

# **MATH 3410**

## **Complex Variables**

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# Preface

These are the first edition of these lecture notes for MATH 2022 (Linear Algebra II). Consequently, there may be several typographical errors, missing exposition on necessary background, and more advanced topics for which there will not be time in class to cover. Future iterations of these notes will hopefully be fairly self-contained provided one has the necessary background. If you come across any typos, errors, omissions, or unclear expositions, please feel free to contact me so that I may continually improve these notes.



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# Chapter 1

## Complex Numbers

The study of complex analysis begins with a deep understanding of complex numbers, which are the foundational objects upon which the entire theory is built. While real numbers suffice for most classical calculus, many mathematical phenomena and physical models demand a more expansive number system that can handle operations like taking square roots of negative numbers. This need leads naturally to the field of complex numbers.

Complex numbers extend the real number system by introducing a new unit, denoted by  $i$ , which satisfies the property  $i^2 = -1$ . The resulting system, denoted by  $\mathbb{C}$ , possesses rich algebraic and geometric structure. Unlike the real line, the complex numbers form a two-dimensional plane, allowing us to study functions, differentiation, and integration in a multidimensional yet highly structured setting.

This chapter introduces the algebraic and geometric properties of complex numbers, explores their representation in both rectangular and polar form, and discusses key operations such as complex conjugation, modulus, and argument. We also explore the complex exponential function and Euler's identity, which provide powerful tools for expressing and manipulating complex quantities. These foundational tools will play a crucial role in the chapters to come.

### 1.1 Complex Numbers

In this section, we give the basics of complex numbers.

**Definition 1.1.1.** The *complex numbers*, denoted  $\mathbb{C}$ , are the set

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

where  $i$  denotes a fixed symbol.

Given a complex number  $z = a + bi$ , where  $a, b \in \mathbb{R}$ , the number  $a$  is called the *real part of  $z$* , and is denoted by  $\operatorname{Re}(z)$ , and the number  $b$  is called the *imaginary part of  $z$* , and is denoted by  $\operatorname{Im}(z)$ .

The symbol  $i$  in a complex number is meant to denote “ $\sqrt{-1}$ ”. To be specific, we will equip  $\mathbb{C}$  with binary operations of addition “ $+$ ” :  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  and multiplication “ $\cdot$ ” :  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  so that  $(0 + 1i)(0 + 1i) = -1$ , so that indeed,  $i$  is a complex solution to  $x^2 + 1 = 0$ .

**Definition 1.1.2.** The binary operations “ $+$ ” :  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  and “ $\cdot$ ” :  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi) \cdot (c + di) &= (ac - bd) + (ad + bc)i\end{aligned}$$

for all  $a, b, c, d \in \mathbb{R}$  are called *complex addition* and *complex multiplication*, respectively.

**Example 1.1.3.** It is not difficult to see that

$$\begin{aligned}(1 + 2i) + (3 + 4i) &= 4 + 6i \\ (1 + 2i)(3 + 4i) &= -5 + 10i.\end{aligned}$$

Moreover, since

$$i^2 = (0 + 1i)(0 + 1i) = -1 + 0i$$

we do indeed have that  $i$  is a complex solution to  $x^2 = -1$ . In addition, it is not difficult to see that  $-i$  is also a complex solution to  $x^2 = -1$ .

In order for  $\mathbb{C}$  to be as nice to work with as  $\mathbb{R}$ , we require complex addition and multiplication to have specific properties. To be specific, we want the following.

**Theorem 1.1.4.** *The set of complex numbers  $\mathbb{C}$  together with complex addition and multiplication is a field; that is,*

1. (*Commutativity of Addition*)  $z + w = w + z$  for all  $z, w \in \mathbb{C}$ .
2. (*Associativity of Addition*)  $z + (w + u) = (z + w) + u$  for all  $z, w, u \in \mathbb{C}$ .
3. (*Additive Unit*) There exists a  $0 \in \mathbb{C}$  such that  $z + 0 = z$  for all  $z \in \mathbb{C}$ .
4. (*Additive Inverses*) For all  $z \in \mathbb{C}$ , there exists a  $-z \in \mathbb{C}$  such that  $z + (-z) = 0$ .

5. (Commutativity of Multiplication)  $zw = wz$  for all  $z, w \in \mathbb{C}$ .
6. (Associativity of Multiplication)  $z(wu) = (zw)u$  for all  $z, w, u \in \mathbb{C}$ .
7. (Multiplicative Unit) There exists a  $1 \in \mathbb{C}$  such that  $1z = z$  for all  $z \in \mathbb{C}$ .
8. (Multiplicative Inverses) For all  $z \in \mathbb{C} \setminus \{0\}$ , there exists a  $z^{-1} \in \mathbb{C}$  such that  $z^{-1}z = 1$ .
9. (Distributivity)  $z(w + u) = zw + zu$  for all  $z, w, u \in \mathbb{C}$ .

*Proof.* Let  $z, w, u \in \mathbb{C}$  be arbitrary. Then there exists  $a, b, c, d, x, y \in \mathbb{R}$  such that

$$z = a + bi, \quad w = c + di, \quad u = x + yi$$

We will examine each of the above nine properties for these arbitrary elements of  $\mathbb{C}$  and demonstrate the properties hold using the analogous properties for real numbers.

To see that (1) holds, notice that

$$z + w = (a + c) + (b + d)i = (c + a) + (d + b)i = w + z$$

due to the commutativity of addition of real numbers. Thus, commutativity of addition of complex numbers has been demonstrated.

To see that (2) holds, notice that

$$\begin{aligned} z + (w + u) &= (a + bi) + [(c + x) + (d + y)i] \\ &= [a + (c + x)] + [b + (d + y)]i \\ &= [(a + c) + x] + [(b + d) + y]i \\ &= [(a + c) + (b + d)i] + (x + yi) \\ &= (z + w) + u \end{aligned}$$

where the third equality holds due to the associativity of addition of real numbers. Thus, associativity of addition of complex numbers has been demonstrated.

To see that (3) holds, notice that with  $0 = 0 + 0i$ , we have

$$z + 0 = (a + 0) + (b + 0)i = a + bi = z$$

due to the property of the zero element of  $\mathbb{R}$ . Thus, the complex numbers have an additive unit.

To see that (4) holds, let  $-z = (-a) + (-b)i$ , where  $-a$  and  $-b$  are the additive inverses of  $a$  and  $b$  in  $\mathbb{R}$ . Then

$$z + (-z) = [a + (-a)] + [b + (-b)]i = 0 + 0i = 0$$

as desired. Thus, the complex numbers have additive inverses.

To see that (5) holds, notice that

$$zw = (ac - bd) + (ad + bc)i = (ca - db) + (cb + da)i = wz$$

due to the commutativity of addition and multiplication of real numbers. Thus, commutativity of multiplication of complex numbers has been demonstrated.

To see that (6) holds, notice that

$$\begin{aligned} z(wu) &= (a + bi)[(c + di)(x + yi)] \\ &= (a + bi)[(cx - dy) + (cy + dx)i] \\ &= [a(cx - dy) - b(cy + dx)] + [a(cy + dx) + b(cx - dy)]i \\ &= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i \\ &= (acx - bdx - ady - bcy) + (acy - bdy + adc + bcx)i \\ &= [(ac - bd)x - (ad + bc)y] + [(ac - bd)y + (ad + bc)x]i \\ &= [(ac - bd) + (ad + bc)i](x + yi) \\ &= [(a + bi)(c + di)](x + yi) \\ &= (zw)u \end{aligned}$$

where commutativity and associativity of addition (fifth equality) and distributivity of real numbers (fourth and sixth equalities) have been used. Thus, associativity of multiplication of complex numbers has been demonstrated.

To see that (7) holds, notice that with  $1 = 1 + 0i$ , we have

$$1z = (1a - 0b) + (1b + 0a)i = a + bi = z$$

due to the property of the one element of  $\mathbb{R}$ . Thus, the complex numbers have a multiplicative unit.

To see that (8) holds, assume that  $z \neq 0$ . Thus, either  $a \neq 0$  or  $b \neq 0$ , so  $a^2 + b^2 > 0$ . Define

$$z^{-1} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i,$$



which makes sense since  $\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \in \mathbb{R}$  since nonzero real numbers have multiplicative inverses. Notice that

$$\begin{aligned} z^{-1}z &= \left[ \left( \frac{a}{a^2+b^2} \right) a - \left( \frac{-b}{a^2+b^2} \right) b \right] + \left[ \left( \frac{a}{a^2+b^2} \right) b + \left( \frac{-b}{a^2+b^2} \right) a \right] i \\ &= \left( \frac{a^2+b^2}{a^2+b^2} \right) + \left( \frac{ab-ba}{a^2+b^2} \right) i \\ &= 1 + 0i \\ &= 1 \end{aligned}$$

due to the commutativity of multiplication and properties of addition of real numbers. Hence,  $z^{-1}$  is indeed the multiplicative inverse of  $z$ . Thus, the existence of multiplicative inverses for nonzero complex numbers has been demonstrated.

Finally, to see that (9) holds, notice that

$$\begin{aligned} z(w+u) &= (a+bi)[(c+di) + (x+yi)] \\ &= (a+bi)[(c+x) + (d+y)i] \\ &= [a(c+x) - b(d+y)] + [a(d+y) + b(c+x)]i \\ &= (ac+ax-bd-dy) + (ad+ay+bc+bx)i \\ &= [(ac-bd) + (ax-dy)] + [(ad+bc) + (ay+bx)]i \\ &= [(ac-bd) + (ad+bc)i] + [(ax-dy) + (ay+bx)i] \\ &= (a+bi)(c+di) + (a+bi)(x+yi) \\ &= zw + zu \end{aligned}$$

where commutativity and associativity of addition (fourth equality) and distributivity of real numbers (third equality) have been used. Thus, distributivity of complex numbers has been demonstrated.

As all nine properties have now been demonstrated for arbitrary complex numbers, the set of complex numbers together with addition and multiplication are a field.  $\square$

**Remark 1.1.5.** Embedded in the proof of Theorem 1.1.4 is the formula for the inverse of a nonzero complex number. Indeed, if  $z = a + bi$ , where  $a, b \in \mathbb{R}$  is such that  $z \neq 0$ , then

$$z^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

For example,

$$(3+4i)^{-1} = \frac{3}{25} - \frac{4}{25}i$$

Note we can also write

$$z^{-1} = \frac{a - bi}{a^2 + b^2}$$

When dealing with complex numbers, both the numerator and denominator are important quantities that are worthy of developing further.

**Definition 1.1.6.** Given a complex number  $z = a + bi$ , where  $a, b \in \mathbb{R}$ , the *absolute value* (or *length* or *modulus*) of  $z$ , denoted  $|z|$  is the quantity

$$|z| = \sqrt{a^2 + b^2}$$

**Definition 1.1.7.** Given a complex number  $z = a + bi$ , where  $a, b \in \mathbb{R}$ , the *complex conjugate* of  $z$ , denoted  $\bar{z}$  is the quantity

$$\bar{z} = a + (-b)i$$

Using our knowledge of the inverse of a nonzero complex number, we have the following.

**Corollary 1.1.8.** If  $z \in \mathbb{C} \setminus \{0\}$ , then  $z^{-1} = \frac{\bar{z}}{|z|^2}$ .

There are many properties of the absolute value and complex conjugates that are worthy of recording.

**Theorem 1.1.9.** For all  $z, w \in \mathbb{C}$ , the following are true.

1.  $\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$ .
2.  $\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$ .
3.  $\overline{z + w} = \bar{z} + \bar{w}$ .
4.  $\overline{zw} = \bar{z}\bar{w}$ .
5.  $|z| = \sqrt{\bar{z}z}$ .
6.  $|\operatorname{Re}(z)| \leq |z|$ .
7.  $|\operatorname{Im}(z)| \leq |z|$ .
8.  $|\bar{z}| = |z|$ .
9.  $|zw| = |z||w|$ .
10. If  $z \neq 0$ , then  $|z^{-1}| = |z|^{-1}$ .

$$11. |z + w| \leq |z| + |w|.$$

$$12. ||z| - |w|| \leq |z - w|.$$

*Proof.* Let  $z, w \in \mathbb{C}$  be arbitrary. Then there exists  $a, b, c, d \in \mathbb{R}$  such that  $z = a + bi$  and  $w = c + di$ .

To see that (1) is true, notice that

$$\frac{z + \bar{z}}{2} = \frac{(a + bi) + (a - bi)}{2} = \frac{2a}{2} = a = \operatorname{Re}(z)$$

as desired.

To see that (2) is true, notice that

$$\frac{z - \bar{z}}{2i} = \frac{(a + bi) - (a - bi)}{2i} = \frac{2bi}{2i} = b = \operatorname{Im}(z)$$

as desired.

To see that (3) is true, notice that

$$\begin{aligned} \overline{z + w} &= \overline{(a + c) + (b + d)i} \\ &= (a + c) + (-(b + d))i \\ &= (a + (-b)i) + (c + (-d)i) \\ &= \bar{z} + \bar{w} \end{aligned}$$

as desired.

To see that (4) is true, notice that

$$\begin{aligned} \overline{zw} &= \overline{(ac - bd) + (ad + bc)i} \\ &= (ac - bd) + (-(ad + bc))i \\ &= (ac - (-b)(-d)) + (a(-d) + (-b)c)i \\ &= (a + (-b)i)(c + (-d)i) \\ &= \bar{z}\bar{w} \end{aligned}$$

as desired.

To see that (5) is true, notice that

$$\begin{aligned} \sqrt{\bar{z}z} &= \sqrt{(a + (-b)i)(a + bi)} \\ &= \sqrt{(a^2 - (-b)b) + (ab + a(-b))i} \\ &= \sqrt{a^2 + b^2} \\ &= |z| \end{aligned}$$

as desired.

To see that (6) and (7) are true, notice that

$$a^2 \leq a^2 + b^2, \quad b^2 \leq a^2 + b^2$$

so

$$\begin{aligned} |\operatorname{Re}(z)| &= |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z| \\ |\operatorname{Im}(z)| &= |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| \end{aligned}$$

as desired.

To see that (8) is true, note that

$$|\bar{z}| = |a + (-b)i| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

as desired.

To see that (9) is true, notice that

$$zw = (ac - bd) + (ad + bc)i$$

so that

$$\begin{aligned} |zw| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2)} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \\ &= |z||w| \end{aligned}$$

as desired.

To see that (10) is true, notice by (9) that

$$|z^{-1}||z| = |z^{-1}z| = |1| = 1$$

so  $|z^{-1}| = |z|^{-1}$  as desired.

To see that (11) is true, notice that

$$\begin{aligned}
|z + w|^2 &= (\overline{z + w})(z + w) && \text{by (5)} \\
&= (\bar{z} + \bar{w})(z + w) && \text{by (3)} \\
&= \bar{z}z + \bar{z}w + \bar{w}z + \bar{w}w \\
&= |z|^2 + \bar{z}w + \overline{\bar{z}w} + |w|^2 && \text{by (4) and (5)} \\
&= |z|^2 + 2\operatorname{Re}(\bar{z}w) + |w|^2 && \text{by (1)} \\
&\leq |z|^2 + 2|\bar{z}w| + |w|^2 && \text{by (6)} \\
&= |z|^2 + 2|\bar{z}||w| + |w|^2 && \text{by (9)} \\
&= |z|^2 + 2|z||w| + |w|^2 && \text{by (8)} \\
&= (|z| + |w|)^2
\end{aligned}$$

Therefore, by taking the square root of both sides of the inequality, the desired result is obtained.

Finally, to see that (12) holds, first notice by (11) that

$$|z| = |(z - w) + w| \leq |z - w| + |w|$$

Therefore,

$$|z| - |w| \leq |z - w|$$

Similarly, notice by (11) and (9) that

$$\begin{aligned}
|w| &= |(w - z) + z| \\
&\leq |w - z| + |z| \\
&= |(-1)(z - w)| + |z| \\
&= |-1||z - w| + |z| \\
&= |z - w| + |z|
\end{aligned}$$

Hence,

$$|w| - |z| \leq |z - w|$$

Therefore, by combining  $|z| - |w| \leq |z - w|$  and  $|w| - |z| \leq |z - w|$ , we obtain that

$$||z| - |w|| \leq |z - w|$$

as desired. □

## 1.2 Polar Form and Euler's Identity

While the rectangular form  $z = x + yi$  of a complex number is often useful for algebraic manipulation, it conceals the elegant geometric structure inherent to the complex plane. A more natural way to describe complex numbers, especially when dealing with multiplication, powers, and roots, is through their polar form. This representation expresses a complex number in terms of its modulus, and its argument, thereby making its geometric behaviour more transparent.

In this section, we develop the polar representation of complex numbers and introduce one of the most profound identities in mathematics: Euler's identity. By linking exponential functions with trigonometric functions, Euler's formula provides a powerful bridge between algebra and geometry.

**Definition 1.2.1.** Given a nonzero complex number  $z = x + yi$ , the *argument of  $z$*  is any real number  $\theta$  such that

$$\cos(\theta) = \frac{x}{|z|}, \quad \sin(\theta) = \frac{y}{|z|}$$

We define the *principal argument of  $z$* , denoted by  $\text{Arg}(z)$ , to be the unique value of  $\theta \in (-\pi, \pi]$  satisfying the above.

**Remark 1.2.2.** Since  $\sin(\theta)$  and  $\cos(\theta)$  are  $2\pi$ -periodic functions, the argument  $\arg(z)$  is only well-defined up to an additive multiple of  $2\pi$ . That is, if  $\theta$  is an argument of  $z$ , then so is  $\theta + 2k\pi$  for any  $k \in \mathbb{Z}$ .

**Definition 1.2.3.** Given a nonzero complex number  $z = x + yi$ , for some  $x, y \in \mathbb{R}$ , we may associate  $z$  two geometric quantities: the modulus of  $z$  and the argument of  $z$ . Moreover, provided that  $z \neq 0$ , we may write  $z$  in the form

$$z = r \cos(\theta) + i \sin(\theta)$$

where  $r = |z|$  and  $\theta = \arg(z)$ . This is known as the *polar form of  $z$* .

**Example 1.2.4.** Let  $z = 1 + i$ . Then

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \cos(\theta) = \frac{1}{\sqrt{2}}, \quad \sin(\theta) = \frac{1}{\sqrt{2}}$$

Thus, one possible value of  $\arg(z)$  is  $\theta = \frac{\pi}{4}$ , and the polar form of  $z$  is

$$z = \sqrt{2} \left( \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right)$$

An advantage of using the polar form is that it provides geometric insight into the multiplication of complex numbers.

**Proposition 1.2.5.** *Let  $z = r_1(\cos(\theta_1) + i \sin(\theta_1))$  and  $w = r_2(\cos(\theta_2) + i \sin(\theta_2))$ . Then the following hold.*

$$1. \quad zw = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

$$2. \quad \frac{z}{w} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$

*Proof.* To see that (1) holds, observe that

$$\begin{aligned} zw &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 ((\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + (\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2))i) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \end{aligned}$$

To see that (2) holds, note that since division is the inverse of multiplication, the result immediately holds.  $\square$

**Remark 1.2.6.** From Proposition 1.2.5, we see that for all complex numbers  $z, w \in \mathbb{C}$ .

$$\arg(zw) = \arg(z) + \arg(w), \quad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

**Definition 1.2.7.** Let  $\tau \in \mathbb{R}$ . The place  $\mathbb{C}_\tau$  given by

$$\mathbb{C}_\tau = \{(r, \theta) : r > 0, \tau < \theta \leq \tau + 2\pi\}$$

is known as the *cut plane along the branch cut*  $\{(r, \tau) : r > 0\}$ . The branch of  $\arg(z)$  that lies in  $(\tau, \tau + 2\pi]$  is denoted by  $\arg_\tau(z)$ .

Of particular importance is the principal branch of  $\arg(z)$ , which we take to be the branch in  $(-\pi, \pi]$ . The principal branch of  $\arg(z)$  is therefore the branch  $\arg_{-\pi}(z) = \text{Arg}(z)$ .

**Example 1.2.8.** Let  $z = 1 + i$ . Then we have

$$\arg(z) = \frac{\pi}{4} + 2k\pi$$

for all  $k \in \mathbb{Z}$ . In particular,

$$\text{Arg}(z) = \frac{\pi}{4}, \quad \arg_{\frac{\pi}{2}}(z) = \frac{3\pi}{4}$$

We all know what  $e^\theta$  is if  $\theta$  is a real number. However, what is  $e^{i\theta}$ ? To answer this question, we again use the rule that complex numbers are treated as real numbers in calculations.

**Definition 1.2.9.** For  $z \in \mathbb{C}$ , the *complex exponential of  $z$* , denoted by  $e^z$ , is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

If we consider  $z = i\theta$ , and using Definition 1.2.9, we obtain

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= \left(1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots\right) + \left(\frac{i\theta}{1!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots\right) \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos(\theta) + i \sin(\theta) \end{aligned}$$

Hence, we have the celebrated Euler's identity to the effect that

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

for all  $\theta \in \mathbb{R}$ .

**Definition 1.2.10.** For any  $\theta \in \mathbb{R}$ , *Euler's identity* is given as

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Recall that the polar form of a nonzero complex number  $z$  is given by

$$z = r(\cos(\theta) + i \sin(\theta))$$

Then by Definition 1.2.10, the polar form becomes

$$z = re^{i\theta}$$

**Example 1.2.11.** If  $\theta = \pi$ , then  $e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1$ .

**Theorem 1.2.12 (De Moivre's Formula).** For all  $n \in \mathbb{N}$  and  $\theta \in \mathbb{R}$ ,

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$$



*Proof.* Using Definition 1.2.10, we have

$$(\cos(\theta) + i \sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

as desired.  $\square$

De Moivre's Formula (Theorem 1.2.12) allows for one to derive trigonometric formulas that are often used in calculus. One of the most common identities used is the double angle formulas.

**Proposition 1.2.13 (Double Angle Formulas).** *For all  $\theta \in \mathbb{R}$ ,*

1.  $\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$ .
2.  $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ .

*Proof.* First observe that by De Moivre's Formula

$$\begin{aligned} e^{i2\theta} &= (\cos(\theta) + i \sin(\theta))^2 \\ &= \cos^2(\theta) + 2i \cos(\theta) \sin(\theta) + i^2 \sin^2(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta) + 2i \sin(\theta) \cos(\theta) \end{aligned}$$

To see that (1) holds, note that

$$\begin{aligned} \sin(2\theta) &= \operatorname{Im}(e^{i2\theta}) \\ &= \operatorname{Im}(\cos^2(\theta) - \sin^2(\theta) + 2i \sin(\theta) \cos(\theta)) \\ &= 2 \sin(\theta) \cos(\theta) \end{aligned}$$

and similarly, to see that (2) holds,

$$\begin{aligned} \cos(2\theta) &= \operatorname{Re}(e^{i2\theta}) \\ &= \operatorname{Re}(\cos^2(\theta) - \sin^2(\theta) + 2i \sin(\theta) \cos(\theta)) \\ &= \cos^2(\theta) - \sin^2(\theta) \end{aligned}$$

as desired.  $\square$

Although we have a nice power series form for  $e^z$  for any complex number  $z$ , another way that we can express is as follows. Let  $z = x + yi$  for some  $x, y \in \mathbb{R}$ . Then

$$e^z = e^{x+yi} = e^x e^{yi} = e^x (\cos(y) + i \sin(y))$$

With Euler's formula, we are now able to solve equations of the form  $z^n = 1$  for some  $n \in \mathbb{N}$ . Let  $z = re^{i\theta}$  be a nonzero complex number. Then for every  $n \in \mathbb{N}$ , the  $n$  power  $z^n$  of  $z$  is given as

$$z^n = r^n e^{in\theta} = r^n (\cos(n\theta) + i \sin(n\theta))$$

However, is there a formula for all the  $n$ th roots of  $z$ ? To solve this problem, let  $\zeta = \sqrt[n]{z}$ . Suppose that  $\zeta = \rho e^{i\varphi}$ . Then

$$\zeta^n = z \implies \rho^n e^{in\varphi} = re^{i\theta}$$

Thus,

$$\begin{aligned} \rho^n &= r \\ \varphi &= \frac{1}{n}(\theta + 2k\pi) \end{aligned}$$

where  $k = 0, 1, 2, \dots, n-1$ . Thus, there are  $n$  distinct  $n$ th roots of  $z$  given by

$$\zeta_k = \sqrt[n]{r} e^{\frac{i(\theta + 2k\pi)}{n}}$$

for all  $k = 0, 1, 2, \dots, n-1$ .

**Example 1.2.14.** Consider the equation  $z^4 = 1$ . Then since  $1 = 1e^0$ , we have  $r = 1$  and  $\theta = 0$ . Thus, the four 4th roots of unity are

$$\zeta_k = e^{\frac{i(2k\pi)}{4}} = e^{\frac{ik\pi}{2}}$$

for all  $k = 0, 1, 2, 3$ . Therefore,

$$\zeta_0 = 1, \quad \zeta_1 = e^{\frac{i\pi}{2}} = i, \quad \zeta_2 = -1, \quad \zeta_3 = e^{\frac{i3\pi}{2}} = -i$$

as desired.

### 1.3 Basic Topology on $\mathbb{C}$

In order to study complex functions and their differentiability, it is important to understand the topological structure of the complex plane. Many fundamental concepts in complex analysis, such as continuity, limits, holomorphicity, and contour integration relies on a precise understanding of open sets, connectedness, and regions in  $\mathbb{C}$ .

In this section, we introduce the basic topological notions required to study functions of a complex variable. We define open disks and open sets,

explore connectedness in the complex plane, and describe what it means for a subset of  $\mathbb{C}$  to be a domain or region.

Usually on a Cartesian plane, we call the axes, the  $x$ -axis and the  $y$ -axis. However, in the complex plane, we have something different.

**Definition 1.3.1.** On the complex plane, we call the  $x$ -axis the *real axis*, and the  $y$ -axis, the *imaginary axis*.

Points in the complex plane are represented as vectors. Given a complex number  $z = x + yi$ , for some  $x, y \in \mathbb{R}$ , we may associate to  $z$  the point  $(x, y)$  in the Cartesian plane. Geometrically, this corresponds to the vector from the origin  $(0, 0)$  to the point  $(x, y)$ , which provides a visual representation of the complex number  $z$ .

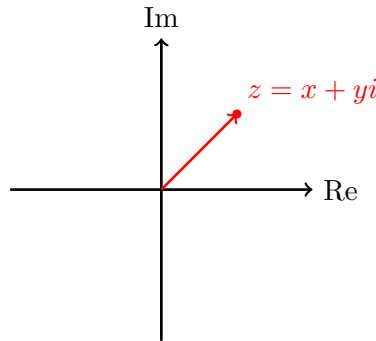


Figure 1.1: The complex number  $z = x + yi$  represented as a vector in the complex plane.

**Definition 1.3.2.** *indexopen disk* Let  $z_0 \in \mathbb{C}$  and let  $r > 0$ . Then we define the set  $D(z_0, r)$  by

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

We call  $D(z_0, r)$  the *open disk with centre  $z_0$  and radius  $r$* . We also call it a *neighbourhood of  $z_0$* . We call a subset  $A \subseteq \mathbb{C}$  an *open set* if for every  $z_0 \in A$  there exists an  $r > 0$  such that  $D(z_0, r) \subseteq A$ .

Analogously, we have closed sets in the complex plane as well.

**Definition 1.3.3.** Let  $z_0 \in \mathbb{C}$  and let  $r > 0$ . Then we define the set  $D[z_0, r]$  by

$$D[z_0, r] = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

We call  $D[z_0, r]$  the *closed disk with centre  $z_0$  and radius  $r$* . We call a subset  $A \subseteq \mathbb{C}$  a *closed set* if its complement  $A^c = \mathbb{C} \setminus A$  is open.

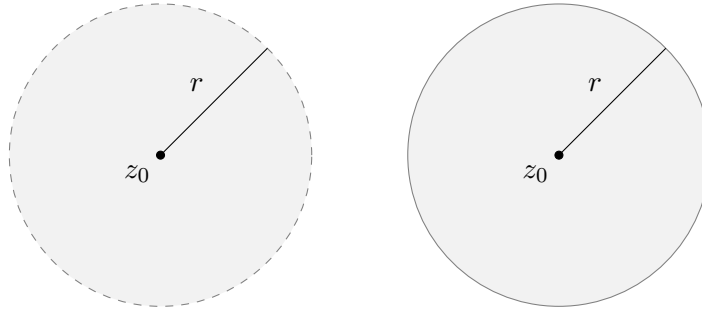


Figure 1.2: Comparison of an open disk (left), which excludes the boundary, and a closed disk (right), which includes the boundary. Both are centered at  $z_0$  with radius  $r$ .

**Remark 1.3.4.** The topology on  $\mathbb{C}$  is inherited from its identification with  $\mathbb{R}^2$ . Just as open intervals form the basic open sets in  $\mathbb{R}$ , open disks form the basic open sets in  $\mathbb{C}$ . Intuitively, a set in  $\mathbb{C}$  is open if around every point in the set, there is a small open disk that is entirely contained within the set, just as in  $\mathbb{R}$ , every point in an open set is surrounded by a small open interval.

**Example 1.3.5.** The complex plane  $\mathbb{C}$  is an open set. Indeed, given any point  $z_0 \in \mathbb{C}$ , we can construct an open disk  $D(z_0, r)$  centered at  $z_0$  with any radius  $r > 0$ , and this disk will be entirely contained in  $\mathbb{C}$ . Therefore, by definition,  $\mathbb{C}$  is open.

**Example 1.3.6.** Let  $A = \{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{C}$ . Then we claim that  $A$  is an open set. To see this, let  $z_0 \in A$  be arbitrary. Then  $|z_0| < 1$ , so there exists an  $r > 0$  such that the open disk  $D(z_0, r) \subseteq A$ . Specifically, we may take  $r = 1 - |z_0| > 0$ , and for any  $z \in D(z_0, r)$ , we have by the triangle inequality,

$$|z| \leq |z - z_0| + |z_0| < r + |z_0| = (1 - |z_0|) + |z_0| = 1$$

Hence,  $z \in A$ , and so  $D(z_0, r) \subseteq A$ . Since  $z_0$  was arbitrary, it follows that  $A$  is open.

**Example 1.3.7.** Let  $B = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\} \subseteq \mathbb{C}$ . Then we claim that  $B$  is not an open set. To see this, consider  $z_0 = 0 \in B$ , but any open disk centered at 0 contains points with  $\operatorname{Im}(z) < 0$ , so no such disk is fully contained in  $B$ , so  $z_0$  is not an interior point of  $B$ .

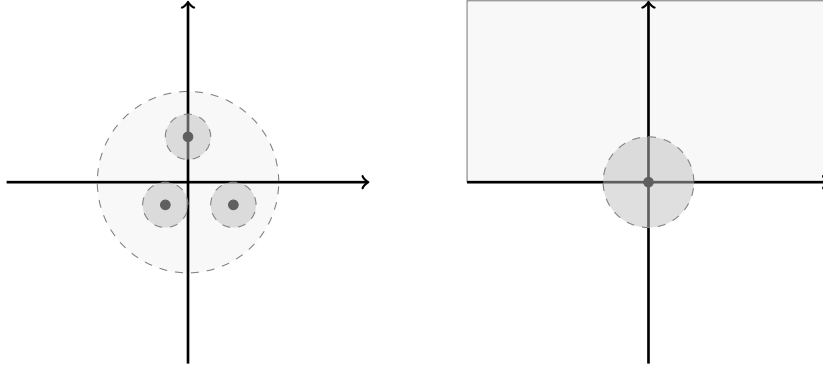


Figure 1.3: The figure on the left illustrates Example 1.3.6 where every point in the open unit disk has a surrounding disk contained entirely within the set. The figure on the right illustrates Example 1.3.7, where a boundary point on the real axis does not admit such a disk, and thus, the set is not open.

Before we can formally define domains and regions, it is important to understand the notion of connectedness. Roughly speaking, a set is connected if it cannot be split into two disjoint open pieces. In the context of the complex plane, this typically means that any two points in the set can be joined by a continuous path lying entirely within the set. Connectedness is essential for studying the behavior of complex functions, especially in results involving analytic continuation and integration.

**Definition 1.3.8.** We say that  $A \subseteq \mathbb{C}$  is a *connected set* if for every pair of points  $z, w \in A$ , there exists a polygonal line that lies entirely in  $A$ .

**Example 1.3.9.** We claim that the set  $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$  is a connected subset of  $\mathbb{C}$ . To see this, let  $z, w \in A$  be arbitrary. Then  $1 < |z|, |w| < 2$ , so both points lie in the annulus centered at the origin with inner radius 1 and outer radius 2. Since  $A$  is an open annular region, we can construct a polygonal path from  $z$  to  $w$  that stays entirely within  $A$ . For example, one can connect  $z$  and  $w$  via line segments to avoid the inner and outer boundaries. Therefore, such a path lies entirely in  $A$ .

**Example 1.3.10.** We claim that the set  $B = \{z \in \mathbb{C} : \operatorname{Re}(z) < 1\} \cup \{z \in \mathbb{C} : \operatorname{Re}(z) > 2\}$  is not a connected subset of  $\mathbb{C}$ . To see this, denote  $B_1 = \{z \in \mathbb{C} : \operatorname{Re}(z) < 1\}$  and  $B_2 = \{z \in \mathbb{C} : \operatorname{Re}(z) > 2\}$ . Let  $z \in B_1$  and  $w \in B_2$ . Then  $z$  and  $w$  lie in separate components of the set  $B$ . Assume for a contradiction that  $B$  is connected. Then there exists a polygonal line

entirely contained in  $B$  that connects  $z$  to  $w$ . However, any such path from  $z$  to  $w$  must pass through a point whose real part lies in the interval  $[1, 2]$ , but this strip is not included in  $B$ , a contradiction. Therefore, no polygonal path between  $z$  and  $w$  can remain entirely within  $B$ .

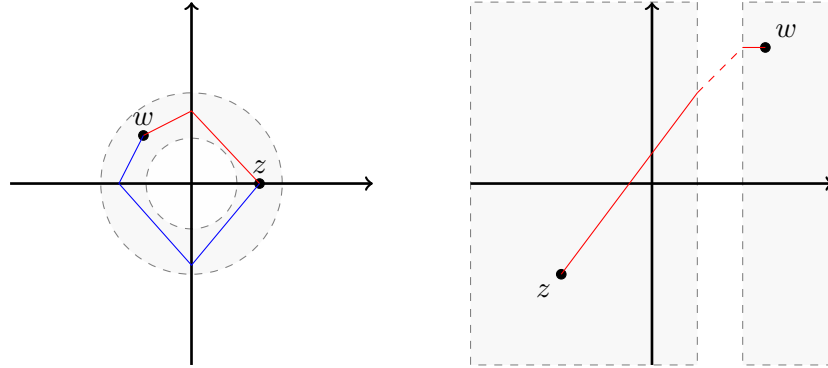


Figure 1.4: The figure on the left illustrates Example 1.3.9, where a polygonal line lies entirely within the annulus, showing that the set is connected. The figure on the right illustrates Example 1.3.10 where no polygonal path connecting  $z$  and  $w$  can remain inside the union of the two disconnected half-planes, so the set is not connected.

The most important sets that are of interest in this course is called the domain.

**Definition 1.3.11.** A subset  $D \subseteq \mathbb{C}$  is said to be a *domain* if it is open and connected.

**Example 1.3.12.** Examples 1.3.5, 1.3.6, and 1.3.9 are domains since each set is open and connected, while Examples 1.3.7 and 1.3.10 are not since it is either not open or not connected.

**Definition 1.3.13.** Let  $A \subseteq \mathbb{C}$ . Let  $z_0 \in \mathbb{C}$  be such that every neighbourhood of  $z_0$  intersects  $A$  and  $\mathbb{C} \setminus A$ . Then we call  $z_0$  a boundary point of  $A$ . The set of all boundary points of  $A$  is denoted by  $\partial A$ .

**Example 1.3.14.** Consider the set  $A = \mathbb{C} \setminus \{0\}$ . Then the only boundary point of  $A$  is the origin 0. Indeed, every open disk centered at 0 contains points in  $A$  (since  $A$  is dense near 0) as well as the point  $0 \notin A$ , so  $0 \in \partial A$ . Any other point  $z \neq 0$  has a neighborhood that lies entirely within  $A$ , so it is not a boundary point.

**Example 1.3.15.** Consider the set  $B = \{z \in \mathbb{C} : 1 < |z| \leq 2\}$ . Then the boundary of  $B$  is given by

$$\partial B = \{z \in \mathbb{C} : |z| = 1\} \cup \{z \in \mathbb{C} : |z| = 2\}.$$

Indeed, any point on either the inner circle  $|z| = 1$  or the outer circle  $|z| = 2$  satisfies the condition that every open disk centered at that point intersects both  $B$  and its complement. Therefore, these points are boundary points. No other points meet this criterion.

**Definition 1.3.16.** Let  $D$  be a domain with some, none, or all of its boundary points. Then  $D$  is called a *region*.





## Chapter 2

# Functions of a Complex Variable

The study of complex analysis centers around functions defined on subsets of the complex plane. While real-valued functions of a real variable form the foundation of single-variable calculus, complex-valued functions of a complex variable exhibit remarkably rich and elegant structure. In fact, the behaviour of complex functions is often far more rigid and constrained than their real counterparts, leading to deep and powerful theorems.

This chapter introduces the basic theory of functions of a complex variable. We begin by defining complex functions, their limits, and continuity. We then investigate the notion of differentiability, which in the complex setting leads to the concept of holomorphicity. Unlike real analysis, complex differentiability implies infinitely many additional properties, such as infinite differentiability, power series expansions, and strong constraints on the geometry and behaviour of the function.

The goal of this chapter is to lay the foundational holomorphic framework for the rest of the course, including the study of contour integration and power series.

### 2.1 Functions of a Complex Variable

According to the following definition, a complex function is characterized as a function that accepts complex numbers as its inputs and also produces them as its outputs.

**Definition 2.1.1.** A *complex function* is a function  $f$  whose domain and range are subsets of  $\mathbb{C}$ .

A complex function is also called a *complex-valued function of a complex variable*. For the most part, we will use the usual symbols  $f, g, h, \dots$  to denote complex functions.

**Example 2.1.2.** The formula  $z^2 - (1+i)z$  can be computed for any complex number  $z$  to produce a distinct complex result, establishing  $f(z) = z^2 - (1+i)z$  as a complex function. The values of  $f$  are determined using the complex number arithmetic covered in Section 1.1. For example, if  $z = i$ , then

$$f(i) = i^2 - (1+i)i = -i$$

**Example 2.1.3.** The expression  $g(z) = z + 2\text{Im}(z)$  also defines a complex function. For example, if  $z = i$ , then

$$g(i) = i + 2\text{Im}(i) = i + 2$$

It is often helpful to express the inputs and the outputs of a complex function in terms of its real and imaginary parts. If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a complex function, then the image of a complex number  $z = x + yi$  under  $f$  is a complex number  $u + iv$ . By simplifying the expression  $f(x + yi)$ , we can write the real and imaginary variables  $u$  and  $v$  in terms of the real variables  $x$  and  $y$ . That is, we can express any complex function in terms of two real functions as

$$f(z) = f(x + yi) = u(x, y) + iv(x, y)$$

The functions  $u(x, y)$  and  $v(x, y)$  are called the real and imaginary parts of  $f$ . Of course, we can also express  $f$  as

$$f(z) = \text{Re}(f(z)) + i\text{Im}(f(z))$$

**Example 2.1.4.** For the complex function  $f(z) = z^2$ , if  $z = x + yi$ , then

$$f(x + yi) = (x + yi)^2 = x^2 + 2xyi - y^2 = (x^2 - y^2) + 2xyi$$

and thus,  $u(x, y) = x^2 - y^2$  and  $v(x, y) = 2xy$ .

Every complex function is determined by the real functions  $u(x, y)$  and  $v(x, y)$ . That is, a function  $f$  can be defined without using the symbol  $z$ . We note that complex functions defined in terms of  $u(x, y)$  and  $v(x, y)$  can always be expressed, if desired, in terms of the operations on the symbols  $z$  and  $\bar{z}$ .

## 2.2 Limits and Continuity

The foundational principle in elementary calculus is the limit. Remember,  $\lim_{x \rightarrow a} f(x) = L$  signifies that the outputs  $f(x)$  of the function  $f$  can be made as near to the real number  $L$  as desired, given that  $x$  is chosen sufficiently near—but not equal—to  $a$ . In real analysis, limits underpin the definitions of continuity, derivatives, and definite integrals.

Similarly, complex limits hold a vital position in complex analysis. The notion of a complex limit mirrors that of a real limit:  $\lim_{z \rightarrow z_0} f(z) = L$  indicates that the values  $f(z)$  of the function  $f$  can approach, but not reach, the complex number  $z_0$ .

**Definition 2.2.1.** Suppose a complex function  $f$  is defined in a deleted neighbourhood of  $z_0$ , i.e.  $f$  is defined on  $\mathbb{C} \setminus \{z_0\}$ , and suppose that  $L \in \mathbb{C}$ . The *limit of  $f$  as  $z$  approaches  $z_0$  exists and is equal to  $L$* , written as  $\lim_{z \rightarrow z_0} f(z) = L$ , if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that if  $|z - z_0| < \delta$ , then  $|f(z) - L| < \varepsilon$ .

For limits of a complex function,  $z$  is allowed to approach  $z_0$  from *any* direction in the complex plane, that is, along any curve or path through  $z_0$ . In order that  $\lim_{z \rightarrow z_0} f(z) = L$ , we require that  $f(z)$  approach the same complex number  $L$  along every possible curve through  $z_0$ .

**Proposition 2.2.2.** *If  $f$  approaches two complex numbers  $L_1 \neq L_2$  for any two different curves or paths through  $z_0$ , then  $\lim_{z \rightarrow z_0} f(z)$  does not exist.*

*Proof.* By definition, the limit  $\lim_{z \rightarrow z_0} f(z)$  exists if and only if  $f(z)$  approaches the same complex number  $L$  along every path approaching  $z_0$ . If  $f(z) \rightarrow L_1$  along one path and  $f(z) \rightarrow L_2$  along another, with  $L_1 \neq L_2$ , then  $f(z)$  does not tend to a single value as  $z \rightarrow z_0$ . Therefore, the limit does not exist.  $\square$

**Example 2.2.3.** We claim that the limit  $\lim_{z \rightarrow 0} \frac{z}{\bar{z}}$  does not exist. To see this, we will find two different ways of letting  $z \rightarrow 0$  that yield different values for  $\lim_{z \rightarrow 0} \frac{z}{\bar{z}}$ . First, let  $z \rightarrow 0$  along the real axis. Then we consider complex numbers of the form  $z = x + 0i$  where  $x \in \mathbb{R}$  approaching 0. Then

$$\lim_{z \rightarrow 0} \frac{z}{\bar{z}} = \lim_{x \rightarrow 0} \frac{x + 0i}{x - 0i} = \lim_{x \rightarrow 0} 1 = 1$$

On the other hand, if we let  $z \rightarrow 0$  along the imaginary axis, then we consider complex numbers of the form  $z = 0 + yi$  where  $y \in \mathbb{R}$  approaching 0. Then

$$\lim_{z \rightarrow 0} \frac{z}{\bar{z}} = \lim_{y \rightarrow 0} \frac{0 + yi}{0 - yi} = \lim_{y \rightarrow 0} (-1) = -1$$

Since both values are not the same, then by Proposition 2.2.2,  $\lim_{z \rightarrow 0} \frac{z}{z}$  does not exist.

Of course, as in real analysis, we can also prove that the limit exists using the  $\varepsilon$ - $\delta$  definition.

**Example 2.2.4.** We claim that  $\lim_{z \rightarrow 1+i} (2+i)z = 1+3i$ . To see this, let  $\varepsilon > 0$  be arbitrary. We seek a  $\delta > 0$  such that if  $|z - (1+i)| < \delta$ , then  $|(2+i)z - (1+3i)| < \varepsilon$ . Indeed, first observe that

$$\begin{aligned} |(2+i)z - (1+3i)| &= |2+i| \left| z - \frac{1+3i}{2+i} \right| \\ &= \sqrt{5} \left| z - \frac{1+3i}{2+i} \right| \\ &= \sqrt{5} |z - (1+i)| \\ &< \sqrt{5}\delta \end{aligned}$$

Therefore, choosing  $\delta = \frac{\varepsilon}{\sqrt{5}}$ , we obtain that

$$|(2+i)z - (1+3i)| < \sqrt{5} \cdot \frac{\varepsilon}{\sqrt{5}} = \varepsilon$$

Therefore, as  $\varepsilon > 0$  was arbitrary,  $\lim_{z \rightarrow 1+i} (2+i)z = 1+3i$ , as desired.

We now present the following result, which relates real limits of  $u(x, y)$  and  $v(x, y)$  with the complex limit of  $f(z) = u(x, y) + iv(x, y)$ .

**Theorem 2.2.5.** Suppose that  $f(z) = u(x, y) + iv(x, y)$ ,  $z_0 = x_0 + iy_0$  and  $L = u_0 + iv_0$ . Then  $\lim_{z \rightarrow z_0} f(z) = L$  if and only if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$$

*Proof.* Assume that  $\lim_{z \rightarrow z_0} f(z) = L$ , and let  $\varepsilon > 0$  be arbitrary. Then there exists a  $\delta > 0$  such that if  $|z - z_0| < \delta$ , then  $|f(z) - L| < \varepsilon$ . Using the identifications  $f(z) = u(x, y) + iv(x, y)$  and  $L = u_0 + iv_0$ , then

$$|f(z) - L| = \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2}$$

Furthermore, since  $0 \leq (v(x, y) - v_0)^2$ , it follows that

$$|u(x, y) - u_0| = \sqrt{(u(x, y) - u_0)^2} \leq \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2}$$

Thus, for all  $z = x + iy$ , we have

$$|u(x, y) - u_0| \leq |f(z) - L|$$

In particular, if  $|f(z) - L| < \varepsilon$ , then  $|u(x, y) - u_0| < \varepsilon$ . Now by making the identifications  $z = x + iy$  and  $z_0 = x_0 + iy_0$ , we also find that

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Therefore, if  $|z - z_0| < \delta$ , we have  $|u(x, y) - u_0| < \varepsilon$ , and thus,  $\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0$ , as desired. The proof in showing the second limit is almost identical.

Conversely, assume that  $\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0$  and  $\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0$ . We need to show that  $\lim_{z \rightarrow z_0} f(z) = L$ . Indeed, let  $\varepsilon > 0$  be arbitrary. Since  $\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0$ , there exists a  $\delta_1 > 0$  such that if  $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1$ , then

$$|u(x, y) - u_0| < \frac{\varepsilon}{2}$$

Similarly, since  $\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0$ , then there exists a  $\delta_2 > 0$  such that if  $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2$ , then

$$|v(x, y) - v_0| < \frac{\varepsilon}{2}$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then if  $|z - z_0| < \delta$

$$|u(x, y) - u_0| + |v(x, y) - v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

On the other hand, if  $f(z) = u(x, y) + iv(x, y)$  and  $L = u_0 + iv_0$ , then by the triangle inequality,

$$|f(z) - L| \leq |u(x, y) - u_0| + |v(x, y) - v_0| < \varepsilon$$

Therefore, as  $\varepsilon > 0$  was arbitrary, we have  $\lim_{z \rightarrow z_0} f(z) = L$ , as desired.  $\square$

**Example 2.2.6.** Since  $f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$ , then by Theorem 2.2.5 with  $u(x, y) = x^2 - y^2$  and  $v(x, y) = 2xy + 1$ , and  $z_0 = 1 + i$ , we have

$$\begin{aligned} u_0 &= \lim_{(x, y) \rightarrow (1, 1)} x^2 - y^2 = 0 \\ v_0 &= \lim_{(x, y) \rightarrow (1, 1)} 2xy + 1 = 3 \end{aligned}$$

Therefore,  $\lim_{z \rightarrow 1+i} z^2 + i = 3i$ .

In addition to computing specific limits, Theorem 2.2.5 is also an important theoretical tool that allows us to derive many properties of complex limits from properties of real limits.

**Theorem 2.2.7.** *Let  $f, g$  be complex functions. If  $\lim_{z \rightarrow z_0} f(z) = L$  and  $\lim_{z \rightarrow z_0} g(z) = M$ , then*

1. For any  $\alpha \in \mathbb{C}$ ,  $\lim_{z \rightarrow z_0} \alpha = \alpha$ .
2.  $\lim_{z \rightarrow z_0} z = z_0$ .
3. For any  $\alpha \in \mathbb{C}$ ,  $\lim_{z \rightarrow z_0} \alpha f(z) = \alpha L$ .
4.  $\lim_{z \rightarrow z_0} f(z) + g(z) = L + M$ .
5.  $\lim_{z \rightarrow z_0} f(z)g(z) = LM$ .
6. If  $M \neq 0$ , then  $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$ .

*Proof.* To see that (1) holds, let  $\varepsilon > 0$  be arbitrary. Since

$$|f(z) - \alpha| = |\alpha - \alpha| = 0 < \varepsilon$$

Then we may choose any  $\delta > 0$  such that if  $|z - z_0| < \delta$ , then  $|f(z) - \alpha| < \varepsilon$ . Therefore, as  $\varepsilon > 0$  was arbitrary, we have  $\lim_{z \rightarrow z_0} \alpha = \alpha$ .

To see that (2) holds, let  $\varepsilon > 0$  be arbitrary. Since

$$|f(z) - z_0| = |z - z_0|$$

We may choose  $\delta = \varepsilon > 0$  so that if  $|z - z_0| < \delta$ , then  $|f(z) - z_0| < \varepsilon$ . Therefore, as  $\varepsilon > 0$  was arbitrary, we have  $\lim_{z \rightarrow z_0} z = z_0$ .

To see that (3) holds, let  $f(z) = u(x, y) + iv(x, y)$ ,  $z_0 = x_0 + iy_0$ ,  $L = u_0 + iv_0$ , and  $c = a + ib$ . Since  $\lim_{z \rightarrow z_0} f(z) = L$ , then by Theorem 2.2.5,  $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0$  and  $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$ . Then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} (au(x, y) - bv(x, y)) = au_0 - bv_0$$

and

$$\lim_{(x,y) \rightarrow (x_0,y_0)} (bu(x, y) - av(x, y)) = bu_0 + av_0$$

But then  $\operatorname{Re}(\alpha f(z)) = au(x, y) - bv(x, y)$  and  $\operatorname{Im}(\alpha f(z)) = bu(x, y) + av(x, y)$ . Therefore,

$$\lim_{z \rightarrow z_0} \alpha f(z) = au_0 - bv_0 + i(bu_0 + av_0) = \alpha L$$

To see that (4) holds, let  $f(z) = u_1(x, y) + iv_1(x, y)$ ,  $g(z) = u_2(x, y) + iv_2(x, y)$ ,  $z_0 = x_0 + iy_0$ ,  $L = u_{1,0} + iv_{1,0}$ , and  $M = u_{2,0} + iv_{2,0}$ . Since  $\lim_{z \rightarrow z_0} f(z) = L$  and  $\lim_{z \rightarrow z_0} g(z) = M$ , then by Theorem 2.2.5, we have for  $i = 1, 2$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u_i(x, y) = u_{i,0}, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v_i(x, y) = v_{i,0}$$

But then  $\operatorname{Re}(f(z) + g(z)) = u_1(x, y) + u_2(x, y)$  and  $\operatorname{Im}(f(z) + g(z)) = v_1(x, y) + v_2(x, y)$ . Therefore,

$$\lim_{z \rightarrow z_0} f(z) + g(z) = (u_{1,0} + iv_{1,0}) + (u_{2,0} + iv_{2,0}) = L + M$$

To see that (5) holds, let  $f(z) = u_1(x, y) + iv_1(x, y)$ ,  $g(z) = u_2(x, y) + iv_2(x, y)$ ,  $z_0 = x_0 + iy_0$ ,  $L = u_{1,0} + iv_{1,0}$ , and  $M = u_{2,0} + iv_{2,0}$ . Since  $\lim_{z \rightarrow z_0} f(z) = L$  and  $\lim_{z \rightarrow z_0} g(z) = M$ , then by Theorem 2.2.5, we have for  $i = 1, 2$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u_i(x, y) = u_{i,0}, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v_i(x, y) = v_{i,0}$$

But then note that

$$\begin{aligned} f(z)g(z) &= (u_1(x, y) + iv_1(x, y))(u_2(x, y) + iv_2(x, y)) \\ &= u_1(x, y)u_2(x, y) + iu_1(x, y)v_2(x, y) + iu_2(x, y)v_1(x, y) - v_1(x, y)v_2(x, y) \end{aligned}$$

So then  $\operatorname{Re}(f(z)g(z)) = u_1(x, y)u_2(x, y) - v_1(x, y)v_2(x, y)$  and  $\operatorname{Im}(f(z)g(z)) = u_1(x, y)v_2(x, y) + u_2(x, y)v_1(x, y)$ . Therefore,

$$\lim_{z \rightarrow z_0} f(z)g(z) = (u_{1,0}u_{2,0} - v_{1,0}v_{2,0}) + i(u_{1,0}v_{2,0} + u_{2,0}v_{1,0}) = LM$$

Finally, to see that (6) holds, assume that  $M \neq 0$ , and let  $f(z) = u_1(x, y) + iv_1(x, y)$ ,  $g(z) = u_2(x, y) + iv_2(x, y)$ ,  $z_0 = x_0 + iy_0$ ,  $L = u_{1,0} + iv_{1,0}$ , and  $M = u_{2,0} + iv_{2,0}$ . Since  $\lim_{z \rightarrow z_0} f(z) = L$  and  $\lim_{z \rightarrow z_0} g(z) = M$ , then by Theorem 2.2.5, we have for  $i = 1, 2$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u_i(x, y) = u_{i,0}, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v_i(x, y) = v_{i,0}$$

But then note that

$$\begin{aligned} \frac{f(z)}{g(z)} &= \frac{u_1(x, y) + iv_1(x, y)}{u_2(x, y) + iv_2(x, y)} \\ &= \frac{u_1(x, y)u_2(x, y) - iu_1(x, y)v_2(x, y) + iu_2(x, y)v_1(x, y) + v_1(x, y)v_2(x, y)}{u_2^2(x, y) + v_2^2(x, y)} \\ &= \frac{u_1(x, y)u_2(x, y) + v_1(x, y)v_2(x, y)}{u_2^2(x, y) + v_2^2(x, y)} + i \frac{u_2(x, y)v_1(x, y) - u_1(x, y)v_2(x, y)}{u_2^2(x, y) + v_2^2(x, y)} \end{aligned}$$

So then  $\operatorname{Re} \left( \frac{f(z)}{g(z)} \right) = \frac{u_1(x,y)u_2(x,y)+v_1(x,y)v_2(x,y)}{u_2^2(x,y)+v_2^2(x,y)}$  and  $\operatorname{Im} \left( \frac{f(z)}{g(z)} \right) = \frac{u_2(x,y)v_1(x,y)-u_1(x,y)v_2(x,y)}{u_2^2(x,y)+v_2^2(x,y)}$ . Therefore,

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{u_{1,0}u_{2,0} + v_{1,0}v_{2,0}}{u_{2,0}^2 + v_{2,0}^2} + i \frac{u_{2,0}v_{1,0} - u_{1,0}v_{2,0}}{u_{2,0}^2 + v_{2,0}^2} = \frac{L}{M}$$

□

**Example 2.2.8.** Using the limit properties in Theorem 2.2.7, we have

$$\lim_{z \rightarrow i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{\lim_{z \rightarrow i} (3+i)z^4 - z^2 + 2z}{\lim_{z \rightarrow i} z+1} = \frac{4+3i}{1+i} = \frac{7}{2} - \frac{1}{2}i$$

**Example 2.2.9.** In order to find  $\lim_{z \rightarrow 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i}$ , note that we cannot directly apply Theorem 2.2.7 just yet, since the limit as  $z \rightarrow 1+\sqrt{3}i$  approaches 0 in both the numerator and the denominator. However, note that  $1+\sqrt{3}i$  is a root of the quadratic polynomial  $z^2 - 2z + 4$ , so we can factor the polynomial to obtain

$$z^2 - 2z + 4 = (z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i)$$

Thus, now we have

$$\lim_{z \rightarrow 1+\sqrt{3}i} \frac{(z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i)}{z - 1 - \sqrt{3}i} = \lim_{z \rightarrow 1+\sqrt{3}i} z - 1 + \sqrt{3}i = 2\sqrt{3}i$$

The definition of continuity for a complex function is, in essence, the same as that for a real function. That is, a complex function  $f$  is continuous at a point  $z_0$  if the limit of  $f$  as  $z$  approaches  $z_0$  exists and is the same as the value of  $f$  at  $z_0$ . This gives us the following definition for complex functions.

**Definition 2.2.10.** A complex function  $f$  is *continuous* at a point  $z_0 \in \mathbb{C}$  if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

A complex function  $f$  is said to be *continuous* if  $f$  is continuous at every point  $z_0 \in \mathbb{C}$ .

If a complex function is not continuous at a point  $z_0$ , then we say that  $f$  is discontinuous at  $z_0$ . For example, the function  $f(z) = \frac{1}{1+z^2}$  is discontinuous at the points  $z = i$  and  $z = -i$ .

Since the concept of continuity is defined using the complex limit, various properties of complex limits can be translated into statements about continuity. Consider Theorem 2.2.5, which describes the connection between the complex limit of  $f(z) = u(x, y) + iv(x, y)$ , and the real limits of  $u$  and  $v$ .



**Theorem 2.2.11.** *Suppose that  $f(z) = u(x, y) + iv(x, y)$  and  $z_0 = x_0 + iy_0$ . Then the complex function  $f$  is continuous at the point  $z_0$  if and only if both real functions  $u(x, y)$  and  $v(x, y)$  are continuous at the point  $(x_0, y_0)$ .*

*Proof.* Assume that the complex function  $f$  is continuous at  $z_0$ . Then by definition, we have

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$$

Then by Theorem 2.2.5, we have

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u(x_0, y_0), \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v(x_0, y_0)$$

Therefore, both  $u$  and  $v$  are continuous at  $(x_0, y_0)$ .

Conversely, if  $u(x, y)$  and  $v(x, y)$  are continuous at  $(x_0, y_0)$ , then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u(x_0, y_0), \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v(x_0, y_0)$$

It then follows from Theorem 2.2.5 that

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) = u(x_0, y_0) + iv(x_0, y_0) = f(z_0)$$

Therefore,  $f$  is continuous. □

**Example 2.2.12.** According to Theorem 2.2.11,  $f(z) = \bar{z} = x - iy$  is a continuous at  $z_0 = x_0 + iy_0$  if both  $u(x, y) = x$  and  $v(x, y) = -y$  are continuous at  $(x_0, y_0)$ . Since  $u(x, y)$  and  $v(x, y)$  are polynomial functions, then it follows

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = x_0, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = -y_0$$

This implies that both  $u(x, y)$  and  $v(x, y)$  are continuous at  $(x_0, y_0)$ , and therefore that  $f$  is continuous at  $z_0$  by Theorem 2.2.11.

The algebraic properties of complex limits from Theorem 2.2.7 can also be restated in terms of the continuity of complex functions.

**Corollary 2.2.13.** *If  $f$  and  $g$  are continuous at the point  $z_0$ , then the following functions are continuous at  $z_0$ .*

1.  $\alpha f$  for all  $\alpha \in \mathbb{C}$ .
2.  $f + g$ .

3.  $fg$ .

4.  $\frac{f}{g}$  if  $g(z_0) \neq 0$ .

Of course, the results of Corollary 2.2.13 (3) and (4) extend to any finite sum or product of continuous functions. We can use these facts to show that polynomials are continuous functions.

**Theorem 2.2.14.** *Polynomial functions are continuous on the entire complex plane  $\mathbb{C}$ .*

*Proof.* Let  $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$  be a polynomial function and let  $z_0 \in \mathbb{C}$  be arbitrary. By Theorem 2.2.7 (2),  $f(z) = z$  is continuous at  $z_0$ , and by repeated application of Corollary 2.2.13, this implies that the power function  $f(z) = z^n$  where  $n \in \mathbb{N}$ , is continuous at this point as well. Moreover, Theorem 2.2.7 (1) implies that every complex constant  $f(z) = c$  is continuous at  $z_0$ , and so it follows from Corollary 2.2.13 that each of the functions  $a_n z^n, a_{n-1} z^{n-1}, \dots, a_0$  are continuous at  $z_0$ . Now from a repeated application of Corollary 2.2.13 (2), we have that  $p(z)$  is continuous at  $z_0$ . As  $z_0 \in \mathbb{C}$  was arbitrary,  $p(z)$  is continuous at every  $z_0 \in \mathbb{C}$ , so  $p(z)$  is continuous.  $\square$

## 2.3 Differentiability and Holomorphic Functions

The calculus concerning complex functions addresses the standard notions of derivatives and integrals associated with these functions. In this segment, we introduce the limit definition of the derivative for a complex function  $f(z)$ . While several concepts, like the product, quotient, and chain differentiation rules, may appear familiar, key distinctions exist between this content and the calculus applicable to real functions.

Suppose  $z = x + iy$  and  $z_0 = x_0 + iy_0$ . Then the change in  $z_0$  is the difference  $\Delta z = z - z_0$  or  $\Delta z = (x - x_0) + i(y - y_0) = \Delta x + i\Delta y$ . If a complex function  $f(z)$  is defined at  $z$  and  $z_0$ , then the corresponding change in the function is the difference

$$\Delta f = f(z_0 + \Delta z) - f(z_0)$$

The derivative of the function  $f$  is defined in terms of a limit of the difference quotient  $\frac{\Delta f}{\Delta z}$  as  $\Delta z \rightarrow 0$ .

**Definition 2.3.1.** Let  $D \subseteq \mathbb{C}$  be a domain, let  $f : D \rightarrow \mathbb{C}$ , and let  $z_0 \in D$ . Then the *derivative of  $f$  at  $z_0$* , denoted by  $f'(z_0)$  is

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists. If the limit exists, then the function  $f$  is said to be *differentiable* at  $z_0$ .

Two symbols denoting the derivative of  $f(z)$  are  $f'(z)$  and  $\frac{df}{dz}$ . If the latter notation is used, then the value of a derivative at a specified point  $z_0$  is written  $\left. \frac{df}{dz} \right|_{z=z_0}$ .

**Example 2.3.2.** If  $f(z) = z^2 - 5z$ , using Definition 2.3.1, first note we have  $f(z + \Delta z) = (z + \Delta z)^2 - 5(z + \Delta z)$ , and

$$f(z + \Delta z) - f(z) = (z + \Delta z)^2 - 5(z + \Delta z) - (z^2 - 5z) = 2z\Delta z + (\Delta z)^2 - 5\Delta z$$

Therefore,

$$\begin{aligned} f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{2z\Delta z + (\Delta z)^2 - 5\Delta z}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\Delta z(2z + \Delta z - 5)}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} 2z + \Delta z - 5 \\ &= 2z - 5 \end{aligned}$$

so  $f'(z) = 2z - 5$ .

Of course, the familiar rules of differentiation in the calculus of real variables carry over to the calculus of complex variables. Using Definition 2.3.1, we obtain the following.

**Proposition 2.3.3 (Differentiation Rules).** *If  $f$  and  $g$  are differentiable at a point  $z$ , and  $\alpha \in \mathbb{C}$ , then*

1. (Constant Rule)  $\frac{d}{dz}\alpha = 0$ .
2. (Scalar Multiple Rule)  $\frac{d}{dz}\alpha f(z) = \alpha f'(z)$ .
3. (Power Rule)  $\frac{d}{dz}z^n = nz^{n-1}$  for all  $n \in \mathbb{Z}$ .
4. (Sum Rule)  $\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z)$ .

$$5. \text{ (Product Rule) } \frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z).$$

$$6. \text{ (Quotient Rule) } \frac{d}{dz} \left( \frac{f(z)}{g(z)} \right) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}.$$

$$7. \text{ (Chain Rule) } \frac{d}{dz}f(g(z)) = f'(g(z))g'(z).$$

**Example 2.3.4.** If  $f(z) = 3z^4 - 5z^3 + 2z$ , then using Proposition 2.3.3, we have

$$f'(z) = 12z^3 - 15z^2 + 2$$

For a complex function  $f$  to be differentiable at a point  $z_0$ , then  $\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$  must exist and equal the same complex number from any direction. This means that in complex analysis, the requirement for differentiability of a function  $f(z)$  at a point  $z_0$  is a far greater demand than in real calculus of functions where we can approach a real number on the number line from only two directions.

**Example 2.3.5.** We claim that the function  $f(z) = x + 4iy$  is not differentiable at any point  $z$ . To see this, let  $z = x + yi \in \mathbb{C}$  be arbitrary with  $\Delta z = \Delta x + i\Delta y$ . Then we have

$$f(z + \Delta z) - f(z) = (x + \Delta x) + 4i(y + \Delta y) - x - 4iy = \Delta x + 4i\Delta y$$

and so

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$$

Now we consider the following cases.

Case 1: If  $\Delta z \rightarrow 0$  along a line parallel to  $x$ -axis. Then  $\Delta y = 0$ ,  $\Delta z = \Delta x$ , and

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta x}{\Delta x} = 1$$

Case 2: If  $\Delta z \rightarrow 0$  along a line parallel to  $y$ -axis. Then  $\Delta x = 0$ ,  $\Delta z = i\Delta y$  and

$$\lim_{\Delta z \rightarrow 0} \frac{4i\Delta y}{i\Delta y} = 4$$

From Cases 1 and 2, the values of the limits are different, so  $f(z) = x + 4iy$  is nowhere differentiable.

Even though the requirements of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements. These functions are called holomorphic functions.

**Definition 2.3.6.** Let  $D \subseteq \mathbb{C}$  be a domain,  $f : D \rightarrow \mathbb{C}$  be a function, and  $z_0 \in D$ . Then  $f$  is said to be *holomorphic at  $z_0$*  if  $f$  is differentiable at  $z_0$  and at every point in some neighbourhood of  $z_0$ . We also say that  $f$  is *holomorphic in  $D$*  if it is holomorphic at every point in  $D$ .

**Remark 2.3.7.** It is important that holomorphicity at a point is *not* the same as differentiability at a point. Holomorphicity at a point is a neighbourhood property. In other words, holomorphicity is a property that is defined over an open set.

**Example 2.3.8.** We claim that the function  $f(z) = |z|^2$  is differentiable at 0, but is nowhere differentiable anywhere else. To see that  $f$  is differentiable, at  $z = 0$ , we have  $f(0) = 0$  and if  $\Delta z = \Delta x + i\Delta y$ ,

$$f(\Delta z) - f(0) = |\Delta x + i\Delta y|^2 - 0 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x + i\Delta y)(\Delta x - i\Delta y)$$

Then,

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{f(\Delta z) - f(0)}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{(\Delta x + i\Delta y)(\Delta x - i\Delta y)}{\Delta x + i\Delta y} \\ &= \lim_{\Delta z \rightarrow 0} \Delta x - i\Delta y \\ &= 0 \end{aligned}$$

Therefore,  $f$  is differentiable at 0. However,  $f$  is not holomorphic at 0 since there exists no neighbourhood of  $z = 0$  through which  $f$  is differentiable; hence, the function  $f(z) = |z|^2$  is nowhere holomorphic.

In contrast, the simple polynomial  $f(z) = z^2$  is holomorphic at every point in the complex plane, so  $f$  is holomorphic everywhere.

As in real analysis, if a function  $f$  is differentiable at a point, the function is necessarily continuous at the point.

**Proposition 2.3.9.** Let  $D \subseteq \mathbb{C}$  be a domain. If  $f : D \rightarrow \mathbb{C}$  is differentiable at  $z_0 \in D$ , then  $f$  is continuous at  $z_0$ .

*Proof.* The limits  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  and  $\lim_{z \rightarrow z_0} z - z_0$  exist and equal  $f'(z_0)$  and 0, respectively. Hence, by Theorem 2.2.7 (5), we can write

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) - f(z_0) &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0) \\ &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \rightarrow z_0} (z - z_0) = f'(z_0) 0 = 0 \end{aligned}$$

From  $\lim_{z \rightarrow z_0} f(z) - f(z_0) = 0$ , we conclude that  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ . Therefore,  $f$  is continuous at  $z_0$ .  $\square$

**Remark 2.3.10.** Of course, the converse of Proposition 2.3.9 is false. It follows from Theorem 2.2.11 and Example 2.3.5 that the function  $f(z) = x + 4iy$  is continuous everywhere, but not differentiable anywhere.

## 2.4 The Cauchy–Riemann Equations

In the previous section, we saw that a function  $f$  of a complex variable is holomorphic at a point whenever  $f$  is differentiable at that point, and differentiable at every point in some neighbourhood of that point. This requirement is more stringent than simply differentiability at a point since a complex function can be differentiable at a point, but nowhere differentiable.

In the following theorem, we see that if  $f(z) = u(x, y) + iv(x, y)$  is differentiable at a point  $z$ , then the functions  $u$  and  $v$  must satisfy a pair of equations that relate their first order partial derivatives.

**Theorem 2.4.1 (The Cauchy–Riemann Equations).** *Suppose  $f(z) = u(x, y) + iv(x, y)$  is differentiable at a point  $z = x + iy$ . Then at  $z$ , the first order partial derivatives of  $u$  and  $v$  exist, and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

*Proof.* By the definition of the derivative, we have at  $z$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

By writing  $f(z) = u(x, y) + iv(x, y)$  and  $\Delta z = \Delta x + i\Delta y$ , the above becomes

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

Since the limit is assumed to exist,  $\Delta z$  can approach 0 from any direction.

Case 1: If  $\Delta \rightarrow 0$  along a horizontal line. Then  $\Delta y = 0$ ,  $\Delta z = \Delta x$ , and

$$\begin{aligned} f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y) + i(v(x + \Delta x, y) - v(x, y))}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \end{aligned}$$

The existence of  $f'(z)$  implies that each limit above exists. These limits are the definitions of the first order partial derivatives with respect to  $x$  of  $u$  and

$v$ , respectively. Hence, we have shown that  $\frac{\partial u}{\partial x}$  and  $\frac{\partial v}{\partial x}$  exist at the point  $z$ , and that the derivative of  $f$  is

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Case 2: If  $\Delta z \rightarrow 0$  along a vertical line. Then  $\Delta x = 0$ ,  $\Delta z = i\Delta y$ , and by a similar computation,

$$f'(z) = \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}$$

In this case, the above shows that  $\frac{\partial u}{\partial y}$  and  $\frac{\partial v}{\partial y}$  exist at  $z$  and that

$$f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Therefore, by Case 1 and 2, we obtain that

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Thus, by equating the real and imaginary parts, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

as desired. □

**Remark 2.4.2.** Since Theorem 2.4.1 states that the Cauchy–Riemann equations hold at  $z$  as a necessary consequence of  $f$  being differentiable at  $z$ , we cannot use the theorem to help us determine where  $f$  is differentiable. However, it is important to realize that Theorem 2.4.1 tells us where a function  $f$  does not possess a derivative. If the Cauchy–Riemann equations are not satisfied at the a point  $z$ , then  $f$  cannot be differentiable at  $z$ .

**Example 2.4.3.** The polynomial function  $f(z) = z^2 + z$  is holomorphic for all  $z$  and can be written as

$$f(z) = (x^2 - y^2 + x) + i(2xy + y)$$

Therefore,  $u(x, y) = x^2 - y^2 + x$  and  $v(x, y) = 2xy + y$ . For any point  $(x, y)$  in the complex plane, we see that the Cauchy–Riemann equations are satisfied:

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x + 1 = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x} \end{aligned}$$

**Example 2.4.4.** We claim that the function  $f(z) = (2x^2 + y) + i(y^2 - x)$  is not holomorphic at any point. Indeed, identifying  $u(x, y) = 2x^2 + y$  and  $v(x, y) = y^2 - x$ , then since

$$\begin{aligned}\frac{\partial u}{\partial x} &= 4x & \frac{\partial v}{\partial y} &= 2y \\ \frac{\partial u}{\partial y} &= 1 & \frac{\partial v}{\partial x} &= -1\end{aligned}$$

we see that  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ , the equality  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  is satisfied only on the line  $y = 2x$ . However, for any point  $z$  on the line, there is no neighbourhood or open disk about  $z$  in which  $f$  is differentiable at every point. Therefore,  $f$  is nowhere holomorphic.

By themselves, the Cauchy–Riemann equations do not ensure holomorphicity of a function  $f(z) = u(x, y) + iv(x, y)$  at a point  $z = x + iy$ . It is possible for the Cauchy–Riemann equations to be satisfied at  $z$ , and yet,  $f(z)$  may not be differentiable at  $z$ , or  $f(z)$  may be differentiable at  $z$ , but nowhere else. In either case,  $f$  is not holomorphic at  $z$ .

However, when we add the condition of continuity to  $u$  and  $v$  and to the four partial derivatives  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$ ,  $\frac{\partial v}{\partial x}$ , and  $\frac{\partial v}{\partial y}$ , it can be shown that the Cauchy–Riemann equations are not only necessary, but also sufficient to guarantee holomorphicity of  $f(z) = u(x, y) + iv(x, y)$ . The proof is long and complicated, so we only state the result.

**Theorem 2.4.5.** Suppose the real functions  $u(x, y)$  and  $v(x, y)$  are continuous and have continuous first order partial derivatives in a domain  $D$ . If  $u(x, y)$  and  $v(x, y)$  satisfy the Cauchy–Riemann equations at all points of  $D$ , then the complex function  $f(z) = u(x, y) + iv(x, y)$  is holomorphic in  $D$ .

**Example 2.4.6.** For the function  $f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ , the real functions  $u(x, y) = \frac{x}{x^2+y^2}$  and  $v(x, y) = -\frac{y}{x^2+y^2}$  are continuous, except at the point where  $x^2 + y^2 = 0$ , that is, at  $z = 0$ . Moreover, the first four first order partial derivatives

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{y^2 - x^2}{(x^2 + y^2)^2} & \frac{\partial u}{\partial y} &= -\frac{2xy}{(x^2 + y^2)^2} \\ \frac{\partial v}{\partial x} &= \frac{2xy}{(x^2 + y^2)^2} & \frac{\partial v}{\partial y} &= \frac{y^2 - x^2}{(x^2 + y^2)^2}\end{aligned}$$

are continuous except at  $z = 0$ . Finally, we see from

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}$$



that the Cauchy–Riemann equations are satisfied except at  $z = 0$ . Thus, we conclude from Theorem 2.4.5 that  $f$  is holomorphic in any domain  $D$  that does not contain the point  $z = 0$ .

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