Basic Definitions

Definition: A probability space $(\Omega, \mathcal{F}, P)$ consists of

(i) A sample space Ω

(ii) A σ-algebra $\mathcal{F}$ of subsets of Ω

(iii) A probability measure P on $(\Omega, \mathcal{F})$

Definition: Let Ω be a sample space and $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a σ-algebra on Ω if

(i) $\emptyset, \Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) If $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

Examples:

(α) The set of all subsets of Ω, $\mathcal{P}(\Omega)$

(β) $\{\emptyset, \Omega\}$

(γ) $\{\emptyset, \mathbb{R}, [3,7], (-\infty,3) \cup (7,\infty)\}$

(δ) $\{A \subset \mathbb{R} : A$ or A^c is countable$\}$.

Proposition: Let $\mathcal{F}$ be a σ-algebra on Ω. Then

(i) $\emptyset \in \mathcal{F}$

(ii) If $A_1, \dots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$

(iii) If $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{F}$, $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$

(iv) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$ and $A_1 \cap A_2 \in \mathcal{F}$.

Proof: (i) follows by definition. To prove (ii), because $\mathcal{F}$ is a σ-algebra, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, so for some $N \in \mathbb{N}$ put $A_n = \emptyset$ for all $n > N$ so that

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{N} A_n \in \mathcal{F}.$$

To see that (iii) holds, note that

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c \in \mathcal{F}$$

Finally, to prove (iv), note $A_1 \cap A_2 \in \mathcal{F}$ is obvious and $A_1 \setminus A_2 = A_1 \cap A_2^c \in \mathcal{F}$.

Question: Can we omit (i) in the definition of σ-algebra?

Suppose (ii) and (iii) hold. For any $B \in \mathcal{F}$, we have $B^c \in \mathcal{F}$ so $B \cup B^c \in \mathcal{F}$, so $\Omega \in \mathcal{F}$.

The reasoning above is incorrect. It fails when $\mathcal{F}$ is the empty set. In this case, (ii) and (iii) hold but (i) does not.

(i) can be replaced by the assertion that $\mathcal{F}$ is nonempty.

Definition: Let $\mathcal{F}$ be a σ-algebra on the set Ω. A **probability measure** $P : \mathcal{F} \to [0,1]$ is a function such that

(i) $P(\Omega) = 1$

(ii) If $(A_n)_{n=1}^{\infty}$ are pairwise disjoint in $\mathcal{F}$, then

$$P\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P(A_n).$$

Proposition: If P is a probability measure on $(\Omega, \mathcal{F})$ then

(i) $P(\emptyset) = 0$

(ii) If $A_1, \dots, A_n \in \mathcal{F}$ are pairwise disjoint then
$$P\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

(iii) If $A, B \in \mathcal{F}$ such that $A \subset B$, then
$$P(B \setminus A) = P(B) - P(A) \text{ and } P(A) \leq P(B).$$

(iv) If $B \in \mathcal{F}$, then $P(B^c) = 1 - P(B)$.

(v) If $(A_n)_{n=1}^{\infty}$ is a collection in $\mathcal{F}$, then
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

Proof: (i) is easy: take $A_n = \emptyset$ for all $n \in \mathbb{N}$ and so
$$P(\emptyset) = P\left(\bigcup_{n=1}^{\infty} \emptyset\right) = \sum_{n=1}^{\infty} P(\emptyset) = 0.$$

To see that (ii) holds, because $\mathcal{F}$ is a σ-algebra, for some $N \in \mathbb{N}$, let $A_n = \emptyset$ for all $n > N$ and so
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n).$$

To see that (iii) holds, note that $B = A \cup (B \setminus A)$ and so $A \cap (B \setminus A) = \emptyset$, so
$$P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$$
which gives $P(B \setminus A) = P(B) - P(A)$. Moreover, $P(A) \leq P(B)$.

To see that (iv) holds, let $B = \Omega$. Then by (iii)

$\Omega = B \cup B^c$ and $B \cap B^c = \emptyset$, so

$$P(\Omega) = P(B \cup B^c) = P(B) + P(B^c)$$
$$1 = P(B) + P(B^c)$$
$$P(B^c) = 1 - P(B).$$

Finally, to prove (v), let $E_1 = A_1$ and for $n \geq 2$, let $E_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i \right)$. Then $E_n \subset A_n$ for all $n \in \mathbb{N}$, $(E_n)_{n=1}^{\infty}$ are pairwise disjoint, and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n$.

Therefore,

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

by monotonicity.

Proposition: Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of σ-algebras of Ω. Then $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-algebra.

Example: Let $\Omega = \{a, b, c, d\}$, $I = \{1, 2\}$. Let $\mathcal{F}_1 = \{\emptyset, \Omega, \{a\}, \{b, c, d\}\}$, $\mathcal{F}_2 = \{\emptyset, \Omega, \{a, b\}, \{c, d\}\}$.

Then $\mathcal{F}_1 \cap \mathcal{F}_2 = \{0, \Omega\}$ is a σ-algebra.

Proof:

(i) Because $\Omega \in \mathcal{F}_i$ for every $i \in I$, $\Omega \in \bigcap_{i \in I} \mathcal{F}_i$

(ii) Let $A \in \bigcap_{i \in I} \mathcal{F}_i$. Then for every $i \in I$, $A \in \mathcal{F}_i$ and because $\mathcal{F}_i$ is a σ-algebra for all $i \in I$, $A^c \in \mathcal{F}_i$ for every $i \in I$, so $A^c \in \bigcap_{i \in I} \mathcal{F}_i$.

(iii) Let $(A_n)_{n=1}^{\infty}$ be a collection in $\bigcap_{i \in I} \mathcal{F}_i$. Then $(A_n)_{n=1}^{\infty}$ is in $\mathcal{F}_i$ for every $i \in I$, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_i$

for all $i \in I$, hence, $\bigcup_{n=1}^{\infty} A_n \in \bigcap_{i \in I} \mathcal{F}_i$.

Exercise: Show that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras $\mathcal{F}_1 \cup \mathcal{F}_2$ may not be a σ-algebra.

Corollary: Let $\mathcal{A}$ be any collection of Ω. Let $\sigma(\mathcal{A})$ be the intersection of all σ-algebras containing $\mathcal{A}$. Then $\sigma(\mathcal{A})$ is a σ-algebra.

Note: We say that $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$. That is, if $\mathcal{B}$ is a σ-algebra containing $\mathcal{A}$, then $\sigma(\mathcal{A}) \subset \mathcal{B}$.

Examples:

(α) For $A \subset \Omega$, let $\mathcal{A} = \{A\}$. Then $\sigma(\mathcal{A}) = \{\phi, \Omega, A, A^c\}$

(β) Let $\mathcal{A}$ be the collection of all singleton sets, $\mathcal{A} = \{\{x\} : x \in \Omega\}$. Then $\sigma(\mathcal{A})$ is the collection of all countable subsets of Ω and their complements. (exercise).

Exercise: If $\mathcal{A} \subset \mathcal{B}$, then $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$. In particular if $\mathcal{A} \subset \sigma(\mathcal{B})$, then $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$

Definition: For $\Omega = \mathbb{R}^d$, $\mathcal{U}$ is the collection of all open balls, then $\sigma(\mathcal{U})$ is called the Borel σ-algebra denoted by $\mathcal{B}(\mathbb{R}^d)$.

Exercise: Let $\Omega = \mathbb{R}$ and let

$\mathcal{A} = \{(a,b) : a < b \in \mathbb{R}\}$ $\mathcal{B} = \{[a,b] : a < b \in \mathbb{R}\}$

and $\mathcal{C} = \{(a,b] : a < b \in \mathbb{R}\}$. Show that

$$\sigma(\mathcal{A}) = \sigma(\mathcal{B}) = \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}).$$

Constructing Probability Spaces

Theorem: Let Ω be a countable (possibly finite)
Let $f : \Omega \to [0,1]$ be a function such that

$$\sum_{x \in \Omega} f(x) = 1.$$

For every $A \subset \Omega$, define

$$P_f(A) = \sum_{x \in A} f(x).$$

Then $(\Omega, \mathcal{P}(\Omega), P_f)$ is a probability triple.

The above theorem deals with discrete distributions. Continuous distributions are somewhat more complicated to deal with. Looking ahead, here is a special case of a result from Chapter 6.

Example: Suppose $g : \mathbb{R} \to [0, \infty)$ is a piecewise continuous function and $\int_{-\infty}^{\infty} g(t)\,dt = 1$. Then we can define a probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_g)$ by

$$P_g(A) = \int_A g(t)\,dt \quad \text{for } A \in \mathcal{B}(\mathbb{R}).$$

In general, for a Borel set A, the above integral needs to be a Lebesgue Integral mentioned later.

Theorem: There exists a probability space $([0,1], \mathcal{M}, P^*)$ such that $\mathcal{M}$ contains every subinterval of $[0,1]$ and $P^*(J)$ is the length of J for every subinterval

$\mathcal{J}$ of $[0,1]$.

Definition: P^* above is called the **Lebesgue Measure on $[0,1]$**.

Coin Tossing

Recall that we talked about tossing three fair coins and defined a sequence of possible outcomes

$$\{H,T\}^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

For example $\{HHT, HTH, THH\}$ is the event of exactly one tail and two heads occur. We can model 3 tosses of a fair coin by the uniform probability measure $P(A) = \frac{|A|}{8}$. for all $A \subset \{H,T\}^3$.

Consider tossing a coin infinitely many times. Each elementary event is an infinite sequence of H, T.

$$\Omega_* = \{H,T\}^{\mathbb{N}} = \{x = (x_n)_{n=1}^{\infty} : x_n \in \{H,T\}\}.$$

Ω_∞ contains uncountably many sequences.

Basic Events: For $n \in \mathbb{N}$, let E_n be the event "nth toss is heads":

$$E_n = \{x \in \Omega_* : x_n = H\}.$$

Example: Let $\mathcal{A}_E = \{E_n\}_{n=1}^{\infty}$, i.e. $E_2 \cap E_5^c$ is the event "2nd toss is heads and 5th toss is tails".

This event is contained in $\sigma(\mathcal{A}_E)$.

Example: The event "all tosses are heads" is the one point set

$$\{(H, H, \dots, H)\} = \bigcap_{n=1}^{\infty} E_n \in \sigma(\mathcal{A}_E).$$

Indeed, every singleton is in $\sigma(\mathcal{A}_E)$.

Let $\mathcal{A}_1$ be the collection of all singleton sets in Ω_*. Note $\mathcal{A}_1 \subset \sigma(\mathcal{A}_E)$ so

$$\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_E) \text{ but } \sigma(\mathcal{A}_1) \neq \sigma(\mathcal{A}_E).$$

Indeed, $E_1 \notin \sigma(\mathcal{A}_1)$ so $\sigma(\mathcal{A}_1)$ is not the right set of events for probability theory.