Math 6461 Lecture 4 (Sept. 16, 2024)

Def: For vector spaces X, Y, we denote

$$L(X,Y) = \{T : X \to Y : T \text{ is linear}\}$$

Remark: $L(X,Y)$ is a vector space with additive identity the zero operator.

- If $S, T \in L(X,Y)$, then $S+T \in L(X,Y)$ is given by
$$(S+T)(x) = S(x) + T(x)$$
- If $T \in L(X,Y)$ and $\alpha \in \mathbb{R}$, then $\alpha T \in L(X,Y)$ is given by $(\alpha T)(x) = \alpha T(x)$.

Linear Functionals

Def: Let X be a vector space, we denote

$$X^{\#} = L(X, \mathbb{R}) = \{f : X \to \mathbb{R} : f \text{ is linear}\}$$

called the algebraic dual of X. A $f \in X^{\#}$ is called a linear functional on X.

Examples:

- For $n \in \mathbb{N}$ and $1 \le i \le n$, $e_i^* : \mathbb{R}^n \to \mathbb{R}$ given by $e_i^*(x) = x_i$
- $\mathbb{E} : C([0,1]) \to \mathbb{R}$, $\mathbb{E}(f) = \int_0^1 f(x)\,dx$.
- For $0 \le t_0 \le 1$, $\delta_{t_0} : C([0,1]) \to \mathbb{R}$ given by $\delta_{t_0}(f) = f(t_0)$
- For $0 < p \le 1$ and recall $\ell^p(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$

Then $S : \ell^p(\mathbb{N}) \to \mathbb{R}$ given by $S(x) = \sum_{n=1}^{\infty} x_n$ is a linear functional.

Remark: If $p>1$, then S is not defined. Show that S is well defined on $\ell^p(\mathbb{N})$ if and only if $0<p\leq 1$.

Coordinate Functionals

Def: Let X be a vector space and B be a Hamel basis of X. The coordinate functionals of B are the subset $\{f_b : b\in B\}$ of $X^\#$ defined as follows: $\forall b\in B$, f_b is the unique operator $f_b : X\to \mathbb{R}$ such that $\forall b'\in B$,

$$f_b(b') = \begin{cases} 1 & \text{if } b=b' \\ 0 & \text{if } b\neq b' \end{cases}$$

This exists by a known theorem.

Remark: For $b_1\neq b_2\in B$, $f_{b_1}\neq f_{b_2}$.

Notation: Let X be a VS and $(x_i)_{i\in I}$ be a collection of vectors on X indexed on some set I. If $\{i\in I : x_i\neq 0_X\}$ is finite, say it is $\{i_1,\dots,i_n\}$. then we write

$$\sum_{i\in I} x_i = \sum_{k=1}^{n} x_{i_k}.$$

Prop: Let X be a VS with Hamel basis B and coordinate function $\{f_b : b\in B\}$

(i) $\forall x\in X$, $x=\sum_{b\in B} f_b(x)b$.

(ii) $\{f_b : b\in B\}$ is linearly independent subset of $X^\#$.

Proof: (i) Fix $x\in X$. Because $\langle B\rangle = X$, there exists distinct $b_1,\dots,b_n\in B$ and $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ such that $x=\sum_{i=1}^{n}\lambda_i b_i$

Then for $b \in B$

$$f_b(x) = \sum_{i=1}^{n} \lambda_i f_b(b_i) = \begin{cases} \lambda_j & \text{if } b = b_j \\ 0 & \text{if } b \notin \{b_1, \dots, b_n\}. \end{cases}$$

Then $x = \sum_{i=1}^{n} \lambda_i b_i = \sum_{i=1}^{n} f_{b_i}(x) b_i = \sum_{b \in B} f_b(x) b$.

(ii) Fix distinct $b_1, \dots, b_n \in B$ and let $(\lambda_i)_{i=1}^{n} \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i f_{b_i} = 0$ (the zero functional).

Take $x = \sum_{i=1}^{n} \lambda_i b_i \Rightarrow 0 = f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j f_{b_i}(b_j) = \sum_{i=1}^{n} \lambda_i^2$

Because $\lambda_i^2 \geq 0$ for $i = 1, \dots, n$, $\lambda_i = 0 \ \forall 1 \leq i \leq n$.

Prop: Let X be a VS with $\dim(X) = n \in \mathbb{N}$ and B be a Hamel basis of X. Then $\forall f \in X^{\#}$, $f = \sum_{b \in B} f(b) f_b$. In particular, $\langle \{f_b : b \in B\} \rangle = X^{\#}$ and $\dim(X) = \dim(X^{\#})$.

Proof: Fix $x \in X$. Then $x = \sum_{b \in B} f_b(x) b$. Then $f(x) = \sum_{b \in B} f_b(x) f(b)$ $= \left(\sum_{b \in B} f_b f(b) \right)(x)$.

Prop: If X is infinite dimensional with a Hamel basis B, then $\langle \{f_b : b \in B\} \rangle \subsetneq X^{\#}$

Proof: Take $\phi : B \to \mathbb{R}$ constantly equal to 1, and extend to a linear functional.

Remark: If $\dim(X) = \infty$, then $\dim(X^{\#}) = 2^{\dim(X)}$

Codimension of a Vector Space.

Def: Let X be a vector space and $Y \subset X$. Then we define the codimension of Y in X is $\operatorname{codim}(Y) = \dim(X/Y)$.

Example:

- Let X be a vector space. Then $\text{codim}(X) = 0$ and $\text{codim}(\{0_X\}) = \dim(X)$.

A subspace Y of X is said to be a codimension one if $\text{codim}(X) = 1$.

Prop: Let X be a vector space and Y is a linear subspace of X. The following are equivalent:

(1) $\text{codim}(X) = 1$

(2) $\forall x_0 \in X \setminus Y$, $\langle \{x_0\} \rangle$ and Y form a linear decomposition of X

(3) $\exists x_0 \in X \setminus Y$, $\langle \{x_0\} \rangle$ and Y form a linear decomposition of X.

Proof: Denote $Q: X \to X/Y$ to be the quotient map, i.e. $\forall x \in X$, $Q(x) = x + Y$. Recall $\ker(Q) = Y$.

(1) $\Rightarrow$ (2) We show that $\langle \{x_0\} \rangle + Y = X$, and $\langle \{x_0\} \rangle \cap Y = \{0_X\}$

Fix $x \in X$. Then $Q(x) = \lambda Q(x_0)$ for some $\lambda \in \mathbb{R}$. Then $y = x - \lambda x_0 \in \ker(Q) = Y$. Then $x = \lambda x_0 + y \in \langle \{x_0\} \rangle + Y$.

Fix $x \in \langle \{x_0\} \rangle \cap Y$. Because $x \in \langle \{x_0\} \rangle$, $x = \lambda x_0$ for some $\lambda \in \mathbb{R}$. Then $-x \in Y = \ker(Q)$, then $\lambda Q(x_0) = Q(x) = 0_{X/Y}$

Then $\lambda = 0 \Rightarrow x = 0_X$.

(3) $\Rightarrow$ (1): Fix $x_0 \in X \setminus Y$ such that $\langle \{x_0\} \rangle$ and Y form

a linear decomposition of X, show $Q(x_0) \neq 0$ and $\langle \{Q(x_0)\} \rangle = X/Y$.

Kernels of Linear Functionals

Prop: Let X be a VS and $f: X \to \mathbb{R}$ is a linear functional. Then $\ker(f)$ is of codimension one or zero. In particular, if f is not the zero functional then $\operatorname{codim}(\ker(f)) = 1$.

Proof: If $f = 0$, easy. If $f \neq 0$, i.e. $\exists x_0 \in X$ s.t. $f(x_0) \neq 0$. We show $\langle \{x_0\} \rangle$ and $\ker(f)$ form a linear decomposition of X.

Take $x \in X$, write $x = \frac{f(x)}{f(x_0)} x_0 + \overbrace{\left(x - \frac{f(x)}{f(x_0)} x_0\right)}^{y}$

when $y \in \ker(f)$ bec. $f(y) = f(x) - \frac{f(x)}{f(x_0)} f(x_0) = 0$.

Prop: Let X be a VS and Y be a linear subspace of X, of codimension 1. Let $x_0 \in X \setminus Y$ and $r_0 \in \mathbb{R}$. Then there exists a unique $f \in X^{\#}$ such that

(1) $f(x_0) = r_0$

(2) $Y \subset \ker(f)$

If furthermore, $r_0 \neq 0$, then $Y = \ker(f)$.

Proof: Fix a Hamel basis A of Y. Put $B = A \cup \{x_0\}$. Bec. $x_0 \in X \setminus \langle A \rangle$, B is linearly independent and $\langle B \rangle = X$ so B is a Hamel basis of X.

Define $\phi: B \to \mathbb{R}$ with $\phi(b) = \begin{cases} 0 & \text{if } b \in A \\ r_0 & \text{if } b = x_0 \end{cases}$ and extend it to a linear functional $f: X \to \mathbb{R}$. Obviously $f(x_0) = r_0$.

For $x \in Y = \langle A \rangle \cdots f(x) = 0 \Rightarrow Y \subset \ker(f)$.

If $g \in X^\#$ is such that $g(x_0) = r_0$ and $Y \subset \ker(g)$, then $f|_B = g|_B \Rightarrow f = g$.

Finally, if $r_0 \neq 0$, we show $Y \subseteq \ker(f)$. Towards a contradiction, assume $\exists w_0 \in \ker(f) \setminus Y$. Then $\langle \{w_0\} \rangle$ and Y form a lin. decomp of X. Then $x_0 = \lambda w_0 + y$ for some $\lambda \in \mathbb{R}$ and $y \in Y \Rightarrow r_0 = f(x_0) = 0$. Absurd.

Theorem: Let X be a vector space and $f, g \in X^\#$. TFAE

(1) $\exists \lambda \in \mathbb{R}$ s.t. $f = \lambda g$.

(2) $\ker(g) \subset \ker(f)$.

Proof: $(2) \Rightarrow (1)$: Consider the following cases

- If $g = 0 \Rightarrow X = \ker(g) \subset \ker(f) \subset X \Rightarrow \ker(f) = X \Rightarrow f = 0$
- $\exists x_0 \in X$ s.t. $g(x_0) \neq 0$. Let $r_0 = f(x_0)$, $Y = \ker(g)$ and apply the above prop. to Y, x_0, r_0 to get a unique $h \in X^\#$ s.t. $h(x_0) = r_0$ and $Y \subset \ker(h)$

Note $\frac{f(x_0)}{g(x_0)} g$ satisfies $\left(\frac{f(x_0)}{g(x_0)} g\right)(x_0) = r_0 \Rightarrow Y \subset \ker \frac{f(x_0)}{g(x_0)} g$ and f satisfies $f(x_0) = r_0$ and $Y = \ker(g) \subset \ker(f)$.

$\Rightarrow f = \frac{f(x_0)}{g(x_0)} g$.

Theorem: Let X be a VS and $g, f_1, \dots, f_n \in X^\#$. TFAE

(1) $g \in \langle \{f_1, \dots, f_n\} \rangle$

(2) $\bigcap_{k=1}^{n} \ker(f_k) \subset \ker(g)$

Proof: (2) $\Rightarrow$ (1) We will by induction on n. the following statement: For all X and for all $g, f_1, \dots, f_n \in X^\#$ such that $\bigcap_{k=1}^{n} \ker(f_k) \subset \ker(g)$. Then $g \in \langle \{f_1, \dots, f_n\} \rangle$

The base case was shown above. Assume for $n \in \mathbb{N}$ the conclusion holds and let X be a VS and $g, f_1, \dots, f_{n+1} \in X^\#$ such that $\bigcap_{k=1}^{n+1} \ker(f_k) \subset \ker(g)$. Let $Y = \ker(f_{n+1})$ and denote $\bar{g} = g|_Y$, $\bar{f}_1 = f_1|_Y, \dots, \bar{f}_n = f_n|_Y$.

Claim: $\bigcap_{k=1}^{n} \ker(\bar{f}_k) \subset \ker(\bar{g})$

Let $x \in Y$, with $x \in \bigcap_{k=1}^{n} \ker(\bar{f}_k)$ Then $f_1(x) = \bar{f}_1(x) = 0, \dots,$ $f_n(x) = \bar{f}_n(x) = 0$ and because $x \in Y = \ker(f_{n+1}) \Rightarrow f_{n+1} = 0$ $\Rightarrow x \in \bigcap_{k=1}^{n} \ker(f_k) \subset \ker(g) \Rightarrow \bar{g}(x) = g(x) = 0$. By IH, $\exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\bar{g} = \sum_{k=1}^{n} \lambda_k \bar{f}_k$ Put $h = g - \sum_{k=1}^{n} \lambda_k f_k$.

Claim: $\ker(f_{n+1}) \subset \ker(g)$

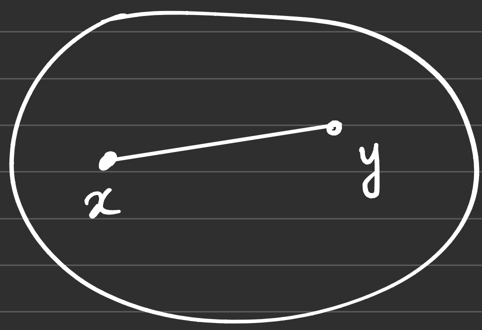
Convex Sets

Def: Let X be a VS

(1) For $x, y \in X$, denote $[x, y] = \{\lambda x + (1-\lambda) y : \lambda \in [0,1]\}$ the linear segment between x and y.

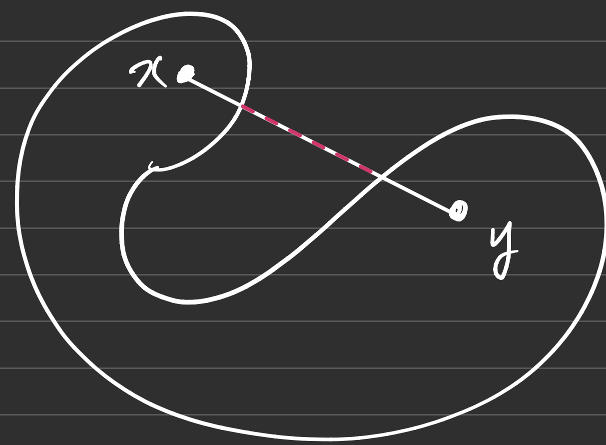
(2) $A \subset X$ is convex if $\forall x, y \in A$ and $\lambda \in [0,1]$,

$\lambda x + (1-\lambda) y \in A$.



Convex

Not Convex



Exercises: 1.4.10, 1.4.14, 1.4.15, 1.4.16, 1.4.21, 1.4.22.