Question 1. *Let X be a vector space, Y be a subspace of X, and $n \in \mathbb{N}$. The following are equivalent:*

(a) Y is of codimension n.

(b) There exists a linearly independent subset A of X with n elements such that $\langle A \rangle$ and Y form a linear decomposition of X.

Question 2. *Let X be a vector space and Y and Z be subspaces of X. Assume that for some $n \in \mathbb{N}_0$, $\operatorname{codim}(Y) = \dim(Z) = n$ and $Y \subset Z$, show that $Y = Z$.*

Question 3. *Let X be a vector space and Y be a subspace of X. Let $Q : X \to X/Y$ denote the quotient map. Prove that there exists a subspace Z of X that satisfies both following properties:*

(a) Y and Z is a linear decomposition of X.

(b) $Q|_Z : Z \to X/Y$ is an algebraic isomorphism.

If X is finite-dimensional, deduce that

$$\dim(X) = \dim(Y) + \dim(X/Y) = \dim(Y) + \operatorname{codim}(Y)$$

Question 4 (Rank-Nullity Theorem)**.** *Let X and Y be vector spaces and $T : X \to Y$ be a linear operator. Show that*

$$\dim(\ker(T)) = \dim(\operatorname{Im}(T)) = \dim(X)$$

Question 5. *Give an example of a vector space X, and linear functionals $g, f_1, f_2, \ldots$ on X such that $\bigcap_{n=1}^{\infty} \ker(f_n) \subset \ker(g)$ and $g \notin \langle \{f_1, f_2, \ldots\} \rangle$*

Question 6. *Let X be a vector space. For every subspace F of $X^{\#}$, define $F_0 = \bigcap_{f \in F} \ker(f)$, which is a subspace of X, called the* pre-annihilator *of F.*

(i) Let $n \in \mathbb{N}$, F be an n-dimensional subspace of $X^{\sharp}$, and $(f_i)_{i=1}^{n}$ be a Hamel basis of F.

(a) Show that $F_0 = \bigcap_{i=1}^{n} \ker(f_i)$.

(b) Show that F_0 is of codimension n.

(ii) Let

$$\mathcal{FD}(X^{\#}) = \{F \text{ is a finite-dimensional subspace of } X^{\#}\}$$
$$\mathcal{FC}(X) = \{Y \text{ is a finite-codimensional subspace of } X\}$$

Show that $F \mapsto F_0$ is a bijection between $\mathcal{FD}(X^{\#})$ and $\mathcal{FC}(X)$.

Question 7. *Let X and Y be vector spaces and $T : X \to Y$ be a linear operator.*

(i) If A is a convex subset of X, show that $T[A]$ is a convex subset of Y.

(ii) If B is a convex subset of Y, show that $T^{-1}[B]$ is a convex subset of X.

Copyright 2024 © Joe Tran