Theorem: (Strong Law of Large Numbers) Assume $\{X_n\}_{n=1}^{\infty}$ are iid RVs with finite mean $E(X_n)=\mu$. For each $n \in \mathbb{N}$, let $\overline{X}_n = \frac{S_n}{n}$, where $S_n = X_1 + \cdots + X_n$. Then $\{\overline{X}_n\}_{n=1}^{\infty}$ converges almost surely to μ.

Proof: We finish step 10 from last time. For each $k \in \mathbb{N}$, define n_k such that $u_{n_k} \le k \le u_{n_k+1}$. Then

$$\frac{S_{u_{n_k}}}{u_{n_k+1}} \le \frac{S_k}{k} \le \frac{S_{u_{n_k+1}}}{u_{n_k}} = \frac{S_{u_{n_k+1}}}{u_{n_k+1}} \cdot \frac{u_{n_k+1}}{u_{n_k}} \longrightarrow \mu \cdot \alpha$$

Thus, $\limsup_{k\to\infty} \frac{S_k}{k} \le \alpha\mu$ a.s. true for all $\alpha > 1$, so
$\limsup_{k\to\infty} \frac{S_k}{k} \le \mu$ a.s.

Similarly, on the left side,

$$\frac{S_{u_{n_k}}}{u_{n_k}} \cdot \frac{u_{n_k}}{u_{n_k+1}} \longrightarrow \mu \cdot \frac{1}{\alpha} \quad \text{so} \quad \liminf_{n\to\infty} \frac{S_k}{k} \ge \frac{\mu}{\alpha} \text{ a.s., so}$$

$$\liminf_{n\to\infty} \frac{S_k}{k} \ge \mu, \quad \text{so} \quad \lim_{k\to\infty} \frac{S_k}{k} = \mu.$$

Corollary: Assume $\{X_n\}_{n=1}^{\infty}$ are iid RVs with $E(X_n) = \mu$ finite. Then $\{\overline{X}_n\}_{n=1}^{\infty}$ converges in probability to μ.

Distributions of RVs

Definition: Let X be a RV on $(\Omega, \mathcal{F}, P)$. The distribution or law of X is the probability measure μ_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\mu_X(A) = P(X \in A) = P(X^{-1}(A)) \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}).$$

We refer to μ_X as $\mathcal{L}(X)$ or PX^{-1}.

Notation: The notation $X \sim \mu$ to denote μ is a distribution of X.

Definition: The cumulative distribution function (cdf) of X is the function $F_X : \mathbb{R} \to [0,1]$ by

$$F_X(t) = P(X \leq t) \quad t \in \mathbb{R}$$

Proposition: Let X be a RV, $X \sim \mu$, $F_X : \mathbb{R} \to [0,1]$ be the cdf of X.

(i) F_X is non decreasing

(ii) F_X is right-continuous

(iii) $\lim_{t \to -\infty} F_X(t) = 0$

(iv) $\lim_{t \to \infty} F_X(t) = 1$

Proof: See Measure Theory Assignment 1.

Proposition: Let X and Y be RVs. Then $\mu_X = \mu_Y$ if and only if $F_X = F_Y$.

Theorem: For a RV X on $(\Omega, \mathcal{F}, P)$, we get another probability space $(\mathbb{R}, \mathcal{B}, \mu_X)$ and it satisfies

$$E_P(g(X)) = E_{\mu_X}(g) \quad \text{for every Borel } g: \mathbb{R} \to \mathbb{R}.$$

In particular

$$E_P(g(X)) = \int_\Omega g(X)\, dP = \int_{\mathbb{R}} g\, d\mu_X = E_{\mu_X}(g)$$

Example: For the probability space being the uniform (Lebesgue) measure on $(0,1)$, $X(\omega) = -\ln(\omega)$. For $t \in \mathbb{R}$

$F_X(t) = P(X \leq t) = 0$ if $t \leq 0$. Otherwise if $t > 0$

$F_X(t) = P(-\ln(\omega) \leq t) = P(\ln(\omega) \geq -t) = P(\omega \geq e^{-t})$

$= \lambda([\bar{e}^{t} - 1]) = 1 - \bar{e}^{t}$. In particular,

$$P(X \leq t) = \int_0^t e^{-s}\, ds$$

Such a μ_X is called the exponential distribution (of parameter 1). For any Borel $A \subseteq \mathbb{R}$,

$$P(X \in A) = \int_A \mathbb{1}_{[0,\infty)}(s)\, e^{-s}\, ds$$

Definition: A RV X has a probability density function $f_X : \mathbb{R} \to [0,\infty)$ if

$$\mu_X(A) = \int_A f_X(s)\, ds \quad \forall A \subseteq \mathbb{R}.$$

Moreover $\mu_X(\mathbb{R}) = 1$.

Claim: For any Borel $g : (0,\infty) \to \mathbb{R}$, and this particular $f_X(s) = e^{-s}$.

$$E_P(g(X)) = \int_{\mathbb{R}} g\, f_X\, dP. = \int_{-\infty}^{\infty} g(s) f_X(s)\, ds.$$

Indeed, we check $g = \mathbb{1}_A$. so

$$E_P(\mathbb{1}_A) = P(X \in A) = \mu_X(A) = E_{\mu_X}(\mathbb{1}_A) = \int_A f_X\, d\mu_X$$

For simple functions $g = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$

$$E_P(g) = \sum_{k=1}^{n} a_k E_P(\mathbb{1}_{A_k}) = \sum_{k=1}^{n} a_k E_{\mu_X}(\mathbb{1}_{A_k}) = E_{\mu_X}(g)$$

For $g \geq 0$, choose $(g_n)_{n=1}^{\infty} \nearrow g$.

$$E_P(g) = \lim_{n\to\infty} E_P(g_n(X)) = \lim_{n\to\infty} E_{\mu_X}(g_n) = E_{\mu_X}(g)$$

For $g = g_+ - g_-$

$$E_P(g) = E_P(g_+) - E_P(g_-) = E_{\mu_X}(g_+) - E_{\mu_X}(g_-) = E_{\mu_X}(g).$$

Example: $X = -\ln(\omega)$ on $(0,1)$. Let $g(x) = x^2$.

$$E_P(g(X)) = \int_0^1 \ln^2(\omega)\, d\omega = E_{\mu_X}(g) = \int_{\mathbb{R}} g(s) f_X(s)\, ds$$
$$= \int_0^\infty s^2 e^{-s}\, ds = \Gamma(3) = 2$$

Proposition: Let X be a RV that has pdf f_X. Then for all $g: \mathbb{R} \to \mathbb{R}$

$$E_P(g(X)) = \int_{-\infty}^{\infty} g(s) f_X(s)\, ds$$

Example: On $(\Omega_*, \sigma_E, P_E)$ let $Y(\omega) = 0, 1, 2, 3$.

$$P(Y = k) = \begin{cases} \binom{3}{k} 2^{-3} & k \in \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$P(Y \in A) = \sum_{k=0}^{3} \binom{3}{k} 2^{-3} \delta_k(A) \quad \text{where}$$

$$\delta_k(A) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{if } k \notin A. \end{cases}$$

(The point-mass measure)