Corollary: If $(x_i)_{i=1}^n \in \mathbb{R}$ and $\{A_i\}_{i=1}^n \subseteq \mathcal{F}$, then

$$\mathbb{E}\left(\sum_{i=1}^n x_i \mathbb{1}_{A_i}\right) = \sum_{i=1}^n x_i P(A_i).$$

Proposition: If X, Z are SRVs,

(i) If $X \geq Z$, then $\mathbb{E}(X) \geq \mathbb{E}(Z)$.

(ii) $\mathbb{E}(X) \leq \mathbb{E}(|X|)$

(iii) If X and Z are independent, $\mathbb{E}(XZ) = \mathbb{E}(X)\mathbb{E}(Z)$.

(iv) $\mathbb{E}(X) = \sup\{\mathbb{E}(Y) : Y \text{ is simple and } Y \leq X\}$.

(note by (i), we get "$\geq$", and for "$=$", take $Y = X$.

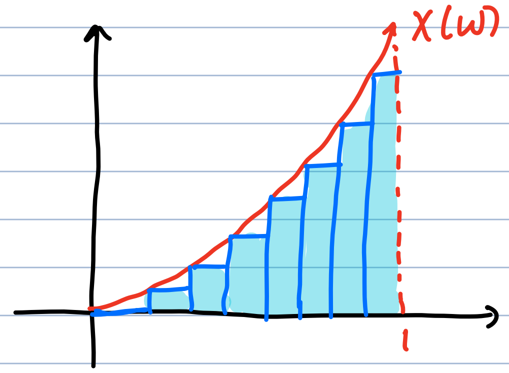
Remark: (i)-(iv) hold for general RVs, but for (iv), we need X nonnegative.

Nonnegative Random Variables

Definition: Let X be a nonnegative RV. Then

$$\mathbb{E}(X) = \sup\{\mathbb{E}(Y) : Y \text{ is simple and } Y \leq X\}.$$

Example: Consider $([0,1], \mathcal{B}([0,1]), \lambda)$. Let $X(\omega) = \omega^2$.



and consider

$$Y_n = \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \mathbb{1}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]} \leq X$$

Then

$$\mathbb{E}(Y_n) = \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \frac{1}{n}$$

$\lim_{n\to\infty} Y_n(\omega) = X(\omega)$ pointwise for all $\omega \in [0,1]$. Also,

$\lim_{n\to\infty} \mathbb{E}(Y_n) = \int_0^1 \omega^2 \, d\omega$. Does it follow $\mathbb{E}(X) = \int_0^1 \omega^2 \, d\omega$?

Example: Suppose $\text{Range}(X) = \mathbb{N} \cup \{0\}$. Does

$$\mathbb{E}(X) = \sum_{n=1}^\infty n P(X = n)$$

We know "$\geq$" since $E(X) \geq E(X \mathbb{1}_{(X\leq n)}) = \sum_{i=1}^{n} i P(X=i)$

$\nearrow \sum_{n=1}^{\infty} n P(X=n)$

Example: Assume X is nonnegative, possibly infinite and $P(X=0)=1$. SRVs must have range in $\mathbb{R}$. If $Y = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$ and $0 \leq Y \leq X$, then $x_i \neq 0$ so $P(A_i)=0$.

Therefore $\mathbb{E}(Y)=0$ a.e.

$$E(X) = \sup\{E(Y) : Y \text{ simple and } Y \leq X\} = 0.$$

Theorem: (Monotone Convergence Theorem For Expected Value)

Let $\{X_n\}_{n=1}^{\infty}$ be nonnegative RVs on $(\Omega, \mathcal{F}, P)$. Assume $\forall n \in \mathbb{N}$, $X_n \leq X_{n+1}$ and let $X = \lim_{n\to\infty} X_n$. Then

$$\lim_{n\to\infty} E(X_n) = E(X).$$

This theorem confirms the formulas in the above examples.

In particular, $X\mathbb{1}_{(X\leq n)} \leq X\mathbb{1}_{(X\leq n+1)}$, so $X\mathbb{1}_{(X\leq n)} \to X$

Proof: By a known proposition, X is a RV and also $E(X_n) \leq E(X_{n+1})$ for all $n \in \mathbb{N}$ and $E(X_n) \leq E(X)$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} E(X_n) \leq E(X)$. For the other direction, we show $\lim_{n\to\infty} E(X_n) \geq E(Y)$ for Y SRV such that $0 \leq Y \leq X$.

Assume $0 \leq Y \leq X$ and $Y = \sum_{i=1}^{m} v_i \mathbb{1}_{A_i}$ for a partition $\{A_i\}_{i=1}^{n}$. Let $\varepsilon > 0$ be arbitrary. Let $B_{i,n} = \{\omega \in A_i : X_n(\omega) \geq v_i - \varepsilon\}$ $v_1 - \varepsilon < v_i = Y(\omega) \leq X(\omega)$, so $X_n(\omega) > v_i - \varepsilon$ for sufficiently large n for all $\omega \in A_i$. That is, $B_{i,n} \nearrow A_i$. Then by the

Monotone Convergence Theorem for Probability Measures

$\lim_{n\to\infty} P(B_{i,n}) = P(A_i)$ for all i, so

$X_n \geq \sum_{i=1}^{n} (v_i - \varepsilon) \mathbb{1}_{A_i} \implies E(X_n) \geq \sum_{i=1}^{m} (v_i - \varepsilon) P(B_{i,n})$.

Now, $\lim_{n\to\infty} E(X_n) \geq \sum_{i=1}^{m} (v_i - \varepsilon) P(A_i) = \sum_{i=1}^{m} v_i P(A_i) - \sum_{i=1}^{m} \varepsilon P(A_i)$

$= E(Y) - \varepsilon$

Because $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} E(X_n) \geq E(Y)$.

Remark: For nonnegative X, Z RVs,

(i) $E(cX) = c\,E(X)$

(ii) If $X \leq Z$, then $E(X) \leq E(Z)$.

(iii) $E(X+Z) = E(X) + E(Z)$ is harder.

Proposition: Let X be a nonnegative RV. Then there exists $\{\bar{X}_n\}_{n=1}^{\infty}$ nonnegative SRVs such that $\bar{X}_n \nearrow X$

Proof: We prove in steps.

<u>Step 1</u>: Let $X_0^{\vee}(\omega) = \lfloor X(\omega) \rfloor$, let $X_n^{\vee}(\omega) = \{X(\omega)\}$ (the nth decimal place). Then for all $n \in \mathbb{N}$, $\bar{X}_n \leq \bar{X}_{n+1} \leq X$ so $\bar{X}_n \nearrow X$ and $\bar{X}_n - X \leq 10^{-k}$