\infty} = \sup_{n\in\mathbb{N}} |x_n|$$

are normed spaces.

(e) *The space $C([0,1])$ of all real-valued continuous functions on the unit interval with the supremum norm given by*

$$\|f\|_{\infty} = \max_{x\in[0,1]} |f(x)|$$

is a normed space.

(f) *The space $C([0,1])$ with the L^1-norm given by*

$$\|f\|_1 = \int_0^1 |f(x)|dx$$

is a normed space.

(g) *The space $C^1([0,1])$ of all real-valued differentiable functions on the unit interval with continuous derivative with the norm given by*

$$\|f\|_{C^1} = \|f\|_{\infty} + \|f'\|_{\infty}$$

is a normed space.

(h) *For an arbitrary nonempty set A, the space*

$$c_{00}(A) = \{f \in \mathbb{R}^A : f(x) \neq 0 \text{ for all but finitely many } x \in A\}$$

with each of the ℓ^1-norm or the ℓ^{∞} norm given by

$$\|f\|_1 = \sum_{x\in A} |f(x)| \quad \|f\|_{\infty} = \max_{x\in A} |f(x)|$$

is a normed space.

Question 6. *(i) Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^{\mathbb{N}}$. Show that*

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_1 \|y\|_\infty$$

(ii) Let $f, g \in C([0,1])$. Show

$$\int_0^1 |f(x)g(x)|dx \leq \|f\|_1 \|g\|_\infty$$

Question 7. *The $\|\cdot\|_p$ and $\|\cdot\|_q$ norms compare, but the direction depends on whether the underlying space is a sequence space or a function space.*

(i) For $x \in \mathbb{R}^n$, for some $n \in \mathbb{N}$, or $x \in \mathbb{R}^{\mathbb{N}}$, prove that for $1 \leq p \leq \infty$,

$$\|x\|_q \leq \|x\|_p$$

(ii) Let $f \in C([0,1])$. Prove that for $1 \leq p \leq \infty$,

$$\|f\|_p \leq \|f\|_q$$

Hint: Use Jensen's inequality from real analysis stating that for a convex function $\phi : \mathbb{R} \to \mathbb{R}$ and $f \in C([0,1])$,

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 \phi(f(x))dx$$

Question 8. *Let $0 < p < 1$. For $n \in \mathbb{N}$, define $\|\cdot\|_p$ on $\mathbb{R}^n$ as follows:*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

Show that $\|\cdot\|_p$ is positively definite and absolutely homogeneous, but fails the triangle inequality, and therefore, is not a norm on $\mathbb{R}^n$.

