Theorem: (Borel-Cantelli Lemma) Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$.

(i) If $\sum_{n=1}^{\infty} P(A_n)$ converges, then $P(A_n) = 0$ i.o.

(ii) If $\sum_{n=1}^{\infty} P(A_n)$ diverges and $\{A_n\}_{n=1}^{\infty}$ are independent, then $P(A_n) = 1$ i.o.

Proof: (ii) Since $\limsup_{n\to\infty} A_n = \left(\liminf_{n\to\infty} A_n^c\right)^c$, it suffices to show that $P\left(\liminf_{n\to\infty} A_n^c\right) = 0$. Indeed,

$$P\left(\liminf_{n\to\infty} A_n^c\right) = P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c\right) \le \sum_{n=1}^{\infty} P\left(\bigcap_{k=n}^{\infty} A_k^c\right)$$

by subadditivity. We need to show $P\left(\bigcap_{k=n}^{\infty} A_k^c\right) = 0 \ \forall n \in \mathbb{N}$. Indeed,

$$\lim_{N\to\infty} P\left(\bigcap_{k=n}^{N} A_n^c\right) = \lim_{N\to\infty} \prod_{k=n}^{N} (1 - P(A_k))$$

Now, we use the fact that $1 + x \le e^x \ \forall x \in \mathbb{R}$, so

$$\lim_{N\to\infty} \prod_{k=n}^{N} e^{-P(A_k)} = \lim_{N\to\infty} e^{-\sum_{k=n}^{N} P(A_k)} = e^{-\sum_{k=n}^{\infty} P(A_k)} = e^{-\infty} = 0$$

As $n \in \mathbb{N}$ was arbitrary, $P\left(\bigcap_{k=n}^{\infty} A_k^c\right) = 0$ and the result follows.

Examples: $(\Omega_*, \sigma_E, P_E)$. Let $A_k = \{\omega : \omega_{5k} = H = \omega_{5k+1} = \omega_{5k+2} = \omega_{5k+3} = \omega_{5k+4}\}$. Note $\{A_n\}_{n=1}^{\infty} \subseteq \sigma_E$ are independent, so $\forall n \in \mathbb{N}$, $P(A_n) = \frac{1}{25}$. Thus, $\sum_{n=1}^{\infty} P(A_n) = \infty$.

So $1 = P(A_n) \le P(5 \text{ consecutive heads}) = 1$, i.o.

Tail Fields

Definition: Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$. Their **tail field** (or tail σ-algebra) is

$$\tau = \bigcap_{n=1}^{\infty} \sigma(\{A_{n+i}\}_{i=0}^{\infty}) = \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, A_{n+2}, \ldots).$$

τ contains events that do not depend on any finite number of A_n's.

Example: $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_n}_{\in \sigma(A_n, A_{n+1}, \ldots)}$, $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} A_n}_{\in \sigma(A_n, \ldots)}$

$\{\omega \in \Omega_* : \lim_{n\to\infty} \frac{1_{A_n}(\omega)}{n} \text{ exists}\} \in \sigma(A_n, A_{n+1}, \ldots)$

Theorem (Kolmogorov Zero-One Law): Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be independent with tail field τ. Let $D \in \tau$. Then either $P(D) = 0$ or $P(D) = 1$.

Proof: We prove for when $D = \limsup_{n\to\infty} A_n$. See notes.

Expected Values

Simple Random Variables

Definition: A simple random variable is a random variable whose range is a finite subset of $\mathbb{R}$.

Note that $R(X) = \{X(\omega) : \omega \in \Omega\} = X[\Omega]$

Example: On $(\{H,T\}^{\mathbb{N}}, \sigma_E, P_E)$, let $M(\omega)$ to be the number of heads in the first four tosses. Then

$$M[\{H,T\}^{\mathbb{N}}] = R(M) = \{0,1,2,3,4\}.$$

We can also write $M = \sum_{k=0}^{4} a_k \mathbb{1}_{(M^{-1}[k])}(\omega)$.

Example: If $Z \sim \text{Uniform}[0,1]$, let

$$Z(\omega) = \begin{cases} 2 & \text{if } \omega < \frac{1}{5} \\ 3 & \text{if } \omega = \frac{1}{5} \\ 4 & \text{if } \omega > \frac{1}{5} \end{cases}$$

$$R(Z) = Z[[0,1]] = \{2,3,4\}$$

$$Z = 2\mathbb{1}_{[0,\frac{1}{5})} + 3\mathbb{1}_{\{\frac{1}{5}\}} + 4\mathbb{1}_{[\frac{1}{5},1]}$$

Definition: Let $X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$, where $\{A_i\}_{i=1}^{n}$ is a partition of Ω, and $A_i \in \mathcal{F}$ $\forall i$, $x_1, \ldots, x_n$ distinct. Then the expected value of X is the function $\mathbb{E} : \Omega \to [0,\infty)$ by

$$\mathbb{E}(X) = \sum_{i=1}^{n} x_i P(A_i).$$

Example: (i) Consider $Z \sim \text{Uniform}[0,1]$ as above.

Then $\mathbb{E}(Z) = 2 \times \frac{1}{5} + 3 \times 0 + 4 \times \frac{4}{5} = \frac{18}{5}$.

(ii) Consider $M \sim$ Coin Tossing,

$$\mathbb{E}(M) = \sum_{k=0}^{4} k P(M = k)$$

(iii) $\mathbb{E}(\mathbb{1}_A) = P(A)$

(iv) For $c \in \mathbb{R}$, $\mathbb{E}(c) = c\,P(\Omega) = c$

Note: We can also write

$$Z = 2\mathbb{1}_{[0,\frac{1}{3})} + 3\mathbb{1}_{\{\frac{1}{3}\}} + 4\mathbb{1}_{(\frac{1}{3},\frac{1}{2}]} + 4\mathbb{1}_{(\frac{1}{2},1]}$$

Proposition: If X, Y SRVs, $c \in \mathbb{R}$, then

(i) $\mathbb{E}(cX) = c\mathbb{E}(X)$

(ii) $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

Proof: Assume $X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$, $Y = \sum_{i=1}^{m} y_i \mathbb{1}_{B_i}$. Then

(i) $cX = c\sum_{i=1}^{n} x_i \mathbb{1}_{A_i} = \sum_{i=1}^{n} cx_i \mathbb{1}_{A_i}$

$$\mathbb{E}(cX) = \sum_{i=1}^{n} cx_i P(A_i) = c\sum_{i=1}^{n} x_i P(A_i) = c\,\mathbb{E}(X).$$

(ii) Note if $\{A_i\}_{i=1}^{n}$, $\{B_i\}_{i=1}^{m}$ are partitions of $\mathcal{F}$, $\{A_i \cap B_j\}_{i,j}$ is another partition in $\mathcal{F}$. In particular,

$$X + Y = \sum_{i=1}^{n}\sum_{j=1}^{m} (x_i + y_j)\mathbb{1}_{A_i \cap B_j}$$
$$= \sum_{j=1}^{m}\sum_{i=1}^{n} x_i \mathbb{1}_{A_i \cap B_j} + \sum_{i=1}^{n}\sum_{j=1}^{m} y_j \mathbb{1}_{A_i \cap B_j}$$

Thus

$$\mathbb{E}(X+Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} (x_i + y_j) P(A_i \cap B_j)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i P(A_i \cap B_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} y_j P(A_i \cap B_j)$$
$$= \mathbb{E}(X) + \mathbb{E}(Y).$$

Corollary: If $(x_i)_{i=1}^{n}$, $\{A_i\}_{i=1}^{n} \subseteq \mathcal{F}$, then

$$\mathbb{E}\left(\sum_{i=1}^{n} x_i \mathbb{1}_{A_i}\right) = \sum_{i=1}^{n} x_i P(A_i)$$

Proposition: Let X, Y, Z be SRVs.

(i) If $X \geq Z$, then $\mathbb{E}(X) \geq \mathbb{E}(Z)$

(ii) $|\mathbb{E}(X)| \leq \mathbb{E}(|X|)$

(iii) If X, Y are independent, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$

(iv) $\mathbb{E}(X) = \sup\{\mathbb{E}(Y) : Y \text{ is simple and } Y \leq X\}$.

Proof: exercise.