Proposition: Assume that $\{\mu_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be nonnegative real numbers such that $\sum_{n=1}^{\infty} \beta_n = 1$. Then $\mu = \sum_{n=1}^{\infty} \beta_n \mu_n$ is a probability distribution and for every Borel function $g: \mathbb{R} \to \mathbb{R}$ we have $E_\mu(g) = \sum_{n=1}^{\infty} \beta_n E_{\mu_n}(g)$ when both sides are defined.

Example: $\mu_Y = \sum_{k=0}^{3} P(Y=k)\delta_k$, where δ_k is the point-mass distribution. Since $E_{\delta_k}(g) = g(k)$, we have then

$$E_Y(g) = \sum_{k=0}^{3} g(k) P(Y=k) = E_P(g(Y))$$

Example: If $X \sim \exp(1)$, then $\mu = 0.6\delta_0 + 0.4\mu_X$ can model amount of rainfall.

Corollary: Let X, Y be RVs. Then $\mu_X = \mu_Y$ if and only if $E(g(X)) = E(g(Y))$ for all Borel $g: \mathbb{R} \to \mathbb{R}$

Theorem: (Fatou's Lemma) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs on $(\Omega, \mathcal{F}, P)$. Assume $X_n \geq C$ for every $n \in \mathbb{N}$ for some $C \in \mathbb{R}$. Then

$$E(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} E(X_n).$$

Theorem: (Dominated Convergence Theorem) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs on $(\Omega, \mathcal{F}, P)$. Assume that $\{X_n\} \to X$ a.s and there exists a RV Y on $(\Omega, \mathcal{F}, P)$ with $E(Y)$ finite such that $|X_n| \leq Y$ for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} E(X_n) = E(X)$

Example: Consider $([0,1], \mathcal{B}([0,1]), \lambda)$ and let $X_n = n\mathbb{1}_{(0,\frac{1}{n})}$

Then $E(X_n) = n \cdot \frac{1}{n} = 1$ for all $n \in \mathbb{N}$, and $X_n \to 0$ a.s.

$E(\liminf_{n\to\infty} X_n) = E(0) = 0 \leq \liminf_{n\to\infty} E(X_n) = 1.$

Example: Consider $([0,1], \mathcal{B}([0,1]), \lambda)$ and for each $n \in \mathbb{N}$

$$X_n = \begin{cases} \mathbb{1}_{[0,\frac{1}{2})} & \text{if } n \text{ is even} \\ \mathbb{1}_{[\frac{1}{2},1]} & \text{if } n \text{ is odd} \end{cases}$$

$\liminf_{n\to\infty} X_n = 0$ a.s. and $E(X_n) = \frac{1}{2}$.

Moment Generating Functions

Definition: The moment generating function (MGF) of a RV X is the function $M_X : \mathbb{R} \to [0,\infty]$ defined by

$$M_X(s) = E(e^{sX})$$

Example: Let $X \sim \text{Uniform}[a,b]$. Then $f_X(t) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(t)$.

Then $M_X(s) = \int_{[a,b]} e^{sx} \frac{1}{b-a} \, dx = \frac{1}{(b-a)s}(e^{bs} - e^{as})$, $s \neq 0$

and $M_X(0) = 1$.

Example: Let $X \sim \text{Poisson}(\lambda)$. Then $P(X=n) = \frac{e^{-\lambda}\lambda^n}{n!}$

for $n \in \mathbb{N}_0$. Then

$$M_X(s) = \sum_{n=0}^{\infty} e^{sn} \cdot \frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda} \sum_{n=1}^{\infty} \frac{(\lambda e^s)^n}{n!} = e^{-\lambda} e^{\lambda e^s}$$

Example: Let $X \sim \exp(\beta)$ with $f_X(t) = \beta e^{-\beta t} \mathbb{1}_{[0,\infty)}(t)$.

Then

$$M_X(s) = \int_0^{\infty} e^{sx} \beta e^{-\beta x} \, dx = \beta \int_0^{\infty} e^{-(\beta - s)x} \, dx = \begin{cases} \infty & \beta - s \leq 0 \\ \frac{\beta}{\beta - s} & \beta - s > 0 \end{cases}$$

Example: $X \sim \text{Cauchy}$ with $f_X(t) = \frac{1}{\pi(1+t)^2}$. Then

$$M_X(s) = \int_{-\infty}^{\infty} \frac{e^{sx}}{\pi(1+x)^2} \, dx = \begin{cases} \infty & \text{if } s \neq 0 \\ 1 & \text{if } s = 0 \end{cases}$$

Proposition: Let X, Y be a RV.

(i) $M_X(0) = 1$

(ii) If X, Y are independent RVs, then $M_{X+Y}(s) = M_X(s) M_Y(s)$

(iii) If $M_X(s) = M_Y(s)$ is finite for all $s \in (-a, a)$ for $a > 0$,

then $\mu_X = \mu_Y$.

Example: If X_1, X_2 are independent RVs with $X_1 \sim \text{Poisson}(\lambda_1)$ $X_2 \sim \text{Poisson}(\lambda_2)$. Then $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

Indeed, for $s \in \mathbb{R}$

$$M_{X_1+X_2}(s) = M_{X_1}(s) M_{X_2}(s) = e^{-\lambda_1 + \lambda_1 e^s} e^{-\lambda_2 + \lambda_2 e^s}$$
$$= e^{-(\lambda_1+\lambda_2) + (\lambda_1+\lambda_2)e^s}$$

which is the MGF of Poisson $(\lambda_1 + \lambda_2)$, by (iii)

$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

Theorem: Let X be a RV. Assume $M_X(s) < \infty \ \forall s \in (-a, a)$ for some $a > 0$. Then

$$M_X(s) = \sum_{n=0}^{\infty} \frac{s^n E(X^n)}{n!} \quad \forall s \in (-a, a)$$

and consequently

$$\left. \frac{d^n M_X(s)}{ds^n} \right|_{s=0} = E(X^n)$$

Proof: Let $s \in (-a, a)$. Then

$M_X(s) = E(e^{sX}) = E\left(\sum_{n=1}^{\infty} \frac{s^n X^n}{n!}\right)$. Let $S_N = \sum_{n=1}^{N} \frac{s^k X^k}{n!} \to e^{sX}$ a.s

$|S_N| \le \sum_{n=1}^{N} \frac{|s^n X^n|}{n!} \le e^{s|X|} \le e^{sX} + e^{-sX} = Y$, and note

$E(e^{sX} + e^{-sX}) < \infty$, so $E(e^{sX}) = \lim_{n \to \infty} E(S_N) = \sum_{n=1}^{\infty} \frac{s^n E(X^n)}{n!}$

by DCT.

Remark: $\frac{d^n}{ds^n} E(e^{sX}) \stackrel{?}{=} E\left(\frac{d^n}{ds^n} e^{sX}\right) = E(X^n e^{sX})$