Theorem: (First Version of Strong Law of Large Numbers)
Assume that $\{X_n\}_{n=1}^{\infty}$ are independent RVs with $E(X_i)=\mu$ and $E((X_n-\mu)^4) \leq A < \infty$ for all $n \in \mathbb{N}$. For each $N \in \mathbb{N}$, let

$$\overline{X}_N = \frac{X_1 + X_2 + \cdots + X_N}{N}$$

Then $\{\overline{X}_N\}_{N=1}^{\infty}$ converges almost surely to μ.

Proof: Last time we showed $E(S_n^4) \leq kn^2$ where $S_n = \sum_{i=1}^{n}(x_i-\mu)$ so $\overline{X}_n - \mu = \frac{S_n}{n}$.

Let $\varepsilon>0$ be arbitrary. Then we have

$$P(|\overline{X}_n - \mu| \geq \varepsilon) = P\left(\frac{|S_n|}{n} \geq \varepsilon\right) = P(S_n^4 \geq n^4\varepsilon^4)$$

Therefore, by Markov's Inequality,

$$P(S_n^4 \geq n^4\varepsilon^4) \leq \frac{E(S_n^4)}{n^4\varepsilon^4} \leq \frac{kn^2}{n^4\varepsilon^4} = \frac{k}{n^2\varepsilon^4}$$ Hence,

$\sum_{n=1}^{\infty} P(|\overline{X}_n - \mu| \geq \varepsilon) \leq \sum_{n=1}^{\infty} \frac{k}{\varepsilon^4} \cdot \frac{1}{n^2} < \infty$, so by the Borel Cantelli Lemma,

$P(|\overline{X}_n - \mu| \geq \varepsilon) = 0$ infinitely often, so $\overline{X}_n \to \mu$ a.s.

Corollary: Assume that for every $\varepsilon>0$, the series $\sum_{n=1}^{\infty} P(|X_n - X| \geq \varepsilon)$ converges. Then $\{X_n\}$ converges almost surely to X.

Definition: A collection of RVs $\{X_i\}_{i\in I}$ are said to be **identically distributed** if for every Borel measurable function $f: \mathbb{R} \to \mathbb{R}$, the value $E(f(X_i))$ is the same for all $i \in I$.

(Equivalently, for every Borel set $V \subseteq \mathbb{R}$, $P(X_i \in V)$ is the same for all $i \in I$).

Definition: A collection of RVs $\{X_i\}_{i \in I}$ are said to be i.i.d if they are independent and identically distributed.

Theorem: (Second Version of Strong Law of Large Numbers)
Assume $\{X_n\}_{n=1}^{\infty}$ are iid RVs with $E(X_n) = \mu < \infty$. For each $n \in \mathbb{N}$, let $\bar{X}_n = \frac{S_n}{n}$ where $S_n = X_1 + \cdots + X_n$. Then $\{\bar{X}_n\}$ converges almost surely to μ

Proof: WLOG, Assume that $\{X_n\}_{n=1}^{\infty}$ are nonnegative.

Step 1: For each $n \in \mathbb{N}$, let $Y_n = X_n \mathbb{1}_{\{X_n \leq n\}}$. We claim that $\{Y_n\}_{n=1}^{\infty}$ are independent and $\mu = \lim_{n \to \infty} E(Y_n)$.

Independence follows from Proposition 3.2.3. For the latter,

$$E(Y_n) = E(X_n \mathbb{1}_{\{X_n \leq n\}}) = E(X_1 \mathbb{1}_{\{X_1 \leq n\}}). \text{ Then as } n \to \infty,$$

$E(X_1 \mathbb{1}_{\{X_1 \leq n\}}) \longrightarrow E(X_1) = \mu$.

Step 2: We claim that $P(X_n = Y_n) = 1$ a.a, and so $\lim_n (\bar{X}_n - \bar{Y}_n) = 0$. Indeed,

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(X_n > n) = \sum_{n=1}^{\infty} P(X_1 > n)$$
$$\leq \sum_{n=1}^{\infty} P(X_n \geq n) = E(\lfloor X_1 \rfloor) \leq E(X_1) < \infty$$

Therefore by Borel-Cantelli Lemma, $P(X_n = Y_n) = 1$ a.a.

Step 3: Let $T_n = Y_1 + Y_2 + \cdots + Y_n$. We claim that

$$\operatorname{Var}(T_n) \leq n E(X_1^2 \mathbb{1}_{\{X_1 \leq n\}})$$

Indeed

$$\mathrm{Var}(T_n) = \sum_{i=1}^{n} \mathrm{Var}(Y_i) \leq \sum_{i=1}^{n} E(X_i^2 \mathbb{1}_{\{X_1 \leq i\}})$$
$$\leq n\, E(X_1^2 \mathbb{1}_{\{X_1 \leq i\}})$$

<u>Step 4</u>: For $m \in \mathbb{N}$ and $\varepsilon > 0$, we claim that

$$P\left(\left| \frac{T_m - E(T_m)}{m} \right| \geq \varepsilon \right) \leq \frac{E(X_1^2 \mathbb{1}_{\{X_1 \leq m\}})}{\varepsilon^2 m}$$

Indeed, by Chebyshev and step 3

$$P\left(\left| \frac{T_m - E(T_m)}{m} \right| \geq \varepsilon \right) \leq \frac{\mathrm{Var}(T_m)}{m^2 \varepsilon^2} \leq \frac{\cancel{m} E(X_1^2 \mathbb{1}_{\{X_1 \leq m\}})}{\varepsilon^2 m^{\cancel{2}}}$$
$$= \frac{E(X_1^2 \mathbb{1}_{\{X_1 \leq m\}})}{\varepsilon^2 \cdot m}$$

<u>Step 5</u>: Fix $\alpha > 1$ and let $m_n = \lfloor \alpha^n \rfloor$ for $n \in \mathbb{N}$. Then we show

$\alpha^n \geq m_n \geq \alpha^n / 2$.

For $\beta > 1$, we have $\beta \geq \lfloor \beta \rfloor \geq \frac{\beta}{2}$, so putting $\beta = \alpha^n$ we get the desired inequality.

<u>Step 6</u>: For $x > 0$, we claim that

$$\sum_{n : m_n \geq x} \frac{1}{m_n} \leq \frac{2}{x(1 - \frac{1}{\alpha})}$$

Indeed, $\displaystyle\sum_{n : m_n \geq x} \frac{1}{m_n} \leq \sum_{n : \alpha^n \geq x} \frac{1}{m_n} = \sum_{n : n \geq \log_\alpha(x)} \frac{1}{m_n} \leq \sum_{n = \lceil \log_\alpha(x) \rceil}^{\infty} \frac{2}{\alpha^n}$

$$= \frac{2}{\alpha^{\lceil \log_\alpha x \rceil}} \left(\frac{1}{1 - \frac{1}{\alpha}} \right) \leq \frac{2}{x(1 - \frac{1}{\alpha})}$$

<u>Step 7</u>: We claim that $\dfrac{T_{m_n} - E(T_{m_n})}{m_n} \longrightarrow 0$ a.s.

For $\varepsilon > 0$, $\displaystyle\sum_{n=1}^{\infty} P\left(\left| \frac{T_{m_n} - E(T_{m_n})}{m_n} \right| \geq \varepsilon \right) \leq \sum_{n=1}^{\infty} \frac{E(X_1^2 \mathbb{1}_{\{X_1 \leq m_n\}})}{m_n \varepsilon^2}$

$$= \frac{1}{\varepsilon^2} E\left(X_1^2 \sum_{n=1}^{\infty} \frac{1}{m_n} \mathbb{1}_{\{X_1 \leq m_n\}} \right)$$
$$= \frac{1}{\varepsilon^2} E\left(X_1^2 \sum_{n=1 : X_1 \leq m_n}^{\infty} \frac{1}{m_n} \right) \leq \frac{1}{\varepsilon^2} E\left(X_1^2 \cdot \frac{2}{X_1 (1 - \frac{1}{\alpha})} \right)$$
$$= \frac{2}{\varepsilon^2 (1 - \frac{1}{\alpha})} E(X_1) < \infty$$

By Borel-Cantelli Lemma, $P\left(\left| \dfrac{T_{m_n} - E(T_{m_n})}{m_n} \right| \geq \varepsilon \right) = 0$ i.o., so

$\dfrac{T_{m_n}}{m_n} - \dfrac{E(T_{m_n})}{m_n} \longrightarrow 0$ a.s.

Step 8: We show $\frac{T_{m_n}}{m_n} \to \mu$ a.s. Indeed, see that

$\frac{E(T_m)}{m} = \frac{\sum_{i=1}^{m} E(Y_i)}{m}$ and because $E(Y_i) \to \mu$ by step 1 and

$Y_i = X_i \mathbb{1}_{\{X_i \le i\}}$ so it follows $\frac{E(T_m)}{m} \to \mu$. Moreover, by Step 7,

$\frac{T_{m_n}}{m_n} - \mu \to 0$ a.s, so $\frac{T_{m_n}}{m_n} \to \mu$ a.s.

Step 9: We claim $\frac{S_{m_n}}{m_n} \to \mu$ a.s. Indeed, because $X_n = Y_n$ a.s. then

$\frac{T_{m_n} - S_{m_n}}{m_n} \to 0$ a.s, so $\frac{S_{m_n}}{m_n} \to \mu$ a.s.

Step 10: $\forall k \in \mathbb{N}$, define $n(k) \in \mathbb{N}$ such that $u_{n(k)} \le k < u_{n(k)+1}$

$\frac{S_{u_{n(k)}}}{u_{n(k)+1}} \le \frac{S_k}{k} \le \frac{S_{u_{n(k)+1}}}{u_{n(k)}} \implies \overline{X}_{u_{n(k)}} \frac{u_{n(k)}}{u_{n(k)+1}} \le \frac{S_k}{k} \le \overline{X}_{u_{n(k)+1}} \frac{u_{n(k)+1}}{u_{n(k)}}$

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