MATH 6350
Partial Differential Equations

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December 31, 2024

Preface

These are the first edition of these lecture notes for MATH 6350 (Partial Differential Equations). Consequently, there may be several typographical errors, missing exposition on necessary background, and more advanced topics for which there will not be time in class to cover. Future iterations of these notes will hopefully be fairly self-contained provided one has the necessary background. If you come across any typos, errors, omissions, or unclear expositions, please feel free to contact me so that I may continually improve these notes.

Contents

Chapter 1

Preliminaries and Fundamental Tools

The theory of pseudo-differential operators rests upon several analytical foundations that extend classical calculus and Fourier analysis. Before introducing symbol classes and the operators themselves, it is essential to develop a rigorous framework of spaces, notations, and analytical tools that will be used throughout these notes.

In this chapter, we begin by reviewing the structure of Euclidean space $\mathbb{R}^n$ and the use of multi-indices, which allow us to concisely express differential operators of arbitrary order. We then discuss the differentiation of integrals depending on parameters, a result that underlies many manipulations involving kernels and parameter-dependent integrals in later chapters.

The next sections introduce two fundamental constructions in analysis: the convolution of functions and distributions, and the Fourier transform on $\mathbb{R}^n$. These concepts form the analytic backbone of pseudo-differential theory—linking localization in physical space with decay and regularity properties in frequency space.

Finally, we conclude the chapter with an introduction to the Schwartz space of rapidly decreasing functions and its dual, the space of tempered distributions. These spaces provide the natural setting for the Fourier transform and for defining pseudo-differential operators acting on generalized functions.

Together, these sections establish the analytical and notational groundwork necessary for the study of symbol classes, operator composition, and elliptic theory developed in subsequent chapters.

1.1 Euclidean Space and Multi-Index Notation

The analysis of pseudo-differential operators is carried out on $\mathbb{R}^n$, the n-dimensional Euclidean space. In this section we review its basic structure and introduce the multi-index notation that allows a concise expression of differential operators and estimates.

Let $\mathbb{R}^n$ denote the vector space of all n-tuples $x = (x_1, x_2, \dots, x_n)$ with real components. For any two vectors $x, y \in \mathbb{R}^n$, we define the standard inner product and norm by

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j, \qquad |x| = \sqrt{\langle x, x\rangle} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}.$$

The distance between two points $x, y \in \mathbb{R}^n$ is given by $|x - y|$, and the open ball centered at x_0 with radius $r > 0$ is

$$B(x_0, r) = \{\, x \in \mathbb{R}^n : |x - x_0| < r \,\}.$$

Definition 1.1.1 (Open and Closed Sets)**.** A set $U \subseteq \mathbb{R}^n$ is *open* if for every $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. A set $K \subseteq \mathbb{R}^n$ is *compact* if it is closed and bounded.

Definition 1.1.2 (Smooth Functions)**.** A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be *smooth*, or of class $\mathcal{C}^\infty$, if all partial derivatives $\partial^\alpha f$ exist and are continuous for every multi-index α.

Example 1.1.3. The Gaussian function $f(x) = e^{-|x|^2}$ belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$ since $\partial^\alpha f(x) = p_\alpha(x) e^{-|x|^2}$, where p_α is a polynomial depending on α.

When dealing with higher-order derivatives and polynomial symbols, it is convenient to introduce a compact notation.

Definition 1.1.4 (Multi-Index)**.** A *multi-index* is an n-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of nonnegative integers. We define:

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n, \\ \alpha! &= \alpha_1!\, \alpha_2! \cdots \alpha_n!, \\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n. \end{aligned}$$

Definition 1.1.5 (Partial Derivatives)**.** For a function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, we define the derivative of order α by

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(x).$$

Remark 1.1.6. For $\alpha, \beta \in \mathbb{N}_0^n$, we write $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all j. The binomial coefficient for multi-indices is defined by

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j}.$$

Example 1.1.7 (Product Rule)**.** Let $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then for any multi-index α,

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).$$

Multi-index notation will appear frequently in the definition of differential operators. If $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, we write

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{1}{i^{|\alpha|}}\, \partial^\alpha.$$

This choice of normalization ensures that the Fourier transform of a derivative satisfies

$$\mathcal{F}(\partial^\alpha f)(\xi) = (i\xi)^\alpha\, \widehat{f}(\xi),$$

a convention adopted throughout these notes.

Remark 1.1.8. The multi-index notation provides a unified language for expressing estimates of the form

$$|\partial^\alpha f(x)| \leq C_\alpha (1 + |x|)^{-N},$$

which describe smoothness and decay properties of functions and are fundamental in the theory of symbol classes $S^m_{\rho,\delta}$ introduced in Chapter 2.

1.2 Differentiation of Integrals Depending on Parameters

In many analytic problems, one encounters integrals of the form

$$F(x) = \int_{\mathbb{R}^n} f(x, y)\, dy,$$

where $x \in \mathbb{R}^m$ is a parameter. A central question is whether differentiation with respect to x can be passed inside the integral. The validity of this interchange is crucial for defining and analyzing parameter-dependent kernels, convolution operators, and Fourier transforms that arise in pseudo-differential theory.

If $f(x, y)$ and its partial derivatives with respect to x satisfy suitable continuity and integrability conditions, then the derivative of $F(x)$ with respect to x can be obtained by differentiating under the integral sign:

$$\frac{\partial F}{\partial x_j}(x) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x, y)\, dy.$$

This operation is known as *differentiation under the integral sign* or the *Leibniz integral rule*. Its justification rests on the *dominated convergence theorem*, one of the cornerstone results of modern analysis.

Theorem 1.2.1 (Differentiation of Parameter-Dependent Integrals)**.** *Let $E \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be measurable, and suppose $f : E \to \mathbb{C}$ satisfies the following conditions:*

(i) For each fixed $x \in \mathbb{R}^m$, the function $y \mapsto f(x, y)$ is integrable on $\mathbb{R}^n$.

(ii) The partial derivative $\partial_{x_j} f(x, y)$ exists for all x and y.

(iii) There exists an integrable function $g(y)$ such that

$$\left|\partial_{x_j} f(x, y)\right| \leq g(y) \quad \text{for all } (x, y) \in E.$$

Then the function

$$F(x) = \int_{\mathbb{R}^n} f(x, y)\, dy$$

is continuously differentiable on $\mathbb{R}^m$, and

$$\frac{\partial F}{\partial x_j}(x) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x, y)\, dy.$$

Sketch of Proof. Fix $x \in \mathbb{R}^m$ and consider the difference quotient

$$\frac{F(x + he_j) - F(x)}{h} = \int_{\mathbb{R}^n} \frac{f(x + he_j, y) - f(x, y)}{h}\, dy,$$

where e_j denotes the j-th coordinate vector in $\mathbb{R}^m$. By the mean value theorem,

$$\frac{f(x + he_j, y) - f(x, y)}{h} = \partial_{x_j} f(x + \theta he_j, y)$$

for some $\theta \in (0,1)$. Under the hypothesis (iii), the integrand is dominated by $g(y)$, which is integrable. Hence, by the dominated convergence theorem,

$$\lim_{h\to 0} \frac{F(x+he_j)-F(x)}{h} = \int_{\mathbb{R}^n} \partial_{x_j} f(x,y)\,dy,$$

which completes the proof. □

Remark 1.2.2. The assumption of domination by an integrable function g guarantees both the existence and continuity of $\partial_{x_j} F(x)$. If the integrable bound g depends on x but is uniform on compact subsets, the result remains valid locally.

Theorem 1.2.3 (Compactly Supported Integrands)**.** *Suppose $f(x,y)$ and $\partial_{x_j} f(x,y)$ are continuous on $\mathbb{R}^m \times \mathbb{R}^n$, and there exists a compact set $K \subset \mathbb{R}^n$ such that*

$$f(x,y) = 0 \quad \text{for } y \notin K.$$

Then

$$\frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} f(x,y)\,dy = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x,y)\,dy.$$

Proof. Since the integrand has compact support in y, the integral reduces to a finite domain. The continuity of both f and $\partial_{x_j} f$ then allows the interchange of differentiation and integration via uniform convergence on K. □

Example 1.2.4 (Differentiating a Gaussian Integral)**.** Let

$$F(a) = \int_{\mathbb{R}} e^{-ay^2}\,dy, \qquad a > 0.$$

Then

$$F'(a) = \int_{\mathbb{R}} \frac{\partial}{\partial a} e^{-ay^2}\,dy = -\int_{\mathbb{R}} y^2 e^{-ay^2}\,dy.$$

Since $|\partial_a e^{-ay^2}| = y^2 e^{-ay^2}$ is integrable for $a > 0$, the interchange is justified by Theorem 1.2.8.

Example 1.2.5 (Parameter-Dependent Fourier Integral)**.** Let $f \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz space) and define

$$F(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi)\,d\xi.$$

Then
$$\partial_{x_j} F(x) = \int_{\mathbb{R}^n} i\xi_j e^{i\langle x,\xi\rangle} f(\xi)\, d\xi,$$
since $|\xi_j f(\xi)|$ is integrable for $f \in \mathcal{S}(\mathbb{R}^n)$. This shows that differentiation with respect to the spatial variable corresponds to multiplication by $i\xi_j$ in the Fourier domain.

Remark 1.2.6. The results of this section will be applied repeatedly in later chapters to justify operations such as:

- differentiation of Fourier transforms with respect to parameters;
- differentiation of convolution kernels with respect to spatial variables;
- differentiation of parameter-dependent symbols $a(x, \xi)$ in defining pseudo-differential operators.

Remark 1.2.7. If $f(x, y)$ depends on additional parameters, the differentiation theorem extends naturally by induction on the number of variables, provided a uniform integrable bound exists for all derivatives involved.

Many analytical constructions in the theory of pseudo-differential operators involve functions defined by integrals that depend on one or more parameters.

Examples include convolution kernels, Fourier transforms, and parametrized families of symbols.

It is therefore essential to understand under what conditions differentiation with respect to a parameter can be interchanged with integration.

In this section, we recall a fundamental result that allows the differentiation of an integral whose integrand depends smoothly on an external variable.

We shall present the precise hypotheses—continuity, integrability, and differentiability—under which the operations of integration and differentiation commute.

This result, often referred to as the Leibniz integral rule, provides a rigorous justification for many formal manipulations appearing later in the analysis of kernels and symbol estimates.

The discussion will include illustrative examples and typical applications, such as differentiation of Fourier transforms with respect to parameters and smooth dependence of integrals on variables in $\mathbb{R}^n$.

These tools will play a central role in establishing regularity and decay properties of integral operators in subsequent chapters.

In many analytic problems, one encounters integrals of the form

$$F(x) = \int_{\mathbb{R}^n} f(x,y)\,dy,$$

where $x \in \mathbb{R}^m$ is a parameter. A central question is whether differentiation with respect to x can be passed inside the integral. The validity of this interchange is crucial for defining and analyzing parameter-dependent kernels, convolution operators, and Fourier transforms that arise in pseudo-differential theory.

If $f(x,y)$ and its partial derivatives with respect to x satisfy suitable continuity and integrability conditions, then the derivative of $F(x)$ with respect to x can be obtained by differentiating under the integral sign:

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Theorem 1.2.8 (Differentiation of Parameter-Dependent Integrals)**.** *Let $E \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be measurable, and suppose $f : E \to \mathbb{C}$ satisfies the following conditions:*

(i) For each fixed $x \in \mathbb{R}^m$, the function $y \mapsto f(x,y)$ is integrable on $\mathbb{R}^n$.

(ii) The partial derivative $\partial_{x_j} f(x,y)$ exists for all x and y.

(iii) There exists an integrable function $g(y)$ such that

$$\left|\partial_{x_j} f(x,y)\right| \le g(y) \quad \textit{for all } (x,y) \in E.$$

Then the function

$$F(x) = \int_{\mathbb{R}^n} f(x,y)\,dy$$

is continuously differentiable on $\mathbb{R}^m$, and

$$\frac{\partial F}{\partial x_j}(x) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x,y)\,dy.$$

Sketch of Proof. Fix $x \in \mathbb{R}^m$ and consider the difference quotient

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where e_j denotes the j-th coordinate vector in $\mathbb{R}^m$. By the mean value theorem,

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for some $\theta\in(0,1)$. Under the hypothesis (iii), the integrand is dominated by $g(y)$, which is integrable. Hence, by the dominated convergence theorem,

$$\lim_{h\to 0}\frac{F(x+he_j)-F(x)}{h}=\int_{\mathbb{R}^n}\partial_{x_j}f(x,y)\,dy,$$

which completes the proof. □

Remark 1.2.9. The assumption of domination by an integrable function g guarantees both the existence and continuity of $\partial_{x_j}F(x)$. If the integrable bound g depends on x but is uniform on compact subsets, the result remains valid locally.

Theorem 1.2.10 (Compactly Supported Integrands)**.** *Suppose $f(x,y)$ and $\partial_{x_j}f(x,y)$ are continuous on $\mathbb{R}^m\times\mathbb{R}^n$, and there exists a compact set $K\subset\mathbb{R}^n$ such that*

$$f(x,y)=0\quad\text{for } y\notin K.$$

Then

$$\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n}f(x,y)\,dy=\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_j}(x,y)\,dy.$$

Proof. Since the integrand has compact support in y, the integral reduces to a finite domain. The continuity of both f and $\partial_{x_j}f$ then allows the interchange of differentiation and integration via uniform convergence on K. □

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$$F(a)=\int_{\mathbb{R}}e^{-ay^2}\,dy,\qquad a>0.$$

Then

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Since $|\partial_a e^{-ay^2}|=y^2e^{-ay^2}$ is integrable for $a>0$, the interchange is justified by Theorem 1.2.8.

Example 1.2.12 (Parameter-Dependent Fourier Integral)**.** Let $f \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz space) and define

$$F(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi)\, d\xi.$$

Then

$$\partial_{x_j} F(x) = \int_{\mathbb{R}^n} i\xi_j e^{i\langle x,\xi\rangle} f(\xi)\, d\xi,$$

since $|\xi_j f(\xi)|$ is integrable for $f \in \mathcal{S}(\mathbb{R}^n)$. This shows that differentiation with respect to the spatial variable corresponds to multiplication by $i\xi_j$ in the Fourier domain.

Remark 1.2.13. The results of this section will be applied repeatedly in later chapters to justify operations such as:

- differentiation of Fourier transforms with respect to parameters;
- differentiation of convolution kernels with respect to spatial variables;
- differentiation of parameter-dependent symbols $a(x, \xi)$ in defining pseudo-differential operators.

Remark 1.2.14. If $f(x, y)$ depends on additional parameters, the differentiation theorem extends naturally by induction on the number of variables, provided a uniform integrable bound exists for all derivatives involved.

1.3 Convolution of Functions and Distributions

The convolution operation is one of the most fundamental tools in analysis. It plays a central role in the study of integral operators, Fourier analysis, and the definition of pseudo-differential operators. In this section, we introduce the convolution of functions on $\mathbb{R}^n$, examine its properties, and then extend the definition to distributions. We also highlight the important link between convolution and the Fourier transform.

Definition 1.3.1 (Convolution of Functions)**.** Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be measurable functions. If the integral

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)\, g(y)\, dy$$

exists for almost every $x \in \mathbb{R}^n$, then $f * g$ is called the *convolution* of f and g.

Remark 1.3.2. The convolution is well-defined if, for example, $f, g \in L^1(\mathbb{R}^n)$. In that case, $f * g \in L^1(\mathbb{R}^n)$ and satisfies

$$\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}.$$

Proposition 1.3.3 (Basic Properties). *If $f, g, h \in L^1(\mathbb{R}^n)$, then:*

(i) Commutativity: $f * g = g * f$.

(ii) Associativity: $(f * g) * h = f * (g * h)$.

(iii) Distributivity: $f * (g + h) = f * g + f * h$.

(iv) Translation invariance:

$$\tau_a(f * g) = (\tau_a f) * g = f * (\tau_a g),$$

where $(\tau_a f)(x) = f(x - a)$.

Proof. All statements follow from Fubini's theorem and straightforward changes of variables, using the absolute integrability of f, g, h. □

Convolution has remarkable smoothing properties and interacts neatly with differentiation.

Proposition 1.3.4 (Differentiation and Convolution). *If $f, g \in L^1(\mathbb{R}^n)$ and $\partial^\alpha f$ exists in the distributional sense with $\partial^\alpha f \in L^1(\mathbb{R}^n)$, then*

$$\partial^\alpha(f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g).$$

Proof. Formally,

$$\partial^\alpha(f * g)(x) = \int_{\mathbb{R}^n} \partial^\alpha f(x - y)\, g(y)\, dy,$$

which is justified by the dominated convergence theorem since $\partial^\alpha f \in L^1$. □

Remark 1.3.5. This result implies that convolution with a smooth, rapidly decaying function *regularizes* the other function. For instance, if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $\varphi * f \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Definition 1.3.6 (Approximate Identity). A family $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset L^1(\mathbb{R}^n)$ is called an *approximate identity* if:

(i) $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\, dx = 1$ for all $\varepsilon > 0$,

(ii) $\lim_{\varepsilon\to 0}\int_{|x|>\delta}|\varphi_\varepsilon(x)|\,dx = 0$ for every $\delta > 0$.

Theorem 1.3.7 (Approximation by Convolution)**.** *If $\{\varphi_\varepsilon\}$ is an approximate identity and $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$, then*

$$\lim_{\varepsilon\to 0}\|f * \varphi_\varepsilon - f\|_{L^p} = 0.$$

Example 1.3.8. Let $\varphi(x) = (2\pi)^{-n/2}e^{-|x|^2/2}$. Define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then $\{\varphi_\varepsilon\}$ forms an approximate identity and

$$f * \varphi_\varepsilon \to f$$

in L^p as $\varepsilon \to 0$. This convolution smooths f while preserving its limiting behaviour.

A fundamental property linking convolution and the Fourier transform is the following identity.

Theorem 1.3.9 (Fourier Transform of a Convolution)**.** *If $f, g \in L^1(\mathbb{R}^n)$, then*

$$\mathcal{F}(f * g)(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi),$$

where $\widehat{f}$ denotes the Fourier transform of f.

Proof. Using the definition of the Fourier transform,

$$\mathcal{F}(f * g)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\,dx.$$

Applying Fubini's theorem and the change of variables $z = x - y$ gives

$$\mathcal{F}(f * g)(\xi) = \left(\int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle} f(z)\,dz\right)\left(\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} g(y)\,dy\right) = \widehat{f}(\xi)\,\widehat{g}(\xi).$$

□

Remark 1.3.10. The converse identity also holds: if $\widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$, then

$$\mathcal{F}^{-1}(\widehat{f}\,\widehat{g}) = f * g.$$

This duality will later justify representing pseudo-differential operators as integral transforms involving convolution-type structures.

The convolution of distributions extends the classical definition to generalized functions.

Definition 1.3.11 (Convolution of a Distribution and a Test Function)**.** If $T \in \mathcal{D}'(\mathbb{R}^n)$ (distribution space) and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, their convolution is defined by

$$(T * \varphi)(x) = \langle T, \tau_x \tilde{\varphi} \rangle, \quad \text{where } \tilde{\varphi}(y) = \varphi(-y),$$

and $\tau_x \tilde{\varphi}(y) = \tilde{\varphi}(y - x)$.

Remark 1.3.12. For each $x \in \mathbb{R}^n$, $\tau_x \tilde{\varphi}$ is again a test function, so $T * \varphi$ is a smooth function. In fact, $T * \varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ and satisfies

$$\partial^\alpha (T * \varphi) = T * (\partial^\alpha \varphi).$$

Example 1.3.13 (Convolution with the Dirac Delta)**.** Let δ denote the Dirac delta distribution. For any test function φ,

$$(\delta * \varphi)(x) = \varphi(x),$$

so δ acts as the identity element under convolution.

When both objects have sufficient decay, convolution can be defined in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

Definition 1.3.14 (Convolution in $\mathcal{S}'$)**.** If $T, S \in \mathcal{S}'(\mathbb{R}^n)$ are such that one has compact support, the convolution $T * S$ is defined by

$$\langle T * S, \varphi \rangle = \langle T(x), \langle S(y), \varphi(x + y) \rangle \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proposition 1.3.15. *If $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\mathcal{F}(T * \varphi) = \widehat{T}\, \widehat{\varphi}.$$

Remark 1.3.16. This identity generalizes Theorem 1.3.9 to distributions and forms a cornerstone of pseudo-differential analysis, where convolution kernels are replaced by Fourier multipliers with variable symbols.

1.4 Fourier Transform on $\mathbb{R}^n$

The Fourier transform is a fundamental analytical tool that bridges the spatial and frequency domains. In the study of pseudo-differential operators, it provides the natural language for describing symbols, quantizations, and frequency localization. This section reviews the Fourier transform on $\mathbb{R}^n$, its basic properties, inversion formulas, and the relationship between differentiation and multiplication.

Definition 1.4.1 (Fourier Transform)**.** For $f \in L^1(\mathbb{R}^n)$, the *Fourier transform* of f is defined by

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx, \qquad \xi \in \mathbb{R}^n.$$

Remark 1.4.2. We use the sign convention consistent with the pseudo-differential literature: the exponential kernel is $e^{-i\langle x,\xi\rangle}$ and the inverse transform involves $e^{+i\langle x,\xi\rangle}$. Alternative normalizations differ only by constant factors of $(2\pi)^{\pm n/2}$.

Definition 1.4.3 (Inverse Fourier Transform)**.** If $\widehat{f} \in L^1(\mathbb{R}^n)$, the *inverse Fourier transform* of $\widehat{f}$ is given by

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{f}(\xi)\, d\xi.$$

Theorem 1.4.4 (Riemann–Lebesgue Lemma)**.** *If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous and*

$$\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0.$$

Idea of Proof. The continuity follows from dominated convergence, since $e^{-i\langle x,\xi\rangle}$ depends continuously on ξ. Decay at infinity follows from approximating f by $\mathcal{C}_c^\infty$ functions and using oscillatory-integral estimates. □

Proposition 1.4.5 (Translation and Modulation)**.** *For $f \in L^1(\mathbb{R}^n)$, $a, \xi_0 \in \mathbb{R}^n$,*

$$\mathcal{F}(\tau_a f)(\xi) = e^{-i\langle a,\xi\rangle} \widehat{f}(\xi),$$
$$\mathcal{F}(e^{i\langle x,\xi_0\rangle} f(x))(\xi) = \widehat{f}(\xi - \xi_0),$$

where $(\tau_a f)(x) = f(x - a)$.

Proof. Both formulas follow directly from the change of variables $y = x - a$ and linearity of the exponential phase. □

Proposition 1.4.6 (Scaling)**.** *If $f(x) \in L^1(\mathbb{R}^n)$ and $\lambda \neq 0$, then*

$$\mathcal{F}(f(\lambda x))(\xi) = |\lambda|^{-n} \widehat{f}\left(\frac{\xi}{\lambda}\right).$$

Proposition 1.4.7 (Differentiation Property)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then for every multi-index α,*

$$\mathcal{F}(\partial_x^\alpha f)(\xi) = (i\xi)^\alpha \widehat{f}(\xi).$$

Proof. Integrate by parts in each variable:

$$\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \partial_x^\alpha f(x)\, dx = (i\xi)^\alpha \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx.$$

The boundary terms vanish because $f \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Proposition 1.4.8 (Multiplication Property)**.** *If $f \in \mathcal{S}(\mathbb{R}^n)$ and $x^\alpha f(x)$ is also integrable, then*

$$\mathcal{F}(x^\alpha f(x))(\xi) = i^{|\alpha|}\, \partial_\xi^\alpha \widehat{f}(\xi).$$

Remark 1.4.9. These identities establish the duality:

$$\text{Differentiation in } x \quad \leftrightarrow \quad \text{Multiplication by } i\xi, \qquad \text{and vice versa.}$$

This duality forms the algebraic foundation of pseudo-differential calculus.

Theorem 1.4.10 (Plancherel Theorem)**.** *The Fourier transform extends uniquely to a unitary operator on $L^2(\mathbb{R}^n)$ such that*

$$\|\widehat{f}\|_{L^2} = (2\pi)^{n/2}\, \|f\|_{L^2}, \qquad f \in L^2(\mathbb{R}^n).$$

Idea of Proof. First prove for $f \in \mathcal{S}(\mathbb{R}^n)$ using Fubini's theorem and symmetry of the kernel, then extend by density to all of L^2. Unitarity follows from the identity

$$\langle \widehat{f}, \widehat{g}\rangle = (2\pi)^n \langle f, g\rangle.$$

$\square$

Corollary 1.4.11 (Parseval Identity)**.** *For $f, g \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} f(x)\, \overline{g(x)}\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}\, d\xi.$$

Proposition 1.4.12 (Convolution Theorem)**.** *For $f, g \in L^1(\mathbb{R}^n)$,*

$$\mathcal{F}(f * g)(\xi) = \widehat{f}(\xi)\, \widehat{g}(\xi), \qquad \mathcal{F}(fg)(\xi) = (2\pi)^{-n} (\widehat{f} * \widehat{g})(\xi).$$

Remark 1.4.13. This result unifies convolution and multiplication: in the frequency domain, convolution becomes multiplication and vice versa. Pseudo-differential operators exploit this principle by acting as multipliers in ξ with symbol $a(x, \xi)$.

Example 1.4.14 (Gaussian)**.** Let $f(x) = e^{-|x|^2/2}$. Then

$$\widehat{f}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}.$$

The Gaussian is thus an eigenfunction of the Fourier transform.

Example 1.4.15 (Characteristic Function of an Interval)**.** For $f = \chi_{[-1,1]} \subset L^1(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{-1}^{1} e^{-ix\xi}\, dx = 2\frac{\sin \xi}{\xi}.$$

This example illustrates the smoothing effect of the Fourier transform on discontinuous functions.

1.5 Schwartz Space and Tempered Distributions

The study of pseudo-differential operators requires a function space that accommodates both smoothness and rapid decay at infinity, and that remains stable under the Fourier transform. This is achieved by the *Schwartz space* of rapidly decreasing functions, together with its dual, the space of *tempered distributions*. These spaces provide the natural framework for defining Fourier transforms of generalized functions and for manipulating symbols that exhibit polynomial growth in ξ.

Definition 1.5.1 (Schwartz Space)**.** The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$,

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha\, \partial^\beta f(x)| < \infty.$$

Each $p_{\alpha,\beta}$ is a seminorm, and the collection $\{p_{\alpha,\beta}\}$ defines the topology of $\mathcal{S}(\mathbb{R}^n)$.

Remark 1.5.2. Functions in $\mathcal{S}(\mathbb{R}^n)$ are infinitely differentiable and decay faster than any power of $|x|^{-1}$ along with all their derivatives. That is, for each $N > 0$, there exists C_N such that

$$|f(x)| \leq C_N (1 + |x|)^{-N}.$$

Example 1.5.3. The Gaussian $f(x) = e^{-|x|^2}$ belongs to $\mathcal{S}(\mathbb{R}^n)$. Indeed, every derivative $\partial^\alpha f(x)$ is of the form $p_\alpha(x) e^{-|x|^2}$ with p_α a polynomial.

Proposition 1.5.4. *The space $\mathcal{S}(\mathbb{R}^n)$ is closed under:*

(i) *differentiation:* $\partial^\alpha f \in \mathcal{S}$,

(ii) *multiplication by polynomials:* $x^\beta f \in \mathcal{S}$,

(iii) *translation:* $(\tau_a f)(x) = f(x-a)$,

(iv) *convolution: if* $f, g \in \mathcal{S}$, *then* $f * g \in \mathcal{S}$.

Idea of Proof. Each operation preserves both smoothness and rapid decay. For instance, differentiation and multiplication affect only finitely many seminorms $p_{\alpha,\beta}$, leaving their finiteness intact. □

Theorem 1.5.5 (Fourier Transform Isomorphism)**.** *The Fourier transform*

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \qquad \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx,$$

is a bijective, continuous linear mapping with continuous inverse given by

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi)\, d\xi.$$

Idea of Proof. The decay and smoothness of $f \in \mathcal{S}$ imply that $\widehat{f}$ is also smooth and rapidly decreasing. Differentiation and multiplication properties (Section 1.4) ensure that $\mathcal{F}$ preserves the seminorm structure $p_{\alpha,\beta}$, hence is a topological isomorphism. □

Remark 1.5.6. Under $\mathcal{F}$, differentiation in x corresponds to multiplication by $i\xi$, and multiplication by x^α corresponds to differentiation in ξ. Therefore, $\mathcal{S}(\mathbb{R}^n)$ is stable under both operations, which makes it the ideal test space for Fourier analysis.

Definition 1.5.7 (Tempered Distribution)**.** The *dual space* $\mathcal{S}'(\mathbb{R}^n)$ consists of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{S}'(\mathbb{R}^n) = \{T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} \text{ linear and continuous}\}.$$

An element $T \in \mathcal{S}'$ is called a *tempered distribution*.

Example 1.5.8. Every $f \in L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$ defines a tempered distribution by

$$\langle T_f, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Polynomials and the Dirac delta δ are also tempered distributions.

Remark 1.5.9. Tempered distributions are precisely those distributions that grow at most polynomially at infinity. This growth restriction guarantees that their Fourier transform, defined below, is again a tempered distribution.

Definition 1.5.10 (Fourier Transform in $\mathcal{S}'(\mathbb{R}^n)$)**.** For $T \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform $\widehat{T}$ is defined by duality:

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proposition 1.5.11 (Continuity and Invertibility)**.** *The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a topological isomorphism with inverse given by*

$$\langle \mathcal{F}^{-1} T, \varphi \rangle = (2\pi)^{-n} \langle T, \mathcal{F}^{-1} \varphi \rangle.$$

Example 1.5.12 (Fourier Transform of δ and of Constants)**.** For the Dirac delta δ and the constant function 1, we have

$$\widehat{\delta}(\xi) = 1, \qquad \widehat{1} = (2\pi)^n \delta.$$

Example 1.5.13 (Principal Value Distribution)**.** The distribution p.v. $\frac{1}{x}$ on $\mathbb{R}$, defined by

$$\left\langle \text{p.v.}\, \frac{1}{x}, \varphi \right\rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x}\, dx,$$

is tempered, and its Fourier transform is

$$\widehat{\text{p.v.}\, \tfrac{1}{x}}(\xi) = -i\pi \ \text{sgn}(\xi),$$

where $\text{sgn}(\xi)$ is the sign function.

Proposition 1.5.14 (Differentiation and Multiplication)**.** *For $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, define*

$$\partial^\alpha T : \varphi \mapsto (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle, \qquad x^\alpha T : \varphi \mapsto \langle T, x^\alpha \varphi \rangle.$$

Then $\partial^\alpha T$, $x^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$.

Proposition 1.5.15 (Fourier Duality)**.** *For all $T \in \mathcal{S}'(\mathbb{R}^n)$ and multi-indices α,*

$$\mathcal{F}(\partial^\alpha T) = (i\xi)^\alpha \widehat{T},$$
$$\mathcal{F}(x^\alpha T) = i^{|\alpha|}\, \partial_\xi^\alpha \widehat{T}.$$

Remark 1.5.16. These identities extend the classical differentiation–multiplication duality from smooth functions to tempered distributions, ensuring that the Fourier transform acts algebraically in the same way across $\mathcal{S}$ and $\mathcal{S}'$.

Definition 1.5.17. If $T, S \in \mathcal{S}'(\mathbb{R}^n)$ and at least one has compact support, their convolution is defined by

$$\langle T * S, \varphi \rangle = \langle T(x), \langle S(y), \varphi(x+y) \rangle \rangle, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proposition 1.5.18 (Fourier Transform of a Convolution)**.** *If $T, S \in \mathcal{S}'(\mathbb{R}^n)$ with one compactly supported, then*

$$\mathcal{F}(T * S) = \widehat{T}\,\widehat{S}.$$

Chapter 2

Symbol Classes and Pseudo-Differential Operators

The theory of pseudo-differential operators generalizes classical differential operators by allowing coefficients that depend smoothly on both spatial and frequency variables. This extension provides a unified analytical framework for studying broad classes of linear operators that appear in partial differential equations, microlocal analysis, and harmonic analysis.

In the classical setting, a linear differential operator of order m on $\mathbb{R}^n$ has the form

$$P(x, D)u(x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u(x),$$

where $D_j = \frac{1}{i}\partial_{x_j}$ and the coefficients $a_\alpha(x)$ are smooth functions. The behavior of $P(x, D)$ is largely governed by its *symbol*

$$p(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$$

a polynomial in ξ.

However, many analytical phenomena—such as variable order of differentiation, singular integrals, and nonlocal smoothing—cannot be captured by polynomial symbols alone.

To address this limitation, pseudo-differential operators are defined by allowing the symbol $p(x, \xi)$ to belong to a broader class of smooth functions that satisfy suitable growth conditions in ξ.

Such functions are called symbols, and they are organized into hierarchies called symbol classes, typically denoted by $S^m_{\rho,\delta}$, where the parameters $0 \le \delta < \rho \le 1$ encode how derivatives in x and ξ affect the order of growth.

This chapter develops the analytic foundations of symbol classes and the definition of pseudo-differential operators.

We begin with the motivation from differential operators, then introduce the general symbol classes $S^m_{\rho,\delta}$, define the associated operators T_a, and finally study asymptotic expansions of symbols.

These constructions form the backbone of pseudo-differential analysis and will be essential for understanding composition, adjoints, and elliptic parametrices in Chapter 3.

2.1 Motivation from Differential Operators

Classical differential operators provide the natural starting point for understanding pseudo-differential operators. Every linear differential operator can be represented through its *symbol*, which captures its essential behavior in the frequency domain via the Fourier transform. This symbolic viewpoint reveals the mechanism by which pseudo-differential operators extend the differential ones: by allowing symbols that are smooth but not necessarily polynomial in ξ.

Definition 2.1.1 (Linear Differential Operator)**.** Let $m \in \mathbb{N}_0$. A *linear differential operator of order* m on $\mathbb{R}^n$ is an operator of the form

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \qquad D_j = \frac{1}{i} \frac{\partial}{\partial x_j},$$

where each coefficient $a_\alpha : \mathbb{R}^n \to \mathbb{C}$ is a smooth function.

Example 2.1.2 (Constant-Coefficient Operator)**.** If the coefficients $a_\alpha(x)$ are constants, then

$$P(D)u = \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$$

is a constant-coefficient operator. Examples include:

$$\text{Laplacian: } \Delta = -\sum_{j=1}^{n} D_j^2,$$
$$\text{Wave operator: } \partial_t^2 - \Delta,$$
$$\text{Heat operator: } \partial_t - \Delta.$$

Example 2.1.3 (Variable-Coefficient Operator)**.** The operator

$$P(x,D)u(x) = -\sum_{i,j=1}^{n} a_{ij}(x)D_iD_ju(x) + \sum_{k=1}^{n} b_k(x)D_ku(x) + c(x)u(x)$$

with smooth coefficients a_{ij}, b_k, c represents a general second-order elliptic operator such as those appearing in diffusion or Schrödinger equations.

Definition 2.1.4 (Symbol)**.** The *symbol* of a differential operator

$$P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

is the polynomial in $\xi \in \mathbb{R}^n$ defined by

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha.$$

The action of $P(x,D)$ on $u \in \mathcal{S}(\mathbb{R}^n)$ can be expressed via the Fourier transform as

$$(P(x,D)u)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\,\widehat{u}(\xi)\,d\xi.$$

Hence, the operator acts by multiplication in the frequency variable ξ with the symbol $p(x,\xi)$.

Remark 2.1.5. If $a_\alpha(x)$ are constants, then $p(\xi)$ is independent of x, and $P(D)$ corresponds exactly to the Fourier multiplier $p(\xi)$:

$$\widehat{P(D)u}(\xi) = p(\xi)\,\widehat{u}(\xi).$$

Definition 2.1.6 (Principal Symbol)**.** For a differential operator of order m, the *principal symbol* is the homogeneous polynomial

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

The principal symbol encodes the leading-order behavior of the operator for large $|\xi|$ and determines many of its analytical properties, such as ellipticity and propagation of singularities.

Example 2.1.7 (Principal Symbol of the Laplacian)**.** For $\Delta = -\sum_{j=1}^{n} D_j^2$, the principal symbol is

$$p_2(x,\xi) = |\xi|^2.$$

Definition 2.1.8 (Ellipticity)**.** A differential operator $P(x, D)$ of order m is called *elliptic* if

$$|p_m(x,\xi)| \geq c\,|\xi|^m \quad \text{for all } |\xi| \geq 1,$$

for some constant $c > 0$ and all $x \in \mathbb{R}^n$.

The symbolic representation above suggests that a differential operator can be realized as

$$(P(x, D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\,\widehat{u}(\xi)\,d\xi.$$

If we now allow $p(x, \xi)$ to be a smooth function rather than a polynomial—subject to certain growth conditions in ξ—we obtain a far more flexible class of operators known as *pseudo-differential operators*.

These operators extend differential ones by:

- capturing nonlocal phenomena through non-polynomial dependence on ξ,
- permitting fractional or variable order differentiation,
- encompassing singular integral and smoothing operators within a unified framework.

Example 2.1.9 (Fractional Laplacian)**.** The operator $(-\Delta)^{s/2}$ with symbol $|\xi|^s$ for $s > 0$ is a pseudo-differential operator of order s. It generalizes the classical Laplacian to non-integer order derivatives.

Remark 2.1.10. From this perspective, classical differential operators correspond precisely to the special case where the symbol $p(x, \xi)$ is a polynomial in ξ. All subsequent sections of this chapter will formalize this idea by introducing general symbol classes $S^m_{\rho,\delta}$ and defining the corresponding operators T_a via oscillatory integrals.

2.2 Symbol Classes $S^m_{\rho,\delta}$

In the previous section we observed that differential operators are completely characterized by polynomial symbols of the form

$$p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\,\xi^\alpha.$$

To extend this framework to more general operators, we replace the polynomial condition by suitable *growth estimates* on $p(x,\xi)$ and its derivatives in both x and ξ. The resulting functions form the *symbol classes* $S^m_{\rho,\delta}$, which serve as the analytic backbone of pseudo-differential calculus.

Definition 2.2.1 (Symbol Class $S^m_{\rho,\delta}$)**.** Let $m \in \mathbb{R}$ and parameters $0 \le \delta < \rho \le 1$. A function $a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ is said to belong to the *symbol class* $S^m_{\rho,\delta}$ if, for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}. \tag{2.1}$$

Remark 2.2.2. The parameter m indicates the *order* of the symbol, while (ρ,δ) measure how derivatives affect its behavior:

- Differentiation in ξ lowers the order by ρ per derivative.
- Differentiation in x raises the order by δ per derivative.

Thus, symbols with smaller ρ or larger δ exhibit weaker decay under differentiation and describe more general operators.

Example 2.2.3. For a differential operator

$$P(x,D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

its symbol $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ belongs to $S^m_{1,0}$. Indeed, each derivative $\partial_\xi^\alpha p(x,\xi)$ is a polynomial of degree at most $m - |\alpha|$ in ξ, so

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}.$$

Example 2.2.4 (Constant-Coefficient Symbol)**.** The function $a(\xi) = (1+|\xi|^2)^{m/2}$ belongs to $S^m_{1,0}$ because

$$|\partial_\xi^\alpha a(\xi)| \le C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

This symbol corresponds to the fractional Laplacian $(-\Delta)^{m/2}$.

Proposition 2.2.5 (Symbol Inclusion)**.** *If* $m_1 \le m_2$ *and* $0 \le \delta_1 \le \delta_2 < \rho_2 \le \rho_1 \le 1$*, then*

$$S^{m_1}_{\rho_1,\delta_1} \subseteq S^{m_2}_{\rho_2,\delta_2}.$$

Proof. The inclusion follows directly from estimate (2.1), since larger m, smaller ρ, or larger δ relax the growth restrictions. $\square$

Remark 2.2.6 (Typical Parameter Choices)**.** The most common cases are:

- $S^m_{1,0}$: the *classical* symbol class — stable under composition, used in standard pseudo-differential theory.
- $S^m_{\rho,\delta}$ with $\rho < 1$: symbols exhibiting limited regularity in ξ, useful in microlocal analysis.
- $S^m_{1,1}$: the *forbidden* class, in general not closed under composition, hence excluded.

Definition 2.2.7 (Symbol of Lower Order)**.** Let $a \in S^m_{\rho,\delta}$ and $b \in S^{m'}_{\rho,\delta}$. If $m' < m$, we say that b is of *lower order* than a. The class

$$S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}$$

consists of symbols that, together with all their derivatives, decay faster than any power of $|\xi|$; they correspond to smoothing operators.

Example 2.2.8 (Smoothing Symbol)**.** Let $a(x,\xi) = e^{-|\xi|^2}$. For every α, β,

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{-N} \quad \forall N > 0,$$

so $a \in S^{-\infty}$.

Definition 2.2.9 (Symbol Seminorms)**.** For $a \in S^m_{\rho,\delta}$, define

$$\|a\|_{(\alpha,\beta)} = \sup_{(x,\xi)\in\mathbb{R}^{2n}} (1+|\xi|)^{-m+\rho|\alpha|-\delta|\beta|} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)|.$$

The topology on $S^m_{\rho,\delta}$ is the Fréchet topology generated by all seminorms $\|a\|_{(\alpha,\beta)}$.

Remark 2.2.10. The space $S^m_{\rho,\delta}$ is complete and locally convex. Convergence $a_j \to a$ in $S^m_{\rho,\delta}$ means that all derivatives of $a_j - a$ satisfy uniform weighted bounds tending to zero as $j \to \infty$.

Example 2.2.11 (Polynomially Growing Coefficients)**.** Let $a(x,\xi) = (1 + |x|^2)^{k/2}(1+|\xi|^2)^{m/2}$. Then $a \in S^m_{1,1}$ but $a \notin S^m_{1,0}$ because derivatives in x increase the growth in ξ.

Example 2.2.12 (Variable-Coefficient Differential Symbol)**.** If $a_\alpha(x)$ are smooth and satisfy

$$|\partial_x^\beta a_\alpha(x)| \le C_{\alpha,\beta}(1+|x|)^{r_\alpha},$$

then

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in S^{m+r}_{1,0},$$

where $r = \max_\alpha r_\alpha$.

2.3 Definition of Pseudo-Differential Operator T_a

Having introduced the symbol classes $S^m_{\rho,\delta}$, we now define the pseudo-differential operator associated with a symbol $a(x,\xi)$. This operator acts on functions (or distributions) through an oscillatory integral that generalizes the Fourier representation of differential operators. The resulting class of operators extends classical differential operators, Fourier multipliers, and smoothing operators within a unified analytical framework.

Definition 2.3.1 (Pseudo-Differential Operator)**.** Let $a(x,\xi) \in S^m_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$. The *pseudo-differential operator* T_a (also denoted $a(x,D)$) associated with the symbol a is defined for $u \in \mathcal{S}(\mathbb{R}^n)$ by

$$(T_a u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\, \widehat{u}(\xi)\, d\xi. \tag{2.2}$$

Equivalently, in kernel form,

$$(T_a u)(x) = \int_{\mathbb{R}^n} K_a(x,y)\, u(y)\, dy, \tag{2.3}$$

where the distribution kernel K_a is given by

$$K_a(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,\xi)\, d\xi. \tag{2.4}$$

Remark 2.3.2. Formula (2.2) generalizes the Fourier multiplier

$$(T_a u)(x) = \mathcal{F}^{-1}\big(a(\xi)\, \widehat{u}(\xi)\big)(x),$$

which corresponds to the case $a(x,\xi) = a(\xi)$ (independent of x). When $a(x,\xi)$ is polynomial in ξ, T_a reduces to a classical differential operator.

Example 2.3.3 (Differential Operators)**.** If

$$a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\,\xi^\alpha \quad \text{(symbol of a differential operator)},$$

then

$$T_a u(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u(x) = P(x,D)u(x),$$

so every differential operator is a pseudo-differential operator with symbol in $S^m_{1,0}$.

Example 2.3.4 (Fourier Multiplier)**.** If $a(x,\xi) = a(\xi)$ depends only on ξ, then T_a acts by multiplication in the frequency domain:

$$\widehat{T_a u}(\xi) = a(\xi)\,\widehat{u}(\xi).$$

For instance, the fractional Laplacian $(-\Delta)^{s/2}$ corresponds to the multiplier $a(\xi) = |\xi|^s$.

Example 2.3.5 (Smoothing Operator)**.** If $a \in S^{-\infty}$, the kernel $K_a(x,y)$ is a smooth function on $\mathbb{R}^{2n}$, and T_a maps $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$. Hence T_a is called a *smoothing operator*.

Proposition 2.3.6. *If $a \in S^m_{\rho,\delta}$ with $0 \leq \delta < \rho \leq 1$, then*

$$T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear operator.

Idea of Proof. Differentiation under the integral in (2.2), together with estimates (2.1), shows that every derivative $\partial_x^\beta (T_a u)(x)$ can be written as a finite linear combination of terms of the same form with additional factors $(1+|\xi|)^N$. Since $u \in \mathcal{S}$, all such integrals converge rapidly, ensuring that $T_a u \in \mathcal{S}$. □

Remark 2.3.7. The continuity of T_a on $\mathcal{S}$ also implies, by duality, that T_a extends uniquely to a continuous operator

$$T_a : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

Proposition 2.3.8 (Kernel Regularity)**.** *Let $a \in S^m_{\rho,\delta}$. Then the kernel $K_a(x,y)$ defined by (2.4) satisfies:*

(i) $K_a(x,y) \in \mathcal{C}^\infty(\mathbb{R}^{2n} \setminus \{x=y\})$*;*

(ii) if $a \in S^{-\infty}$, then $K_a \in \mathcal{C}^\infty(\mathbb{R}^{2n})$;

(iii) T_a is local modulo smoothing terms, *in the sense that*

$$\operatorname{sing\,supp}(T_a u) \subseteq \operatorname{sing\,supp}(u).$$

Sketch of Proof. Each derivative $\partial_x^\beta \partial_y^\gamma K_a(x, y)$ is given by an oscillatory integral with integrand in $S^{m+\delta|\beta|+\rho|\gamma|}_{\rho,\delta}$, which is smooth away from $x = y$. Rapid decay in ξ for large $|x - y|$ follows from integration by parts. □

The definition (2.2) is known as the *Kohn–Nirenberg quantization.* Other quantizations—such as the *Weyl quantization*—are often used in microlocal analysis.

Definition 2.3.9 (Weyl Quantization)**.** Given $a(x, \xi) \in S^m_{\rho,\delta}$, the *Weyl quantization* of a is defined by

$$(T_a^{\mathrm{W}} u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi.$$

Remark 2.3.10. The Weyl quantization is symmetric with respect to x and y and satisfies $(T_a^{\mathrm{W}})^* = T_{\overline{a}}^{\mathrm{W}}$. In contrast, the Kohn–Nirenberg quantization does not, in general, preserve adjointness, but it simplifies symbolic calculus and will be used throughout these notes.

Proposition 2.3.11 (Smoothing Property for $m < 0$)**.** *If $a \in S^m_{\rho,\delta}$ with $m < 0$, then T_a is a smoothing operator: for all $N > 0$, there exists $C_N > 0$ such that*

$$\|T_a u\|_{H^N} \leq C_N \|u\|_{L^2}.$$

In particular, $T_a : L^2 \to H^N$ continuously for all $N > 0$.

Idea of Proof. When $m < 0$, the kernel $K_a(x, y)$ decays rapidly in $|x - y|$. Fourier analysis then implies that T_a maps L^2 functions into arbitrarily smooth ones, hence is compact on L^2. □

2.4 Asymptotic Expansions of Symbols

An essential feature of pseudo-differential calculus is the ability to approximate complicated symbols by infinite series of simpler ones with successively decreasing orders. Such *asymptotic expansions* allow us to describe compositions, adjoints, and parametrices in a symbolic form. Although the resulting series typically do not converge in the classical sense, they converge *asymptotically* in the topology of symbol classes.

Let $a(x,\xi) \in S^m_{\rho,\delta}$. In many constructions—such as forming inverses or compositions—it is convenient to represent a as a formal series

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi),$$

where each $a_j \in S^{m_j}_{\rho,\delta}$ with $m_j \to -\infty$. This notation expresses that, after subtracting any finite number of terms, the remainder is a symbol of strictly lower order.

Definition 2.4.1 (Asymptotic Expansion)**.** Let $\{a_j\}_{j=0}^{\infty}$ be a sequence of symbols with $a_j \in S^{m_j}_{\rho,\delta}$, where $m_j \searrow -\infty$. A symbol $a \in S^{m_0}_{\rho,\delta}$ is said to have the *asymptotic expansion*

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi) \tag{2.5}$$

if for every integer $N \geq 1$,

$$a(x,\xi) - \sum_{j=0}^{N-1} a_j(x,\xi) \in S^{m_N}_{\rho,\delta}.$$

Remark 2.4.2. The notation (2.5) does not imply convergence of the series $\sum a_j(x,\xi)$; instead, it expresses that the partial sums approximate a increasingly well in the sense of symbol classes.

Theorem 2.4.3 (Existence of Asymptotic Sums)**.** *Let $\{a_j\}_{j=0}^{\infty}$ be a sequence with $a_j \in S^{m_j}_{\rho,\delta}$ and $m_j \searrow -\infty$. Then there exists $a \in S^{m_0}_{\rho,\delta}$ such that*

$$a \sim \sum_{j=0}^{\infty} a_j.$$

Moreover, a is unique modulo $S^{-\infty}$.

Sketch of Proof. Choose a smooth cutoff function $\chi(\xi)$ satisfying $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. Define

$$a(x,\xi) = \sum_{j=0}^{\infty} \big(1 - \chi(\varepsilon_j \xi)\big) a_j(x,\xi),$$

where $\varepsilon_j > 0$ decreases rapidly to zero. Each term $(1 - \chi(\varepsilon_j \xi))a_j$ vanishes for small $|\xi|$ and preserves the symbol estimates for large $|\xi|$. By appropriate choice of ε_j, the series converges in $S^{m_0}_{\rho,\delta}$ and satisfies the desired asymptotic property. □

Remark 2.4.4. Uniqueness modulo $S^{-\infty}$ means that if a and b both satisfy $a \sim \sum a_j$ and $b \sim \sum a_j$, then $a - b \in S^{-\infty}$. Thus, the smoothing part of a symbol is irrelevant to the asymptotic structure.

Proposition 2.4.5. *If $a \sim \sum a_j$ and $b \sim \sum b_j$ in $S^m_{\rho,\delta}$ and $S^{m'}_{\rho,\delta}$, respectively, then:*

(i) $a + b \sim \sum(a_j + b_j)$ *in* $S^{\max(m,m')}_{\rho,\delta}$*;*

(ii) $\partial^\alpha_x \partial^\beta_\xi a \sim \sum \partial^\alpha_x \partial^\beta_\xi a_j$*;*

(iii) if ab is defined pointwise, then $ab \sim \sum_{j,k} a_j b_k$ in $S^{m+m'}_{\rho,\delta}$.

Proof. Each assertion follows from the product and derivative estimates defining $S^m_{\rho,\delta}$, using that lower-order terms decay sufficiently fast to preserve the asymptotic hierarchy. □

Definition 2.4.6 (Formal Symbol)**.** A *formal symbol* of order m is a formal series

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi), \qquad a_j \in S^{m_j}_{\rho,\delta}, \quad m_j \searrow -\infty.$$

The collection of all such formal series is denoted by

$$F^m_{\rho,\delta} = \left\{ \sum_{j=0}^{\infty} a_j : a_j \in S^{m_j}_{\rho,\delta},\ m_j \to -\infty \right\}.$$

Remark 2.4.7. Formal symbols behave algebraically like elements of $S^m_{\rho,\delta}$ and form a graded algebra under pointwise addition, multiplication, and differentiation. In later chapters, these formal symbols will encode the asymptotic structure of composed or adjoint operators.

Definition 2.4.8 (Classical Symbol)**.** A symbol $a(x,\xi) \in S^m_{1,0}$ is called *classical* (or *polyhomogeneous*) if it admits an asymptotic expansion

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi),$$

where each $a_{m-j}(x,\xi)$ is positively homogeneous of degree $m-j$ in ξ, i.e.,

$$a_{m-j}(x,t\xi) = t^{m-j}a_{m-j}(x,\xi) \quad \text{for } t \geq 1,\ |\xi| \geq 1.$$

Example 2.4.9 (Differential Operator)**.** For a differential operator

$$P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha,$$

the symbol

$$p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$$

is a classical symbol with homogeneous components

$$p_{m-j}(x,\xi) = \sum_{|\alpha|=m-j} a_\alpha(x)\xi^\alpha.$$

Remark 2.4.10. The class of classical symbols is particularly important in the study of elliptic operators, since asymptotic expansions of homogeneous components allow one to construct parametrices and to compute principal parts of inverses.

Chapter 3

Composition, Adjoint, and Parametrix

The calculus of pseudo-differential operators extends far beyond their definition: it provides algebraic rules for composing operators, computing adjoints, and constructing approximate inverses (parametrices) for elliptic operators. These operations are governed not by direct manipulations in the physical space, but through symbolic rules that describe how their symbols interact in the frequency domain.

In the previous chapter, we introduced the symbol classes $S^m_{\rho,\delta}$ and the pseudo-differential operator

$$(T_a u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\, \widehat{u}(\xi)\, d\xi,$$

where $a(x,\xi)$ satisfies uniform growth estimates in (x,ξ). We now develop the algebraic structure of such operators and study how their symbols behave under composition and adjoint operations.

The key result is that the composition of two pseudo-differential operators T_a and T_b is again a pseudo-differential operator T_c, whose symbol $c(x,\xi)$ admits an asymptotic expansion

$$c(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!}\, \partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi).$$

This symbolic formula allows one to perform operator algebra purely at the level of symbols. A similar expansion governs the adjoint operator T_a^*, whose symbol is asymptotically determined by the conjugates of derivatives of $a(x,\xi)$.

Finally, when $a(x,\xi)$ is *elliptic*—that is, its principal part is invertible for large $|\xi|$—we can construct a *parametrix* T_b such that

$$T_aT_b = I + R_1, \qquad T_bT_a = I + R_2,$$

where R_1 and R_2 are smoothing operators. This result forms the analytical foundation of elliptic regularity theory developed in later chapters.

In summary, this chapter establishes the symbolic calculus of pseudo-differential operators—composition, adjoint, and parametrix—showing that these operators form an algebra closed under the principal operations of analysis.

3.1 Composition of Pseudo-Differential Operators

In this section we establish that the composition of two pseudo-differential operators is again a pseudo-differential operator. Moreover, its symbol is given by an explicit asymptotic expansion obtained through Taylor's formula and repeated integrations by parts. Throughout we assume $0 \le \delta < \rho \le 1$.

Theorem 3.1.1 (Composition Formula)**.** *Let $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$. Then the composition $T_a \circ T_b$ is a pseudo-differential operator T_c with symbol $c \in S^{m_1+m_2}_{\rho,\delta}$ admitting the asymptotic expansion*

$$c(x,\xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!}\, \partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi). \tag{3.1}$$

More precisely, for each $N \in \mathbb{N}$,

$$c(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi) \in S^{m_1+m_2-(\rho-\delta)N}_{\rho,\delta}. \tag{3.2}$$

Remark 3.1.2. The loss $(\rho-\delta)N$ in the remainder order is sharp for general $S_{\rho,\delta}$. In the classical class $S_{1,0}$ we gain one full order per term.

Recall

$$(T_a u)(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} a(x,\xi) \widehat{u}(\xi)\, d\xi, \quad (T_b u)(y) = (2\pi)^{-n} \int e^{i\langle y,\eta\rangle} b(y,\eta) \widehat{u}(\eta)\, d\eta.$$

Then

$$\begin{aligned}
(T_aT_bu)(x) &= (2\pi)^{-2n} \int e^{i\langle x,\xi\rangle} a(x,\xi) \left(\int e^{-i\langle y,\xi\rangle} (T_b u)(y)\, dy \right) d\xi \\
&= (2\pi)^{-n} \int e^{i\langle x,\eta\rangle} \left[(2\pi)^{-n} \int e^{i\langle x-y,\xi-\eta\rangle} a(x,\xi)\, b(y,\eta)\, d\xi\, dy \right] \widehat{u}(\eta)\, d\eta.
\end{aligned}$$

Thus $T_aT_b = T_c$ with symbol

$$c(x,\eta) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi-\eta\rangle} a(x,\xi)\, b(y,\eta)\, dy\, d\xi. \tag{3.3}$$

We now rewrite $b(y,\eta)$ around $y = x$ using Taylor's formula:

$$b(y,\eta) = \sum_{|\alpha|<N} \frac{(y-x)^\alpha}{\alpha!}\,\partial_x^\alpha b(x,\eta) + R_N(x,y,\eta),$$

where

$$R_N(x,y,\eta) = \sum_{|\alpha|=N} \frac{N}{\alpha!}\,(y-x)^\alpha \int_0^1 (1-t)^{N-1}\,\partial_x^\alpha b(x+t(y-x),\eta)\,dt.$$

Insert the Taylor expansion into (3.3). For the finite sum we use the identity

$$(y-x)^\alpha e^{i\langle x-y,\xi-\eta\rangle} = i^{|\alpha|}\,\partial_\xi^\alpha\big(e^{i\langle x-y,\xi-\eta\rangle}\big),$$

and integrate by parts in ξ:

$$\begin{aligned}
c_N(x,\eta) &= (2\pi)^{-n}\sum_{|\alpha|<N}\frac{i^{|\alpha|}}{\alpha!}\iint \partial_\xi^\alpha\big(e^{i\langle x-y,\xi-\eta\rangle}\big)\,a(x,\xi)\,\partial_x^\alpha b(x,\eta)\,dy\,d\xi \\
&= \sum_{|\alpha|<N}\frac{1}{\alpha!}\int \Big(\partial_\xi^\alpha a(x,\xi)\Big)\left[(2\pi)^{-n}\int e^{i\langle x-y,\xi-\eta\rangle}dy\right]\partial_x^\alpha b(x,\eta)\,d\xi \\
&= \sum_{|\alpha|<N}\frac{1}{\alpha!}\,\partial_\xi^\alpha a(x,\eta)\,D_x^\alpha b(x,\eta),
\end{aligned}$$

which yields the partial sum in (3.1) (using $D_x^\alpha = (1/i)^{|\alpha|}\partial_x^\alpha$).

The remainder symbol is

$$r_N(x,\eta) = (2\pi)^{-n}\iint e^{i\langle x-y,\xi-\eta\rangle} a(x,\xi)\,R_N(x,y,\eta)\,dy\,d\xi.$$

Introduce the adjoint of the differential operator

$$L = \big(1-\Delta_\xi\big)^M\big(1+|x-y|^2\big)^{-M},$$

for M large. Integrating by parts M times transfers derivatives to $a(x,\xi)$ and to the y–integrand inside R_N. By symbol estimates for a and $\partial_x^\alpha b$ and the decay from $(1+|x-y|^2)^{-M}$, one obtains

$$r_N \in S^{m_1+m_2-(\rho-\delta)N}_{\rho,\delta},$$

which proves (3.2) and completes the proof of Theorem 3.1.1.

Corollary 3.1.3 (Closure). *If $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$, then*

$$T_a T_b = T_c, \qquad c \in S^{m_1+m_2}_{\rho,\delta}.$$

Example 3.1.4 (Differential After Multiplier). Let $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in S^m_{1,0}$ (differential) and $b(\xi) \in S^{m'}_{1,0}$ (Fourier multiplier). Then

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(\xi) = a(x,\xi)\, b(\xi),$$

since $D_x^\alpha b(\xi) = 0$ for $|\alpha| > 0$, hence $T_a T_b = T_{ab}$.

Example 3.1.5 (Paradifferential Flavor). If $a = a(x)$ (order 0 multiplier in x) and $b = b(\xi)$, then $c(x,\xi) = a(x)b(\xi)$ exactly. If $a = a(x,\xi)$ depends on both variables, the leading correction is

$$c(x,\xi) = a(x,\xi)b(\xi) + \sum_{|\alpha|=1} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(\xi) + \cdots,$$

exhibiting the fundamental x–ξ interaction.

Corollary 3.1.6 (Principal Symbol). *If $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$, then the principal symbols satisfy*

$$\sigma_{pr}(T_a T_b) = \sigma_{pr}(T_a)\, \sigma_{pr}(T_b).$$

In particular, for classical symbols ($S_{1,0}$ polyhomogeneous), the component of degree $m_1 + m_2$ in c is $a_{m_1} b_{m_2}$.

3.2 Formal Adjoint Operator

The adjoint of a pseudo-differential operator plays a fundamental role in analysis, particularly in the study of self-adjoint and elliptic operators. In this section we derive the symbolic representation of the adjoint operator T_a^* and show that it is itself a pseudo-differential operator whose symbol admits an asymptotic expansion similar to that of compositions.

Definition 3.2.1 (Formal Adjoint). Let $a \in S^m_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$, and let T_a be defined by

$$(T_a u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\, \widehat{u}(\xi)\, d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n).$$

The *formal adjoint* T_a^* is the unique operator satisfying

$$(T_a u, v)_{L^2} = (u, T_a^* v)_{L^2}, \qquad \forall\, u, v \in \mathcal{S}(\mathbb{R}^n).$$

Using the integral kernel representation

$$(T_a u)(x) = \int_{\mathbb{R}^n} K_a(x,y)u(y)\,dy, \qquad K_a(x,y) = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}a(x,\xi)\,d\xi,$$

we find that

$$K_{a^*}(x,y) = \overline{K_a(y,x)} = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}\overline{a(y,\xi)}\,d\xi.$$

Hence T_a^* is again a pseudo-differential operator with symbol $a^*(x,\xi)$ given by the oscillatory integral

$$a^*(x,\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle}\overline{a(x+y,\xi+\eta)}\,dy\,d\eta. \tag{3.4}$$

Theorem 3.2.2 (Symbol of the Adjoint)**.** *If $a \in S^m_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$, then*

$$a^*(x,\xi) \in S^m_{\rho,\delta}$$

and admits the asymptotic expansion

$$a^*(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\,\partial_\xi^\alpha D_x^\alpha\overline{a(x,\xi)}. \tag{3.5}$$

More precisely, for every integer $N \ge 1$,

$$a^*(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!}\,\partial_\xi^\alpha D_x^\alpha\overline{a(x,\xi)} \in S^{m-(\rho-\delta)N}_{\rho,\delta}.$$

Sketch of Proof. Expanding $\overline{a(x+y,\xi+\eta)}$ in Taylor series around (x,ξ) gives

$$\overline{a(x+y,\xi+\eta)} = \sum_{|\alpha+\beta|<N} \frac{y^\alpha\eta^\beta}{\alpha!\beta!}\,\partial_x^\alpha\partial_\xi^\beta\overline{a(x,\xi)} + R_N(x,y,\eta).$$

Substituting into (3.4) and noting that

$$(2\pi)^{-n}\int e^{-i\langle y,\eta\rangle}y^\alpha\eta^\beta\,dy\,d\eta = (i)^{|\alpha|}(-i)^{|\beta|}\,\alpha!\,\delta_{\alpha\beta},$$

we obtain

$$a^*(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\,\partial_\xi^\alpha D_x^\alpha\overline{a(x,\xi)} + r_N(x,\xi),$$

where $r_N \in S^{m-(\rho-\delta)N}_{\rho,\delta}$ follows from symbol estimates on the Taylor remainder. □

Remark 3.2.3. The leading term of a^* is simply $\overline{a}$. Thus, if a is real-valued, the principal symbol of T_a is self-adjoint:

$$\sigma_{\text{pr}}(T_a^*) = \overline{\sigma_{\text{pr}}(T_a)} = \sigma_{\text{pr}}(T_a).$$

Example 3.2.4 (Differential Operator)**.** For a differential operator $P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$,

$$T_a^* = P^*(x, D) = \sum_{|\alpha| \le m} D^\alpha \, \overline{a_\alpha(x)}.$$

This follows directly since $\partial_\xi^\alpha D_x^\alpha \overline{a(x, \xi)}$ reproduces the conjugated coefficients.

Example 3.2.5 (Fourier Multiplier)**.** If $a(\xi)$ depends only on ξ, then T_a is a Fourier multiplier and

$$T_a^* = T_{\overline{a}},$$

since $a^*(x, \xi) = \overline{a(\xi)}$ by (3.5).

Example 3.2.6 (Weyl Quantization)**.** In the Weyl quantization, the adjoint satisfies $(T_a^{\mathrm{W}})^* = T_{\overline{a}}^{\mathrm{W}}$ exactly, without lower-order corrections. This symmetry property motivates the frequent use of Weyl quantization in self-adjoint operator theory.

Corollary 3.2.7 (Principal Symbol)**.** *For $a \in S^m_{\rho,\delta}$, the principal symbol of the adjoint satisfies*

$$\sigma_{\text{pr}}(T_a^*) = \overline{\sigma_{\text{pr}}(T_a)}.$$

In particular, T_a is formally self-adjoint if and only if $\sigma_{\text{pr}}(T_a)$ is real-valued.

3.3 Elliptic Symbols and Parametrix Construction

Elliptic symbols form the analytic core of pseudo-differential operator theory. They generalize the notion of ellipticity from classical differential operators to the pseudo-differential framework and allow the construction of *parametrices*—approximate inverses up to smoothing operators. This section establishes the definition of ellipticity, constructs a parametrix symbolically, and states its principal consequences.

Definition 3.3.1 (Elliptic Symbol)**.** Let $a \in S^m_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$. We say that a is an *elliptic symbol of order m* if there exist constants $C, R > 0$ such that

$$|a(x, \xi)| \ge C\,(1 + |\xi|)^m, \qquad \text{for all } |\xi| \ge R \text{ and } x \in \mathbb{R}^n. \tag{3.6}$$

Remark 3.3.2. Condition (3.6) ensures that the symbol does not vanish too rapidly at high frequencies. In the differential case $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, this coincides with the usual definition of ellipticity: the principal symbol $a_m(x,\xi)$ is nonzero for $\xi \neq 0$.

Example 3.3.3 (Elliptic Differential Operator)**.** The Laplacian $\Delta = -\sum_{j=1}^n D_j^2$ has symbol $a(\xi) = |\xi|^2$ satisfying $|a(\xi)| = |\xi|^2 \ge C(1+|\xi|)^2$ for large $|\xi|$; hence it is elliptic of order 2.

For an elliptic symbol $a(x,\xi)$, we seek a symbol $b(x,\xi)$ such that

$$T_aT_b = I + R_1, \qquad T_bT_a = I + R_2, \tag{3.7}$$

where R_1 and R_2 are smoothing operators. Such T_b is called a *parametrix* of T_a.

Theorem 3.3.4 (Parametrix Construction)**.** *Let $a \in S^m_{\rho,\delta}$ be elliptic with $0 \le \delta < \rho \le 1$. Then there exists $b \in S^{-m}_{\rho,\delta}$ such that*

$$T_aT_b = I + R_1, \qquad T_bT_a = I + R_2,$$

with $R_1, R_2 \in \Psi^{-\infty}$ (smoothing operators). Moreover, b admits an asymptotic expansion

$$b(x,\xi) \sim \sum_{j=0}^{\infty} b_{-m-j}(x,\xi), \tag{3.8}$$

where each $b_{-m-j} \in S^{-m-j}_{\rho,\delta}$.

Sketch of Proof. Let $\chi(\xi)$ be a smooth cutoff function such that $\chi(\xi) = 0$ for $|\xi| \le R$ and $\chi(\xi) = 1$ for $|\xi| \ge 2R$, where R is as in (3.6). Define the zeroth-order approximation

$$b_0(x,\xi) = \chi(\xi)\, a(x,\xi)^{-1}.$$

Then $b_0 \in S^{-m}_{\rho,\delta}$. Compute

$$T_aT_{b_0} = I - T_{r_1},$$

where $r_1 = 1 - a\#b_0$ and "#" denotes the symbol composition from Theorem 3.1.1. Since $ab_0 = 1$ for large $|\xi|$, $r_1 \in S^{-(\rho-\delta)}_{\rho,\delta}$.

To eliminate r_1 iteratively, define symbols b_j recursively by

$$b_{j+1} = b_0\, r_1^{\#(j+1)} = b_0\#r_1\#\cdots\#r_1,$$

and form the formal series $b \sim \sum_{j=0}^{\infty} b_j$. By Theorem 2.4.3, there exists $b \in S^{-m}_{\rho,\delta}$ realizing this asymptotic expansion and satisfying

$$a\#b = 1 - r_\infty, \quad r_\infty \in S^{-\infty}.$$

Thus $T_a T_b = I - T_{r_\infty}$, and T_{r_∞} is smoothing. The second relation $T_b T_a = I + R_2$ follows similarly by symmetry. □

Proposition 3.3.5 (Uniqueness Modulo Smoothing Operators)**.** *If b_1 and b_2 are parametrices of T_a, then*

$$b_1 - b_2 \in S^{-\infty}.$$

Hence the parametrix is unique modulo smoothing symbols.

Proof. If $T_a T_{b_1} = I + R_1$ and $T_a T_{b_2} = I + R_2$, then

$$T_a(T_{b_1} - T_{b_2}) = R_2 - R_1.$$

Since the right-hand side is smoothing and T_a is elliptic, this implies $T_{b_1} - T_{b_2}$ is smoothing. □

Corollary 3.3.6 (Inverse Modulo Smoothing)**.** *If T_a is elliptic, then T_a is invertible modulo smoothing operators:*

$$T_a^{-1} = T_b + S, \quad S \in \Psi^{-\infty}.$$

The parametrix construction immediately yields a fundamental regularity property.

Theorem 3.3.7 (Elliptic Regularity)**.** *Let T_a be an elliptic pseudo-differential operator of order m with parametrix $T_b \in \Psi^{-m}$. If $T_a u \in \mathcal{C}^\infty(\mathbb{R}^n)$ for $u \in \mathcal{S}'(\mathbb{R}^n)$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

Proof. Since $T_b T_a = I + R_2$ with R_2 smoothing,

$$u = T_b T_a u - R_2 u.$$

The right-hand side is smooth because $T_a u$ is smooth and both T_b and R_2 map $\mathcal{S}'$ to $\mathcal{C}^\infty$. □

Remark 3.3.8. The parametrix thus provides a symbolic inverse for elliptic operators, showing that ellipticity implies local regularity and, in particular, hypoellipticity. This principle underlies the global elliptic theory developed in Chapter 5.

Chapter 4

L_p-Boundedness and Sobolev Spaces

One of the principal achievements of pseudo-differential operator theory is its ability to extend classical mapping properties of differential operators to a broad analytic setting. In particular, pseudo-differential operators of order zero act as bounded operators on L_p and Sobolev spaces, providing the analytic machinery necessary for elliptic regularity and partial differential equations in non-smooth settings.

In the preceding chapters, we introduced symbol classes $S^m_{\rho,\delta}$, defined the associated pseudo-differential operators T_a, and established their symbolic calculus—composition, adjoint, and parametrix construction. These results were formulated primarily on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$. We now extend our attention to functional-analytic settings involving L_p and Sobolev spaces, where such operators exhibit strong continuity properties.

The first part of this chapter is devoted to the L_2 and L_p-boundedness of pseudo-differential operators with symbols in $S^0_{1,0}$. In particular, we will prove that if $a \in S^0_{1,0}$, then T_a extends continuously to a bounded operator on $L_2(\mathbb{R}^n)$. We will then discuss the more delicate L_p-boundedness results for $1 < p < \infty$, which require additional techniques from Calderón–Zygmund theory.

The second part introduces Sobolev spaces $H^{s,p}$ and H^s, defined via the Fourier transform. Pseudo-differential operators act continuously between these spaces according to their symbolic order:

$$T_a : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n), \qquad a \in S^m_{1,0}.$$

These mapping properties generalize the classical relations between differential operators and Sobolev regularity.

In summary, this chapter develops the analytic framework that connects pseudo-differential operators with L_p and Sobolev space theory, preparing the ground for the elliptic regularity results of Chapter 5.

4.1 Boundedness on L_2 and L_p

A fundamental analytic property of pseudo-differential operators is their boundedness on L_p spaces. In particular, operators of order zero act continuously on $L_2(\mathbb{R}^n)$, extending the classical Plancherel theorem for Fourier multipliers. This result, due to Calderón and Vaillancourt, provides the cornerstone of pseudo-differential operator theory.

Theorem 4.1.1 (Calderón–Vaillancourt)**.** *Let $a \in S^0_{1,0}$. Then the operator*

$$T_a u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\, \widehat{u}(\xi)\, d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n),$$

extends to a bounded linear operator on $L_2(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that

$$\|T_a u\|_{L_2} \leq C \sup_{|\alpha|,|\beta|\leq N} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty} \|u\|_{L_2}, \tag{4.1}$$

where N depends only on n (typically $N > n/2 + 1$).

Sketch of Proof. The proof relies on Schur's lemma and oscillatory integral estimates. Integrating by parts in ξ and exploiting the decay generated by derivatives of the exponential $e^{i\langle x-y,\xi\rangle}$, one shows that the kernel

$$K(x,y) = (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} a(x,\xi)\, d\xi$$

satisfies

$$|K(x,y)| \leq C_N (1 + |x-y|)^{-N}$$

uniformly in x, y. Schur's test then gives the L_2-boundedness. Full details can be found in Calderón and Vaillancourt (1971) or Hörmander's *Analysis of Linear Partial Differential Operators.* □

Remark 4.1.2. If $a(\xi)$ is independent of x, then T_a reduces to a Fourier multiplier operator:

$$\widehat{T_a u}(\xi) = a(\xi)\, \widehat{u}(\xi),$$

and L_2-boundedness follows immediately from Plancherel's theorem with $\|T_a\|_{L_2\to L_2} = \|a\|_{L^\infty}$. Thus, Theorem 4.1.1 generalizes this to variable coefficients $a(x,\xi)$.

The L_p theory is more delicate. While T_a is always bounded on L_2, boundedness on L_p for $p \neq 2$ requires additional structure on the symbol or decay in x.

Theorem 4.1.3 (Mihlin–Hörmander–Calderón–Vaillancourt Type)**.** *Let $a \in S^0_{1,0}$. Then T_a extends to a bounded operator on $L_p(\mathbb{R}^n)$ for all $1 < p < \infty$. More precisely, there exists a constant $C_p > 0$ such that*

$$\|T_a u\|_{L_p} \le C_p \|u\|_{L_p}.$$

Idea of Proof. The proof follows from the Calderón–Zygmund singular integral theory. The kernel

$$K(x,y) = (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} a(x,\xi)\, d\xi$$

is smooth off the diagonal $x = y$ and satisfies

$$|K(x,y)| \le C(1+|x-y|)^{-n-\varepsilon}, \qquad |\nabla_x K(x,y)|+|\nabla_y K(x,y)| \le C(1+|x-y|)^{-n-1}.$$

Hence K defines a Calderón–Zygmund kernel, and T_a is an L_p-bounded singular integral operator. □

Remark 4.1.4. The class $S^0_{1,0}$ is optimal for L_p-boundedness. If $\rho < 1$, one can still obtain boundedness for $p = 2$, but not necessarily for all p.

Example 4.1.5 (Fourier Multiplier of Order Zero)**.** Let $a(\xi) = (1 + |\xi|^2)^{i\lambda/2}$ for $\lambda \in \mathbb{R}$. Then $a \in S^0_{1,0}$, and T_a is an isometry on L_2 because $|a(\xi)| = 1$. In fact, T_a is unitary since $\widehat{T_a u} = a(\xi)\widehat{u}$.

Example 4.1.6 (Variable Coefficient Operator)**.** Let $a(x,\xi) = \frac{1}{1+|x|^2}(1 + |\xi|^2)^0$. Then $a \in S^0_{1,0}$. The decay in x guarantees that T_a is bounded on L_p for all $1 < p < \infty$ and compact on L_2.

Example 4.1.7 (Non-L_p-Bounded Symbol)**.** If $a(x,\xi) = e^{i|x|^2|\xi|}$, then $a \notin S^0_{1,0}$ since derivatives in x increase the order. In this case, T_a fails to be bounded on L_2.

Proposition 4.1.8. *If $a \in S^m_{1,0}$ with $m < 0$, then T_a is bounded from $L_p(\mathbb{R}^n)$ to the Sobolev space $H^{|m|,p}(\mathbb{R}^n)$ for $1 < p < \infty$. In particular, $T_a : L_2 \to H^{|m|}$ continuously.*

Idea of Proof. Since $(1+|\xi|^2)^{m/2}$ acts as a Fourier multiplier from L_p to $H^{|m|,p}$, and $a(x,\xi)/(1+|\xi|^2)^{m/2} \in S^0_{1,0}$, the result follows from Theorem 4.1.3. □

4.2 Sobolev Spaces $H^{s,p}$ and H^s

Sobolev spaces provide the natural functional framework for studying the mapping properties of pseudo-differential operators. They measure smoothness in terms of integrability and the decay of the Fourier transform, allowing differential and pseudo-differential operators to act continuously between spaces of varying regularity.

Definition 4.2.1 (Bessel Potential Operator)**.** For $s \in \mathbb{R}$, define the operator

$$J^s = (I - \Delta)^{s/2},$$

whose symbol is $(1+|\xi|^2)^{s/2}$. Thus, for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{J^s u}(\xi) = (1+|\xi|^2)^{s/2}\widehat{u}(\xi).$$

Definition 4.2.2 (Sobolev Space $H^{s,p}$)**.** For $1 < p < \infty$ and $s \in \mathbb{R}$, the Sobolev space $H^{s,p}(\mathbb{R}^n)$ is defined by

$$H^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : J^s u \in L_p(\mathbb{R}^n) \right\},$$

with norm

$$\|u\|_{H^{s,p}} = \|J^s u\|_{L_p}.$$

When $p = 2$, we simply write $H^s = H^{s,2}$.

Remark 4.2.3. The operator J^s acts as an isomorphism

$$J^s : H^{t,p} \to H^{t-s,p},$$

so the parameter s represents a shift in smoothness.

Proposition 4.2.4. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then:*

(i) $H^{s,p}$ coincides with the set of tempered distributions u whose Fourier transform satisfies

$$(1+|\xi|^2)^{s/2}\widehat{u}(\xi) \in L_p(\mathbb{R}^n).$$

(ii) For integer $s = k \geq 0$, $H^{k,p}$ is equivalent to the classical Sobolev space $W^{k,p}$ with norm

$$\|u\|_{H^{k,p}} \approx \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_p}.$$

Idea of Proof. For (i), this follows from the definition of J^s as a Fourier multiplier. For (ii), note that $(1 + |\xi|^2)^{k/2}$ expands into a polynomial of degree k in ξ; Plancherel's identity and standard multiplier theorems then yield the equivalence of norms. □

Remark 4.2.5. The case $s < 0$ corresponds to negative-order Sobolev spaces, which are defined as the duals of $H^{-s,p'}$ where $1/p + 1/p' = 1$. They are useful in the weak formulation of PDEs.

Theorem 4.2.6 (Sobolev Embedding)**.** *If $s > t$ and $1 < p < \infty$, then*

$$H^{s,p}(\mathbb{R}^n) \hookrightarrow H^{t,p}(\mathbb{R}^n).$$

If $s > n/p$, then $H^{s,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}_b(\mathbb{R}^n)$ continuously.

Theorem 4.2.7 (Interpolation)**.** *For $s_0, s_1 \in \mathbb{R}$ and $0 < \theta < 1$,*

$$[H^{s_0,p}, H^{s_1,p}]_\theta = H^{(1-\theta)s_0+\theta s_1,p},$$

where $[\cdot,\cdot]_\theta$ denotes complex interpolation.

Remark 4.2.8. Sobolev spaces form a continuous scale under interpolation and differentiation. This scale interacts naturally with pseudo-differential operators through their order m.

Theorem 4.2.9 (Continuity on Sobolev Spaces)**.** *Let $a \in S^m_{1,0}$. Then the pseudo-differential operator T_a extends continuously as*

$$T_a : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n), \qquad \forall s \in \mathbb{R},\ 1 < p < \infty.$$

Idea of Proof. We use that $T_a J^s = J^{s-m} T_{\tilde{a}}$, where $\tilde{a}(x,\xi) = (1+|\xi|^2)^{m/2} a(x,\xi)(1+|\xi|^2)^{-s/2}$ is again a symbol in $S^0_{1,0}$. By Theorem 4.1.3, $T_{\tilde{a}}$ is bounded on L_p, which implies the desired mapping property. □

Remark 4.2.10. Theorem 4.2.9 generalizes the classical result for differential operators: if $P(x,D)$ is of order m, then $P : H^{s,p} \to H^{s-m,p}$ continuously.

Example 4.2.11 (Fractional Laplacian)**.** The operator $(-\Delta)^{s/2}$ has symbol $|\xi|^s \in S^s_{1,0}$. Hence

$$(-\Delta)^{s/2} : H^{t,p} \to H^{t-s,p},$$

and $(-\Delta)^{-s/2}$ provides the inverse map $H^{t-s,p} \to H^{t,p}$.

Example 4.2.12 (Smoothing Operator)**.** If $a \in S^{-\infty}$, then T_a maps $\mathcal{S}'$ continuously into $\mathcal{C}^\infty$. In particular,

$$T_a : H^{s,p} \to H^{t,p}$$

for all $s, t \in \mathbb{R}$ and $1 < p < \infty$.

Proposition 4.2.13. *If $a(x,\xi)$ is elliptic of order m, then*

$$\|u\|_{H^{s,p}} \approx \|T_a u\|_{H^{s-m,p}} + \|u\|_{H^{-N,p}},$$

for any $N > 0$.

Idea of Proof. Since T_a has a parametrix $T_b \in \Psi^{-m}$ such that $T_b T_a = I + R$, $R \in \Psi^{-\infty}$, applying T_b gives equivalence of norms up to smoothing terms. □

4.3 Action of Pseudo-Differential Operators on Sobolev Spaces

Having introduced Sobolev spaces and the L_p-boundedness of pseudo-differential operators, we now establish the general mapping properties of such operators between Sobolev spaces of different orders. These results extend the classical continuity of differential operators and form the analytic foundation for elliptic regularity theory.

Theorem 4.3.1 (Mapping Property on Sobolev Spaces)**.** *Let $a \in S^m_{\rho,\delta}$ with $0 \le \delta < \rho \le 1$, and let $1 < p < \infty$. Then for every $s \in \mathbb{R}$, the operator*

$$T_a : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n)$$

is continuous.

Idea of Proof. Let $J^t = (I - \Delta)^{t/2}$. Then

$$T_a J^s = J^{s-m} T_{a_s}, \qquad a_s(x,\xi) = (1 + |\xi|^2)^{(m-s)/2}\, a(x,\xi)\, (1 + |\xi|^2)^{s/2}.$$

The symbol a_s belongs to $S^0_{\rho,\delta}$. Hence T_{a_s} is bounded on L_p by Theorem 4.1.3, implying

$$\|T_a u\|_{H^{s-m,p}} = \|T_{a_s} J^s u\|_{L_p} \le C\|J^s u\|_{L_p} = C\|u\|_{H^{s,p}}.$$

□

Remark 4.3.2. This result recovers all previously established cases:

- For $\rho = 1$, $\delta = 0$, it agrees with the standard $S^m_{1,0}$ calculus.
- For $\rho < 1$, it provides continuity but with weaker symbolic decay.

Proposition 4.3.3 (Compactness)**.** *If $a \in S^m_{\rho,\delta}$ with $m < 0$ and $0 \leq \delta < \rho \leq 1$, then*

$$T_a : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n)$$

is compact.

Idea of Proof. For $m < 0$, the symbol satisfies rapid decay estimates, implying that T_a has a kernel $K(x, y)$ smooth in both variables and vanishing rapidly as $|x - y| \to \infty$. The image of a bounded set in $H^{s,p}$ under T_a is therefore precompact in $H^{s-m,p}$. □

Corollary 4.3.4 (Smoothing Operators)**.** *If $a \in S^{-\infty}$, then T_a is infinitely smoothing:*

$$T_a : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{t,p}(\mathbb{R}^n) \quad \text{continuously for all } s, t \in \mathbb{R}.$$

Proof. Since all derivatives of a decay faster than any power of $|\xi|$, repeated integration by parts in the kernel representation of T_a yields a smooth kernel. □

Theorem 4.3.5 (Adjoint and Composition Properties)**.** *Let $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$. Then:*

(i) The adjoint T_a^* extends continuously as

$$T_a^* : H^{s,p} \longrightarrow H^{s-m_1,p}.$$

(ii) The composition $T_a T_b$ satisfies

$$T_a T_b = T_c, \quad c(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) \, D_x^\alpha b(x, \xi),$$

and acts continuously

$$H^{s,p} \to H^{s-(m_1+m_2),p}.$$

Idea of Proof. For (i), the adjoint symbol a^* given by Theorem 3.2.2 belongs to the same class $S^{m_1}_{\rho,\delta}$. The continuity follows from Theorem 4.3.1. For (ii), the composition formula from Theorem 3.1.1 shows that $c \in S^{m_1+m_2}_{\rho,\delta}$, and the mapping follows by the same reasoning. □

Theorem 4.3.6 (Elliptic Isomorphism)**.** *Let $a \in S^m_{1,0}$ be elliptic and let T_a denote the associated operator. Then for all $s \in \mathbb{R}$ and $1 < p < \infty$,*

$$T_a : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n)$$

is a Fredholm operator with finite-dimensional kernel and cokernel. If a is elliptic with parametrix $T_b \in \Psi^{-m}$, then

$$T_b T_a = I + R, \quad R \in \Psi^{-\infty}.$$

Idea of Proof. Since $T_b T_a = I + R$ with R smoothing, both T_a and T_b are bounded between the stated Sobolev spaces. The smoothing term R is compact, and hence T_a is Fredholm. □

Remark 4.3.7. Theorem 4.3.6 generalizes classical elliptic regularity: if $T_a u = f$ with $f \in H^{s-m,p}$, then $u \in H^{s,p}$. The parametrix T_b provides a symbolic inverse up to smoothing errors.

Chapter 5

Elliptic Regularity and Solution Theory

The fundamental goal of pseudo-differential operator theory is to understand the regularity and solvability of partial differential equations. Among these, *elliptic equations* occupy a central role: they model steady-state phenomena, and their solutions exhibit remarkable smoothness and stability properties. Pseudo-differential calculus provides a unified framework for analyzing such equations through their symbols and parametrices.

In previous chapters, we developed the symbolic calculus of pseudo-differential operators, including composition, adjoint, and parametrix constructions, and established their boundedness on L_p and Sobolev spaces. We are now prepared to apply these tools to study elliptic operators and the regularity of their solutions.

This chapter focuses on three main themes:

1. **Global Regularity of Elliptic Operators.** We will show that if a pseudo-differential operator T_a is elliptic of order m, then every distributional solution u of $T_a u = f$ inherits the smoothness of f. More precisely, if $f \in H^{s-m,p}$, then $u \in H^{s,p}$, a result that follows from the existence of a parametrix for T_a.

2. **Weak Solutions and A Priori Estimates.** For elliptic operators acting on Sobolev spaces, we will derive fundamental inequalities—such as Gårding's inequality—that provide lower bounds for the real part of $(T_a u, u)$ in L_2. These inequalities yield coercivity and uniqueness results for weak solutions.

3. **Elliptic Regularity Theorems.** Combining the symbolic calculus and Sobolev mapping properties, we will establish that elliptic pseudo-differential operators are *hypoelliptic* and enjoy global regularity: smooth right-hand sides imply smooth solutions. This extends the classical theory of elliptic differential operators to the broader pseudo-differential setting.

The analytic results of this chapter serve as the culmination of the pseudo-differential framework: they connect symbolic properties of operators with the functional-analytic regularity of their solutions. These principles will be illustrated by concrete examples, such as the Laplacian and fractional powers of elliptic operators, and will set the stage for deeper studies of microlocal analysis and propagation of singularities.

5.1 Global Regularity of Elliptic Operators

A central feature of elliptic pseudo-differential operators is their regularizing effect: if T_a is elliptic and $T_a u = f$, then the smoothness of f transfers to u. This property generalizes classical elliptic regularity for differential operators and follows directly from the existence of a parametrix.

Recall that a symbol $a(x,\xi) \in S^m_{1,0}$ is said to be *elliptic* if there exist constants $C, R > 0$ such that

$$|a(x,\xi)| \geq C(1+|\xi|)^m, \qquad |\xi| \geq R.$$

For such a, Theorem 3.3.4 guarantees the existence of a symbol $b(x,\xi) \in S^{-m}_{1,0}$ and corresponding operator T_b satisfying

$$T_a T_b = I + R_1, \qquad T_b T_a = I + R_2,$$

where $R_1, R_2 \in \Psi^{-\infty}$ are smoothing operators.

The operator T_b is called a *parametrix* of T_a and serves as a symbolic inverse modulo smoothing terms. This provides an analytic framework for solving equations of the form $T_a u = f$.

Theorem 5.1.1 (Global Elliptic Regularity). *Let $a \in S^m_{1,0}$ be elliptic, and let T_a denote the associated pseudo-differential operator. Then for every $s \in \mathbb{R}$ and $1 < p < \infty$,*

$$T_a : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n)$$

is a Fredholm operator. Moreover, if $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$T_a u = f \in H^{s-m,p}(\mathbb{R}^n),$$

then $u \in H^{s,p}(\mathbb{R}^n)$.

Proof. Let T_b be a parametrix of T_a, so that $T_b T_a = I + R_2$ with R_2 smoothing. Applying T_b to both sides of $T_a u = f$ gives

$$u = T_b f - R_2 u.$$

Since $T_b : H^{s-m,p} \to H^{s,p}$ continuously and R_2 is smoothing (hence maps any distribution into $\mathcal{C}^\infty$), it follows that $u \in H^{s,p}$. Fredholmness follows because R_2 is compact on $H^{s,p}$. □

Remark 5.1.2. The result shows that elliptic pseudo-differential operators are *hypoelliptic*: if $T_a u$ is smooth, then u is smooth. Indeed, if $T_a u = f \in \mathcal{C}^\infty$, then applying the same argument yields $u \in \mathcal{C}^\infty$.

The following inequality provides a quantitative version of elliptic regularity.

Theorem 5.1.3 (A Priori Estimate)**.** *Let T_a be an elliptic operator of order m. Then for all $u \in H^{s,p}(\mathbb{R}^n)$,*

$$\|u\|_{H^{s,p}} \leq C\big(\|T_a u\|_{H^{s-m,p}} + \|u\|_{H^{-N,p}}\big), \tag{5.1}$$

for some constants $C > 0$ and integer $N > 0$.

Proof. Let T_b be the parametrix of T_a, so that $T_b T_a = I + R$ with $R \in \Psi^{-\infty}$. Then

$$u = T_b T_a u - Ru.$$

Taking norms and using boundedness of T_b and R gives

$$\|u\|_{H^{s,p}} \leq C_1 \|T_a u\|_{H^{s-m,p}} + C_2 \|Ru\|_{H^{s,p}}.$$

Since R is smoothing, $\|Ru\|_{H^{s,p}} \leq C\|u\|_{H^{-N,p}}$ for large N, giving (5.1). □

Remark 5.1.4. The remainder term $\|u\|_{H^{-N,p}}$ accounts for possible elements of the kernel of T_a. If T_a is injective on $H^{s,p}$, then the weaker term can be omitted.

Example 5.1.5 (Elliptic Differential Operator)**.** Let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \qquad a_\alpha \in \mathcal{C}_b^\infty(\mathbb{R}^n),$$

be an elliptic differential operator. Then $P(x,D) = T_p$ with $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in S^m_{1,0}$. The elliptic regularity theorem implies that

$$P(x,D)u = f \in H^{s-m,p} \quad \Rightarrow \quad u \in H^{s,p}.$$

This recovers the classical result that elliptic differential operators are hypoelliptic.

Example 5.1.6 (Fractional Laplacian)**.** For $T_a = (-\Delta)^{m/2}$ with $a(\xi) = |\xi|^m$, ellipticity is immediate. If $(-\Delta)^{m/2}u = f \in H^{s-m,p}$, then $u \in H^{s,p}$. In particular, if $f \in L_2$, then $u \in H^m$.

Corollary 5.1.7 (Smooth Regularity)**.** *If T_a is elliptic and $T_a u = f$ with $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

Proof. Since f is smooth, $f \in H^{s-m,p}$ for all s. By Theorem 5.1.1, $u \in H^{s,p}$ for all s, hence $u \in \mathcal{C}^\infty$. □

Theorem 5.1.8 (Fredholm Property of Elliptic Operators)**.** *Let T_a be elliptic of order m. Then for every $s \in \mathbb{R}$ and $1 < p < \infty$, the operator*

$$T_a : H^{s,p} \to H^{s-m,p}$$

is Fredholm with index independent of s and p.

Proof. Since T_a admits a parametrix T_b such that $T_b T_a = I + R$ and R is compact, T_a is Fredholm. The index is invariant under compact perturbations and continuous deformations in s, p. □

5.2 Weak Solutions and Gårding's Inequality

In many applications, elliptic equations are not solved in the classical sense but in a weaker distributional sense. Pseudo-differential operators provide a natural framework for defining and analyzing such *weak solutions*, since their action extends continuously to Sobolev spaces. An essential tool for proving existence, uniqueness, and stability of weak solutions is *Gårding's inequality*, which gives a coercive lower bound on the real part of the operator.

Consider an elliptic pseudo-differential operator T_a of order m with $a \in S^m_{1,0}$. We study the equation

$$T_a u = f, \qquad f \in H^{s-m,p}(\mathbb{R}^n), \tag{5.2}$$

where the solution u is sought in $H^{s,p}(\mathbb{R}^n)$.

Definition 5.2.1 (Weak Solution)**.** Let $a \in S^m_{1,0}$ and $1 < p < \infty$. A function $u \in H^{s,p}(\mathbb{R}^n)$ is called a *weak solution* of (5.2) if

$$(T_a u, \varphi) = (f, \varphi), \qquad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where $(\cdot, \cdot)$ denotes the distributional dual pairing extending the L_2 inner product.

Remark 5.2.2. When $p = 2$, the pairing coincides with the L_2 inner product, and the notion of weak solution corresponds to that of a variational or energy solution in the Hilbert space setting.

Suppose $a(x, \xi)$ is real-valued and elliptic of order m. Then T_a is formally self-adjoint (up to lower-order terms). The energy functional associated with T_a is

$$E[u] = (T_a u, u)_{L_2}.$$

Coercivity of this quadratic form is expressed by the following inequality.

Theorem 5.2.3 (Gårding's Inequality)**.** *Let $a \in S^m_{1,0}$ be real-valued and elliptic. Then there exist constants $C_1, C_2 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$\mathrm{Re}(T_a u, u)_{L_2} \geq C_1 \|u\|^2_{H^{m/2}} - C_2 \|u\|^2_{L_2}. \tag{5.3}$$

Idea of Proof. Decompose $a(x, \xi) = a_m(x, \xi) + r(x, \xi)$ where a_m is the principal part, homogeneous of degree m, and $r \in S^{m-1}_{1,0}$. For large $|\xi|$, ellipticity gives $a_m(x, \xi) \geq c|\xi|^m$. Let $b(x, \xi) = a(x, \xi) - c(1 + |\xi|^2)^{m/2}$; then $b \in S^{m-1}_{1,0}$ and T_b is bounded on L_2. Hence

$$(T_a u, u) = c\, \|J^{m/2} u\|^2_{L_2} + (T_b u, u),$$

and $|(T_b u, u)| \leq C_2 \|u\|^2_{L_2}$, giving (5.3). $\square$

Remark 5.2.4. Inequality (5.3) generalizes the classical elliptic estimate

$$(\Delta u, u) = \|\nabla u\|^2_{L_2}$$

to general pseudo-differential operators. The first term measures the high-frequency (elliptic) coercivity, while the lower-order correction ensures boundedness.

Theorem 5.2.5 (Existence of Weak Solutions)**.** *Let T_a be real-valued, elliptic of order $m > 0$, and satisfy Gårding's inequality (5.3). Then for every $f \in H^{s-m,2}(\mathbb{R}^n)$, there exists a unique weak solution $u \in H^{s,2}(\mathbb{R}^n)$ of*

$$T_a u = f.$$

Moreover,

$$\|u\|_{H^s} \leq C\|f\|_{H^{s-m}}. \tag{5.4}$$

Sketch of Proof. Let T_b be a parametrix of T_a. Then $u = T_b f - Ru$, where R is smoothing. Applying Gårding's inequality yields coercivity of $(T_a u, u)$, allowing one to apply the Lax–Milgram theorem in the Hilbert space $H^{m/2}$. The parametrix provides the regularity estimate (5.4). □

Corollary 5.2.6 (Uniqueness). *If $T_a u = 0$ and a is real-valued, elliptic, and satisfies (5.3), then $u = 0$.*

Proof. By Gårding's inequality,

$$0 = \mathrm{Re}(T_a u, u) \geq C_1 \|u\|_{H^{m/2}}^2 - C_2 \|u\|_{L_2}^2.$$

Hence $\|u\|_{H^{m/2}} \leq C\|u\|_{L_2}$. Ellipticity then implies u is smooth and compactly supported only if $u = 0$. □

A refinement of Theorem 5.2.3 gives a sharper lower bound when the symbol is not necessarily positive but has positive real part.

Theorem 5.2.7 (Sharp Gårding Inequality). *Let $a \in S^m_{1,0}$ satisfy $\mathrm{Re}\, a(x,\xi) \geq 0$. Then there exists a constant $C > 0$ such that*

$$\mathrm{Re}(T_a u, u)_{L_2} \geq -C\, \|u\|^2_{H^{(m-1)/2}}, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n). \tag{5.5}$$

Idea of Proof. The proof uses a microlocal symmetrization argument. Decompose $a = a_1 + ia_2$ with a_1 real and a_2 imaginary. Apply Weyl quantization to ensure self-adjointness of $a_1(x,D)$ and use commutator estimates to control the contribution of a_2. The main point is that $\mathrm{Re}\, a(x,\xi) \geq 0$ implies positivity modulo $S^{m-1}_{1,0}$. □

Remark 5.2.8. The sharp form (5.5) implies that the real part of T_a is nonnegative up to a lower-order perturbation, providing a crucial tool for stability and spectral estimates.

Example 5.2.9 (Laplacian). For the Laplacian $\Delta = -\sum_j D_j^2$, we have

$$(\Delta u, u) = \|\nabla u\|^2_{L_2} \geq 0,$$

which is a special case of Gårding's inequality with $m = 2$, $C_1 = 1$, and $C_2 = 0$.

Example 5.2.10 (Fractional Laplacian). For $T_a = (-\Delta)^{s/2}$ with $a(\xi) = |\xi|^s$, $s > 0$, we obtain

$$((-\Delta)^{s/2} u, u) = \|J^{s/2} u\|^2_{L_2} \geq 0,$$

which satisfies the sharp form of Gårding's inequality.