Proposition: If $\{X_n\}_{n=1}^{\infty}$ are nonnegative RVs, then

$$E\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} E(X_n).$$

Proof: $E\left(\sum_{n=1}^{\infty} X_n\right) = E\left(\lim_{N\to\infty} \sum_{n=1}^{N} X_n\right) \overset{MCT}{=} \lim_{N\to\infty} E\left(\sum_{n=1}^{N} X_n\right)$

$= \lim_{N\to\infty} \sum_{n=1}^{N} E(X_n) = \sum_{n=1}^{\infty} E(X_n).$

General Random Variables

Definition: Let X be a RV. We define the positive and negative parts of X as

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0 \end{cases} \qquad X^- = \begin{cases} -X & \text{if } X < 0 \\ 0 & \text{if } X \geq 0 \end{cases}$$

Then $X = X^+ - X^-$. Define the expected value of X as $E(X) = E(X^+) - E(X^-)$, except for the case where both $E(X^+)$ and $E(X^-)$ are infinite, then $E(X)$ is undefined.

Proposition: Let X and Y be RVs with defined values.

(i) If $X \leq Y$, then $E(X) \leq E(Y)$

(ii) If $E(X) + E(Y)$ is defined, $E(X+Y) = E(X) + E(Y)$

(iii) If X and Y are independent, $E(XY) = E(X)E(Y)$.

Proof: (i) and (iii) Exercise.

(ii) Let $W = X+Y$ and write $W^+ - W^- = (X^+ - X^-) + (Y^+ - Y^-)$

Then $W^+ + X^- + Y^- = X^+ + Y^+ + W^-$

Observe that $W^+ \leq X^+ + Y^+$ and $W^- \leq X^- + Y^-$. Thus

$E(W^+ + X^- + Y^-) = E(X^+ + Y^+ + W^-)$

$E(W^+) + E(X^-) + E(Y^-) = E(X^+) + E(Y^+) + E(W^-)$

Case 1: If $E(X)$ and $E(Y)$ are finite, then the above are all finite, so

$$E(W) = E(W^+) - E(W^-) = E(X^+) - E(X^-) + E(Y^+) - E(Y^-)$$

Case 2: If $E(X^+) = \infty$. Then $E(X^-) < \infty$, $E(Y^-) < \infty$, and $E(W^-) < \infty$. Then $E(W^+) = \infty$, so we get the desired result.

Corollary: If X, Y RVs and $P(X = Y) = 1$. Then $E(X) = E(Y)$.

Proof: Let $W = X - Y$. It suffices to show $E(W) = 0$. Then write $W = W^+ - W^-$. Then $P(W = 0) = 1 = P(W^+ = 0) = P(W^- = 0)$

Recall: $E(W^+) = \sup\{0\} = 0$ and $E(W^-) = \sup\{0\}$, so

$$E(W) = 0$$

Put $X = Y + W$, so

$$E(X) = E(Y + W) = E(Y) + \cancel{E(W)}^{\,0} = E(Y)$$

Note: The above proof works for W taking values from $[-\infty, \infty]$. If $0 \le Z \le \infty$, $P(Z = 0) = 1$, then $E(Z) = 0$.

If $0 \le Z \le \infty$ and $P(Z = \infty) > 0$, then $E(Z) = \infty$.

Definition: Let X and Y be RVs with finite expected values.

The covariance of X and Y is

$$\begin{aligned} \text{Cov}(X, Y) &= E((X - E(X))(Y - E(Y))) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

The variance of X and Y is

$$\begin{aligned} \text{Var}(X) &= E((X - E(X))^2) \\ &= E(X^2) - E(X)^2 \end{aligned}$$

Proposition:

(i) If X and Y are independent RVs, then $Cov(X,Y)=0$. (converse is false),

(ii) $Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{i,j\,:\,1\le i<j\le m} Cov(X_i, X_j)$

(iii) $Var(X)$ and $E(X)$ are finite if and only if $E(X^2)$ is.

(iv) If $E(X^2)$ is finite and $a,b \in \mathbb{R}$, $Var(aX+b) = a^2 Var(X)$

Proof: (i), (ii) exercise.

(iii) If $E(X^2)<\infty$, note $0 \le (|X|-1)^2 = |X|^2 - 2|X| + 1$

so $|X| = \frac{X^2+1}{2}$, so $E(|X|) = E(X^+) + E(X^-) < \infty$,

so $E(X)<\infty$. For the latter,

$E((X-E(X))^2) =: E(X^2 - 2XE(X) + E(X)^2) \Rightarrow$ finite.

Now, $X^2 = (X-E(X))^2 + 2XE(X) - E(X)^2$

(iv) If $E(X^2)<\infty$, $a,b\in\mathbb{R}$. $Var(aX+b) = a^2 Var(X)$.

Theorem: Let $H:[0,1]\to\mathbb{R}$ be a Riemann integrable function. Then H is a random variable on $([0,1], \mathcal{F}, \lambda)$ and $E(H) = \int_0^1 H\, d\omega$.

Inequalities and Convergence

Proposition (Markov's Inequality) Let X be a nonnegative RV and $\alpha > 0$. Then $P(X \geq \alpha) \leq \frac{E(X)}{\alpha}$.

Proof: Let $Z = \alpha \mathbb{1}_{\{X \geq \alpha\}}$. Then $X \geq Z$ because if $\omega \in \{X \geq \alpha\}$, trivial, $Z = \alpha$. Otherwise, $Z = 0$.

$E(X) \geq E(Z) = \alpha P(X \geq \alpha) \Rightarrow P(X \geq \alpha) \leq \frac{E(X)}{\alpha}$.

Proposition (Chebyshev's Inequality) Let X be a random variable with $E(X) < \infty$. Let $\alpha > 0$. Then

$$P(|X - E(X)| \geq \alpha) \leq \frac{Var(X)}{\alpha^2}.$$

Proof: $P(|X - E(X)| \geq \alpha) = P(|X - E(X)|^2 \geq \alpha^2)$

By Markov's Inequality,

$$P(|X - E(X)|^2 \geq \alpha^2) \leq \frac{E((X - E(X))^2)}{\alpha^2} = \frac{Var(X)}{\alpha^2}.$$

Example: Assume $\{X_n\}_{n=1}^{\infty}$ are independent RVs, such that $E(X_n) = \mu$, $Var(X_n) = \sigma^2$ $\forall i$. Then let $\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$

$E(\bar{X}_n) = \frac{1}{n} E(X_1 + \cdots + X_n) = \frac{1}{n} n \cdot \mu = \mu$.

By Markov,

$$P(\bar{X}_n \geq 1.1\mu) \leq \frac{\mu}{1.1\mu} = \frac{1}{1.1}$$

By Chebyshev: $Var(\bar{X}_n) = \frac{1}{n^2} Var(X_1 + \cdots + X_n) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$

$$P(|\bar{X}_n - \mu| \geq 0.01) \leq \frac{\sigma^2}{0.0001n}$$