Recall:

- If X is an inner product space, then
$$\|x\| = \sqrt{\langle x, x\rangle}$$
defines a norm on X.
- $\forall x, y \in X$, $|\langle x,y\rangle| \le \|x\|\cdot\|y\|$ (Cauchy-Schwarz)
- $\forall x, y \in X$, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Parallelogram)
- $x, y \in X$ are orthogonal if $\langle x, y\rangle = 0$.
- If $x, y \in X$ are orthogonal, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ (Pythagorean)
- A complete inner product space is called a Hilbert space, $\mathcal{H}$.
  - $(\mathbb{R}^n, \|\cdot\|_2)$ and $(\ell^2(\mathbb{N}), \|\cdot\|_2)$ are Hilbert spaces.
  - $(C[0,1], \|\cdot\|_2)$ is not a Hilbert space.

Orthogonal Projections

Def: Let $\mathcal{H}$ be a Hilbert space and $A \subset \mathcal{H}$. The orthogonal set of A is
$$A^\perp = \{x \in \mathcal{H} : \forall y \in A.\ \langle x, y\rangle = 0\}$$

Prop: Let $\mathcal{H}$ be a Hilbert space and $A \subset \mathcal{H}$, then $A^\perp$ is a closed subspace of $\mathcal{H}$.

Proof: If $x, z \in A^\perp$ and $\lambda \in \mathbb{R}$, we will show that $x + \lambda z \in A^\perp$.
Fix $y \in A$. Then $\langle x+\lambda z, y\rangle = \langle x, y\rangle + \lambda\langle z, y\rangle = 0 + \lambda\cdot 0 = 0$
Therefore, $x + \lambda z \in A^\perp$.

If $x \in \overline{A^\perp}$, take a sequence $(x_n)_{n=1}^\infty$ in $A^\perp$ such that $x_n \to x$ as $n \to \infty$. Then, for some $y \in A$,
$$\langle x, y\rangle = \langle \lim_{n\to\infty} x_n, y\rangle = \lim_{n\to\infty} \langle x_n, y\rangle = \lim_{n\to\infty} 0 = 0$$
$\Rightarrow x \in \overline{A^\perp}$ ∎

Recall: If X is a vector space, then two subspaces Y, Z of X form a linear decomposition of X if

(a) $Y \cap Z = \{0_X\}$

(b) $Y + Z = X$

Prop: If X is a vector space and Y and Z form a linear decomposition of X, then there exists a unique linear projection $P: X \to X$ such that $P(X) = Y$ and $\ker(P) = Z$.

Def: Let X be a Banach space and Y and Z be subspaces of X. Then we say X is a direct sum of Y and Z and write $X = Y \oplus Z$, if

(a) Y and Z form a linear decomposition of X,

(b) Y and Z are closed subspaces of X.

Theorem: (Existence of Orthogonal Projections) Let $\mathcal{H}$ be a Hilbert space and Y closed subspace of $\mathcal{H}$. The following hold:

(i) $\mathcal{H} = Y \oplus Y^{\perp}$

(ii) There exists a unique linear projection $P: \mathcal{H} \to \mathcal{H}$ such that $P[X] = Y$ and $\ker(P) = Y^{\perp}$. Furthermore, P is bounded and $\|P\| = 1$ unless Y is zero-dimensional.

This P is called the orthogonal projection onto Y.

Proof:

Step 1: Let $\mathcal{H}$ be a Hilbert space and F be a closed and convex subset of $\mathcal{H}$. Then $\forall x \in \mathcal{H}$, there exists a unique $y_0 \in F$ such that $\|x - y_0\| = \operatorname{dist}(x, F)$

Fix $(y_n)_{n=1}^{\infty}$ in F such that $\|x - y_n\| \to \operatorname{dist}(x, F) = \delta > 0$. We will show that $(y_n)_{n=1}^{\infty}$ is Cauchy. Note that for every $n, m \in \mathbb{N}$, $\frac{1}{2}(y_n + y_m) \in F$, then $\|x - \frac{1}{2}(y_n + y_m)\| \geq \delta$ and so $\frac{1}{4}\|2x - (y_n + y_m)\|^2 \geq \delta^2$. $(*)$

Fix $n, m \in \mathbb{N}$. Then we have, by the parallelogram law,

$$2\|x - y_n\|^2 + 2\|y_m - x\|^2 = \|y_n - y_m\|^2 + \|2x - (y_n + y_m)\|^2.$$

$$\geq \|y_n - y_m\|^2 + 4\delta^2$$

$$\Rightarrow \|y_n - y_m\|^2 \leq 2\underbrace{\|x - y_n\|^2}_{\to \delta^2} + 2\underbrace{\|x - y_m\|^2}_{\to \delta^2} - 4\delta^2 \to 0$$

as $n, m \to \infty$. Therefore, $(y_n)_{n=1}^{\infty}$ is Cauchy. Because $\mathcal{H}$ is a Hilbert space, then there exists $y_0 \in \mathcal{H}$ such that $y_n \to y_0$ as $n \to \infty$, and because F is closed, $y_0 \in F$. Then

$$\|x - y_0\| = \lim_{n\to\infty} \|x - y_n\| = \text{dist}(x, F).$$

Next, let $y' \in F$ such that $\|x - y'\| = \text{dist}(x, F)$, we will show $y_0 = y'$. By the parallelogram law,

$$\begin{aligned} 2\|x-y'\|^2 + 2\|y_0 - x\|^2 &= \|y_0 - y'\|^2 + \|2x - (y' + y_0)\|^2 \\ &= \|y_0 - y'\|^2 + 4\|x - \tfrac{1}{2}(y' + y_0)\|^2 \\ &\geq \|y_0 - y'\|^2 + 4\delta^2 \end{aligned}$$

$$\Rightarrow \|y_0 - y'\|^2 \leq 2\delta^2 + 2\delta^2 - 4\delta^2 = 0 \Rightarrow y_0 = y'. \qquad \blacksquare$$

Step 2: If $\mathcal{H}$ is a Hilbert space and Y is a closed subspace of $\mathcal{H}$ and $x \in \mathcal{H}$. For $y \in Y$, the following are equivalent.

(a) $\|x - y\| = \text{dist}(x, Y)$

(b) $x - y \in Y^{\perp}$

Assume (b). We will show for $z \in Y$ arbitrary, $\|x - y\| \leq \|x - z\|$. Because $x - y \in Y^{\perp}$,

$$\begin{aligned} \|x - z\|^2 = \|(x-y) - (z-y)\|^2 &= \|x-y\|^2 + \|z - y\|^2 \\ &\geq \|x-y\|^2 \end{aligned}$$

Assume (a): For all $z \in Y$, $\|x - y\| \geq \|x - z\|$. Fix $w \in Y$ arbitrary. We will show that $\langle x - y, w\rangle = 0$. For $\lambda \in \mathbb{R}$, observe that

$$\begin{aligned} \|x-y\|^2 \leq \|x - (y + \lambda w)\|^2 &= \|(x-y) + \lambda w\|^2 \\ &= \|x-y\|^2 + \|\lambda w\|^2 + 2\langle x-y, \lambda w\rangle \\ &= \|x-y\|^2 + \lambda^2\|w\|^2 + 2\lambda\langle x-y, w\rangle \end{aligned}$$

Then $\forall \lambda \in \mathbb{R}$, $\lambda^2\|w\|^2 + 2\lambda\langle x-y, w\rangle \geq 0$ which is a quadratic function that is nonnegative, so by the discriminant, so $2\langle x-y, w\rangle = 0 \Rightarrow \langle x-y, w\rangle = 0 \Rightarrow x - y \in Y^{\perp}$ $\blacksquare$

Step 3 (Main Proof): Let $\mathcal{H}$ be a Hilbert space and Y be a closed subspace of $\mathcal{H}$.

(i) $\mathcal{H} = Y \oplus Y^{\perp}$. Because Y and $Y^{\perp}$ are closed, we check that Y and $Y^{\perp}$ form a linear decomposition of $\mathcal{H}$.

Take $x \in Y \cap Y^{\perp}$. Then $x \perp x \Rightarrow \|x\|^2 = \langle x, x \rangle = 0$.

$\Rightarrow x = 0_{\mathcal{H}}$.

Let $x \in \mathcal{H}$ and let $y \in Y$ such that $\|x-y\| = \text{dist}(x, Y)$. By Step 2, because $\|x-y\| = \text{dist}(x, F)$, $x-y \in Y^{\perp}$, then $x = y + (x-y)$.

(ii) Uniqueness of linear projection $P : \mathcal{H} \to \mathcal{H}$ such that $P[\mathcal{H}] = Y$ and $\ker(P) = Y^{\perp}$. Because Y and $Y^{\perp}$ form a linear decomposition of $\mathcal{H}$, such P exists and is unique. We will show $\|P\| = 1$. For $x \in \mathcal{H}$, then

$$\|x\|^2 = \|Px + (x - Px)\|^2 = \|Px\|^2 + \|x - Px\|^2 \geq \|Px\|^2$$

$\Rightarrow \|Px\|^2 \leq \|x\|^2 \Rightarrow P$ is bounded with $\|P\| \leq 1$.

To show $\|P\| \geq 1$, take $y \in Y$ with $\|y\| = 1$. Then $\|P\| \geq \|Py\| = \|y\| = 1$ (bec. $y \in P[\mathcal{H}]$). $\blacksquare$

Remarks: If $\mathcal{H}$ is a Hilbert space and Y is a closed subspace of $\mathcal{H}$, denote P the orthogonal projection onto Y.

(i) $\forall x \in \mathcal{H}$. Px is the closest vector to x in Y, i.e.

$\|x - Px\| = \text{dist}(x, Y)$.

(ii) $\forall x \in \mathcal{H}$ and $y \in Y$, $\langle y, x \rangle = \langle y, Px \rangle$. Indeed,

$$\begin{aligned} \langle y, x \rangle &= \langle y, Px + (x - Px) \rangle \\ &= \langle y, Px \rangle + \overset{0}{\cancel{\langle y, \underbrace{x - Px}_{\in \ker(P)} \rangle}} \\ &= \langle y, Px \rangle \end{aligned}$$

The Riesz Representation Theorem for Hilbert Spaces

Theorem: Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{H} \equiv \mathcal{H}^*$. More precisely, the map $\phi : \mathcal{H} \to \mathcal{H}^*$ given by $(\phi x)(y) = \langle x, y \rangle$ for $x \in \mathcal{H}$, $y \in \mathcal{H}$, is an onto linear isometry.

Remark: If $\mathcal{H}$ is a Hilbert space. This can be identified with a bounded linear functional on $\mathcal{H}$ such that $(\phi x)(y) = \langle x, y \rangle$. Furthermore, for every $f \in \mathcal{H}^*$ is of the above form.

Remark: $\ell^2(\mathbb{N})$ is a Hilbert space and $\ell^2(\mathbb{N}) \equiv \ell^2(\mathbb{N})^*$

Proof: For $x \in \mathcal{H}$. ϕx is well-defined because for $y \in \mathcal{H}$,

$|(\phi x)(y)| = |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$. $\Rightarrow \phi x \in \mathcal{H}^*$ and $\|\phi x\| \leq \|x\|$.

We also show for $x \in \mathcal{H}$, $\|\phi x\| \geq \|x\|$.

If $x = 0_{\mathcal{H}}$, then this is trivial. Otherwise, $\|\phi x\| \geq \|(\phi x)(\|x\|^{-1} x)\|$

$= \langle x, \|x\|^{-1} x \rangle = \frac{1}{\|x\|} \langle x, x \rangle = \|x\|$

We will show ϕ is onto. Fix $f \in \mathcal{H}^*$. We seek $x \in \mathcal{H}$ such that $\phi x = f$. If $f = 0_{\mathcal{H}^*}$, easy. Otherwise, if f is a nonzero bounded linear functional, then $\ker(f)$ is a closed subspace of codimension 1.

By a previous proposition, $\mathcal{H} = Y \oplus Y^\perp$ and because $Y \subsetneq \mathcal{H}$, then $Y^\perp \neq \{0_{\mathcal{H}}\}$. Take $x_0 \in Y^\perp$ with $\|x_0\| = 1$. Define $x = f(x_0) x_0$.

We show $f = \phi x$. Recall f is the unique linear functional on $\mathcal{H}$ such that $Y \subset \ker(f)$ and it has value $f(x_0)$ at x_0. Observe that $Y \subset \ker(\phi x)$ and $(\phi x)(x_0) = f(x_0)$. If $y \in Y$, then

$$(\phi x)(y) = \langle x, y \rangle = \langle f(x_0) x_0, y \rangle = 0.$$

Also,

$$(\phi x)(x_0) = \langle x, x_0 \rangle = \langle f(x_0) x_0, x_0 \rangle = f(x_0) \|x_0\|^2 = f(x_0). \quad \blacksquare$$

Orthonormal Sets

Def: Let $\mathcal{H}$ be a Hilbert space. A subspace O of $\mathcal{H}$ is called an orthonormal set if it satisfies the following:

(i) $\forall x \in O$, $\|x\| = 1$

(ii) $\forall x \neq y \in O$, $\langle x, y \rangle = 0$.

Remark: If $\mathcal{H}$ is a Hilbert space and O is an orthonormal set. Then $\forall x_1, \ldots, x_n \in O$ (distinct) and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$,

$$\|\sum_{i=1}^{n} \lambda_i x_i\|^2 = \left(\sum_{i=1}^{n} \lambda_i^2\right)^{\frac{1}{2}}$$

In particular, O is linearly independent.

Examples:

- In any Hilbert space $\emptyset$ is orthonormal.
- In $\mathbb{R}^n$ with standard inner product, $\{e_1, \dots, e_n\}$ is orthonormal.
- In $\ell^2(\mathbb{N})$ with standard inner product, $\{e_n\}_{n=1}^{\infty}$ of $C_{00}(\mathbb{N})$ is an orthonormal set.
- In $C[0,1]$ with the standard inner product, $\{p_0, p_1, \dots\}$ with $p_0(x) = 1$, $p_1(x) = x$, ... is linearly independent, but is not orthonormal.