Material:

(1) Review of Infinite Dimensional Linear Algebra (Chapter 1)

(2) Normed spaces and Banach spaces (Chapter 2 - 7)

(3) Topological Vector Spaces and Weak Topologies.

(4) The Riesz Representation Theorem for Continuous Functions (Chapter 12)

(5) Compact operators and Spectral Theory (Chapter 14, 15, 24)

Infinite - Dimensional Linear Algebra

Def: A set V endowed with two operations

(1) $+ : V \times V \to V$ (vector addition)

(2) $\cdot : \mathbb{R} \times V \to V$ (scalar multiplication)

is called a vector space over $\mathbb{K}$ if

(1) $\forall x, y, z \in V$, $(x+y)+z = x+(y+z)$

(2) $\exists 0_V \in V$ s.t $\forall x \in V$, $x + 0_V = x$.

(3) $\forall x, y \in V$, $x+y = y+x$

(4) $\forall x \in V$, $\exists -x \in V$ s.t. $x+(-x) = 0_V$

(5) $\forall \alpha \in \mathbb{K}$, $\forall x, y \in V$, $\alpha(x+y) = \alpha x + \alpha y$

(6) $\forall \alpha, \beta \in \mathbb{K}$, $\forall x \in V$, $(\alpha+\beta)x = \alpha x + \beta x$

(7) $\forall \alpha, \beta \in \mathbb{K}$, $\forall x \in V$, $(\alpha\beta)x = \alpha(\beta x)$

(8) $\forall x \in V$, $1 \cdot x = x$

Example: If V is a vector space, then for $x, y \in V$

Examples:

- $\mathbb{R}$ with usual addition and scalar multiplication.
- $\mathbb{R}^n = \{x = (x_i)_{i=1}^n : x_i \in \mathbb{R}\}$ with coordinate wise addition and scalar multiplication.
- $\mathbb{R}^{\mathbb{N}} = \{x = (x_n)_{n=1}^{\infty} : x_n \in \mathbb{R} : n \in \mathbb{N}\}$ with coordinatewise addition and scalar multiplication.
- $C_{00}(\mathbb{R}) = \{x \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N} \text{ such that } x_n = 0 \ \forall n \geq N\}$
- $\ell^{\infty}(\mathbb{R}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sup_{n \in \mathbb{N}} |x_n| < \infty\}$
- For $1 \leq p < \infty$, $\ell^p(\mathbb{R}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$
- $C([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$ with pointwise addition and scalar multiplication.
- $C^1([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuously differentiable}\}$
- $\mathcal{P}_n([0,1]) = \{p \in C([0,1]) : p \text{ is a polynomial of degree at most } n\}$
- $\mathcal{P}([0,1]) = \{p \in C([0,1]) : p \text{ is a polynomial}\}$

Remark: For $1 \leq p < \infty$ and $x, y \in \ell^p(\mathbb{R})$ then $x + y \in \ell^p(\mathbb{R})$.

Then for $n \in \mathbb{N}$,

$$\begin{aligned} |x_n + y_n|^p &\leq (|x_n| + |y_i|)^p \leq [2 \max\{|x_n|, |y_n|\}]^p \\ &= 2^p \max\{|x_n|^p, |y_n|^p\} \\ &\leq 2^p (|x_n|^p + |y_n|^p) \end{aligned}$$

Then $\sum_{n=1}^{\infty} |x_n + y_n|^p \leq 2^p \left(\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p \right)$

Subspaces

Def: Let V be a vector space. A nonempty subset W of V is called a subspace of V if

(1) For $x, y \in W$, $x + y \in W$ (closed under addition)

(2) For $x \in W$, $\alpha \in \mathbb{K}$, $\alpha x \in W$ (closed under scalar multiplication)

Remark: If V is a vector space and W is a subspace of V, then $0_V \in W$.

Examples:

- If V is a vector space, $\{0_V\}$ and V are subspaces.
- $c_{00}(\mathbb{K})$ is a subspace of $\ell^\infty(\mathbb{K})$
- $\mathcal{P}_n([0,1])$ is a subspace of $C([0,1])$
- In $\mathcal{P}_n([0,1])$, the subset of all polynomials of degree exactly n is not a subspace of $\mathcal{P}_n([0,1])$.

Proposition: If V is a vector space and $(W_i)_{i\in I}$ is a collection of subspaces of V, then $W = \bigcup_{i\in I} W_i$ is a subspace of V.

Def: Let V be a vector space and $E \subset V$. Then the linear span of E is the set

$$\mathrm{span}(E) = \bigcap \{W : W \text{ is a subspace of } V \text{ and } E \subset W\}$$

Proposition: Let V be a vector space and $E \subset V$. Then

(1) $\mathrm{span}(E)$ is a subspace of V and $E \subset \mathrm{span}(E)$

(2) If W is a subspace of V such that $E \subset W$, then $\mathrm{span}(E) \subset W$.

Exercise: If V is a vector space, then $\mathrm{span}(\emptyset) = \{0_V\}$

Notation: If V is a vector space,

- $x+y+z$ means $(x+y)+z$ or $x+(y+z)$.
- For $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n \in V$, $\sum_{i=1}^{n} x_i$ is defined inductively.
- For $I = \{i_1, i_2, \dots, i_n\}$ with $i_1, \dots, i_n$ are pairwise different, and $(x_i)_{i\in I}$ in V $\sum_{i\in I} x_i = \sum_{k=1}^{n} x_{i_k}$
- For $x_1, \dots, x_n \in V$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$, $x = \sum_{i=1}^{n} \alpha_i x_i$ is called a linear combination of $x_1, \dots, x_n$.

Remark: If V is a vector space and $(x_n)_{n=1}^{\infty}$ is a sequence in V, $\sum_{n=1}^{\infty} x_n$ is not defined

Remark: If V is a vector space and W is a subspace of V, and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, then

$\sum_{i=1}^{n} \alpha_i x_i \in W$ (by induction on n)

Theorem: If V is a vector space and $E \subset V$ nonempty, then

$$\text{span}(E) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i, \dots, \alpha_n \in \mathbb{R},\ x_1, \dots, x_n \in V \right\}$$

Notes:

- If $Z = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i, \dots, \alpha_n \in \mathbb{R},\ x_1, \dots, x_n \in V \right\}$, then because span$(E)$ is a subspace and $E \subset \text{span}(E)$, then by the above remark, $Z \subset \text{span}(E)$.
- For $\text{span}(E) \subset Z$, we show Z is a subspace of V and $E \subset Z$.

With this, by previous proposition, $\text{span}(E) \subset Z$

Examples:

- If V is a vector space and $x_0 \in V$, $\text{span}(\{x_0\}) = \{\alpha x_0 : \alpha \in \mathbb{R}\}$

  If $x_1, x_2 \in V$, $\text{span}(\{x_0, x_1\}) = \{\alpha_1 x_1 + \alpha_2 x_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$
- If $X = C([0,1])$, $E = \{1, t, t^2, \dots, t^n\}$, then $\text{span}(\{1, \dots, t^n\})$

  $= \mathcal{P}_n([0,1])$
- Let $n \in \mathbb{N}$ and $X = \mathbb{R}^n$. For $1 \le i \le n$, denote

  $e_i = (0, 0, \dots, \underset{\uparrow \text{ ith}}{1}, 0, \dots)$, let $B = \{e_i\}_{i=1}^{n}$. Then $\text{span}(B) = \mathbb{R}^n$

  B is called the standard basis.

Linear Decomposition of a Vector Space

You know in $\mathbb{R}^n$ if Y is a subspace, then $\mathbb{R}^n = Y + Y^{\perp}$

Def: Let V be a vector space

(1) If A, B are subsets of V, then the Minkowski sum of A and B is the set

$A+B=\{a+b : a\in A,\ b\in B\}$

(2) If $x\in V$ and $A\subset V$, then $x+A=\{x+a : a\in A\}$

(3) If $\alpha\in\mathbb{R}$ and $A\subset V$, $\alpha A=\{\alpha\cdot a : a\in A\}$

<u>Def:</u> If V is a vector space. Two subspaces W, U of V form a linear decomposition if

(1) $W\cap U=\{0_V\}$

(2) $W\cup U=V$.

<u>Example:</u> In $\mathbb{R}^2$, $\{(x,0) : x\in\mathbb{R}\}$, $\{(0,y) : y\in\mathbb{R}\}$

<u>Proposition:</u> Let V be a vector space, W, U subspace of V. The following are equivalent

(1) W, U form a linear decomposition of V.

(2) For all $x\in V$, there exists a unique $w\in W$ and $u\in U$ such that $x=w+u$.

<u>Proof:</u> (1) $\Rightarrow$ (2): Let $x\in V$. Because $V=W+U$, there exists $w\in W$ and $u\in U$ such that $x=w+u$. To show uniqueness let $w'\in W$, $u'\in U$ such that $x=w'+u'$. Then

$x=w+u=w'+u'$ so $w-w'=u-u'\in W\cap U=\{0_V\}$

so $w-w'=0_V \Rightarrow w=w'$ Similarly for $u-u'$

(2) $\Rightarrow$ (1): For any $x\in V$, there exists $w\in W$, $u\in U$, such that $x=w+u$, then $W+U=X$

Next we show $W\cap U=\{0_V\}$. Take $x\in W\cap U$. By assumption, there are unique $w\in W$ and $u\in U$ such that $x=w+u$

At the same time