Recall: $\Omega_* = \{H, T\}^{\mathbb{N}} = \{x : x_n \in \{H, T\}\}$

For $k \in \mathbb{N}$, $E_k = \{x \in \Omega_* : x_k = H\}$. Denote $\mathcal{E} = (E_n)_{n=1}^{\infty}$ and denote $\sigma_E = \sigma(\mathcal{E})$. Then σ_E is a σ-algebra.

For example, σ_E contains events such as $E_1 \cap E_2 \cap E_3^c \cap E_4^c$ is the event that the first four tosses are H, H, T, T.

For a fair coin, this should have probability $\frac{1}{2^5} = \frac{1}{32}$. More generally, a fair coin should satisfy

(i) $P(\bigcap_{i=1}^{n} \tilde{E}_i) = \frac{1}{2^n}$ where $\tilde{E}_i \in \{E_i, E_i^c\}$.

Theorem: There exists a (unique) probability measure $P = P_E$ on (Ω_*, σ_E) such that

$$P\left(\bigcap_{i=1}^{n} \tilde{E}_i\right) = \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N},\ \tilde{E}_i \in \{E_i, E_i^c\}.$$

Proof: omitted.

Remark: Each $y \in [0,1]$ can be written as

$$y = \sum_{n=1}^{\infty} \frac{y_n}{2^n} \quad \text{where} \quad y_n \in \{0,1\}$$

for each $n \in \mathbb{N}$. We define $g : \Omega_* \to [0,1]$ by

$$g(x) = \sum_{n=1}^{\infty} \frac{z(x_n)}{2^n} \quad \text{where} \quad z(x_n) = \begin{cases} 1 & \text{if } x_n = H \\ 0 & \text{if } x_n = T. \end{cases}$$

Then

$$g(E_1) = \{y \in [0,1] : y_1 = 1\} = [\tfrac{1}{2}, 1]$$

$$g(E_1 \cap E_2) = \{y \in [0,1] : y_1 = y_2 = 1\} = [\tfrac{3}{4}, 1]$$

$$g(E_1^c \cap E_2) = \{y \in [0,1] : y_1 = 0, y_2 = 1\} = [\tfrac{1}{4}, \tfrac{1}{2}]$$

Then P_E on (Ω_*, σ_E) corresponds to the Lebesgue measure (uniform distribution) on $([0,1], \mathcal{B}([0,1]))$.

Product Measures

Definition: Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ be probability spaces. The product measure space is $(\Omega_1 \times \Omega_2, \mathcal{F}_{12}, P_{12})$ where $\mathcal{F}_{12}$ is the σ-algebra generated by the class $\mathcal{A}_{12} = \{A \times B, A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$ of measurable rectangles, and P_{12} is the probability measure on $(\Omega_1 \times \Omega_2, \mathcal{F}_{12})$ such that

$$P_{12}(A \times B) = P_1(A) P_2(B)$$

for $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$.

Proposition: Let $(\Omega_1 \times \Omega_2, \mathcal{F}_{12}, P_{12})$ be a product measure space. Then the probability measure P_{12} on $(\Omega_1 \times \Omega_2, \mathcal{F}_{12})$ is unique.

In general, we can extend the product measure to $\Omega = \prod_{i=1}^{n} \Omega_i$, $\mathcal{F} = \mathcal{F}_{1,2,\ldots,n}$, $P = P_{1 \cdots n}$ so $(\Omega, \mathcal{F}, P)$ is a probability triple.

<u>Special Case</u>: Product of Uniform$([0,1])$ with itself gives Uniform$([0,1]^2)$.

Random Variables

Definition: A random variable (RV) on $(\Omega, \mathcal{F}, P)$ is a function $X : \Omega \to \mathbb{R}$ such that $\{x \in \Omega : X(x) \leq t\}$ belongs to $\mathcal{F}$ for all $t \in \mathbb{R}$.

Notation: For $D \subset \Omega$, the characteristic function (or indicator function) is

$$\mathbb{1}_D(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

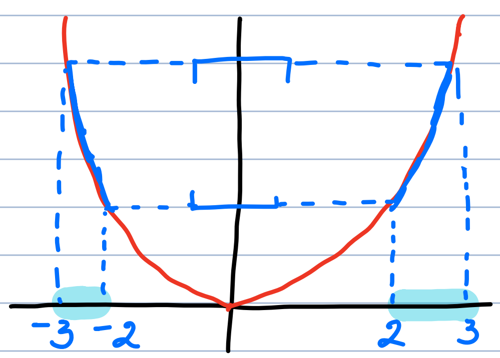
<u>Technical Requirement:</u> X must be compatible with $\mathcal{F}$.

Notation: Let $f : A \to B$ be a function. For $U \subset B$, the inverse image of U under f is

$$f^{-1}[U] = \{a \in A : f(a) \in U\}$$

The definition of a RV says that X is a RV if $X^{-1}((-\infty, t]) \in \mathcal{F}$.

<u>e.g</u> if $A = B = \mathbb{R}$, $f(x) = x^2$, $f^{-1}([4,9]) = [-3,2] \cup [2.3]$



Consider $\mathbb{1}_D$. Then note $(\mathbb{1}_D)^{-1}((-\infty, 0]) = D^c$. Also

$$(\mathbb{1}_D)^{-1}((-\infty, x]) = \begin{cases} \emptyset & : x < 0 \\ D & : 0 \leq x < 1 \\ \Omega & : x \geq 1. \end{cases}$$

Proposition: Let $f: A \to B$, $(U_i)_{i \in I} \subset A$, $(V_i)_{i \in I} \subset B$.

(i) $f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i)$

(ii) $f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i)$

(iii) $f^{-1}(U^c) = [f^{-1}(D)]^c$

(iv) $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$.

Proof: Exercise.

Lemma: Let $g: A \to B$ be a function and let $\mathcal{F}$ be a σ-algebra of A. Let $\mathcal{A} = \{T \subset B : g^{-1}(T) \in \mathcal{F}\}$

Then $\mathcal{A}$ is a σ-algebra of B.

Proof: $g^{-1}(B) = A \in \mathcal{F}$, i.e. $B \in \mathcal{A}$.

Let $T \in \mathcal{A}$. Then $g^{-1}(T^c) = [g^{-1}(T)]^c \in \mathcal{F}$ so $T^c \in \mathcal{A}$.

Let $(T_n)_{n=1}^{\infty}$ be a collection in $\mathcal{A}$. Then

$$g^{-1}\left(\bigcup_{n=1}^{\infty} T_n\right) = \bigcup_{n=1}^{\infty} g^{-1}(T_n) \in \mathcal{F} \quad \text{so} \quad \bigcup_{n=1}^{\infty} T_n \in \mathcal{A}.$$

Corollary: Let X be a RV on $(\Omega, \mathcal{F})$. Then $X^{-1}(V) \in \mathcal{F}$ for every Borel set V.

Proof: By previous lemma, because X is a RV, $\mathcal{A}$ contains $(-\infty, t]$ $\forall t \in \mathbb{R}$. Since $\mathcal{A}$ is a σ-algebra, it contains the σ-algebra generated by these intervals, which is $\mathcal{B}(\mathbb{R})$.

Proposition: Let X, Y RVs and let $c \in \mathbb{R}$. Then $X + Y$, cX,

X^2, XY are RVs.