MATH 3410
Complex Variables

Joe Tran

July 16, 2025

Preface

These are the first edition of these lecture notes for MATH 3410 (Complex Variables). Consequently, there may be several typographical errors, missing exposition on necessary background, and more advanced topics for which there will not be time in class to cover. Future iterations of these notes will hopefully be fairly self-contained provided one has the necessary background. If you come across any typos, errors, omissions, or unclear expositions, please feel free to contact me so that I may continually improve these notes.

Contents

Chapter 1

Complex Numbers

The study of complex analysis begins with a deep understanding of complex numbers, which are the foundational objects upon which the entire theory is built. While real numbers suffice for most classical calculus, many mathematical phenomena and physical models demand a more expansive number system that can handle operations like taking square roots of negative numbers. This need leads naturally to the field of complex numbers.

Complex numbers extend the real number system by introducing a new unit, denoted by i, which satisfies the property $i^2 = -1$. The resulting system, denoted by $\mathbb{C}$, possesses rich algebraic and geometric structure. Unlike the real line, the complex numbers form a two-dimensional plane, allowing us to study functions, differentiation, and integration in a multidimensional yet highly structured setting.

This chapter introduces the algebraic and geometric properties of complex numbers, explores their representation in both rectangular and polar form, and discuss key operations such as complex conjugation, modulus, and argument. We also explore the complex exponential function and Euler's identity, which provide powerful tools for expressing and manipulating complex quantities. These foundational tools will play a crucial role in the chapters to come.

1.1 Complex Numbers

In this section, we give the basics of complex numbers.

Definition 1.1.1. The *complex numbers*, denoted $\mathbb{C}$, are the set

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

where i denotes a fixed symbol.

Given a complex number $z = a + bi$, where $a, b \in \mathbb{R}$, the number a is called the *real part of* z, and is denoted by $\mathrm{Re}(z)$, and the number b is called the *imaginary part of* z, and is denoted by $\mathrm{Im}(z)$.

The symbol i in a complex number is meant to denote "$\sqrt{-1}$". To be specific, we will equip $\mathbb{C}$ with binary operations of addition "$+$" $: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and multiplication "$\cdot$" $: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ so that $(0 + 1i)(0 + 1i) = -1$, so that indeed, i is a complex solution to $x^2 + 1 = 0$.

Definition 1.1.2. The binary operations "$+$" $: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and "$\cdot$" $: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi) \cdot (c + di) &= (ac - bd) + (ad + bc)i\end{aligned}$$

for all $a, b, c, d \in \mathbb{R}$ are called *complex addition* and *complex multiplication*, respectively.

Example 1.1.3. It is not difficult to see that

$$\begin{aligned}(1 + 2i) + (3 + 4i) &= 4 + 6i \\ (1 + 2i)(3 + 4i) &= -5 + 10i.\end{aligned}$$

Moreover, since

$$i^2 = (0 + 1i)(0 + 1i) = -1 + 0i$$

we do indeed have that i is a complex solution to $x^2 = -1$. In addition, it is not difficult to see that $-i$ is also a complex solution to $x^2 = -1$.

In order for $\mathbb{C}$ to be as nice to work with as $\mathbb{R}$, we require complex addition and multiplication to have specific properties. To be specific, we want the following.

Theorem 1.1.4. *The set of complex numbers $\mathbb{C}$ together with complex addition and multiplication is a field; that is,*

1. *(Commutativity of Addition) $z + w = w + z$ for all $z, w \in \mathbb{C}$.*
2. *(Associativity of Addition) $z + (w + u) = (z + w) + u$ for all $z, w, u \in \mathbb{C}$.*
3. *(Additive Unit) There exists a $0 \in \mathbb{C}$ such that $z + 0 = z$ for all $z \in \mathbb{C}$.*
4. *(Additive Inverses) For all $z \in \mathbb{C}$, there exists a $-z \in \mathbb{C}$ such that $z + (-z) = 0$.*

5. *(Commutativity of Multiplication)* $zw = wz$ *for all* $zw \in \mathbb{C}$.
6. *(Associativity of Multiplication)* $z(wu) = (zw)u$ *for all* $z, w, u \in \mathbb{C}$.
7. *(Multiplicative Unit) There exists a* $1 \in \mathbb{C}$ *such that* $1z = z$ *for all* $z \in \mathbb{C}$.
8. *(Multiplicative Inverses) For all* $z \in \mathbb{C} \setminus \{0\}$*, there exists a* $z^{-1} \in \mathbb{C}$ *such that* $z^{-1}z = 1$.
9. *(Distributivity)* $z(w + u) = zw + zu$ *for all* $z, w, u \in \mathbb{C}$.

Proof. Let $z, w, u \in \mathbb{C}$ be arbitrary. Then there exists $a, b, c, d, x, y \in \mathbb{R}$ such that

$$z = a + bi, \quad w = c + di, \quad u = x + yi$$

We will examine each of the above nine properties for these arbitrary elements of $\mathbb{C}$ and demonstrate the properties hold using the analogous properties for real numbers.

To see that (1) holds, notice that

$$z + w = (a + c) + (b + d)i = (c + a) + (d + b)i = w + z$$

due to the commutativity of addition of real numbers. Thus, commutativity of addition of complex numbers has been demonstrated.

To see that (2) holds, notice that

$$\begin{aligned} z + (w + u) &= (a + bi) + [(c + x) + (d + y)i] \\ &= [a + (c + x)] + [b + (d + y)]i \\ &= [(a + c) + x] + [(b + d) + y]i \\ &= [(a + c) + (b + d)i] + (x + yi) \\ &= (z + w) + u \end{aligned}$$

where the third equality holds due to the associativity of addition of real numbers. Thus, associativity of addition of complex numbers has been demonstrated.

To see that (3) holds, notice that with $0 = 0 + 0i$, we have

$$z + 0 = (a + 0) + (b + 0)i = a + bi = z$$

due to the property of the zero element of $\mathbb{R}$. Thus, the complex numbers have an additive unit.

To see that (4) holds, let $-z = (-a) + (-b)i$, where $-a$ and $-bb$ are the additive inverses of a and b in $\mathbb{R}$. Then

$$z + (-z) = [a + (-a)] + [b + (-b)]i = 0 + 0i = 0$$

as desired. Thus, the complex numbers have additive inverses.

To see that (5) holds, notice that

$$zw = (ac - bd) + (ad + bc)i = (ca - db) + (cb + da)i = wz$$

due to the commutativity of addition and multiplication of real numbers. Thus, commutativity of multiplication of complex numbers has been demonstrated.

To see that (6) holds, notice that

$$\begin{aligned} z(wu) &= (a + bi)[(c + di)(x + yi)] \\ &= (a + bi)[(cx - dy) + (cy + dx)i] \\ &= [a(cx - dy) - b(cy + dx)] + [a(cy + dx) + b(cx - dy)]i \\ &= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i \\ &= (acx - bdx - ady - bcy) + (acy - bdy + adc + bcx)i \\ &= [(ac - bd)x - (ad + bc)y] + [(ac - bd)y + (ad + bc)x]i \\ &= [(ac - bd) + (ad + bc)i](x + yi) \\ &= [(a + bi)(c + di)](x + yi) \\ &= (zw)u \end{aligned}$$

where commutativity and associativity of addition (fifth equality) and distributivity of real numbers (fourth and sixth equalities) have been used. Thus, associativity of multiplication of complex numbers has been demonstrated.

To see that (7) holds, notice that with $1 = 1 + 0i$, we have

$$1z = (1a - 0b) + (1b + 0a)i = a + bi = z$$

due to the property of the one element of $\mathbb{R}$. Thus, the complex numbers have a multiplicative unit.

To see that (8) holds, assume that $z \neq 0$. Thus, either $a \neq 0$ or $b \neq 0$, so $a^2 + b^2 > 0$. Define

$$z^{-1} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i,$$

which makes sense since $\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \in \mathbb{R}$ since nonzero real numbers have multiplicative inverses. Notice that

$$\begin{aligned} z^{-1}z &= \left[\left(\frac{a}{a^2+b^2}\right)a - \left(\frac{-b}{a^2+b^2}\right)b\right] + \left[\left(\frac{a}{a^2+b^2}\right)b + \left(\frac{-b}{a^2+b^2}\right)a\right]i \\ &= \left(\frac{a^2+b^2}{a^2+b^2}\right) + \left(\frac{ab-ba}{a^2+b^2}\right)i \\ &= 1+0i \\ &= 1 \end{aligned}$$

due to the commutativity of multiplication and properties of addition of real numbers. Hence, z^{-1} is indeed the multiplicative inverse of z. Thus, the existence of multiplicative inverses for nonzero complex numbers has been demonstrated.

Finally, to see that (9) holds, notice that

$$\begin{aligned} z(w+u) &= (a+bi)[(c+di)+(x+yi)] \\ &= (a+bi)[(c+x)+(d+y)i] \\ &= [a(c+x)-b(d+y)]+[a(d+y)+b(c+x)]i \\ &= (ac+ax-bd-dy)+(ad+ay+bc+bx)i \\ &= [(ac-bd)+(ax-dy)]+[(ad+bc)+(ay+bx)]i \\ &= [(ac-bd)+(ad+bc)i]+[(ax-dy)+(ay+bx)i] \\ &= (a+bi)(c+di)+(a+bi)(x+yi) \\ &= zw+zu \end{aligned}$$

where commutativity and associativity of addition (fourth equality) and distributivity of real numbers (third equality) have been used. Thus, distributivity of complex numbers has been demonstrated.

As all nine properties have now been demonstrated for arbitrary complex numbers, the set of complex numbers together with addition and multiplication are a field. $\square$

Remark 1.1.5. Embedded in the proof of Theorem 1.1.4 is the formula for the inverse of a nonzero complex number. Indeed, if $z = a + bi$, where $a, b \in \mathbb{R}$ is such that $z \neq 0$, then

$$z^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

For example,

$$(3+4i)^{-1} = \frac{3}{25} - \frac{4}{25}i$$

Note we can also write

$$z^{-1} = \frac{a - bi}{a^2 + b^2}$$

When dealing with complex numbers, both the numerator and denominator are important quantities that are worthy of developing further.

Definition 1.1.6. Given a complex number $z = a + bi$, where $a, b \in \mathbb{R}$, the *absolute value* (or *length* or *modulus*) of z, denoted $|z|$ is the quantity

$$|z| = \sqrt{a^2 + b^2}$$

Definition 1.1.7. Given a complex number $z = a + bi$, where $a, b \in \mathbb{R}$, the *complex conjugate of* z, denoted $\bar{z}$ is the quantity

$$\bar{z} = a + (-b)i$$

Using our knowledge of the inverse of a nonzero complex number, we have the following.

Corollary 1.1.8. *If* $z \in \mathbb{C} \setminus \{0\}$*, then* $z^{-1} = \frac{\bar{z}}{|z|^2}$.

There are many properties of the absolute value and complex conjugates that are worthy of recording.

Theorem 1.1.9. *For all* $z, w \in \mathbb{C}$*, the following are true.*

1. $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$.
2. $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$.
3. $\overline{z + w} = \bar{z} + \bar{w}$.
4. $\overline{zw} = \bar{z}\bar{w}$.
5. $|z| = \sqrt{\bar{z}z}$.
6. $|\mathrm{Re}(z)| \leq |z|$.
7. $|\mathrm{Im}(z)| \leq |z|$.
8. $|\bar{z}| = |z|$.
9. $|zw| = |z||w|$.
10. *If* $z \neq 0$*, then* $|z^{-1}| = |z|^{-1}$.

11. $|z+w| \leq |z| + |w|$.

12. $||z| - |w|| \leq |z - w|$.

Proof. Let $z, w \in \mathbb{C}$ be arbitrary. Then there exists $a, b, c, d \in \mathbb{R}$ such that $z = a + bi$ and $w = c + di$.

To see that (1) is true, notice that

$$\frac{z+\bar{z}}{2} = \frac{(a+bi)+(a-bi)}{2} = \frac{2a}{a} = a = \mathrm{Re}(z)$$

as desired.

To see that (2) is true, notice that

$$\frac{z-\bar{z}}{2i} = \frac{(a+bi)-(a-bi)}{2i} = \frac{2bi}{2i} = b = \mathrm{Im}(z)$$

as desired.

To see that (3) is true, notice that

$$\begin{aligned}
\overline{z+w} &= \overline{(a+c)+(b+d)i} \\
&= (a+c) + (-(b+d))i \\
&= (a+(-b)i) + (c+(-d)i) \\
&= \bar{z} + \bar{w}
\end{aligned}$$

as desired.

To see that (4) is true, notice that

$$\begin{aligned}
\overline{zw} &= \overline{(ac-bd)+(ad+bc)i} \\
&= (ac-bd) + (-(ad+bc))i \\
&= (ac-(-b)(-d)) + (a(-d)+(-b)c)i \\
&= (a+(-b)i)(c+(-d)i) \\
&= \bar{z}\bar{w}
\end{aligned}$$

as desired.

To see that (5) is true, notice that

$$\begin{aligned}
\sqrt{\bar{z}z} &= \sqrt{(a+(-b)i)(a+bi)} \\
&= \sqrt{(a^2-(-b)b)+(ab+a(-b))i} \\
&= \sqrt{a^2+b^2} \\
&= |z|
\end{aligned}$$

as desired.

To see that (6) and (7) are true, notice that

$$a^2 \leq a^2 + b^2, \quad b^2 \leq a^2 + b^2$$

so

$$\begin{aligned} |\text{Re}(z)| &= |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z| \\ |\text{Im}(z)| &= |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| \end{aligned}$$

as desired.

To see that (8) is true, note that

$$|\bar{z}| = |a + (-b)i| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

as desired.

To see that (9) is true, notice that

$$zw = (ac - bd) + (ad + bc)i$$

so that

$$\begin{aligned} |zw| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2)} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\ &= |z||w| \end{aligned}$$

as desired.

To see that (10) is true, notice by (9) that

$$|z^{-1}||z| = |z^{-1}z| = |1| = 1$$

so $|z^{-1}| = |z|^{-1}$ as desired.

To see that (11) is true, notice that

$$\begin{aligned}
|z+w|^2 &= (\overline{z+w})(z+w) && \text{by (5)}\\
&= (\bar{z}+\bar{w})(z+w) && \text{by (3)}\\
&= \bar{z}z+\bar{z}w+\bar{w}z+\bar{w}w\\
&= |z|^2+\bar{z}w+\overline{\bar{z}w}+|w|^2 && \text{by (4) and (5)}\\
&= |z|^2+2\mathrm{Re}(\bar{z}w)+|w|^2 && \text{by (1)}\\
&\le |z|^2+2|\bar{z}w|+|w|^2 && \text{by (6)}\\
&= |z|^2+2|\bar{z}||w|+|w|^2 && \text{by (9)}\\
&= |z|^2+2|z||w|+|w|^2 && \text{by (8)}\\
&= (|z|+|w|)^2
\end{aligned}$$

Therefore, by taking the square root of both sides of the inequality, the desired result is obtained.

Finally, to see that (12) holds, first notice by (11) that

$$|z| = |(z-w)+w| \le |z-w|+|w|$$

Therefore,

$$|z|-|w| \le |z-w|$$

Similarly, notice by (11) and (9) that

$$\begin{aligned}
|w| &= |(w-z)+z|\\
&\le |w-z|+|z|\\
&= |(-1)(z-w)|+|z|\\
&= |-1||z-w|+|z|\\
&= |z-w|+|z|
\end{aligned}$$

Hence,

$$|w|-|z| \le |z-w|$$

Therefore, by combining $|z|-|w| \le |z-w|$ and $|w|-|z| \le |z-w|$, we obtain that

$$||z|-|w|| \le |z-w|$$

as desired. $\square$

1.2 Polar Form and Euler's Identity

While the rectangular form $z = x + yi$ of a complex number is often useful for algebraic manipulation, it conceals the elegant geometric structure inherent to the complex plane. A more natural way to describe complex numbers, especially when dealing with multiplication, powers, and roots, is through their polar form. This representation expresses a complex number in terms of its modulus, and its argument, thereby making its geometric behaviour more transparent.

In this section, we develop the polar representation of complex numbers and introduce one of the most profound identities in mathematics: Euler's identity. By linking exponential functions with trigonometric functions, Euler's formula provides a powerful bridge between algebra and geometry.

Definition 1.2.1. Given a nonzero complex number $z = x+yi$, the *argument of z* is any real number θ such that

$$\cos(\theta) = \frac{x}{|z|}, \quad \sin(\theta) = \frac{y}{|z|}$$

We define the *principal argument of z*, denoted by $\operatorname{Arg}(z)$, to be the unique value of $\theta \in (-\pi, \pi]$ satisfying the above.

Remark 1.2.2. Since $\sin(\theta)$ and $\cos(\theta)$ are 2π-periodic functions, the argument $\arg(z)$ is only well-defined up to an additive multiple of 2π. That is, if θ is an argument of z, then so is $\theta + 2k\pi$ for any $k \in \mathbb{Z}$.

Definition 1.2.3. Given a nonzero complex number $z = x + yi$, for some $x, y \in \mathbb{R}$, we may associate z two geometric quantities: the modulus of z and the argument of z. Moreover, provided that $z \neq 0$, we may write z in the form

$$z = r\cos(\theta) + i\sin(\theta))$$

where $r = |z|$ and $\theta = \arg(z)$. This is known as the *polar form of z*.

Example 1.2.4. Let $z = 1 + i$. Then

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \cos(\theta) = \frac{1}{\sqrt{2}}, \quad \sin(\theta) = \frac{1}{\sqrt{2}}$$

Thus, one possible value of $\arg(z)$ is $\theta = \frac{\pi}{4}$, and the polar form of z is

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

An advantage of using the polar form is that it provides geometric insight into the multiplication of complex numbers.

Proposition 1.2.5. *Let* $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$ *and* $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$. *Then the following hold.*

1. $zw = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$.
2. $\frac{z}{w} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$.

Proof. To see that (1) holds, observe that

$$\begin{aligned} zw &= r_1r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1r_2((\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + (\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))i) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

To see that (2) holds, note that since division is the inverse of multiplication, the result immediately holds. ☐

Remark 1.2.6. From Proposition 1.2.5, we see that for all complex numbers $z, w \in \mathbb{C}$.

$$\arg(zw) = \arg(z) + \arg(w), \quad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

Definition 1.2.7. Let $\tau \in \mathbb{R}$. The place $\mathbb{C}_\tau$ given by

$$\mathbb{C}_\tau = \{(r,\theta) : r > 0, \tau < \theta \leq \tau + 2\pi\}$$

is known as the *cut plane along the branch cut* $\{(r,\tau) : r > 0\}$. The branch of $\arg(z)$ that lies in $(\tau, \tau+2\pi]$ is denoted by $\arg_\tau(z)$.

Of particular importance is the principal branch of $\arg(z)$, which we take to be the branch in $(-\pi, \pi]$. The principal branch of $\arg(z)$ is therefore the branch $\arg_{-\pi}(z) = \operatorname{Arg}(z)$.

Example 1.2.8. Let $z = 1 + i$. Then we have

$$\arg(z) = \frac{\pi}{4} + 2k\pi$$

for all $k \in \mathbb{Z}$. In particular,

$$\operatorname{Arg}(z) = \frac{\pi}{4}, \quad \arg_{\frac{\pi}{2}}(z) = \frac{3\pi}{4}$$

We all know what e^θ is if θ is a real number. However, what is $e^{i\theta}$? To answer this question, we again use the rule that complex numbers are treated as real numbers in calculations.

Definition 1.2.9. For $z \in \mathbb{C}$, the *complex exponential of* z, denoted by e^z, is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

If we consider $z = i\theta$, and using Definition 1.2.9, we obtain

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\
&= \left(1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots\right) + \left(\frac{i\theta}{1!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots\right) \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}$$

Hence, we have the celebrated Euler's identity to the effect that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

for all $\theta \in \mathbb{R}$.

Definition 1.2.10. For any $\theta \in \mathbb{R}$, *Euler's identity* is given as

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Recall that the polar form of a nonzero complex number z is given by

$$z = r(\cos(\theta) + i\sin(\theta))$$

Then by Definition 1.2.10, the polar form becomes

$$z = re^{i\theta}$$

Example 1.2.11. If $\theta = \pi$, then $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$.

Theorem 1.2.12 (**De Moivre's Formula**). *For all* $n \in \mathbb{N}$ *and* $\theta \in \mathbb{R}$,

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Proof. Using Definition 1.2.10, we have

$$(\cos(\theta) + i\sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

as desired. $\square$

De Moivre's Formula (Theorem 1.2.12) allows for one to derive trigonometric formulas that are often used in calculus. One of the most common identities used is the double angle formulas.

Proposition 1.2.13 (Double Angle Formulas). *For all $\theta \in \mathbb{R}$,*

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.
2. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

Proof. First observe that by De Moivre's Formula

$$\begin{aligned} e^{i2\theta} &= (\cos(\theta) + i\sin(\theta))^2 \\ &= \cos^2(\theta) + 2i\cos(\theta)\sin(\theta) + i^2\sin^2(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta) + 2i\sin(\theta)\cos(\theta) \end{aligned}$$

To see that (1) holds, note that

$$\begin{aligned} \sin(2\theta) &= \mathrm{Im}(e^{i2\theta}) \\ &= \mathrm{Im}(\cos^2(\theta) - \sin^2(\theta) + 2i\sin(\theta)) \\ &= 2\sin(\theta)\cos(\theta) \end{aligned}$$

and similarly, to see that (2) holds,

$$\begin{aligned} \cos(2\theta) &= \mathrm{Re}(e^{i2\theta}) \\ &= \mathrm{Re}(\cos^2(\theta) - \sin^2(\theta) + 2i\sin(\theta)) \\ &= \cos^2(\theta) - \sin^2(\theta) \end{aligned}$$

as desired. $\square$

Although we have a nice power series form for e^z for any complex number z, another way that we can express is as follows. Let $z = x + yi$ for some $x, y \in \mathbb{R}$. Then

$$e^z = e^{x+yi} = e^x e^{yi} = e^x(\cos(y) + i\sin(y))$$

With Euler's formula, we are now able to solve equations of the form $z^n = 1$ for some $n \in \mathbb{N}$. Let $z = re^{i\theta}$ be a nonzero complex number. Then for every $n \in \mathbb{N}$, the n power z^n of z is given as

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$$

However, is there a formula for all the nth roots of z? To solve this problem, let $\zeta = \sqrt[n]{z}$. Suppose that $\zeta = \rho e^{i\varphi}$. Then

$$\zeta^n = z \implies \rho^n e^{in\theta} = re^{i\theta}$$

Thus,

$$\begin{aligned} \rho^n &= r \\ \varphi &= \frac{1}{n}(\theta + 2k\pi) \end{aligned}$$

where $k = 0, 1, 2, ..., n-1$. Thus, there are n distinct nth roots of z given by

$$\zeta_k = \sqrt[n]{r}e^{\frac{i(\theta+2k\pi)}{n}}$$

for all $k = 0, 1, 2, ..., n-1$.

Example 1.2.14. Consider the equation $z^4 = 1$. Then since $1 = 1e^0$, we have $r = 1$ and $\theta = 0$. Thus, the four 4th roots of unity are

$$\zeta_k = e^{\frac{i(2k\pi)}{4}} = e^{\frac{ik\pi}{2}}$$

for all $k = 0, 1, 2, 3$. Therefore,

$$\zeta_0 = 1, \quad \zeta_1 = e^{\frac{i\pi}{2}} = i, \quad \zeta_2 = -1, \quad \zeta_3 = e^{\frac{i3\pi}{2}} = -i$$

as desired.

1.3 Basic Topology on $\mathbb{C}$

In order to study complex functions and their differentiability, it is important to understand the topological structure of the complex plane. Many fundamental concepts in complex analysis, such as continuity, limits, holomorphicity, and contour integration relies on a precise understanding of open sets, connectedness, and regions in $\mathbb{C}$.

In this section, we introduce the basic topological notions required to study functions of a complex variable. We define open disks and open sets, explore connectedness in the complex plane, and describe what it means for a subset of $\mathbb{C}$ to be a domain or region.

Usually on a Cartesian plane, we call the axes, the x-axis and the y-axis. However, in the complex plane, we have something different.

Definition 1.3.1. On the complex plane, we call the x-axis the *real axis*, and the y-axis, the *imaginary axis*.

Points in the complex plane are represented as vectors. Given a complex number $z = x + yi$, for some $x, y \in \mathbb{R}$, we may associate to z the point (x, y) in the Cartesian plane. Geometrically, this corresponds to the vector from the origin $(0, 0)$ to the point (x, y), which provides a visual representation of the complex number z.

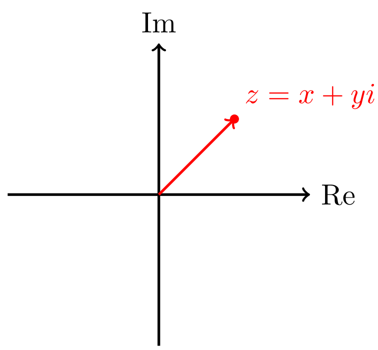

Figure 1.1: The complex number $z = x + yi$ represented as a vector in the complex plane.

Definition 1.3.2. indexopen disk Let $z_0 \in \mathbb{C}$ and let $r > 0$. Then we define the set $D(z_0, r)$ by

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

We call $D(z_0, r)$ the *open disk with centre* z_0 *and radius* r. We also call it a *neighbourhood of* z_0. We call a subset $A \subseteq \mathbb{C}$ an *open set* if for every $z_0 \in A$ there exists an $r > 0$ such that $D(z_0, r) \subseteq A$.

Analogously, we have closed sets in the complex plane as well.

Definition 1.3.3. Let $z_0 \in \mathbb{C}$ and let $r > 0$. Then we define the set $D[z_0, r]$ by

$$D[z_0, r] = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

We call $D[z_0, r]$ the *closed disk with centre* z_0 *and radius* r. We call a subset $A \subseteq \mathbb{C}$ a *closed set* if its complement $A^c = \mathbb{C} \setminus A$ is open.

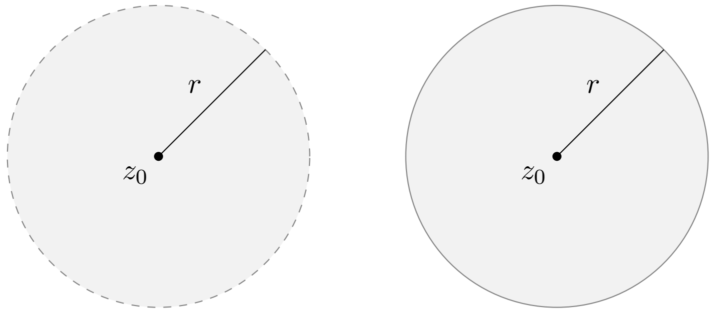

Figure 1.2: Comparison of an open disk (left), which excludes the boundary, and a closed disk (right), which includes the boundary. Both are centered at z_0 with radius r.

Remark 1.3.4. The topology on $\mathbb{C}$ is inherited from its identification with $\mathbb{R}^2$. Just as open intervals form the basic open sets in $\mathbb{R}$, open disks form the basic open sets in $\mathbb{C}$. Intuitively, a set in $\mathbb{C}$ is open if around every point in the set, there is a small open disk that is entirely contained within the set, just as in $\mathbb{R}$, every point in an open set is surrounded by a small open interval.

Example 1.3.5. The complex plane $\mathbb{C}$ is an open set. Indeed, given any point $z_0 \in \mathbb{C}$, we can construct an open disk $D(z_0, r)$ centered at z_0 with any radius $r > 0$, and this disk will be entirely contained in $\mathbb{C}$. Therefore, by definition, $\mathbb{C}$ is open.

Example 1.3.6. Let $A = \{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{C}$. Then we claim that A is an open set. To see this, let $z_0 \in A$ be arbitrary. Then $|z_0| < 1$, so there exists an $r > 0$ such that the open disk $D(z_0, r) \subseteq A$. Specifically, we may take $r = 1 - |z_0| > 0$, and for any $z \in D(z_0, r)$, we have by the triangle inequality,

$$|z| \leq |z - z_0| + |z_0| < r + |z_0| = (1 - |z_0|) + |z_0| = 1$$

Hence, $z \in A$, and so $D(z_0, r) \subseteq A$. Since z_0 was arbitrary, it follows that A is open.

Example 1.3.7. Let $B = \{z \in \mathbb{C} : \text{Im}(z) \geq 0\} \subseteq \mathbb{C}$. Then we claim that B is not an open set. To see this, consider $z_0 = 0 \in B$, but any open disk centered at 0 contains points with $\text{Im}(z) < 0$, so no such disk is fully contained in B, so z_0 is not an interior point of B.

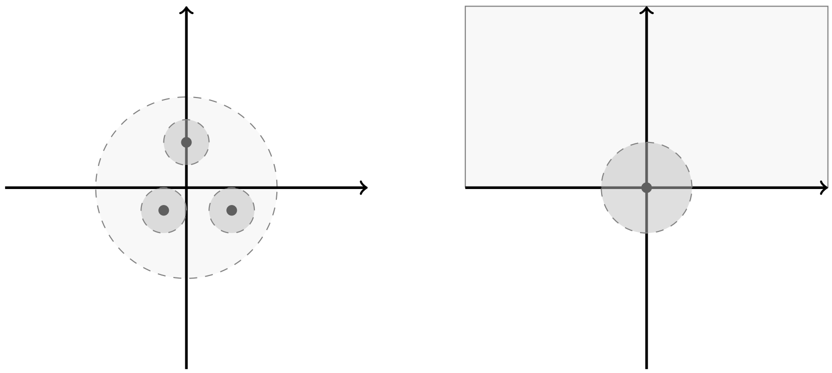

Figure 1.3: The figure on the left illustrates Example 1.3.6 where every point in the open unit disk has a surrounding disk contained entirely within the set. The figure on the right illustrates Example 1.3.7, where a boundary point on the real axis does not admit such a disk, and thus, the set is not open.

Before we can formally define domains and regions, it is important to understand the notion of connectedness. Roughly speaking, a set is connected if it cannot be split into two disjoint open pieces. In the context of the complex plane, this typically means that any two points in the set can be joined by a continuous path lying entirely within the set. Connectedness is essential for studying the behavior of complex functions, especially in results involving analytic continuation and integration.

Definition 1.3.8. We say that $A \subseteq \mathbb{C}$ is a *connected set* if for every pair of points $z, w \in A$, there exists a polygonal line that lies entirely in A.

Example 1.3.9. We claim that the set $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$ is a connected subset of $\mathbb{C}$. To see this, let $z, w \in A$ be arbitrary. Then $1 < |z|, |w| < 2$, so both points lie in the annulus centered at the origin with inner radius 1 and outer radius 2. Since A is an open annular region, we can construct a polygonal path from z to w that stays entirely within A. For example, one can connect z and w via line segments to avoid the inner and outer boundaries. Therefore, such a path lies entirely in A.

Example 1.3.10. We claim that the set $B = \{z \in \mathbb{C} : \mathrm{Re}(z) < 1\} \cup \{z \in \mathbb{C} : \mathrm{Re}(z) > 2\}$ is not a connected subset of $\mathbb{C}$. To see this, denote $B_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) < 1\}$ and $B_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) > 2\}$. Let $z \in B_1$ and $w \in B_2$. Then z and w lie in separate components of the set B. Assume for a contradiction that B is connected. Then there exists a polygonal line entirely contained in B that connects z to w. However, any such path from z to w must pass through a point whose real part lies in the interval $[1, 2]$, but this strip is not included in B, a contradiction. Therefore, no polygonal path between z and w can remain entirely within B.

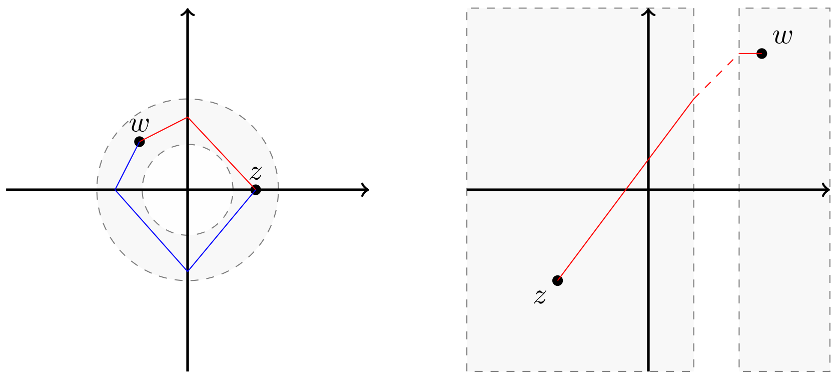

Figure 1.4: The figure on the left illustrates Example 1.3.9, where a polygonal line lies entirely within the annulus, showing that the set is connected. The figure on the right illustrates Example 1.3.10 where no polygonal path connecting z and w can remain inside the union of the two disconnected half-planes, so the set is not connected.

The most important sets that are of interest in this course is called the domain.

Definition 1.3.11. A subset $D \subseteq \mathbb{C}$ is said to be a *domain* if it is open and connected.

Example 1.3.12. Examples 1.3.5, 1.3.6, and 1.3.9 are domains since each set is open and connected, while Examples 1.3.7 and 1.3.10 are not since it is either not open or not connected.

Definition 1.3.13. Let $A \subseteq \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be such that every neighbourhood of z_0 intersects A and $\mathbb{C} \setminus A$. Then we call z_0 a boundary point of A. The set of all boundary points of A is denoted by ∂A.

Example 1.3.14. Consider the set $A = \mathbb{C} \setminus \{0\}$. Then the only boundary point of A is the origin 0. Indeed, every open disk centered at 0 contains points in A (since A is dense near 0) as well as the point $0 \notin A$, so $0 \in \partial A$. Any other point $z \neq 0$ has a neighborhood that lies entirely within A, so it is not a boundary point.

Example 1.3.15. Consider the set $B = \{z \in \mathbb{C} : 1 < |z| \leq 2\}$. Then the boundary of B is given by

$$\partial B = \{z \in \mathbb{C} : |z| = 1\} \cup \{z \in \mathbb{C} : |z| = 2\}.$$

Indeed, any point on either the inner circle $|z| = 1$ or the outer circle $|z| = 2$ satisfies the condition that every open disk centered at that point intersects both B and its complement. Therefore, these points are boundary points. No other points meet this criterion.

Definition 1.3.16. Let D be a domain with some, none, or all of its boundary points. Then D is called a *region*.

Chapter 2

Functions of a Complex Variable

The study of complex analysis centers around functions defined on subsets of the complex plane. While real-valued functions of a real variable form the foundation of single-variable calculus, complex-valued functions of a complex variable exhibit remarkably rich and elegant structure. In fact, the behaviour of complex functions is often far more rigid and constrained than their real counterparts, leading to deep and powerful theorems.

This chapter introduces the basic theory of functions of a complex variable. We begin by defining complex functions, their limits, and continuity. We then investigate the notion of differentiability, which in the complex setting leads to the concept of holomorphicity. Unlike real analysis, complex differentiability implies infinitely many additional properties, such as infinite differentiability, power series expansions, and strong constraints on the geometry and behaviour of the function.

The goal of this chapter is to lay the foundational holomorphic framework for the rest of the course, including the study of contour integration and power series.

2.1 Functions of a Complex Variable

According to the following definition, a complex function is characterized as a function that accepts complex numbers as its inputs and also produces them as its outputs.

Definition 2.1.1. A *complex function* is a function f whose domain and range are subsets of $\mathbb{C}$.

A complex function is also called a *complex-valued function of a complex variable*. For the most part, we will use the usual symbols f, g, h,... to denote complex functions.

Example 2.1.2. The formula $z^2-(1+i)z$ can be computed for any complex number z to produce a distinct complex result, establishing $f(z)=z^2-(1+i)z$ as a complex function. The values of f are determined using the complex number arithmetic covered in Section 1.1. For example, if $z=i$, then

$$f(i)=i^2-(1+i)i=-i$$

Example 2.1.3. The expression $g(z)=z+2\mathrm{Im}(z)$ also defines a complex function. For example, if $z=i$, then

$$g(i)=i+2\mathrm{Im}(i)=i+2$$

It is often helpful to express the inputs and the outputs of a complex function in terms of its real and imaginary parts. If $f:\mathbb{C}\to\mathbb{C}$ is a complex function, then the image of a complex number $z=x+yi$ under f is a complex number $u+iv$. By simplifying the expression $f(x+yi)$, we can write the real and imaginary variables u and v in terms of the real variables x and y. That is, we can express any complex function in terms of two real functions as

$$f(z)=f(x+yi)=u(x,y)+iv(x,y)$$

The functions $u(x,y)$ and $v(x,y)$ are called the real and imaginary parts of f. Of course, we can also express f as

$$f(z)=\mathrm{Re}(f(z))+\mathrm{Im}(f(z))$$

Example 2.1.4. For the complex function $f(z)=z^2$, if $z=x+yi$, then

$$f(x+yi)=(x+yi)^2=x^2+2xyi-y^2=(x^2-y^2)+2xyi$$

and thus, $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$.

Every complex function is determined by the real functions $u(x,y)$ and $v(x,y)$. That is, a function f can be defined without using the symbol z. We note that complex functions defined in terms of $u(x,y)$ and $v(x,y)$ can always be expressed, if desired, in terms of the operations on the symbols z and $\bar{z}$.

Up to this point, the real and imaginary parts of a complex function were determined using the Cartesian description $x+iy$ of the complex variable

z using either the polar form $z = r(\cos(\theta) + i\sin(\theta))$, or, equivalently, the exponential form $z = re^{i\theta}$. Given a complex function, if we replace the symbol z with $r(\cos(\theta) + i\sin(\theta))$, then we can write this function as

$$f(z) = u(r, \theta) + iv(r, \theta)$$

We still call the functions $u(r, \theta)$ and $v(r, \theta)$ the real and imaginary parts of f, respectively.

Example 2.1.5. If $f(z) = z^2$ and $z = r(\cos(\theta) + i\sin(\theta))$, then we have by De Moivre's Formula (Theorem 1.2.12),

$$f(z) = z^2 = r^2\cos(2\theta) + ir^2\sin(2\theta)$$

Then using the polar form of z, we have shown that the real and imaginary parts of $f(z) = z^2$ are

$$u(r, \theta) = r^2\cos(2\theta), \quad v(r, \theta) = r^2\sin(2\theta)$$

It is important to note that the functions $u(r, \theta)$ and $v(r, \theta)$ are not the same as the functions $u(x, y)$ and $v(x, y)$.

2.2 Limits and Continuity

The foundational principle in elementary calculus is the limit. Remember, $\lim_{x\to a} f(x) = L$ signifies that the outputs $f(x)$ of the function f can be made as near to the real number L as desired, given that x is chosen sufficiently near—but not equal—to a. In real analysis, limits underpin the definitions of continuity, derivatives, and definite integrals.

Similarly, complex limits hold a vital position in complex analysis. The notion of a complex limit mirrors that of a real limit: $\lim_{z\to z_0} f(z) = L$ indicates that the values $f(z)$ of the function f can approach, but not reach, the complex number z_0.

Definition 2.2.1. Suppose a complex function f is defined in a deleted neighbourhood of z_0, i.e. f is defined on $\mathbb{C} \setminus \{z_0\}$, and suppose that $L \in \mathbb{C}$. The *limit of f as z approaches z_0 exists and is equal to L*, written as $\lim_{z\to z_0} f(z) = L$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - L| < \varepsilon$.

For limits of a complex function, z is allowed to approach z_0 from *any* direction in the complex plane, that is, along any curve or path through z_0. In order that $\lim_{z\to z_0} f(z) = L$, we require that $f(z)$ approach the same complex number L along every possible curve through z_0.

Proposition 2.2.2. *If f approaches two complex numbers $L_1 \neq L_2$ for any two different curves or paths through z_0, then $\lim_{z\to z_0} f(z)$ does not exist.*

Proof. By definition, the limit $\lim_{z\to z_0} f(z)$ exists if and only if $f(z)$ approaches the same complex number L along every path approaching z_0. If $f(z) \to L_1$ along one path and $f(z) \to L_2$ along another, with $L_1 \neq L_2$, then $f(z)$ does not tend to a single value as $z \to z_0$. Therefore, the limit does not exist. □

Example 2.2.3. We claim that the limit $\lim_{z\to 0} \frac{z}{\bar{z}}$ does not exist. To see this, we will find two different ways of letting $z \to 0$ that yield different values for $\lim_{z\to 0} \frac{z}{\bar{z}}$. First, let $z \to 0$ along the real axis. Then we consider complex numbers of the form $z = x + 0i$ where $x \in \mathbb{R}$ approaching 0. Then

$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{x\to 0} \frac{x+0i}{x-0i} = \lim_{x\to 0} 1 = 1$$

On the other hand, if we let $z \to 0$ along the imaginary axis, then we consider complex numbers of the form $z = 0 + yi$ where $y \in \mathbb{R}$ approaching 0. Then

$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{y\to 0} \frac{0+yi}{0-yi} = \lim_{y\to 0}(-1) = -1$$

Since both values are not the same, then by Proposition 2.2.2, $\lim_{z\to 0} \frac{z}{\bar{z}}$ does not exist.

Of course, as in real analysis, we can also prove that the limit exists using the ε-δ definition.

Example 2.2.4. We claim that $\lim_{z\to 1+i}(2+i)z = 1+3i$. To see this, let $\varepsilon > 0$ be arbitrary. We seek a $\delta > 0$ such that if $|z-(1+i)| < \delta$, then $|(2+i)z-(1+3i)| < \varepsilon$. Indeed, first observe that

$$\begin{aligned}
|(2+i)z-(1+3i)| &= |2+i|\left|z - \frac{1+3i}{2+i}\right| \\
&= \sqrt{5}\left|z - \frac{1+3i}{2+i}\right| \\
&= \sqrt{5}|z-(1+i)| \\
&< \sqrt{5}\delta
\end{aligned}$$

Therefore, choosing $\delta = \frac{\varepsilon}{\sqrt{5}}$, we obtain that

$$|(2+i)z-(1+3i)| < \sqrt{5}\cdot\frac{\varepsilon}{\sqrt{5}} = \varepsilon$$

Therefore, as $\varepsilon > 0$ was arbitrary, $\lim_{z\to 1+i}(2+i)z = 1+3i$, as desired.

We now present the following result, which relates real limits of $u(x, y)$ and $v(x, y)$ with the complex limit of $f(z) = u(x, y) + iv(x, y)$.

Theorem 2.2.5. *Suppose that* $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ *and* $L = u_0 + iv_0$. *Then* $\lim_{z \to z_0} f(z) = L$ *if and only if*

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

Proof. Assume that $\lim_{z\to z_0} f(z) = L$, and let $\varepsilon > 0$ be arbitrary. Then there exists a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - L| < \varepsilon$. Using the identifications $f(z) = u(x, y) + iv(x, y)$ and $L = u_0 + iv_0$, then

$$|f(z) - L| = \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2}$$

Furthermore, since $0 \leq (v(x, y) - v_0)^2$, it follows that

$$|u(x, y) - u_0| = \sqrt{(u(x, y) - u_0)^2} \leq \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2}$$

Thus, for all $z = x + iy$, we have

$$|u(x, y) - u_0| \leq |f(z) - L|$$

In particular, if $|f(z) - L| < \varepsilon$, then $|u(x, y) - u_0| < \varepsilon$. Now by making the identifications $z = x + iy$ and $z_0 = x_0 + iy_0$, we also find that

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Therefore, if $|z-z_0| < \delta$, we have $|u(x, y)-u_0| < \varepsilon$, and thus, $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$, as desired. The proof in showing the second limit is almost identical.

Conversely, assume that $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$. We need to show that $\lim_{z\to z_0} f(z) = L$. Indeed, let $\varepsilon > 0$ be arbitrary. Since $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$, there exists a $\delta_1 > 0$ such that if $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1$, then

$$|u(x, y) - u_0| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$, then there exists a $\delta_2 > 0$ such that if $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2$, then

$$|v(x, y) - v_0| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|z - z_0| < \delta$

$$|u(x, y) - u_0| + |v(x, y) - v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

On the other hand, if $f(z) = u(x, y) + iv(x, y)$ and $L = u_0 + iv_0$, then by the triangle inequality,

$$|f(z) - L| \leq |u(x, y) - u_0| + |v(x, y) - v_0| < \varepsilon$$

Therefore, as $\varepsilon > 0$ was arbitrary, we have $\lim_{z\to z_0} f(z) = L$, as desired. □

Example 2.2.6. Since $f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$, then by Theorem 2.2.5 with $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy + 1$, and $z_0 = 1 + i$, we have

$$u_0 = \lim_{(x,y)\to(1,1)} x^2 - y^2 = 0$$
$$v_0 = \lim_{(x,y)\to(1,1)} 2xy + 1 = 3$$

Therefore, $\lim_{z\to 1+i} z^2 + i = 3i$.

In addition to computing specific limits, Theorem 2.2.5 is also an important theoretical tool that allows us to derive many properties of complex limits from properties of real limits.

Theorem 2.2.7. *Let f, g be complex functions. If $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then*

1. *For any $\alpha \in \mathbb{C}$, $\lim_{z\to z_0} \alpha = \alpha$.*
2. *$\lim_{z\to z_0} z = z_0$.*
3. *For any $\alpha \in \mathbb{C}$, $\lim_{z\to z_0} \alpha f(z) = \alpha L$.*
4. *$\lim_{z\to z_0} f(z) + g(z) = L + M$.*
5. *$\lim_{z\to z_0} f(z)g(z) = LM$.*
6. *If $M \neq 0$, then $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$.*

Proof. To see that (1) holds, let $\varepsilon > 0$ be arbitrary. Since

$$|f(z) - \alpha| = |\alpha - \alpha| = 0 < \varepsilon$$

Then we may choose any $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - \alpha| < \varepsilon$. Therefore, as $\varepsilon > 0$ was arbitrary, we have $\lim_{z\to z_0} \alpha = \alpha$.

To see that (2) holds, let $\varepsilon > 0$ be arbitrary. Since

$$|f(z) - z_0| = |z - z_0|$$

We may choose $\delta = \varepsilon > 0$ so that if $|z - z_0| < \delta$, then $|f(z) - z_0| < \varepsilon$. Therefore, as $\varepsilon > 0$ was arbitrary, we have $\lim_{z\to z_0} z = z_0$.

To see that (3) holds, let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, $L = u_0 + iv_0$, and $c = a + ib$. Since $\lim_{z\to z_0} f(z) = L$, then by Theorem 2.2.5, $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$. Then

$$\lim_{(x,y)\to(x_0,y_0)} (au(x, y) - bv(x, y)) = au_0 - bv_0$$

and

$$\lim_{(x,y)\to(x_0,y_0)} (bu(x, y) - av(x, y)) = bu_0 + av_0$$

But then $\mathrm{Re}(\alpha f(z)) = au(x, y) - bv(x, y)$ and $\mathrm{Im}(\alpha f(z)) = bu(x, y) + av(x, y)$. Therefore,

$$\lim_{z\to z_0} \alpha f(z) = au_0 - bv_0 + i(bu_0 + av_0) = \alpha L$$

To see that (4) holds, let $f(z) = u_1(x, y) + iv_1(x, y)$, $g(z) = u_2(x, y) + iv_2(x, y)$, $z_0 = x_0 + iy_0$, $L = u_{1,0} + iv_{1,0}$, and $M = u_{2,0} + iv_{2,0}$. Since $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then by Theorem 2.2.5, we have for $i = 1, 2$

$$\lim_{(x,y)\to(x_0,y_0)} u_i(x, y) = u_{i,0}, \qquad \lim_{(x,y)\to(x_0,y_0)} v_i(x, y) = v_{i,0}$$

But then $\mathrm{Re}(f(z) + g(z)) = u_1(x, y) + u_2(x, y)$ and $\mathrm{Im}(f(z) + g(z)) = v_1(x, y) + iv_2(x, y)$. Therefore,

$$\lim_{z\to z_0} f(z) + g(z) = (u_{1,0} + iv_{1,0}) + (u_{2,0} + iv_{2,0}) = L + M$$

To see that (5) holds, let $f(z) = u_1(x, y) + iv_1(x, y)$, $g(z) = u_2(x, y) + iv_2(x, y)$, $z_0 = x_0 + iy_0$, $L = u_{1,0} + iv_{1,0}$, and $M = u_{2,0} + iv_{2,0}$. Since $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then by Theorem 2.2.5, we have for $i = 1, 2$

$$\lim_{(x,y)\to(x_0,y_0)} u_i(x, y) = u_{i,0}, \qquad \lim_{(x,y)\to(x_0,y_0)} v_i(x, y) = v_{i,0}$$

But then note that

$$\begin{aligned} f(z)g(z) &= (u_1(x, y) + iv_1(x, y))(u_2(x, y) + iv_2(x, y)) \\ &= u_1(x, y)u_2(x, y) + iu_1(x, y)v_2(x, y) + iu_2(x, y)v_1(x, y) - v_1(x, y)v_2(x, y) \end{aligned}$$

So then $\mathrm{Re}(f(z)g(z)) = u_1(x, y)u_2(x, y) - v_1(x, y)v_2(x, y)$ and $\mathrm{Im}(f(z)g(z)) = u_1(x, y)v_2(x, y) + u_2(x, y)v_1(x, y)$. Therefore,

$$\lim_{z\to z_0} f(z)g(z) = (u_{1,0}u_{2,0} - v_{1,0}v_{2,0}) + i(u_{1,0}v_{2,0} + u_{2,0}v_{1,0}) = LM$$

Finally, to see that (6) holds, assume that $M \neq 0$, and let $f(z) = u_1(x,y) + iv_1(x,y)$, $g(z) = u_2(x,y) + iv_2(x,y)$, $z_0 = x_0 + iy_0$, $L = u_{1,0} + iv_{1,0}$, and $M = u_{2,0} + iv_{2,0}$. Since $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then by Theorem 2.2.5, we have for $i = 1, 2$

$$\lim_{(x,y)\to(x_0,y_0)} u_i(x,y) = u_{i,0}, \qquad \lim_{(x,y)\to(x_0,y_0)} v_i(x,y) = v_{i,0}$$

But then note that

$$\begin{aligned}\frac{f(z)}{g(z)} &= \frac{u_1(x,y) + iv_1(x,y)}{u_2(x,y) + iv_2(x,y)} \\ &= \frac{u_1(x,y)u_2(x,y) - iu_1(x,y)v_2(x,y) + iu_2(x,y)v_1(x,y) + v_1(x,y)v_2(x,y)}{u_2^2(x,y) + v_2^2(x,y)} \\ &= \frac{u_1(x,y)u_2(x,y) + v_1(x,y)v_2(x,y)}{u_2^2(x,y) + v_2^2(x,y)} + i\frac{u_2(x,y)v_1(x,y) - u_1(x,y)v_2(x,y)}{u_2^2(x,y) + v_2^2(x,y)}\end{aligned}$$

So then $\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) = \frac{u_1(x,y)u_2(x,y)+v_1(x,y)v_2(x,y)}{u_2^2(x,y)+v_2^2(x,y)}$ and $\operatorname{Im}\left(\frac{f(z)}{g(z)}\right) = \frac{u_2(x,y)v_1(x,y)-u_1(x,y)v_2(x,y)}{u_2^2(x,y)+v_2^2(x,y)}$. Therefore,

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{u_{1,0}u_{2,0} + v_{1,0}v_{2,0}}{u_{2,0}^2 + v_{2,0}^2} + i\frac{u_{2,0}v_{1,0} - u_{1,0}v_{2,0}}{u_{2,0}^2 + v_{2,0}^2} = \frac{L}{M}$$

□

Example 2.2.8. Using the limit properties in Theorem 2.2.7, we have

$$\lim_{z\to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{\lim_{z\to i}(3+i)z^4 - z^2 + 2z}{\lim_{z\to i} z+1} = \frac{4+3i}{1+i} = \frac{7}{2} - \frac{1}{2}i$$

Example 2.2.9. In order to find $\lim_{z\to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}}$, note that we cannot directly apply Theorem 2.2.7 just yet, since the limit as $z \to 1 + \sqrt{3}i$ approaches 0 in both the numerator and the denominator. However, note that $1 + \sqrt{3}i$ is a root of the quadratic polynomial $z^2 - 2z + 4$, so we can factor the polynomial to obtain

$$z^2 - 2z + 4 = \left(z - 1 + \sqrt{3}i\right)\left(z - 1 - \sqrt{3}i\right)$$

Thus, now we have

$$\lim_{z\to 1+\sqrt{3}i} \frac{\left(z - 1 + \sqrt{3}i\right)\left(z - 1 - \sqrt{3}i\right)}{z - 1 - \sqrt{3}i} = \lim_{z\to 1+\sqrt{3}i} z - 1 + \sqrt{3}i = 2\sqrt{3}i$$

The definition of continuity for a complex function is, in essence, the same as that for a real function. That is, a complex function f is continuous at a point z_0 if the limit of f as z approaches z exists and is the same as the value of f at z_0. This gives us the following definition for complex functions.

Definition 2.2.10. A complex function f is *continuous* at a point $z_0 \in \mathbb{C}$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

A complex function f is said to be *continuous* if f is continuous at every point $z_0 \in \mathbb{C}$.

If a complex function is not continuous at a point z_0, then we say that f is discontinuous at z_0. For example, the function $f(z) = \frac{1}{1+z^2}$ is discontinuous at the points $z = i$ and $z = -i$.

Since the concept of continuity is defined using the complex limit, various properties of complex limits can be translated into statements about continuity. Consider Theorem 2.2.5, which describes the connection between the complex limit of $f(z) = u(x, y) + iv(x, y)$, and the real limits of u and v.

Theorem 2.2.11. *Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions $u(x, y)$ and $v(x, y)$ are continuous at the point (x_0, y_0).*

Proof. Assume that the complex function f is continuous at z_0. Then by definition, we have

$$\lim_{z \to z_0} f(z) = f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$$

Then by Theorem 2.2.5, we have

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u(x_0, y_0), \qquad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v(x_0, y_0)$$

Therefore, both u and v are continuous at (x_0, y_0).

Conversely, if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u(x_0, y_0), \qquad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v(x_0, y_0)$$

It then follows from Theorem 2.2.5 that

$$\lim_{z \to z_0} f(z) = f(z_0) = u(x_0, y_0) + iv(x_0, y_0) = f(z_0)$$

Therefore, f is continuous. □

Example 2.2.12. According to Theorem 2.2.11, $f(z) = \bar{z} = x - iy$ is a continuous at $z_0 = x_0 + iy_0$ if both $u(x, y) = x$ and $v(x, y) = -y$ are continuous at (x_0, y_0). Since $u(x, y)$ and $v(x, y)$ are polynomial functions, then it follows

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = x_0, \quad \lim_{(x,y)\to(x_0,y_0} v(x, y) = -y_0$$

This implies that both $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), and therefore that f is continuous at z_0 by Theorem 2.2.11.

The algebraic properties of complex limits from Theorem 2.2.7 can also be restated in terms of the continuity of complex functions.

Corollary 2.2.13. *If f and g are continuous at the point z_0, then the following functions are continuous at z_0.*

1. *αf for all $\alpha \in \mathbb{C}$.*
2. *$f + g$.*
3. *fg.*
4. *$\frac{f}{g}$ if $g(z_0) \neq 0$.*

Of course, the results of Corollary 2.2.13 (3) and (4) extend to any finite sum or product of continuous functions. We can use these facts to show that polynomials are continuous functions.

Theorem 2.2.14. *Polynomial functions are continuous on the entire complex plane $\mathbb{C}$.*

Proof. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial function and let $z_0 \in \mathbb{C}$ be arbitrary. By Theorem 2.2.7 (2), $f(z) = z$ is continuous at z_0, and by repeated application of Corollary 2.2.13, this implies that the power function $f(z) = z^n$ where $n \in \mathbb{N}$, is continuous at this point as well. Moreover, Theorem 2.2.7 (1) implies that every complex constant $f(z) = c$ is continuous at z_0, and so it follows from Corollary 2.2.13 that each of the functions $a_n z^n$, $a_{n-1} z^{n-1}$,..., a_0 are continuous at z_0. Now from a repeated application of Corollary 2.2.13 (2), we have that $p(z)$ is continuous at z_0. As $z_0 \in \mathbb{C}$ was arbitrary, $p(z)$ is continuous at every $z_0 \in \mathbb{C}$, so $p(z)$ is continuous. $\square$

2.3 Differentiability and Holomorphic Functions

The calculus concerning complex functions addresses the standard notions of derivatives and integrals associated with these functions. In this segment, we introduce the limit definition of the derivative for a complex function $f(z)$. While several concepts, like the product, quotient, and chain differentiation rules, may appear familiar, key distinctions exist between this content and the calculus applicable to real functions.

Suppose $z = x + iy$ and $z_0 = x_0 + iy_0$. Then the chance in z_0 is the difference $\Delta z = z - z_0$ or $\Delta z = (x - x_0) + i(y - y_0) = \Delta x + i\Delta y$. If a complex function $f(z)$ is defined at z and z_0, then the corresponding change in the function is the difference

$$\Delta f = f(z_0 + \Delta z) - f(z_0)$$

The derivative of the function f is defined in terms of a limit of the difference quotient $\frac{\Delta f}{\Delta z}$ as $\Delta z \to 0$.

Definition 2.3.1. Let $D \subseteq \mathbb{C}$ be a domain, let $f : D \to \mathbb{C}$, and let $z_0 \in D$. Then the *derivative of f at z_0*, denoted by $f'(z_0)$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists. If the limit exists, then the function f is said to be *differentiable* at z_0.

Two symbols denoting the derivative of $f(z)$ are $f'(z)$ and $\frac{df}{dz}$. If the latter notation is used, then the value of a derivative at a specified point z_0 is written $\left.\frac{df}{dz}\right|_{z=z_0}$.

Example 2.3.2. If $f(z) = z^2 - 5z$, using Definition 2.3.1, first note we have $f(z + \Delta) = 5z - 5\Delta z$, and

$$f(z + \Delta z) - f(z) = 2z\Delta z + (\Delta z)^2 - 5\Delta z$$

Therefore,

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2 - 5\Delta z}{\Delta z} \\ &= \lim_{\Delta \to 0} \frac{\Delta z(2z + \Delta z - 5)}{\Delta z} \\ &= \lim_{\Delta \to 0} 2z + \Delta z - 5 \\ &= 2z - 5 \end{aligned}$$

so $f'(z) = 2z - 5$.

Of course, the familiar rules of differentiation in the calculus of real variables carry over to the calculus of complex variables. Using Definition 2.3.1, we obtain the following.

Proposition 2.3.3 (Differentiation Rules). *If f and g are differentiable at a point z, and $\alpha \in \mathbb{C}$, then*

1. *(Constant Rule)* $\frac{d}{dz}\alpha = 0$.
2. *(Scalar Multiple Rule)* $\frac{d}{dz}\alpha f(z) = \alpha f'(z)$.
3. *(Power Rule)* $\frac{d}{dz}z^n = nz^{n-1}$ *for all* $n \in \mathbb{Z}$.
4. *(Sum Rule)* $\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z)$.
5. *(Product Rule)* $\frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z)$.
6. *(Quotient Rule)* $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{f'(z)g(z)-f(z)g'(z)}{g^2(z)}$.
7. *(Chain Rule)* $\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$.

Example 2.3.4. If $f(z) = 3z^4 - 5z^3 + 2z$, then using Proposition 2.3.3, we have

$$f'(z) = 12z^3 - 15z^2 + 2$$

For a complex function f to be differentiable at a point z_0, then $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ must exist and equal the same complex number from any direction. This means that in complex analysis, the requirement for differentiability of a function $f(z)$ at a point z_0 is a far greater demand than in real calculus of functions where we can approach a real number on the number line from only two directions.

Example 2.3.5. We claim that the function $f(z) = x + 4iy$ is not differentiable at any point z. To see this, let $z = x + yi \in \mathbb{C}$ be arbitrary with $\Delta z = \Delta x + i\Delta y$. Then we have

$$f(z + \Delta z) - f(z) = (x + \Delta x) + 4i(y + \Delta y) - x - 4iy = \Delta x + 4i\Delta y$$

and so

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$$

Now we consider the following cases.

Case 1: If $\Delta z \to 0$ along a line parallel to x-axis. Then $\Delta y = 0$, $\Delta z = \Delta x$, and

$$\lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = 1$$

Case 2: If $\Delta z \to 0$ along a line parallel to y-axis. Then $\Delta x = 0$, $\Delta z = i\Delta y$ and

$$\lim_{\Delta z \to 0} \frac{4i\Delta y}{i\Delta y} = 4$$

From Cases 1 and 2, the values of the limits are different, so $f(z) = x + 4iy$ is nowhere differentiable.

Even though the requirements of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements. These functions are called holomorphic functions.

Definition 2.3.6. Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ be a function, and $z_0 \in D$. Then f is said to be *holomorphic at* z_0 if f is differentiable at z_0 and at every point in some neighbourhood of z_0. We also say that f is *holomorphic in* D if it is holomorphic at every point in D.

Remark 2.3.7. It is important that holomorphicity at a point is *not* the same as differentiability at a point. Holomorphicity at a point is a neighbourhood property. In other words, holomorphicity is a property that is defined over an open set.

Example 2.3.8. We claim that the function $f(z) = |z|^2$ is differentiable at 0, but is nowhere differentiable anywhere else. To see that f is differentiable, at $z = 0$, we have $f(0) = 0$ and if $\Delta z = \Delta x + i\Delta y$,

$$f(\Delta z) - f(0) = |\Delta x + i\Delta y|^2 - 0 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x + i\Delta y)(\Delta x - i\Delta y)$$

Then,

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(\Delta x + i\Delta y)(\Delta x - i\Delta y)}{\Delta x + i\Delta y} \\ &= \lim_{\Delta z \to 0} \Delta x - i\Delta y \\ &= 0 \end{aligned}$$

Therefore, f is differentiable at 0. However, f is not holomorphic at 0 since there exists no neighbourhood of $z = 0$ through which f is differentiable; hence, the function $f(z) = |z|^2$ is nowhere holomorphic.

In contrast, the simple polynomial $f(z) = z^2$ is holomorphic at every point in the complex plane, so f is holomorphic everywhere.

As in real analysis, if a function f is differentiable at a point, the function is necessarily continuous at the point.

Proposition 2.3.9. *Let $D \subseteq \mathbb{C}$ be a domain. If $f : D \to \mathbb{C}$ is differentiable at $z_0 \in D$, then f is continuous at z_0.*

Proof. The limits $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ and $\lim_{z\to z_0} z - z_0$ exist and equal $f'(z_0)$ and 0, respectively. Hence, by Theorem 2.2.7 (5), we can write

$$\begin{aligned}\lim_{z\to z_0} f(z) - f(z_0) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}(z-z_0) \\ &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \lim_{z\to z_0}(z-z_0) = f'(z_0)0 = 0\end{aligned}$$

From $\lim_{z\to z_0} f(z) - f(z_0) = 0$, we conclude that $\lim_{z\to z_0} f(z) = f(z_0)$. Therefore, f is continuous at z_0. □

Remark 2.3.10. Of course, the converse of Proposition 2.3.9 is false. It follows from Theorem 2.2.11 and Example 2.3.5 that the function $f(z) = x + 4iy$ is continuous everywhere, but not differentiable anywhere.

2.4 The Cauchy–Riemann Equations

In the previous section, we saw that a function f of a complex variable is holomorphic at a point whenever f is differentiable at that point, and differentiable at every point in some neighbourhood of that point. This requirement is more stringent than simply differentiability at a point since a complex function can be differentiable at a point, but nowhere differentiable.

In the following theorem, we see that if $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point z, then the functions u and v must satisfy a pair of equations that relate their first order partial derivatives.

Theorem 2.4.1 (The Cauchy–Riemann Equations). *Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z, the first order partial derivatives of u and v exist, and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof. By the definition of the derivative, we have at z

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

By writing $f(z) = u(x,y) + iv(x,y)$ and $\Delta z = \Delta x + i\Delta y$, the above becomes

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

Since the limit is assumed to exist, Δz can approach 0 from any direction.

Case 1: If $\Delta \to 0$ along a horizontal line. Then $\Delta y = 0$, $\Delta z = \Delta x$, and

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i(v(x+\Delta x, y) - v(x,y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta y} \end{aligned}$$

The existence of $f'(z)$ implies that each limit above exists. These limits are the definitions of the first order partial derivatives with respect to x of u and v, respectively. Hence, we have shown that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exist at the point z, and that the derivative of f is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Case 2: If $\Delta z \to 0$ along a vertical line. Then $\Delta x = 0$, $\Delta z = i\Delta y$, and by a similar computation,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

In this case, the above shows that $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ exist at z and that

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Therefore, by Case 1 and 2, we obtain that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Thus, by equating the real and imaginary parts, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

as desired. □

Remark 2.4.2. Since Theorem 2.4.1 states that the Cauchy–Riemann equations hold at z as a necessary consequence of f being differentiable at z, we cannot use the theorem to help us determine where f is differentiable. However, it is important to realize that Theorem 2.4.1 tells us where a function f does not possess a derivative. If the Cauchy–Riemann equations are not satisfied at the a point z, then f cannot be differentiable at z.

Example 2.4.3. The polynomial function $f(z) = z^2 + z$ is holomorphic for all z and can be written as

$$f(z) = (x^2 - y^2 + x) + i(2xy + y)$$

Therefore, $u(x, y) = x^2 - y^2 + x$ and $v(x, y) = 2xy + y$. For any point (x, y) in the complex plane, we see that the Cauchy–Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Example 2.4.4. We claim that the function $f(z) = (2x^2 + y) + i(y^2 - x)$ is not holomorphic at any point. Indeed, identifying $u(x, y) = 2x^2 + y$ and $v(x, y) = y^2 - x$, then since

$$\frac{\partial u}{\partial x} = 4x \quad \frac{\partial v}{\partial y} = 2y$$
$$\frac{\partial u}{\partial y} = 1 \quad \frac{\partial v}{\partial x} = -1$$

we see that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, the equality $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ is satisfied only on the line $y = 2x$. However, for any point z on the line, there is no neighbourhood or open disk about z in which f is differentiable at every point. Therefore, f is nowhere holomorphic.

By themselves, the Cauchy–Riemann equations do not ensure holomorphicity of a function $f(z) = u(x, y) + iv(x, y)$ at a point $z = x + iy$. It is possible for the Cauchy–Riemann equations to be satisfied at z, and yet, $f(z)$ may not be differentiable at z, or $f(z)$ may be differentiable at z, but nowhere else. In either case, f is not holomorphic at z.

However, when we add the condition of continuity to u and v and to the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$, it can be shown that the Cauchy–Riemann equations are not only necessary, but also sufficient to guarantee holomorphicity of $f(z) = u(x, y) + iv(x, y)$. The proof is long and complicated, so we only state the result.

Theorem 2.4.5. *Suppose the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first order partial derivatives in a domain D. If $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is holomorphic in D.*

Example 2.4.6. For the function $f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$, the real functions $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = -\frac{y}{x^2+y^2}$ are continuous, except at the point where $x^2 + y^2 = 0$, that is, at $z = 0$. Moreover, the first four first order partial derivatives

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}$$
$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2} \quad \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

are continuous except at $z = 0$. Finally, we see from

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}$$

that the Cauchy–Riemann equations are satisfied except at $z = 0$. Thus, we conclude from Theorem 2.4.5 that f is holomorphic in any domain D that does not contain the point $z = 0$.

We saw in Section 2.1 that a complex function can be expressed in terms of polar coordinates. Indeed, the form $f(z) = u(r, \theta) + iv(r, \theta)$ is often more convenient to use.

Theorem 2.4.7 (Cauchy–Riemann Equations, Polar Form). *Suppose $f(z) = u(r, \theta) + iv(r, \theta)$ is differentiable at a point $z = r(\cos(\theta) + i\sin(\theta))$. Then at z, the first order partial derivatives of u and v exist, and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof. Suppose $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The first order partial derivatives of x and y, with respect to r and θ are given by

$$\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta), \quad \frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r\cos(\theta)$$

If $f(z) = u(x, y) + iv(x, y)$, we have by the chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta), \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin(\theta) + \frac{\partial u}{\partial y}r\cos(\theta), \\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta), \\
\frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}r\sin(\theta) + \frac{\partial v}{\partial y}r\cos(\theta)
\end{aligned}$$

By the Cauchy–Riemann equations for Cartesian coordinates (Theorem 2.4.1), we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and so, we have

$$\begin{aligned}
\frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial y}\cos(\theta) + \frac{\partial u}{\partial x}\sin(\theta) \\
\frac{\partial v}{\partial \theta} &= r\left(\frac{\partial u}{\partial y}\sin(\theta) + \frac{\partial u}{\partial x}\cos(\theta)\right)
\end{aligned}$$

The second expression implies

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}\sin(\theta) + \frac{\partial u}{\partial x}\cos(\theta) = \frac{\partial u}{\partial r}$$

Similarly, the first expression implies

$$\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos(\theta) + \frac{\partial u}{\partial x}\sin(\theta) = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Therefore, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

as desired. □

In the proof of the Cauchy–Riemann equations in Cartesian coordinates, we have described the derivative of f as

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

We can also derive the derivative of f in terms of its polar coordinates as well.

Proposition 2.4.8. *If $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at a point $z = r(\cos(\theta) + i\sin(\theta))$, then the derivative of f at (r,θ) is*

$$f'(z) = (\cos(\theta) - i\sin(\theta))\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Proof. Using the expressions derived above

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta),\\
\frac{\partial u}{\partial \theta} &= -\frac{\partial u}{\partial x}r\sin(\theta) + \frac{\partial u}{\partial y}r\cos(\theta),\\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta),\\
\frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x}r\sin(\theta) + \frac{\partial v}{\partial y}r\cos(\theta)
\end{aligned}$$

We first solve for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Using the first two equations,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta),\\
\frac{\partial u}{\partial \theta} &= -\frac{\partial u}{\partial x}r\sin(\theta) + \frac{\partial u}{\partial y}r\cos(\theta)
\end{aligned}$$

we can determine that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cos(\theta) - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin(\theta)\\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin(\theta) + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos(\theta)
\end{aligned}$$

Similarly, solving for $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ using the last two equations

$$\begin{aligned}
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta),\\
\frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x}r\sin(\theta) + \frac{\partial v}{\partial y}r\cos(\theta)
\end{aligned}$$

we can determine that

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\cos(\theta) - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin(\theta)\\
\frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\sin(\theta) + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos(\theta)
\end{aligned}$$

Now, from the expression $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, we have by Theorem 2.4.7,

$$\begin{aligned}
f'(z) &= \left(\frac{\partial u}{\partial r}\cos(\theta) - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin(\theta)\right) + i\left(\frac{\partial v}{\partial r}\cos(\theta) - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin(\theta)\right) \\
&= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos(\theta) + \left(-\frac{1}{r}\frac{\partial u}{\partial \theta} - i\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin(\theta) \\
&= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos(\theta) + \left(\frac{\partial v}{\partial r} - i\frac{\partial u}{\partial r}\right)\sin(\theta) \\
&= (\cos(\theta) - i\sin(\theta))\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \\
&= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)
\end{aligned}$$

as desired. $\square$

Chapter 3

Elementary Functions

In the previous chapter, we defined a class of functions that is of the most interest in complex analysis, the holomorphic functions. In this chapter, we will investigate the complex exponential, logarithmic, power, and trigonometric functions.

3.1 Complex Exponential Function

We briefly mentioned the complex exponential function in Section 1.2. We recall the definition of the complex exponential function.

Definition 3.1.1. The function e^z defined by

$$e^z = e^x \cos(y) + ie^x \sin(y)$$

is called the *complex exponential function*.

Remark 3.1.2. We call e^z the complex exponential because it agrees with the real exponential function when z is real, i.e. if $z = x + 0i$, then

$$e^x = e^x(\cos(0) + i\sin(0)) = e^x$$

The complex exponential function also shares important differential properties of the real exponential functions.

Proposition 3.1.3. *The exponential function e^z is holomorphic everywhere in $\mathbb{C}$ and its derivative is given by*

$$\frac{d}{dz}e^z = e^z$$

Proof. To see that e^z is holomorphic everywhere in $\mathbb{C}$, we use Theorem 2.2.14. We first note that $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$ which are continuous real functions and have continuous first-order partial derivatives for all (x,y). Moreover,

$$\frac{\partial u}{\partial x} = e^x\cos(y) = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x\sin(y) = -\frac{\partial v}{\partial x}$$

So the Cauchy–Riemann equations are satisfied. Therefore, e^z is holomorphic everywhere in $\mathbb{C}$. Moreover, the derivative of f is given as

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x\cos(y) + ie^x\sin(y) = e^z$$

□

Example 3.1.4. Of course, if $f(z) = iz^4(z^2 - e^z)$, then the derivative is easily determined by using the familiar differentiation rules

$$f'(z) = iz^4(2z - e^z) + 4iz^3(z^2 - e^z) = 6iz^5 - iz^4e^z - 4iz^3e^z$$

The modulus, argument, and conjugate of the exponential function are also easily determined. If we express $f(z) = e^z$ as

$$f(z) = e^x\cos(y) + ie^x\sin(y) = r(\cos(\theta) + i\sin(\theta))$$

then we see that $r = e^x$ and $\theta = y + 2k\pi$ for $k \in \mathbb{Z}$. Since r is the modulus and θ is the argument of $f(z)$, we have

$$|e^z| = e^x, \quad \arg(e^z) = y + 2k\pi$$

Furthermore, from calculus, $e^x > 0$ for all $x \in \mathbb{R}$, so $|e^z| > 0$ for all $z \in \mathbb{C}$.

A formula for the conjugate of the complex exponential e^z is found using properties of real cosine and sine functions. Since the real cosine function is even, we have $\cos(x) = \cos(-x)$ for all x and since the real sine function is odd, we have $-\sin(x) = \sin(-x)$ for all x, and so

$$\overline{e^z} = e^x\cos(y) - ie^x\sin(y) = e^x\cos(-y) + ie^x\sin(-y) = e^{x-iy} = e^{\bar{z}}$$

Therefore, for all $z \in \mathbb{C}$,

$$\overline{e^z} = e^{\bar{z}}$$

In Proposition 3.1.3, we have shown that differentiating the complex exponential function is, in essence, the same as differentiating the real exponential function. Of course, these two functions also share the following properties.

Proposition 3.1.5. *For all $z, w \in \mathbb{C}$,*

1. $e^0 = 1$.
2. $e^z e^w = e^{z+w}$.
3. $\frac{e^z}{e^w} = e^{z-w}$.
4. $(e^z)^n = e^{nz}$ *for all* $n \in \mathbb{Z}$.

Proof. The proof of (1) follows from the observation that the complex exponential agrees with the real exponential function for real input. That is, we have $e^{0+0i} = e^0$, and we know that for the real exponential function $e^0 = 1$.

To see that (2) holds, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Then

$$\begin{aligned} e^z e^w &= (e^{x_1}\cos(y_1) + ie^{x_1}\sin(y_1))(e^{x_2}\cos(y_2) + ie^{x_2}\sin(y_2)) \\ &= e^{x_1+x_2}(\cos(y_1)\cos(y_2) - \sin(y_1)\sin(y_2)) + ie^{x_1+x_2}(\sin(y_1)\cos(y_2) + \cos(y_1)\sin(y_2)) \end{aligned}$$

Using the addition formulas for real and cosine functions, we have

$$e^z e^w = e^{x_1+x_2}\cos(y_1+y_2) + ie^{x_1+x_2}\sin(y_1+y_2)$$

Thus, the right side of the above is e^{z+w}.

Note that (3) follows from (2) simply by replacing w with $-w$.

To see that (4) holds, first note that if $n = 0$, then by using De Moivre's Formula, with modulus $r = e^x$ and argument $\theta = y$,

$$(e^z)^0 = (e^x\cos(y) + ie^x\sin(y))^0 = (e^x)^0\cos(0 \cdot y) + i(e^x)^0\sin(0 \cdot y) = 1$$

Now we apply induction on $n \in \mathbb{N}$ to show that $(e^z)^n = e^{nz}$.

Base Case: If $n = 1$. Then we have by De Moivre's Formula,

$$\begin{aligned} (e^z)^1 &= (e^x\cos(y) + ie^x\sin(y))^1 \\ &= (e^x)^1\cos(1 \cdot y) + (e^x)^1\sin(1 \cdot y) \\ &= e^x\cos(y) + ie^x\sin(y) \\ &= e^{1 \cdot z} \end{aligned}$$

Inductive Step: Assume that for $n \in \mathbb{N}$, we have $(e^z)^n = e^{nz}$. Then for $n + 1$, we have that by (2)

$$\begin{aligned} e^{(n+1)z} &= e^{nz+z} \\ &= e^{nz}e^z \\ &= (e^z)^n e^z \\ &= (e^z)^{n+1} \end{aligned}$$

as desired.

It can be shown that for $n \in \mathbb{N}$, $(e^z)^{-n} = e^{-nz}$ also by using induction, in a similar process as above. $\square$

3.2 Complex Logarithmic Function

In real analysis, the natural logarithm $\ln(x)$ is often defined as an inverse function of the real exponential function e^x. We will use $\ln(x)$ to denote the real exponential function. Since the real exponential function is injective on its domain $\mathbb{R}$, there is no ambiguity involved in defining its inverse function. The situation is very different however in complex analysis, since the complex exponential function e^z is not one-to-one on its domain $\mathbb{C}$.

Lemma 3.2.1. *Given a nonzero complex number $z \in \mathbb{C}$, the equation $e^w = z$ has infinitely many solutions.*

Proof. Suppose $w = u + iv$ is a solution of $e^w = z$. Then it follows that $|e^w| = |z|$ and $\arg(e^w) = \arg(z)$. Furthermore, $e^u = |z|$ and $v = \arg(z)$, or equivalently, $u = \ln(|z|)$ and $v = \arg(z)$. Therefore, given a nonzero complex number z, we have

$$w = \ln(|z|) + i \arg(z)$$

Since there are infinitely many arguments of z, the above gives us infinitely many solutions w to the equation $e^w = z$. $\square$

The following definition summarizes this discussion.

Definition 3.2.2. The multiple-valued function $\log(z)$ defined by

$$\log(z) = \ln(|z|) + i \arg(z)$$

is called the *complex logarithm*.

Example 3.2.3. Given the equation $e^w = i$, we see that $|z| = 1$ and $\arg(z) = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$, so the solutions of the equation $e^w = i$ are given by

$$w = \log(i) = \ln(1) + i\left(\frac{\pi}{2} + 2k\pi\right) = \left(\frac{\pi}{2} + 2k\pi\right) i$$

for all $k \in \mathbb{Z}$.

Definition 3.2.2 can be used to prove that the complex logarithm satisfies the following identities, which are analogous to identities for the real logarithm.

Theorem 3.2.4. *If z and w are nonzero complex numbers and $n \in \mathbb{Z}$, then*

1. $\log(zw) = \log(z) + \log(w)$.
2. $\log\left(\frac{z}{w}\right) = \log(z) - \log(w)$.
3. $\log(z^n) = n\log(z)$.

Proof. To see that (1) is true, note that by Theorem 1.1.9 (9) and Remark 1.2.6, we have

$$|zw| = |z||w|, \quad \arg(zw) = \arg(z) + \arg(w)$$

Therefore, by definition of the complex logarithm, we have

$$\begin{aligned}\log(zw) &= \ln(|zw|) + i\arg(zw)\\ &= \ln(|z||w|) + i(\arg(z) + \arg(w))\\ &= \ln(|z|) + \ln(|w|) + i(\arg(z) + \arg(w))\\ &= \log(z) + \log(w)\end{aligned}$$

Note that (2) follows from (1) and (3) by replacing w with $\frac{1}{w}$, and noting that when $n = -1$, we have $\log(z^{-1}) = -\log(z)$. Thus, it suffices to show that (3) holds. First note that when $n = 0$, we have

$$\log(z^0) = \log(1) = \ln(1) + i\arg(1) = 2k\pi$$

for $k \in \mathbb{Z}$. In particular, when $k = 0$, we have $\log(z^0) = 0 = 0\log(z)$.

Now we will show by induction on $n \in \mathbb{N}$ that $\log(z^n) = n\log(z)$.

Base Case: $n = 1$. Then we have

$$\begin{aligned}\log(z^1) &= \ln(|z^1|) + i\arg(z^1)\\ &= \ln(|z|) + i\arg(z)\\ &= \log(z)\end{aligned}$$

Inductive Step: Assume for $n \in \mathbb{N}$, we have $\log(z^n) = n\log(z)$. Then for $n+1$, we have by (1),

$$\log(z^{n+1}) = \log(z^n z) = \log(z^n) + \log(z) = n\log(z) + \log(z) = (n+1)\log(z)$$

as desired. $\square$

It is interesting to note that the complex logarithm of a positive real number has infinitely many values. For example, $\log(5)$ is the number $1.6094 + 2k\pi i$ for $k \in \mathbb{Z}$, whereas the real logarithm $\ln(5)$ has a single value 1.6094. The unique value of $\ln(5)$ corresponding to $n = 0$ is the same as the value of the real logarithm $\ln(5)$. In general, this value of the complex logarithm is called the principal value of the logarithm, since it is found by using the principal argument $\text{Arg}(z)$ in place of the argument $\arg(z)$. We denote the principal value of the logarithm by $\text{Log}(z)$. Thus, the expression $f(z) = \text{Log}(z)$ defines a function, whereas $F(z) = \log(z)$ defines a multiple-valued function.

Definition 3.2.5. The complex function $\text{Log}(z)$ defined by

$$\text{Log}(z) = \ln(|z|) + i\,\text{Arg}(z)$$

is called the *principal logarithm*.

Example 3.2.6. For $z = i$, we have $|z| = 1$ and $\text{Arg}(z) = \frac{\pi}{2}$, so

$$\text{Log}(i) = \ln(1) + \frac{\pi}{2}i$$

For $z = -2$, we have $|z| = 2$ and $\text{Arg}(z) = \pi$, and so

$$\text{Log}(-2) = \ln(2) + \pi i$$

Remark 3.2.7. It is important to note that the identities for the complex logarithm in Theorem 3.2.4 are not necessarily satisfied by the principal logarithm. For example, it is not true that $\text{Log}(zw) = \text{Log}(z) + \text{Log}(w)$ for all complex numbers z and w.

Remark 3.2.8. Since $\text{Log}(z)$ is one of the values of the complex logarithm $\log(z)$, it follows that $e^{\text{Log}(z)} = z$ for all $z \neq 0$. This suggests that the logarithmic function $\text{Log}(z)$ is an inverse of the exponential function e^z. Since the complex exponential function is not one-to-one on its domain, this statement is not completely accurate. The exponential function must be restricted to a domain on which it is one-to-one in order to have a well-defined inverse. In particular e^z is a one-to-one function on the fundamental region $x \in \mathbb{R}$, $y \in (-\pi, \pi]$.

Proposition 3.2.9. *Let $z = x + iy$ be such that $x \in \mathbb{R}$ and $y \in (-\pi, \pi]$. If e^z is restricted to the fundamental region, then the principal logarithm* $\text{Log}(z)$ *is its inverse function.*

Proof. Let $z = x + yi$ be in the fundamental region. Since $|e^z| = e^x$ and $\arg(e^z) = y + 2k\pi$ for $k \in \mathbb{Z}$, then y is argument of e^z. Since y is in the fundamental region, then y is the principal argument of e^z, so $\operatorname{Arg}(e^z) = y$. In particular, since $\ln(e^x) = x$, we have

$$\begin{aligned}\operatorname{Log}(e^z) &= \ln(|e^z|) + i \operatorname{Arg}(z) \\ &= \ln(e^x) + iy \\ &= x + iy\end{aligned}$$

Therefore, $\operatorname{Log}(e^z) = z$ if z is in the fundamental region. □

Our aim is to identify the points of discontinuity for the function $f(z) = \operatorname{Log}(z)$. Specifically, the principal logarithm exhibits discontinuity at $z = 0$ due to being undefined at this point. Additionally, this function is discontinuous along the entire negative real axis. Intuitively, this makes sense because for a point z located near the negative x-axis in the second quadrant, $\operatorname{Log}(z)$ has an imaginary component approaching π, while for a neighouring point closer to the third quadrant, its imaginary component approaches $-\pi$.

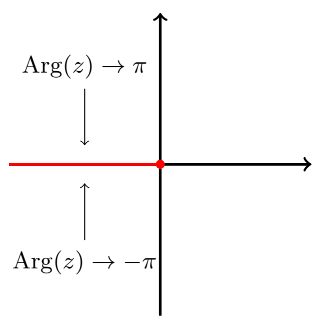

Figure 3.1: $\operatorname{Log}(z)$ is discontinuous at $z = 0$ and on the negative real axis.

However, there is a certain region where $\operatorname{Log}(z)$ is continuous.

Theorem 3.2.10. *The principal logarithm function $f(z) = \operatorname{Log}(z)$ is continuous on the set consisting of the complex plane excluding the nonpositive real axis.*

Proof. First, note that the real and imaginary parts of $\operatorname{Log}(z)$ are given by the functions

$$\begin{aligned}u(x, y) &= \ln(|z|) = \ln(\sqrt{x^2 + y^2}) \\ v(x, y) &= \operatorname{Arg}(z)\end{aligned}$$

From multivariable calculus, we have that $u(x, y)$ is continuous at all points in the plane except at $(0, 0)$. Also, we have that $v(x, y) = \text{Arg}(z)$ is continuous on the domain $|z| > 0$ with $-\pi < \arg(z) < \pi$. Therefore, by Theorem 2.2.11, it follows that $\text{Log}(z)$ is a continuous function on the domain $|z| > 0$ and $-\pi < \arg(z) < \pi$. Put it another way, define the function

$$f_1(z) = \ln(r) + i\theta$$

then f_1 is continuous on the domain $|z| > 0$ and $-\pi < \arg(z) < \pi$. □

Since the function f_1 agrees with the principal logarithm $\text{Log}(z)$ where they are both defined, it follows that f_1 assigns to the input z one of the values of the multiple-valued function $F(z) = \log(z)$. The function f_1 defined above is a branch of the multiple-valued function $F(z) = \log(z)$. Indeed, we have introduced the branch cut back in Section 1.2.

Theorem 3.2.11. *The principal branch of f_1 of the complex logarithm defined by $f_1(z) = \ln(r) + i\theta$ is a holomorphic function and its derivative is given by*

$$f_1'(z) = \frac{1}{z}$$

Proof. We prove that f_1 is holomorphic using the Cauchy–Riemann equations in polar coordinates (Theorem 2.4.7). Since f_1 is defined on the domain $r > 0$ and $-\pi < \theta < \pi$, if z is a point in this domain, then we can write $z = re^{i\theta}$ with $-\pi < \theta < \pi$. Since the real and imaginary parts of f_1 are $u(r, \theta) = \ln(r)$ and $v(r, \theta) = \theta$, we find that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial v}{\partial \theta} = 1$$
$$\frac{\partial v}{\partial r} = 0 \quad \frac{\partial u}{\partial \theta} = 0$$

Thus, u and v satisfy the Cauchy–Riemann equations in polar coordinates.

Since u and v and the first partial derivatives of u and v are continuous at all points in the domain described above, it follows from Theorem 2.4.5 that f_1 is holomorphic in this domain.

In particular, the derivative of f_1 is given by

$$f_1'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

□

3.3 Complex Powers

Complex powers such as $(1+i)^i$ are defined in terms of the complex exponential and logarithmic functions. In order to motivate this definition, recall that $z = e^{\log(z)}$ for all nonzero complex numbers z. Thus, when n is an integer, we can write z^n as $z^n = e^{n\log(z)}$. This formula, which holds for integer exponents, suggests the following formula used to define the complex power z^α for any complex exponent α.

Definition 3.3.1. If $\alpha \in \mathbb{C}$ and $z \neq 0$, then the *complex power* z^α is defined to be

$$z^\alpha = e^{\alpha \log(z)}$$

In general, the definition gives an infinite set of values since the complex logarithm $\log(z)$ is multi-valued. When n is an integer, the expression becomes a single-valued. Indeed, observe that

$$z^n = e^{n\log(z)} = e^{n(\ln(|z|)+i\arg(z))} = e^{n\ln(|z|)}e^{n\arg(z)i}$$

If $\theta = \mathrm{Arg}(z)$, then $\arg(z) = \theta + 2k\pi$, where $k \in \mathbb{Z}$, so then

$$e^{n\arg(z)i} = e^{n(\theta+2k\pi)i} = e^{n\theta i}e^{2nk\pi i}$$

Thus, from Definition 3.1.1, we have

$$e^{2nk\pi i} = \cos(2nk\pi) + i\sin(2nk\pi) = 1$$

and thus,

$$z^n = e^{n\ln(|z|)}e^{n\,\mathrm{Arg}(z)i}$$

which is single-valued.

Complex powers satisfy the following properties that are analogous to properties of real powers.

Proposition 3.3.2. *Let $\alpha, \beta \in \mathbb{C}$ and let z be a nonzero complex power. Then*

1. $z^\alpha z^\beta = z^{\alpha+\beta}$.
2. $\frac{z^\alpha}{z^\beta} = z^{\alpha-\beta}$.
3. *For all* $n \in \mathbb{Z}$, $(z^\alpha)^n = z^{n\alpha}$.

Proof. To see that (1) holds, note that by definition we have

$$\begin{aligned} z^\alpha z^\beta &= e^{\alpha \log(z)} e^{\beta \log(z)} \\ &= e^{\alpha \log(z) + \beta \log(z)} \\ &= e^{(\alpha+\beta) \log(z)} \\ &= z^{\alpha+\beta} \end{aligned}$$

Note that (2) holds by replacing β with $-\beta$.

To see that (3) holds, note that for any $n \in \mathbb{Z}$,

$$(z^\alpha)^n = (e^{\alpha \log(z)})^n = e^{n\alpha \log(z)} = z^{n\alpha}$$

as desired. □

Of course, as the complex power is multi-valued, we can also assign a unique value to z^α by using the principal logarithm.

Definition 3.3.3. If $\alpha \in \mathbb{C}$ and $z \neq 0$, then the function defined by

$$z^\alpha = e^{\alpha \operatorname{Log}(z)}$$

is called the *principal power of z*.

Remark 3.3.4. The notation z^α will be used to denote both the multi-valued power function and the principal power of z. In context, it will be clear which of these two we are referring to.

In general, the principal power z^α is not a continuous function on the complex plane since $\operatorname{Log}(z)$ is not a continuous function on the complex plane. However, since the function $e^{\alpha z}$ is continuous on the entire complex plane, and since the function $\operatorname{Log}(z)$ is continuous on the domain $|z| > 0$ and $-\pi < \arg(z) < \pi$, it follows that z^α is also continuous on the domain$|z| > 0$ and $-\pi < \arg(z) < \pi$. In terms of polar coordinates, the function defined by

$$f_1(z) = e^{\alpha(\ln(r) + i\theta)}$$

is a branch of the multi-valued function $F(z) = z^\alpha$, which is the principal branch of z^α.

Theorem 3.3.5. *The principal branch of f_1 of the complex power defined by $f_1(z) = z^\alpha = e^{\alpha \operatorname{Log}(z)}$ is a holomorphic function and its derivative is given by*

$$f_1'(z) = \alpha z^{\alpha-1}$$

Proof. Recall that the principal branch of the complex logarithm $\mathrm{Log}(z) = \ln(|z|) + i\,\mathrm{Arg}(z)$ is holomorphic on the domain $|z| > 0$ and $-\pi < \arg(z) < \pi$, and satisfies

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}$$

Since the composition of holomorphic functions is holomorphic, e^w is holomorphic for all $w \in \mathbb{C}$, it follows that $f_1(z) = e^{\alpha\,\mathrm{Log}(z)}$ is holomorphic on the domain $|z| > 0$ and $-\pi < \arg(z) < \pi$.

By the chain rule, we have

$$f_1'(z) = \frac{d}{dz}e^{\alpha\,\mathrm{Log}(z)} = e^{\alpha\,\mathrm{Log}(z)}\frac{d}{dz}\alpha\,\mathrm{Log}(z) = e^{\alpha\,\mathrm{Log}(z)}\frac{\alpha}{z}$$

Since $z^\alpha = e^{\alpha\,\mathrm{Log}(z)}$, then the expression simplifies to

$$f_1'(z) = \frac{\alpha z^\alpha}{z} = \alpha z^{\alpha-1}$$

as desired. □

3.4 Complex Trigonometric Functions

If x is a real variable, it follows from Definition 3.1.1 that

$$e^{ix} = \cos(x) + i\sin(x) \quad \text{and} \quad e^{-ix} = \cos(x) - i\sin(x)$$

By adding these equations and simplifying, we obtain

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

and similarly, by subtracting these equations, we obtain

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

The formulas for the real cosine and sine are given as above, and we can use them to define the complex sine and cosine functions.

Definition 3.4.1. The *complex sine* and *cosine* functions are defined by

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Analogous to real trigonometric ratios, we can also define the complex tangent, cotangent, secant, and cosecant functions using the definition of the complex sine and cosine functions.

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \quad \cot(z) = \frac{\cos(z)}{\sin(z)}, \quad \sec(z) = \frac{1}{\cos(z)}, \quad \csc(z) = \frac{1}{\sin(z)}$$

These functions also agree with their real counterparts for real input.

Most of the familiar identities for real trigonometric functions hold for the complex trigonometric functions. This follows from Definition 3.4.1 and properties of the complex exponential function. We now list some of the more useful of the trigonometric identities.

Proposition 3.4.2. *For every $z, w \in \mathbb{C}$,*

1. $\sin(-z) = -\sin(z)$.
2. $\cos(-z) = \cos(z)$.
3. $\cos^2(z) + \sin^2(z) = 1$.
4. $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$.
5. $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$.
6. $\sin(2z) = 2\sin(z)\cos(z)$.
7. $\cos(2z) = \cos^2(z) - \sin^2(z)$.

Proof. To see that (1) holds, note that by Definition 3.4.1

$$\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = \frac{e^{-iz} - e^{iz}}{2i} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z)$$

as desired.

To see that (2) holds, note that by Definition 3.4.1,

$$\cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2} = \frac{e^{-iz} + e^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

as desired.

To see that (3) holds, note that we have

$$\cos^2(z) = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} + 2e^{iz}e^{-iz} + e^{-2iz}}{4} = \frac{e^{2iz} + e^{-2iz} + 2}{4}$$

$$\sin^2(z) = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} - 2e^{iz}e^{-iz} + e^{-2iz}}{-4} = \frac{-e^{2iz} - e^{-2iz} + 2}{4}$$

Therefore,

$$\cos^2(z)+\sin^2(z)=\frac{e^{2iz}+e^{-2iz}+2}{4}+\frac{-e^{2iz}-e^{-2iz}+2}{4}=1$$

as desired.

To see that (4) holds, first note that

$$\begin{aligned}
\sin(z)\cos(w) &= \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}+e^{-iw}}{2} \\
&= \frac{e^{iz}e^{iw}+e^{iz}e^{-iw}-e^{-iz}e^{iw}-e^{-iz}e^{-iw}}{4i} \\
&= \frac{e^{i(z+w)}+e^{i(z-w)}-e^{-i(z-w)}-e^{-i(z+w)}}{4i}
\end{aligned}$$

and

$$\begin{aligned}
\cos(z)\sin(w) &= \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}-e^{-iw}}{2i} \\
&= \frac{e^{iz}e^{iw}-e^{iz}e^{-iw}+e^{-iz}e^{iw}-e^{-iz}e^{-iw}}{4i} \\
&= \frac{e^{i(z+w)}-e^{i(z-w)}+e^{-i(z-w)}-e^{-i(z+w)}}{4i}
\end{aligned}$$

Therefore, adding the two expressions yield

$$\begin{aligned}
\sin(z)\cos(w)+\cos(z)\sin(w) &= \frac{2e^{i(z+w)}-2e^{-i(z+w)}}{4i} \\
&= \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} \\
&= \sin(2z)
\end{aligned}$$

as desired.

To see that (5) holds, first note that

$$\begin{aligned}
\cos(z)\cos(w) &= \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}+e^{-iw}}{2} \\
&= \frac{e^{i(z+w)}+e^{i(z-w)}+e^{-i(z-w)}+e^{-i(z+w)}}{4}
\end{aligned}$$

and

$$\begin{aligned}
\sin(z)\sin(w) &= \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}-e^{-iw}}{2i} \\
&= \frac{-e^{i(z+w)}+e^{i(z-w)}+e^{-i(z-w)}-e^{-i(z+w)}}{4}
\end{aligned}$$

Therefore, subtracting the two expressions yield

$$\begin{aligned}\cos(z)\cos(w) - \sin(z)\sin(w) &= \frac{2e^{i(z+w)} + 2e^{-i(z+w)}}{4} \\ &= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} \\ &= \cos(z+w)\end{aligned}$$

Note that (6) and (7) follows from (4) and (5), respectively. □

Remark 3.4.3. It is important to recognize that some properties of real trigonometric functions are not satisfied by their complex counterparts. For example, $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$, but $|\cos(i)| > 1$ and $|\sin(2+i)| > 1$, so these inequalities, in general, are not satisfied for complex input.

Proposition 3.4.4. *The complex sine and cosine functions are 2π-periodic. That is,*

1. $\sin(z + 2\pi) = \sin(z)$.
2. $\cos(z + 2\pi) = \cos(z)$.

Proof. To see that (1) holds, note that

$$\begin{aligned}\sin(z+2\pi) &= \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} \\ &= \frac{e^{iz}e^{2\pi i} - e^{-iz}e^{2\pi i}}{2i} \\ &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \sin(z)\end{aligned}$$

as desired.

To see that (2) holds, note that

$$\begin{aligned}\cos(z+2\pi) &= \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} \\ &= \frac{e^{iz}e^{2\pi i} + e^{-iz}e^{2\pi i}}{2} \\ &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \cos(z)\end{aligned}$$

as desired. □

Put another way, Proposition 3.4.4 shows that the complex sine and cosine are periodic functions with a real period of 2π. The periodicity of the secant and cosecant follows immediately. Furthermore, it can be shown that $\sin(z+\pi) = -\sin(z)$ and $\cos(z+\pi) = -\cos(z)$ can be used to show that the complex tangent and cotangent are periodic with real period π.

We now turn our attention to solving simple trigonometric equations. Since the complex sine and cosine functions are periodic, there are always infinitely many solutions to equations of the form $\sin(z) = w$ or $\cos(z) = w$. One approach to solving such equations is by using Definition 3.4.1 and the quadratic formula.

Example 3.4.5. Consider the complex sine equation $\sin(z) = 5$. By Definition 3.4.1, we have

$$\frac{e^{iz} - e^{-iz}}{2i} = 5 \implies e^{2iz} - 10ie^{iz} - 1 = 0$$

This is a quadratic equation in e^{iz}, so by the quadratic formula,

$$e^{iz} = \frac{10i \pm \sqrt{-96}}{2} = 5i \pm 2\sqrt{6}i = \left(5 \pm 2\sqrt{6}\right) i$$

In order to find the values of z satisfying the above, we solve the two exponential equations using the complex logarithm.

<u>Case 1: If $e^{iz} = \left(5+2\sqrt{6}\right) i$.</u> Then

$$iz = \log\left(5i + 2\sqrt{6}i\right) \implies z = -\log\left(\left(5+2\sqrt{6}\right) i\right)$$

Since $\left(5+2\sqrt{6}\right) i$ is imaginary and $5+2\sqrt{6} > 0$, then $\arg\left(\left(5+2\sqrt{6}\right) i\right) = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$. Then

$$\begin{aligned} z &= -\log\left(\left(5+2\sqrt{6}\right) i\right) \\ &= -i\left(\left(\ln(5+2\sqrt{6}\right) + i\left(\frac{\pi}{2} + 2k\pi\right)\right) \\ &= \frac{(4n+1)\pi}{2} - i\ln(5+2\sqrt{6}) \end{aligned}$$

for $k \in \mathbb{Z}$.

<u>Case 2: If $e^{iz} = \left(5-2\sqrt{6}\right) i$.</u> Then similar to Case 1, we can determine that

$$z = \frac{(4n+1)\pi}{2} - i\ln(5-2\sqrt{6})$$

for $k \in \mathbb{Z}$.

The modulus of a complex trigonometric function can be helpful in solving trigonometric equations. To find a formula for the modulus in terms of x and y for the modulus of the sine and cosine functions, we must express these functions in terms of their real and imaginary parts. Observe that if $z = x + yi$, then

$$\begin{aligned}\sin(z) &= \frac{e^{-y+ix} - e^{y-ix}}{2i} \\ &= \frac{e^{-y}(\cos(x) + i\sin(x)) - e^{y}(\cos(x) - i\sin(x))}{2i} \\ &= \sin(x)\left(\frac{e^{y}+e^{-y}}{2}\right) + i\cos(x)\left(\frac{e^{y}-e^{-y}}{2}\right)\end{aligned}$$

Since the real hyperbolic sine and cosine functions are defined by

$$\sinh(y) = \frac{e^{y}-e^{-y}}{2} \quad \text{and} \quad \cosh(y) = \frac{e^{y}+e^{-y}}{2}$$

then

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

A similar computation shows that the complex cosine can be expressed in terms of its real and imaginary parts.

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

Now we can use both $\sin(z)$ and $\cos(z)$ above to derive formulas for the modulus of the complex sine and cosine functions. Indeed, we have

$$|\sin(z)| = \sqrt{\sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y)}$$

which can be simplified using identities $\cos^2(x) + \sin^2(x) = 1$ and $\cosh^2(y) = 1 + \sinh^2(y)$, so then

$$\begin{aligned}|\sin(z)| &= \sqrt{\sin^2(x)(1 + \sinh^2(y) + \cos^2(x)\sinh^2(y)} \\ &= \sqrt{\sin^2(x) + (\cos^2(x) + \sin^2(x))\sinh^2(y)} \\ &= \sqrt{\sin^2(x) + \sinh^2(y)}\end{aligned}$$

After a similar computation, we obtain the modulus for the complex cosine

$$|\cos(z)| = \sqrt{\cos^2(x) + \sinh^2(y)}$$

The derivatives of the complex sine and cosine functions are found using the chain rule. Analogous to the real sine and cosine function, the derivative of the complex sine is the complex cosine

$$\frac{d}{dz}\sin(z) = \cos(z)$$

and the derivative of complex cosine is the negative complex sine

$$\frac{d}{dz}\cos(z) = -\sin(z)$$

Of course, these derivatives can be used to find the derivatives of other trigonometric functions. As it turns out, the complex sine and cosine are holomorphic on all of $\mathbb{C}$.

Index