Recall: Let X be a vector space.

(i) $X^{\#} = \{ f : X \to \mathbb{R}, f \text{ is linear} \}$

(ii) If $f \in X^{\#}$, then $\ker(f)$ is a subspace of codimension one or zero

(iii) If Y is a subspace of codimension one, $x_0 \in X \setminus Y$ and $r_0 \in \mathbb{R}$, there exists a unique $f \in X^{\#}$ such that

(a) $Y \subset \ker(f)$

(b) $f(x_0) = r_0$

If $r_0 \neq 0$, then $\ker(f) = Y$

(iv) If $f, g \in X^{\#}$, $\ker(g) \subset \ker(f)$ if and only if $\exists \lambda \in \mathbb{R}$ such that $f = \lambda g$.

Definition: Let X be a normed space. The dual of X is the following:

$$X^* = \{ f : X \to \mathbb{R}, f \text{ linear and bounded} \} = \mathcal{L}(X, \mathbb{R})$$

Remark: Because $\mathbb{R}$ is a Banach space, X^* with the operator norm of $\mathcal{L}(X, \mathbb{R})$ is a Banach space.

Kernels of Bounded Linear Spaces

Remark: Let X be a normed space and $f \in X^*$, then $\ker(f)$ is a closed subspace of codimension one or zero.

Proposition: Let X be a normed space and Y be a closed subspace of codimension one and $x_0 \in X \setminus Y$. Then there exists a unique $f \in X^*$ such that

(i) $\ker(f) = Y$

(ii) $f(x_0) = \mathrm{dist}(x_0, Y)$.

Additionally, $\|f\|_{X^*} = 1$.

Proof: By a previous proposition, there exists a unique $f \in X^\#$ such that $Y = \ker(f)$, and

$f(x_0) = \mathrm{dist}(x_0, Y) = \delta > 0$ (because Y is closed and $x_0 \notin Y$)

We need to show that $f \in X^*$ and $\|f\|_{X^*} = 1$.

Step 1: $\forall x \in X$, $|f(x)| \leq \|x\|$

Because Y is of codimension one, and $x_0 \notin Y$, then $\langle \{x_0\} \rangle$ and Y form a linear decomposition of X.

For $x \in X$ $\exists \lambda \in \mathbb{R}$ and $y \in Y$ such that $x = \lambda x_0 + y$.

Then

$$|f(x)| = |\lambda f(x_0) + f(y)| = |\lambda| \mathrm{dist}(x_0, Y) = \mathrm{dist}(\lambda x_0, Y)$$
$$\leq \|\lambda x_0 - (-y)\| = \|x\|.$$

Step 2: $\|f\|_{X^*} \geq 1$. Let $(y_n)_{n=1}^{\infty}$ be a sequence in Y such that $\|x_0 - y_n\| \to \mathrm{dist}(x_0, Y)$. Let $x_n = \dfrac{x_0 - y_n}{\|x_0 - y_n\|}$

Then $$\|f\|_{X^*} \geq |f(x_n)| = \frac{1}{\|x_0 - y_n\|} |f(x_0) - f(y_n)|$$
$$= \frac{\mathrm{dist}(x_0, Y)}{\|x_0 - y_n\|} \longrightarrow 1.$$

Corollary: Let X be a normed space and $f \in X^*$. Then $\forall x \in X$, $|f(x)| = \|f\|_{X^*} \operatorname{dist}(x, \ker(f))$.

Proof: If $x \in \ker(f)$, then $0 = 0$.

Otherwise, let $g \in X^*$ be such that $\ker(f) = \ker(g)$. and $g(x) = \operatorname{dist}(x, \ker(f))$ and recall $\|g\|_{X^*} = 1$. Then $\exists \lambda \in \mathbb{R}$ such that $f = \lambda g \Rightarrow \|f\| = \|\lambda g\| = |\lambda|$

$\Rightarrow |f(x)| = |\lambda g(x)| = \|f\| \operatorname{dist}(x, \ker(f))$.

Proposition: Let X be a normed space and $f \in X^\#$. The following are equivalent:

(i) $f \in X^*$

(ii) $\ker(f)$ is closed.

Proof: (ii) $\Rightarrow$ (i) If $\ker(f) = X$, then $f = 0_{X^*} \in X^*$.

Otherwise let $x_0 \in X \setminus \ker(f)$ and let $g \in X^*$ such that $\ker(g) = \ker(f)$ and $g(x_0) = \operatorname{dist}(x, \ker(f))$ and $\|g\|_{X^*} = 1$.

Then $\exists \lambda \in \mathbb{R}$ such that $f = \lambda g \Rightarrow f \in X^*$.

Automatic Continuity of Linear Operators on Finite-Dimensional Normed Spaces.

Lemma: Let X be a normed space and Y subspace of X that is a Banach space with the norm inherited for X. Then Y is a closed subspace of X.

Theorem: Let X be a finite-dimensional normed space

(i) X is a Banach space.

(ii) $X^{\#} = X^*$

Proof: By induction on $\dim(X) = n$.

- For $n=0$, then $X = \{0_X\}$ and $X^{\#} = \{0_{X^*}\}$ so the conclusion holds.
- Let $n \in \mathbb{N}$ and assume the conclusion holds for normed space Y with $\dim(Y) < n$. Let X be an n-dimensional normed space. We first show $f \in X^{\#} \Rightarrow f \in X^*$. If $f=0$, easy. Otherwise, $\ker(f)$ is a proper subspace of X $\Rightarrow \dim(\ker(f)) < n \Rightarrow \ker(f)$ is a Banach space $\Rightarrow \ker(f)$ is closed $\Rightarrow f$ is bounded.

We now prove that X is a Banach space. Fix a basis $(x_i)_{i=1}^n$ in X and its bounded coordinate functionals $(f_i)_{i=1}^n$. Take a Cauchy sequence $(y_m)_{m=1}^\infty$ in X. Then $\forall 1 \le i \le n$ $(f_i(y_m))_{m=1}^\infty$ is Cauchy

$(|f_i(y_m) - f_i(y_\ell)| \le \|f_i\| \|y_m - y_\ell\|)$

$\Rightarrow \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\forall 1 \le i \le n$,

$\lim_m f_i(y_m) = \lambda_i$. Let $y = \sum_{i=1}^n \lambda_i x_i$. Then

$$\|y_m - y\| = \left\| \sum_{i=1}^n f_i(y_m) x_i - \sum_{i=1}^n \lambda_i x_i \right\|$$
$$= \left\| \sum_{i=1}^n (f_i(y_m) - \lambda_i) x_i \right\| \le \sum_{i=1}^n \|x_i\| \cdot |f_i(y_m) - \lambda_i| \longrightarrow 0.$$

Corollary: Every finite dimensional subspace of a normed space is closed.

Automatic Continuity Theorem: Let X be a finite-dimensional normed space, Y normed space and $T: X \to Y$ linear. Then T is bounded.

Proof: Fix a Hamel basis $(x_i)_{i=1}^n$ of X with coordinate functionals $(f_i)_{i=1}^n$. For $x \in \mathbb{B}_X$

$$\|Tx\| = \left\| T\left(\sum_{i=1}^n f_i(x) x_i\right)\right\| = \left\| \sum_{i=1}^n f_i(x) Tx_i \right\|$$
$$\leq \sum_{i=1}^n |f_i(x)| \cdot \|Tx_i\| \leq \sum_{i=1}^n \|f_i\| \|x\| \|Tx_i\|$$
$$= \left(\sum_{i=1}^n \|f_i\| \|Tx_i\| \right) \|x\|$$

$\Rightarrow T$ is bounded and $\|T\| \leq \sum_{i=1}^n \|f_i\| \cdot \|Tx_i\|$.

Corollary: Let X be a finite-dimensional normed space. Then $X \simeq Y \Rightarrow \dim(X) = \dim(Y)$.

Proof: If $X \simeq Y \Rightarrow \exists$ algebraic isomorphism $T: X \to Y$ but T, T^{-1} are automatically continuous.

Corollary: If X is an n-dimensional normed space, then $X \simeq (\mathbb{R}^n, \|\cdot\|_2)$.

Corollary: If X is a finite-dimensional vector space, then all norms on X are equivalent, i.e. $id: (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is an isomorphism.

Recall: A metric space is compact if and only if every sequence in it has a convergent subsequence.

Theorem: For a normed space X, the following are equivalent:

(i) X is finite-dimensional

(ii) $\mathbb{B}_X = \{x \in X : \|x\| \le 1\}$ is compact.

Lemma (Riesz): Let X be a normed space and Y be a proper closed subspace of X. Then $\forall\, 0<\varepsilon<1$, $\exists\, x \in X \setminus Y$ such that

(i) $\|x\| = 1$

(ii) $\operatorname{dist}(x, Y) > 1-\varepsilon$.

Proof of Lemma: Bec. Y is a proper closed subspace, X/Y is a nonzero dimensional normed space.

Recall the quotient $Q: X \to X/Y$ satisfies $Q(\mathbb{B}_X^\circ) = \mathbb{B}_{X/Y}^\circ$

Fix $\varepsilon > 0$, take $x' \in \mathbb{B}_X^\circ$ ($\|x'\| < 1$) such that $\|Qx'\| > 1-\varepsilon$. Let $x = \frac{x'}{\|x'\|}$. Then $\operatorname{dist}(x, Y) = \|Qx\|$ $= \frac{1}{\|x'\|} \|Qx'\| > 1-\varepsilon$.

Proof of Theorem: Let X be a normed space. If X is finite-dimensional, then take an isomorphism $T: X \to (\mathbb{R}^n, \|\cdot\|_2)$. Then $T(\mathbb{B}_X)$ is a bounded closed subset of $\mathbb{R}^n$. By Heine-Borel, $T(\mathbb{B}_X^\circ)$ i compact $\Rightarrow \mathbb{B}_X = T^{-1}(T(\mathbb{B}_X))$ is compact.

Assume that X is infinite-dimensional. We will construct a sequence in $\mathbb{B}_X$, without a convergent subsequence.

Fix $x_1 \in \mathbb{B}_X$ with $\|x_1\| = 1$. Assume we have chosen $x_1, \ldots, x_n \in \mathbb{B}_X$ such that for $1 \leq i < j \leq n$, $\|x_i - x_j\| > \frac{1}{2}$. Let $F = \langle \{x_1, \ldots, x_n\} \rangle$ which is finite dimensional, and thus closed. By Riesz's Lemma, $\exists\, x_{n+1} \in X$ such that $\|x_{n+1}\| = 1$ and $\operatorname{dist}(x_{n+1}, F) > \frac{1}{2}$. In particular, for $1 \leq i \leq n$, $\|x_{n+1} - x_i\| \geq \operatorname{dist}(x_{n+1}, F) > \frac{1}{2}$. The sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence.

Duals of Classical Spaces

We will prove the following:

- $c_0^*(\mathbb{N}) \equiv \ell^1(\mathbb{N})$
- $(\ell^1(\mathbb{N}))^* \equiv \ell^\infty(\mathbb{N})$
- For $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $(\ell^p(\mathbb{N}))^* \equiv \ell^q(\mathbb{N})$

Theorem: Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $\Phi : \ell^q(\mathbb{N}) \to (\ell^p(\mathbb{N}))^*$ given by $(\Phi x)(y) = \sum_{n=1}^{\infty} x_i y_i$ where $x \in \ell^q(\mathbb{N})$, $y \in \ell^p(\mathbb{N})$ is an onto linear isometry.

Proof:

<u>Step 1</u>: Φ is well-defined, i.e. $\forall x \in \ell^q(\mathbb{N})$, $\Phi(x)$ is a bounded linear functional on $\ell^p(\mathbb{N})$.

By Hölder's, if $y \in \ell^p(\mathbb{N})$, then

$(\Phi x)(y) = \sum_{n=1}^{\infty} x_i y_i \leq \|x\|_q \|y\|_p$.

Φx is linear and Φ is bounded with $\|\Phi x\| \leq \|x\|_q$.

<u>Step 2</u>: Φ is linear, i.e. for $x, y \in \ell^q(\mathbb{N})$, $\lambda \in \mathbb{R}$

$\Phi(x + \lambda y) = \Phi x + \lambda \Phi y$

<u>Step 3</u>: Φ is an injective isometry

$\forall x, y \in \ell^q(\mathbb{N})$, $\|\Phi x\| = \|x\|_q$. We know $\|\Phi x\| \geq \|x\|_q$

we will show $\|\Phi x\| \geq \|x\|_q$. Recall bec. $\frac{1}{p} + \frac{1}{q} = 1$,

for $x \in \ell^q(\mathbb{N})$ $\exists y \in \ell^p(\mathbb{N})$ s.t.

(i) $\|y\|_p = 1$

(ii) $\sum_{i=1}^{\infty} x_i y_i = \|x\|_q$

$\Longrightarrow \|\Phi x\| \geq (\Phi x)(y) = \sum_{i=1}^{\infty} x_i y_i = \|x\|_q$.

<u>Step 4</u>: Φ is onto, i.e. $\forall f \in (\ell^p(\mathbb{N}))^*$ $\exists x \in \ell^q(\mathbb{N})$

such that $\Phi x = f$.