Theorem: Let $\{\mu, \mu_n\}_{n=1}^{\infty}$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

TFAE:

(i) $\mu_n \to \mu$

(ii) $\lim_{n\to\infty} E_{\mu_n}(f) = E_\mu(f)$ for every bounded continuous function

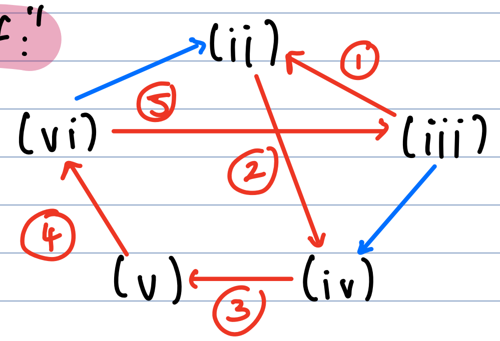
(iii) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all $A \in \mathcal{B}(\mathbb{R})$ such that $\mu(\partial A) = 0$

(iv) $\lim_{n\to\infty} \mu_n((-\infty, x]) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$ such that $\mu(\{x\}) = 0$

(v) There exists a probability space $(\Omega, \mathcal{F}, P)$ with RVs $\{Y, Y_n\}_{n=1}^{\infty}$ such that $\mathcal{L}(Y) = \mu$, $\mathcal{L}(Y_n) = \mu_n$ $\forall n$ and $Y_n \to Y$ a.s.

(vi) $\lim_{n\to\infty} E_{\mu_n}(f) = E_\mu(f)$ for every bounded Borel measurable function $f: \mathbb{R} \to \mathbb{R}$ such that $\mu(D_f) = 0$.

"**Proof:**"



(vi) $\Rightarrow$ (iii) because if $f = \mathbb{1}_A$,

$\int_{\mathbb{R}} \mathbb{1}_A \, d\mu_n = \mu_n(A)$

$\partial A = \{x : \mathbb{1}_A \text{ is discontinuous}\}$

Example: For each $n \in \mathbb{N}$, let μ_n be the uniform distribution on $\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$, i.e. $\mu_n = \sum_{k=1}^{n} \frac{1}{n} \delta_{\frac{k}{n}}$. Then the cdf of uniform distribution on $[0,1]$

What about $A = [0,1] \setminus \mathbb{Q}$? Then $\mu_n(A) = 0$, but $\mu(A) = 1$.

Proof: (ii) $\Rightarrow$ (iv) If $\mu(\{x\}) = 0$, let $\varepsilon > 0$ be arbitrary.

Then $F(x) = E_\mu(\mathbb{1}_{(\infty, x]})$. Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be continuous such that

$$g_1(t) = \begin{cases} 1 & \text{if } t \leq x - \varepsilon \\ 0 & \text{if } t \geq x \end{cases} \qquad g_2(t) = \begin{cases} 1 & \text{if } t \leq x \\ 0 & \text{if } t \geq x + \varepsilon \end{cases}$$

Then $\mathbb{1}_{(-\infty, x-\varepsilon]}(t) \leq g_1(t) \leq \mathbb{1}_{(-\infty, x]}(t) \leq g_2(t) \leq \mathbb{1}_{(-\infty, x+\varepsilon]}$, so

$E_{\mu_n}(g_1) \leq E_{\mu_n}(\mathbb{1}_{(-\infty, x]}) \leq E_{\mu_n}(g_2)$, so by (ii)

$F(x-\varepsilon) \leq E_\mu(g_1) \leq \limsup_{n\to\infty} F_n(x) \leq E_{\mu_n}(g_1) \leq F(x+\varepsilon)$ sending $\varepsilon \to 0$,

F is continuous at x, so $\limsup_{n\to\infty} F_n(x) = F(x)$. Similar for

$\liminf_{n\to\infty} F_n(x) = F(x)$, so we are done