Question 1. *Let X be a vector space and Y be a subspace of X.*

(i) Prove that for every $x, z \in X$, $[x]_Y \tilde{+} [z]_Y = [x]_Y + [z]_Y$, i.e. addition on X/Y coincides with the Minkowski sum of sets.

(ii) Prove that for $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in X$ $\lambda \tilde{\cdot} [x]_Y = \lambda [x]_Y$, i.e. when $\lambda \neq 0$, multiplication is the same.

(iii) Prove that if $Y \neq \{0_X\}$, for $x \in X$, $0 \tilde{\cdot} [x]_Y \neq 0[x]_Y$. That is, for $\lambda = 0$, multiplication fails.

Question 2. *If $X = \mathcal{C}([0,1])$, show that $A = \{p_0, p_1, ..., p_n, ...\}$ where $p_0(x) = 1$, and for $n \in \mathbb{N}$, $p_n(x) = x^n$ for $x \in [0,1]$ is linearly independent.*

Question 3. *Let $X = \mathcal{C}([0,1])$ and $A = \{p_0, f, g\}$, where $p_0(x) = 1$, $f(x) = \sin^2(x)$ and $g(x) = \cos^2(x)$ for $x \in [0,1]$. Show that A is linearly dependent.*

Question 4. *Let X be a vector space and A be a countable (finite or infinite) subset of X. Prove by induction that there exists a linearly independent subset C of A such that $\langle A \rangle = \langle C \rangle$.*

Question 5. *Let X be a vector space and $A = \{x_1, x_2, ..., x_n\}$ be a linearly independent subset of X with $x_1, ..., x_n$ distinct.*

(i) For $1 \leq k \leq n$, let $s_k = \sum_{i=1}^{k} x_i$. Prove that $S = \{s_1, ..., s_n\}$ is linearly independent and $\langle A \rangle = \langle S \rangle$.

(ii) Let $d_1 = x_1$ and for $2 \leq k \leq n$ let $d_k = x_k - x_{k-1}$. Prove that $D = \{d_1, ..., d_n\}$ is linearly independent and $\langle A \rangle = \langle D \rangle$.

Question 6. *Let X be a vector space and A be a subset of X. Prove that there exists a linearly independent subset $C \subset A$ such that $\langle A \rangle = \langle C \rangle$.*

Question 7. *Let X be a vector space.*

(i) If A is a Hamel basis of X and $B \subset A$, put $Y = \langle A \rangle$ and $Z = \langle B \setminus A \rangle$, show that Y and Z are a linear composition of X.

(ii) Let Y and Z be a linear decomposition of X. Let A be a Hamel basis of Y and B be a Hamel basis of Z. Prove that $A \cup B$ is a Hamel basis of X.

Question 8. *Let X be a vector space and $A \subset X$. Use Zorn's lemma to prove that there exists a linearly independent $A' \subset A$ such that $\langle A' \rangle = \langle A \rangle$.*