Question 1. *Prove that each of the following spaces X is a vector space. More precisely, show the following:*

- *For $x, y \in X$, show $x + y$ is in X, where $+$ is the given operation of addition.*
- *For $\lambda \in \mathbb{R}$ and $x \in X$, show λx is in X, where $\cdot$ is the given operation of scalar multiplication.*
- *Show that the axioms of a vector space are satisfied. In particular, identify the additive identity element of 0_X.*

(a) *The real line $\mathbb{R}$ with usual addition and scalar multiplication is a vector space. The additive identity is zero.*

(b) *For $n \in \mathbb{N}$, the n-dimensional Euclidean space $\mathbb{R}^n$ with usual coordinate-wise vector addition and scalar multiplication is a vector space:*

$$\mathbb{R}^n = \{x = (x_1, ..., x_n) : x_1, ..., x_n \in \mathbb{R}\}$$

The additive identity is the zero vector $(0, ..., 0)$. In this particular vector space, the zero vector space is sometimes also called the origin.

(c) *The space of all real sequences $\mathbb{R}^{\mathbb{N}}$ with coordinate-wise vector addition and scalar multiplication is a vector space:*

$$\mathbb{R}^{\mathbb{N}} = \{x = (x_n)_{n=1}^{\infty} : \textit{for } n \in \mathbb{N},\ x_n \in \mathbb{R}\}$$

The additive identity is the zero sequence $(0, ..., 0, ...)$.

(d) *The space of eventually null real sequences $c_{00}(\mathbb{N})$ with coordinate-wise vector addition and scalar multiplication is a vector space:*

$$c_{00}(\mathbb{N}) = \{x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \textit{there exists an } N \in \mathbb{N} \textit{ such that } x_n = 0 \textit{ for all } n \geq N\}$$

The additive identity is the zero sequence.

(e) *The space of real sequences that converge to zero $c_0(\mathbb{N})$ with coordinate-wise vector addition and scalar multiplication is a vector space:*

$$c_0(\mathbb{N}) = \left\{x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\right\}$$

The additive identity is the zero sequence.

(f) *The space of bounded real sequences $\ell^{\infty}(\mathbb{N})$ with coordinate-wise vector addition and scalar multiplication is a vector space:*

$$\ell^{\infty}(\mathbb{N}) = \left\{x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} |x_n| < \infty\right\}$$

The additive identity is the zero sequence.

(g) *For $0 < p < \infty$, the space of p-summable real sequences $\ell^p(\mathbb{N})$ with coordinate-wise vector addition and scalar multiplication is a vector space.*

$$\ell^p(\mathbb{N}) = \left\{ x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

The additive identity is the zero sequence.

(h) *The space of all real-valued continuous functions on the unit interval $\mathcal{C}([0,1])$ with pointwise addition and scalar multiplication is a vector space:*

$$\mathcal{C}([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$$

The additive identity is the zero function.

(i) *The space of all real-valued continuously differentiable functions on the unit interval $\mathcal{C}^1([0,1])$ with pointwise addition and scalar multiplication is a vector space:*

$$\mathcal{C}^1([0,1]) = \{f \in \mathcal{C}([0,1]) : f \text{ is differentiable on } [0,1] \text{ and } f' \text{ is continuous}\}$$

The additive identity is the zero function.

(j) *For $n \in \mathbb{N}$, the space of all real polynomials $p : [0,1] \to \mathbb{R}$ of degree at most n, $\mathcal{P}_n([0,1])$ with pointwise addition and scalar multiplication is a vector space:*

$$\mathcal{P}_n([0,1]) = \{p \in \mathcal{C}([0,1]) : \exists a_0, ..., a_n \in \mathbb{R} \text{ such that } \forall x \in [0,1],\, p(x) = a_n x^n + \cdots + a_0\}$$

The additive identity is the zero function.

(k) *The space of all real polynomials $p : [0,1] \to \mathbb{R}$, $\mathcal{P}([0,1]) = \bigcup_{n=1}^{\infty} \mathcal{P}_n([0,1])$ with pointwise addition and scalar multiplication is a vector space. The additive identity is the zero function.*

(l) *For an arbitrary nonempty set A, the space of all real-valued functions with domain A, $\mathbb{R}^A$ with pointwise addition and scalar multiplication is a vector space:*

$$\mathbb{R}^A = \{f : A \to \mathbb{R}\}$$

(m) *For an arbitrary nonempty set A, the space of all finitely supported real-valued functions with domain A, $c_{00}(A)$ with pointwise addition and scalar multiplication is a vector space:*

$$c_{00}(A) = \{f \in \mathbb{R}^A : f(x) \neq 0 \text{ for only finitely many } x \in A\}$$

The additive identity is the zero function.

(n) *For a collection of vector spaces $(X_i)_{i \in I}$ indexed over a set I, the Cartesian product $\prod_{i \in I} X_i$ with pointwise addition and scalar multiplication is a vector space:*

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} : \text{for all } i \in I,\, x_i \in X_i\}$$

The additive identity is $(0_{X_i})_{i \in I}$.

Question 2. *Let X be a vector space.*

(i) Prove that the additive identity $0_X \in X$ is unique.

(ii) For every $x \in X$, prove that its additive inverse $-x$ is unique.

(iii) For every $x \in x$, prove that $0x = 0_X$.

(iv) Prove that, for $x \in X$, $(-1)x = -x$.

(v) Prove that, for all $\lambda \in \mathbb{R}$, $\lambda 0_X = 0_X$

(vi) For $\lambda, \mu \in \mathbb{R}$ and $x \in X$ such that $\lambda x = \mu$ prove that $\lambda = \mu$ or $x = 0_X$.

Question 3. *Prove that the following statements are true:*

(a) If X is a vector space, then $Y = \{0_X\}$ and $Y = X$ are subspaces of X.

(b) For $0 < p < \infty$, $c_{00}(\mathbb{N})$ is a subspace of $\ell^p(\mathbb{N})$.

(c) For $0 < p \leq q < \infty$, $\ell^p(\mathbb{N})$ is a subspace of $\ell^q(\mathbb{N})$

(d) For $0 < p < \infty$, $\ell^p(\mathbb{N})$ is a subspace of $c_0(\mathbb{N})$

(e) $c_0(\mathbb{N})$ is a subspace of $\ell^\infty(\mathbb{N})$.

(f) The space of real polynomials $\mathcal{P}([0,1]) = \bigcup_{n=1}^\infty \mathcal{P}_n([0,1])$ is a subspace of $\mathcal{C}([0,1])$.

(g) For $n \in \mathbb{N}$, the subset of $\mathcal{P}_n([0,1])$ consisting of all real polynomials precisely n, denote by $\mathcal{P}_n^*([0,1])$, is not a subspace of $\mathcal{P}_n([0,1])$.

(h) For two nonempty sets $Y \subset X$, $\mathbb{R}^Y$ is not a subspace of $\mathbb{R}^X$.

Question 4. *Unlike intersections, unions of vector spaces are not necessarily vector spaces unless special additional restrictions are imposed.*

(i) Give an example of a vector space X and two subspaces Y_1 and Y_2 of X such that $Y_1 \cup Y_2$ is not a subspace of X.

(ii) Let X be a vector space and $(Y_i)_{i \in I}$ be a collection of subspaces of X, where I is an arbitrary index set with the following property: for every $i, j \in I$, there exists a $k \in I$ such that $Y_i \cup Y_j \subset Y_k$. Show that

$$Y = \bigcup_{i \in I} Y_i$$

is a subspace of X.

(iii) Show that $Y = \bigcup_{0<p<\infty} \ell^p(\mathbb{N})$ is a subspace of $c_0(\mathbb{N})$, but $Y \subsetneq c_0(\mathbb{N})$.

Question 5. *Let X be a vector space and A be a nonempty subset of X. Show that*

$$\langle A \rangle = \left\{ \sum_{i=1}^n \lambda_i x_i : n \in \mathbb{N}, x_1, ..., x_n \in A \text{ distinct}, \lambda_1, ..., \lambda_n \in \mathbb{R} \right\}$$

Question 6. *Prove the following statements are true:*

(a) If X is a vector space and $x_0 \in X$, then

$$\langle \{x_0\} \rangle = \{\lambda x_0 : \lambda \in \mathbb{R}\}$$

(b) If X is a vector space, and $x_0, y_0 \in X$, then

$$\langle \{x_0, y_0\} \rangle = \{\lambda x_0 + \mu y_0 : \lambda, \mu \in \mathbb{R}\}$$

(c) If $X = \mathcal{C}([0,1])$ and $A = \{p_0, p_1, ..., p_n\}$ where $p_0(x) = 1$, $p_1(x) = x$,..., $p_n(x) = x^n$, then $\langle A \rangle = \mathcal{P}_n([0,1])$.

(d) Let $n \in \mathbb{N}$ and $X = \mathbb{R}^n$. For $0 \leq i \leq n$, denote

$$e_i = (0, 0, 0, ..., 0, \underbrace{1}_{ith\ position}, 0, ..., 0)$$

let $B = \{e_i\}_{i=1}^n$. Then $\langle B \rangle = \mathbb{R}^n$.

