Projections and Complemented Subspaces in Banach Spaces

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Preface

This text grew out of a graduate-level survey paper exploring structural aspects of Banach space theory, with a particular focus on projections, complemented subspaces, and Schauder bases. Motivated by a desire to understand how finite-dimensional approximations inform the global geometry of Banach spaces, this exposition builds toward the celebrated Lindenstrauss–Tzafriri Theorem, which characterizes Hilbert spaces as those in which every closed subspace is complemented.

The text is intended for graduate students and researchers with a background in real and functional analysis, especially those interested in the geometry of Banach spaces. Familiarity with normed vector spaces, bounded linear operators, and basic topology is assumed, although key definitions and results are recalled throughout for convenience.

The presentation is structured to be self-contained and pedagogical. Early chapters develop foundational tools such as the relationship between projections and direct sums, complemented subspaces, and the theory of Schauder bases. Later sections examine more advanced topics including finite representability, spreading models, and quantitative versions of Dvoretzky's Theorem. The final chapter presents a detailed proof of the Lindenstrauss–Tzafriri Theorem, integrating ideas developed throughout the book.

I have benefited greatly from discussions with mentors, colleagues, and peers during the writing of this work. In particular, I would like to thank my supervisor for their valuable feedback, and for their encouragement and insight. I also thank the mathematical community for the inspiring literature that helped shape this exposition.

It is my hope that this book will serve not only as a reference, but also as an invitation to further study in the rich and intricate landscape of Banach space theory.

Contents

Chapter 1

Schauder Bases

In linear algebra, we are introduced to the concept of a Hamel basis, which serves as a coordinate system for a vector space. Formally, a Hamel basis is a linearly independent spanning set, which means that every vector in the space can be represented uniquely as a finite linear combination of basis elements. Although this concept works well in finite-dimensional spaces, it becomes problematic in the infinite-dimensional setting, regardless of whether the dimension is countable or uncountable. The issue is that Hamel bases are typically too large to handle constructively and, more importantly, do not interact well with the topology of normed spaces, making them unsuitable for analytic purposes.

In order to address this problem, a Polish mathematician, Juliusz Schauder (1899-1943)—known for his contributions to functional analysis, partial differential equations, and mathematical physics [17]—introduced the concept now known as a Schauder basis. Schauder's work emerged during a period of rapid development in functional analysis alongside the foundational results of Banach, Hilbert, and Hahn. His ideas helped shift the field toward a more geometric and topological viewpoint. Unlike a Hamel basis, a Schauder basis is a countable sequence of vectors such that every element in a Banach space can be written as a convergent infinite linear combination of those vectors.

In order to build the theory for Lindenstrauss–Tzafriri's Theorem, we are required to build knowledge on projections, complementability, and Schauder bases.

1.1 Projections and Complementability

In the study of Banach spaces, projections and decompositions play a central role in understanding the structure of subspaces. Many arguments in functional analysis rely on the ability to isolate parts of a space via linear projections and to decompose elements uniquely with respect to these subspaces.

Consider a vector space V and a subspace $W \subseteq V$. A natural question is whether V can be written as a direct sum $V = W \oplus U$ for some other subspace $U \subseteq V$. If such a decomposition exists, we say that U is an *algebraic complement* of W.

Definition 1.1.1. Let V be a vector space, and let $W, U \subseteq V$ be subspaces. We say that V is the *algebraic direct sum* of W and U, denoted as $V = W \oplus U$, if $V = W + U$ and $W \cap U = \{0\}$.

The first part of the definition means that for any $v \in V$, there exists a unique $w \in W$ and $u \in U$ such that $v = w + u$.

Definition 1.1.2. Let V be a vector space. A linear operator $P : V \to V$ is said to be a *linear projection* if $P^2 = P$.

We can observe that if V is a vector space and P is a linear projection on V, then V is the algebraic direct sum of its range $P(V)$ and its kernel $\ker(P)$. To see this, observe that for any $v \in V$, we can write

$$v = Pv + (v - Pv)$$

in which it is not difficult to see that $Pv \in P(V)$. On the other hand, we note that

$v - Pv \in \ker(P)$ because

$$P(v - Pv) = Pv - P^2v = Pv - Pv = 0$$

as P is a linear projection on V. Also, if $w \in P(V) \cap \ker(P)$, then there exists $v \in V$ such that $w = Pv$, but since $w \in \ker(P)$, then $Pv = 0$, and since P is a linear projection on V, then

$$0 = Pv = PPv = Pw$$

and thus, we have $w = 0$. Thus, we have that $V = P(V) \oplus \ker(P)$.

While every vector subspace has an algebraic complement, the situation is far more delicate when V is equipped with a norm, and we require W and U to be closed subspaces. This leads to the notion of topological complements and complemented subspaces.

Definition 1.1.3. Let X be a Banach space, and let Y be a closed subspace of X.

1. We say that Y is *complemented* in X if there exists a bounded linear projection $P : X \to X$ with $P(X) = Y$.
2. For $\lambda \geq 1$, we say that Y is λ-*complemented* if $P : X \to X$ is a projection with $P(X) = Y$ and $\|P\| \leq \lambda$.
3. We say that a subspace Z of X is a *topological complement* of Y in X, if $X = Y \oplus Z$ (that is, X is an algebraic direct sum of Y and Z) and Z is a closed subspace of X.

Fact 1.1.4. *Let X be a Banach space and let Y be a complemented subspace of X. Then, Y is closed.*

Proof. Since Y is a complemented subspace of X, then there exists a bounded linear projection $P : X \to X$ with $P(X) = Y$. Thus, since P is bounded, it is continuous, so $P(X) = Y = \ker(I - P) = (I - P)^{-1}(\{0\})$, which is the preimage of a closed set under a continuous map. Therefore, Y is closed. $\square$

Proposition 1.1.5. *Let X be a Banach space and let Y be a closed subspace of X. Then Y is complemented in X if and only if there exists a topological complement of Y in X.*

Proof. First we assume that Y is complemented in X. Then by Definition 1.1.3, there exists a bounded linear projection $P : X \to X$ with $P(X) = Y$. Take $Z = \ker(P)$, and we claim that Z is a topological complement of Y. Indeed, note that P is a bounded linear operator, so it is continuous, and thus, Z is closed. The fact that $X = Y \oplus Z$ follows from the observation after Definition 1.1.2.

Conversely, assume that there exists a topological complement Z of Y in X. Then by Definition 1.1.3, we have $X = Y \oplus Z$ and that Z is closed. Let $P : X \to X$ be the unique linear projection such that $P(X) = Y$ and $Z = \ker(P)$, which exists since $X = Y \oplus Z$. We need to show that P is bounded. To see this, let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $(x_n) \to 0$ and $(Px_n) \to y \in X$. We claim that $y = 0$. Indeed, for each $n \in \mathbb{N}$, we have $Px_n \in Y$, and as Y is a closed subspace of X, we have $y \in Y$. For each $n \in \mathbb{N}$, let $z_n = x_n - Px_n$. Then $(z_n)_{n=1}^{\infty}$ is a sequence in $Z = \ker(P)$ as for each $n \in \mathbb{N}$, we have

$$Pz_n = Px_n - P^2x_n = Px_n - Px_n = 0$$

and furthermore,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} Px_n = -y.$$

Since Z is a closed subspace, $-y \in Z$, and thus, $y \in Z$. Thus, as $y \in Y \cap Z$, and $X = Y \oplus Z$ by assumption, then $y = 0$, as claimed. $\square$

Example 1.1.6. Let $\mathcal{H}$ be a Hilbert space and let F be a closed subspace of $\mathcal{H}$. Then the orthogonal complement $F^{\perp}$ of F is also a closed subspace, and we have the orthogonal decomposition $\mathcal{H} = F \oplus F^{\perp}$, that is, $F^{\perp}$ is a topological complement of F in the sense of Definition 1.1.3. Moreover, there exists a unique bounded linear projection $P : \mathcal{H} \to \mathcal{H}$ with $P(\mathcal{H}) = F$, $\ker(P) = F^{\perp}$, and $\|P\| = 1$. In particular, F is a 1-complemented subspace of $\mathcal{H}$.

The following fact illustrates that being a complemented subspace is an isomorphic invariant in the category of Banach spaces.

Fact 1.1.7. *Let X and Y be Banach spaces and let $T : X \to Y$ be an isomorphism. Then X_1 is complemented in X with topological complement X_2 if and only if $T(X_1)$ is complemented in Y with topological complement $T(X_2)$.*

Proof. We prove that if X_1 is complemented in X with topological complement X_2, then $T(X_1)$ is complemented in Y with topological complement $T(X_2)$, as the proof for the converse follows by taking inverses. Let $T : X \to Y$ be an isomorphism, and assume that X_1 is complemented in X with topological complement X_2. Then, by Definition 1.1.3, we have that $X = X_1 \oplus X_2$ and that X_1 and X_2 are closed subspaces of X.

Let $y \in Y$ be arbitrary. Since T is an isomorphism, there exists a unique $x \in X$ such that $y = Tx$. Then by assumption x can be written uniquely as a sum of $x_1 \in X_1$ and $x_2 \in X_2$, so $x = x_1 + x_2$, and thus,

$$y = Tx = Tx_1 + Tx_2.$$

Then we have $Tx_1 \in T(X_1)$ and $Tx_2 \in T(X_2)$, so every $y \in Y$ can be written uniquely as $y = y_1 + y_2$, where $y_1 \in T(X_1)$ and $y_2 \in T(X_2)$, so we have $Y = T(X_1) \oplus T(X_2)$. The fact that X_1 and X_2 are closed, and T is an isomorphism implies that $T(X_1)$ and $T(X_2)$ are also closed subspaces of Y. Therefore, $T(X_1)$ is complemented in Y with topological complement $T(X_2)$. □

Remark 1.1.8. From Fact 1.1.7, if $P : X \to X_1$ is a projection with $\ker(P) = X_2$, then $Q = TPT^{-1}$ is a projection of Y onto $T(X_1)$ with $\ker(Q) = T(X_2)$.

Proposition 1.1.9. *Let X be a Banach space, and let Y and Z be closed subspaces of X. Assume that Y is complemented in X with topological complement Z. Then, the following holds.*

1. *$X/Y \simeq Z$. Moreover, if $P : X \to X$ is the projection onto Z parallel to Y with $\|P\| = 1$, then $X/Y \equiv Z$.*
2. *$X^* \simeq Y^* \oplus Z^*$; in short, $(Y \oplus Z)^* \simeq Y^* \oplus Z^*$. If $P : X \to X$ is the projection onto Y parallel to Z, then $Y^* \simeq P^*(X^*)$, and if $\|P\| = 1$, then $Y^* \equiv P^*(X^*)$.*

Proof. To prove the first statement, let $Q : Z \to X/Y$ be the map given by

$$Qz = [z]_Y$$

where $[z]_Y$ denotes the equivalence class containing z. We claim that Q is an isomorphism, so we show that Q is bounded, injective, and surjective.

To see that Q is bounded, note that for any $z \in Z$, by definition of the norm of quotient space,

$$\|Qz\| = \|[z]_Y\| = \inf\{\|y\| : y \in [z]_Y\} \leq \|z\|$$

so Q is bounded with $\|Q\| \leq 1$.

To see that Q is injective, let $z \in Z$ be such that $Qz = 0_{X/Y} = [0_X]_Y$. Then we have $[z]_Y = [0_X]_Y = Y$, which implies that $z \in Y$. Thus, since $X = Y \oplus Z$ by assumption, $z \in Y \cap Z$ implies that $z = 0$, so Q is injective.

To see that Q is surjective, let $[x]_Y \in X/Y$ be arbitrary where $x \in X$. Then since $X = Y \oplus Z$, there exists $y \in Y$ and $z \in Z$ such that $x = y + z$, so then

$$[x]_Y = [y + z]_Y = [y]_Y + [z]_Y = [0_X]_Y + [z]_Y = [z]_Y = Qz.$$

Therefore, we have shown that Q is surjective, and thus, $Q : Z \to X/Y$ is an isomorphism as claimed.

To prove the "moreover" statement, using our map Q, we need to show that for all $z \in Z$, $\|Qz\| = \|z\|$. Since Q is bounded and $\|Q\| \leq 1$, we have $\|Qz\| \leq \|z\|$. On the other hand, let $z \in Z$ and let $x \in X$ be such that $x \in [z]_Y$. Since $X = Y \oplus Z$, there exists a $y \in Y$ such that $x = y + z$, and thus, write $z = x - y$. Since P is a projection onto Z parallel to Y and $\|P\| = 1$, we have

$$\|z\| = \|x - y\| = \|P(x - y)\| = \|Px\| \leq \|P\|\|x\| = \|x\|$$

and thus, as $x \in [z]_Y$ was arbitrary, this implies that $\|z\| \leq \|Qz\|$, so we have shown that $\|Qz\| = \|z\|$, i.e. $Q : Z \to X/Y$ is an isometry, as desired.

In order to prove the second assertion, we make the following claim.

Claim. *Let X be a Banach space, and let $P : X \to P(X) \subseteq X$ be a bounded linear projection. Then $P(X)^* \simeq P^*(X^*)$, and furthermore, if $\|P\| = 1$, then $P(X)^* \equiv P^*(X^*)$.*

Proof of Claim. Since $P : X \to X$ is a bounded linear projection onto $P(X)$, define $\tilde{P} : X \to P(X)$ given by $\tilde{P}x = Px$. Then we have the dual operators $P^* : X^* \to X^*$ and $\tilde{P}^* : P(X)^* \to X^*$. We claim that $\tilde{P}^*$ is an isomorphism onto $P^*(X^*)$.

Note that since P is a bounded linear operator, it follows that $\tilde{P}^*$ is also a bounded linear operator.

To see that $\tilde{P}^*$ is injective, let $y^* \in P(X)^*$ be such that $P^*y^* = 0$. Then by definition of the dual operator, we have $y^* \circ P = 0$, so for any $x \in X$, we have $y^*(Px) = 0$, which implies that $y^* = 0$ on $P(X)$, so $\tilde{P}^*$ is injective.

To see that $\tilde{P}^*$ is surjective, we first need the following subclaim. Let $f \in P^*(X^*)$. Then there exists $x^* \in X^*$ such that $f = P^*x^* = x^* \circ P$. Let $g = x^*|_{P(X)} \in P(X)^*$. Then observe that for any $x \in X$,

$$(P^*g)(x) = g(Px) = x^*(Px) = (P^*x^*)(x) = f(x).$$

Thus, $P^*g = f$, so P^* is surjective. Therefore, we have shown that $P^* : P(X)^* \to P^*(X^*)$ is an isomorphism.

To prove the "furthermore" statement, note that since $\|P\| = 1$ by assumption, then we have $\|P^*\| = 1$ also, so for any $g \in P(X)^*$, we have $\|P^*g\| \leq \|g\|$. On the other hand, since $P^*g = g \circ P$ and $g \in P(X)^*$, by the Hahn-Banach Theorem (Theorem A.0.3), there exists $\tilde{g} \in X^*$ such that $\tilde{g}|_{P(X)} = g$, i.e. $\tilde{g}$ is a linear extension of g to X, and that $\|\tilde{g}\| = \|g\|$, thus,

$$\|P^*g\| = \|g \circ P\| \geq \|\tilde{g}\| = \|g\|.$$

Therefore, by combining both inequalities, we obtain that $\|P^*g\| = \|g\|$, and thus, P^* is an isometric isomorphism. $\square$

With the claim, let $P : X \to X$ be a bounded linear projection onto Y and let $Q = I_X - P : X \to X$ be a bounded linear projection onto Z. Then by the claim, we have that $Y^* \simeq P^*(X^*)$ and $Z^* \simeq Q^*(X^*)$. In particular, since Y is complemented in X with topological complement Z, we have $I_X = P + Q$, so $I_{X^*} = P^* + Q^*$, which implies that $X^* = P^*(X^*) + Q^*(X^*)$. Now we need to show that $P^*(X^*) \cap Q^*(X^*) = \{0\}$. To see this, let $f \in P^*(X^*) \cap Q^*(X^*)$ be arbitrary. Since $f \in Q^*(X^*)$, for any $y \in Y$, we have

$$f(y) = (Q^*f)(y) = f(Qy) = 0$$

and also, since $f \in P^*(X^*)$, for any $z \in Z$, we have

$$f(z) = (P^*f)(z) = f(Pz) = 0.$$

Therefore, it follows that $f = 0$. From the claim, we note that $\ker(P) = Q(X) = Z$, and thus, $X^* = P^*(X^*) \oplus Q^*(X^*)$, and thus, we also have $Z^* = Q(X)^* \simeq Q^*(X^*)$. Furthermore, by Proposition 1.1.5, it follows that $P^*(X^*)$ and $Q^*(X^*)$ are closed subspaces of X^*, and so we have $X^* = P^*(X^*) \oplus Q^*(X^*)$.

The "moreover" part of the second statement follows from the second part of the claim. $\square$

The following result is a direct consequence of Proposition 1.1.9, applied to the setting of Hilbert spaces, where orthogonal complements yield norm-one projections.

Corollary 1.1.10. *Let $\mathcal{H}$ be a Hilbert space, and let Y be a closed subspace of $\mathcal{H}$. If Z is the orthogonal complement of Y in $\mathcal{H}$, then $\mathcal{H}/Y \equiv Z$.*

Having established the relationship between projections, complemented subspaces, and dual space decompositions, we now turn to a concrete example of a particularly well-behaved basis in Banach space theory, known as an *Auerbach basis*. This concept is closely related to the idea of biorthogonal systems and serves as a helpful tool in studying finite-dimensional Banach spaces, particularly in the context of Lindenstrauss–Tzafriri's Theorem.

The Auerbach basis provides a coordinate-like structure in an arbitrary finite-dimensional Banach space that imitates the standard bases of ℓ_p^n or $\mathbb{R}^n$, with the additional property that both the bases and their associated coordinate functionals are norm one.

Definition 1.1.11. Let X be a Banach space and let $\{e_1, e_2, ..., e_n\} \subseteq X$. The associated coordinate functionals $\{f_1, f_2, ..., f_n\} \subseteq X^*$ are said to be *biorthogonal* to $\{e_1, e_2, ..., e_n\}$ if for all $1 \le i, j \le n$, $f_i(e_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We denote $\{e_i; f_i\}_{i=1}^n$ to be a *biorthogonal system* in $X \times X^*$. Furthermore, if $\{e_i; f_i\}_{i=1}^n$ is a biorthogonal system of $X \times X^*$, $\{e_1, e_2, ..., e_n\}$ is a basis of X, and $\|e_i\| = \|f_i\| = 1$ for all $1 \le i \le n$, then we say that $\{e_i; f_i\}_{i=1}^n$ is an *Auerbach basis* of X.

In general, not every closed subspace of a Banach space X is complemented. However, two important exceptions are worth noting: every finite-dimensional subspace of X is complemented, and so is every closed subspace of finite codimension. The following theorem by Auerbach establishes the connection between finite-dimensional Banach spaces, Auerbach bases, and bounded linear projections.

Theorem 1.1.12 (Auerbach's Lemma)**.** *Every finite-dimensional Banach space admits an Auerbach basis.*

Proof. This proof follows the classical volume-maximization argument; see, for instance, [5, Theorem 4.5]. Assume that X is a finite-dimensional Banach space with $\dim(X) = n < \infty$, and let $\{x_1, x_2, ..., x_n\}$ be a Hamel basis of X. The idea is to maximize the volume of the parallelepiped generated by the vectors $u_1, u_2, ..., u_n \in B_X$. Indeed, let $|\det(u_1, u_2, ..., u_n)|$ be the volume of the parallelepiped generated by $u_1, u_2, ..., u_n \in B_X$ (the matrix of the jth formed by the coordinates of u_j in the Hamel basis $\{x_1, x_2, ..., x_n\}$).

Here, we note that $|\det|$ is continuous on B_X^n, which is compact since X is finite-dimensional, so B_X is compact, and thus, B_X^n is compact. Therefore, we choose $(e_1, e_2, ..., e_n) \in B_X^n$ such that

$$|\det(e_1, e_2, ..., e_n)| = \max\{|\det(x_1, x_2, ..., x_n)| : (x_1, x_2, ..., x_n) \in B_X^n\}.$$

Therefore, as determinants are homogeneous, we have that $\|e_i\| = 1$ for all $1 \leq i \leq n$. Therefore, this implies that $\det(e_1, e_2, ..., e_n) \neq 0$, and so the vectors $\{e_1, e_2, ..., e_n\}$ are linearly independent, and thus, form a Hamel basis of X.

Next, for each $1 \leq i \leq n$, let $f_i : X \to \mathbb{R}$ be defined by

$$f_i(x) = \frac{\det(e_1, e_2, ..., e_{i-1}, x, e_{i+1}, ..., e_n)}{\det(e_1, e_2, ..., e_n)}$$

for $x \in X$. This definition uses the multilinearity and alternating property of the determinant. It is easy to verify that each f_i is linear. We claim that f_i is bounded for all $1 \leq i \leq n$. Indeed, fix $1 \leq i \leq n$ and fix $x \in B_X$. Then observe that we have

$$|\det(e_1, e_2, ..., e_{i-1}, x, e_{i+1}, ..., e_n)| \leq |\det(e_1, e_2, ..., e_n)|$$

since we have chosen $(e_1, e_2, ..., e_n) \in B_X^n$ to give us the largest volume of the parallelepiped, so then

$$|f_i(x)| = \left|\frac{\det(e_1, e_2, ..., e_{i-1}, x, e_{i+1}, ..., e_n)}{\det(e_1, e_2, ..., e_n)}\right| \leq \left|\frac{\det(e_1, e_2, ..., e_n)}{\det(e_1, e_2, ..., e_n)}\right| = 1.$$

Therefore, f_i is bounded with $\|f_i\| \leq 1$. Moreover, by the alternating property of the determinant, we also have that $f_i(e_j) = \delta_{i,j}$ for $1 \leq i, j \leq n$. Therefore, we have shown that $\{e_i, f_i\}_{i=1}^n$ is a biorthogonal system. Furthermore, it is easy to note that $\|f_i\| \geq |f_i(e_i)| = 1$ for any $1 \leq i \leq n$, which then shows that $\|f_i\| = 1$. Therefore, $\{e_i; f_i\}_{i=1}^n$ is an Auerbach basis as claimed. $\square$

Using Auerbach's Lemma (Theorem 1.1.12) and Proposition 1.1.5, we obtain the following results.

Corollary 1.1.13. *Let X be a Banach space.*

1. *If Y is a finite-dimensional subspace of X, then Y is n-complemented; that*

is, there exists a bounded linear projection $P : X \to X$ with $P(X) = Y$ such that $\|P\| \leq n$.

2. *If Y is a closed cofinite-dimensional subspace of X, then Y is complemented.*

Proof. To see that (1) holds, let $\{e_i; f_i\}_{i=1}^n$ be an Auerbach basis of Y, where $\{e_i\}_{i=1}^n$ is a basis of Y, $\{f_i\}_{i=1}^n$ are the associated coordinate functionals in Y, and $\|e_i\| = \|f_i\| = 1$ for all $1 \leq i \leq n$. Using the Hahn-Banach Theorem (Theorem A.0.3), we extend each $f_i \in Y^*$ to a functional in X^*; that is, there exists $\tilde{f}_i \in X^*$ such that $\|\tilde{f}_i\| = \|f_i\| = 1$ and $\tilde{f}_i|_Y = f_i$. Then define $P : X \to X$ by

$$Px = \sum_{i=1}^{n} \tilde{f}_i(x) e_i$$

for $x \in X$. We claim that P is a bounded linear projection. To see that P is a projection, note that if $x \in Y$, write $x = \sum_{j=1}^n \alpha_j e_j$ for $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$. Then

$$Px = P\left(\sum_{j=1}^{n} \alpha_j e_j\right) = \sum_{j=1}^{n} \alpha_j P e_j = \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{n} f_i(e_j) e_i = \sum_{j=1}^{n} \alpha_j e_j = x.$$

Therefore, P is a projection as claimed. Next, to see that P is bounded, let $x \in X$. Then we have that by the triangle inequality

$$\|Px\| = \left\|\sum_{i=1}^{n} \tilde{f}_i(x) e_i\right\| \leq \sum_{i=1}^{n} |\tilde{f}_i(x)| \|e_i\| \leq \sum_{i=1}^{n} \|\tilde{f}_i\| \|e_i\| \|x\| = \|x\| \sum_{i=1}^{n} 1 = n\|x\|.$$

Therefore, P is bounded with $\|P\| \leq n$. Therefore, we have shown that Y is n-complemented in X.

To see that (2) is true, we will show that Y has a topological complement of X. Assume that $\dim(X/Y) = n < \infty$. Let $Q : X \to X/Y$ denote the quotient map, and let $\{[e_1]_Y, [e_2]_Y, ..., [e_n]_Y\}$ be a Hamel basis of X/Y, where each $e_i \in X$. Since Q is onto, for each $1 \leq i \leq n$, choose $x_i \in X$ such that $Qx_i = [e_i]_Y$, and let $Z = \operatorname{span}\{x_1, x_2, ..., x_n\}$. We claim that Z is a topological complement of Y in X. To see this, we need to show that $X = Y \oplus Z$ and that Z is closed. Since Z is a finite-dimensional subspace of X, it is closed. Let $x \in X$ be arbitrary. Then, we have by the quotient map

$$Qx = \sum_{i=1}^{n} \alpha_i [e_i]_Y$$

for some $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ as $\{[e_1]_Y, [e_2]_Y, ..., [e_n]_Y\}$ is a Hamel basis of X/Y. Then let $z = \sum_{i=1}^{n} \alpha_i x_i$ so that

$$Qz = \sum_{i=1}^{n} \alpha_i Q x_i = \sum_{i=1}^{n} \alpha_i [e_i]_Y = Qx.$$

Therefore, we have then $Q(x - z) = 0$ or $x - z \in Y$, and thus, $x \in Y + z \subseteq Y + Z$. Next, to see that $Y \cap Z = \{0\}$, let $x \in Y \cap Z$. Since $x \in Z$, write $x = \sum_{i=1}^{n} \alpha_i x_i$ for some $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$. Also, since $x \in Y$, we have $Qx = [0]_Y$. Therefore,

$$[0]_Y = Qx = \sum_{i=1}^{n} \alpha_i Q x_i = \sum_{i=1}^{n} \alpha_i [e_i]_Y.$$

Therefore, since $\{[e_1]_Y, [e_2]_Y, ..., [e_n]_Y\}$ is a Hamel basis of X/Y, it is linearly independent, so it follows that $\alpha_i = 0$ for all $1 \leq i \leq n$, and so

$$x = \sum_{i=1}^{n} \alpha_i x_i = 0.$$

Therefore, $Y \cap Z = \{0\}$. $\square$

[5] outlined another approach for (2) of Corollary 1.1.13; by taking an Auerbach basis $\{f_i; F_i\}_{i=1}^n$, where $\{f_i\}_{i=1}^n$ is a Hamel basis of $Y^{\perp}$, where $Y^{\perp}$ is the annihilator of Y, and $\{F_i\}_{i=1}^n \subseteq (Y^{\perp})^*$. Note that in this case, $(X/Y)^* \equiv Y^{\perp}$, and that $(X/Y)^{**} \equiv (Y^{\perp})^*$. Then one can show that for any $\varepsilon > 0$, there exists a bounded linear projection $P : X \to X$ onto Y such that $\|P\| < n + 1 + \varepsilon$, in which case, it shows that Y is complemented in X.

1.2 Schauder Bases

In the study of vector spaces, Hamel bases serve as fundamental tools for understanding the algebraic structure, while in Hilbert spaces, orthonormal bases play a similar central role in analysis. Recall that, given a Hilbert space $\mathcal{H}$, a sequence of vectors $\{e_n\}_{n=1}^{\infty}$ is called an orthonormal basis of $\mathcal{H}$ if every $x \in \mathcal{H}$ has a unique representation in terms of the coefficients described by the sequence $(\langle x, e_n \rangle)_{n=1}^{\infty}$, that is,

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

where the convergence is in norm. Orthonormal bases are useful and relatively easy to construct using tools such as the Gram-Schmidt Orthogonalization Process.

The idea of a Schauder basis generalizes the coordinate representation property of orthonormal bases in Hilbert spaces to arbitrary Banach spaces.

Definition 1.2.1. Let X be a infinite-dimensional normed space. A sequence of vectors $\{e_n\}_{n=1}^{\infty}$ of X is called a *Schauder basis* if for every $x \in X$, there exists a unique sequence of scalars $(a_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ such that

$$x = \sum_{n=1}^{\infty} a_n e_n$$

where the series converges in the norm topology of X.

Remark 1.2.2. Every Banach space with a Schauder basis is separable. Indeed, if $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of a Banach space X, consider the set

$$D = \left\{ \sum_{i=1}^{n} q_i e_i : n \in \mathbb{N}, q_i \in \mathbb{Q} \right\}$$

which consists of finite linear combinations of the basis vectors with rational coefficients. Then D is countable, as a countable union of finite-dimensional vector spaces over $\mathbb{Q}$. Furthermore, since span $\{e_n : n \in \mathbb{N}\}$ is dense in X, by the definition of a Schauder basis, D is also dense in X, so X is separable.

Remark 1.2.3. If X is a finite-dimensional Banach space, then every Hamel basis of X is a Schauder basis of X. This is because every finite linear combination is trivially convergent in the norm topology, so every element has a unique representation with respect to any Hamel basis of X, which satisfies the definition of a Schauder basis.

Remark 1.2.4. If $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of a Banach space X, then $\{e_n\}_{n=1}^{\infty}$ is linearly independent. Indeed, if there exists some nontrivial finite linear combination of the e_n such that

$$\sum_{i=1}^{n} a_i e_i = 0$$

for some $n \in \mathbb{N}$ and $a_i \neq 0$, this would contradict the unique representation of the zero vector.

For a Schauder basis, the approximations of vectors by finite linear combinations of basis elements are naturally captured by projection operators. These are known as the canonical projections associated with the basis.

Definition 1.2.5. Let X be a Banach space, and let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of X. For $n \in \mathbb{N}$, we denote the *nth canonical projection* $\Pi_n : X \to X$ by

$$\Pi_n \left(\sum_{i=1}^{\infty} a_i e_i \right) = \sum_{i=1}^{n} a_i e_i.$$

These projections approximate any $x \in X$ in the sense that

$$\Pi_n x = \sum_{i=1}^{n} a_i e_i \to x$$

in norm as $n \to \infty$. Such canonical projections satisfy the following properties:

Lemma 1.2.6. *Let X be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of X. If for each $n \in \mathbb{N}$, $\Pi_n : X \to X$ denotes the nth canonical projection of X, then the following holds*

1. $\dim(\Pi_n(X)) = n$ *for all* $n \in \mathbb{N}$.

2. $\Pi_n \Pi_m = \Pi_m \Pi_n = \Pi_{\min(n,m)}$ *for all* $n, m \in \mathbb{N}$*.*

3. $\Pi_n x \to x$ *in norm for all* $n \in \mathbb{N}$ *and* $x \in X$*.*

We omit the proof of Lemma 1.2.6 as it is easy.

Conversely, a sequence of projections satisfying the properties of Lemma 1.2.6 generates a Schauder basis. That is, if a sequence of projections satisfies the same structural properties as the canonical projections of a Schauder basis, then it actually arises from one. This leads to the following characterization, which provides a method for constructing Schauder bases from sequences of projections.

Lemma 1.2.7. *Let* X *be a Banach space. If* $\{\Pi_n\}_{n=1}^{\infty}$ *is a sequence of bounded linear projections that satisfies (1)-(3) of Lemma 1.2.6, then* $\{\Pi_n\}_{n=1}^{\infty}$ *are canonical projections associated with some Schauder basis of* X*.*

Proof. Let $\Pi_0 = 0$, and for each $n \in \mathbb{N}$, since $\dim(\Pi_n(X)) = n$ and $\Pi_{n-1}(X) \subseteq \Pi_n(X)$ with $\dim(\Pi_n(X)) = \dim(\Pi_{n-1}(X)) + 1$, then the subspace

$$E_n = \Pi_n(X) \cap \ker(\Pi_{n-1})$$

is one-dimensional. Therefore, for each $n \in \mathbb{N}$, let $e_n \in E_i$ be a nonzero vector. Then $\{e_i\}_{i=1}^{\infty}$ is a linearly independent sequence for any $n \in \mathbb{N}$ and

$$\Pi_n(X) = \operatorname{span}\{e_1, e_2, ..., e_n\}.$$

For each $x \in X$, note that $x = \Pi_n x - \Pi_{n-1} x \in E_n$, so there exists an $a_n \in \mathbb{R}$ such that

$$\Pi_n x - \Pi_{n-1} x = a_n e_n.$$

Then using the telescopic sum,

$$x = \lim_{n\to\infty} \Pi_n x = \lim_{n\to\infty} (\Pi_n x - \Pi_0 x) = \lim_{n\to\infty} \sum_{i=1}^{n} (\Pi_i x - \Pi_{i-1} x) = \sum_{i=1}^{\infty} a_i e_i.$$

Now we need to show that the scalars $(a_i)_{i=1}^{\infty}$ are unique. Assume that $(b_i)_{i=1}^{\infty}$ is another sequence of scalars such that $x = \sum_{i=1}^{\infty} b_i e_i$. Then for each $n \in \mathbb{N}$, we have

$$\Pi_n x = \sum_{i=1}^{n} b_i e_i = \sum_{i=1}^{n} a_i e_i.$$

Since for each $n \in \mathbb{N}$, $\{e_1, e_2, ..., e_n\}$ are linearly independent, and passing to the limit as $n \to \infty$, it follows that $a_i = b_i$ for all $i \in \mathbb{N}$, and so the scalars are unique. $\square$

Fact 1.2.8. *Let X be a normed space with Schauder basis $\{e_i\}_{i=1}^{\infty}$ and canonical projections $\{\Pi_n\}_{n=1}^{\infty}$. If $\sup_{n\in\mathbb{N}} \|\Pi_n\| < \infty$ ($\{\Pi_n\}_{n=1}^{\infty}$ are uniformly bounded), then $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis of the completion $\mathscr{X}$ of X.*

Proof. For each $n \in \mathbb{N}$, let $\tilde{\Pi}_n$ be the extensions of Π_n on $\mathscr{X}$. We will show that $\{\tilde{\Pi}_n\}_{n=1}^{\infty}$ satisfies the conditions of Lemma 1.2.7.

To see that (1) holds, first note that since $\Pi_n(X)$ is finite-dimensional, it is closed and contained in $\mathscr{X}$, so then $\tilde{\Pi}_n(\mathscr{X}) = \Pi_n(X)$, and thus, $\dim(\tilde{\Pi}_n(\mathscr{X})) = n$.

To see that (2) holds, note that for all $n, m \in \mathbb{N}$, we have that $\Pi_n \Pi_m = \Pi_{\min(n,m)}$ holds on X. Since Π_n are uniformly bounded and X is dense in $\mathscr{X}$, this identity extends to $\tilde{\Pi}_n$ on $\mathscr{X}$ by continuity, so $\tilde{\Pi}_n \tilde{\Pi}_m = \tilde{\Pi}_{\min(n,m)}$ on $\mathscr{X}$.

Finally, to see that (3) holds, since for any $x \in X$, we have $\Pi_n x \to x$ in norm, and since $\{\Pi_n\}_{n=1}^{\infty}$ are uniformly bounded on X and X is dense in $\mathscr{X}$, it follows that $\tilde{\Pi}_n \tilde{x} \to \tilde{x}$ for all $\tilde{x} \in \mathscr{X}$.

Furthermore, note that for each $n \in \mathbb{N}$, $e_n \in \Pi_n(X) \cap \ker(\Pi_{n-1})$, and hence, $e_n \in \tilde{\Pi}_n(\mathscr{X}) \cap \ker(\tilde{\Pi}_{n-1})$. Therefore, by Lemma 1.2.7, it follows that $\{e_i\}_{i=1}^{\infty}$ forms a Schauder basis for $\mathscr{X}$. $\square$

Given a vector space V and a Hamel basis B, there is an association between the Hamel basis and a sequence of coordinate functionals $\{f_b\}_{b\in B}$. In particular, for any $x \in V$, we have

$$x = \sum_{b \in B} f_b(x) b$$

which is well-defined since $f_b(x) = 0$ for all but finitely many $b \in B$. These functionals are linear and uniquely determined by the coordinates of x with respect to the Hamel basis B, but they are not necessarily bounded when V is equipped with a norm.

With this information, let X be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of X. Since $\Pi_n(X)$ is finite-dimensional with Hamel basis $\{e_1, e_2, ..., e_n\}$, we can associate the Hamel basis to a sequence of coordinate functionals given by $\{e_1^\sharp, e_2^\sharp, ..., e_n^\sharp\}$ so that for any $x \in X$,

$$\Pi_n x = \sum_{i=1}^{n} e_i^\sharp(x) e_i$$

and since $\Pi_n x \to x$ in norm as $n \to \infty$, we obtain

$$x = \sum_{i=1}^{\infty} e_i^\sharp(x) e_i$$

where these functionals $\{e_i^\sharp\}_{i=1}^{\infty}$ are linear, and are uniquely determined by the coordinates of x with respect to the Schauder basis $\{e_i\}_{i=1}^{\infty}$.

A natural question that arises is the following: Can we expect these functionals to be bounded? Such a sequence would form a biorthogonal system with $\{e_i\}_{i=1}^{\infty}$, and each coordinate functional would act as a coordinate functional extracting the ith coefficient in the expansion of x. In finite-dimensional Banach spaces, these coordinate functionals are automatically continuous. Otherwise, for this representation to be valid and well-behaved, we must ensure that the coordinate functionals $e_i^* = e_i^\sharp$ are bounded, that is, each $e_i^* \in X^*$, and that the corresponding canonical projections are uniformly bounded. In order to answer this question, we need to know if whether the canonical projections are uniformly bounded.

Theorem 1.2.9 (Banach's Theorem)**.** *Let X be a Banach space and let $\{\Pi_n\}_{n=1}^{\infty}$ be a sequence of canonical projections on X. Then $\{\Pi_n\}_{n=1}^{\infty}$ are uniformly bounded* *on X, that is,*

$$\sup_{n\in\mathbb{N}} \|\Pi_n\| < \infty.$$

Before proving the theorem, we need the following lemma.

Lemma 1.2.10. *Let $(X, \|\cdot\|)$ be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis on $(X, \|\cdot\|)$. Define the function $|||\cdot||| : X \to [0, \infty)$ by*

$$|||x||| = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} a_i e_i \right\|$$

for $x = \sum_{i=1}^{\infty} a_i e_i \in X$. The following statements hold.

1. *$|||\cdot|||$ is a norm on X.*
2. *$\{e_i\}_{i=1}^{\infty}$ is a Schauder basis on $(X, |||\cdot|||)$.*
3. *$\{\Pi_n\}_{n=1}^{\infty}$ are uniformly bounded by 1 on $(X, |||\cdot|||)$.*
4. *$(X, |||\cdot|||)$ is a Banach space.*
5. *$(X, \|\cdot\|) \sim (X, |||\cdot|||)$.*
6. *If $\{e_i^{\sharp}\}_{i=1}^{\infty}$ are coordinate functionals associated to $\{e_i\}_{i=1}^{\infty}$, then $\{e_i^{\sharp}\}_{i=1}^{\infty}$ are bounded.*

Proof. To see that (1) holds, first note that if $x = \sum_{i=1}^{\infty} a_i e_i = 0$, then for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} a_i e_i = 0$. Then $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^{n} a_i e_i\| = 0$. Conversely, if $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^{n} a_i e_i\| = 0$, then for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} a_i e_i = 0$, and hence, $x = \sum_{i=1}^{\infty} a_i e_i = 0$.

Next, for any $x = \sum_{i=1}^{\infty} a_i e_i \in X$ and $\alpha \in \mathbb{R}$, observe that

$$|||\alpha x||| = \left|\left|\left| \sum_{i=1}^{\infty} \alpha a_i e_i \right|\right|\right| = \sup_{n\in\mathbb{N}} \left\| \alpha \sum_{i=1}^{n} a_i e_i \right\| = |\alpha| \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} a_i e_i \right\| = |\alpha|\ |||x|||.$$

Finally, for any $x = \sum_{i=1}^{\infty} a_i e_i, y = \sum_{i=1}^{\infty} b_i e_i \in X$, we have $x + y = \sum_{i=1}^{\infty}(a_i + b_i)e_i$ and

$$\begin{aligned}
|||x+y||| &= \left|\left|\left|\sum_{i=1}^{\infty}(a_i+b_i)e_i\right|\right|\right| = \sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n}(a_i+b_i)e_i\right\| = \sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n}a_ie_i + \sum_{i=1}^{n}b_ie_i\right\| \\
&\leq \sup_{n\in\mathbb{N}}\left(\left\|\sum_{i=1}^{n}a_ie_i\right\| + \left\|\sum_{i=1}^{n}b_ie_i\right\|\right) \leq \sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n}a_ie_i\right\| + \sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n}b_ie_i\right\| \\
&= |||x||| + |||y|||.
\end{aligned}$$

Therefore, we have shown that $|||\cdot|||$ is a norm on X. As an observation, note that since $\Pi_n x \to x$ in norm for all $x \in X$, we have $|||x||| \geq \|x\|$ for all $x \in X$.

To see that (2) holds, let $\{\Pi_n\}_{n=1}^{\infty}$ be the sequence of canonical projections on $(X, |||\cdot|||)$. We verify the properties of Lemma 1.2.7 with respect to the norm $|||\cdot|||$. The first two properties are obvious, as $\{\Pi_n\}_{n=1}^{\infty}$ is a sequence of canonical projections associated to $\{e_i\}_{i=1}^{\infty}$ with respect to the norm $\|\cdot\|$. To see that the third property holds, note that for any $x \in X$, we have

$$\lim_{m\to\infty} |||x - \Pi_m x||| = \lim_{m\to\infty}\sup_{n\in\mathbb{N}}\|\Pi_n x - \Pi_n\Pi_m x\| = \lim_{m\to\infty}\sup_{n\geq m}\|\Pi_n x - \Pi_m x\| = 0.$$

Therefore, $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis on $(X, |||\cdot|||)$.

To see that (3) holds, denoting $B_{(X,|||\cdot|||)} = \{x \in B_X : \sup_{i\in\mathbb{N}}\|\Pi_i x\| \leq 1\}$, we have for each $m \in \mathbb{N}$,

$$\begin{aligned}
|||\Pi_m||| &= \sup_{x\in B_{(X,|||\cdot|||)}} |||\Pi_m x||| \\
&= \sup_{x\in B_{(X,|||\cdot|||)}}\sup_{n\in\mathbb{N}}\|\Pi_n\Pi_m x\| \\
&= \sup_{n\in\mathbb{N}}\sup_{x\in B_{(X,|||\cdot|||)}}\|\Pi_n\Pi_m x\| \\
&\leq 1.
\end{aligned}$$

Therefore, $\{\Pi_n\}_{n=1}^{\infty}$ are uniformly bounded by 1 on $(X, |||\cdot|||)$.

To see that (4) holds, let $\varepsilon > 0$ be arbitrary, and let $(x_m)_{m=1}^{\infty}$ be a Cauchy sequence in $(X, |||\cdot|||)$. Note that we have $x_m \xrightarrow{\|\cdot\|} x \in X$. We need to show that $x_m \xrightarrow{|||\cdot|||} x$.

Note that for each $n \in \mathbb{N}$, we have $\Pi_n x_m \xrightarrow{\|\cdot\|} y_n \in X$ and that $(\Pi_n x_m)_{m=1}^{\infty}$ is contained on the finite-dimensional subspace $[\{e_1, e_2, ..., e_n\}]$. Moreover, the functionals $e_i^\sharp$ are continuous on any finite-dimensional subspace, so for each $1 \leq i \leq n$,

$$\lim_{m\to\infty} e_i^\sharp(x_m) = e_i^\sharp(y_n).$$

Now we need to show that $\sum_{i=1}^{n} e_i^\sharp(y_n)e_i \xrightarrow{\|\cdot\|} x$.

Indeed, since $(x_m)_{m=1}^{\infty}$ is a Cauchy sequence, choose $m \in \mathbb{N}$ such that for all $k \geq m$,

$$|||x_k - x_m||| < \frac{\varepsilon}{3}.$$

Also, since $\Pi_n x_m \to x_m$, choose $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|x_m - \Pi_n x_m\| < \frac{\varepsilon}{3}.$$

Therefore, by the triangle inequality, we have

$$\begin{aligned}
\|y_n - x\| &\leq \lim_{k\to\infty} \|\Pi_n x_k - \Pi_n x_m\| + \|\Pi_n x_m - x_m\| + \lim_{k\to\infty} \|x_k - x_m\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|y_n - x\| = 0$, and so by the uniqueness of the expansion of x, we have $\Pi_n x = y_n$. Furthermore,

$$|||x_m - x||| = \sup_{n\in\mathbb{N}} \|\Pi_n x_m - \Pi_n x\| \leq \limsup_{k\to\infty} \sup_{n\in\mathbb{N}} \|\Pi_n x_m - \Pi_n x_k\|$$

and so $x_m \xrightarrow{|||\cdot|||} x$, showing that $(X, |||\cdot|||)$ is complete.

To see that (5) holds, note that by the Closed Graph Theorem (Theorem A.0.7), the identity map $I_X : (X, \|\cdot\|) \to (X, |||\cdot|||)$ is bounded, since $\lim_{n\to\infty} \|x - \Pi_n x\| = 0$

and that $\|x\| \leq |||x|||$ for all $x \in X$, so there exists a $K > 0$ such that $\|\Pi_n x\| \leq K\|x\|$ for all $x \in X$ and $n \in \mathbb{N}$.

To see that (6) holds, note that for any $i \in \mathbb{N}$ and $x \in X$, by the proof of (5), we have

$$|e_i^\sharp(x)|\|e_i\| = \|\Pi_n x - \Pi_{n-1}x\| \leq 2\|x\|$$

which shows that $e_i^\sharp \in X^*$, and moreover,

$$\|e_i^*\| \leq 2K\|e_i\|^{-1}$$

as desired. $\square$

To answer the question posed above, it is indeed possible to associate a Schauder basis $\{e_i\}_{i=1}^\infty$ of a Banach space X to a sequence of bounded linear coordinate functionals $\{e_i^*\}_{i=1}^\infty \subseteq X^*$ such that for every $x \in X$,

$$x = \sum_{i=1}^{\infty} e_i^*(x)e_i$$

with the convergence in norm. Moreover, the system $\{e_i; e_i^*\}_{i=1}^\infty$ forms a biorthogonal system, that is, $e_i^*(e_j) = \delta_{i,j}$ for all $i, j \in \mathbb{N}$.

Proof of Theorem 1.2.9. By Lemma 1.2.7, we have that $\{\Pi_n\}_{n=1}^\infty$ are bounded, and for every $x \in X$, $\Pi_n x \to x$ in norm with $\sup_{n\in\mathbb{N}} \|\Pi_n x\| < \infty$. Therefore, by the Uniform Boundedness Principle,

$$\sup_{n\in\mathbb{N}} \|\Pi_n\| < \infty.$$

$\square$

By Banach's Theorem, we associate a Schauder basis to an important quantity that is often used in the study of Schauder bases.

Definition 1.2.11. Let X be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of X. We denote the *basis constant* of $\{e_i\}_{i=1}^{\infty}$ as

$$\mathrm{bc}(\{e_i\}) = \sup_{n\in\mathbb{N}} \|\Pi_n\|.$$

Definition 1.2.12. Let X be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of X.

1. If $\|e_i\| = 1$ for all $i \in \mathbb{N}$, then we say that $\{e_i\}_{i=1}^{\infty}$ is a *normalized* Schauder basis.

2. If

   $$0 < \inf_{i\in\mathbb{N}} \|e_i\| \leq \sup_{i\in\mathbb{N}} \|e_i\| < \infty$$

   then $\{e_i\}_{i=1}^{\infty}$ is called a *seminormalized* Schauder basis.

3. If $\mathrm{bc}(\{e_i\}) = 1$, then we say that $\{e_i\}_{i=1}^{\infty}$ is a *monotone* Schauder basis.

Example 1.2.13. Of course, as mentioned in the beginning of the section, every orthonormal basis of a separable Hilbert space is a Schauder basis.

Example 1.2.14. The Banach space $\ell_\infty(\mathbb{N})$ does not admit a Schauder basis. Indeed, every Banach space with a Schauder basis is separable, but $\ell_\infty(\mathbb{N})$ is not separable, so $\ell_\infty(\mathbb{N})$ cannot have a Schauder basis.

Example 1.2.15. Let $c(\mathbb{N}) = \{x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n \text{ converges}\}$ be the space of all convergent sequences, equipped with the ∞-norm. For each $n \in \mathbb{N}$, define projections Π_n for $x \in c(\mathbb{N})$ by

$$\Pi_n x = (x_1, x_2, x_3, ..., x_n, x_n, x_n, ...).$$

That is, $\Pi_n x$ agrees on x on the first n coordinates, and then repeats the nth term afterwards. We verify the properties of Lemma 1.2.7 as follows.

First, note that for all $n \in \mathbb{N}$, $\dim(\Pi_n(c(\mathbb{N})) = n$, since the map $T : \Pi_n(c(\mathbb{N})) \to \mathbb{R}^n$ given by $Ty = (y_1, y_2, ..., y_n)$ is a surjection. Also, for $n, m \in \mathbb{N}$ and $x \in c(\mathbb{N})$, if $n \leq m$, then

$$\Pi_n \Pi_m x = \Pi_n(x_1, x_2, ..., x_n, x_{n+1}, ..., x_m, x_m, ...) = (x_1, x_2, ..., x_n, x_n, ...) = \Pi_n x$$

and similarly, if $n \geq m$, then $\Pi_m \Pi_n x = \Pi_m x$. Finally, observe that for any $x \in c(\mathbb{N})$ and $n \in \mathbb{N}$, we have that

$$x - \Pi_n x = (0, 0, ..., 0, x_{n+1} - x_n, x_{n+2} - x_n, ...).$$

Then

$$\lim_{n\to\infty} \|x - \Pi_n x\|_\infty = \lim_{n\to\infty} \sup_{i>n} |x_i - x_n| = 0$$

so $\Pi_n x \to x$ in norm as $n \to \infty$. Thus, by Lemma 1.2.7, the canonical projections $\{\Pi_n\}_{n=1}^\infty$ generate a Schauder basis given as follows:

$$\begin{aligned}
x_1 &= (1, 1, 1, 1, 1, ...) \\
x_2 &= (0, 1, 1, 1, 1, ...) \\
x_3 &= (0, 0, 1, 1, 1, ...) \\
&\vdots \\
x_n &= (0, 0, ..., 0, \underset{\substack{\uparrow \\ n\text{th}}}{1}, 1, 1, ...)
\end{aligned}$$

To understand how any $y = (y_1, y_2, ...) \in c(\mathbb{N})$ are represented in terms of such a Schauder basis, suppose $\Pi_n y = \sum_{i=1}^n a_i x_i$. Then comparing the coordinates, we see that

$$\begin{aligned}
y_1 &= a_1 \\
y_2 &= a_1 + a_2 \\
y_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
y_n &= a_1 + a_2 + \cdots + a_n
\end{aligned}$$

so that $a_i = y_i - y_{i-1}$ with the convention $y_0 = 0$. Hence,

$$\|\Pi_n y\|_\infty = \max_{1\le i\le n} |y_i| = \max_{1\le i\le n} \left| \sum_{j=1}^{i} a_j \right|.$$

The basis $\{x_i\}_{i=1}^\infty$ is called the *summing basis* of $c(\mathbb{N})$. It differs from the unit vector basis in $\ell_\infty(\mathbb{N})$, which is not a Schauder basis of $c(\mathbb{N})$.

Given a Banach space X and a sequence of vectors $\{e_i\}_{i=1}^\infty$ of X, it may happen that this sequence does not form a Schauder basis for the entire space X. A natural question that arises is: Can the sequence still be considered a Schauder basis for some subspace of X? In particular, does it form a basis for the closed linear span it generates? The following definition addresses this question.

Definition 1.2.16. Let X be a Banach space and let $\{e_i\}_{i=1}^\infty$ be a sequence of vectors in X. We say that $\{e_i\}_{i=1}^\infty$ is a *basic sequence* of X if it is a Schauder basis of $[\{e_n : n \in \mathbb{N}\}]$.

The definition of a basic sequence is useful, but not always convenient to verify directly. Grümblum's Criterion [7] provides a useful characterization: it gives a sufficient condition for a sequence to be basic, based only on how linear combinations of the vectors behave.

Proposition 1.2.17 (Grümblum's Criterion)**.** *Let X be a Banach space and let $\{e_i\}_{i=1}^\infty$ be a sequence of nonzero vectors in X. Then $\{e_i\}_{i=1}^\infty$ is a basic sequence of X if and only if there exists a $K > 0$ such that for any $n, m \in \mathbb{N}$ with $n \le m$ and $a_1, a_2, ..., a_m \in \mathbb{R}$,*

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| \le K \left\| \sum_{i=1}^{m} a_i e_i \right\|.$$

Moreover, the smallest such K is the basis constant $\mathrm{bc}(\{e_i\})$*.*

Before we prove Grümblum's Criterion, let us see how we can use it.

Example 1.2.18. Consider the unit vector basis of $c_{00}(\mathbb{N})$ given as $\{e_i\}_{i=1}^{\infty}$, where

$$e_i = (0, 0, ..., 0, \underset{\substack{\uparrow \\ i\text{th}}}{1}, 0, 0, ...).$$

Here, we note that $c_{00}(\mathbb{N})$ is dense subspace of $\ell_p(\mathbb{N})$, $1 \leq p < \infty$ and $c_0(\mathbb{N})$. We will show that $\{e_i\}_{i=1}^{\infty}$ is a basic sequence of $\ell_p(\mathbb{N})$ (a similar approach follows for $c_0(\mathbb{N})$ using the ∞-norm). Observe that for any $n, m \in \mathbb{N}$ with $n \leq m$ and $a_1, a_2, ..., a_m \in \mathbb{R}$, we have

$$\left\|\sum_{i=1}^{n} a_i e_i\right\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m} |a_i|\right)^{\frac{1}{p}} = \left\|\sum_{i=1}^{m} a_i e_i\right\|_p.$$

Therefore, choosing $K = 1$ allows us to conclude that $\{e_i\}_{i=1}^{\infty}$ is a basic sequence of $\ell_p(\mathbb{N})$. Moreover, as span $\{e_n : n \in \mathbb{N}\} = c_{00}(\mathbb{N})$, then

$$[\{e_n : n \in \mathbb{N}\}] = \overline{c_{00}(\mathbb{N})} = \ell_p(\mathbb{N}).$$

In particular, $\{e_i\}_{i=1}^{\infty}$ is a (normalized) Schauder basis of $\ell_p(\mathbb{N})$.

Moreover, we claim that $\{e_i\}_{i=1}^{\infty}$ is monotone. For each $n \in \mathbb{N}$ and $x = \sum_{i=1}^{\infty} a_i e_i \in \ell_p(\mathbb{N})$, denoting

$$\Pi_n x = \sum_{i=1}^{n} a_i e_i$$

to be the nth canonical projection of $\ell_p(\mathbb{N})$, from the above, we have that $\|\Pi_n x\| \leq \|\Pi_m x\|$, so then by extending $m \to \infty$, observe that

$$\|\Pi_n x\|_p \leq \lim_{m\to\infty} \|\Pi_m x\|_p = \|x\|_p$$

and thus, $\|\Pi_n\| \leq 1$, so the canonical projections are uniformly bounded by 1, and so the unit vector basis of $\ell_p(\mathbb{N})$ is monotone, as claimed.

Proof of Proposition 1.2.17. Observe that if $\{e_i\}_{i=1}^{\infty}$ is a basic sequence, then choosing $K = \mathrm{bc}(\{e_i\})$ so that for any $n, m \in \mathbb{N}$ and $a_1, a_2, ..., a_m \in \mathbb{R}$, we have

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \left\| \Pi_n \left(\sum_{i=1}^{m} a_i e_i \right) \right\| \leq \|\Pi_n\| \left\| \sum_{i=1}^{m} a_i e_i \right\| \leq \mathrm{bc}(\{e_i\}) \left\| \sum_{i=1}^{m} a_i e_i \right\|.$$

Conversely, assume that there exists a $K > 0$ such that for all $n, m \in \mathbb{N}$ with $n \leq m$ and $a_1, a_2, ..., a_m \in \mathbb{R}$

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| \leq K \left\| \sum_{i=1}^{m} a_i e_i \right\|.$$

Let $E = \mathrm{span}\,\{e_i : i \in \mathbb{N}\}$ and for each $m \in \mathbb{N}$, let $\pi_m : E \to [\{e_i\}_{i=1}^m]$ be defined by

$$\pi_m \left(\sum_{i=1}^{n} a_i e_i \right) = \sum_{i=1}^{\min(n,m)} a_i e_i.$$

We verify the statements of Lemma 1.2.7 to show that $\{e_i\}_{i=1}^\infty$ is a basic sequence of X. Indeed, first note that by density, each π_m extends to $\Pi_m : \overline{E} \to [\{e_i\}_{i=1}^m]$ with $\|\Pi_m\| = \|\pi_m\| \leq K$ showing that (1) holds. Next to see that (2) holds, note that for any $x \in E$ and $n, m \in \mathbb{N}$

$$\Pi_n \Pi_m x = \Pi_m \Pi_n x = \Pi_{\min(n,m)} x.$$

Therefore, by density, the above holds for all $x \in \overline{E}$. Finally, to see that (3) holds, note that $\Pi_n x \to x$ for all $x \in \overline{E}$ and thus, it can be shown that $\{x \in \overline{E} : \Pi_m x \to x\}$ is closed and contains E which is dense in $\overline{E}$. Therefore, by Lemma 1.2.7, we obtain that $\{e_i\}_{i=1}^\infty$ is a basic sequence with canonical projections $\{\Pi_n\}_{n=1}^\infty$. $\square$

Every Schauder basis naturally gives rise to a biorthogonal system in the dual-second dual pairing. In particular, if X is a Banach space with a Schauder basis $\{e_i\}_{i=1}^\infty$, and its associated coordinate functionals $\{e_i^*\}_{i=1}^\infty \subseteq X^*$, then identifying X canonically with its image in X^{**} via the natural embedding, we may view each e_i as an element of X^{**}. In this setting, the pair $\{e_i^*; e_i\}_{i=1}^\infty$ forms a biorthogonal system in $X^* \times X^{**}$ satisfying $e_i^*(e_j) = \delta_{i,j}$.

Fact 1.2.19. *Let X be a Banach space with Schauder basis $\{e_i; e_i^*\}_{i=1}^{\infty}$ and canonical projections $\{\Pi_n\}_{n=1}^{\infty}$.*

1. *For each $n \in \mathbb{N}$ and $f \in X^*$, $\Pi_n^* f = \sum_{i=1}^{n} f(e_i) e_i^* = \sum_{i=1}^{n} e_i(f) e_i^*$.*
2. *For all $f \in X^*$, $\Pi_n^* f \overset{*}{\rightharpoonup} f$ in X^*.*
3. *$\{e_i^*; e_i\}_{i=1}^{\infty}$ is a basic sequence of X^* with canonical projections $\{\Pi_n^*\}_{n=1}^{\infty}$. In particular, $\Pi_n^* f \to f$ for all $f \in [\{e_i^* : i \in \mathbb{N}\}]$.*

Proof. To see that (1) holds, let $n \in \mathbb{N}$, $f \in X^*$, and $x = \sum_{i=1}^{\infty} e_i^*(x) e_i \in X$. Then observe that

$$\begin{aligned}
\Pi_n^* f(x) &= f(\Pi_n x) \\
&= f\left(\sum_{i=1}^{n} e_i^*(x) e_i\right) \\
&= \sum_{i=1}^{n} e_i^*(x) f(e_i) \\
&= \left(\sum_{i=1}^{n} f(e_i) e_i^*\right)(x) \\
&= \left(\sum_{i=1}^{n} e_i(f) e_i^*\right)(x).
\end{aligned}$$

Therefore, as $x \in X$ was arbitrary, we have $\Pi_n^* f = \sum_{i=1}^{n} f(e_i) e_i^* = \sum_{i=1}^{n} e_i(f) e_i^*$.

To see that (2) holds, let $f \in X^*$. Since f is continuous, we have for any $x = \sum_{i=1}^{\infty} e_i^*(x) e_i \in X$, by (1)

$$\begin{aligned}
\lim_{n\to\infty} \Pi_n^* f(x) &= \lim_{n\to\infty} \sum_{i=1}^{n} e_i^*(x) f(e_i) \\
&= f\left(\lim_{n\to\infty} \sum_{i=1}^{\infty} e_i^*(x) e_i\right) \\
&= f\left(\sum_{i=1}^{\infty} e_i^*(x) e_i\right)
\end{aligned}$$

$$= f(x).$$

Finally, to see that (3) holds, we verify the properties of Lemma 1.2.7 as follows. Indeed, by using (1), we obtain that for any $f \in X^*$, $x = \sum_{i=1}^{\infty} e_i^*(x) e_i \in X$ and $n, m \in \mathbb{N}$ with, say $n \leq m$,

$$\Pi_n^* \Pi_m^* f(x) = \Pi_n^* \left(\sum_{i=1}^{m} e_i^*(x) f(e_i) \right) = \sum_{i=1}^{n} e_i^*(x) f(e_i) = \Pi_n^* f(x).$$

So we have $\Pi_n^* \Pi_m^* f = \Pi_n^* f$, and similarly, if $n > m$, $\Pi_n^* \Pi_m^* f = \Pi_m^* f$. Next, observe that if $f \in \operatorname{span}\{e_i^* : i \in \mathbb{N}\}$, then $\Pi_n^* f = f$ for sufficiently large $n \in \mathbb{N}$, and so

$$\lim_{n \to \infty} \|\Pi_n^* f - f\| = 0.$$

Moreover, since $\|\Pi_n\| = \|\Pi_n^*\|$, and $\{\Pi_n\}_{n=1}^{\infty}$ are uniformly bounded, so then by Lemma 1.2.10 and Fact 1.2.8, it follows that $\{e_i^*; e_i\}_{i=1}^{\infty}$ is a Schauder basis of $[\{e_i^* : i \in \mathbb{N}\}]$. $\square$

To conclude this section, we recall from Remark 1.2.2 that every Banach space with a Schauder basis is separable. However, the converse is false; not every separable Banach space admits a Schauder basis. Nevertheless, the following classical result of Mazur ensures that every infinite-dimensional Banach space contains a basic sequence.

Theorem 1.2.20 (Mazur's Theorem)**.** *Every infinite-dimensional Banach space admits a basic sequence.*

The proof of Mazur's Theorem relies on a geometric lemma about finite-dimensional subspaces, which will also be used later in the proof of the Lindenstrauss–Tzafriri Theorem.

Lemma 1.2.21. *Let X be an infinite-dimensional Banach space and let Y be a finite-dimensional subspace of X. For every $\varepsilon > 0$, there exists a $x \in S_X$ such that for all $y \in Y$ and $\lambda \in \mathbb{R}$,*

$$\|y\| \leq (1 + \varepsilon)\|y + \lambda x\|.$$

Proof. The proof idea follows from the diagram shown in Figure 1.1, but was also used in the proof in [5]. Let $0 < \varepsilon < 1$ and let $\{y_i\}_{i=1}^n$ be an $\varepsilon/2$-net in S_Y. Then by the Hahn-Banach Theorem (Theorem A.0.3), for each $1 \le i \le n$, let $y_i^* \in S_{X^*}$ such that $y_i^*(y_i) = 1$, as shown in the red planes. Since X is infinite-dimensional, there is an $x \in S_X$ such that $y_i^*(x) = 0$ for all $1 \le i \le n$, as shown in the blue planes. We need to show that x satisfies the above property. Indeed, if $y \in S_Y$, choose $1 \le i \le n$ so that $\|y_i - y\| < \frac{\varepsilon}{2}$ (so we are choosing a y_i so that the distance between y and y_i is appropriately small). For instance, take the point y in blue, and choose y_1 to be the closer vector to y, and let $\lambda \in \mathbb{R}$. Then observe that

$$\|y + \lambda x\| \ge \|y_i + \lambda x\| - \frac{\varepsilon}{2} \ge y_i^*(y_i + \lambda x) - \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} \ge \frac{1}{1+\varepsilon}.$$

Therefore, for any nonzero $y \in Y$ and $\lambda \in \mathbb{R}$, we have

$$\frac{1}{\|y\|}\|y + \lambda x\| \ge \frac{1}{1+\varepsilon}.$$

and therefore,

$$\|y\| \le (1+\varepsilon)\|y + \lambda x\|$$

as desired. $\square$

Proof of Theorem 1.2.20. Let $\varepsilon > 0$ be arbitrary and for each $n \in \mathbb{N}$, choose $\varepsilon_n > 0$ such that $\prod_{n=1}^{\infty}(1+\varepsilon_n) \le 1 + \varepsilon$. Using Lemma 1.2.21, we need to construct a sequence $\{x_n\}_{n=1}^{\infty}$ in S_X so that for all $n \ge 1$ and for all $y \in \operatorname{span}\{x_1, x_2, ..., x_n\}$

$$\|y\| \le (1+\varepsilon_n)\|y + \lambda x_{n+1}\|.$$

Indeed, for first step, set $x_1 \in S_X$. Now assume that we have constructed $x_1, x_2, ..., x_n \in S_X$ such that for all $1 \le k < n$ and for all $y \in \operatorname{span}\{x_1, x_2, ..., x_k\}$ and $\lambda \in \mathbb{R}$,

$$\|y\| \le (1+\varepsilon_k)\|y + \lambda x_{k+1}\|.$$

Then define $Y_n = \operatorname{span}\{x_1, x_2, ..., x_n\}$, which is finite-dimensional. Thus, by Lemma

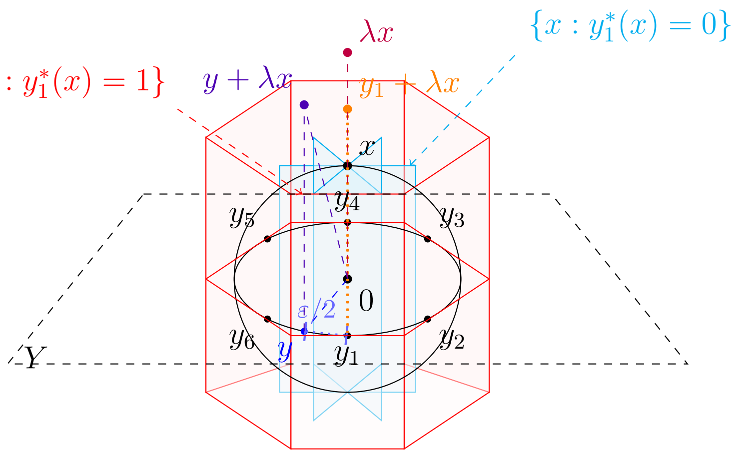

Figure 1.1: Geometric idea behind Lemma 1.2.21. The vector $x \in S_X$ lies in the intersection of $\ker(y_i^*)$ making x "invisible" to each y_i^* and inflating the norm of $y + \lambda x$.

1.2.21, there exists $x_{n+1} \in S_X$ such that for all $y \in Y_n$ and $\lambda \in \mathbb{R}$,

$$\|y\| \leq (1 + \varepsilon_n)\|y + \lambda x_{n+1}\|.$$

By induction, we construct a sequence $\{x_n\}_{n=1}^{\infty} \subseteq S_X$ with the above property. We now claim that $\{x_n\}_{n=1}^{\infty}$ is a basic sequence, and we will show this using Grümblum's Criterion (Proposition 1.2.17). Indeed, for $n, m \in \mathbb{N}$ such that $n \leq m$ and $a_1, a_2, ..., a_m \in \mathbb{R}$, see that if $y = \sum_{i=1}^{n} a_i x_i$

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leq \prod_{i=n}^{m-1}(1 + \varepsilon_i)\left\|\sum_{i=1}^{m} a_i x_i\right\| \leq (1 + \varepsilon)\left\|\sum_{i=1}^{m} a_i x_i\right\|.$$

Therefore, by Grümblum's Criterion $\{x_n\}_{n=1}^{\infty}$ is indeed a basic sequence with $\mathrm{bc}(\{x_n\}) \leq 1 + \varepsilon$. Furthermore, as $\varepsilon > 0$ was arbitrary, this shows that every infinite-dimensional Banach space contains a basic sequence whose basis constant is arbitrarily close to 1. $\square$

1.3 Equivalent Bases, Perturbation, and the Bessaga-Pełczyński Selection Principle

In this section, we present several fundamental properties of basic sequences, including the notion of equivalent bases, the Small Perturbation Lemma, and the Bessaga–Pełczyński Selection Principle. These results are central to the study of Schauder bases and play a crucial role in understanding the stability and structural properties of Banach spaces.

Definition 1.3.1. Let X and Y be Banach spaces with basic sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$, respectively. We say that $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are *equivalent bases* (denoted $\{x_i\} \sim \{y_i\}$) if for every sequence of scalars $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$, $\sum_{i=1}^{\infty} a_i x_i$ converges if and only if $\sum_{i=1}^{\infty} a_i y_i$ converges.

Since X and Y are Banach spaces and $[\{x_i\}]$ and $[\{y_i\}]$ are closed subspaces of X and Y, respectively, it suffices to check convergence via the Cauchy criterion. That is, $\sum_{i=1}^{\infty} a_i x_i$ converges if and only if it is Cauchy in X, and similarly for $\sum_{i=1}^{\infty} a_i y_i$ in Y. Thus, $\{x_i\} \sim \{y_i\}$ if and only if $\sum_{i=1}^{\infty} a_i x_i$ is Cauchy in X if and only if $\sum_{i=1}^{\infty} a_i y_i$ is Cauchy in Y.

By the Closed Graph Theorem (Theorem A.0.7), if $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are equivalent, then the spaces X and Y must be isomorphic. In particular,

Theorem 1.3.2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces with basic sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ of X and Y, respectively. The following assertions are equivalent.*

1. *$\{x_i\} \sim \{y_i\}$.*
2. *There exists an isomorphism $T : [\{x_i\}] \to [\{y_i\}]$ such that for all $i \in \mathbb{N}$, $Tx_i = y_i$.*
3. *There exist $A, B > 0$ such that for all $n \in \mathbb{N}$ and $a_1, a_2, ..., a_n \in \mathbb{R}$,*

$$A \left\| \sum_{i=1}^{n} a_i x_i \right\|_X \le \left\| \sum_{i=1}^{n} a_i y_i \right\|_Y \le B \left\| \sum_{i=1}^{n} a_i x_i \right\|_X .$$

Proof. Assume that (1) holds, and we show (2). Let $T : [\{x_i\}] \to [\{y_i\}]$ be defined by

$$T\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{\infty} a_i y_i$$

for $\sum_{i=1}^{\infty} a_i x_i \in [\{x_i\}]$. Observe that for each $i \in \mathbb{N}$, we have $Tx_i = y_i$.

We first verify that T is well-defined. Indeed, since $\{x_i\}_{i=1}^{\infty}$ is a basic sequence in X, any vector in $[\{x_i\}]$ has a unique representation in terms of x_i, that is, if $\sum_{i=1}^{\infty} a_i x_i = \sum_{i=1}^{\infty} b_i x_i$, then $a_i = b_i$ for all $i \in \mathbb{N}$. But then observe that $T\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{\infty} a_i y_i$ and $T\left(\sum_{i=1}^{\infty} b_i x_i\right) = \sum_{i=1}^{\infty} b_i y_i$. Since $a_i = b_i$ for all $i \in \mathbb{N}$, the corresponding series in Y are also the same, that is, if $\sum_{i=1}^{\infty} a_i x_i = \sum_{i=1}^{\infty} b_i x_i$, then $\sum_{i=1}^{\infty} a_i y_i = \sum_{i=1}^{\infty} b_i y_i$, so the definition of T does not depend on the choice of the coefficients, so T is well-defined.

To see that T is injective, let $\sum_{i=1}^{\infty} a_i x_i \in [\{x_i\}]$ be such that $T\left(\sum_{i=1}^{\infty} a_i x_i\right) = 0$, that is, $\sum_{i=1}^{\infty} a_i y_i = 0$. Since $\{y_i\}_{i=1}^{\infty}$ is a basic sequence in Y, then every vector in $[\{y_i\}]$ has a unique representation, and consequently, for each $i \in \mathbb{N}$, we have $a_i = 0$. Therefore, it follows that $\sum_{i=1}^{\infty} a_i x_i = 0$ in X, so T is injective, as claimed.

To see that T is onto, let $\sum_{i=1}^{\infty} a_i y_i \in [\{y_i\}]$ by (1), $\sum_{i=1}^{\infty} a_i x_i$ converges in $[\{x_i\}]$ and by the definition of T, $T\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{\infty} a_i y_i$.

To see that T is bounded, we will show that T has a closed graph. Indeed, let for each $m \in \mathbb{N}$, $\sum_{i=1}^{\infty} a_{i,m} x_i \in [\{x_i\}]$ be such that $\sum_{i=1}^{\infty} a_{i,m} x_i \to \sum_{i=1}^{\infty} a_i x_i$ and $T\left(\sum_{i=1}^{\infty} a_{i,m} x_i\right) \to \sum_{i=1}^{\infty} b_i y_i$. Since the coordinate functionals are continuous, we have $a_{i,m} \to a_i$ and $a_{i,m} \to b_i$, and thus, we have $a_i = b_i$ and

$$\lim_{m\to\infty} T\left(\sum_{i=1}^{\infty} a_{i,m} x_i\right) = \sum_{i=1}^{\infty} a_i y_i = T\left(\sum_{i=1}^{\infty} a_i x_i\right).$$

Since $T : [\{x_i\}] \to [\{y_i\}]$ is a bounded linear bijection between Banach spaces, then its inverse T^{-1} is also bounded, by Corollary A.0.6.

Therefore, we have shown that $T : [\{x_i\}] \to [\{y_i\}]$ is an isomorphism with $Tx_i = y_i$ for all $i \in \mathbb{N}$.

Assume (2) holds, i.e. let $T : [\{x_i\}] \to [\{y_i\}]$ be an isomorphism such that $Tx_i = y_i$ for all $i \in \mathbb{N}$, and we show (3). First observe that $\|T\|$ and $\|T^{-1}\|$ are finite as T is an isomorphism. Let $n \in \mathbb{N}$ and $a_1, a_2, ..., a_n \in \mathbb{R}$.

$$\left\|\sum_{i=1}^{n} a_i y_i\right\|_Y = \left\|\sum_{i=1}^{n} a_i T x_i\right\|_Y = \left\|T\left(\sum_{i=1}^{n} a_i x_i\right)\right\|_Y \leq \|T\| \left\|\sum_{i=1}^{n} a_i x_i\right\|_X .$$

Thus, choose $B = \|T\|$ for the upper bound. On the other hand, since T is a bijection with inverse $T^{-1}y_i = x_i$ for each $i \in \mathbb{N}$, then $T^{-1}Tx_i = x_i$ for all $i \in \mathbb{N}$, and thus,

$$\left\|\sum_{i=1}^{n} a_i x_i\right\|_X = \left\|\sum_{i=1}^{n} a_i T^{-1} T x_i\right\|_X = \left\|T^{-1}\left(\sum_{i=1}^{n} a_i y_i\right)\right\|_X \leq \|T^{-1}\| \left\|\sum_{i=1}^{n} a_i y_i\right\|_Y .$$

Therefore,

$$\frac{1}{\|T^{-1}\|} \left\|\sum_{i=1}^{n} a_i x_i\right\|_X \leq \left\|\sum_{i=1}^{n} a_i y_i\right\|_Y .$$

Thus, choose $A = \|T^{-1}\|^{-1}$ for the lower bound. Therefore, we have found $A, B > 0$ such that for all $a_1, a_2, ..., a_n \in \mathbb{R}$

$$A \left\|\sum_{i=1}^{n} a_i x_i\right\|_X \leq \left\|\sum_{i=1}^{n} a_i y_i\right\|_Y \leq B \left\|\sum_{i=1}^{n} a_i x_i\right\|_X .$$

Finally, assume (3) holds. By assumption and since $\{y_i\}_{i=1}^{\infty}$ is a basic sequence, we have that $\sum_{i=1}^{\infty} a_i y_i$ is Cauchy if and only if $\sum_{i=1}^{\infty} a_i x_i$ is Cauchy. Moreover, since X and Y are Banach spaces with closed subspaces $[\{x_i\}]$ and $[\{y_i\}]$, respectively, both series converge simultaneously. $\square$

The James space, denoted by J, is a classical example of a Banach space constructed to have certain properties that contrast with ℓ_p spaces. The James space consists of all sequences of bounded 2-variation. In the theory of Banach spaces, the James space provides an insightful example of a Banach space without an unconditional basis (see [1, 5] for more details).

Definition 1.3.3. The *James space* J is defined as

$$J = \left\{ x = (x_i)_{i=1}^{\infty} \in c_0(\mathbb{N}) : \sup_{p_1 < p_2 < \cdots < p_k} \sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2 < \infty \right\}$$

and we define the James norm $\|\cdot\|_J$ to be

$$\|x\|_J = \sup_{p_1 < p_2 < \cdots < p_k} \left(\sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2 \right)^{\frac{1}{2}}.$$

Remark 1.3.4. The unit vector basis $\{e_i\}_{i=1}^{\infty}$ is a basis of J. In the case when $x = (a_1, a_2, ..., a_n, 0, 0, ...) \in c_{00}(\mathbb{N})$, we have

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_J = \sup_{p_1 < p_2 < \cdots < p_k \leq n} \left(\sum_{i=1}^{k-1} |a_{p_i} - a_{p_{i+1}}|^2 + |a_{p_k}|^2 \right)^{\frac{1}{2}}$$

To see this, let $p_1 < p_2 < \cdots < p_k$, let $l = \max\{1 \leq i \leq k : p_i \leq n\}$. We consider the following cases.

<u>Case 1: If $l = k$.</u> Let $q_i = p_i$ for $1 \leq i \leq k$, and $q_{k+1} = n + 1$. Then $a_{q_{k+1}} = 0$ and so

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_J \geq \left(\sum_{i=1}^{k} |a_{q_i} - a_{q_{i+1}}|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{k-1} |a_{p_i} - a_{p_{i+1}}|^2 + |a_{p_k}|^2 \right)^{\frac{1}{2}}$$

<u>Case 2: If $l < k$.</u> Then for all $l < i \leq k$, we have $a_{p_i} = 0$. Then observe that

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_J \geq \left(\sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{k-1} |a_{p_i} - a_{p_{i+1}}|^2 + |a_{p_k}|^2 \right)^{\frac{1}{2}}$$

since $a_{p_k} = 0$ and adding zero inside the square root does not change the value.

Therefore, from Case 1 and Case 2, we have

$$\sup_{p_1<p_2<\cdots<p_k\leq n}\left(\sum_{i=1}^{k-1}|a_{p_i}-a_{p_{i+1}}|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{i=1}^{n} a_i e_i\right\|_J$$

On the other hand, for every $p_1 < p_2 < \cdots < p_k$, there exists $q_1 < q_2 < \cdots < q_m \leq n$ such that

$$\left(\sum_{i=1}^{k-1}|a_{p_i}-a_{p_{i+1}}|^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{m-1}|a_{q_i}-a_{q_{i+1}}|^2+|a_{q_m}|^2\right)^{\frac{1}{2}}$$

Therefore,

$$\left\|\sum_{i=1}^{n} a_i e_i\right\|_J \leq \sup_{q_1<q_2<\cdots<q_m\leq n}\left(\sum_{i=1}^{m-1}|a_{q_i}-a_{q_{i+1}}|^2+|a_{q_m}|^2\right)^{\frac{1}{2}}$$

Before we proceed with an example, we first need to make sure that the James space is a Banach space, and that it has a Schauder basis being the unit vector basis of J.

Proposition 1.3.5. *The James space $(J, \|\cdot\|_J)$ is a Banach space with a monotone Schauder basis, being the unit vector basis $\{e_i\}_{i=1}^{\infty}$.*

Proof. We first verify that $\|\cdot\|_J$ is a norm. It is easy to note that $x = 0$ if and only if $\|x\|_J = 0$. Let $x \in J$ and let $\alpha \in \mathbb{R}$. Observe that

$$\begin{aligned}
\|\alpha x\|_J &= \sup_{p_1<p_2<\cdots<p_k}\left(\sum_{i=1}^{k-1}|\alpha x_{p_i}-\alpha x_{p_{i+1}}|^2\right)^{\frac{1}{2}} \\
&= \sup_{p_1<p_2<\cdots<p_k}\left(|\alpha|^2\sum_{i=1}^{k-1}|x_{p_i}-x_{p_{i+1}}|^2\right)^{\frac{1}{2}} \\
&= |\alpha|\sup_{p_1<p_2<\cdots<p_k}\left(\sum_{i=1}^{k-1}|x_{p_i}-x_{p_{i+1}}|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$= |\alpha|\|x\|_J.$$

Finally, let $x, y \in J$. Then observe that

$$\begin{aligned}
\|x+y\|_J &= \sup_{p_1<p_2<\cdots<p_k} \left(\sum_{i=1}^{k-1} |(x_{p_i}+y_{p_i}) + (x_{p_{i+1}} - y_{p_{i+1}})|^2\right)^{\frac{1}{2}} \\
&\leq \sup_{p_1<p_2<\cdots<p_k} \left(\sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2 + \sum_{i=1}^{k-1} |y_{p_i} - y_{p_{i+1}}|^2\right)^{\frac{1}{2}} \\
&\leq \sup_{p_1<p_2<\cdots<p_k} \left(\sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2\right)^{\frac{1}{2}} + \sup_{p_1<p_2<\cdots<p_k} \left(\sum_{i=1}^{k-1} |y_{p_i} - y_{p_{i+1}}|^2\right)^{\frac{1}{2}} \\
&= \|x\|_J + \|y\|_J.
\end{aligned}$$

Next, we show that $(J, \|\cdot\|_J)$ is complete, and we will show by absolute summability. For each $i \in \mathbb{N}$, let $x_i \in J$ be such that $\sum_{i=1}^{\infty} \|x_i\|_J < \infty$. Then choosing $p_1 = n$ and $p_2 = m$, we have

$$\|x_i\|_J \geq |x_{i,n} - x_{i,m}|$$

for all $m \geq n+1$. Thus, taking $m \to \infty$ yields

$$\|x_i\|_J \geq \lim_{m\to\infty} |x_{i,n} - x_{i,m}| = |x_{i,n}|.$$

Therefore, $\sum_{i=1}^{\infty} |x_{i,n}| < \infty$.

Now, for each $n \in \mathbb{N}$, let $a_n = \sum_{i=1}^{\infty} x_{i,n}$. Observe that

$$\sum_{i=1}^{\infty} x_i = \left(\sum_{i=1}^{\infty} x_{i,n}\right)_{n=1}^{\infty} = (a_n)_{n=1}^{\infty}.$$

We will show that $x = (a_n)_{n=1}^{\infty} \in J$. Indeed, for any $p_1 < p_2 < \cdots < p_k$, we have

$$\left(\sum_{j=1}^{k-1} |a_{p_j} - a_{p_{j+1}}|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{k-1} \left|\sum_{i=1}^{\infty} (x_{i,p_j} - x_{i,p_{j+1}})\right|^2\right)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{k-1} |x_{i,p_j} - x_{i,p_{j+1}}|^2 \right)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{\infty} \|x_i\|_J.$$

Therefore,

$$\|(a_n)_{n=1}^{\infty}\|_J \leq \sum_{i=1}^{\infty} \|x_i\|_J < \infty.$$

To complete the proof, we need to show that $\sum_{i=1}^{\infty} x_i$ converges in the $\|\cdot\|_J$ norm. For each $N \in \mathbb{N}$, define the partial sum

$$S_N = \sum_{i=1}^{N} x_i.$$

Then for $M > N$,

$$\|S_M - S_N\|_J = \left\| \sum_{i=N+1}^{M} x_i \right\|_J \leq \sum_{i=N+1}^{M} \|x_i\|_J$$

Since $\sum_{i=1}^{\infty} \|x_i\|_J < \infty$, it follows that the tail $\sum_{i=N+1}^{\infty} \|x_i\|_J \to 0$ as $N \to \infty$. Hence, $\{S_N\}_{N=1}^{\infty}$ is a Cauchy sequence in the Banach space $(J, \|\cdot\|_J)$, and therefore converges. That is, $x = \sum_{i=1}^{\infty} x_i \in J$.

To see that the unit vector basis $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis, let $(x_i)_{i=1}^{\infty} \in J$ and for each $N \in \mathbb{N}$, let

$$x_N = x - \sum_{i=1}^{N} x_i e_i.$$

Let $\varepsilon > 0$ be arbitrary, and let $p_1 < p_2 < \cdots < p_k$ be such that

$$\sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2 > \|x\|_J^2 - \varepsilon^2.$$

To estimate the norm of x_N when $N > p_k$, it suffices to consider $q_1 < q_2 < \cdots < q_l$

with $N \leq q_l$. Then for $p_1 < p_2 < \cdots < p_k < q_1 < q_2 < \cdot < q_l$, we have

$$\begin{aligned}\|x\|_J^2 &\geq \sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2 + |x_{q_1} - x_{p_k}|^2 + \sum_{i=1}^{l-1} |x_{q_i} - x_{q_{i+1}}|^2 \\ &\geq \sum_{i=1}^{k-1} |x_{p_i} - x_{p_{i+1}}|^2 + \sum_{i=1}^{l-1} |x_{q_i} - x_{q_{i+1}}|^2.\end{aligned}$$

In particular,

$$\sum_{i=1}^{l-1} |x_{q_i} - x_{q_{i+1}}|^2 \leq \varepsilon^2.$$

Therefore, $\|x_N\|_J < \varepsilon$ for all $N > p_k$.

Finally, to see that $\{e_i\}_{i=1}^{\infty}$ is monotone, let $n \leq m$ and let $a_1, a_2, ..., a_m \in \mathbb{R}$. First, we have by Remark **??**,

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_J = \sup_{p_1 < p_2 < \cdots < p_k \leq n} \left(\sum_{i=1}^{k-1} |a_{p_i} - a_{p_{i+1}}|^2 + |a_{p_k}|^2 \right)^{\frac{1}{2}}$$

Let $q_i = p_i$ if $i = 1, 2, ..., k$, and let $q_{k+1} = m + 1 \geq n + 1$. Then

$$\left\| \sum_{i=1}^{m} a_i e_i \right\|_J \geq \left(\sum_{i=1}^{k} |a_{q_i} - a_{q_{i+1}}|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{k-1} |a_{p_i} - a_{p_{i+1}}|^2 + |a_{p_k}|^2 \right)^{\frac{1}{2}}$$

Therefore, taking the supremum over all $p_1 < p_2 < \cdots < p_k \leq n$, we get

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_J \leq \left\| \sum_{i=1}^{m} a_i e_i \right\|_J .$$

That is, the canonical projections $\{\Pi_n\}_{n=1}^{\infty}$ associated to the Schauder basis $\{e_i\}_{i=1}^{\infty}$ are norm non-decreasing and hence, $\|\Pi_n\| = 1$ for all $n \in \mathbb{N}$, so $\{e_i\}_{i=1}^{\infty}$ is a monotone Schauder basis as desired. $\square$

Example 1.3.6. Let $\{e_{2i}\}_{i=1}^{\infty}$ be the subsequence of the unit vector basis of J consisting of vectors supported on even coordinates only, i.e. for all $i, j \in \mathbb{N}$, $e_{2i}^j = \delta_{2i,j}$, and let $\{u_i\}_{i=1}^{\infty}$ be a unit vector basis of $\ell_2(\mathbb{N})$. We claim that $\{e_{2i}\} \sim \{u_i\}$. To see this, we will show Theorem 1.3.2 (3) holds. Let $n \in \mathbb{N}$, and let $a_1, a_2, ..., a_n \in \mathbb{R}$. We find $A, B > 0$ such that

$$A \left\| \sum_{i=1}^{n} a_i e_{2i} \right\|_J \leq \left\| \sum_{i=1}^{n} a_i u_i \right\|_2 \leq B \left\| \sum_{i=1}^{n} a_i e_{2i} \right\|_J .$$

First, write $b_i = 0$ if i is odd and $b_i = a_{i/2}$ if i is even. Consider the partition $p_1 < p_2 < \cdots < p_k \leq n$ where $p_i = i$ for all $1 \leq i \leq k$. Then observe that

$$\begin{aligned}
\sum_{i=1}^{k-1} |b_{p_i} - b_{p_{i+1}}|^2 &= \sum_{i=1}^{k-1} |b_i - b_{i+1}|^2 \\
&= \sum_{\substack{i=1 \\ i \text{ is odd}}}^{k-1} |b_i - b_{i+1}|^2 + \sum_{\substack{i=2 \\ i \text{ is even}}}^{k} |b_i - b_{i+1}|^2 \\
&= \sum_{\substack{i=1 \\ i \text{ is odd}}}^{k-1} |a_{(i+1)/2}|^2 + \sum_{\substack{i=2 \\ i \text{ is even}}}^{k} |a_{i/2}|^2 \\
&\geq 2 \sum_{i=1}^{\lfloor k/2 \rfloor} |a_i|^2.
\end{aligned}$$

Therefore, by taking the supremum over all indices, we have

$$\left\| \sum_{i=1}^{n} a_i e_{2i} \right\|_J^2 \geq 2 \sum_{i=1}^{n} |a_i|^2 = 2 \left\| \sum_{i=1}^{n} a_i u_i \right\|_2^2 .$$

Hence, by choosing $B = \frac{1}{\sqrt{2}}$, we obtain that $\left\| \sum_{i=1}^{n} a_i u_i \right\|_2 \leq \frac{1}{\sqrt{2}} \left\| \sum_{i=1}^{n} a_i e_{2i} \right\|_J$.

Next, to find A, write

$$\sum_{i=1}^{n} a_i e_{2i} = \sum_{i=1}^{n} b_i e_i$$

where $b_i = 0$ if i is odd and $b_i = a_{i/2}$ if i is even. Let $q_1 < q_2 < \cdots < q_k$ be an increasing partition such that

$$\left\| \sum_{i=1}^{n} b_i e_i \right\|_J = \sup_{p_1 < p_2 < \cdots < p_k} \left(\sum_{i=1}^{k-1} |b_{p_i} - b_{p_{i+1}}|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{k-1} |b_{q_i} - b_{q_{i+1}}|^2 \right)^{\frac{1}{2}}$$

Then notice by Minkowski's Inequality (Theorem A.0.2),

$$\begin{aligned}\left\|\sum_{i=1}^{n} b_i e_i\right\|_J &\leq \left(\sum_{i=1}^{k-1} |b_{q_i} - b_{q_{i+1}}|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^{k-1} |b_{q_i}|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{k-1} |b_{q_{i+1}}|^2\right)^{\frac{1}{2}} \\ &\leq 2\left\|\sum_{i=1}^{n} a_i u_i\right\|_2\end{aligned}$$

Hence, by choosing $A = \frac{1}{2}$, we obtain that $\frac{1}{2}\left\|\sum_{i=1}^{n} a_i e_{2i}\right\|_J \leq \left\|\sum_{i=1}^{n} a_i u_i\right\|_2$. Thus,

$$\frac{1}{2}\left\|\sum_{i=1}^{n} a_i e_{2i}\right\|_J \leq \left\|\sum_{i=1}^{n} a_i u_i\right\|_2 \leq \frac{1}{\sqrt{2}}\left\|\sum_{i=1}^{n} a_i e_{2i}\right\|_J .$$

Therefore, by Theorem 1.3.2, we have $\{e_{2i}\} \sim \{u_i\}$, as desired. By the equivalence, the linear operator $T : \langle\{e_{2i}\}_{i=1}^{\infty}\rangle \to \langle\{u_i\}_{i=1}^{\infty}\rangle$ such that $Te_{2i} = u_i$ for all $i \in \mathbb{N}$ extends uniquely to a bounded linear operator $\tilde{T} : [\{e_{2i}\}] \to \ell_2(\mathbb{N})$ with a bounded inverse.

Definition 1.3.7. Let X be a Banach space and let $\{x_i\}_{i=1}^{\infty}$ be a basic sequence of X. We say that $\{x_i\}_{i=1}^{\infty}$ is *complemented* if $[\{x_i\}]$ is a complemented subspace of X.

The following stability result from 1940 by Krein, Milman, and Rutman [13] states that if we have a basic sequence $\{x_i\}_{i=1}^{\infty}$ in a Banach space X, and $\{y_i\}_{i=1}^{\infty}$ is another sequence in X such that $\|x_i - y_i\| \to 0$ fast enough, then $\{y_i\}_{i=1}^{\infty}$ is equivalent to $\{x_i\}_{i=1}^{\infty}$.

Theorem 1.3.8 (Small Perturbation Lemma)**.** *Let X be a Banach space with a basic sequence $\{x_i\}_{i=1}^{\infty}$ and $\{x_i^*\}_{i=1}^{\infty}$ be the associated coefficient functionals of $\{x_i\}_{i=1}^{\infty}$ of $[\{x_i\}]$. If $\{y_i\}_{i=1}^{\infty}$ is another sequence in X such that*

$$\sum_{i=1}^{\infty} \|x_i - y_i\|\|x_i^*\| = \theta < 1,$$

then the following results hold.

1. *$\{y_i\}_{i=1}^{\infty}$ is a basic sequence in X equivalent to $\{x_i\}_{i=1}^{\infty}$. In particular, there exists an isomorphism $T : X \to X$ such that $\|T\| \leq 1 + \theta$, $\|T^{-1}\| \leq \frac{1}{1-\theta}$, and for all $i \in \mathbb{N}$, $Tx_i = y_i$.*

2. *If $[\{x_i\}]$ is complemented in X, then so is $[\{y_i\}]$.*

3. *If $\{x_i\}_{i=1}^{\infty}$ is a Schauder basis of X, then so is $\{y_i\}_{i=1}^{\infty}$. Moreover, if $\{y_i^*\}_{i=1}^{\infty}$ are the coefficient functionals associated to $\{y_i\}_{i=1}^{\infty}$ of X, then $[\{x_i^*\}] = [\{y_i^*\}]$.*

Proof. We first show that (1) holds. Indeed, by the Hahn-Banach Theorem (Theorem A.0.3), we extend x_i^* to functionals on X with the same norm, that is, for each $i \in \mathbb{N}$, there exists $\tilde{x}_i^* \in X^*$ such that $\tilde{x}_i^*|_{[\{x_i^*\}]} = x_i^*$ and $\|\tilde{x}_i^*\| = \|x_i^*\|$. Then for any $x \in X$, observe that

$$\sum_{i=1}^{\infty} \|\tilde{x}_i^*(x)(x_i - y_i)\| \leq \|x\| \sum_{i=1}^{\infty} \|\tilde{x}_i^*\| \|x_i - y_i\| = \theta \|x\|.$$

Therefore, the operator $Sx = \sum_{i=1}^{\infty} \tilde{x}_i^*(x)(x_i - y_i)$ defines a bounded operator from X into X with $\|S\| \leq \theta < 1$. Now let $T : X \to X$ be the map defined by $Tx = x - Sx$ for $x \in X$. Then observe that

$$\|x - Tx\| = \|Sx\| \leq \theta \|x\|.$$

Therefore, $\|Tx\| \leq (1 + \theta)\|x\|$ and also, $\|Tx\| \geq (1 - \theta)\|x\|$.

Claim. *Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator. Assume there exists a $\delta > 0$ such that $\|Tx\| \geq \delta \|x\|$ for all $x \in X$. Then $T(X)$ is closed in Y, and T is an isomorphism from X onto $T(X)$.*

Proof of Claim. The inequality above implies that T is injective, so $T^{-1} : T(X) \to X$ is well-defined and $\|T^{-1}\| \leq 1/\delta$. The mapping $T : X \to T(X)$ is an isomorphism, so $T(X)$ is a Banach subspace of Y, hence, closed. $\square$

With the claim, since $1-\theta > 0$, the claim implies that T is an isomorphism from X onto $T(X)$ with $\|T\| \leq 1+\theta$ and $\|T^{-1}\| \leq \frac{1}{1-\theta}$.

Next, we need to show that $T(X) = X$. If not, then as $\theta < 1$, Riesz's lemma implies there exists an $x \in S_X$ such that $\operatorname{dist}(x, T(X)) > \theta$, which contradicts the inequality $\|x - Tx\| \leq \theta$. Therefore, $T : X \to X$ is an isomorphism. Since $Tx_i = y_i$ for all $i \in \mathbb{N}$, we have that $T : [\{x_i\}] \to [\{y_i\}]$.

The proof of (2) follows from the proof of (1) using the isomorphism T, Fact 1.1.7, and Corollary 1.3.10.

Finally, to show that (3) holds, assume that $\{x_i\}_{i=1}^{\infty}$ is a Schauder basis of X. Let $T : X \to X$ be the isomorphism obtained from (1). Since $\{x_i\}_{i=1}^{\infty}$ is a Schauder basis of X, then

$$X = T(X) = T([\{x_i\}]) = [\{y_i\}]$$

and thus, $\{y_i\}_{i=1}^{\infty}$ is also a Schauder basis of X.

To prove the moreover part, let $i \in \mathbb{N}$, and for each $j \in \mathbb{N}$, let

$$f_{i,j} = \sum_{k=1}^{j} y_i^*(x_k) x_k^*.$$

Observe that we have $f_{i,j} \in [\{x_i^*\}]$ and also, by Fact 1.2.19, $f_{i,j} \xrightarrow{*} y_i^*$ as $j \to \infty$. Let $x_j' \in B_X$ and let $\varepsilon > 0$ be such that

$$\sup_{x \in B_X} |(y_i^* - f_{i,j})(x)| \leq |(y_i^* - f_{i,j})(x_j')| + \varepsilon$$

Then observe that as $j \to \infty$,

$$\begin{aligned}
\lim_{j\to\infty} \sup_{x \in B_X} |(y_i^* - f_{i,j})(x)| &\leq \lim_{j\to\infty} |(y_i^* - f_{i,j})(x_j')| + \varepsilon \\
&= \lim_{j\to\infty} \left| \sum_{k=j+1}^{\infty} y_i^*(x_k) x_k^*(x_j') \right| + \varepsilon
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{j\to\infty} \left| \sum_{k=j+1}^{\infty} y_i^*(x_k - y_k)(x_j') \right| + \varepsilon \\
&\le \lim_{j\to\infty} \|y_i^*\| \sum_{k=j+1}^{\infty} \|x_k - y_k\| \|x_j'\| + \varepsilon \\
&= \varepsilon.
\end{aligned}
$$

Therefore, as $\varepsilon > 0$ was arbitrary, we have $\lim_{j\to\infty} \sup_{x\in B_X} |(y_i^* - f_{i,j})(x)| = 0$, and thus, $\|y_i^* - f_{i,j}\| \to 0$ as $j \to \infty$, and thus, $y_i^* \in [\{x_i^*\}]$, so $[\{y_i^*\}] \subseteq [\{x_i^*\}]$. For the other inclusion, note that $(T^{-1})^*(x_i^*) = y_i^*$ for each $i \in \mathbb{N}$, so $\|y_i^*\| \le \|(T^{-1})^*\| \|x_i^*\|$ for all $i \in \mathbb{N}$.

Therefore, $\sum_{i=1}^{\infty} \|y_i^*\| \|x_i - y_i\|$ converges and we can reverse the roles of x_i and y_i from above, so that $[\{x_i\}] \subseteq [\{y_i\}]$. $\square$

Remark 1.3.9. Given the assumptions of the Small Perturbation Lemma, we claim that [1]

$$
\sum_{i=1}^{\infty} \|x_i - y_i\| \|x_i^*\| \le 2\,\mathrm{bc}(\{x_i\}) \sum_{i=1}^{\infty} \frac{\|x_i - y_i\|}{\|x_i\|}.
$$

To see this, for any $x \in X$, observe that by the definition of the basis constant

$$
\begin{aligned}
\|x_n^*(x)x_n\| &= \left\| \sum_{i=1}^{n} x_i^*(x)x_i - \sum_{i=1}^{n-1} x_i^*(x)x_i \right\| \\
&\le \left\| \sum_{i=1}^{n} x_i^*(x)x_i \right\| + \left\| \sum_{i=1}^{n-1} x_i^*(x)x_i \right\| \\
&\le 2\,\mathrm{bc}(\{x_i\})\|x\|.
\end{aligned}
$$

Thus, $\|x_i^*\| \|x_i\| \le 2\,\mathrm{bc}(\{x_i\})$, or $\|x_i^*\| \le \frac{2\,\mathrm{bc}(\{x_i\})}{\|x_i\|}$. Therefore,

$$
\sum_{i=1}^{\infty} \|x_i - y_i\| \|x_i^*\| \le \sum_{i=1}^{\infty} \|x_i - y_i\| \frac{2\,\mathrm{bc}(\{e_i\})}{\|x_i\|} = 2\,\mathrm{bc}(\{x_i\}) \sum_{i=1}^{\infty} \frac{\|x_i - y_i\|}{\|x_i\|}.
$$

Corollary 1.3.10. *Let X be a Banach space and let Y be a finite-dimensional complemented subspace in X with projection P. If $\{y_1, y_2, ..., y_n\}$ is a basis of Y, then there exists an $\varepsilon > 0$ so that if, for each $1 \le i \le n$, $x_i \in X$ satisfies*

$\|y_i - x_i\| < \varepsilon$, *then* $\operatorname{span}\{x_1, x_2, ..., x_n\}$ *is complemented in* X *with projection of norm at most* $2\|P\|$.

Proof. Assume that $\{y_1, y_2, ..., y_n\}$ is a basis of Y. Let $\{y_1^*, y_2^*, ..., y_n^*\}$ be the associated coefficient functionals of the basis $\{y_1, y_2, ..., y_n\}$ of Y. Since the function $f(x) = \frac{1+x}{1-x} \leq 2$ if and only if $x \in \left(0, \frac{1}{3}\right)$, for each $1 \leq i \leq n$, choose $\varepsilon = \frac{1}{3nM}$, where $M = \max_{1\leq i \leq n} \|y_i^*\|$ and let $x_i \in X$ be such that $\|y_i - x_i\| < \frac{1}{3nM}$. Then observe that

$$\theta = \sum_{i=1}^{n} \|y_i - x_i\|\|y_i^*\| < \sum_{i=1}^{n} \varepsilon\|y_i^*\| = \sum_{i=1}^{n} \varepsilon\|y_i^*\| = \frac{1}{3} < 1.$$

Therefore, by the Small Perturbation Lemma (Theorem 1.3.8), $\{x_1, x_2, ..., x_n\}$ is also a basic sequence that is complemented in X, as Y is a complemented subspace in X. Furthermore, $\{y_1, y_2, ..., y_n\}$ is equivalent to $\{x_1, x_2, ..., x_n\}$, so there exists an isomorphism $T : X \to X$ with $Ty_i = x_i$ for all $1 \leq i \leq n$ with $\|T\| \leq 1 + \theta \leq \frac{4}{3}$ and $\|T^{-1}\| \leq \frac{1}{1-\theta} \leq \frac{3}{2}$.

By Remark 1.1.8, let $Q = TPT^{-1} : X \to X$ be the bounded linear projection onto $\langle\{x_1, x_2, ..., x_n\}\rangle$. Now we show that $\|Q\| \leq 2\|P\|$. Indeed, observe that

$$\|Q\| = \|TPT^{-1}\| \leq \|T\|\|P\|\|T^{-1}\| \leq \frac{4}{3} \cdot \frac{3}{2} \cdot \|P\| = 2\|P\|.$$

This completes the proof. □

We present a fundamental application of the Small Perturbation Lemma, known as the Bessaga–
Pełczyński Selection Principle. It was first introduced in [2]. This principle is a cornerstone of Banach space theory, frequently used in the analysis of basic sequences and spreading models. The main technical tool in the proof is the gliding hump method (also called the sliding hump argument), which provides a constructive way to extract well-behaved subsequences from weakly null sequences.

Before stating and proving the Selection Principle, we introduce the following foundational concept.

Definition 1.3.11. Let X be a Banach space with basic sequence $\{e_i\}_{i=1}^{\infty}$. If $(p_i)_{i=1}^{\infty}$ is an increasing sequence of integers and $(a_i)_{n=1}^{\infty} \subseteq \mathbb{R}$, then a sequence of nonzero vectors $\{u_i\}_{i=1}^{\infty}$ in X of the form

$$u_i = \sum_{j=p_i+1}^{p_{i+1}} a_j e_j$$

is called a *block basic sequence* of $\{e_i\}_{i=1}^{\infty}$.

Lemma 1.3.12. *Let X be a Banach space with basic sequence $\{e_i\}_{i=1}^{\infty}$ and let $\{u_j\}_{j=1}^{\infty}$ be a block basic sequence of $\{e_i\}_{i=1}^{\infty}$. Then $\{u_j\}_{j=1}^{\infty}$ is a basic sequence with basis constant at most* $\mathrm{bc}(\{e_i\})$.

Proof. For each $j \in \mathbb{N}$, write $u_j = \sum_{i=p_j+1}^{p_{j+1}} a_i e_i$ for $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$. Then for any $(b_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ and $n, m \in \mathbb{N}$ with $n \leq m$, we have

$$\begin{aligned} \left\| \sum_{j=1}^{n} b_j u_j \right\| &= \left\| \sum_{j=1}^{n} b_j \sum_{i=p_j+1}^{p_{j+1}} a_i e_i \right\| = \left\| \sum_{j=1}^{n} \sum_{i=p_j+1}^{p_{j+1}} b_j a_i e_i \right\| \\ &\leq \mathrm{bc}(\{e_i\}) \left\| \sum_{j=1}^{m} \sum_{i=p_j+1}^{p_{j+1}} b_j a_i e_i \right\| = \mathrm{bc}(\{e_i\}) \left\| \sum_{j=1}^{m} b_j u_j \right\|. \end{aligned}$$

Therefore, $\{u_j\}_{j=1}^{\infty}$ satisfies Grümblum's Criterion (Proposition 1.2.17) so $\{u_j\}_{j=1}^{\infty}$ is a basic sequence with basis constant at most $\mathrm{bc}(\{e_i\})$. □

Theorem 1.3.13 (Bessaga-Pełczyński Selection Principle)**.** *Let X be a Banach space with Schauder basis $\{e_i\}_{i=1}^{\infty}$ and associated coordinate functionals $\{e_i^*\}_{i=1}^{\infty}$. If $\{x_i\}_{i=1}^{\infty}$ is a sequence in X such that*

1. $\inf_{i \in \mathbb{N}} \|x_i\| > 0$
2. $\lim_{i \to \infty} e_j^*(x_i) = 0$ *for all* $j \in \mathbb{N}$

Then $\{x_i\}_{i=1}^{\infty}$ contains a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ that is equivalent to some block basic sequence $\{u_k\}_{k=1}^{\infty}$ of $\{e_i\}_{i=1}^{\infty}$.

Proof. Denote $\varepsilon = \inf_{i \in \mathbb{N}} \|x_i\| > 0$ and denote $K = \mathrm{bc}(\{e_i\})$. For convenience, set $p_0 = 0$. Let $i_1 \in \mathbb{N}$ and $p_1 \in \mathbb{N}$ be such that

$$\left\| \sum_{j=p_1+1}^{\infty} e_j^*(x_{i_1}) e_j \right\| = \left\| x_{i_1} - \sum_{j=1}^{p_1} e_j^*(x_{i_1}) e_j \right\| < \frac{\varepsilon}{4K}.$$

Since $\lim_{i \to \infty} \left\| \sum_{j=1}^{p_1} e_j^*(x_i) e_j \right\| = 0$, there exists an $i_2 > i_1$ such that

$$\left\| \sum_{j=1}^{p_1} e_j^*(x_{i_2}) e_j \right\| < \frac{\varepsilon}{8K}.$$

Then choose $p_2 > p_1$ so that

$$\left\| \sum_{j=p_2+1}^{\infty} e_j^*(x_{i_2}) e_j \right\| = \left\| x_{i_2} - \sum_{j=1}^{p_2} e_j^*(x_{i_2}) e_j \right\| < \frac{\varepsilon}{8K}.$$

Again, since $\lim_{i \to \infty} \left\| \sum_{j=1}^{p_2} e_j^*(x_i) e_j \right\| = 0$, there exists $i_3 > i_2$ such that

$$\left\| \sum_{j=1}^{p_2} e_j^*(x_{i_3}) e_j \right\| < \frac{\varepsilon}{16K}.$$

Proceeding inductively, we obtain a sequence $\{x_{i_k}\}_{k=1}^{\infty}$ of X and an increasing sequence of positive integers $(p_k)_{k=0}^{\infty}$ so that

$$\left\| \sum_{j=1}^{p_{k-1}} e_j^*(x_{i_k}) e_j \right\| < \frac{\varepsilon}{2^{k+1}K}, \quad \left\| \sum_{j=p_k+1}^{\infty} e_j^*(x_{i_k}) e_j \right\| < \frac{\varepsilon}{2^{k+1}K}.$$

Thus for each $k \in \mathbb{N}$, define

$$u_k = \sum_{j=p_{k-1}+1}^{p_k} e_j^*(x_{i_k}) e_i.$$

Then by Definition 1.3.11, $\{u_k\}_{k=1}^{\infty}$ is a block basic sequence of $\{e_i\}_{i=1}^{\infty}$. Also, by Lemma 1.3.12, $\{u_k\}_{k=1}^{\infty}$ is a basic sequence with basis constant at most $\mathrm{bc}(\{e_i\})$.

Observe that for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_{i_k} - u_k\| &= \left\| \sum_{j=1}^{\infty} e_j^*(x_{i_k}) e_j - \sum_{j=p_{k-1}+1}^{p_k} e_j^*(x_{i_k}) e_j \right\| \\
&= \left\| \sum_{j=1}^{p_{k-1}} e_j^*(x_{i_k}) e_j + \sum_{j=p_k+1}^{\infty} e_j^*(x_{i_k}) e_j \right\| \\
&\leq \left\| \sum_{j=1}^{p_{k-1}} e_j^*(x_{i_k}) e_j \right\| + \left\| \sum_{j=p_k+1}^{\infty} e_j^*(x_{i_k}) e_j \right\| \\
&< \frac{\varepsilon}{2^{k+1}K} + \frac{\varepsilon}{2^{k+1}K} \\
&= \frac{\varepsilon}{2^k K}.
\end{aligned}$$

Therefore,

$$\|u_k\| > \varepsilon - \frac{\varepsilon}{2^k K} \geq \frac{\varepsilon}{2}.$$

Finally, by Remark 1.3.9, we have

$$2K \sum_{k=1}^{\infty} \frac{\|x_{i_k} - u_k\|}{\|u_k\|} < 2K \sum_{k=1}^{\infty} \frac{\varepsilon/2^k K}{\varepsilon/2} = \sum_{i=1}^{\infty} \frac{1}{2^{k+2}} = \frac{1}{4} < 1.$$

Therefore, by the Small Perturbation Lemma (Theorem 1.3.8), $\{x_{i_k}\}_{k=1}^{\infty}$ is a basic sequence equivalent to $\{u_k\}_{k=1}^{\infty}$. □

We conclude this section by briefly introducing the concept of finite-dimensional decomposition (FDDs).

Definition 1.3.14. Let X be a Banach space. A sequence of finite-dimensional subspaces $\{X_i\}_{i=1}^{\infty}$ of X is called a *finite-dimensional decomposition* of X (abbreviated as FDD), if every $x \in X$ has a unique representation of the form

$$x = \sum_{i=1}^{\infty} x_i$$

where for each $i \in \mathbb{N}$, $x_i \in X_i$.

Remark 1.3.15. Let X be a Banach space and for each $i \in \mathbb{N}$ suppose $\dim(X_i) = 1$. Then for each $i \in \mathbb{N}$, there exists an $x_i \in X_i$ such that $X_i = \operatorname{span}\{x_i\}$. Then $\{X_i\}_{i=1}^{\infty}$ is a finite-dimensional decomposition if and only if $\{x_i\}_{i=1}^{\infty}$ is a Schauder basis of X.

Indeed, for any $x \in X$, there exists a unique sequence of scalars $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ such that

$$x = \sum_{i=1}^{\infty} a_i x_i$$

with the convergence in norm. This is precisely the definition of a Schauder basis. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is a Schauder basis of X, then setting $X_i = \operatorname{span}\{x_i\}$ gives one-dimensional subspaces whose canonical projection yields the original vector, verifying that $\{X_i\}_{i=1}^{\infty}$ is a finite-dimensional decomposition.

As we have canonical projections for Schauder bases, we can come up with a similar concept for finite-dimensional decompositions.

Definition 1.3.16. Let X be a Banach space and let $\{X_i\}_{i=1}^{\infty}$ be a finite-dimensional decomposition of X. For each $n \in \mathbb{N}$, define projections Π_n on X by

$$\Pi_n \left(\sum_{i=1}^{\infty} x_i \right) = \sum_{i=1}^{n} x_i.$$

The following result is the finite-dimensional decomposition version of Lemma 1.2.6 and Lemma 1.2.7.

Proposition 1.3.17. *Let X be a Banach space. If $\{X_i\}_{i=1}^{\infty}$ is a finite-dimensional decomposition of X, then*

$$\sup_{i \in \mathbb{N}} \|\Pi_n\| < \infty.$$

Conversely, if $\{\Pi_n\}_{n=1}^{\infty}$ is a sequence of finite rank projections on X such that

1. $\Pi_n \Pi_m = \Pi_{\min(n,m)}$
2. $\Pi_n x \to x$ *in norm for all* $x \in X$

then $\{\Pi_n\}_{n=1}^{\infty}$ *determines a unique finite-dimensional decomposition on* X *by putting* $X_1 = \Pi_1(X)$ *and* $X_n = (\Pi_n - \Pi_{n-1})(X)$ *for* $i \in \mathbb{N}$ *with* $i \geq 2$.

1.4 Unconditional Bases

In this section, we highlight one more key property that basic sequences have, known as unconditional bases. The idea of unconditional convergence is that when we take any permutation $\sigma \subseteq \mathbb{N}$ and a sequence in a Banach space $(x_i)_{i=1}^{\infty}$, the series $\sum_{i=1}^{\infty} x_{\sigma(i)}$ converges, in which case, we say that the series $\sum_{i=1}^{\infty} x_i$ converges unconditionally [8]. There are many other equivalent formulations of unconditional convergence. For instance, the one we will work with is the following from [5].

Definition 1.4.1. Let X be a Banach space, and let $\{x_i\}_{i=1}^{\infty}$ be a sequence in X. We say that $\sum_{i=1}^{\infty} x_i$ converges *unconditionally* if for every choice of signs $\varepsilon_i = \pm 1$, the series $\sum_{i=1}^{\infty} \varepsilon_i x_i$ converges.

Thus, we extend the notion of unconditional convergence to unconditional bases as follows.

Definition 1.4.2. Let X be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of X. We say that $\{e_i\}_{i=1}^{\infty}$ is an *unconditional basis* of X, if for every $x \in X$ its expansion $x = \sum_{i=1}^{\infty} a_i e_i$ converges unconditionally, that is, for every choice of signs $\varepsilon_i = \pm 1$, $\sum_{i=1}^{\infty} \varepsilon_i a_i x_i$ converges.

Example 1.4.3. The unit vector basis $\{e_i\}_{i=1}^{\infty}$ of the classical sequence spaces $\ell_p(\mathbb{N})$, $1 \leq p < \infty$ and $c_0(\mathbb{N})$ are unconditional. We will show this for $\ell_p(\mathbb{N})$. Let $x \in \ell_p(\mathbb{N})$ and $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$. Then we have

$$\|x\|_p = \left\| \sum_{i=1}^{\infty} a_i e_i \right\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{\frac{1}{p}} < \infty.$$

Therefore, for any $\delta > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} |a_i|^p < \delta^p.$$

Then for all $n > m \geq N$ and any choices of signs $\varepsilon_i = \pm 1$, we have

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \varepsilon_i a_i e_i - \sum_{i=1}^{m} \varepsilon_i a_i e_i \right\|_p &= \left\| \sum_{i=m+1}^{n} \varepsilon_i a_i e_i \right\|_p \\
&= \left(\sum_{i=m+1}^{n} |a_i|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=m+1}^{\infty} |a_i|^p \right)^{\frac{1}{p}} \\
&< \delta
\end{aligned}$$

which shows that the sequence of partial sums is Cauchy, so the series $\sum_{i=1}^{\infty} \varepsilon_i a_i e_i$ is convergent, and thus, $x = \sum_{i=1}^{\infty} a_i e_i$ converges unconditionally. The same argument applies for $c_0(\mathbb{N})$ by taking the ∞-norm.

Example 1.4.4. Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathcal{H}$ is isometrically isomorphic to $\ell_2(\mathbb{N})$, so it admits an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Since any orthonormal basis in a Hilbert space is automatically a Schauder basis, it follows that $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis of $\mathcal{H}$. Moreover, orthonormal bases in Hilbert spaces are unconditional, as we have established in Example 1.4.3 for $\ell_p(\mathbb{N})$ when $p = 2$.

Proposition 1.4.5. *Let X and Y be Banach spaces and let $\{e_i\}_{i=1}^{\infty}$ be a basic sequence in X. If $\{e_i\}_{i=1}^{\infty}$ is equivalent to some unconditional basic sequence $\{u_i\}_{i=1}^{\infty}$ of Y, then $\{e_i\}_{i=1}^{\infty}$ is also unconditional.*

This is easy to show; if $x = \sum_{i=1}^{\infty} a_i e_i$ converges, then for any $\varepsilon_i = \pm 1$, the sum $\sum_{i=1}^{\infty} \varepsilon_i a_i x_i$ converges as well.

Example 1.4.6. We provide an example of a basis that is not unconditional, i.e. conditional. Let $\{e_n\}_{n=1}^{\infty}$ denote the unit vector basis of $c_0(\mathbb{N})$ with biorthogonal functions $\{e_n^*\}_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, let

$$x_n = e_1 + e_2 + \cdots + e_n.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ is called the *summing basis of* $c_0(\mathbb{N})$. We claim that $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis of $c_0(\mathbb{N})$ that is not unconditional.

We first show that $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis of $c_0(\mathbb{N})$. Indeed, we show that for any $a = (a_n)_{n=1}^{\infty} \in c_0(\mathbb{N})$

$$a = \sum_{n=1}^{\infty} x_n^*(a) x_n$$

where $x_n^* = e_n^* - e_{n+1}^*$ are the biorthogonal functions of $\{x_n\}_{n=1}^{\infty}$. Indeed, for any $m \in \mathbb{N}$, observe that

$$\begin{aligned}
\sum_{n=1}^{m} x_n^*(a) x_n &= \sum_{n=1}^{m} (e_n^*(a) - e_{n+1}^*(a)) x_n \\
&= \sum_{n=1}^{m} (a_n - a_{n+1}) x_n \\
&= \sum_{n=1}^{m} a_n x_n - \sum_{n=2}^{m+1} a_n x_{n-1} \\
&= \sum_{n=1}^{m} a_n (x_n - x_{n+1}) - a_{m+1} x_m \\
&= \sum_{n=1}^{m} a_n e_n - a_{m+1} x_m
\end{aligned}$$

where we used the convention $x_0 = 0$. Therefore, now by taking $m \to \infty$,

$$\begin{aligned}
\lim_{m\to\infty} \left\| a - \sum_{n=1}^{m} x_n^*(a) x_n \right\|_{\infty} &= \lim_{m\to\infty} \left\| a - \sum_{n=1}^{m} (e_n^*(a) - e_{n+1}^*(a)) x_n \right\|_{\infty} \\
&= \lim_{m\to\infty} \left\| \sum_{n=1}^{\infty} e_n^*(a) e_n - \sum_{n=1}^{m} (a_n - a_{n+1}) x_n \right\|_{\infty} \\
&= \lim_{m\to\infty} \left\| \sum_{n=1}^{\infty} a_n e_n - \sum_{n=1}^{m} a_n e_n - a_{m+1} x_m \right\| \\
&= \lim_{m\to\infty} \left\| \sum_{n=m+1}^{\infty} a_n e_n - a_{m+1} x_m \right\|_{\infty} \\
&\leq \lim_{m\to\infty} \left\| \sum_{n=m+1}^{\infty} a_n e_n \right\|_{\infty} + \lim_{m\to\infty} |a_{m+1}| \|x_m\|_{\infty} \\
&= 0
\end{aligned}$$

where the convergence tends to zero, as $a \in c_0(\mathbb{N})$ and $\|x_m\|_\infty = 1$ for all $m \in \mathbb{N}$. Therefore, we have $a = \sum_{n=1}^{\infty} x_n^*(a)x_n$. The coordinate functionals x_n^* above show that the coefficients are unique, so indeed, $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis of $c_0(\mathbb{N})$.

To see that $\{x_n\}_{n=1}^{\infty}$ is not unconditional, consider $a = (a_n)_{n=1}^{\infty} \in c_0(\mathbb{N})$ given as

$$a_n = \ln(2) + \sum_{i=1}^{n} \frac{(-1)^i}{i}.$$

Note that we choose $\ln(2)$ since $\sum_{i=1}^{\infty} \frac{(-1)^i}{i} = -\ln(2)$ (for a quick proof, recall that $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, so taking $x = -1$ yields the desired quantity). The series

$$a = \sum_{n=1}^{\infty} (a_n - a_{n+1})x_n$$

converges in $c_0(\mathbb{N})$ since for each $N \in \mathbb{N}$, consider the sequence of partial sums $\{S_N\}_{N=1}^{\infty}$ given as

$$S_N = \sum_{n=1}^{N} (a_n - a_{n+1})x_n = \left(\sum_{n=1}^{N} a_n e_n\right) - \left(a_{N+1} \sum_{n=1}^{N} e_n\right)$$

Now both $\sum_{n=1}^{N} a_n e_n$ and $a_{N+1} \sum_{n=1}^{N} e_n$ belong to $c_0(\mathbb{N})$ and as $(a_n) \to 0$, we have $\sum_{n=1}^{N} a_n e_n$ converges in $c_0(\mathbb{N})$, and $a_{N+1} \sum_{n=1}^{N} e_n \to 0$ in norm since $a_{N+1} \to 0$ and $\left\|\sum_{n=1}^{N} e_n\right\|_\infty = 1$. Therefore, the sequence $\{S_N\}_{N=1}^{\infty}$ converges in $c_0(\mathbb{N})$ and thus, $\sum_{n=1}^{\infty} (a_n - a_{n+1})x_n$ converges in $c_0(\mathbb{N})$.

However, since

$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$$

so the series is not absolutely convergent. Moreover, there exist signs $\varepsilon_n = \pm 1$ such that the series

$$\sum_{n=1}^{\infty} \varepsilon_n (a_n - a_{n+1})x_n = \sum_{n=1}^{\infty} |a_n - a_{n+1}| x_n$$

diverges in norm since

$$\begin{aligned}
\lim_{N\to\infty}\left\|\sum_{n=1}^{N}|a_n-a_{n+1}|x_n\right\|_\infty &= \lim_{N\to\infty}\left\|\sum_{n=1}^{N}\left(\sum_{i=n}^{N}|a_i-a_{i+1}\right)e_n\right\|_\infty \\
&= \lim_{N\to\infty}\sum_{n=1}^{N}|a_n-a_{n+1}| \\
&= \infty.
\end{aligned}$$

Therefore, the summing basis $\{x_n\}_{n=1}^\infty$ of $c_0(\mathbb{N})$ is not an unconditional basis, despite being a Schauder basis.

Before we prove the main result of this section, we first need to establish some terminology of convergence, and see how they interact with one another.

Definition 1.4.7. Let X be a Banach space and let $\sum_{i=1}^\infty x_i$ be a series in X.

1. We say that $\sum_{i=1}^\infty x_i$ is *unconditionally Cauchy* if for every $\varepsilon > 0$, there exists $E \subseteq \mathbb{N}$ such that $\left\|\sum_{i\in F} x_i\right\| < \varepsilon$ for every $F \subseteq \mathbb{N}$ such that $E \cap F = \emptyset$.

2. We say that $\sum_{i=1}^\infty x_i$ is *subseries convergent* if for every increasing sequence $(i_k)_{k=1}^\infty \subseteq \mathbb{N}$, the series $\sum_{k=1}^\infty x_{i_k}$ is convergent.

Lemma 1.4.8. *Let X be a Banach space and let $\sum_{i=1}^\infty x_i$ be a series in X. The following are equivalent.*

1. *$\sum_{i=1}^\infty x_i$ is unconditionally convergent.*

2. *$\sum_{i=1}^\infty x_i$ is unconditionally Cauchy.*

3. *$\sum_{i=1}^\infty x_i$ is subseries convergent.*

Proof. We will first show that (2) and (3) are equivalent. Assume that $\sum_{i=1}^\infty x_i$ is unconditionally Cauchy. Let $(i_k)_{k=1}^\infty \subseteq \mathbb{N}$ be an increasing sequence of indices.

For every $\varepsilon > 0$, since the series is unconditionally Cauchy, choose a finite subset $E \subseteq \mathbb{N}$ such that for every $F \subseteq \mathbb{N}$ with $E \cap F = \emptyset$, we have

$$\left\| \sum_{i \in F} x_i \right\| < \varepsilon.$$

Since only finitely many i_k lie in E, the tail of the subseries $\sum_{k=1}^{\infty} x_{i_k}$ lies in $\mathbb{N} \setminus E$, and thus has Cauchy tails. Therefore, $\sum_{k=1}^{\infty} x_{i_k}$ is Cauchy.

Assume for a contradiction that $\sum_{i=1}^{\infty} x_i$ is subseries convergent but not unconditionally Cauchy. Then there exists an $\varepsilon > 0$ and a sequence $\{F_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} F_n < \inf_{n \in \mathbb{N}} F_{n+1}$ and

$$\left\| \sum_{i \in F_n} x_i \right\| \geq \varepsilon$$

for all $n \in \mathbb{N}$. By listing the elements in $\bigcup_{n=1}^{\infty} F_n$ in increasing order, we obtain a non-Cauchy subseries of $\sum_{i=1}^{\infty} x_i$, which is a contradiction.

We will next prove that (2) implies (1). Assume that $\sum_{i=1}^{\infty} x_i$ is unconditionally Cauchy, so for every $\varepsilon > 0$, there exists a finite subset $E \subseteq \mathbb{N}$ such that for all $F \subseteq \mathbb{N}$ with $E \cap F = \emptyset$, we have $\left\| \sum_{i \in F} x_i \right\| < \varepsilon$. For each $n \in \mathbb{N}$ and for any choices of signs $\varepsilon_n = \pm 1$, let

$$S_n = \sum_{i=1}^{n} \varepsilon_i x_i.$$

We show that $(S_n)_{n=1}^{\infty}$ is Cauchy in X. Choose $N = \max(E)$ since E is finite, so that for any $n > m \geq N$ and if $F = \{m < i \leq n : \varepsilon_i = \pm 1\}$, which is disjoint from E,

$$\left\| \sum_{i=m+1}^{n} \varepsilon_i x_i \right\| \leq \left\| \sum_{i \in F} x_i \right\| < \varepsilon.$$

Therefore, as $\varepsilon > 0$ was arbitrary, $(S_n)_{n=1}^{\infty}$ is Cauchy in X, and thus, $\sum_{i=1}^{\infty} \varepsilon_i x_i$ converges.

To complete the proof, we will show (1) implies (3). Indeed, assume that $\sum_{i=1}^{\infty} x_i$ is unconditionally convergent, and let $(i_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ be an increasing sequence. We will show that $\sum_{k=1}^{\infty} x_{i_k}$ is Cauchy. For each $i \in \mathbb{N}$, define

$$\varepsilon_i = \begin{cases} 1 & \text{if there exists } k \in \mathbb{N} \text{ such that } i = i_k \\ -1 & \text{if for all } k \in \mathbb{N},\ i \neq i_k \end{cases}$$

Then observe that

$$\sum_{k=1}^{\infty} x_{i_k} = \frac{1}{2}\left(\sum_{i=1}^{\infty} x_i + \sum_{i=1}^{\infty} \varepsilon_i x_i\right)$$

In particular, this implies that $\sum_{k=1}^{\infty} x_{i_k}$ also converges, since the series on the right side converges. Thus, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n > m \geq N$, $\left\|\sum_{k=m+1}^{n} x_{i_k}\right\| < \varepsilon$, so $\sum_{k=1}^{\infty} x_{i_k}$ is subseries convergent, as desired. $\square$

Proposition 1.4.9. *Let X be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be a sequence in X. The following assertions are equivalent.*

1. *$\{e_i\}_{i=1}^{\infty}$ is an unconditional basic sequence.*

2. *There exists a $K > 0$ such that for all $a_1, a_2, ..., a_n \in \mathbb{R}$ and signs $\varepsilon_i = \pm 1$,*

$$\left\|\sum_{i=1}^{n} \varepsilon_i a_i e_i\right\| \leq K \left\|\sum_{i=1}^{n} a_i e_i\right\|.$$

3. *There exists a $L > 0$ such that for all $a_1, a_2, ..., a_n \in \mathbb{R}$ and every subset $\sigma \subseteq \{1, 2, ..., n\}$,*

$$\left\|\sum_{i\in\sigma} a_i e_i\right\| \leq L \left\|\sum_{i=1}^{n} a_i e_i\right\|.$$

Proof. We will first start by showing that (2) and (3) are equivalent statements. Indeed, first let us assume (2), i.e. that there exists a $K > 0$ such that for all $a_1, a_2, ..., a_n \in \mathbb{R}$ and signs $\varepsilon_i = \pm 1$, that

$$\left\|\sum_{i=1}^{n} \varepsilon_i a_i e_i\right\| \leq K \left\|\sum_{i=1}^{n} a_i e_i\right\|.$$

Claim. *For any* $\sigma \subseteq \{1,2,...,n\}$*, if* $\varepsilon_i = 1$ *for* $i \in \sigma$ *and* $\varepsilon_i = -1$ *if* $i \notin \sigma$*, then*

$$\sum_{i\in\sigma} a_i e_i = \frac{1}{2}\left(\sum_{i=1}^{n} \varepsilon_i a_i e_i + \sum_{i=1}^{n} a_i e_i\right).$$

Proof of Claim. Consider the right side of the equality. We split the sums into parts that include $i \in \sigma$ and $i \notin \sigma$, so

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon_i a_i e_i + \sum_{i=1}^{n} a_i e_i &= \sum_{i\in\sigma} \varepsilon_i a_i e_i + \sum_{i\notin\sigma} \varepsilon_i a_i e_i + \sum_{i\in\sigma} a_i e_i + \sum_{i\notin\sigma} a_i e_i \\ &= \left(\sum_{i\in\sigma} (\varepsilon_i + 1) a_i e_i\right) + \left(\sum_{i\notin\sigma} (\varepsilon_i + 1) a_i e_i\right) \\ &= \left(\sum_{i\in\sigma} 2 a_i e_i\right) + \left(\sum_{i\notin\sigma} 0 a_i e_i\right) \\ &= 2\sum_{i\in\sigma} a_i e_i.\end{aligned}$$

Hence,

$$\sum_{i\in\sigma} a_i e_i = \frac{1}{2}\left(\sum_{i=1}^{n} \varepsilon_i a_i e_i + \sum_{i=1}^{n} a_i e_i\right).$$

as claimed. □

Thus, with the claim, let $\sigma \subseteq \{1,2,...,n\}$ be such that $\varepsilon_i = 1$ if $i \in \sigma$ and $\varepsilon_i = -1$ if $i \notin \sigma$. Then see that

$$\begin{aligned}\left\|\sum_{i\in\sigma} a_i e_i\right\| &= \frac{1}{2}\left\|\sum_{i=1}^{n} \varepsilon_i a_i e_i + \sum_{i=1}^{n} a_i e_i\right\| \leq \frac{1}{2}\left\|\sum_{i=1}^{n} \varepsilon_i a_i e_i\right\| + \frac{1}{2}\left\|\sum_{i=1}^{n} a_i e_i\right\| \\ &\leq \frac{1}{2}K\left\|\sum_{i=1}^{n} a_i e_i\right\| + \frac{1}{2}\left\|\sum_{i=1}^{n} a_i e_i\right\| = \frac{1}{2}(K+1)\left\|\sum_{i=1}^{n} a_i e_i\right\|.\end{aligned}$$

Thus, letting $L = \frac{1}{2}(K+1) > 0$, there is such a constant $L > 0$ so that for any

$\sigma \subseteq \{1, 2, ..., n\}$ and $a_1, a_2, ..., a_n \in \mathbb{R}$ that

$$\left\| \sum_{i \in \sigma} a_i e_i \right\| \leq L \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

Now assume that (3) holds, i.e. there exists $L > 0$ such that for all $a_1, a_2, ..., a_n \in \mathbb{R}$ and $\sigma \subseteq \{1, 2, ..., n\}$,

$$\left\| \sum_{i \in \sigma} a_i e_i \right\| \leq L \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

Claim. *For any $a_1, a_2, ..., a_n \in \mathbb{R}$ and signs $\varepsilon_i = \pm 1$, if $\sigma = \{i : \varepsilon_i = 1\}$ and $\tau = \{1, 2, ..., n\} \setminus \sigma$,*

$$\sum_{i=1}^{n} \varepsilon_i a_i e_i = \sum_{i \in \sigma} a_i e_i - \sum_{i \in \tau} a_i e_i.$$

Proof of Claim. Starting with the left side, we have note that τ is equivalent to saying $\tau = \{i : \varepsilon_i = -1\}$, so

$$\sum_{i=1}^{n} \varepsilon_i a_i e_i = \sum_{i \in \sigma} \varepsilon_i a_i e_i + \sum_{i \in \tau} \varepsilon_i a_i e_i = \sum_{i \in \sigma} a_i e_i - \sum_{i \in \tau} a_i e_i.$$

as claimed. $\square$

Thus, with the claim, if σ and τ are as in the claim and $a_1, a_2, ..., a_n \in \mathbb{R}$, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} \varepsilon_i a_i e_i \right\| &= \left\| \sum_{i \in \sigma} a_i e_i - \sum_{i \in \tau} a_i e_i \right\| \leq \left\| \sum_{i \in \sigma} a_i e_i \right\| + \left\| \sum_{i \in \tau} a_i e_i \right\| \\ &\leq L \left\| \sum_{i=1}^{n} a_i e_i \right\| + L \left\| \sum_{i=1}^{n} a_i e_i \right\| = 2L \left\| \sum_{i=1}^{n} a_i e_i \right\|. \end{aligned}$$

Therefore, letting $K = 2L > 0$, there is a $K > 0$ such that for any $a_1, a_2, ..., a_n \in \mathbb{R}$ and signs $\varepsilon_i = \pm 1$,

$$\left\| \sum_{i=1}^{n} \varepsilon_i a_i e_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

Now that we have established that (2) and (3) are equivalent, we will show that (1) and (3) are equivalent.

Assume (1) that $\{e_i\}_{i=1}^\infty$ is an unconditional basic sequence, and put $E = [\{e_i\}]$. Let $\sigma \subseteq \mathbb{N}$, and let define $P_\sigma : E \to E$ by

$$P_\sigma x = \sum_{i\in\sigma} a_i e_i$$

for $x = \sum_{i=1}^\infty a_i e_i \in E$. By assumption and Lemma 1.4.8, P_σ is well-defined, as $\sum_{i\in\sigma} a_i e_i$ converges whenever $\sum_{i=1}^\infty a_i e_i$ converges. To show that P_σ is bounded, we will show that P_σ has a closed graph. For each $n \in \mathbb{N}$, let $x_n = \sum_{i=1}^\infty a_{i,n} e_i$ and let $x = \sum_{i=1}^\infty a_i e_i$ be such that $x_n \to x$ in E and $P_\sigma x_n = \sum_{i\in\sigma} a_{i,n} e_i \to y = \sum_{i\in\sigma} b_i e_i$. Then since the coordinate functionals are continuous, we have $a_{i,n} \to a_i$ and $a_{i,n} \to b_i$ for every $i \in \mathbb{N}$, and thus, $a_i = b_i$ for all $i \in \mathbb{N}$, and $P_\sigma x = y$, so P_σ has a closed graph, so by the Closed Graph Theorem (Theorem A.0.7), P_σ is bounded.

Now define the family of operators P_σ where σ runs through all subsets of $\mathbb{N}$. We claim that for every $x = \sum_{i=1}^\infty a_i e_i \in X$, $\{P_\sigma x\}$ is bounded. By assumption, since $\{e_i\}_{i=1}^\infty$ is unconditional, for every $\varepsilon > 0$, there exists a finite subset $F \subseteq \mathbb{N}$ such that for all $G \subseteq \mathbb{N}$ with $F \cap G = \emptyset$, $\left\|\sum_{i\in G} a_i e_i\right\| < \varepsilon$. Therefore, for all $\sigma \subseteq \mathbb{N}$, since $\sigma = (\sigma \cap F) \cup (\sigma \setminus F)$ are disjoint,

$$\sum_{i\in\sigma} a_i e_i = \sum_{i\in\sigma\cap F} a_i e_i + \sum_{i\in\sigma\setminus F} a_i e_i.$$

Also, since $F \cap (\sigma \setminus F) = \emptyset$, $\left\|\sum_{i\in\sigma\setminus F} a_i e_i\right\| < \varepsilon$, so

$$\begin{aligned}
\|P_\sigma x\| &= \left\| \sum_{i\in\sigma\cap F} a_i e_i + \sum_{i\in\sigma\setminus F} a_i e_i \right\| \\
&\leq \left\| \sum_{i\in\sigma\cap F} a_i e_i \right\| + \left\| \sum_{i\in\sigma\setminus F} a_i e_i \right\| \\
&\leq \sum_{i\in\sigma\cap F} |a_i| \|e_i\| + \varepsilon
\end{aligned}$$

$$\leq \sum_{i\in F} |a_i|\|e_i\| + \varepsilon.$$

Therefore, as $\varepsilon > 0$ was arbitrary, the boundedness of $\{P_\sigma x\}$ follows, so by the Banach-Steinhaus Theorem (Theorem A.0.9), $\{P_\sigma\}$ are uniformly bounded by some L.

Finally, assume that (3) holds, i.e. there exists $L > 0$ such that for all $a_1, a_2, ..., a_n \in \mathbb{R}$ and $\sigma \subseteq \{1, 2, ..., n\}$,

$$\left\| \sum_{i\in\sigma} a_i e_i \right\| \leq L \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

Let $n, m \in \mathbb{N}$ be such that $n \leq m$ and $a_1, a_2, ..., a_m \in \mathbb{R}$. Let $\sigma = \{1, 2, ..., n\}$ so that by Grümblum's Criterion (Proposition 1.2.17), we have

$$\left\| \sum_{i\in\sigma} a_i e_i \right\| = \left\| \sum_{i=1}^{n} a_i e_i \right\| \leq L \left\| \sum_{i=1}^{m} a_i e_i \right\|$$

so $\{e_i\}_{i=1}^{\infty}$ is a basic sequence with $\mathrm{bc}(\{e_i\}) \leq L$.

To show that $\{e_i\}_{i=1}^{\infty}$ is an unconditional basis, assume that $\sum_{i=1}^{\infty} a_i e_i$ is a convergent series. Then for any choice of signs $\varepsilon_i = \pm 1$, since (2) and (3) are equivalent, by (2), there exists a $K > 0$ so that

$$\left\| \sum_{i=n}^{m} \varepsilon_i a_i e_i \right\| \leq 2K \left\| \sum_{i=n}^{m} a_i e_i \right\|$$

and thus, $\sum_{i=1}^{\infty} \varepsilon_i a_i e_i$ is Cauchy, so convergent. Therefore, $\sum_{i=1}^{\infty} a_i e_i$ converges unconditionally. $\square$

Definition 1.4.10. Let X be a Banach space and let $\{e_i\}_{i=1}^{\infty}$ be an unconditional basis of X. The *unconditional basis constant* $\mathrm{ubc}(\{e_i\})$ is the smallest K so that for any $a_1, a_2, ..., a_n \in \mathbb{R}$ and signs $\varepsilon_i = \pm 1$,

$$\left\| \sum_{i=1}^{n} \varepsilon_i a_i e_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

Remark 1.4.11. In the proof of Proposition 1.4.9, we have $L \leq K$, so in general, we have $\mathrm{bc}(\{e_i\}) \leq \mathrm{ubc}(\{e_i\})$.

Remark 1.4.12. From Mazur's Theorem (Theorem 1.2.20), we have that every infinite-dimensional Banach space has a basic sequence. However, a natural question that one may ask is whether every Banach space contains an unconditional basic sequence. The answer to this was false, as it was shown by Gowers and Maurey [6].

To conclude this section, we will show a relationship between basic sequences and bounded sequences of scalars.

Proposition 1.4.13. *Let X be a Banach space with Schauder basis $\{e_i\}_{i=1}^{\infty}$. If $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ is such that $\sum_{i=1}^{\infty} a_i e_i$ converges and $(b_i)_{i=1}^{\infty}$ is a bounded sequence of scalars, then*

$$\left\| \sum_{i=1}^{\infty} b_i a_i e_i \right\| \leq \mathrm{ubc}(\{e_i\}) \left(\sup_{i \in \mathbb{N}} |b_i| \right) \left\| \sum_{i=1}^{\infty} a_i e_i \right\|.$$

Proof. Let $n \in \mathbb{N}$. By the Hahn-Banach Theorem (Theorem A.0.4), choose $x^* \in S_{X^*}$ such that $x^* \left(\sum_{i=1}^{n} b_i a_i e_i \right) = \|\sum_{i=1}^{n} b_i a_i e_i\|$, and define the signs ε_i so that $\varepsilon_i = 1$ if $a_i x^*(e_i) \geq 0$, and $\varepsilon_i = -1$ if $a_i x^*(e_i) < 0$. Then by Proposition 1.4.9 observe that

$$\begin{aligned}
\left\| \sum_{i=1}^{n} b_i a_i e_i \right\| &\leq \sum_{i=1}^{n} |b_i| |a_i x^*(e_i)| \\
&\leq \left(\sup_{1 \leq i \leq n} |b_i| \right) \sum_{i=1}^{n} \varepsilon_i a_i x^*(e_i) \\
&\leq \left(\sup_{1 \leq i \leq n} |b_i| \right) \|x^*\| \left\| \sum_{i=1}^{n} \varepsilon_i a_i e_i \right\| \\
&\leq \left(\sup_{1 \leq i \leq n} |b_i| \right) \mathrm{ubc}(\{e_i\}) \left\| \sum_{i=1}^{n} a_i e_i \right\|.
\end{aligned}$$

The result then follows by extending $n \to \infty$. $\square$

1.5 Complemented Subspaces of Classical Spaces: $c_0(\mathbb{N})$, $\ell_p(\mathbb{N})$, $L_p[0,1]$, $1 \le p < \infty$

In this section, we turn our attention to the classical Banach spaces $c_0(\mathbb{N})$, $\ell_p(\mathbb{N})$, and $L_p[0,1]$, $1 \le p < \infty$, and study how Schauder bases behave in these settings. These spaces are among the most fundamental in analysis and provide a natural context for understanding projections, complemented subspaces, and structural properties of bases.

There are two highlights in this section. The first result is due to Pełczyński, which asserts that if a closed infinite-dimensional subspace of either $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ is complemented, then it is isomorphic to the whole space.

The second highlight is Khintchine's inequality, a useful tool that plays a crucial role in the structure theory of L_p spaces. It characterizes the behaviour of Rademacher functions in $L_p[0,1]$, and is extremely useful in identifying complemented subspaces of L_p isomorphic to ℓ_2, even when $p \neq 2$.

In this section, to simplify the notations, we will denote $\mathcal{X} = c_0(\mathbb{N})$ or $\mathcal{X} = \ell_p(\mathbb{N})$, $1 \le p < \infty$.

Proposition 1.5.1. *If $\{u_i\}_{i=1}^{\infty}$ is a normalized (see Definition 1.2.12) block basic sequence of the unit vector basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{X}$, then the following holds.*

1. *$\{u_i\} \sim \{e_i\}$. In particular, $[\{u_i\}] \equiv \mathcal{X}$.*
2. *There exists a norm-one projection of $\mathcal{X}$ onto $[\{u_i\}]$.*

Proof. We will prove the case for when $\mathcal{X} = \ell_p(\mathbb{N})$, $1 \le p < \infty$, as the proof for the case when $\mathcal{X} = c_0(\mathbb{N})$ is similar. Assume that $\{u_i\}_{i=1}^{\infty}$ is a normalized block basic sequence of the unit vector basis $\{e_i\}_{i=1}^{\infty}$ of $\ell_p(\mathbb{N})$, so for scalars $(a_j)_{j=p_i+1}^{p_{i+1}} \subseteq \mathbb{R}$

$$\|u_i\|_p = \left\| \sum_{j=p_i+1}^{p_{i+1}} a_j e_j \right\|_p = \left(\sum_{j=p_i+1}^{p_{i+1}} |a_j|^p \right)^{\frac{1}{p}} = 1.$$

To see that (1) holds, let $n \in \mathbb{N}$, and let $b_1, b_2, ..., b_n \in \mathbb{R}$. Then we have

$$\left\|\sum_{i=1}^{n} b_i u_i\right\|_p = \left\|\sum_{i=1}^{n} b_i \sum_{j=p_i+1}^{p_{i+1}} a_j e_j\right\|_p = \left\|\sum_{i=1}^{n} \sum_{j=p_i+1}^{p_{i+1}} b_i a_j e_j\right\|_p = \left(\sum_{i=1}^{n} \sum_{j=p_i+1}^{p_{i+1}} |b_i|^p |a_j|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}} \left(\sum_{j=p_i+1}^{p_{i+1}} |a_j|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}} = \left\|\sum_{i=1}^{n} b_i e_i\right\|_p .$$

Therefore, we have shown that $\{u_i\} \sim \{e_i\}$. Thus, there exists an isomorphism $T : [\{u_i\}] \to \ell_p(\mathbb{N})$ such that $Tu_i = e_i$ for all $i \in \mathbb{N}$. In particular, the above computation shows that T is in fact an isometric isomorphism.

To see that (2) holds, since $\{u_i\}_{i=1}^{\infty}$ is a block basic sequence, we have

$$u_i = \sum_{j=p_i+1}^{p_{i+1}} a_j e_j$$

and $\|u_i\| = 1$. By the Hahn-Banach Theorem, for each $i \in \mathbb{N}$, there exists $u_i^* \in \ell_p^*(\mathbb{N}) \equiv \ell_q(\mathbb{N})$ such that $\|u_i^*\| = 1$ and $u_i^*(u_i) = \|u_i\| = 1$. Then since $u_i^* \in \ell_q(\mathbb{N})$, then $u_i^* = (b_j)_{j=1}^{\infty}$ and so we can write

$$u_i^*(u_i) = \sum_{j=1}^{\infty} a_j b_j = \sum_{j=p_i+1}^{p_{i+1}} a_j b_j$$

Let $\tilde{u}_i^* = (\tilde{b}_i)_{i=1}^{\infty}$ with

$$\tilde{b}_j = \begin{cases} b_j & \text{if } p_i < j \leq p_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{u}_i^*(u_i) = u_i^*(u_i) = 1$ and

$$\tilde{u}_i = \sum_{j=p_i+1}^{p_{i+1}} b_j e_j^*$$

Furthermore, observe that by the formula of the q-norm,

$$1 = \|u_i^*\|_q \geq \|\tilde{u}_i^*\|_q \geq \tilde{u}_i^*(u_i) = 1$$

Therefore, $\|\tilde{u}_i^*\|_q = \|\tilde{u}_i\|_q = 1$. Define $P : \ell_p(\mathbb{N}) \to [\{u_i\}]$ by

$$Px = \sum_{i=1}^{\infty} \tilde{u}_i^*(x) u_i$$

for $x \in \ell_p(\mathbb{N})$. We claim that P is the desired norm-one bounded linear projection. To see that P is a projection, let $x = \sum_{j=1}^{\infty} a_j u_j$. Then see that

$$Px = \sum_{i=1}^{\infty} \tilde{u}_i^* \left(\sum_{j=1}^{\infty} a_j u_j \right) u_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_j \tilde{u}_i^*(u_j) u_i = \sum_{i=1}^{\infty} a_i u_i = x.$$

Therefore, P is a projection. To see that $\|P\| = 1$, first note that for any $i \in \mathbb{N}$,

$$\|P\| = \|P\| \|u_i\|_p \geq \|Pu_i\|_p = |\tilde{u}_i^*(u_i)| = 1.$$

On the other hand, for any $x = \sum_{i=1}^{\infty} a_i e_i \in \ell_p(\mathbb{N})$, first observe by Hölder's inequality (Theorem A.0.1) that

$$\begin{aligned} |\tilde{u}_i^*(x)| &= \left| \sum_{j=p_i+1}^{p_{i+1}} b_j e_j^*(x) \right| = \left| \sum_{j=p_i+1}^{p_{i+1}} b_j e_j^* \left(\sum_{i=1}^{\infty} a_i e_i \right) \right| = \left| \sum_{j=p_i+1}^{p_{i+1}} \sum_{i=1}^{\infty} b_j a_i e_j^*(e_i) \right| = \left| \sum_{j=p_i+1}^{p_{i+1}} b_j a_j \right| \\ &\leq \left(\sum_{j=p_i+1}^{p_{i+1}} |b_j|^q \right)^{\frac{1}{q}} \left(\sum_{j=p_i+1}^{p_{i+1}} |a_j|^p \right)^{\frac{1}{p}} = \left(\sum_{j=p_i+1}^{p_{i+1}} |a_j|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore,

$$\|Px\|_p = \left(\sum_{i=1}^{\infty} |\tilde{u}_i^*(x)|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} \sum_{j=p_i+1}^{p_{i+1}} |a_j|^p \right)^{\frac{1}{p}} = \|x\|_p.$$

Therefore, $\|P\| = 1$, as desired. $\square$

Before we prove the following result, we borrow a statement from [1]. This proof uses the sliding hump argument as in the Bessaga-Pełczyński Selection Principle.

Lemma 1.5.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a normalized weakly null sequence in $\mathcal{X}$. Then there exists a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ that is a basic sequence equivalent to the unit vector basis of $\mathcal{X}$ and complemented in $\mathcal{X}$.*

Proof. Since $\|x_i\| = 1$ for all $i \in \mathbb{N}$ and $x_i \rightharpoonup 0$, by the Bessaga-Pełczyński Selection Principle (Theorem 1.3.13), there exists a basic subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ that is equivalent to a block basic sequence $\{u_k\}_{k=1}^{\infty}$ of the unit vector basis of $\mathcal{X}$. Furthermore, we have that $\{x_{i_k}\}_{k=1}^{\infty}$ is complemented in $\mathcal{X}$ whenever $\{u_k\}_{k=1}^{\infty}$ is complemented in $\mathcal{X}$. $\square$

The following results are due to Pełczyński (1960) [18] which demonstrate the uses of basic sequence techniques established in previous sections. The following follows from Proposition 1.5.1 and Lemma 1.5.2.

Proposition 1.5.3. *Every infinite-dimensional closed subspace Y of $\mathcal{X}$ contains a closed subspace Z that is isomorphic to $\mathcal{X}$ and complemented in $\mathcal{X}$.*

Proof. We first prove the following claim.

Claim. *If Y is an infinite-dimensional subspace of $\mathcal{X}$, then for every $n \in \mathbb{N}$, there exists $y_n \in S_Y$ such that for each $1 \leq i \leq n$, $e_i^*(y_n) = 0$.*

Proof of Claim. Assume for a contradiction that the statement was false. Then there exists an $n \in \mathbb{N}$ such that the nth canonical projection restricted to Y $\Pi_n|_Y$, where

$$\Pi_n\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{n} a_i e_i$$

is injective, as $y \neq 0$ implies $\Pi_n y \neq 0$. Then this would imply that $\Pi_n|_Y : Y \to \Pi_n(Y)$ is an isomorphism, which is a contradiction, since Y is infinite-dimensional. $\square$

Therefore, by the Claim and Proposition 1.5.1, the sequence $\{y_n\}_{n=1}^{\infty}$ has a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ that is basic and equivalent to the unit vector basis of $\mathcal{X}$, and that $Z = [\{y_{n_k}\}]$ is complemented in $\mathcal{X}$. $\square$

Before we introduce the following result, we need to define the following spaces.

Definition 1.5.4. Let X be a Banach space.

1. For $1 \leq p < \infty$, the space $\ell_p(X) = \left(\bigoplus_{i=1}^{\infty} X\right)_p$ is called the *infinite direct sum of X in the sense of ℓ_p*, and it consists of all sequences $x = (x_i)_{i=1}^{\infty}$ with values in X so that $(\|x_i\|_X)_{i=1}^{\infty} \in \ell_p(\mathbb{N})$ with the norm

$$\|x\|_{\ell_p(X)} = \|(\|x_i\|_X)_{i=1}^{\infty}\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|_X^p\right)^{\frac{1}{p}}.$$

2. The space $c_0(X) = \left(\bigoplus_{i=1}^{\infty} X\right)_{c_0}$ is called the *infinite direct sum of X in the sense of* $c_0(\mathbb{N})$, and it consists of all sequences $x = (x_i)_{i=1}^{\infty}$ with values in X so that $(\|x_i\|_X)_{i=1}^{\infty} \in c_0(\mathbb{N})$ with the norm

$$\|x\|_{c_0(X)} = \|(\|x_i\|_X)_{i=1}^{\infty}\|_\infty = \sup_{i \in \mathbb{N}} \|x_i\|_X.$$

Example 1.5.5. Take $X = \ell_p(\mathbb{N})$, and consider the space $\ell_p(\ell_p(\mathbb{N})) = \left(\bigoplus_{i=1}^{\infty} \ell_p(\mathbb{N})\right)_p$. This space consists of all sequences $x = (x_i)_{i=1}^{\infty}$ with, for each $i \in \mathbb{N}$, $x_i \in \ell_p(\mathbb{N})$ so that $(\|x_i\|_p)_{i=1}^{\infty} \in \ell_p(\mathbb{N})$. Observe that when we compute the norm, we have

$$\|x\|_{\ell_p(\ell_p(\mathbb{N}))} = \left(\sum_{i=1}^{\infty} \|x_i\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |x_{i,j}|^p\right)^{\frac{p}{p}}\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |x_{i,j}|^p\right)^{\frac{1}{p}}.$$

Based on this observation, we identify that $\ell_p(\ell_p(\mathbb{N}))$ is simply $\ell_p(\mathbb{N} \times \mathbb{N})$, and moreover, since there exists a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, we conclude that

$$\ell_p(\ell_p(\mathbb{N})) \equiv \ell_p(\mathbb{N} \times \mathbb{N}) \equiv \ell_p(\mathbb{N}).$$

In a similar observation, taking $X = c_0(\mathbb{N})$, for any $x = (x_i)_{i=1}^{\infty}$, where for each

$i \in \mathbb{N}$, $x_i \in c_0(\mathbb{N})$ so that $(\|x_i\|_\infty)_{i=1}^\infty \in c_0(\mathbb{N})$, we have that

$$\|x\|_{c_0(c_0(\mathbb{N}))} = \sup_{i\in\mathbb{N}} \|x_i\|_\infty = \sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} |x_{i,j}|.$$

Thus, we identify that $c_0(c_0(\mathbb{N}))$ is $c_0(\mathbb{N}\times\mathbb{N})$, and moreover, as there exists a bijection from $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$, we conclude that

$$c_0(c_0(\mathbb{N})) \equiv c_0(\mathbb{N}\times\mathbb{N}) \equiv c_0(\mathbb{N}).$$

Remark 1.5.6. For any Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ (can be extended to more than two) and $1 \le p < \infty$, we claim that

$$\ell_p(X \oplus Y) \simeq \ell_p(X) \oplus \ell_p(Y).$$

To see this, note that for any $(x_i, y_i)_{i=1}^\infty \in \ell_p(X\oplus Y)$, where for each $i \in \mathbb{N}$, $x_i \in X$ and $y_i \in Y$, we have

$$\begin{aligned}
\|(x_i, y_i)_{i=1}^\infty\|_{\ell_p(X\oplus Y)}^p &= \sum_{i=1}^\infty \|(x_i, y_i)\|_{X\oplus Y}^p \\
&= \sum_{i=1}^\infty (\|x_i\|_X^p + \|y_i\|_Y^p) \\
&= \sum_{i=1}^\infty \|x_i\|_X^p + \sum_{i=1}^\infty \|y_i\|_Y^p \\
&= \|(x_i)_{i=1}^\infty\|_{\ell_p(X)}^p + \|(y_i)_{i=1}^\infty\|_{\ell_p(Y)}^p \\
&= \|((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty)\|_{\ell_p(X)\oplus\ell_p(Y)}^p.
\end{aligned}$$

Therefore, this shows that $\ell_p(X \oplus Y) \simeq \ell_p(X) \oplus \ell_p(Y)$.

Theorem 1.5.7 (Pełczyński Decomposition Method)**.** *Let X and Y be Banach spaces so that X is isomorphic to a complemented subspace of Y, and Y is isomorphic to a complemented subspace of X. Assume that either*

1. *$X \simeq X \oplus X$ and $Y \simeq Y \oplus Y$, or*

2. *$X \simeq c_0(X)$ or $X \simeq \ell_p(X)$, $1 \le p < \infty$.*

Then $X \simeq Y$.

Proof. Assume that X is isomorphic to a complemented subspace of Y and that Y is isomorphic to a complemented subspace of X. So there exist closed subspaces E and F such that $X \simeq Y \oplus E$ and $Y \simeq X \oplus F$.

Case 1: If $X \simeq X \oplus X$ and $Y \simeq Y \oplus Y$. Observe that

$$X \simeq Y \oplus E \simeq (Y \oplus Y) \oplus E \simeq Y \oplus (Y \oplus E) \simeq Y \oplus X \simeq Y \oplus X.$$

Also, by symmetry

$$Y \simeq X \oplus F \simeq (X \oplus X) \oplus F \simeq X \oplus (X \oplus F) \simeq X \oplus Y.$$

Therefore, since $X \simeq Y \oplus X$ and $Y \simeq X \oplus Y$, then $X \simeq Y$ as desired.

Case 2: If $X \simeq c_0(X)$ or $X \simeq \ell_p(X)$, $1 \leq p < \infty$. We show for the case when $X \simeq \ell_p(X)$, and the proof for $X \simeq c_0(X)$ is similar. First, observe that if $X \simeq \ell_p(X)$, then by Remark 1.5.6,

$$X \simeq \ell_p(X) \simeq \ell_p(X \oplus X) \simeq \ell_p(X) \oplus \ell_p(X) \simeq X \oplus X.$$

so by Case 1, $Y \simeq X \oplus Y$. Observe that

$$X \simeq \ell_p(X) \simeq \ell_p(Y \oplus E) \simeq \ell_p(Y) \oplus \ell_p(E) \simeq Y \oplus \ell_p(Y) \oplus \ell_p(E) \simeq Y \oplus \ell_p(X) \simeq Y \oplus X.$$

Therefore, since $X \simeq Y \oplus X$ and $Y \simeq X \oplus Y$, then $X \simeq Y$, as desired. □

Using the results above, we are ready to prove one of the main highlights of this section by Pełczyński [18], which had a significant impact on the development of Banach space theory.

Theorem 1.5.8 (Pełczyński's Theorem)**.** *If Y is a complemented infinite-dimensional subspace of $\mathcal{X}$, then $Y \simeq \mathcal{X}$.*

Proof. By Proposition 1.5.3, since Y is an infinite-dimensional subspace of $\mathcal{X}$, then

Y contains a further closed subspace Z that is complemented and isomorphic to $\mathcal{X}$. Furthermore, $\mathcal{X}$ is isomorphic to a complemented subspace of Y, in which Y is a complemented subspace of $\mathcal{X}$ itself. By Example 1.5.5, since $\ell_p(\ell_p(\mathbb{N})) \equiv \ell_p(\mathbb{N})$ and $c_0(c_0(\mathbb{N})) \equiv c_0(\mathbb{N})$, by (2) of Pełczyński Decomposition Theorem (Theorem 1.5.7), the proof is complete. $\square$

For the second half of this section, we will now focus on L_p spaces, and recognize that the structure of subspaces of L_p differs from that of ℓ_p spaces. These results are due to Khintchine.

Theorem 1.5.9 (Khintchine's Theorem)**.**

1. *For $1 < p < \infty$, $L_p[0,1]$ contains a complemented subspace isomorphic to $\ell_2(\mathbb{N})$.*
2. *$L_1[0,1]$ contains a subspace isomorphic to $\ell_2(\mathbb{N})$.*

Before proving Khintchine's Theorem (Theorem 1.5.9), we first need to develop the notion of Rademacher functions.

Definition 1.5.10. For each $n \in \mathbb{N}$, we define the *Rademacher functions* by $r_n : [0,1] \to \{-1,1\}$ where

$$r_n(x) = \operatorname{sgn}(\sin(2^n \pi x))$$

for $x \in [0,1]$.

Alternatively, the Rademacher functions can be described as [1]

$$r_n(x) = \begin{cases} 1 & x \in \bigcup_{i=1}^{2^n} \left[\frac{2i-2}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\right) \\ -1 & x \in \bigcup_{i=1}^{2^n} \left[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\right) \end{cases}.$$

The Rademacher functions, introduced by Rademacher in 1922 [19], came from his study of conditions under which the randomly signed series $\sum_{i=1}^{\infty} \pm a_i$ converges almost surely. He showed that convergence is guaranteed if $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. The converse, namely, that almost sure convergence implies square summability, was subsequently established by Khintchine and Kolmogoroff in 1925 [12].

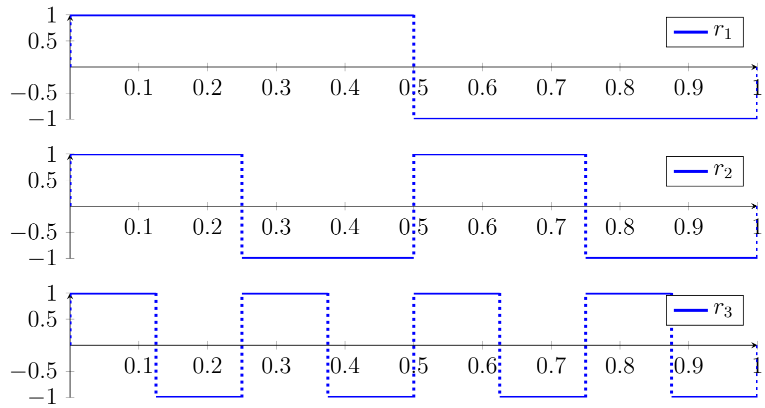

Figure 1.2: The first three Rademacher functions.

Whenever we need to make a distinction between the norms of ℓ_p and norms of L_p, we will denote $\|\cdot\|_{\ell_p}$ and $\|\cdot\|_{L_p}$, respectively, rather than the usual notation $\|\cdot\|_p$.

Proposition 1.5.11. *Let $\{r_n\}_{n=1}^{\infty}$ be the sequence of Rademacher functions in $[0,1]$. The following properties hold.*

1. *For all $n, m \in \mathbb{N}$, $\int_0^1 r_n(x)r_m(x)\ dx = \delta_{n,m}$.*
2. *$\{r_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $L_2[0,1]$.*
3. *$\{r_n\}_{n=1}^{\infty}$ considered in $L_2[0,1]$ is a monotone basic sequence equivalent to the unit vector basis of $\ell_2(\mathbb{N})$.*

Proof. To see that (1) holds, we consider the following cases.

Case 1: If $n = m$. Then we have $r_n(x)r_m(x) = r_n(x)^2 = 1$ since we have either $1^2 = (-1)^2 = 1$. Thus,

$$\int_0^1 r_n(x)r_m(x)\ dx = \int_0^1\ dx = 1.$$

Case 2: If $n \neq m$. Without loss of generality, assume that $n < m$. Then observe that

$$\int_0^1 r_n(x)r_m(x)\ dx = \int_0^1 \operatorname{sgn}(\sin(2^n\pi x))\operatorname{sgn}(\sin(2^m\pi x))\ dx$$
$$= \sum_{i=0}^{2^n-1}\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \operatorname{sgn}(\sin(2^n\pi x))\operatorname{sgn}(\sin(2^m\pi x))\ dx.$$

In each summand, $\operatorname{sgn}(\sin(2^n\pi x))$ is constant and $\operatorname{sgn}(\sin(2^m\pi x))$ runs over 2^{m-n} full periods, so the summand would be zero.

To see that (2) holds, it suffices to show that $\|r_n\|_2 = 1$ for all $n \in \mathbb{N}$. Indeed, note that

$$\|r_n\|_{L_2} = \left(\int_0^1 |r_n(x)|^2\ dx\right)^{\frac{1}{2}} = \left(\int_0^1\ dx\right)^{\frac{1}{2}} = 1.$$

Combining (1) with the inner product $\langle r_n, r_m\rangle = \int_0^1 r_n(x)r_m(x)\ dx$ shows that $\{r_n\}_{n=1}^\infty$ is an orthonormal sequence of $L_2[0.1]$.

Finally, to see that (3) holds, we note that $L_2[0,1]$ is a Hilbert space and from (1) and (2), $\{r_n\}_{n=1}^\infty$ is an orthonormal basis of $L_2[0,1]$, so it is a basic sequence of $L_2[0,1]$. To see that $\{r_n\} \sim \{e_n\}$, where $\{e_n\}_{n=1}^\infty$ denotes the unit vector basis of $\ell_2(\mathbb{N})$, observe that since $\{r_n\}_{n=1}^\infty$ is an orthonormal basis, observe that for any $m \in \mathbb{N}$, by the Pythagorean Theorem,

$$\left\|\sum_{n=1}^m a_n r_n\right\|_{L_2}^2 = \sum_{n=1}^m \|a_n r_n\|_2^2 = \sum_{n=1}^m |a_n|^2\|r_n\|_{L_2}^2 = \sum_{n=1}^m |a_n|^2 = \left\|\sum_{n=1}^m a_n e_n\right\|_{\ell_2}^2$$

Since the above holds for any $m \in \mathbb{N}$, by extending $m \to \infty$, there exists an isometric isomorphism $T : [\{r_n\}] \to [\{e_n\}]$, showing that $\{r_n\}_{n=1}^\infty$ is monotone and equivalent to $\{e_n\}_{n=1}^\infty$. □

On the other hand, Rademacher functions behave slightly differently in L_p spaces for when $1 \le p < \infty$.

Theorem 1.5.12 (Khintchine's Inequality). *Let $\{r_n\}_{n=1}^\infty$ be the sequence of Rademacher functions on $[0,1]$. For every $p \in [1,\infty)$, there exist constants A_p and B_p such that* *for all $m \in \mathbb{N}$ and $a_1, a_2, ..., a_m \in \mathbb{R}$,*

$$A_p \left(\sum_{n=1}^{m} |a_n|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{n=1}^{m} a_n r_n(x) \right|^p \, dx \right)^{\frac{1}{p}} \leq B_p \left(\sum_{n=1}^{m} |a_n|^2 \right)^{\frac{1}{2}}.$$

The best possible constants A_p and B_p of the inequality are called Khintchine's constants.

Remark 1.5.13. From Proposition 1.5.11, we have shown that when $p = 2$, $[\{r_n\}] \equiv \ell_2(\mathbb{N})$. In this case, Khintchine's inequality becomes an equality with constants $A_2 = B_2 = 1$.

Remark 1.5.14. By Hölder's inequality (Theorem A.0.1), if $p > q$, we have

$$\left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}} \geq \left(\int_0^1 |f(x)|^q \, dx \right)^{\frac{1}{q}}$$

and thus, we obtain $A_q \leq A_p$ and $B_q \leq B_p$.

In the proof of Khintchine's inequality, we consider multinomial coefficients: For any $(\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{N}^m$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = p$, denote

$$A_{\alpha_1, \alpha_2, ..., \alpha_m} = \frac{\left(\sum_{j=1}^{m} \alpha_j \right)!}{\prod_{j=1}^{m} (\alpha_j)!}.$$

Proof of Theorem 1.5.12. We follow the proof as in [5]. By Remark 1.5.14, we will show that there exists $A_1 > 0$ and $B_{2k} < \infty$ for all $k \in \mathbb{N}$.

To see that $B_{2k} < \infty$, first observe that

$$\begin{aligned} \int_0^1 \left| \sum_{i=1}^{n} a_i r_i(x) \right|^{2k} \, dx &= \int_0^1 \left(\sum_{i=1}^{n} a_i r_i(x) \right)^{2k} \, dx \\ &= \sum A_{\alpha_1, \alpha_2, ..., \alpha_j} a_{i_1}^{\alpha_1} a_{i_2}^{\alpha_2} \cdots a_{i_j}^{\alpha_j} \int_0^1 r_{i_1}^{\alpha_1}(x) r_{i_2}^{\alpha_2}(x) \cdots r_{i_j}^{\alpha_j}(x) \, dx \end{aligned}$$

where the sum runs through all multi-indices $(\alpha_1, \alpha_2, ..., \alpha_j) \in \mathbb{N}^j$ with $\sum_{l=1}^{j} \alpha_l = 2k$ and $1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq n$. Note that if $\alpha_1, \alpha_2, ..., \alpha_j$ are all even, then

$\int_0^1 \prod_{i_1<i_2<\cdots<i_k} r_{i_l}^{\alpha_l} = 1$, and is equal to zero otherwise. Thus, for each $1 \le l \le j$, setting $\beta_l = \alpha_l/2$, and observe that if $\sum_{l=1}^{j} \beta_l = k$ and $1 \le i_1 \le i_2 \le \cdots \le i_j \le n$, we have

$$\int_0^1 \left| \sum_{i=1}^{n} a_i r_i(x) \right|^{2k} dx = \sum A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j} a_{i_1}^{2\beta_1} a_{i_2}^{2\beta_2} \cdots a_{i_j}^{2\beta_j}.$$

Therefore,

$$\begin{aligned}
\left(\sum_{i=1}^{n} |a_i|^2 \right)^k &= \sum A_{\beta_1, \beta_2, \ldots, \beta_j} a_{i_1}^{2\beta_1} a_{i_2}^{2\beta_2} \cdots a_{i_j}^{2\beta_j} \\
&= \sum \frac{A_{\beta_1, \beta_2, \ldots, \beta_j}}{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}} A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j} a_{i_1}^{2\beta_1} a_{i_2}^{2\beta_2} \cdots a_{i_j}^{2\beta_j} \\
&\ge \min \left\{ \frac{A_{\beta_1, \beta_2, \ldots, \beta_j}}{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}} \right\} \sum A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j} a_{i_1}^{2\beta_1} a_{i_2}^{2\beta_2} \cdots a_{i_j}^{2\beta_j} \\
&= \min \left\{ \frac{A_{\beta_1, \beta_2, \ldots, \beta_j}}{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}} \right\} \int_0^1 \left| \sum_{i=1}^{n} a_i r_i(x) \right|^{2k} dx
\end{aligned}$$

where the minimum is taken over all multi-indices with the same total exponent $2k$, and since the sum is finite, the minimum exists and is positive. Therefore, putting $B_{2k} = \min \left\{ \frac{A_{\beta_1, \beta_2, \ldots, \beta_j}}{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}} \right\}^{-1}$, we have $B_{2k} < \infty$, as claimed.

Now we show the existence of $A_1 > 0$. Denote $f(x) = \sum_{i=1}^{n} a_i r_i(x)$ for $x \in [0, 1]$. Note that by Hölder's inequality (Theorem A.0.1) with $p = \frac{3}{2}$ and $q = 3$, by the proof of $B_{2k} < \infty$ with $k = 2$, and Proposition 1.5.11,

$$\begin{aligned}
\int_0^1 |f(x)|^2 \, dx &= \int_0^1 |f(x)|^{\frac{2}{3}} |f(x)|^{\frac{4}{3}} \, dx \\
&\le \left(\int_0^1 |f(x)| \, dx \right)^{\frac{2}{3}} \left(\int_0^1 |f(x)|^4 \, dx \right)^{\frac{1}{3}} \quad \text{Hölder's Inequality (Theorem A.0.1)} \\
&\le \left(\int_0^1 |f(x)| \, dx \right)^{\frac{2}{3}} B_4^{\frac{4}{3}} \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{2}{3}} \quad B_{2k} < \infty \text{ when } k = 2 \\
&= \left(\int_0^1 |f(x)| \, dx \right)^{\frac{2}{3}} B_4^{\frac{4}{3}} \left(\int_0^1 |f(x)|^2 \right)^{\frac{2}{3}}. \quad \text{Proposition 1.5.11}
\end{aligned}$$

Therefore, since $\left(\int_0^1 |f(x)|\ dx\right)^{\frac{2}{3}} \geq B_4^{-\frac{4}{3}} \left(\int_0^1 |f(x)|^2\ dx\right)^{\frac{1}{3}}$, we conclude that

$$\int_0^1 |f(x)|\ dx \geq B_4^{-2} \left(\int_0^1 |f(x)|^2\ dx\right)^{\frac{1}{2}} = B_4^{-2} \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}}.$$

Thus, $A_1 \geq B_4^{-2} > 0$, and the proof is complete. □

With Khintchine's inequality, we are now ready to prove Khintchine's Theorem (Theorem 1.5.9).

Proof of Theorem 1.5.9. To see that (1) holds, let $T : \ell_2(\mathbb{N}) \to L_p[0,1]$ be defined by

$$Ta = \sum_{i=1}^{\infty} a_i r_i$$

for $a = (a_i)_{i=1}^{\infty} \in \ell_2(\mathbb{N})$, where $\{r_i\}_{i=1}^{\infty}$ are the Rademacher functions. By Khintchine's inequality (Theorem 1.5.12), T is an isomorphism from $\ell_2(\mathbb{N})$ into $L_p[0,1]$.

We consider the following cases.

Case 1: If $p \geq 2$. Then $L_p[0,1]$ is a subspace of $L_2[0,1]$. Since in Hilbert spaces, every closed subspace has a unique orthogonal projection, let $\tilde{P} : L_2[0,1] \to [\{r_n\}]$, and let $P = \tilde{P}|_{L_p[0,1]} : L_p[0,1] \to [\{r_n\}]$. Then for each $f \in L_p[0,1]$ and $x \in [0,1]$, see that

$$(Pf)(x) = \sum_{i=1}^{\infty} \left(\int_0^1 f(t) r_i(t)\ dt\right) r_i(x)$$

which is the formula for the Fourier expansion of f in $\{r_i\}_{i=1}^{\infty}$, which is well-defined for $f \in L_p[0,1]$. Observe that for any $f \in L_p[0,1]$, we have by Hölder's Inequality (Theorem A.0.1) and Khintchine's Inequality (Theorem 1.5.12),

$$\|Pf\|_{L_p} \leq B_p \|Pf\|_{L_2} \leq B_p \|f\|_{L_2} \leq B_p \|f\|_{L_p}$$

which shows that P is a bounded linear projection of $L_p[0,1]$ onto $\ell_2(\mathbb{N})$.

Case 2: If $1 < p < 2$. Choose $q \in [2,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Since $q \geq 2$, Case

1 implies that there exists a complemented (by some projection P) subspace Y of $L_q[0,1]$ that is isomorphic to $\ell_2(\mathbb{N})$. In particular, $P^*(Y^*)$ is a complemented subspace of $L_p[0,1]$ and is isomorphic to $\ell_2(\mathbb{N})$, by the Claim in the proof of Proposition 1.1.9.

To see that (2) holds, note that by Khintchine's inequality (Theorem 1.5.12), for any $(a_i)_{i=1}^{\infty} \in \ell_2(\mathbb{N})$, $\sum_{i=1}^{\infty} a_i r_i$ converges in $L_1[0,1]$, and for any $n \in \mathbb{N}$

$$A_1 \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{i=1}^{n} a_i r_i \right\|_{L_1} \leq B_1 \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} .$$

Therefore, the mapping $T : \ell_2(\mathbb{N}) \to [\{r_n\}]$ from (1) defines a bounded linear embedding of $\ell_2(\mathbb{N})$ into $L_1[0,1]$, and the image is isomorphic to $\ell_2(\mathbb{N})$. $\square$

Chapter 2

Finite Dimensional Spaces

The key highlight of this chapter is the Lindenstrauss–Tzafriri Theorem, which characterizes Hilbert spaces by complementation properties. In this chapter, we first introduce the concept of finite representability and then build onto spreading models. Then we introduce the crude Dvoretsky Theorem, which is relevant to proving the Lindenstrauss–Tzafriri Theorem at the end of the chapter.

In the first section, we study finite representability, which describes the relationship between finite-dimensional subspaces of Banach spaces and how close they are related, in terms of the Banach–Mazur distance.

In the second section, we study spreading models, which provide a way to understand the asymptotic behaviour of sequences in a Banach space. A spreading model of a Banach space is essentially a new Banach space constructed from a sequence in the original space, reflecting the "spreading" or "asymptotic" properties of that sequence.

In the third section, we introduce Tzafriri's Theorem, without proof, which characterizes sequences of finite-dimensional complemented subspaces (that behave like ℓ_p-spaces) in Banach spaces with an unconditional basis. Although we omit the proof, we remark that it utilizes techniques from spreading models and involves the use of Rademacher functions, reflecting deeper structural properties of Banach spaces. However, we include a consequence of Tzafriri's Theorem, which states that every Hilbert space is crudely finitely representable in every infinite-dimensional Banach space.

We then close this chapter by introducing the heart of this paper: the Lindenstrauss–Tzafriri Theorem, which characterizes Hilbert spaces by their complementability and their subspaces.

2.1 Finite Representability

In this section, we introduce a fundamental notion known as finite representability. The ideas of finite representability emerged as a concept in Banach space theory in the late 1960s by James [9]. Before we introduce the definition of finite representability, we first need to introduce the Banach-Mazur distance.

Definition 2.1.1. Let X and Y be isomorphic Banach spaces. We define the *Banach-Mazur distance* between X and Y, denoted by $d_{\mathrm{BM}}(X, Y)$, is defined by

$$d_{\mathrm{BM}}(X, Y) = \inf\{\|T\|\|T^{-1}\| : T : X \to Y \text{ is an isomorphism}\}.$$

with the convention that $d_{\mathrm{BM}}(X, Y) = \infty$, if no such isomorphism from X to Y exists.

Remark 2.1.2. While the Banach–Mazur distance $d_{\mathrm{BM}}(X, Y)$ between isomorphic Banach spaces is called a "distance", it is *not* a metric in the strict sense: it does not satisfy the triangle inequality in the usual way. However, its logarithmic version given as

$$\delta_{\mathrm{BM}}(X, Y) = \log(d_{\mathrm{BM}}(X, Y))$$

behaves more like a metric and is often used in applications. In particular, δ_{BM} satisfies the triangle inequality and induces a semimetric on equivalence classes of Banach spaces under isomorphism.

Instead of the usual triangle inequality, the Banach–Mazur distance is able to satisfy the *multiplicative triangle inequality* on the class of isomorphic Banach spaces, that is, if $X \simeq Y \simeq Z$ are Banach spaces, then

$$d_{\mathrm{BM}}(X,Z) \leq d_{\mathrm{BM}}(X,Y) d_{\mathrm{BM}}(Y,Z).$$

Remark 2.1.3. If X and Y are isometric Banach spaces, then $d_{\mathrm{BM}}(X,Y) = 1$, but it is not necessarily true that if $d_{\mathrm{BM}}(X,Y) = 1$, then X and Y are isometric Banach spaces.

Definition 2.1.4. Let X and Y be Banach spaces.

1. We say that Y is *crudely finitely representable* in X if there exists a $K > 0$ such that for every finite-dimensional subspace F of Y, there exists an isomorphism $T : F \to T(F) \subseteq X$ such that $\|T\|\|T^{-1}\| < K$; that is, in terms of the Banach-Mazur distance $d_{\mathrm{BM}}(F, T(F)) < K$. Alternatively, we say that Y is *K-finitely representable* in X.

2. We say that Y is *finitely representable* in X if for every $\varepsilon > 0$, Y is crudely finitely representable with $K = 1 + \varepsilon$.

Thus, Y is finitely representable in X if and only if Y is crudely finitely representable in X with constant K for all $K > 1$. The concept of finite representability was studied by James [10, 11].

Proposition 2.1.5. *Let X, Y, and Z be Banach spaces. If Z is crudely finitely representable in Y, and Y is crudely finitely representable in X, then Z is finitely representable in X.*

Proof. Since Z is crudely finitely representable in Y, let $K > 0$ be such that for any $E \subseteq Z$, there is an isomorphism $T : E \to T(E) \subseteq Y$ such that $\|T\|\|T^{-1}\| < K$, and since Y is crudely finitely representable in X, let $M > 0$ be such that for any $F \subseteq Y$, there exists an isomorphism $S : F \to S(F) \subseteq X$ such that $\|S\|\|S^{-1}\| < M$. Letting

$L = KM$, define $R = ST : E \to ST(E)$. Since the composition of isomorphisms is also an isomorphism, R is an isomorphism. In particular,

$$\|R\|\|R^{-1}\| = \|ST\|\|(ST)^{-1}\| \leq \|S\|\|T\|\|S^{-1}\|\|T\|^{-1} < KM = L.$$

Therefore, Z is crudely finitely representable in X, as desired. □

Theorem 2.1.6. *Every Banach space is finitely representable in* $c_0(\mathbb{N})$ *and* $\ell_2(\ell_\infty^n)$.

Proof. Let Y be a Banach space, let $\varepsilon > 0$ be arbitrary, and let $E \subseteq Y$ be finite-dimensional. Choose $\delta > 0$ so that $\{y_i\}_{i=1}^n \subseteq S_E$ is a δ-net and that $\frac{1}{1-\delta} < 1 + \varepsilon$.

By the Hahn-Banach Theorem (Theorem A.0.4), for each $1 \leq i \leq n$, choose $y_i^* \in S_{Y^*}$ such that $y_i^*(y_i) = 1$. Let $T : E \to \ell_\infty^n$ be defined by

$$Ty = (y_i^*(y))_{i=1}^n$$

for $y \in E$. We will show that T is an isomorphism. Indeed, for any $y \in E$, first observe that

$$\|Ty\|_\infty = \max_{1 \leq i \leq n} |y_i^*(y)| \leq \max_{1 \leq i \leq n} \|y_i^*\|\|y\| \leq \|y\|_Y$$

and so $\|T\| \leq 1$. On the other hand, let $y \in S_E$ and choose $1 \leq i \leq n$ such that $\|y - y_i\|_Y < \delta$. Then see that

$$\|Ty\|_\infty \geq |y_i^*(y)| = |y_i^*(y_i) + y_i^*(y - y_i)| \geq |y_i^*(y)| - |y_i^*(y - y_i)| \geq 1 - \|y_i^*\|\|y - y_i\| \geq 1 - \delta.$$

Therefore, we have $\|T\| \geq 1 - \delta$, and hence, for any $y \in E$,

$$(1 - \delta)\|y\|_Y \leq \|Ty\|_\infty \leq \|y\|_Y$$

and thus, $T : E \to T(E) \subseteq \ell_\infty^n$ is an isomorphism onto its image, with $\|T^{-1}\| \leq \frac{1}{1-\delta}$.

<u>Case 1: Y is finitely representable in $c_0(\mathbb{N})$.</u> Now let $S : \ell_\infty^n \to c_0(\mathbb{N})$ be defined by

$$S(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, 0, 0, ...)$$

for $(x_1, x_2, ..., x_n) \in \ell_\infty^n$. We claim that S is a linear isometry. Indeed, for any

$x = (x_1, x_2, ..., x_n) \in \ell_\infty^n$, we have

$$\begin{aligned} \|Sx\|_\infty &= \|(x_1, x_2, ..., x_n, 0, 0, ...)\|_\infty \\ &= \max_{1\le i\le n} |x_i| \\ &= \|x\|_\infty. \end{aligned}$$

Therefore, S is a linear isometry. Moreover, $\|S\|\|S^{-1}\| \le 1$.

Now define $R = ST : E \to c_0(\mathbb{N})$. Note that R is an isomorphism as T is an isomorphism and S is a linear isometry. Moreover,

$$\|R\|\|R^{-1}\| \le \|ST\|\|T^{-1}S^{-1}\| \le \|S\|\|T\|\|T^{-1}\|\|S^{-1}\| < 1 + \varepsilon.$$

Therefore, we have constructed a linear isomorphism $R : E \to R(E) = c_0(\mathbb{N})$ such that $\|R\|\|R^{-1}\| < 1 + \varepsilon$, and therefore, Y is finitely representable in $c_0(\mathbb{N})$.

<u>Case 2: Y is finitely representable in $\ell_2(\ell_\infty^n)$.</u> Let $U : \ell_\infty^n \to \ell_2(\ell_\infty^n)$ be defined by

$$U((x_i)_{i=1}^n) = (0, 0, ..., 0, \underset{\substack{\uparrow \\ n\text{th block}}}{(x_i)_{i=1}^n}, 0, 0, ...)$$

where $(x_i)_{i=1}^n$ is the nth block in $\ell_2(\ell_\infty^n)$. We claim that U is a linear isometry. Indeed, observe that for any $(x_i)_{i=1}^n \in \ell_\infty^n$, we have

$$\|U((x_i)_{i=1}^n)\|_{\ell_2(\ell_\infty^n)} = \|(0, 0, ..., 0, (x_i)_{i=1}^n, 0, 0, ...)\|_{\ell_2(\ell_\infty^n)} = \|(x_i)_{i=1}^n\|_\infty$$

so U is a linear isometry. Moreover, $\|U\|\|U^{-1}\| \le 1$.

Now define $V = UT : E \to \ell_2(\ell_\infty^n)$. Note that V is an isomorphism as T is an isomorphism and U is a linear isometry. Moreover,

$$\|V\|\|V^{-1}\| \le \|UT\|\|T^{-1}U^{-1}\| \le \|U\|\|T\|\|T^{-1}\|\|U^{-1}\| < 1 + \varepsilon.$$

Therefore, we have constructed a linear isomorphism $V : E \to V(E) = \ell_2(\ell_\infty^n)$ such that $\|V\|\|V^{-1}\| < 1 + \varepsilon$, and therefore, Y is finitely representable in $\ell_2(\ell_\infty^n)$. $\square$

Theorem 2.1.7. *For every $p \ge 1$, $L_p[0,1]$ is finitely representable in $\ell_p(\mathbb{N})$.*

Proof. Let $E \subseteq L_p[0,1]$ be a finite-dimensional subspace, and let $\varepsilon > 0$ be arbitrary, and let $\{f_i; f_i^*\}_{i=1}^n$ be a biorthogonal system of E. Let $\delta > 0$ be small so that $\frac{1+\delta}{1-\delta} < 1 + \varepsilon$.

By Theorem A.0.11 applied to $([0,1], \mathfrak{B}([0,1]), \lambda)$, for each $1 \leq i \leq n$, there exists a simple function $\varphi_i : [0,1] \to \mathbb{R}$ such that

$$\|\varphi_i - f_i\|_{L_p} \leq \frac{\delta}{\sum_{i=1}^n \|f_i^*\|}.$$

Note that for each $1 \leq i \leq n$, we have

$$\varphi_i = \sum_{j=1}^N a_{i,j} \chi_{A_j}$$

where $\{A_j\}_{j=1}^N \subseteq \mathfrak{B}([0,1])$ are pairwise disjoint and $(a_{i,j})_{j=1}^N \subseteq \mathbb{R}$, so that $\{\varphi_1, \varphi_2, ..., \varphi_n\} \subseteq L_p[0,1]$.

Define $F = \operatorname{span}\{\varphi_1, \varphi_2, ..., \varphi_n\}$, and let $T : E \to F = T(E) \subseteq L_p[0,1]$ be defined by

$$T\left(\sum_{i=1}^n f_i^*(f) f_i\right) = \sum_{i=1}^n f_i^*(f)\varphi_i.$$

Note that for each $1 \leq i \leq n$, we have $Tf_i = \varphi_i$. We claim that T is an isomorphism. To see this, note that for any $f = \sum_{i=1}^n f_i^*(f) f_i \in E$, we have

$$\begin{aligned}
\|Tf\|_{\ell_p} &= \left\| \sum_{i=1}^n f_i^*(f)\varphi_i \right\|_{\ell_p} \\
&= \left\| \sum_{i=1}^n f_i^*(f) f_i + \sum_{i=1}^n f_i^*(f)(\varphi_i - f_i) \right\|_{\ell_p} \\
&\leq \|f\|_{L_p} + \sum_{i=1}^n \|f_i^*\| \|f\|_{L_p} \|\varphi_i - f_i\|_{L_p} \\
&= \|f\|_{L_p} \left(1 + \sum_{i=1}^n \|f_i^*\| \|\varphi_i - f_i\|_{L_p}\right) \\
&\leq \|f\|_{L_p}(1+\delta).
\end{aligned}$$

Therefore, $\|T\| \leq 1 + \delta$. On the other hand, we have similarly, by using the lower triangle inequality,

$$\begin{aligned}
\|Tf\|_{\ell_p} &= \left\| \sum_{i=1}^{n} f_i^*(f)\varphi_i \right\|_{\ell_p} \\
&= \left\| \sum_{i=1}^{n} f_i^*(f) f_i + \sum_{i=1}^{n} f_i^*(f)(\varphi_i - f_i) \right\|_{\ell_p} \\
&\geq \|f\|_{L_p} - \sum_{i=1}^{n} \|f_i^*\| \|f\|_{L_p} \|\varphi_i - f_i\| \\
&\geq \|f\|_{L_p}(1-\delta).
\end{aligned}$$

Therefore, $\|T\| \geq (1-\delta)$, so for any $f \in E$ we have

$$(1-\delta)\|f\|_{L_p} \leq \|Tf\|_{\ell_p} \leq (1+\delta)\|f\|_{L_p}$$

so $T : E \to F = T(E) \subseteq L_p[0,1]$ is an isomorphism, and moreover, $\|T^{-1}\| \leq \frac{1}{1-\delta}$. Moreover, observe that by the choice of $\delta > 0$, we have

$$\|T\|\|T^{-1}\| \leq \frac{1+\delta}{1-\delta} < 1 + \varepsilon.$$

Now define $S : F \to \ell_p^N$ given by

$$S\left(\sum_{i=1}^{n} b_i \varphi_i\right) = \left(\sum_{i=1}^{n} b_i a_{i,j} \lambda(A_j)^{\frac{1}{p}}\right)_{j=1}^{N}.$$

We claim that S is a linear isometry. To see this, note that for any $f = \sum_{i=1}^{n} b_i \varphi_i \in F$, that

$$f = \sum_{i=1}^{n} b_i \sum_{j=1}^{N} a_{i,j} \chi_{A_j} = \sum_{j=1}^{N} \sum_{i=1}^{n} b_i a_{i,j} \chi_{A_j}.$$

On one hand, we have

$$\begin{aligned}\|f\|_{L_p}^p &= \int_0^1 \left| \sum_{j=1}^N \sum_{i=1}^n b_i a_{i,j} \chi_{A_j} \right|^p \, d\lambda \\ &= \sum_{j=1}^N \int_{A_j} \left| \sum_{i=1}^n b_i a_{i,j} \right|^p \, d\lambda \\ &= \sum_{j=1}^N \left| \sum_{i=1}^n b_i a_{i,j} \right|^p \lambda(A_j).\end{aligned}$$

On the other hand, we have

$$\|Sf\|_{\ell_p}^p = \sum_{j=1}^N \left| \sum_{i=1}^n b_i a_{i,j} \lambda(A_j)^{\frac{1}{p}} \right|^p = \sum_{j=1}^N \left| \sum_{i=1}^n b_i a_{i,j} \right|^p \lambda(A_j) = \|f\|_{L_p}^p.$$

Therefore, we have shown that S is a linear isometry, and thus, $\|S\| = 1$.

Finally, define $R = ST : E \to \ell_p^N \subseteq \ell_p(\mathbb{N})$. Then note that R is an isomorphism as $T : E \to F$ is an isomorphism and $S : F \to \ell_p^N$ is a linear isometry. Furthermore,

$$\|R\|\|R^{-1}\| \leq \|S\|\|T\|\|T^{-1}\|\|S^{-1}\| < 1 + \varepsilon.$$

Therefore, we conclude that $L_p[0,1]$ is finitely representable in $\ell_p(\mathbb{N})$ as desired. $\square$

Theorem 2.1.8. *If X is a Banach space that is crudely finitely representable in a Hilbert space, then X is isomorphic to a Hilbert space.*

Proof. We prove for the case when X is a separable Banach space as in [5]. Let $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$ be subspaces of X such that

$$X = \overline{\bigcup_{n=1}^{\infty} X_n}$$

and X_n is n-dimensional.

Since X is K-finitely representable, for each $n \in \mathbb{N}$, define $T_n : X_n \to \ell_2^n$ such that $\|T_n\| \leq 1$ and $\|T_n^{-1}\| \leq K$. Then for each $n \in \mathbb{N}$, let $G_n : X_n \times X_n \to \mathbb{R}$ be the function defined by

$$G_n(x, y) = \langle T_n x, T_n y \rangle$$

for $x, y \in X_n$. Note that X_n with the above function G_n, is a Hilbert space, since X_n is finite dimensional. Also observe that for any $x \in X_n$, we have by the Cauchy-Schwarz inequality,

$$G_n(x, x) \leq \|x\|^2 \leq K^2 G_n(x, x).$$

To complete the proof, we seek an infinite subsequence $(G_{n_k})_{k=1}^{\infty}$ such that for every $x, y \in \bigcup_{n=1}^{\infty} X_n$, $\lim_{k\to\infty} G_{n_k}(x, y)$ exists. To see this, for each $n \in \mathbb{N}$, let $\{x_i^{(n)}\}_{i=1}^{n}$ be a Hamel basis of X_n. Observe that for each $n \in \mathbb{N}$, there exists a subsequence $(G_{n_k^{(n)}})_{k=1}^{\infty}$ of $(G_{n_k^{(n-1)}})_{k=1}^{\infty}$ such that for all $1 \leq i, j \leq n$, $\lim_{k\to\infty} G_{n_k^{(n)}}(x_i^{(n)}, x_j^{(n)})$ exists. Then by using Cantor's diagonalization argument, take the diagonal of these subsequences given by $(G_{n_k})_{k=1}^{\infty}$, where we have set $n_k = n_k^{(k)}$. Then note that for each $m \in \mathbb{N}$, $\lim_{k\to\infty} G_{n_k}(x_i^{(m)}, x_j^{(m)})$ exists.

Now we claim that for all $x, y \in \bigcup_{n=1}^{\infty} X_n$, $\lim_{k\to\infty} G_{n_k}(x, y)$ exists. To see this, note that if $x, y \in \bigcup_{n=1}^{\infty} X_n$, then there exists an $N \in \mathbb{N}$ such that $x, y \in X_N$, and so, there exists $a_1, a_2, ..., a_N, b_1, b_2, ..., b_N \in \mathbb{R}$ such that $x = \sum_{i=1}^{N} a_i x_i^{(N)}$ and $y = \sum_{j=1}^{N} b_j x_j^{(N)}$. Then

$$\begin{aligned}
\lim_{k\to\infty} G_{n_k}(x, y) &= \lim_{k\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j G_{n_k}(x_i^{(N)}, x_j^{(N)}) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j \lim_{k\to\infty} G_{n_k}(x_i^{(N)}, x_j^{(N)})
\end{aligned}$$

which exists.

On $Z = \bigcup_{n=1}^{\infty} X_n$ with $G(x, y) = \lim_{k\to\infty} G_{n_k}(x, y)$ for all $x, y \in X$, the identity map $I_Z : (Z, \|\cdot\|) \to (Z, \|\cdot\|_G)$ is an isomorphism with

$$\frac{1}{K}\|x\|_G \leq \|x\| \leq \|x\|_G$$

and thus, $I_Z : (Z, \|\cdot\|) \to \mathcal{H}$ extends to $\overline{I}_Z : (X, \|\cdot\|) \to \mathcal{H}$, and the extension of an isomorphism is an isomorphism. $\square$

2.2 Spreading Models

We introduce the notion of spreading models. If we are given two sequences $(x_i)_{i=1}^\infty$ and $(e_i)_{i=1}^\infty$, we say roughly that $(x_i)_{i=1}^\infty$ generates $(e_i)_{i=1}^\infty$ as a spreading model, if asymptotically, $(x_i)_{i=1}^\infty$ behaves like $(e_i)_{i=1}^\infty$. In order to present the main idea of a spreading model, we first need to introduce a preliminary result before proving what is known as the Brunel–Sucheston Theorem. Recall that a basic sequence $\{e_n\}_{n=1}^\infty$ is said to be seminormalized if

$$0 < \inf_{i\in\mathbb{N}} \|e_i\| \le \sup_{i\in\mathbb{N}} \|e_i\| < \infty$$

(see Definition 1.2.12).

Lemma 2.2.1. *Let X be a Banach space and let $\{x_n\}_{n=1}^\infty$ be a seminormalized basic sequence in X. For every $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists a subsequence $\{y_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that for all $a_1, a_2, ..., a_k \in \mathbb{R}$,*

$$(1-\varepsilon)\left\|\sum_{i=1}^k a_i y_i\right\| \le \left\|\sum_{i=1}^k a_i y_{n_i}\right\| \le (1+\varepsilon)\left\|\sum_{i=1}^k a_i y_i\right\| \tag{2.2.1}$$

for all $n_1 < n_2 < \cdots < n_k$.

Proof. Since $\{x_n\}_{n=1}^\infty$ is seminormalized and basic, choose $C = \mathrm{bc}(\{x_n\}) > 0$ that can satisfy the inequality in Grümblum's Criterion (Proposition 1.2.17), and choose $R > 0$ such that $-R \le \|x_n\| \le R$ for all $n \in \mathbb{N}$. Then, for any $a_1, a_2, ..., a_k \in \mathbb{R}$ and $n_1 < n_2 < \cdots < n_k$, we have for each $1 \le i \le k$,

$$2^{-1}C^{-1}R^{-1}|a_i| \le \left\|\sum_{i=1}^k a_i x_{n_i}\right\| \le Rk \max_{1\le i\le k} |a_i|$$

Since the sequence is basic and normalized, norm equivalence over k-tuples holds with a constant $K > 0$ depending on k and the basis constants. Thus, there exists a $K = K(k, \{x_n\}) > 0$ such that for all scalars $a_1, a_2, ..., a_k \in \mathbb{R}$ and all subsequences $\{x_{n_i}\}_{i=1}^{k}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$K^{-1} \max_{1 \le i \le k} |a_i| \le \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| \le K \max_{1 \le i \le k} |a_i|. \tag{2.2.2}$$

Let $\delta > 0$, and let $\{(a_{1,j}, a_{2,j}, ..., a_{k,j})\}_{j=1}^{M} \subseteq \mathbb{R}^k$ be a finite δ-net in the unit ball of ℓ_∞^k, that is, $\max_{1 \le i \le k} |a_{i,j}| \le 1$ for all $1 \le j \le M$. We inductively select a subsequence of $\{x_n\}_{n=1}^{\infty}$ indexed by sets N_j for $1 \le j \le M$.

For the basis step, choose $N_0 = \mathbb{N}$. Assume that $N_0, N_1, ..., N_{j-1}$ have been defined. Then partition all increasing k-tuples $(n_1, n_2, ..., n_k)$ from N_{j-1} into finitely many sets $\{S_m\}_{m=1}^{p}$ where
$p = \left\lceil \delta^{-1}(K - K^{-1}) \max_{1 \le i \le k} |a_{i,j}| \right\rceil$ and

$$\begin{aligned} S_m = \{&(n_1, n_2, ..., n_k) : n_1 < n_2 < \cdots < n_k, \\ &\left\| \sum_{i=1}^{k} a_{i,j} x_{n_i} \right\| \in \left[K^{-1} \max_{1 \le i \le k} |a_{i,j}| + (m-1)\delta, K^{-1} \max_{1 \le i \le k} |a_{i,j}| + m\delta \right) \right\} \end{aligned}$$

for $1 \le m \le p$. By the Pigeonhole principle and Ramsey's Theorem (Theorem A.0.14), one of the S_m contains infinitely many k-tuples. Choose an infinite subset $N_j \subseteq N_{j-1}$ such that all k-tuples from N_j lie in some S_m.

After M steps, let $N_M = \{m_i\}_{i=1}^{\infty}$, and define $y_i = x_{m_i}$. Assume that $\|(a_i)_{i=1}^{k} - (a_{i,j})_{i=1}^{k}\| < \delta$. First, note that we have by the definition of S_m,

$$\left| \left\| \sum_{i=1}^{k} a_{i,j} y_i \right\| - \left\| \sum_{i=1}^{k} a_{i,j} y_{n_i} \right\| \right| < \delta. \tag{2.2.3}$$

Also,

$$\left| \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\| - \left\| \sum_{i=1}^{k} a_{i,j} y_{n_i} \right\| \right| \le \left\| \sum_{i=1}^{k} (a_i - a_{i,j}) y_{n_i} \right\| \le Rk\delta \tag{2.2.4}$$

and similarly,

$$\left| \left\| \sum_{i=1}^{k} a_i y_i \right\| - \left\| \sum_{i=1}^{k} a_{i,j} y_i \right\| \right| \leq \left\| \sum_{i=1}^{k} (a_i - a_{i,j}) y_i \right\| \leq Rk\delta. \tag{2.2.5}$$

Therefore, by combining (2.2.3)-(2.2.5), we obtain

$$\begin{aligned}
\left| \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\| - \left\| \sum_{i=1}^{k} a_i y_i \right\| \right| &= \left| \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\| - \left\| \sum_{i=1}^{k} a_{i,j} y_{n_i} \right\| + \left\| \sum_{i=1}^{k} a_{i,j} y_{n_i} \right\| \right. \\
&\quad \left. - \left\| \sum_{i=1}^{n} a_{i,j} y_i \right\| + \left\| \sum_{i=1}^{k} a_{i,j} y_i \right\| - \left\| \sum_{i=1}^{k} a_i y_i \right\| \right| \\
&\leq \left| \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\| - \left\| \sum_{i=1}^{k} a_{i,j} y_{n_i} \right\| \right| + \left| \left\| \sum_{i=1}^{k} a_{i,j} y_{n_i} \right\| - \left\| \sum_{i=1}^{k} a_{i,j} y_i \right\| \right| \\
&\quad + \left| \left\| \sum_{i=1}^{k} a_{i,j} y_i \right\| - \left\| \sum_{i=1}^{k} a_i y_i \right\| \right| \\
&\leq Rk\delta + \delta + Rk\delta \\
&= \delta(1 + 2Rk)
\end{aligned} \tag{2.2.6}$$

Now let $(a_1, a_2, ..., a_k) \in \ell_\infty^k$ and let $A = \max_{1 \leq i \leq k} |a_i|$. Then (2.2.6) implies that

$$\frac{1}{A} \left\| \sum_{i=1}^{k} a_i y_i \right\| - \delta(1 + 2Rk) \leq \frac{1}{A} \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\| \leq \frac{1}{A} \left\| \sum_{i=1}^{k} a_i y_i \right\| + \delta(1 + 2Rk)$$

and so, multiplying all sides by A yields

$$\left\| \sum_{i=1}^{k} a_i y_i \right\| - A\delta(1 + 2Rk) \leq \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\| \leq \left\| \sum_{i=1}^{k} a_i y_i \right\| + A\delta(1 + 2Rk).$$

Then by (2.2.2), we obtain

$$(1 - K\delta(1 + 2Rk)) \left\| \sum_{i=1}^{k} a_i y_i \right\| \leq \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\| \leq (1 + K\delta(1 + 2Rk)) \left\| \sum_{i=1}^{k} a_i y_i \right\|.$$

Therefore, for $\varepsilon > 0$ sufficiently small, choose $\delta = \frac{\varepsilon}{K(1+2Rk)}$ so that

$$(1-\varepsilon)\left\|\sum_{i=1}^{k} a_i y_i\right\| \leq \left\|\sum_{i=1}^{k} a_i y_{n_i}\right\| \leq (1+\varepsilon)\left\|\sum_{i=1}^{k} a_i y_i\right\|.$$

Therefore, we obtain (2.2.1). $\square$

Definition 2.2.2. Let X be a Banach space and let $\{e_n\}_{n=1}^{\infty}$ be a basis of X. We say that $\{e_n\}_{n=1}^{\infty}$ is *subsymmetric* if

$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\| = \left\|\sum_{i=1}^{\infty} a_i e_{k_i}\right\|$$

for every increasing sequence of integers $(k_i)_{i=1}^{\infty}$.

Theorem 2.2.3 (Brunel–Sucheston Theorem)**.** *Let $(X, \|\cdot\|)$ be a Banach space, let $(\varepsilon_n) \searrow 0$, and let $\{x_n\}_{n=1}^{\infty}$ be a seminormalized basic sequence in X. Then there exists a subsequence $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a Banach space $(Y, \|\cdot\|_Y)$ with a subsymmetric basis $\{e_n\}_{n=1}^{\infty}$ such that for all $k \in \mathbb{N}$ and $a_1, a_2, ..., a_k \in \mathbb{R}$,*

$$(1-\varepsilon_k)\left\|\sum_{i=1}^{k} a_i e_i\right\|_Y \leq \left\|\sum_{i=1}^{k} a_i y_{n_i}\right\| \leq (1+\varepsilon_k)\left\|\sum_{i=1}^{k} a_i e_i\right\|_Y \tag{2.2.7}$$

whenever $k \leq n_1 < n_2 < \cdots < n_k$.

Proof. By applying Lemma 2.2.1 repeatedly, we obtain a decreasing sequence of infinite subsets $M_1 \supseteq M_2 \supseteq \cdots$ of the integers $M_k = (m_{i,k})_{i=1}^{\infty}$ so that

$$(1-\varepsilon_k)\left\|\sum_{i=1}^{k} a_i x_{m_{i,k}}\right\| \leq \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\| \leq (1+\varepsilon_k)\left\|\sum_{i=1}^{k} a_i x_{m_{i,k}}\right\| \tag{2.2.8}$$

whenever $n_1 < n_2 < \cdots < n_k$ are in M_k. Then let $M = (m_{i,i})_{i=1}^{\infty}$ be the diagonal sequence of the system $\{M_n\}_{n=1}^{\infty}$, and denote for all $i \in \mathbb{N}$, as in Lemma 2.2.1, $y_i = x_{m_{i,i}}$.

Let $(Y, \|\cdot\|_Y)$ be the completion of the normed space of all finitely supported elements of $c_0(\mathbb{N})$ where the norm is defined as

$$\left\| \sum_{i=1}^{k} a_i e_i \right\|_Y = \lim_{\substack{n_1<n_2<\cdots<n_k \\ n_1\to\infty}} \left\| \sum_{i=1}^{k} a_i y_{n_i} \right\|. \tag{2.2.9}$$

Then from (2.2.8), the limit of (2.2.9) exists, and (2.2.7) is satisfied. □

Definition 2.2.4. The Banach space $(Y, \|\cdot\|_Y)$ constructed above is called the *spreading model built from* $\{x_n\}_{n=1}^{\infty}$. We also say that Y is a spreading model of X if Y results as a spreading model built on some basic sequence in X.

Theorem 2.2.5. *Let X be a Banach space and let $(Y, \|\cdot\|_Y)$ be a spreading model of X generated by a seminormalized Schauder basic sequence $\{x_n\}_{n=1}^{\infty}$ in X.*

1. *Y is finitely representable in X.*
2. *If $\{x_n\}_{n=1}^{\infty}$ is weakly null, then $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis of Y, where $\{e_n\}_{n=1}^{\infty}$ is the basis constructed in Theorem 2.2.3.*

Proof. The finite representability of the spreading model Y in X follows directly from (2.2.7). Indeed, for any $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{R}$, the inequality

$$(1-\varepsilon_k) \left\| \sum_{i=1}^{k} a_i e_i \right\|_Y \leq \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| \leq (1+\varepsilon_k) \left\| \sum_{i=1}^{k} a_i e_i \right\|_Y$$

ensures that finite-dimensional subspaces in Y are realized up to arbitrarily small distortion by finite linear combinations in X. By definition, this means that Y is finitely representable in X.

To see that the second statement holds, we require the following claim.

Claim. *For $N \in \mathbb{N}$, $\delta > 0$, and $n_0 \in \mathbb{N}$, if $\{y_i\}_{i=1}^{\infty}$ is weakly null, there exists $n > n_0$ such that for all $f \in B_{X^*}$,*

$$|\{n_0 < i < n : |f(y_i)| < \delta\}| > N.$$

Proof of Claim. Assume for a contradiction that for all $n > n_0$, there exists $f_n \in B_{X^*}$ such that

$$|\{n_0 < i < n : |f(y_i)| < \delta\}| \geq N. \tag{2.2.10}$$

Since B_{X^*} is weakly-$*$ compact, take a limit point of $(f_n)_{n=1}^{\infty}$, call it $f_0 \in B_{X^*}$, i.e. for every weakly-$*$ open set U containing f_0, U contains infinitely many f_n's.

Subclaim. *$|f_0(y_i)| \geq 2\delta$ for all but at most $n_0 + N - 1$ values of $i \in \mathbb{N}$.*

Proof of Subclaim. If the subclaim is false, then $|f_0(y_i)| \geq 2\delta$ for at least N terms, say $y_{i_1}, y_{i_2}, ..., y_{i_N}$ with $n_0 \leq i_1 < i_2 < \cdots < i_N$. Consider

$$U = \{f \in X^* : |f_0(y_{i_k}) - f(y_{i_k})| < \delta, 1 \leq k \leq N\}.$$

The U contains many f_n's, in particular, it contains an f_n with $n \geq i_N$.

For $1 \leq k \leq N$, we have

$$|f_n(y_{i_k})| \geq |f_0(y_{i_k})| - |f_0(y_{i_k}) - f(y_{i_k})| \geq 2\delta - \delta = \delta$$

which implies that

$$|\{n_0 < i < n : |f(y_i)| < \delta\}| \geq |\{n_0 < i < i_N : |f(y_i)| < \delta\}| \geq N$$

which contradicts (2.2.10), therefore, the subclaim holds. □

The subclaim contradicts $y_i \xrightarrow{w} 0$ which completes the proof of the claim. □

By Proposition 1.4.9, it suffices to show that for any finitely supported $v = \sum_{i=1}^{N} a_i e_i$ and $S \subseteq \mathbb{N}$,

$$\left\| \sum_{i \in S} a_i e_i \right\|_Y \leq \left\| \sum_{i=1}^{N} a_i e_i \right\|_Y .$$

To that end, let $\{y_i\}_{i=1}^{\infty}$ be the subsequence of $\{x_i\}_{i=1}^{\infty}$ used to construct the spreading model Y in Theorem 2.2.3, let $\varepsilon > 0$ be arbitrary, and choose $n_0 \in \mathbb{N}$

such that for all $n_0 \leq m_1 < m_2 < \cdots < m_N$,

$$\left| \left\| \sum_{i=1}^{N} a_i y_{m_i} \right\| - \left\| \sum_{i=1}^{N} a_i e_i \right\|_Y \right| < \frac{\varepsilon}{3}$$

and

$$\left| \left\| \sum_{i \in S} a_i y_{m_i} \right\| - \left\| \sum_{i \in S} a_i e_i \right\|_Y \right| < \frac{\varepsilon}{3}.$$

Let $\delta > 0$ be such that $\delta < \frac{\varepsilon}{3\sum_{i=1}^{N} |a_i|}$. By the Claim, construct a sequence of indices $n_0 < n_1 < \cdots < n_N$ such that for each $j = 1, 2, ..., N$ and all $f \in B_{X^*}$,

$$|\{n_{j-1} < i < n_j : |f(y_i)| < \delta\}| \geq N + 1$$

Let $S = \{s_1, s_2, ..., s_k\}$ and $\{1, 2, ..., N\} = S \cup \left(\bigcup_{i=1}^{k-1} (s_{i-1}, s_i) \right)$ with the convention $s_0 = 0$ and $s_{k+1} = N + 1$. Define

$$v_S = \sum_{i \in S} a_i y_{n_{s_i}}$$

By the Hahn–Banach Theorem (Theorem A.0.4), there exists $f_S \in B_{X^*}$ such that $f_S(v_S) = \|v_S\|$. For each $1 \leq i \leq k+1$, pick $\{l_j : j \in (s_{i-1}, s_i)\}$ in $(n_{s_{i-1}}, n_{s_i})$ such that $|f_S(y_{l_j})| < \delta$ for $j \in (s_{i-1}, s_i)$. Define

$$v = \sum_{i \in S} a_i y_{n_{s_i}} + \sum_{i=1}^{k+1} \sum_{j \in (s_{i-1}, s_i)} a_i y_{l_j}$$

Then

$$|f_S(v)| \geq |f_S(v_S)| - \sum_{i=1}^{k+1} \sum_{j \in (s_{i-1}, s_i)} |a_j| |f_S(y_{l_j})| \geq \|v_S\| - \frac{\varepsilon}{3}$$

In particular, by the spreading model,

$$\left\| \sum_{i=1}^{N} a_i e_i \right\|_U \geq \|v\| - \frac{\varepsilon}{3} \geq \|v_S\| - \frac{2\varepsilon}{3} \geq \left\| \sum_{i \in S} a_i e_i \right\|_Y - \varepsilon$$

Therefore, as $\varepsilon > 0$ was arbitrary, we conclude that

$$\left\| \sum_{i \in S} a_i e_i \right\|_Y \leq \left\| \sum_{i=1}^{N} a_i e_i \right\|_Y$$

completing the proof. $\square$

Corollary 2.2.6. *Let X be an infinite-dimensional Banach space. Then there exists an infinite-dimensional Banach space with an unconditional Schauder basis which is crudely finitely representable in X.*

Proof. Since X is an infinite-dimensional Banach space, by Theorem A.0.12, one of the following occurs.

Case 1: X contains a basic sequence that is equivalent to the unit vector basis of $\ell_1(\mathbb{N})$. Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in X that is equivalent to the unit vector basis of $\ell_1(\mathbb{N})$. Since the unit vector basis of $\ell_1(\mathbb{N})$ is unconditional (Example 1.4.3), then $\{x_n\}_{n=1}^{\infty}$ is also an unconditional Schauder basis of $Y = [\{x_n\}]$ which is an infinite-dimensional Banach space with an unconditional Schauder basis. Furthermore, Y is crudely finitely representable in X since Y is the closed linear span of $\{x_n\}_{n=1}^{\infty}$.

Case 2: X contains a normalized weakly null basic sequence. Let $\{x_n\}_{n=1}^{\infty}$ be a normalized weakly null basic sequence in X. By the Bessaga–Pełczyński Selection Principle (Theorem 1.3.13), we may pass $\{x_n\}_{n=1}^{\infty}$ to a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ so that it remains basic. Then by the Brunel–Sucheston Theorem (Theorem 2.2.3), this basic sequence yields a Banach space with an unconditional Schauder basis that is crudely finitely representable in X. $\square$

2.3 Tzafriri's Theorem and Dvoretzky's Theorem

In this section, we present two fundamental results in finite dimensional Banach spaces. The first result that we mention is known as Tzafriri's Theorem [21], which asserts that for any Banach space X with an unconditional basis, there exists a sequence of projections $\{\Pi_n\}_{n=1}^{\infty}$ with uniformly bounded norm at most some $M > 0$ such that $\|\Pi_N\| \leq M$ and $d_{\mathrm{BM}}(\Pi_n(X), \ell_p^n) \leq M$ for all $n \in \mathbb{N}$ and some $p \in \{1, 2, \infty\}$..

Following Tzafriri's Theorem, we also present results from Dvoretzky, which characterizes Hilbert spaces being finitely representable in every infinite-dimensional Banach space. This is a powerful result, but Dvoretzky proved a more qualitative version, before Krivine's Theorem came to be.

Theorem 2.3.1 (Tzafriri's Theorem)**.** *Let X be an infinite-dimensional Banach space with an unconditional basis. Then there exists $p \in \{1, 2, \infty\}$, a constant $M > 0$, and a sequence of projections $\{\Pi_n\}_{n=1}^{\infty}$ in X such that $\|\Pi_n\| \leq M$ and $d_{\mathrm{BM}}(\Pi_n(X), \ell_p^n) \leq M$ for all $n \in \mathbb{N}$.*

The proof will be carried out for real Banach spaces. The complex version follows with some minor adjustments.

Proposition 2.3.2. *Let X be a Banach space with a normalized monotone Schauder basis $\{e_n\}_{n=1}^{\infty}$. Then for every $\varepsilon > 0$, there exists a sequence $\{\lambda(j)\}_{j=1}^{\infty}$ of positive real numbers and a subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ of $\{e_n\}_{n=1}^{\infty}$ such that for every set of indices $j < k_1 < k_2 < \cdots < k_j$, for $j \in \mathbb{N}$,*

$$0 < \lambda(j) - \|e_{n_{k_1}} + e_{n_{k_2}} + \cdots + e_{n_{k_j}}\| < \varepsilon \tag{2.3.1}$$

In addition, for every $j, m \in \mathbb{N}$ with $j < m$,

$$1 \leq \lambda(j) < \lambda(m) + \varepsilon \tag{2.3.2}$$

Proof. Let $\varepsilon > 0$ be arbitrary. For every $k \in \mathbb{N}$, we consider a fixed partition of the interval $[1, k+1]$

$$\lambda_0^{(k)} = 1 < \lambda_1^{(k)} < \lambda_2^{(k)} < \cdots < \lambda_{s(k)}^{(k)} = k+1$$

with the property that $\lambda_j^{(k)} < \lambda_{j-1}^{(k)} < \varepsilon$ for $j = 1, 2, ..., s(k)$. Define a function φ_k from all the unordered k-tuples of different positive integers into thhe set $\{1, 2, ..., s(k)\}$ by setting $\varphi_k(\{n_1, n_2, ..., n_k\}) = j$ if $\lambda_{j-1}^{(k)} \leq \|e_{n_1} + e_{n_2} + \cdots + e_{n_k}\| < \lambda_j^{(k)}$.

Now proceed inductively: put $N^{(1)} = \mathbb{N}$ and $\lambda(1) = 1$. Apply Ramsey's Theorem (Theorem A.0.14) to the function φ_2 to obtain an infinite set $N^{(2)} \subseteq N^{(1)}$ such that φ_2 restricted to all unordered couples from $N^{(2)}$ is constant, say j, for some $j \in \{1, 2, ..., s(2)\}$. In this case, put $\lambda(2) = \lambda_j^{(2)}$. Apply Ramsey's Theorem (Theorem A.0.14) again to the set $N^{(2)}$ and the function φ_3 restricted to unordered triples from $N^{(3)}$ is constant, say k, for some $k \in \{1, 2, ..., s(3)\}$. In this case, put $\lambda(3) = \lambda_k^{(3)}$. Continue in this way to obtain a sequence $N^{(1)} \supseteq N^{(2)} \supseteq N^{(3)} \supseteq \cdots$ of infinite sets and a sequence $\{\lambda(j)\}_{j=1}^{\infty}$. A standard diagonal argument gives the sequence $n_1 < n_2 < \cdots$ and the result. The last estimate follows from the monotonicity of the basis. $\square$

Remark 2.3.3. Two estimates follows from (2.3.1), the monotonicity of the basis, and the fact that all of its vectors are normalized. We single them out for future references.

$$1 = \|e_{n_{k_1}}\| \leq \|e_{n_{k_1}} + e_{n_{k_2}} + \cdots + e_{n_{k_j}}\| \leq \lambda(j) \tag{2.3.3}$$

and if $\varepsilon \leq 1$,

$$\lambda(j) \leq \|e_{n_{k_1}} + e_{n_{k_2}} + \cdots + e_{n_{k_j}}\| + \varepsilon \leq 2\|e_{n_{k_1}} + e_{n_{k_2}} + \cdots + e_{n_{k_j}}\| \tag{2.3.4}$$

for all $j < k_1 < k_2 < \cdots < k_j$, $j \in \mathbb{N}$.

Proposition 2.3.4. *Let $0 < \varepsilon < \frac{1}{2}$, let $\{\lambda(j)\}_{j=1}^{\infty}$ and $\{e_{n_k}\}_{k=1}^{\infty}$ be as in Proposition 2.3.2. Assume that there exists $h \in \mathbb{N}$ such that*

$$\frac{\lambda(hn)}{\lambda(n)} \geq 1 + \varepsilon \tag{2.3.5}$$

for $n \in \mathbb{N}$. Then there exists $A > 0$ and a number $q > 2$ such that

$$\frac{1}{\lambda(n)} \left\| \sum_{j=1}^{n} a_j e_{n_{m+k_j}} \right\| \leq n^{-\frac{1}{q}} A \left(\sum_{j=1}^{n} |a_j|^q \right)^{\frac{1}{q}} \tag{2.3.6}$$

for every $n, m \in \mathbb{N}$ such that $n < m$, integers $0 < k_1 < k_2 < \cdots < k_n$, and every $a_1, a_2, ..., a_n$.

Proof. Find $r > 2$ such that $1 < h^{\frac{1}{r}} < 1 + \varepsilon$. Then $\frac{\lambda(hn)}{\lambda(n)} > h^{\frac{1}{r}}$ for all $n \in \mathbb{N}$. It follows that for $n, s \in \mathbb{N}$,

$$\lambda(h^s n) > h^{\frac{1}{r}} \lambda(h^{s-1} n) > (h^2)^{\frac{1}{r}} \lambda(h^{s-2} n) > \cdots > (h^{s-1})^{\frac{1}{r}} \lambda(hn) > (h^s)^{\frac{1}{r}} \lambda(n)$$

that is

$$\frac{\lambda(h^s n)}{\lambda(n)} > (h^s)^{\frac{1}{r}} \tag{2.3.7}$$

Let $\alpha, \beta \in \mathbb{Z}$ be such that $h^{j-1} < \beta \leq h^j$ and $h^{i-1} < \alpha \leq h^i$ for some $i \leq j$ in $\mathbb{N}$. We consider the following cases.

Case 1: If $i < j$. We get

$$h^{j-1} < \beta \leq h^j \leq h^{i-1} < \alpha \leq h^i$$

Then we have $\lambda(\beta) < \lambda(h^j) + \varepsilon$ and $\lambda(h^{i-1}) < \lambda(\alpha) + \varepsilon$. Hence, if $\rho = \frac{\lambda(h^{i-1})}{\lambda(h^j)}$, then

$$\begin{aligned}
\frac{\lambda(\alpha)}{\lambda(\beta)} &> \frac{\lambda(h^{i-1}) - \varepsilon}{\lambda(h^j) + \varepsilon} = \frac{\rho - \frac{\varepsilon}{\lambda(h^j)}}{1 + \frac{\varepsilon}{\lambda(h^j)}} \geq \frac{\rho - \varepsilon}{1 + \varepsilon} \\
&> \frac{\rho - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{2\rho - 1}{3} > \frac{2\rho - \rho}{3} = \frac{\rho}{3} > \frac{\rho}{4} \\
&= \frac{1}{4} \frac{\lambda(h^{i-j-1} h^j)}{\lambda(h^j)} \geq \frac{1}{4} (h^{i-j-1})^{\frac{1}{r}} \geq \frac{1}{4h^{\frac{2}{r}}} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{r}}
\end{aligned}$$

where the last but one inequality comes from (2.3.7) and the last one from the fact that $\frac{\alpha}{\beta} \leq \frac{h^i}{h^{j-1}}$.

Case 2: If $i = j$. Then

$$h^{j-1} < \beta \leq \alpha \leq h^j$$

and we get

$$\frac{\lambda(\alpha)}{\lambda(\beta)} \geq \frac{\lambda(\beta) - \varepsilon}{\lambda(\beta)} = 1 - \frac{\varepsilon}{\lambda(\beta)} \geq 1 - \varepsilon \geq \frac{1}{2} \geq \frac{1}{2h^{\frac{1}{r}}} \left(\frac{h^j}{h^{j-1}} \right)^{\frac{1}{r}} \geq \frac{1}{2h^{\frac{1}{r}}} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{r}}$$

where the first inequality follows from (2.3.2), and the last one from the fact that $\frac{\alpha}{\beta} \leq \frac{h^j}{h^{j-1}}$.

Thus, in both cases, we have

$$\frac{\lambda(\alpha)}{\lambda(\beta)} \geq \frac{1}{4h^{\frac{2}{r}}} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{r}} \tag{2.3.8}$$

Fix positive integers $n < m$ and $k_1 < k_2 < \cdots < k_n$, and a sequence of scalars $a_1, a_2, ..., a_n \in \mathbb{R}$. Put $a_j = b_j^{(1)} b_j^{(2)}$, where $0 \leq b_j^{(s)} \leq |a_j|$ for $s = 1, 2$ and $j = 1, 2, ..., n$. Then

$$\left\| \sum_{j=1}^{n} a_j e_{n_{m+k_j}} \right\| \leq \sum_{s=1}^{2} \left\| \sum_{j=1}^{n} b_j^{(s)} e_{n_{m+k_j}} \right\| \tag{2.3.9}$$

Let $\sum_{j=1}^{n} b_j e_{n_{m+k_j}}$ be one of the two sums in the right hand side of (2.3.9). Let σ be a permutation of $\{1, 2, ..., n\}$ such that $b_{\sigma(1)} \geq b_{\sigma(2)} \geq \cdots \geq b_{\sigma(n)} \geq 0$. Then by (2.3.3),

$$\begin{aligned}
\left\| \sum_{j=1}^{n} b_j e_{n_{m+k_j}} \right\| &= \left\| \sum_{j=1}^{n} b_{\sigma(j)} e_{n_{m+k_{\sigma(j)}}} \right\| \\
&= \left\| (b_{\sigma(1)} - b_{\sigma(2)} \sum_{s=1}^{1} e_{n_{m+k_{\sigma(s)}}} + (b_{\sigma(2)} - b_{\sigma(3)} \sum_{s=1}^{2} e_{n_{m+k_{\pi(s)}}} + \cdots \right. \\
&\quad \left. + (b_{\sigma(n-1)} - b_{\sigma(n)}) \sum_{s=1}^{n-1} e_{n_{m+k_{\sigma(s)}}} + b_{\sigma(n)} \sum_{s=1}^{n} e_{n_{m+k_{\sigma(s)}}} \right\| \\
&\leq (b_{\sigma(1)} - b_{\sigma(2)}) \lambda(1) + (b_{\sigma(2)} - b_{\sigma(3)}) \lambda(2) + \cdots
\end{aligned}$$

$$+ (b_{\sigma(n-1)} - b_{\sigma(n)})\lambda(n-1) + b_{\sigma(n)}\lambda(n)$$

Thus, using (2.3.8),

$$\begin{aligned}
\frac{1}{\lambda(n)}\left\|\sum_{j=1}^{n} b_j e_{n_{m+k_j}}\right\| &\leq 4h^{\frac{2}{r}}\left((b_{\sigma(1)} - b_{\sigma(2)})\left(\frac{1}{n}\right)^{\frac{1}{r}} + (b_{\sigma(2)} - b_{\sigma(3)})\left(\frac{2}{n}\right)^{\frac{1}{r}} + \cdots\right. \\
&\qquad \left. +(b_{\sigma(n-1)} + b_{\sigma(n)})\left(\frac{n-1}{n}\right)^{\frac{1}{r}} + b_{\sigma(n)}\right) \\
&= 4h^{\frac{2}{r}}\sum_{j=1}^{n} b_{\sigma(j)}\left(\left(\frac{j}{n}\right)^{\frac{1}{r}} - \left(\frac{j-1}{n}\right)^{\frac{1}{r}}\right) \qquad (2.3.10)
\end{aligned}$$

Set $q > r$ and let q' and r' be the conjugate indices to q and r, respectively. Observe that for $j \in \mathbb{N}$, we have

$$j^{\frac{1}{r}} - (j-1)^{\frac{1}{r}} \leq j^{-\frac{1}{r'}} \qquad (2.3.11)$$

Carry (2.3.11) to (2.3.10) and use Hölder's Inequality (Theorem A.0.1) for q and q' to obtain

$$\begin{aligned}
\frac{1}{\lambda(n)}\left\|\sum_{j=1}^{n} b_j e_{n_{m+k_j}}\right\| &\leq \frac{4h^{\frac{2}{r}}}{n^{\frac{1}{r}}}\sum_{j=1}^{n}\frac{b_{\sigma(j)}}{j^{\frac{1}{r'}}} \\
&\leq \frac{4h^{\frac{2}{r}}}{n^{\frac{1}{r}}}\left(\sum_{j=1}^{n}|b_{\sigma(j)}|^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{n}\frac{1}{j^{\frac{q'}{r'}}}\right)^{\frac{1}{q'}} \\
&\leq \frac{4h^{\frac{2}{r}}}{n^{\frac{1}{r}}}\left(\sum_{j=1}^{n}|a_j|^q\right)^{\frac{1}{q}}\left(\frac{1}{1-\frac{q'}{r'}}\right)^{\frac{1}{q'}} n^{\frac{1}{q'}-\frac{1}{r'}} \qquad (2.3.12)
\end{aligned}$$

where the last inequality is obtained by integrating the function $\frac{1}{x^{\frac{q'}{r}}}$ between 0 and

n. This amounts to

$$\frac{\left\|\sum_{j=1}^{n} b_j e_{n_{m+k_j}}\right\|}{\lambda(n)} \leq \frac{4h^{\frac{2}{r}}}{\left(1-\frac{q'}{r'}\right)^{\frac{1}{q'}}}\left(\sum_{j=1}^{n}|a_j|^q\right)^{\frac{1}{q}} n^{-\frac{1}{q}} \tag{2.3.13}$$

Since this is true for any of the sums at the right side of (2.3.9), by putting $A = \frac{8h^{\frac{2}{r}}}{\left(1-\frac{q'}{r'}\right)^{\frac{1}{q'}}}$, we obtain (2.3.6). $\square$

Proposition 2.3.5. *Let V be a 2^n-dimensional Banach space generated by a system of vectors $\{v_1, v_2, ..., v_{2^n}\}$. Suppose that there exist constants $K > 1$ and $p > 2$ such that*

$$K^{-1}\left(\sum_{j=1}^{2^n}|a_j|^{p'}\right)^{\frac{1}{p'}}(2^{-n})^{\frac{1}{p}} \leq \left\|\sum_{j=1}^{2^n} a_j v_j\right\|\left\|\sum_{j=1}^{2^n} v_j\right\|^{-1} \leq K\left(\sum_{j=1}^{2^n}|a_j|^{p}\right)^{\frac{1}{p}}(2^{-n})^{\frac{1}{p}} \tag{2.3.14}$$

for every $a_1, a_2, ..., a_{2^n}$ where p' is the conjugate index to p. Then there exists a constant $M = M(K,p)$, and a projection $P : V \to V$ such that $\|P\| \leq M$ and $d_{\mathrm{BM}}(P(V), \ell_2^n) \leq M$.

Proof. For $j = 1, 2, ..., 2^n$, $k = 1, 2, ..., n$, and $h = 1, 2, ..., 2^{k-1}$, put

$$\varepsilon_{k,j} = \begin{cases} 1 & \text{if } (2h-2)2^{n-k}+1 \leq j \leq (2h-1)2^{n-k} \\ -1 & \text{if } (2h-1)2^{n-k}+1 \leq j \leq 2h2^{n-k} \end{cases} \tag{2.3.15}$$

Let χ_S denote the characteristic function of a set $S \subseteq [0,1]$. Observe that the functions

$$r_k = \sum_{j=1}^{2^n} \varepsilon_{k,j}\chi_{\left[\frac{j-1}{2^n},\frac{j}{2^n}\right]} \tag{2.3.16}$$

for $k = 1, 2, ..., n$ are the first n Rademacher functions. Put

$$w_k = \sum_{j=1}^{2^n} \varepsilon_{k,j} v_j$$

for $k = 1, 2, ..., n$. Vectors $w_1, w_2, ..., w_n$ are the first n elements of the so-called Rademacher system in V associated to the basis $\{v_1, v_2, ..., v_n\}$.

We shall prove that $W = \operatorname{span}\{w_1, w_2, ..., w_n\}$ is the range of a projection in V that satisfies the requirements.

Khintchine's inequality (Theorem 1.5.12) for this index p say that there exists $K_p > 0$ such that

$$K_p^{-1}\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{k=1}^{n}a_k r_k\right\|_p \leq K_p\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}} \tag{2.3.17}$$

Observe that

$$\sum_{k=1}^{n}a_k w_k = \sum_{k=1}^{n}a_k\sum_{j=1}^{2^n}\varepsilon_{k,j}v_j = \sum_{j=1}^{2^n}\left(\sum_{k=1}^{n}\varepsilon_{k,j}a_k\right)v_j \tag{2.3.18}$$

Hence, putting together (2.3.14), (2.3.17), and (2.3.18), we get

$$\begin{aligned}\left\|\sum_{k=1}^{n}a_k w_k\right\|\left\|\sum_{j=1}^{2^n}v_j\right\|^{-1} &= \left\|\sum_{j=1}^{2^n}\left(\sum_{k=1}^{n}\varepsilon_{k,j}a_k\right)v_j\right\|\left\|\sum_{j=1}^{2^n}v_j\right\|^{-1}\\ &\leq K\left(\sum_{j=1}^{2^n}\left|\sum_{k=1}^{n}\varepsilon_{k,j}a_k\right|^p\right)^{\frac{1}{p}}(2^{-n})^{\frac{1}{p}}\\ &= K\left\|\sum_{k=1}^{n}a_k r_k\right\|_p\\ &= KK_p\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}}\end{aligned} \tag{2.3.19}$$

Analogously,

$$\left\|\sum_{k=1}^{n}a_k w_k\right\|\left\|\sum_{j=1}^{2^n}v_j\right\|^{-1} \geq K^{-1}K_{p'}^{-1}\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}} \tag{2.3.20}$$

Inequalities (2.3.19) and (2.3.20) together show that $d_{\mathrm{BM}}(W, \ell_2^n) \leq K^2 K_p K_{p'}$.

Let Q be the orthogonal projection in $L_2[0,1]$ whose range is span $\{r_1, r_2, ..., r_n\}$. In the proof of Theorem 1.5.9, we showed that Q acts as a bounded linear projection in every $L_r[0,1]$, $r > 1$, and that the norm $\|Q\|_r$ of Q in $L_r[0,1]$ is independent of n. If

$$Q\left(\sum_{j=1}^{2^n} a_j \chi_{\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]}\right) = \sum_{k=1}^{n} b_k r_k \tag{2.3.21}$$

then we set

$$P\left(\sum_{j=1}^{2^n} a_j v_j\right) = \sum_{k=1}^{n} b_k w_k$$

The mapping P so defined is a linear projection in V whose range is W. Moreover, using consecutively (2.3.19), (2.3.17), (2.3.21), and (2.3.14), we obtain

$$\begin{aligned}
\left\| P\left(\sum_{j=1}^{2^n} a_j v_j\right)\right\| &= \left\|\sum_{k=1}^{n} b_k w_k\right\| \\
&\leq K K_p \left\|\sum_{j=1}^{2^n} v_j\right\| \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}} \\
&\leq K K_p K_{p'} \left\|\sum_{j=1}^{2^n} v_j\right\| \left\|\sum_{k=1}^{n} b_k r_k\right\|_{p'} \\
&\leq \|Q\|_{p'} K K_p K_{p'} \left\|\sum_{j=1}^{2^n} v_j\right\| \left(\sum_{j=1}^{2^n} |a_j|^{p'}\right)^{\frac{1}{p'}} (2^{-n})^{\frac{1}{p'}} \\
&\leq \|Q\|_{p'} K^2 K_p K_{p'} \left\|\sum_{j=1}^{2^n} a_j v_j\right\|
\end{aligned}$$

In conclusion, we proved that for $M = \|Q\|_{p'} K^2 K_p K_{p'}$, we have $\|P\| \leq M$ and $d_{\mathrm{BM}}(V, \ell_2^n) \leq M$. $\square$

Proof of Tzafriri's Theorem (Theorem 2.3.1). Assume without loss of generality that the unconditional basis constant is 1. Let $0 < \varepsilon < \frac{1}{2}$. The basic sequence

$\{e_n^*\}_{n=1}^\infty$ formed by the functional coefficients is also unconditional with the same unconditional basis constant. By Proposition 2.3.2, obtain two sequences of real numbers $\{\lambda(j)\}_{j=1}^\infty$ and $\{\mu(j)\}_{j=1}^\infty$, and two subsequences $\{z_i = e_{n_i}\}_{i=1}^\infty$ and $\{z_i^* = e_{n_i}^*\}_{i=1}^\infty$ so that

$$0 < \lambda(j) - \|e_{n_{k_1}} + e_{n_{k_2}} + \cdots + e_{n_{k_j}}\| < \varepsilon$$
$$0 < \mu(j) - \|e_{n_{k_1}}^* + e_{n_{k_2}}^* + \cdots + e_{n_{k_j}}^*\| < \varepsilon$$

for every set of indices $j < k_1 < k_2 < \cdots < k_j$, $j \in \mathbb{N}$.

We shall distinguish three cases. Each of them will produce one of the three situations in the statement of the theorem.

Case 1: If $p = \infty$. Assume that for every $h \in \mathbb{N}$, there exists $n = n(h) \in \mathbb{N}$ such that $\frac{\lambda(hn)}{\lambda(n)} < 1 + \varepsilon$. Fix $h \in \mathbb{N}$ and let $n = n(h)$. Consider the following vectors:

$$u_1 = \frac{e_{n_{hn+1}} + \cdots + e_{n_{hn+n}}}{\lambda(n)}$$
$$u_2 = \frac{e_{n_{hn+n+1}} + \cdots + e_{n_{hn+2n}}}{\lambda(n)}$$
$$\vdots$$
$$u_h = \frac{e_{n_{hn+(h-1)n+1}} + \cdots + e_{n_{hn+hn}}}{\lambda(n)}$$

From (2.3.1), we get $0 < 1 - \|u_n\| < \frac{\varepsilon}{\lambda(n)} < \frac{1}{2}$, hence $\|u_n\| \geq \frac{1}{2}$ for $n = 1, 2, ..., h$. The basis $\{x_n\}_{n=1}^\infty$ is unconditional. Use monotonicity of the basis, then Lemma 1.4.8, (2.3.3), and the assumption on λ to get, for every $a_1, a_2, ..., a_h$,

$$\begin{aligned}
\frac{1}{2} \max_{1\leq i\leq h} |a_i| &\leq \left\| \sum_{i=1}^h a_i u_i \right\| \\
&\leq \max_{1\leq i\leq h} |a_i| \left\| \sum_{i=1}^h u_i \right\| \\
&= \max_{1\leq i\leq h} |a_i| \left\| \sum_{j=1}^{hn} e_{n_{hn+j}} \right\| \frac{1}{\lambda(n)}
\end{aligned}$$

$$
\begin{aligned}
&\le \max_{1\le i\le h} |a_i| \frac{\lambda(hn)}{\lambda(n)} \\
&\le 2 \max_{1\le i\le h} |a_i| \qquad (2.3.22)
\end{aligned}
$$

This shows that $d_{\mathrm{BM}}(X_h, \ell_\infty^h) \le 4$ where $X_h = \mathrm{span}\,\{u_1, u_2, ..., u_j\}$. Since ℓ_∞ is an injective space, it follows immediately that there are projections P_h in X such that $P_h(X) = X_h$ and $\|P_h\| \le 4$ for all $h \in \mathbb{N}$.

Case 2: If $p = 1$. Assume that for every $h \in \mathbb{N}$, there exists an integer $n = n(h) \in \mathbb{N}$ such that $\frac{\mu(hn)}{\mu(n)} < 1 + \varepsilon$. The basic sequence $\{x_n^*\}_{n=!}^\infty$ is also unconditional, monotone, and has unconditional basis constant 1. Case 1 applied to $\{e_{n_i}^*\}_{i=1}^\infty$ gives, for every $h \in \mathbb{N}$, $\{y_1^*, y_2^*, ..., y_h^*\}$ in X^* which have disjoint supports relatively to the basis $\{x_n\}_{n=1}^\infty$ and such that

$$
\frac{1}{2} \max_{1\le i\le h} |a_i| \le \left\| \sum_{i=1}^h a_i y_i^* \right\| \le 2 \max_{1\le i\le h} |a_i| \qquad (2.3.23)
$$

for every sequence $a_1, a_2, ..., a_h$ of scalars (see (2.3.22). Due again, to the unconditionality of the basis, find for every $i = 1, 2, ..., h$, a vector $y_i \in S_X$ such that its support with respect to $\{x_n\}_{n=1}^\infty$ is contained in the support of y_i^* and such that $\frac{1}{2} \le \langle y_i^*, y_i \rangle \le 2$. We get, for every sequence $b_1, b_2, ..., b_h$ of scalars,

$$
\begin{aligned}
\left\| \sum_{i=1}^h b_i y_i \right\| &\le \sum_{i=1}^h |b_i| \\
&\le 2 \left\langle \sum_{i=1}^h \mathrm{sgn}(b_i) y_i^*, \sum_{i=1}^h b_i y_i \right\rangle \\
&\le 4 \left\| \sum_{i=1}^h b_i y_i \right\|
\end{aligned}
$$

where we used the last inequality in (2.3.23). This shows that $d_{\mathrm{BM}}(Y_h, \ell_1^h) \le 4$, where $Y_h = \mathrm{span}\,\{y_1, y_2, ..., y_h\}$.

For $x \in X$, put

$$P_h x = \sum_{i=1}^{h} \frac{\langle y_i^*, x \rangle}{\langle y_i^*, y_i \rangle} y_i$$

Then P_h is a projection from X onto Y_h for which

$$\|P_h x\| \leq \sum_{i=1}^{h} \frac{|\langle y_i^*, x \rangle|}{\langle y_i^*, y_i \rangle} = \left(\sum_{i=1}^{h} \frac{\text{sgn}(\langle y_i^*, x \rangle)}{\langle y_i^*, y_i \rangle} y_i^* \right) \leq 4\|x\|$$

again $\|P_h\| \leq 4$ for $h \in \mathbb{N}$.

Case 3: If $p = 2$. We can then use Proposition 2.3.4 for $\{z_k\}_{k=1}^{\infty}$ in X and then for $\{z_k^*\}_{k=1}^{\infty}$ in X^* to obtain constants $B > 1$ and $p > 2$ such that simultaneously

$$\frac{1}{\lambda(n)} \left\| \sum_{j=1}^{n} a_j z_{n+j} \right\| \leq B \left(\sum_{j=1}^{n} |a_j|^p \right)^{\frac{1}{p}} n^{-\frac{1}{p}} \tag{2.3.24}$$

and

$$\frac{1}{\mu(n)} \left\| \sum_{j=1}^{n} a_j z_{n+j}^* \right\| \leq B \left(\sum_{j=1}^{n} |a_j|^p \right)^{\frac{1}{p}} n^{-\frac{1}{p}} \tag{2.3.25}$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, ..., a_n$.

Fix $n \in \mathbb{N}$. To obtain $\left\| \sum_{j=1}^{n} z_{n+j}^* \right\|$, we compute $\sup_{x \in S_X} \langle \sum_{j=1}^{n} z_{n+j}^*, x \rangle$. It is certainly enough to take the supremum on all $x \in S_X$ having a support with respect to $\{z_n\}_{n=1}^{\infty}$ contained in the support of $\sum_{j=1}^{n} z_{n+j}^*$. Those vectors are in a finite dimensional. Those vectors are in a finite-dimensional subspace of X, hence, there exists $x = \sum_{j=1}^{n} b_j z_{n+j} \in S_X$ such that

$$\left\| \sum_{j=1}^{n} z_{n+j}^* \right\| = \left\langle \sum_{j=1}^{n} z_{n+j}^*, x \right\rangle = \sum_{j=1}^{n} b_j$$

Choose a positive integer $C > 0$ such that $C > (8B)^P$.

Claim. *If* $\eta = \left\{ j : 1 \leq j \leq n, |b_j| \geq \frac{8C}{\lambda(n)} \right\}$ *and* s *denotes the cardinal of* η, *then*

$n > Cs$.

Proof of Claim. Indeed, if $n \le Cs$, then

$$\lambda(n) \le 2 \left\| \sum_{j=1}^{n} z_{Cs+j} \right\| \le 2 \left\| \sum_{j=1}^{Cs} z_{Cs+j} \right\| \le 2C\lambda(s) \tag{2.3.26}$$

where the first inequality follows from (2.3.4), the second from the monotonicity of the basis, and the third observing that $Cs \ge n \ge s$, that

$$\left\| \sum_{j=1}^{Cs} z_{Cs+j} \right\| \le \left\| \sum_{j=1}^{s} z_{Cs+j} \right\| + \left\| \sum_{j=s+1}^{2s} z_{Cs+j} \right\| + \cdots + \left\| \sum_{j=(C-1)s+1}^{Cs} z_{Cs+j} \right\|$$

and then using (2.3.3).

On the other hand,

$$1 = \|x\| \ge \left\| \sum_{j\in\eta} |b_j| z_{n+j} \right\| \ge \frac{8C}{\lambda(n)} \left\| \sum_{j\in\eta} z_{n+j} \right\| \ge 4C \frac{\lambda(s)}{\lambda(n)} \tag{2.3.27}$$

where the first and second inequalities follow from the unconditionality of the basis, and the third from (2.3.1), and the fact that $\lambda(s) - \varepsilon \ge \frac{\lambda(s)}{2}$. Certainly, (2.3.26) and (2.3.27) are in contradiction, so the claim holds. $\square$

We have

$$\frac{\mu(s)}{\mu(n)} \le \frac{2}{\mu(n)} \left\| \sum_{j=1}^{s} z^*_{n+j} \right\| \le 2B \left(\frac{s}{n} \right)^{\frac{1}{p}} \le \frac{2B}{C^{\frac{1}{p}}} < \frac{1}{4} \tag{2.3.28}$$

where the first inequality follows from (2.3.4), the second one from (2.3.25), the third one from the Claim, and the last one from the condition on C above.

From (2.3.4) again,

$$\mu(n) \le 2 \left\| \sum_{j=1}^{n} z_{n+j}^{*} \right\| = 2 \sum_{j=1}^{n} b_j$$

Hence,

$$\begin{aligned}
\mu(n) &\le 2 \sum_{j=1}^{n} b_j \\
&\le 16C \frac{n}{\lambda(n)} + 2 \sum_{j\in\eta} b_j \\
&\le 16C \frac{n}{\lambda(n)} + 2 \left\langle \sum_{j\in\eta} z_{n+j}^{*}, x \right\rangle \\
&\le 16C \frac{n}{\lambda(n)} + 2\mu(s) \\
&\le 16C \frac{n}{\lambda(n)} + \frac{\mu(n)}{2}
\end{aligned}$$

that is,

$$\mu(n) \le 32C \frac{n}{\lambda(n)}$$

for $n \in \mathbb{N}$. Observe too that

$$\begin{aligned}
\sum_{j=1}^{n} |a_j|^{p'} &\le \left\langle \sum_{j=1}^{n} |a_j|^{p'-1} \operatorname{sgn}(a_j) z_{n+j}^{*}, \sum_{j=1}^{n} a_j z_{n+j} \right\rangle \\
&\le \left\| \sum_{j=1}^{n} a_j z_{n+j} \right\| \left\| \sum_{j=1}^{n} |a_j|^{p'-1} \operatorname{sgn}(a_j) z_{n+j}^{*} \right\|
\end{aligned}$$

Consequently for $\frac{1}{p} + \frac{1}{p'} = 1$, we hvae by using (2.3.25) and (2.3),

$$\left\| \sum_{j=1}^{n} a_j z_{n+j} \right\| \ge \frac{\sum_{j=1}^{n} |a_j|^{p'}}{\left\| \sum_{j=1}^{n} |a_j|^{p'-1} \operatorname{sgn}(a_j) z_{n+j}^{*} \right\|}$$

$$
\begin{aligned}
&\geq \frac{1}{B} n^{\frac{1}{p}} \left(\sum_{j=1}^{n} |a_j|^{p'} \right) \frac{1}{\mu(n)} \left(\sum_{j=1}^{n} |a_j|^{\frac{p'-1}{p}} \right)^{-\frac{1}{p}} \\
&\geq \frac{1}{32BC} \frac{\lambda(n)}{n^{\frac{1}{p'}}} \left(\sum_{j=1}^{n} |a_j|^{p'} \right)^{\frac{1}{p'}}
\end{aligned}
$$

To finish the proof, it is enough to apply Proposition 2.3.5, since B, C and p are independent of n. □

Tzafriri's Theorem (Theorem 2.3.1) provides a structural insight into Banach spaces with an unconditional basis. Specifically, it shows that in any such space, one can find finite-dimensional subspaces that are uniformly isomorphic to the canonical spaces ℓ_p^n for some $p \in \{1, 2, \infty\}$, and this isomorphism is controlled uniformly by the Banach–Mazur distance. The existence of uniformly bounded projections $\{\Pi_n\}_{n=1}^{\infty}$ ensures that these subspaces are not only well-behaved geometrically but also canonically embedded in X.

This result foreshadows deeper structure theorems in the geometry of Banach spaces such as Krivine's Theorem [14] (see, e.g. [1, 5]), which asserts that every infinite-dimensional Banach space contains finite-dimensional subspaces almost isometric to ℓ_p^n for some $1 \leq p \leq \infty$. Together, these results motivate a broader investigation into finite representability and asymptotic structure in Banach space theory.

Tzafriri's Theorem also leads to a natural question of whether every infinite-dimensional Banach space contains a finite-dimensional subspace that resembles Hilbert spaces, regardless of whether they admit an unconditional basis. This is precisely known as Dvoretzky's Theorem [4]. By Theorems 2.2.3 and 2.2.5, we have the following.

Theorem 2.3.6 (Crude Dvoretzky's Theorem). *The Hilbert space is crudely finitely representable in every infinite-dimensional Banach space X.*

Proof. By Corollary 2.2.6, there exists a Banach space Y with an unconditional basis that is crudely finitely representable in X. By Tzafriri's Theorem, we have that either $c_0(\mathbb{N})$, $\ell_2(\mathbb{N})$ or $\ell_1(\mathbb{N})$ is K-finitely representable in Y for some $K > 0$, so is also finitely representable in X.

By Theorem 2.1.6, we have that $\ell_2(\mathbb{N})$ is finitely representable in $c_0(\mathbb{N})$. On the other hand, by Khintchine's inequality (Theorem 1.5.12), $\ell_2(\mathbb{N})$ is isomorphic to a subspace of $L_1[0, 1]$, and the latter space is then finitely representable in $\ell_1(\mathbb{N})$ by Theorem 2.1.7. □

Dvoretzky's Theorem stands as one of the most renowned results in the theory of Banach spaces. However, the proof outlined above is not the original or standard approach. The theorem was first established by Dvoretzky in 1961, prior to the development of methods such as those found in Krivine's Theorem [3, 4]. Consequently, the original formulation of the result leads to a stronger, more quantitative version as discussed in [1].

Theorem 2.3.7 (Dvoretzky Theorem: Quantitative Version)**.** *For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $N = N(n, \varepsilon)$ such that if X is a Banach space of dimension N, then there exists a subspace $E \subseteq X$ with $\dim(E) = n$ and $d_{\mathrm{BM}}(E, \ell_2^n) < 1 + \varepsilon$.*

2.4 The Lindenstrauss–Tzafriri Theorem

We now arrive at the central result of this paper, known as the Lindenstrauss–Tzafriri Theorem [15]. This theorem provides a geometric characterization of Hilbert spaces among Banach spaces, formulated through the notion of complemented subspaces. It asserts that a Banach space in which every closed subspace is complemented must be isomorphic to a Hilbert space.

At its heart, the theorem connects an abstract structural property, complementability of subspaces, with the geometry of Hilbert spaces. This result stands in contrast to the general behaviour of Banach spaces, where complemented subspaces are rare and often difficult to characterize. By identifying a condition that forces the entire space to exhibit Hilbertian structure, the Lindenstrauss–Tzafriri Theorem offers a richer insight into the interplay of algebraic decomposability and geometric isomorphism.

The proof of the theorem combines several essential ideas: the geometry of projections in Banach spaces, finite representability, and the minimalization of projection norms. It also leverages results discussed earlier in this paper, such as Auerbach's Lemma (Theorem 1.1.12), Dvoretzky's Theorem (Theorem 2.3.6 and 2.3.7), and the theory of Schauder bases.

Theorem 2.4.1 (The Lindenstrauss–Tzafriri Theorem)**.** *Every closed subspace of a Banach space X is complemented in X if and only if X is isomorphic to a Hilbert space.*

Before we prove the Lindenstrauss–Tzafriri Theorem, we need to introduce two results.

Lemma 2.4.2. *Let X be a Banach space. For every finite-dimensional subspace E of X, there exists a projection with minimal norm. That is, there exists a projection $P : X \to X$ onto E such that*

$$\|P\| = \inf\{\|Q\| : Q : X \to X \textit{ is a projection onto } E\}.$$

Proof. Let E be a finite-dimensional subspace and denote $\mathcal{P}(E)$ to be the collection of all bounded linear projections from X onto E. First note that since E is finite-dimensional, it is closed, and thus, by Corollary 1.1.13, E is complemented in X. Thus, we have $\mathcal{P}(E) \neq \emptyset$ and

$$\lambda(E) := \inf\{\|Q\| : Q \in \mathcal{P}(E)\} < \infty.$$

Let $\{e_i; e_i^*\}_{i=1}^k$ be a biorthogonal system of E. For each $n \in \mathbb{N}$, find $P_n \in \mathcal{P}(E)$ such that $\|P_n\| \leq \lambda(E) + \frac{1}{n} = L_n$ and define P_n as

$$P_n x = \sum_{i=1}^{k} e_i^*(P_n x) e_i$$

for $x \in X$. Note that for each $x \in X$, $1 \leq i \leq k$, and $n \in \mathbb{N}$, denoting $M = \max_{1\leq i\leq k} \|e_i^*\|$, we have

$$|e_i^*(P_n x)| \leq \|e_i^*\|\|P_n\|\|x\| \leq ML_n\|x\|$$

so $\|e_i^* \circ P_n\| \leq ML_n$. Moreover, since $L = \sup_{n\in\mathbb{N}} L_n < \infty$, we have for each $1 \leq i \leq k$, $\{e_i^* \circ P_n\}_{n=1}^{\infty}$ is a bounded subset of X^*. In particular, for each $1 \leq i \leq k$, $\{e_i^* \circ P_n\}_{n=1}^{\infty} \subseteq MLB_{X^*}$. By the Banach–Alaoglu Theorem (Theorem A.0.10), B_{X^*} is w^*-compact, and scalar multiples of it are also w^*-compact. Hence, for each $1 \leq i \leq k$, $\{e_i^* \circ P_n\}_{n=1}^{\infty}$ lies in a w^*-compact subset of X^*. Now we need to find a single subsequence $\{P_{n_j}\}_{j=1}^{\infty}$ that works simultaneously for $1 \leq i \leq k$. That is,

Claim. *For each $1 \leq i \leq k$, $\{e_i^* \circ P_n\}_{n=1}^{\infty}$ has a common w^*-convergent subsequence $\{P_{n_j}\}_{j=1}^{\infty}$ such that $e_i^* \circ P_{n_j} \overset{*}{\rightharpoonup} e_i^\star \in X^*$.*

Proof of Claim. Since $\{e_i^* \circ P_n\}_{n=1}^{\infty}$ is bounded by X^*, by the Banach–Alaoglu Theorem (Theorem A.0.10), it has a w^*-convergent subsequence $\{P_{n,1}\}_{n=1}^{\infty}$ such that $e_1^* \circ P_{n,1} \overset{*}{\rightharpoonup} e_1^\star \in X^*$. Now consider $\{e_2^* \circ P_{n,1}\}_{n=1}^{\infty}$, a subsequence of the original. Again, it is bounded, so we can extract a further subsequence $\{P_{n,2}\}_{n=1}^{\infty} \subseteq \{P_{n,1}\}_{n=1}^{\infty}$ such that $e_2^* \circ P_{n,2} \overset{*}{\rightharpoonup} e_2^\star \in X^*$. Proceeding in the same way, at the kth step, extract a subsequence $\{P_{n,k}\}_{n=1}^{\infty} \subseteq \{P_{n,k-1}\}_{n=1}^{\infty}$ such that $e_k^* \circ P_{n,k} \overset{*}{\rightharpoonup} e_k^\star$.

Then for each $j \in \mathbb{N}$, define $P_{n_j} := P_{j,j}$, the jth term of the jth subsequence. Then for each $1 \leq i \leq k$, $\{e_i^* \circ P_{n_j}\}_{j=1}^{\infty}$ is a subsequence of a w^*-convergent sequence such that $e_i^* \circ P_{n_j} \overset{*}{\rightharpoonup} e_i^\star$. The diagonal sequence $\{P_{n_j}\}_{j=1}^{\infty}$ ensures that each $e_i^* \circ P_{n_j}$ is a subsequence of a w^*-subsequence and hence, converges to $e_i^\star$. □

Define $P : X \to E$ by

$$Px = \sum_{i=1}^{k} e_i^\star(x) e_i$$

for $x \in X$. Observe that for each $x \in E$,

$$Px = \sum_{i=1}^{k} e_i^\star(x) e_i = \lim_{j\to\infty} \sum_{i=1}^{k} e_i^*(P_{n_j} x) e_i = \lim_{j\to\infty} P_{n_j} x = x.$$

Therefore, P is a linear projection. Furthermore, for each $x \in X$, by a similar computation as above, we have $Px = \lim_{j\to\infty} P_{n_j}x$, so by the Banach–Steinhaus Theorem (Theorem A.0.9),

$$\|P\| \leq \liminf_{j\to\infty} \|P_{n_j}\| = \lambda(E).$$

Thus, P is bounded with $\|P\| \leq \lambda(E)$. Furthermore, we also have $\|P\| \geq \lambda(E)$, so in fact, $\|P\| = \lambda(E)$, so P is the desired bounded linear projection. $\square$

Lemma 2.4.3 (Main Lemma)**.** *Let X be a Banach space in which every closed subspace is complemented in X. Then there exists a $\lambda \geq 1$ such that every finite-dimensional subspace E is λ-complemented in X.*

Proof. For a finite-dimensional subspace $E \subseteq X$, let $\lambda(E)$ be the minimal norm of projections from X onto E (which exists by Lemma 2.4.2). We need to show that $\sup\{\lambda(E) : \dim(E) < \infty\} < \infty$.

Assume for a contradiction that $\sup\{\lambda(E) : \dim(E) < \infty\} = \infty$.

Claim. *For every subspace X_0 of X of finite codimension,*

$$\sup\{\lambda(E) : \dim(E) < \infty, E \subseteq X_0\} = \infty. \tag{2.4.1}$$

Proof of Claim. Assume for a contradiction that for some subspace X_0 of X of codimension k, we have

$$M = \sup\{\lambda(E) : \dim(E) < \infty, E \subseteq X_0\} < \infty.$$

We will prove that $\sup\{\lambda(E) : \dim(E) < \infty, E \subseteq X\} < \infty$. Let E be a finite-dimensional subspace of X, let $E_0 = E \cap X$ and let $P : X \to E_0$ be such that $\|P\| \leq M$. Let $F = \ker(P) \cap E$. Note that for any $x \in E$, we have $x = Px + (I - P)x$, where $Px \in E_0$ and $(I - P)x \in F$ as $x, Px \in E$ and that

$$P(I - P)x = P(x - Px) = Px - PPx = Px - Px = 0.$$

Here, $\dim(F) \leq k$ since X is of codimension k, so then by Corollary 1.1.13, there exists a projection $R : X \to F$ with $\|R\| \leq \dim(F) = k$.

Define $Q = P + R - RP$. We claim that Q is a linear projection, observe that for any $x \in E$, $Qx \in E$ since

$$Qx = \underset{E_0 \subseteq E}{\underset{\uparrow}{Px}} + \underset{F \subseteq E}{\underset{\uparrow}{Rx}} - \underset{F \subseteq E}{\underset{\uparrow}{RPx}}.$$

Furthermore, note that since $x = Px + (I - P)x$ we have

$$\begin{aligned} Qx &= P(Px - (I - P)x) + R(Px + (I - P)x) - RP(Px + (I - P)x) \\ &= Px + RPx + R(I - P)x - RPx \\ &= Px + R(I - P)x \\ &= Px + (I - P)x \\ &= x \end{aligned}$$

where the second last equality holds since $(I - P)x \in F$, so $R(I - P)x = (I - P)x$. Therefore, Q is a linear projection from X onto E with

$$\|Q\| \leq \|P + R - RP\| \leq \|P\| + \|R\| + \|RP\| \leq M + k + Mk \leq (M + 1)(k + 1)$$

Therefore, $\lambda(E) \leq \|Q\| \leq (M + 1)(k + 1)$ (where the M and k do not depend on the choice of E). In particular, $\sup\{\lambda(E) : \dim(E) < \infty, E \subseteq X\} \leq (M+1)(k+1)$, which is a contradiction. Therefore, (2.4.2) holds. $\square$

Let E be a finite-dimensional subspace of X. By the proof of Lemma 1.2.21 and since E is a finite-dimensional subspace of X, for any $\varepsilon > 0$, there exists a finite-codimensional subspace X_0 of X such that

$$\|e + x\| \geq (1 - \varepsilon)\|x\|$$

for $e \in E$ and $x \in X_0$.

Claim. *There exists a sequence $\{E_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces and a* *sequence $\{X_n\}_{n=1}^{\infty}$ of finite-codimensional subspaces such that for all $n \in \mathbb{N}$,*

1. $\lambda(E_n) > n$.
2. $\|e + x\| \geq \frac{1}{2}\|e\|$ *for all* $e \in E_1 + E_2 + \cdots + E_n$ *and* $x \in X_n$.
3. $E_{n+1} \subseteq X_n$.
4. $X_{n+1} \subseteq X_n$.

Proof of Claim. Take $X_0 = X$ and find E_1 finite-dimensional such that $\lambda(E_1) > 1$. By Lemma 1.2.21, there exists X_1 of finite codimension such that for all $e \in E_1$ and $x \in X_1$,

$$\|e + x\| \geq \frac{1}{2}\|e\|.$$

Then there exists $E_2 \subseteq X_1$ such that $\lambda(E_2) > 2$. Take $E_1 + E_2$ which is finite-dimensional. By Lemma 1.2.21 again, there exists $Y_2 \subseteq X$ of finite codimension such that for all $e \in E_1 + E_2$ and $x \in Y_2$,

$$\|e + x\| \geq \frac{1}{2}\|e\|.$$

Then take $X_2 = X_1 \cap Y_2$, which is still finite-codimensional. Then there exists $E_3 \subseteq X_2$ such that $\lambda(E_3) > 3$.

Continuing inductively, we obtain that for all $n \in \mathbb{N}$, $\lambda(E_n) > n$ and for all $e \in E_1 + E_2 + \cdots + E_n$ and all $x \in X_{n+1}$, $\|e + x\| \geq \frac{1}{2}\|e\|$, $E_{n+1} \subseteq X_n$, and $X_{n+1} \subseteq X_n$. $\square$

Finally, if $e_1 \in E_1$, $e_2 \in E_2$, ..., $e_N \in E_N$ for some $N \in \mathbb{N}$ with $1 \leq m \leq N$, observe that

$$\begin{aligned}\|e_1 + e_2 + \cdots + e_N\| &= \|\underbrace{e_1 + e_2 + \cdots + e_m}_{\in E_1 + E_2 + \cdots + E_m} + \underbrace{e_{m+1} + e_{m+2} + \cdots + e_N}_{\in X_{m+1}}\| \\ &\geq \frac{1}{2}\|e_1 + e_2 + \cdots + e_m\|\end{aligned}$$

and so

$$\|e_1 + e_2 + \cdots + e_m\| \leq 2\|e_1 + e_2 + \cdots + e_N\|.$$

Let $Y = [\{E_n\}_{n=1}^{\infty}]$ and for each $m \in \mathbb{N}$, let $P_m : Y \to E_1 + E_2 + \cdots + E_m$ be a projection such that $\|P_m\| \leq 2$. Then see that $\|P_m - P_{m-1}\| \leq 4$. Define $Q_m = P_m - P_{m-1} : Y \to E_m$. Then Q_m is also a projection with $\|Q_m\| \leq 4$. By assumption, there exists a projection $R : X \to Y$ such that $\|R\| < \infty$. Then for all $m \in \mathbb{N}$, $Q_m R : X \to E_m$ is a projection and

$$m < \lambda(E_m) \leq \|Q_m R\| \leq 4\|R\|$$

but then for m sufficiently large, we obtain $\|R\| = \infty$, which is a contradiction. $\square$

Proof of Lindenstrauss–Tzafriri Theorem (Theorem 2.4.1). Assume that $X \simeq \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space, and let E be a closed subspace of X. Let $T : X \to \mathcal{H}$ be an isomorphism, and define $T|_E : E \to T(E)$. Then $T|_E$ is also an isomorphism, and furthermore, $T(E)$ is a closed subspace of $\mathcal{H}$, so by Example 1.2.18, there exists a norm one projection $P : \mathcal{H} \to T(E)$, that is, $T(E)$ is complemented in $\mathcal{H}$. Then, by Fact 1.1.7, complementations are preserved under isomorphisms, then E is also complemented, as required.

The proof of the converse requires several steps, but we follow the approach presented in [5]. Let X be a Banach space such that every closed subspace is complemented in X. Then by Dvoretzky's Theorem (Theorem 2.3.6), there exists a $K > 0$ such that the Hilbert space $\mathcal{H}$ is crudely finitely representable in X. Let X_0 be a subspace of X of finite codimension n. For $m \in \mathbb{N}$, we seek $F \subseteq X_0$ such that $d_{\mathrm{BM}}(F, \ell_2^m) \leq K$. By assumption, there exists $G \subseteq X$ with $d_{\mathrm{BM}}(G, \ell_2^{n+m}) \leq K$. Since $\dim(G) = n + m$ and X_0 is of codimension n, letting $F' = G \cap X_0$, we have $\dim(F') \geq m$. Take $F \subseteq F'$ arbitrary with $\dim(F) = m$. Then $F \subseteq X_0$ and $d_{\mathrm{BM}}(F, \ell_2^m) \leq K$. With the above, we may assume that the same K applies uniformly to all subspaces of finite codimension. Let λ be the same as in Lemma 2.4.3. In this case, we need to show that X is crudely finitely represented in $\mathcal{H}$, so that Theorem 2.1.8 will give us the conclusion.

Let E be a finite-dimensional subspace of X with $\dim(E) = n$. Denote

$$\alpha = d_{\mathrm{BM}}(E, \ell_2^n). \tag{2.4.2}$$

We claim that $\alpha \leq \lambda^4 2^6 K^3$.

Let $Q : X \to E$ be a projection so that $\|Q\| \leq \lambda$. Let $F \subseteq \ker(Q)$ be such that

$$d_{\mathrm{BM}}(F, \ell_2^n) \leq K. \tag{2.4.3}$$

By Remark 1.3.15, we have by (2.4.2) and (2.4.3),

$$d_{\mathrm{BM}}(E, F) \leq d_{\mathrm{BM}}(E, \ell_2^n) d_{\mathrm{BM}}(F, \ell_2^n) = \alpha K.$$

In particular, there exists an isomorphism $T : E \to F$ such that

$$\frac{1}{\alpha K}\|x\| \leq \|Tx\| \leq \|x\|. \tag{2.4.4}$$

Define $G = \{x + \mu Tx : x \in E\}$ where $\mu = 2^4\lambda^2 K^2$, so that any $g \in G$ has a unique representation given as $g = x + \mu Tx$ for all $x \in E$. Let $P : X \to G$ be a projection with $\|P\| \leq \lambda$.

Define $V : E \to E$ by

$$Vx = QPTx$$

Note that the map V is well defined since

$$\underset{\in E}{x} \xrightarrow{T} \underset{\in F}{Tx} \xrightarrow{P} \underset{\in G}{PTx} \xrightarrow{Q} \underset{\in E}{QPTx} = Vx$$

We claim that

$$PTx = Vx + \mu TVx \tag{2.4.5}$$

To see this, recall that by the definition of G, every $g \in G$ has a unique representation as $g = x + \mu Tx$ for some $x \in E$. Thus, if $y = x + \mu Tx$, since P is a projection onto G, we have $Py = y$ as P fixes all elements of G. Now consider the definition of V

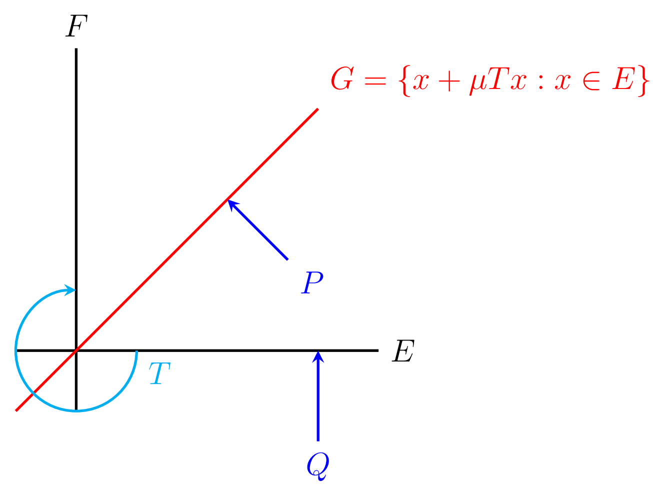

Figure 2.1: Geometric representation of the subspaces and operators in the proof of the Lindenstrauss–Tzafriri Theorem. Here, E is a finite-dimensional subspace of X, $F \subseteq \ker(Q)$ is a subspace of the complement of E, and $G = \{x + \mu Tx : x \in E\}$ is used to interpolate between E and F. The operators Q, T, and P represent key components used to construct an embedding of E into a Hilbert space with a controlled distortion.

above. Putting $y = Tx \in F$, then there exists $x' \in E$ such that

$$PTx = x' + \mu Tx'$$

Then applying Q to both sides yields

$$QPTx = Q(x' + \mu Tx') = Qx' + \mu QTx' = x'$$

since $Q : X \to E$ is a projection onto E and $QTx' = 0$ as $Tx' \in F \subseteq \ker(Q)$. Therefore, $Vx = x'$, and thus, we obtain (2.4.5).

Observe that for any $x \in E$, we have $Vx = QPTx$, and so

$$\|Vx\| = \|QPTx\| \leq \|Q\|\|P\|\|Tx\| \leq \lambda^2\|Tx\|. \tag{2.4.6}$$

Then observe that for any $x \in E$, we have by (2.4.5),

$$\begin{aligned} Px &= P(x + \mu Tx) - \mu PTx \\ &= x + \mu Tx - \mu(Vx + \mu TVx) \\ &= (x - \mu Vx) + \mu(T(x - \mu Vx)) \end{aligned}$$

and we note that $x - \mu Vx \in E = Q(X)$ and that $\mu(T(x - \mu Vx)) \in F \subseteq \ker(Q)$, so then

$$(I - Q)Px = \mu(T(x - \mu Vx)) \in F \tag{2.4.7}$$

for all $x \in E$. Then observe that for all $x \in E$, by (2.4.4), (2.4.6), and (2.4.7), we have

$$\begin{aligned} \|(I - Q)Px\| &= \mu\|T(x - \mu Vx)\| \quad \text{by (2.4.7)} \\ &\geq \frac{\mu}{\alpha K}\|x - \mu Vx\| \quad \text{by (2.4.4)} \\ &\geq \frac{\mu}{\alpha K}(\|x\| - \mu\|Vx\|) \quad \text{by lower triangle inequality} \\ &\geq \frac{\mu}{\alpha K}(\|x\| - \mu\lambda^2\|Tx\|) \quad \text{by (2.4.6).} \end{aligned} \tag{2.4.8}$$

By (2.4.3), find $U : F \to \ell_2^n$ such that

$$\frac{1}{K}\|y\| \leq \|Uy\| \leq \|y\| \tag{2.4.9}$$

for all $y \in F$.

Note that $\ell_2^n \oplus \ell_2^n$ is a Hilbert space. Let $W : E \to \ell_2^n \oplus \ell_2^n$ be defined by

$$Wx = \left[\tfrac{1}{2}UTx, \quad \tfrac{1}{4\lambda^2}U(I - Q)Px\right]. \tag{2.4.10}$$

We need to show that W is an isomorphism. First see that for any $x \in E$,

$$\begin{aligned} \|Wx\| &\leq \left\|\frac{1}{2}UTx\right\| + \left\|\frac{1}{4\lambda^2}U(I - Q)Px\right\| \\ &\leq \frac{1}{2}\underbrace{\|U\|}_{\leq 1}\underbrace{\|T\|}_{\leq 1}\|x\| + \frac{1}{4\lambda^2}\underbrace{\|U\|}_{\leq 1}\underbrace{\|I - Q\|}_{\leq 1+\lambda}\underbrace{\|P\|}_{\leq \lambda}\|x\| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\|x\| + \frac{\lambda(\lambda+1)}{4\lambda^2}\|x\| \\
&= \frac{1}{2}\|x\| + \frac{\lambda+1}{4\lambda}\|x\| \\
&\le \frac{1}{2}\|x\| + \frac{1}{2}\|x\| \quad \text{since } \frac{\lambda+1}{4\lambda} \le \frac{1}{2} \text{ for all } \lambda \ge 1. \\
&= \|x\|. \qquad (2.4.11)
\end{aligned}
$$

On the other hand, observe by (2.4.8), and (2.4.9), we have for any $x \in E$,

$$
\begin{aligned}
\|Wx\| &\ge \max\left\{\left\|\frac{1}{2}UTx\right\|, \left\|\frac{1}{4\lambda^2}U(I-Q)Px\right\|\right\} \\
&\ge \max\left\{\frac{1}{2K}\|Tx\|, \frac{1}{4K\lambda^2}\|(I-Q)Px\|\right\} \quad \text{by (2.4.9)} \\
&\ge \max\left\{\frac{1}{2K}\|Tx\|, \frac{\mu}{4\alpha K^2\lambda^2}(\|x\| - \mu\lambda^2\|Tx\|)\right\}. \qquad (2.4.12)
\end{aligned}
$$

Now we consider the following cases.

Case 1: If $\lambda^4 2^5 K^2\|Tx\| \ge \|x\|$. Then

$$\|Wx\| \ge \frac{1}{2K}\|Tx\| \ge \frac{1}{\lambda^4 2^6 K^3}\|x\|. \qquad (2.4.13)$$

Case 2: If $\lambda^4 2^5 K^2\|Tx\| < \|x\|$. Then by (2.4.8) and (2.4.12), we have

$$\|Wx\| \ge \frac{\mu}{4\alpha K^2\lambda^2}(\|x\| - \mu\lambda^2\|Tx\|) \ge \frac{\mu}{4\alpha K^2\lambda^2}\left(\|x\| - \frac{1}{2}\|x\|\right) = \frac{\mu}{8\alpha K^2\lambda^2}\|x\| = \frac{2}{\alpha}\|x\|. \qquad (2.4.14)$$

From (2.4.13) and (2.4.14), we have for all $x \in E$

$$\|Wx\| \ge \min\left\{\frac{2}{\alpha}, \frac{1}{\lambda^4 2^6 K^3}\right\}\|x\|. \qquad (2.4.15)$$

Therefore, $W : E \to \ell_2^n \oplus \ell_2^n$ is an $\max\left\{\frac{\alpha}{2}, \lambda^4 2^6 K^3\right\}$-isomorphism. However, since

α is defined to be the smallest and W is an isomorphism, we have

$$\alpha \leq \|W\|\|W^{-1}\| \leq \max\left\{\frac{\alpha}{2}, \lambda^4 2^6 K^3\right\}$$

Clearly, α cannot be at most $\frac{\alpha}{2}$, so we have $\alpha \leq \lambda^4 2^6 K^3$, as desired. $\square$

Appendix A

Key Theorems

This appendix chapter is to list the important theorems that are often used in this project, including the Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle, Hahn-Banach Theorem, etc. Majority of these are from Functional Analysis Course Notes from Fall 2024 [16], but other references include Measure Theory Course Notes from Winter 2025 [20].

Theorem A.0.1 (Hölder's Inequality)**.** *Let $p, q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.*

1. *If $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, then*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

2. *If $(x_i)_{i=1}^{\infty} \in \ell_p(\mathbb{N})$ and $(y_i)_{i=1}^{\infty} \in \ell_q(\mathbb{N})$, then $(x_i y_i)_{i=1}^{\infty} \in \ell_1(\mathbb{N})$ and*

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}}$$

3. *If $f \in L_p[0,1]$ and $g \in L_q[0,1]$, then $fg \in L_1[0,1]$ and*

$$\int_0^1 |f(x)g(x)|\ dx = \left(\int_0^1 |f(x)|^p\ dx\right)^{\frac{1}{p}} \left(\int_0^1 |g(x)|^q\ dx\right)^{\frac{1}{q}}$$

Theorem A.0.2 (Minkowski's Inequality)**.** *Let $p \in [1,\infty)$.*

1. *If $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, then*

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

2. *If $(x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty} \in \ell_p(\mathbb{N})$, then*

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{\frac{1}{p}}$$

3. *If $f, g \in L_p[0,1]$, then*

$$\left(\int_0^1 |f(x) + g(x)|^p\ dx\right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x)|^p\ dx\right)^{\frac{1}{p}} + \left(\int_0^1 |g(x)|^p\ dx\right)^{\frac{1}{p}}$$

Theorem A.0.3 (Hahn-Banach: Norm Preserving Extension)**.** *Let X be a normed space and let Y be a subspace of X with $f \in Y^*$. Then there exists a $\tilde{f} \in X^*$ such that $\tilde{f}|_Y = f$ (that is, $\tilde{f}$ is a linear extension of f to X), and $\|\tilde{f}\| = \|f\|$.*

Theorem A.0.4 (Hahn-Banach: Norm Attaining Functional)**.** *Let X be a normed space and let $x_0 \in X \setminus \{0\}$. Then there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

Theorem A.0.5 (Open Mapping Theorem)**.** *A bounded linear operator $T : X \to Y$ between Banach spaces X and Y is a surjection if and only if it is open.*

Corollary A.0.6. *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear bijection. Then T is an isomorphism.*

Theorem A.0.7 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a linear operator. Then T is bounded if and only if $\mathrm{Gr}(T)$ is closed.*

Theorem A.0.8 (Uniform Boundedness Principle)**.** *Let X be a Banach space and Y be a normed space and let I be an arbitrary index set. For each $i \in I$, let $T_i : X \to Y$ be a bounded linear operator such that for all $x \in X$*

$$\sup_{i \in I} \|T_i x\| < \infty$$

Then $\sup_{i \in I} \|T_i\| < \infty$.

Theorem A.0.9 (Banach-Steinhaus Theorem)**.** *Let X be a Banach space and let Y be a normed space. Let $\{T_n\}_{n=1}^{\infty}$ be a family of bounded linear operators from X to Y such that for all $x \in X$, $\lim_{n\to\infty} T_n x$ exists. Then the following hold.*

1. $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$.
2. *The map $T : X \to Y$ given by $Tx = \lim_{n\to\infty} T_n x$ is a bounded linear operator.*
3. $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$.

Theorem A.0.10 (Banach–Alaoglu Theorem)**.** *If X is a Banach space, then B_{X^*} is compact in the w^*-topology.*

Theorem A.0.11. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The set*

$$\mathcal{F} = \operatorname{span}\{\varphi : X \to \mathbb{R} : \varphi \text{ is simple and there exists } A \in \mathcal{A} \text{ such that } \mu(A) < \infty \text{ and } \varphi|_{A^c} = 0\}$$

is dense in $L_p(X, \mu)$.

Theorem A.0.12 (Characterization of Schur Spaces)**.** *Let X be an infinite-dimensional Banach space. Then either*

1. *X contains a normalized weakly null basic sequence.*
2. *X contains a basic sequence that is equivalent to the unit vector basis of $\ell_1(\mathbb{N})$.*

Definition A.0.13. Let X be a nonempty set and let $X^{(n)}$ be the set of all subsets of X with cardinality n. We say that a system of k disjoint sets $\{S_i\}_{i=1}^{k}$ forms a *partitioning of $X^{(n)}$* if

$$X^{(n)} = \bigcup_{i=1}^{k} S_i$$

Theorem A.0.14 (Ramsey's Theorem)**.** *Let $k, n \in \mathbb{N}$. Then for every partitioning $\{S_i\}_{i=1}^k$ of $\mathbb{N}^{(n)}$ there exists $1 \leq i \leq k$ and an infinite subset $M \subseteq \mathbb{N}$ such that $M^{(n)} \subseteq S_i$.*

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