Proposition: Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables. Let $X(x) = \lim_{n\to\infty} X_n(x)$ for $x \in \Omega$, and assume that the limit exists $\forall x$. Then X is a RV.

Proof: Fix $t \in \mathbb{R}$, and consider $\{x : X(x) \leq t\}$. That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $X_n(x) \leq t+\varepsilon$ $\forall n \geq N$. Then take $\varepsilon = \frac{1}{m}$. Then $\forall m \in \mathbb{N}, \exists N \in \mathbb{N}$ such that $X_n(x) \leq t + \frac{1}{m}$. In particular,

$$\bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \{X_n \leq t + \tfrac{1}{m} \ \forall n \geq N\} = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x : X_n(x) \leq t + \tfrac{1}{m}\}.$$

Because each X_n is a random variable, the set $\{x : X_n(x) \leq t + \frac{1}{m}\} \in \mathcal{F}$, which would then show that $\{x : X(x) \leq t\} = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x : X_n(x) \leq t + \frac{1}{m}\}$ is in $\mathcal{F}$, so X is a RV.

Remark: The proof above holds if $(X_n)_{n=1}^{\infty}$ is a sequence of RV, then $\limsup_{n\to\infty} X_n$ is a RV.

Proposition: Let X be a RV, let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $h(X)$ is a RV.

Proof: Let X be a RV on $(\Omega, \mathcal{F}, P)$. If V is an open subset of V, because h is a continuous function, then $h^{-1}(V)$ is also open.

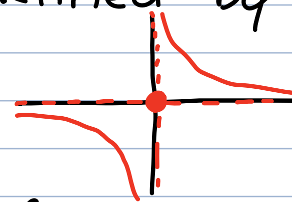
By Lemma 3.1, $h^{-1}(V)$ is a Borel set for every Borel set V. In particular, $h^{-1}((-\infty, t])$ is a Borel set for every $t \in \mathbb{R}$. Then, $(h(X))^{-1}((-\infty, t])$

$= X^{-1}(h^{-1}((-\infty,t]))$, and because $h^{-1}((-\infty,t])$ is a B-set, it follows that X^{-1} of a B-set is a B-set, so $X^{-1}(h^{-1}((-\infty,t])) \in \mathcal{F}$, so $h(X)$ is a RV.

More generally, it suffices to assume that h is a B-measurable function (Borel measurable), that is $h^{-1}(V) \in \mathcal{B}(\mathbb{R})$ $\forall\, V \in \mathcal{B}(\mathbb{R})$.

Example: Let $h: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$h(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



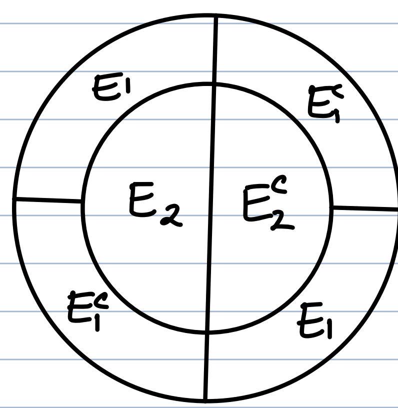
Then h is B-measurable. So if X is a RV s.t. $X(x) \neq 0$ $\forall x$, then $h(X) = \frac{1}{X}$ is a RV.

Example: Let $\Omega_* = \{H,T\}^{\mathbb{N}} = \{x = (x_n)_{n=1}^{\infty} : x_n \in \{H,T\}\}$.

let $E_k = \{x : x_k = H\}$, let $\mathcal{F}_4 = \sigma(E_1, E_2, E_3, E_4)$,

let $X_k(x) = \begin{cases} 1 & \text{if } x_k = H \\ 0 & \text{if } x_k = T \end{cases} = \mathbb{1}_{E_k}(x)$. Then X_3 is a RV on $(\Omega_*, \mathcal{F}_4, P)$ because $E_3 \in \mathcal{F}_4$. However, X_5 is not a RV bec. $E_5 \notin \mathcal{F}_4$.

Generally, for $n > N$, X_n is not measurable with respect to $\sigma(E_1, \ldots, E_N)$.



Independence

Definition: Let $(\Omega, \mathcal{F}, P)$ be a PS.

(i) Two events A, B are independent if

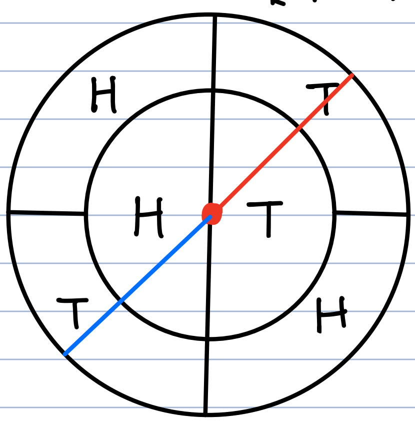
$$P(A \cap B) = P(A)P(B).$$

(ii) A collection $(A_i)_{i\in I}$ is independent if $\forall F \subset I$ finite,

$$P\left(\bigcap_{i\in F} A_i\right) = \prod_{i\in F} P(A_i)$$

More generally, we could reindex so that for every finite $F = \{1, 2, \dots, n\} = \{i_1, \dots, i_n\}$, then

$$P\left(\bigcap_{k=1}^{n} A_{i_k}\right) = \prod_{k=1}^{n} P(A_{i_k}).$$



$[P(T\cap T) = P(T)P(T)]$

$[P(H\cap T) = P(H)P(T)]$

(iii) Two collections $\mathcal{A}, \mathcal{B}$ are independent if A and B are independent for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(iv) Two RVs X, Y are independent if

$$P(X\in A, Y\in B) = P(X\in A)\,P(Y\in B)$$

$\forall X, Y \in \mathcal{B}(\mathbb{R})$.

Equivalently, the σ-algebras $\{X^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\}$

and $\{Y^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$ are independent. Thus,

$$P(X^{-1}(A) \cap Y^{-1}(B)) = P(X^{-1}(A))P(Y^{-1}(B))$$

(v) A collection of RVs $(X_i)_{i \in I}$ are independent if $\forall F \subset I$ finite,

$$P(X_i \in A_i \ \forall i \in F) = \prod_{i \in F} P(X_i \in A_i).$$

Proposition: Let X, Y be ind. RVs on $(\Omega, \mathcal{F}, P)$ PS. Let $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}$-measurable functions. Then $f(X)$ and $g(Y)$ are ind RVs.

Proof: (see notes)

Proposition: Let X, Y be RVs on $(\Omega, \mathcal{F}, P)$ PS. TFAE:

(a) X and Y are independent.

(b) $\forall s, t \in \mathbb{R}$, $P(X \le s, Y \le t) = P(X \le s)P(Y \le t)$. (*)

Proof: (a) $\Rightarrow$ (b) easy; by definition.

Claim: Let P; Q be prob. measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Assume $P((-\infty, t]) = Q((-\infty, t])$ $\forall t \in \mathbb{R}$. Then $P = Q$. i.e. $P(A) = Q(A)$ $\forall A \in \mathcal{B}(\mathbb{R})$.

(b) $\Rightarrow$ (a) Assume that (b) holds. Fix $t \in \mathbb{R}$. Define $\mu(A) = P(X \in A)$. Then μ is a prob. measure on $\mathcal{B}(\mathbb{R})$. Define

$$P_t(A) = \frac{P(X \in A, Y \le t)}{P(Y \le t)}$$

for $A \in \mathcal{B}(\mathbb{R})$. Then $P_t((-\infty, s]) = \dfrac{P(X \in (-\infty, s], Y \le t)}{P(Y \le t)}$

$\overset{(b)}{=} P(X \in (-\infty, s])$ $(\forall s \in \mathbb{R}.)$ $= \mu((-\infty, s])$. Thus,

$P_t = \mu$, i.e. $\forall A \in \mathcal{B}(\mathbb{R})$, $P_t(A) = \mu(A)$. Moreover,

$$P(X \in A, Y \le t) = P(X \in A)\, P(Y \le t)$$

for all $A \in \mathcal{B}(\mathbb{R})$ and $t \in \mathbb{R}$. Now fix $B \in \mathcal{B}(\mathbb{R})$

Use same argument $P(X \le s, Y \in B) = P(X \le s) P(Y \in B)$.