Continuity of Probabilities

Theorem (Monotone Convergence Theorem For Probability Measures):

Let $\{A_n\}_{n=1}^{\infty}$ be a collection in the probability space $(\Omega, \mathcal{F}, P)$.

(i) If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$$

(ii) If $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, then
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Example: Let X be a RV with $X(\omega) \in \mathbb{R}$ for all $\omega \in \Omega$

Let $A_n = \{\omega : X(\omega) \geq n\} = [n, \infty)$. Then clearly for all $n \in \mathbb{N}$, $[n+1, \infty) \subseteq [n, \infty)$, so by MCT,
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right) = P(\emptyset) = 0.$$

Let $B_n = \{\omega : X(\omega) \geq \frac{1}{n}\} = [\frac{1}{n}, \infty)$. Then clearly, for all $n \in \mathbb{N}$, $[\frac{1}{n}, \infty) \subseteq [\frac{1}{n+1}, \infty)$, so by MCT
$$\lim_{n\to\infty} P(B_n) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = P((0,\infty)) > 0.$$

Note: If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then for all $\omega \in \Omega$ $\mathbb{1}_{A_n}(\omega) \leq \mathbb{1}_{A_{n+1}}(\omega)$ so $\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega)$ exists and equals $\mathbb{1}_A(\omega)$, where $A = \bigcup_{n=1}^{\infty} A_n$.

Limit Events

Definition: Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

(i) We define the limit superior as $n \to \infty$ of the sequence $(x_n)_{n=1}^{\infty}$ by

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{k \geq n} x_k$$

(ii) We define the limit inferior as $n \to \infty$ of the sequence $(x_n)_{n=1}^{\infty}$ by

$$\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{k \geq n} x_k.$$

Example: Let $(x_n)_{n=1}^{\infty}$ be the sequence

$$x_n = (-1)^n \left(\frac{n+1}{n}\right) \quad \forall n \in \mathbb{N}.$$

$x_1 = -2$, $x_2 = \frac{3}{2}$, $x_3 = -\frac{4}{3}$, $x_4 = \frac{5}{4}$, ...

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

$\sup_{k \geq 1} x_1 = \frac{3}{2}$ $\sup_{k \geq 2} x_2 = \frac{3}{2}$ $\sup_{k \geq 3} x_3 = \frac{5}{4}$ $\sup_{k \geq 4} x_4 = \frac{5}{4}$

Thus, taking $n \to \infty$, $\limsup_{n \to \infty} x_n = 1$.

Example: $\limsup_{n \to \infty} (-n^3) = -\infty$, $\limsup_{n \to \infty} (n^3) = \infty$.

Example: Let $L = \left\{ x \in [-\infty, \infty] : \lim_{k \to \infty} x_{n_k} = x \text{ for some subsequence } (x_{n_k})_{k=1}^{\infty} \right\}$.

L is called the space of all extended real numbers with a convergent subsequence. Note

$$\limsup_{n \to \infty} c_n = \sup L$$

Remark: TFAE about limit superiors

(a) $\limsup_{n \to \infty} x_n = \bar{x}$

(b) For every $\varepsilon > 0$, $\{ k : x_k > \bar{x} + \varepsilon \}$ is finite and

$\{k : x_k > \bar{x} - \varepsilon\}$ are finite.

Remark: $-\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} (-x_n)$

Remark: $\lim_{n\to\infty} x_n$ exists if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$

Definition: Let $\{A_n\} \subseteq \mathcal{F}$. Define

(i) $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \in \mathcal{F}$

(ii) $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \in \mathcal{F}$

$\omega \in \limsup_{n\to\infty} A_n \Leftrightarrow \forall n \in \mathbb{N} \quad \omega \in \bigcup_{k=n}^{\infty} A_k$

$\Leftrightarrow \exists k \geq n$ s.t. $\omega \in A_k$

$\Leftrightarrow \{k : \omega \in A_k\}$ is infinite.

so $\limsup_n A_n$ is the set of all ω s.t. A_n occurs infinitely often (i.o.). Similarly.

$\omega \in \liminf_{n\to\infty} A_n \Leftrightarrow \exists n \in \mathbb{N}$ such that $\omega \in \bigcap_{k=n}^{\infty} A_n$

$\Leftrightarrow \exists n \in \mathbb{N}$ s.t. $\omega \in A_k \ \forall k \geq n$

$\Leftrightarrow \{k : \omega \notin A_k\}$ is finite

so $\liminf_n A_n$ is the set of all ω such that A_n occurs for all but finitely many n. (a.a). ABFM

Remark: $\limsup_{n\to\infty} A_n^c = (\liminf_{n\to\infty} A_n)^c$

Proposition: Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, let

$\bar{A} = \limsup_n A_n \qquad \underline{A} = \liminf_n A_n$

Then $\forall \omega \in \Omega$

$\mathbb{1}_{\bar{A}}(\omega) = \limsup_n \mathbb{1}_{A_n}(\omega)$

$\mathbb{1}_{\underline{A}}(\omega) = \liminf_n \mathbb{1}_{A_n}(\omega)$.

Proof: Let $\omega \in \Omega$. The set $L \subseteq \{0,1\}$. So

- $\limsup\limits_{n} \mathbb{1}_{A_n}(\omega) = 1 \iff$ there exists a subsequence n_k such that $\mathbb{1}_{n_k}(\omega) = 1$. $\iff$ ω is in infinitely many A_n's $\iff \mathbb{1}_{\bar{A}}(\omega) = 1$
- $\liminf\limits_{n} \mathbb{1}_{A_n}(\omega) = 1 \iff$ there is no subsequence n_k such that $\mathbb{1}_{n_k}(\omega) = 0 \iff \omega \in \{A_n : aa\}$

$$\iff \mathbb{1}_{\underline{A}}(\omega) = 1$$

Proposition: Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$

$$P(\liminf_n A_n) \underset{\text{text}}{\leq} \underbrace{\liminf_n P(A_n) \leq \boxed{\limsup_n P(A_n)}}_{\text{obvious}} \boxed{\underset{\text{show}}{\leq} P(\limsup_n A_n)}$$

$\limsup\limits_{n\to\infty} A_n = \bigcap\limits_{n=1}^{\infty} B_n$, where $B_n = \bigcap\limits_{k=n}^{\infty} A_k$, so

$B_{n+1} \subseteq B_n \ \forall n \in \mathbb{N}$, so by MCT

$$\lim_{n\to\infty} P(B_n) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = P(\limsup_n A_n)$$

$$= \limsup_n P(B_n) \geq \limsup_n P(A_n) \quad (A_n \subseteq B_n)$$

Theorem (Borel-Cantelli Lemma): Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$

(i) If $\sum\limits_{n=1}^{\infty} P(A_n)$ converges, then $P(A_n) = 0$ i.o.

(ii) If $\sum\limits_{n=1}^{\infty} P(A_n)$ diverges and $\{A_n\}_{n=1}^{\infty}$ are independent.

Then $P(A_n) = 1$ i.o.

Proof: (i) $\limsup\limits_{n\to\infty} A_n = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k \subseteq \bigcup\limits_{k=N}^{\infty} A_n \ \forall N \in \mathbb{N}$. So now

$$P(\limsup_n A_n) \leq P\left(\bigcup_{k=N}^{\infty} A_k\right) \leq \sum_{k=N}^{\infty} P(A_k) \longrightarrow 0, \ N \to \infty.$$