MATH 2022
Linear Algebra II

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Preface

These are the first edition of these lecture notes for MATH 2022 (Linear Algebra II). Consequently, there may be several typographical errors, missing exposition on necessary background, and more advanced topics for which there will not be time in class to cover. Future iterations of these notes will hopefully be fairly self-contained provided one has the necessary background. If you come across any typos, errors, omissions, or unclear expositions, please feel free to contact me so that I may continually improve these notes.

Contents

Chapter 1

Vector Spaces

Linear algebra is fundamentally concerned with the study of vector spaces and linear transformations between them. A vector space is a mathematical structure formed by a collection of objects, called vectors, which can be added together and multiplied ("scaled") by elements of a field, such as the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

The abstraction of vector spaces allows us to unify and generalize many familiar mathematical objects—such as geometric vectors, sequences, matrices, polynomials, and functions—under a single framework. Despite their varied appearances, these objects all obey a shared set of algebraic rules.

Understanding vector spaces is crucial not only in pure mathematics but also in applied fields such as physics, engineering, computer science, and data science. Many problems in these domains can be reduced to questions about vectors, subspaces, and transformations.

In this chapter, we begin by defining vector spaces formally, exploring key examples and counterexamples, and laying the groundwork for deeper study of linear structure, spanning sets, linear independence, basis, and dimension.

1.1 Vector Spaces

A vector space consists of a nonempty set V and two operations, one called vector addition, and the other called scalar multiplication. In these notes, we will be more focused on vector spaces over $\mathbb{F}$, where $\mathbb{F} = \mathbb{Q}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Later on, we will be more focused on scalars of real or complex numbers, so we will denote $\mathbb{K}$ to be either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 1.1.1. A *vector space* (or *linear space*) over a field $\mathbb{F}$ is a set V equipped with:

- A binary operation $+ : V \times V \to V$ (called *vector addition*).
- An operation $\cdot : \mathbb{F} \times V \to V$ (called *scalar multiplication*)

such that the following axioms hold for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha, \beta \in \mathbb{F}$:

1. (Associativity of Vector Addition) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
2. (Commutativity of Vector Addition) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. (Additive Identity) There exists a $\vec{0}_V \in V$ such that for all $\vec{v} \in V$, $\vec{v} + \vec{0}_V = \vec{0}_V + \vec{v} = \vec{v}$.
4. (Additive Inverse) For every $\vec{v} \in V$, there exists a $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = -\vec{v} + \vec{v} = \vec{0}_V$.
5. (Multiplicative Identity) $\vec{v} \in V$, $1 \cdot \vec{v} = \vec{v}$.
6. (Distributivity over Vector Addition) $\alpha \cdot (\vec{u} + \vec{v}) = \alpha \cdot \vec{u} + \alpha \cdot \vec{v}$.
7. (Distributivity over Scalar Addition) $(\alpha + \beta) \cdot \vec{v} = \alpha \cdot \vec{v} + \beta \cdot \vec{v}$.
8. (Compatibility of Scalar Multiplication) $(\alpha\beta) \cdot \vec{v} = \alpha \cdot (\beta\vec{v})$.

Example 1.1.2. For $n \in \mathbb{N}$, we define the n-dimensional Euclidean space $\mathbb{R}^n$ by

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, x_2, ..., x_n \in \mathbb{R} \right\}$$

We equip $\mathbb{R}^n$ with the usual vector addition and scalar multiplication defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad \alpha \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

It is elementary to verify that $\mathbb{R}^n$ with the defined operations is a vector space. Note that instead of $\mathbb{R}^n$, we can also have $\mathbb{C}^n$ or $\mathbb{Q}^n$ with the same operations defined.

Example 1.1.3. For $m, n \in \mathbb{N}$, we define the set of all $m \times n$ matrices with entries in $\mathbb{F}$ by

$$\mathcal{M}_{mn}(\mathbb{F}) = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} : x_{ij} \in \mathbb{F} \text{ for all } 1 \leq i \leq m,\ 1 \leq j \leq n \right\}$$

We equip $\mathcal{M}_{mn}(\mathbb{F})$ with the usual matrix addition defined by

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} = \begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} & \cdots & x_{1n}+y_{1n} \\ x_{21}+y_{21} & x_{22}+y_{22} & \cdots & x_{2n}+y_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1}+y_{m1} & x_{m2}+y_{m2} & \cdots & x_{mn}+y_{mn} \end{bmatrix}$$

and scalar multiplication defined by

$$\alpha \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} = \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \cdots & \alpha x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \cdots & \alpha x_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha x_{m1} & \alpha x_{m2} & \cdots & \alpha x_{mn} \end{bmatrix}$$

It is elementary to verify that $\mathcal{M}_{mn}(\mathbb{F})$ with the defined operations is a vector space over $\mathbb{F}$. Note that in the case where we are working with $n \times n$ matrices over $\mathbb{F}$, we will simply denote it as $\mathcal{M}_n(\mathbb{F})$.

Example 1.1.4. For $n \in \mathbb{N}$, we define the set of all functions on the interval $[0, 1]$ over $\mathbb{F}$ by

$$\mathcal{F}[0,1] = \{f : [0,1] \to \mathbb{F} : f \text{ is a function}\}$$

We equip $\mathcal{F}[0,1]$ with pointwise addition and scalar multiplication defined by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x) \end{aligned}$$

It is elementary to verify that $\mathcal{F}[0,1]$ with the defined operations is a vector space over $\mathbb{F}$.

Example 1.1.5. For $n \in \mathbb{N}$, we define the set of all polynomials over $\mathbb{F}$ of degree at most n on the interval $[0,1]$ by

$$\mathcal{P}_n[0,1] = \left\{ p : [0,1] \to \mathbb{F} : n \in \mathbb{N}, a_0, a_1, ..., a_n \in \mathbb{F}, p(x) = \sum_{k=0}^{n} a_k x^k \right\}$$

We equip $\mathcal{P}_n[0,1]$ with pointwise addition and scalar multiplication as in Example 1.1.4. It is elementary to verify that $\mathcal{P}_n[0,1]$ with the defined operations is a vector space over $\mathbb{F}$.

Example 1.1.6. The set of all $\mathbb{F}$-valued sequences defined by

$$\mathbb{F}^{\mathbb{N}} = \{(x_n)_{n=1}^{\infty} : \text{for all } n \in \mathbb{N},\, x_n \in F\}$$

We equip $\mathbb{F}^{\mathbb{N}}$ with the usual vector addition and scalar multiplication defined by

$$(x_n)_{n=1}^{\infty} + (y_n)_{n=1}^{\infty} = (x_n + y_n)_{n=1}^{\infty}$$
$$\alpha (x_n)_{n=1}^{\infty} = (\alpha x_n)_{n=1}^{\infty}$$

It is elementary to verify that $\mathbb{F}^{\mathbb{N}}$ with the defined operations is a vector space over $\mathbb{F}$.

We now present some examples of non-vector spaces.

Example 1.1.7. Let $V = \{x \in \mathbb{R} : x \geq 0\}$. We claim that V is not a vector space over $\mathbb{R}$. To see this, note that for any $x > 0$, there is no additive inverse $-x \in \mathbb{R}$ such that $x + (-x) = -x + x = 0$. For example, $2 \in V$, but $-2 \notin V$.

Example 1.1.8. Consider the 2-dimensional Euclidean space $\mathbb{R}^2$, but say we equip it with addition defined by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \boxplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and usual scalar multiplication. We claim that $\mathbb{R}^2$ with this form of addition and scalar multiplication, is not a vector space over $\mathbb{R}$. To see this, we show that $\mathbb{R}^2$ does not satisfy the associativity property. Observe that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \boxplus \left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \boxplus \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \boxplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

On the other hand, we have

$$\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \boxplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) \boxplus \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \boxplus \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + z_1 \\ x_2 + z_2 \end{bmatrix}$$

Clearly, the two resulting vectors are different, so $\mathbb{R}^n$ with such addition and scalar multiplication is not a vector space over $\mathbb{R}$.

1.2 Subspaces of a Vector Space

Subspaces form the foundational building blocks within a vector space. Just as subsets of a set inherit structure from the parent set, subspaces inherit the algebraic structure of a vector space. A subspace is a subset that is itself a vector space under the same operations of addition and scalar multiplication.

Understanding subspaces is central to linear algebra because many important constructs—such as the null space and column space of a matrix, the space of solutions to a homogeneous system, and spaces of polynomials or continuous functions—are all subspaces of some ambient vector space. Much of the power of linear algebra lies in analyzing the structure and relationships between these subspaces.

An essential aspect of studying subspaces involves understanding how they are generated. Given a subset of vectors, we are often interested in the smallest subspace that contains them. This leads naturally to the concept of the *span* of a set of vectors: the set of all linear combinations of those vectors. A spanning set provides a way to describe the entirety of a subspace in terms of simpler components.

In this section, we explore the definition and properties of subspaces, techniques to verify whether a given subset is a subspace, important examples and counterexamples, the concept of the span of a set of vectors, and the relationship between spanning sets and subspaces. These ideas form the theoretical foundation for understanding linear dependence, bases, and dimension in later sections.

Definition 1.2.1. Let V be a vector space over a field $\mathbb{F}$. A subset $W \subseteq V$ is called a *subspace* of V if W is itself a vector space under the same operations of vector addition and scalar multiplication defined on V.

To verify that a nonempty subset $W \subseteq V$ is a subspace, we require the following proposition.

Theorem 1.2.2 (Subspace Test). *Let V be a vector space over $\mathbb{F}$ and let $W \subseteq V$. Then W is a subspace of V if and only if*

1. *(Zero element) $\vec{0}_V \in W$*
2. *(Closed under vector addition) If $\vec{u}, \vec{v} \in W$, then $\vec{v} + \vec{w} \in W$.*
3. *(Closed under scalar multiplication) If $\vec{v} \in W$ and $\alpha \in \mathbb{F}$, then $\alpha\vec{v} \in W$.*

Proof. Follows directly from checking the vector space axioms restricted to the subset W. $\square$

Example 1.2.3. If V is a vector space, then $\{\vec{0}_V\}$ and V are subspaces of V.

Example 1.2.4. Let $V = \mathbb{R}^3$ be a vector space over $\mathbb{R}$, and define

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V : x + y + z = 0 \right\}$$

We claim that W is a subspace of V. To see this, we need to verify the conditions of Theorem 1.2.2.

To see that (1) holds, note that the zero vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ satisfies $x+y+z = 0+0+0 = 0$, so $\vec{0} \in W$.

To see that (2) holds, let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in W. Then we have $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$. Then observe that

$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0$$

Therefore, the vector

$$\vec{u} + \vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

belongs to W.

Finally, to see that (3) holds, let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and let $\alpha \in \mathbb{R}$. Then we have $x + y + z = 0$. Then observe that

$$\alpha x + \alpha y + \alpha z = \alpha(x + y + z) = 0$$

Therefore, the vector

$$\alpha\vec{v} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \end{bmatrix}$$

belongs to W.

Therefore, by Theorem 1.2.2, we have shown that W is a subspace of V.

Example 1.2.5. Let $V = \mathcal{F}[0,1]$ be a vector space over $\mathbb{F}$, and let $W = \{f \in V : f(0) = 0\}$. We claim that W is a subspace of V. To see this, we need to verify the conditions of Theorem 1.2.2.

To see that (1) holds, it is easy to note that the zero function is a member of W.

To see that (2) holds, let $f, g \in W$. Then $f(0) = 0$ and $g(0) = 0$ and so

$$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$$

Therefore, $f + g \in W$.

Finally, to see that (3) holds, let $f \in W$ and $\alpha \in \mathbb{F}$. Then $f(0) = 0$ and so

$$(\alpha f)(0) = \alpha f(0) = \alpha \cdot 0 = 0$$

Therefore, $\alpha f \in W$.

Therefore, by Theorem 1.2.2, we have shown that W is a subspace of V.

Example 1.2.6. Let $V = \mathcal{M}_n(\mathbb{R})$, and for $A \in \mathcal{M}_n(\mathbb{R})$, define

$$Z_n(A) = \{B \in \mathcal{M}_n(\mathbb{R}) : AB = BA\}$$

Then $Z_n(A)$ is a subspace of $\mathcal{M}_n(\mathbb{R})$. To see this, we need to verify the conditions of Theorem 1.2.2.

To see that (1) holds, first note that the zero matrix $O_n \in \mathcal{M}_n(\mathbb{R})$ satisfies

$$AO_n = O_nA = O_n$$

so $O_n \in Z_n(A)$.

To see that (2) holds, let $B, C \in Z_n(\mathbb{R})$. Then we have $AB = BA$ and $AC = CA$. Consequently,

$$A(B+C) = AB + AC = BA + CA = (B+C)A$$

which shows that $B + C \in Z_n(\mathbb{R})$.

Finally, to see that (3) holds, let $B \in Z_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then we have $AB = BA$. Consequently,

$$A(\alpha B) = \alpha AB = \alpha BA = (\alpha B)A$$

which shows that $\alpha B \in Z_n(\mathbb{R})$.

Therefore, by Theorem 1.2.2, we have shown that $Z_n(\mathbb{R})$ is a subspace of $\mathcal{M}_n(\mathbb{R})$.

Example 1.2.7. Let $V = \mathcal{C}[0,1]$ be the space of all continuous functions from the unit interval $[0,1]$ to $\mathbb{F}$. It is elementary to verify that $\mathcal{C}[0,1]$ with pointwise addition and scalar multiplication is a vector space over $\mathbb{F}$. Define

$$\mathcal{D}[0,1] = \{f \in \mathcal{C}[0,1] : f \text{ is differentiable on } [0,1]\}$$

We claim that $\mathcal{D}[0,1]$ is a subspace of $\mathcal{C}[0,1]$. To see this, we need to verify the conditions of Theorem 1.2.2.

To see that (1) holds, note that clearly, the zero function $O(x) = 0$ is differentiable on $[0,1]$, so $O \in \mathcal{D}[0,1]$.

To see that (2) holds, let $f, g \in \mathcal{D}[0,1]$, so f and g are differentiable on $[0,1]$, so f', g' exists. Then from calculus, it follows that $f+g$ is differentiable on $[0,1]$ and

$$(f+g)' = f' + g'$$

so $f+g \in \mathcal{D}[0,1]$.

To see that (3) holds, let $f \in \mathcal{D}[0,1]$ and $\alpha \in \mathbb{F}$, so f is differentiable on $[0,1]$, so f' exists. From calculus, again, it follows that αf is differentiable on $[0,1]$, and

$$(\alpha f)' = \alpha f'$$

so $\alpha f \in \mathcal{D}[0,1]$.

Therefore, by Theorem 1.2.2, we have shown that $\mathcal{D}[0,1]$ is a subspace of $\mathcal{C}[0,1]$.

In linear algebra, one of the central goals is to describe vector spaces using simpler, more manageable building blocks. A key concept in this process is the idea of a *spanning set*.

Definition 1.2.8. Let V be a vector space over $\mathbb{F}$ and let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a collection of vectors in V.

1. A vector $\vec{v}$ is said to be a *linear combination* of the vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ if there are coefficients $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{F}$ such that

$$\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

2. The set of all linear combinations of $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is the *span*, and is denoted by

$$\text{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} = \left\{\sum_{k=1}^{n} \alpha_k\vec{v}_k : \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{F}\right\}$$

3. We say that a nonempty set $S \subseteq V$ is a *spanning set* for V if
$$\text{span}(S) = V$$

Example 1.2.9. Let $V = \mathbb{R}^2$, and consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Then the vector $\vec{v} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$ is a linear combination of $\vec{v}_1$ and $\vec{v}2$ because
$$\vec{v} = 3\vec{v}_1 + 2\vec{v}_2 = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Example 1.2.10. Let $V = \mathcal{P}_2(\mathbb{R})$ and let $p(x) = 1 + x + 4x^2$. We claim that $p(x)$ belongs to
$$\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$$
To see this, let $a, b \in \mathbb{R}$ be such that
$$p(x) = a(1 + 2x - x^2) + b(3 + 5x + 2x^2)$$
Then
$$1 + x + 4x^2 = (a + 3b) + (2a + 5b)x + (-a + 2b)x^2$$
Equating coefficients of powers of x, we have
$$\begin{cases} a + 3b = 1 \\ 2a + 5b = 1 \\ -a + 2b = 4 \end{cases}$$
These equations have the solutions $a = -2$ and $b = 1$, so $p(x)$ is indeed in $\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Example 1.2.11. Let $V = \mathbb{R}^3$ and let
$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$
Then $\text{span}(S) = \mathbb{R}^3$ since every vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ can be written as
$$\vec{v} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In this case, S is a spanning set for $\mathbb{R}^3$. In general, this can be extended to n-dimensional. That is, if

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^n$$

Then $\text{span}(S) = \mathbb{R}^n$.

Example 1.2.12. Let $V = \mathcal{M}_2(\mathbb{R})$ and let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq \mathcal{M}_2(\mathbb{R})$$

Then $\text{span}(S) = \mathcal{M}_2(\mathbb{R})$ since every vector $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathcal{M}_2(\mathbb{R})$ can be written as

$$A = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In this case, S is a spanning set for $\mathcal{M}_2(\mathbb{R})$. In general, this can be extended to $\mathcal{M}_n(\mathbb{R})$. In this case, if

$$S = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, ..., \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \right\} \subseteq \mathcal{M}_n(\mathbb{R})$$

Then $\text{span}(S) = \mathcal{M}_n(\mathbb{R})$.

Example 1.2.13. Let $V = \mathcal{P}_2(\mathbb{R})$ and let

$$S = \{1, x, x^2\} \subseteq \mathcal{P}_2(\mathbb{R})$$

Then $\text{span}(S) = \mathcal{P}_2(\mathbb{R})$ since every vector $p(x) = a + bx + cx^2$ in $\mathcal{P}_2(\mathbb{R})$ can be written as

$$p(x) = a \cdot 1 + b \cdot x + c \cdot x^2$$

In this case, S is a spanning set for $\mathcal{P}_2(\mathbb{R})$. In general, this can be extended to $\mathcal{P}_n(\mathbb{R})$. In this case, if

$$S = \{1, x, x^2, ..., x^n\} \subseteq \mathcal{P}_n(\mathbb{R})$$

Then $\text{span}(S) = \mathcal{P}_n(\mathbb{R})$.

We now present a theorem that says that the span of any set of vectors is a subspace, and in fact, it is the smallest subspace containing such vectors.

Theorem 1.2.14. *Let V be a vector space over $\mathbb{F}$ and let $W = \text{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$. Then*

1. *W is a subspace of V containing each of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$.*
2. *W is the smallest subspace containing these vectors in the sense that any subspace that contains each of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ must contain W.*

The following illustrates how (2) of Theorem 1.2.14 works.

Example 1.2.15. Let $V = \mathcal{P}_3(\mathbb{R})$ and let

$$S = \{x^2 + x^3, x, 2x^2 + 1, 3\}$$

We claim that $\text{span}(S) = \mathcal{P}_3(\mathbb{R})$. To see this, we need to show that $\text{span}(S) \subseteq \mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R}) \subseteq \text{span}(S)$.

For the first, note that any linear combination of vectors in S belong to $\mathcal{P}_3(\mathbb{R})$ because if $a, b, c, d \in \mathbb{R}$ are such that

$$p(x) = a(x^2 + x^3) + bx + c(2x^2 + 1) + 3d$$

Then see that

$$\begin{aligned} p(x) &= ax^2 + ax^3 + bx + 2cx^2 + c + 3d \\ &= ax^3 + (a + c)x^2 + bx + (c + 3d) \end{aligned}$$

and we note from Example 1.2.13, $p(x)$ is a linear combination of $\{1, x, x^2, x^3\}$, so we have shown that $\text{span}(S) \subseteq \mathcal{P}_3(\mathbb{R})$.

For the second, let $p(x) = ax^3 + bx^2 + cx + d \in \mathcal{P}_3(\mathbb{R})$ be arbitrary. We need to show that $p(x)$ can be expressed as a linear combination of vectors in S. So we have for some $e, f, g, h \in \mathbb{R}$,

$$e(x^2 + x^3) + fx + g(2x^2 + 1) + 3h = ax^3 + bx^2 + cx + d$$

Using the above, we have on the left side

$$ex^3 + (e + g)x^2 + fx + (g + 3h) = ax^3 + bx^2 + cx + d$$

So comparing the coefficients, we have $a = e$, $b = e + g$, $c = f$, and $d = g + 3h$. This shows that it is possible to express any arbitrary polynomial $p(x) = ax^3 + bx^2 + cx + d$ as a linear combination of vectors in S, so we have $\mathcal{P}_3(\mathbb{R}) \subseteq \text{span}(S)$.

Therefore, since $\text{span}(S) \subseteq \mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R}) \subseteq \text{span}(S)$, then $\text{span}(S) = \mathcal{P}_3(\mathbb{R})$, as desired.

Example 1.2.16. Let V be a vector space over $\mathbb{F}$, and let $\vec{u}, \vec{v} \in V$. We claim that

$$\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u} + 2\vec{v}, \vec{u} - \vec{v}\}$$

To see this we need to show "$\subseteq$" and "$\supseteq$".

To see the latter, it is obvious that $\text{span}\{\vec{u} + 2\vec{v}, \vec{u} - \vec{v}\} \subseteq \text{span}\{\vec{u}, \vec{v}\}$ by Theorem 1.2.14 since both $\vec{u} + 2\vec{v}$ and $\vec{u}, \vec{v}$ lie in $\text{span}\{\vec{u}, \vec{v}\}$.

To see the former, note that

$$\vec{u} = \frac{1}{3}(\vec{u} + 2\vec{v}) + \frac{2}{3}(\vec{u} - \vec{v})$$

and

$$\vec{v} = \frac{1}{3}(\vec{u} + 2\vec{v}) - \frac{1}{3}(\vec{u} - \vec{v})$$

so $\text{span}\{\vec{u}, \vec{v}\} \subseteq \text{span}\{\vec{u} + 2\vec{v}, \vec{u} - \vec{v}\}$ by Theorem 1.2.14, as desired.

1.3 Linear Independence and Bases

In the previous sections, we studied how vector spaces can be constructed from smaller subsets using linear combinations and spanning sets. However, not all spanning sets are equally efficient or informative. Some may contain redundant vectors—vectors that can be written as combinations of others in the set. To identify the most efficient way to describe a vector space, we turn to the concept of *linear independence*.

Definition 1.3.1. Let V be a vector space over $\mathbb{F}$ and let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \subseteq V$. We say that $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is *linearly independent* if for $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{F}$ are such that

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}_V \implies \alpha_k = 0 \text{ for all } k = 1, 2, ..., n$$

If a set of vectors are not linearly independent, then we say that the set is *linearly dependent.*

Example 1.3.2. Let $V = \mathbb{R}^2$, and let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We claim that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. To see this, assume that for $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$$

Then

$$\begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Comparing entries, we see that $\alpha_1 = 0$ and $\alpha_2 = 0$. In general, the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^n$$

is linearly independent.

Example 1.3.3. Let $V = \mathcal{M}_2(\mathbb{R})$ and let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq \mathcal{M}_2(\mathbb{R})$$

We claim that S is a linearly independent subset of $\mathcal{M}_2(\mathbb{R})$. To see this, assume that for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, we have that

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So comparing entries, we see that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, and $\alpha_4 = 0$. In general, the set

$$S = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \right\} \subseteq \mathcal{M}_n(\mathbb{R})$$

is linearly independent.

Example 1.3.4. Let $V = \mathcal{P}_2(\mathbb{R})$ and let

$$S = \{1, x, x^2\} \subseteq \mathcal{P}_2(\mathbb{R})$$

We claim that S is a linearly independent subset of $\mathcal{P}_2(\mathbb{R})$. To see this, assume that for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we have

$$\alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3 \cdot x^2 = 0$$

Then observe that

$$\alpha_1 + \alpha_2 x + \alpha_3 x^2 = 0 + 0x + 0x^2$$

So comparing coefficients, we see that $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$. In general, the set

$$S = \{1, x, x^2, ..., x^n\} \subseteq \mathcal{P}_n(\mathbb{R})$$

is linearly independent.

Example 1.3.5. Let $V = \mathcal{P}_2(\mathbb{R})$, and let $S = \{1 + x, 3x + x^2, 2 + x - x^2\} \subseteq \mathcal{P}_2(\mathbb{R})$. We claim that S is a linearly independent subset of $\mathcal{P}_2(\mathbb{R})$. To see this, assume that for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we have

$$\alpha_1(1 + x) + \alpha_2(3x + x^2) + \alpha_3(2 + x - x^2) = 0$$

Then rearranging the equation so that

$$(\alpha_2 - \alpha_3)x^2 + (\alpha_1 + 3\alpha_2 + \alpha_3)x + (\alpha_1 + 2\alpha_3) = 0x^2 + 0x + 0$$

Now comparing the coefficients, we have the system of equations given by

$$\begin{cases} \alpha_2 - \alpha_3 = 0 \\ \alpha_1 + 3\alpha_2 + \alpha_3 = 0 \\ \alpha_1 + 2\alpha_3 = 0 \end{cases}$$

One can check that the unique solution to this system is when $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$. So the set S is linearly independent.

Example 1.3.6. Let $V = \mathcal{C}[0, 2\pi]$, and let $S = \{\sin(x), \cos(x)\} \subseteq \mathcal{C}[0, 2\pi]$. We claim that S is linearly independent. To see this, assume that for $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\alpha_1 \sin(x) + \alpha_2 \cos(x) = 0$$

This must hold for all values of $x \in [0, 2\pi]$. Taking $x = 0$ yields $\alpha_2 = 0$ and taking $x = \frac{\pi}{2}$ yields $\alpha_1 = 0$.

Example 1.3.7. Let $V = \mathbb{R}^2$, and let $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$. We claim that S is linearly dependent. To see this, assume that for $\alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 + 4\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So comparing entries, we see that $\alpha_1 + 2\alpha_2 = 0$ and $2\alpha_1 + 4\alpha_2 = 0$, but then $\alpha_1 = -2\alpha_2$ and not always zero, so S has to be linearly dependent.

Example 1.3.8. We claim that the set of polynomials of distinct degrees is independent. To see this, let $p_1, p_2, ..., p_n$ be polynomials such that $\deg(p_k) = d_k$ for each k. By relabelling if necessary, we may assume that $d_1 > d_2 > \cdots > d_n$. Suppose that for $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$,

$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n = 0$$

Since $\deg(p_1) = d_1$, let ax^{d_1} be the term in p_1 of highest degree with $a \neq 0$. Since $d_1 > d_2 > \cdots > d_n$, it follows that $\alpha_1 a x^{d_1}$ is the only term of degree d_1 in the linear combination $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n = 0$. This implies that $\alpha_1 a x^{d_1} = 0$, so $\alpha_1 = 0$. But then $\alpha_2 p_2 + \alpha_3 p_3 + \cdots + \alpha_n p_n = 0$, so we can repeat the argument to show that $\alpha_2 = 0$, and so on.

A set of vectors is linearly independent if $\vec{0}$ is a linear combination in a unique way. The following proposition shows that every linear combination of these vectors has uniquely determined coefficients.

Proposition 1.3.9. *Let V be a vector space over $\mathbb{F}$ and let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \subseteq V$ be linearly independent. If $\vec{v}$ has two representations*

$$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$$
$$\vec{v} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n$$

as a linear combination of these vectors, then for all $k = 1, 2, ..., n$, we have $\alpha_k = \beta_k$.

Proof. Note that by subtracting both equations we yield

$$\vec{0} = (\alpha_1 - \beta_1)\vec{v}_1 + (\alpha_2 - \beta_2)\vec{v}_2 + \cdots + (\alpha_n - \beta_n)\vec{v}_n$$

Then by linear independence, we have $\alpha_k - \beta_k = 0$ for each k, so $\alpha_k = \beta_k$ for each k, as desired. $\square$

The following theorem is one of the most useful results in linear algebra.

Theorem 1.3.10. *Let V be a a vector space over $\mathbb{F}$. Assume that V can be spanned by n vectors. If any set of m vectors in V is linearly independent, then $m \leq n$.*

Proof. Since V can be spanned by n vectors, let $V = \text{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$, and assume that $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_m\}$ is a linearly independent set in V. Then we can write

$$\vec{u}_1 = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$$

where each $\alpha_k \in \mathbb{R}$. As $\vec{u}_1 \neq \vec{0}_V$, not all of the α_k are zero. Without loss of generality, say that $\alpha_1 \neq 0$, after relabelling the $\vec{v}_k$. Then $V = \text{span}\{\vec{u}_1, \vec{v}_2, ..., \vec{v}_n\}$. Similarly, write

$$\vec{u}_2 = \beta_1 \vec{v}_1 + \gamma_2 \vec{v}_2 + \cdots + \gamma_n \vec{v}_n$$

Then some $\gamma_k \neq 0$ since $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent, so $V = \text{span}\{\vec{u}_1, \vec{u}_2, ..., \vec{v}_n\}$. Proceeding inductively, if $m > n$, this procedure continues until all the vectors $\vec{v}_k$ are replaced by the vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$. In particular, $V = \text{span}\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_n\}$. But then $\vec{u}_{n+1}$ is a linear combination of $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$, contrary to the linear independence of the $\vec{u}_k$. Hence, we cannot have $m > n$, so $m \leq n$, as desired. □

We now introduce the concept of a basis of a vector space.

Definition 1.3.11. Let V be a vector space over $\mathbb{F}$. A subset $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\} \subseteq V$ is called a *basis* for V if

1. $\mathcal{B}$ is linearly independent.
2. $V = \text{span}(\mathcal{B})$.

In other words, $\mathcal{B}$ is a basis of V if $\mathcal{B}$ is a linearly independent spanning set.

Example 1.3.12. Let $V = \mathbb{F}^n$. The set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^n$$

is a basis for $\mathbb{F}^n$. From Example 1.2.11, we have shown that $\mathcal{B}$ is a spanning set for $\mathbb{R}^n$, and in Example 1.3.2, we have shown that $\mathcal{B}$ is linearly independent. We call $\mathcal{B}$ the *standard basis* of $\mathbb{F}^n$.

Example 1.3.13. Let $V = \mathcal{M}_n(\mathbb{F})$. The set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, ..., \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \right\} \subseteq \mathcal{M}_n(\mathbb{R})$$

is a basis for $\mathcal{M}_n(\mathbb{F})$. From Example 1.2.12, we have shown that $\mathcal{B}$ is a spanning set for $\mathcal{M}_n(\mathbb{F})$, and in Example 1.3.3, we have shown that $\mathcal{B}$ is linearly independent. We call $\mathcal{B}$ the *standard basis* of $\mathcal{M}_n(\mathbb{F})$.

Example 1.3.14. Let $V = \mathcal{P}_n(\mathbb{F})$. The set

$$\mathcal{B} = \{1, x, x^2, ..., x^n\} \subseteq \mathcal{P}_n(\mathbb{R})$$

is a basis for $\mathcal{P}_n(\mathbb{F})$. From Example 1.2.13, we have shown that $\mathcal{B}$ is a spanning set for $\mathcal{P}_n(\mathbb{F})$, and in Example 1.3.4, we have shown that $\mathcal{B}$ is linearly independent. We call $\mathcal{B}$ the *standard basis* of $\mathcal{P}_n(\mathbb{F})$.

Thus, based on the examples above, if $\mathcal{B}$ is a basis for V, then every vector in V can be written as a linear combination of these vectors in a unique way. But even more, any two bases of V must contain the same number of vectors.

Proposition 1.3.15. *Let V be a vector space over $\mathbb{F}$, and suppose that $\mathcal{B}_1$ is a basis for V that has n elements and $\mathcal{B}_2$ is a basis for V that has m elements. Then $n = m$.*

Proof. Since $\mathcal{B}_1$ is a basis for V that contains n elements, and $\mathcal{B}_2$ is a basis for V that contains m elements, it follows from Theorem 1.3.10 that $m \leq n$. Similarly, $n \leq m$, so $n = m$, as asserted. $\square$

Proposition 1.3.15 guarantees that no matter which basis V is chosen it contains the same number of vectors as any other basis. Hence, there is no ambiguity about the following definition.

Definition 1.3.16. Let V be a vector space over $\mathbb{F}$ and let $\mathcal{B}$ be a basis of V. The number of vectors in the basis $\mathcal{B}$ is called the *dimension* of V, and is denoted by

$$\dim(V) = |\mathcal{B}|$$

If $\mathcal{B}$ contains a finite number of elements, then we say that V is a *finite dimensional vector space*. Otherwise, if $\mathcal{B}$ contains an infinite number of elements, then we say that V is an *infinite dimensional vector space.*

Knowing more about infinite dimensional vector spaces would be nice, but it is not the focus of the course, but we mention it for our own understanding that if we have a vector space over $\mathbb{F}$, it is possible that we could have an infinite dimensional vector space. For example, the sequence space $\mathbb{F}^{\mathbb{N}}$ is an example of an infinite dimensional vector space.

Example 1.3.17. If $V = \mathbb{R}^n$, then the set $\mathcal{B}$ in Example 1.3.12 contains n elements, so $\dim(V) = n$.

Example 1.3.18. If $V = \mathcal{M}_n(\mathbb{R})$, then the set $\mathcal{B}$ in Example 1.3.13 contains n^2 elements, so $\dim(V) = n^2$.

Example 1.3.19. If $V = \mathcal{P}_n(\mathbb{R})$, then the set $\mathcal{B}$ in Example 1.3.14 contains $n + 1$ elements, so $\dim(V) = n + 1$.

The question that one may ask is, how do we construct a basis for a vector space? Proposition 1.3.15 tells us that we can have more than one basis for a vector space, on the condition that we make sure that this new basis contains the same number of elements. The following example illustrates how we can construct a basis for a given vector space.

Example 1.3.20. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and consider the subspace of $\mathcal{M}_2(\mathbb{R})$ given by

$$U(A) = \{B \in \mathcal{M}_2(\mathbb{R}) : AB = BA\} \subseteq \mathcal{M}_2(\mathbb{R})$$

We first need to find a basis for $\mathcal{M}_2(\mathbb{R})$. Start with an arbitrary 2×2 matrix in $U(A)$, so say $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(A)$. Then by the given condition that $AB = BA$, we have that

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

Thus,

$$\begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

Now, this would imply that $c = 0$, $a = b + d$, so

$$\begin{aligned} B &= \begin{bmatrix} b+d & b \\ 0 & d \end{bmatrix} \\ &= \begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \\ &= b \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Therefore, as $B \in U(A)$ was arbitrary and B is a linear combination of matrices $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we claim that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $U(A)$. We have already showed that $\mathcal{B}$ is a spanning set for $U(A)$, so it suffices to show that $\mathcal{B}$ is linearly independent.

Assume that for $\alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$\alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} \alpha_1 + \alpha_2 & \alpha_1 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing entries, we easily see that $\alpha_1 = 0$ and $\alpha_2 = 0$, so $\mathcal{B}$ is indeed linearly independent. Therefore, $\mathcal{B}$ is indeed a basis of $U(A)$. Since $\mathcal{B}$ contains two elements, then $\dim(U(A)) = 2$.

Example 1.3.21. Let $V = \mathcal{M}_2(\mathbb{R})$, and let

$$W = \{A \in \mathcal{M}_2(\mathbb{R}) : A^T = A\}$$

i.e. W is the set of all 2×2 matrices such that A is symmetric. We want to find a basis for W and find its dimension, so let $A \in W$ be arbitrary, so $A^T = A$. Such matrices A takes the form

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$. Then observe that

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, A is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, define

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

We claim that $\mathcal{B}$ is a basis for W. Indeed, since any matrix in W is a linear combination of matrices in $\mathcal{B}$, $\mathcal{B}$ is a spanning set for W. To see that $\mathcal{B}$ is linearly independent, assume that for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we have

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So then,

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and we easily see that by comparing the entries, $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$, so $\mathcal{B}$ is indeed linearly independent. Since $\mathcal{B}$ has three elements, $\dim(W) = 3$.

Up until this point, we only had a look at examples involving vector spaces over $\mathbb{R}$. Now let us have a look at some examples that involve vector spaces over $\mathbb{C}$.

Example 1.3.22. Let $V = \mathbb{C}^3$, and define a subspace

$$W = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3 : z_1 + iz_2 = 0 \right\}$$

We want to find a basis for W and the dimension of W, so let $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ be arbitrary, so $z_1 + iz_2 = 0$, and thus, $z_1 = -iz_2$. Then observe that

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -iz_2 \\ z_2 \\ z_3 \end{bmatrix} = z_2 \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} + z_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus, define

$$\mathcal{B} = \left\{ \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We claim that $\mathcal{B}$ is a basis for W. Indeed, since any vector in W is a linear combination of the vectors in $\mathcal{B}$, $\mathcal{B}$ is a spanning set for W. To see that $\mathcal{B}$ is linearly independent, assume that for $\alpha_1, \alpha_2 \in \mathbb{C}$, we have

$$\alpha_1 \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} -i\alpha_1 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, comparing entries, we easily see that $\alpha_1 = 0$ and $\alpha_2 = 0$, and so $\mathcal{B}$ is linearly independent, as claimed. Since $\mathcal{B}$ has two elements, $\dim(W) = 2$.

Example 1.3.23. Let $V = \mathcal{P}_3(\mathbb{C})$ and consider the subspace

$$U = \{p(x) \in \mathcal{P}_3(\mathbb{C}) : p(1+i) = 0 \text{ and } p(1-i) = 0\}$$

We want to find a basis for U, and the dimension of U. Let $p(x) \in U$ be arbitrary, so $p(1+i) = 0$ and $p(1-i) = 0$. Then $1+i$ and $1-i$ are factors of the polynomial $p(x)$, so there is some polynomial $q(x)$ such that

$$p(x) = (x - (1+i))(x - (1-i))q(x)$$

where $\deg(q(x)) \leq 1$ as $\deg(p(x)) \leq 3$. In this case, we can write $q(x) = ax+b$ for some $a, b \in \mathbb{C}$, and so

$$\begin{aligned} p(x) &= (x^2 - 2x + 2)(ax + b) \\ &= ax^3 - 2ax^2 + 2ax + bx^2 - 2bx + 2b \\ &= a(x^3 - 2x^2 + 2x) + b(x^2 - 2x + 2) \end{aligned}$$

Therefore, $p(x)$ is a linear combination of the polynomials $x^3 - 2x^2 + 2x$ and $x^2 - 2x + 2$. Thus, define

$$\mathcal{B} = \{x^3 - 2x^2 + 2x, x^2 - 2x + 2\}$$

We claim that $\mathcal{B}$ is a basis for U. Indeed, since any polynomial in U is a linear combination of polynomials in $\mathcal{B}$, then $\mathcal{B}$ is indeed a spanning set for U. To see that $\mathcal{B}$ is linearly independent, assume that for $\alpha_1, \alpha_2 \in \mathbb{C}$, we have

$$\alpha_1(x^3 - 2x^2 + 2x) + \alpha_2(x^2 - 2x + 2) = 0$$

Then rearranging

$$\alpha_1 x^3 + (-2\alpha_1 + \alpha_2)x^2 + (2\alpha_1 - 2\alpha_2)x + 2\alpha_2 = 0$$

So comparing coefficients, we have

$$\begin{cases} \alpha_1 = 0 \\ -2\alpha_1 + \alpha_2 = 0 \\ 2\alpha_1 - 2\alpha_2 = 0 \\ 2\alpha_2 = 0 \end{cases}$$

Therefore, we easily see that then $\alpha_1 = 0$ and $\alpha_2 = 0$, so $\mathcal{B}$ is indeed linearly independent. Since $\mathcal{B}$ has two elements, $\dim(U) = 2$.

1.4 Finite Dimensional Bases

We have introduced the definition of finite dimensional and infinite dimensional vector spaces in the previous section, but up to this point, we had no guarantee that an arbitrary vector space has a basis, and hence, no guarantee that one can speak at all of the dimension of V. The following theorem shows that any space that is spanned by a finite set of vectors has a finite basis.

Lemma 1.4.1. *Let V be a vector space over $\mathbb{F}$ and let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be an independent set of vectors in V. If $\vec{u} \in V \setminus \operatorname{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$, then $\{\vec{u}, \vec{v}_1, ..., \vec{v}_n\}$ is linearly independent.*

Proof. Assume that for $\alpha, \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{F}$, we have

$$\alpha\vec{u} + \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = \vec{0}_V$$

First, note that $\alpha = 0$ since otherwise,

$$\vec{u} = -\frac{\alpha_1}{\alpha}\vec{v}_1 - \frac{\alpha_2}{\alpha}\vec{v}_2 - \cdots - \frac{\alpha_n}{\alpha}\vec{v}_n \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$$

which is a contradiction. Therefore, we then have $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}_V$, and by assumption, $\alpha_k = 0$ for all $k = 1, 2, ..., n$. □

Remark 1.4.2. Note that the converse of Lemma 1.4.1 is also true, so if $\{\vec{u}, \vec{v}_1, ..., \vec{v}_n\}$ is linearly independent, then $\vec{u} \notin \operatorname{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$.

The question that we may ask ourselves is: if we have a linearly independent set that contains fewer elements than the dimension of a vector space, can we keep adding vectors to this set so that it eventually becomes a basis of the space?

The answer is yes! In finite-dimensional vector spaces, every linearly independent set can be extended to a basis. This powerful and foundational result guarantees that any ”partial basis” (a linearly independent set) can be completed into a full basis.

To do so, we require the following lemma.

Lemma 1.4.3. *Let V be a finite-dimensional vector space over $\mathbb{F}$. If W is a subspace of V, then any independent subset of W can be enlarged to a finite basis of W.*

Proof. Suppose that A is a linearly independent subset of W. If $\text{span}(A) = W$, then A is already a basis of W. Otherwise, if $\text{span}(A) \neq W$, then choose $\vec{w}_1 \in W$ such that $\vec{w}_1 \notin \text{span}(A)$. Then the set $A \cup \{\vec{w}_1\}$ is linearly independent by Lemma 1.4.1.

Now if $\text{span}(A \cup \{\vec{w}_1\}) = W$, then we are done. Otherwise, if $\text{span}(A \cup \{\vec{w}_1\}) \neq W$, then choose $\vec{w}_2 \in W$ such that $\vec{w}_2 \notin \text{span}(A \cup \{\vec{w}_1\})$. Then the set $A \cup \{\vec{w}_1, \vec{w}_2\}$ is linearly independent by Lemma 1.4.1.

By proceeding inductively, we claim that a basis of W will be reached eventually. To see this, if no basis of W is ever reached, then the process would create arbitrary large independent sets in V. This is not possible by Theorem 1.3.10 since V is finite-dimensional and so is spanned by a finite set of vectors. □

Theorem 1.4.4. *Let V be a finite-dimensional vector space over $\mathbb{F}$ spanned by $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$. The following hold.*

1. *V has a finite basis and* $\dim(V) \leq n$.
2. *Every independent set of vectors in V can be enlarged to a basis of V by adding vectors from any fixed basis of V.*
3. *If W is a subspace of V, then*
   - *(a) W is a finite dimensional subspace and* $\dim(W) \leq \dim(V)$.
   - *(b) If* $\dim(W) = \dim(V)$, *then* $W = V$.

Proof. To see that (1) holds, if $V = \{\vec{0}_V\}$, then V has an empty basis, so $\dim(V) = 0 \leq n$. Otherwise, if $\vec{v} \neq \vec{0}_V$, then $\{\vec{v}\}$ is linearly independent, so (1) follows from Lemma 1.4.3 with $W = V$.

To see that (2) holds, let $\mathcal{B}$ be a basis of V and let A be a linearly independent subset of V. If $\text{span}(A) = V$, then A is a basis for V. Otherwise, $\mathcal{B}$ is not contained in A since $\mathcal{B}$ spans V, so choose $\vec{w}_1 \in \mathcal{B} \setminus \text{span}(A)$ so that

$A \cup \{\vec{w}_1\}$ is linearly independent by Lemma 1.4.1. If $\operatorname{span}(A \cup \{\vec{w}_1\}) = V$, then we are done. Otherwise, a similar argument shows that $A \cup \{\vec{w}_1, \vec{w}_2\}$ is linearly independent for some $\vec{w}_2 \in \mathcal{B}$. Continuing this process, as in the proof of Lemma 1.4.3, a basis of V will be reached eventually.

To see that (3) part (a) holds, note that if $W = \{\vec{0}_V\}$, then this is easy. Otherwise, let $\vec{w} \neq \vec{0}_V \in W$. Then $\{\vec{w}\}$ can be enlarged to a finite basis $\mathcal{B}$ of W by Lemma 1.4.3, so W is finite-dimensional. But $\mathcal{B}$ is also linearly independent, so $\dim(W) \leq \dim(V)$ by Theorem 1.3.10. To see that part (b) holds, if $W = \{\vec{0}_V\}$, then this is trivial since V has a basis. Otherwise, it follows from (2). $\square$

Example 1.4.5. Let $V = \mathcal{M}_2(\mathbb{R})$ and consider the set

$$A = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \right\}$$

Right now, this is not a basis for $\mathcal{M}_2(\mathbb{R})$ just yet since $\dim(\mathcal{M}_2(\mathbb{R})) = 4$, we are missing one element. Recall that the standard basis for $\mathcal{M}_2(\mathbb{R})$ is given by

$$\mathcal{SB}_{\mathcal{M}_2(\mathbb{R})} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

So including one of these in A will produce a basis by Theorem 1.4.4. In fact, *any* one of these matrices in A produces a linearly independent set, and hence, a basis, as we will mention later below.

Example 1.4.6. Let $V = \mathcal{M}_2(\mathbb{C})$ and consider the set

$$A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \right\}$$

Right now, this is not a basis for $\mathcal{M}_2(\mathbb{C})$ just yet, since $\dim(\mathcal{M}_2(\mathbb{C})) = 4$, we are missing two elements. To make it into a basis, we can simply add two standard basis elements to A to make it into a basis, for example,

$$\mathcal{B} = A \cup \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\mathcal{B}$ is still a linearly independent set, and moreover, it is now a basis for $\mathcal{M}_2(\mathbb{C})$.

Example 1.4.7. Let $V = \mathcal{P}_3(\mathbb{R})$ and consider the linearly independent set $A = \{1+x, 1+x^2\}$. Suppose we want to find a basis containing A. Recall that the standard basis for $\mathcal{P}_3(\mathbb{R})$ is given by

$$\mathcal{SB}_{\mathcal{P}_3(\mathbb{R})} = \{1, x, x^2, x^3\}$$

so including two of these vectors will do. If we use 1 and x^3, then we have $\mathcal{B} = \{1, 1+x, 1+x^2, x^3\}$, and this is independent because the polynomials have distinct degrees, and so $\mathcal{B}$ is a basis by Theorem 1.4.4. However, note that if we add $\{1, x\}$ or $\{1, x^2\}$ instead, this would not work!

Example 1.4.8. Let $\mathcal{P}(\mathbb{F}) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathbb{F})$ be the set of all polynomials. It is elementary to verify that $\mathcal{P}(\mathbb{F})$ is a vector space over $\mathbb{F}$. We claim that $\mathcal{P}(\mathbb{F})$ is infinite-dimensional. To see this, note that for each $n \in \mathbb{N}$, $\mathcal{P}(\mathbb{F})$ has a subspace $\mathcal{P}_n(\mathbb{F})$ of dimension $n+1$.

Suppose for a contradiction that $\mathcal{P}(\mathbb{F})$ is finite dimensional, say $\dim(\mathcal{P}(\mathbb{F})) = m$. Then $\dim(\mathcal{P}_n(\mathbb{F})) \leq \dim(\mathcal{P}(\mathbb{F}))$ by Theorem 1.4.4, so $n+1 \leq m$. This is absurd since n is arbitrary, so $\mathcal{P}(\mathbb{F})$ must be infinite dimensional.

Proposition 1.4.9. *Let V be a finite-dimensional vector space over $\mathbb{F}$, and let U and W be subspaces of V.*

1. *If $U \subseteq W$, then $\dim(U) \leq \dim(W)$.*
2. *If $U \subseteq W$ and $\dim(U) = \dim(W)$, then $U = W$.*

Proof. Since W is finite-dimensional, (1) follows by taking $V = W$ in part (3) of Theorem 1.4.4. Now, assume that $\dim(U) = \dim(W) = n$, and let $\mathcal{B}$ be a basis for U. Then $\mathcal{B}$ is a linearly independent set in W. If $U \neq W$, then $\operatorname{span}(\mathcal{B}) \neq W$, so $\mathcal{B}$ can be extended to a linearly independent set of $n+1$ vectors in W, by Lemma 1.4.1. This contradicts Theorem 1.3.10, since W is spanned by $\dim(W) = n$ vectors. Hence, $U = W$, proving (2). □

Example 1.4.10. Let $V = \mathcal{P}_n(\mathbb{R})$, and let

$$W = \{p(x) \in \mathcal{P}_n(\mathbb{R}) : p(1) = 0\}$$

We claim that $\mathcal{B} = \{(x-1), (x-1)^2, ..., (x-1)^n\}$ is a basis for W. To see this, note that $(x-1), (x-1)^2, ..., (x-1)^n$ are all members of W, and that they are linearly independent because they have distinct degrees. Observe then that $\operatorname{span}(\mathcal{B}) \subseteq W \subseteq \mathcal{P}_n(\mathbb{R})$ and $\dim(\operatorname{span}(\mathcal{B})) = n$, and $\dim(\mathcal{P}_n(\mathbb{R})) = n+1$, so $n \leq \dim(W) \leq n+1$ by Proposition 1.4.9. Since $\dim(W)$ is an integer, we must have $\dim(W) = n$ or $\dim(W) = n+1$, but then $W = \operatorname{span}(\mathcal{B})$ or $W = \mathcal{P}_n(\mathbb{R})$ by Proposition 1.4.9. Since $W \neq \mathcal{P}_n(\mathbb{R})$, it follows that $W = \operatorname{span}(\mathcal{B})$, as required.

Lemma 1.4.11. *Let V be a vector space over $\mathbb{F}$ and let $A = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \subseteq V$. Then A is linearly dependent if and only if there exists some vector in A that is a linear combination of the others.*

Proof. Suppose $\vec{v}_2$ is a linear combination of $\vec{v}_1, \vec{v}_3, ..., \vec{v}_n$, so for some $\alpha_1, \alpha_3, ..., \alpha_n \in \mathbb{F}$,

$$\vec{v}_2 = \alpha_1\vec{v}_1 + \alpha_3\vec{v}_3 + \cdots + \alpha_n\vec{v}_N$$

Then

$$\alpha_1\vec{v}_1 + (-1)\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}_V$$

is a nontrivial linear combination that vanishes, so A is linearly dependent.

Conversely, if A is linearly dependent, let

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}_V$$

where some coefficients is nonzero. If, say $\alpha_2 \neq 0$, then

$$\vec{v}_2 = -\frac{\alpha_1}{\alpha_2}\vec{v}_1 - \frac{\alpha_3}{\alpha_2}\vec{v}_3 - \cdots - \frac{\alpha_n}{\alpha_2}\vec{v}_n$$

is a linear combination of the others. □

Lemma 1.4.1 gives us a way to enlarge linearly independent sets to a basis, Theorem 1.4.12 shows that spanning sets can be cut down to a basis.

Theorem 1.4.12. *Let V be a finite-dimensional vector space over $\mathbb{F}$. Any spanning set for V can be cut down to a basis of V.*

Proof. Since V is finite-dimensional, it has a finite spanning set S. Among all spanning sets contained in S, choose S_0 containing the smallest number of vectors. It suffices to show that S_0 is linearly independent.

Assume for a contradiction that S_0 is linearly dependent. Then by Lemma 1.4.11, there exists a vector $\vec{u} \in S_0$ that is a linear combination of the set $S_1 = S_0 \setminus \{\vec{u}\}$ of vectors in S_0 other than $\vec{u}$. It follows that $\text{span}(S_0) = \text{span}(S_1)$, so $V = \text{span}(S_1)$, but S_1 has fewer elements than S_0, so this contradicts the choice of S_0, so S_0 is linearly independent, as claimed. □

Example 1.4.13. Let $V = \mathcal{P}_3(\mathbb{R})$, and consider

$$S = \{1, x + x^2, 2x - 3x^2, 1 + 3x - 2x^2, x^3\}$$

Right now, we have 5 elements in this set, and we know that $\dim(\mathcal{P}_3(\mathbb{R})) = 4$, so we need to eliminate an element from S. However, we note that we cannot remove x^3, since the span of the rest of S is contained in $\mathcal{P}_2(\mathbb{R})$, but by eliminating $1 + 3x - 2x^2$, we can get a basis, since $1 + 3x - 2x^2$ is the sum of the first three polynomials in S.

Theorem 1.4.4 and 1.4.12 have the following consequence.

Corollary 1.4.14. *Let V be a finite-dimensional vector space over $\mathbb{F}$ with $\dim(V) = n$ and let S be a set of exactly n vectors in V. Then S is linearly independent if and only if S spans V.*

Proof. Assume that S is linearly independent. Then by Theorem 1.4.4, S is contained in a basis $\mathcal{B}$ of V. Hence, $|S| = n = |\mathcal{B}|$ so, as $S \subseteq \mathcal{B}$, it follows that $S = B$, so S spans V.

Conversely, if S spans V, then S contains a basis $\mathcal{B}$ by Theorem 1.4.12. Thus, as $|S| = n = |\mathcal{B}|$, then $S \supseteq \mathcal{B}$, so $S = \mathcal{B}$. Therefore, S is linearly independent. □

Example 1.4.15. Let $V = \mathcal{S}_2(\mathbb{R})$ denote the set of all 2×2 symmetric matrices with real entries and let $GL_2(\mathbb{R})$ denote the subspace of $\mathcal{S}_2(\mathbb{R})$ consisting of invertible matrices with real entries. Suppose we want to find a basis for $GL_2(\mathbb{R})$. From Example 1.3.21, we showed that $\dim(V) = 3$, so what is needed is a set of three invertible matrices that (using Corollary 1.4.14) is either independent or spans V. The set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

is indeed a linearly independent set, and spans $GL_2(\mathbb{R})$. We call $GL_n(\mathbb{R})$ the space of all general linear matrices with real entries, or all $n \times n$ matrices that are invertible.

Let V be a vector space over $\mathbb{F}$ and let U and W be subspaces of V. There are two subspaces that are of interest, their *sum* $U + W$, and their *intersection* $U \cap W$, defined by

$$U + W = \{\vec{u} + \vec{w} : \vec{u} \in U, \vec{w} \in W\}$$

and

$$U \cap W = \{\vec{v} \in V : \vec{v} \in U \text{ and } \vec{v} \in W\}$$

It is routine to verify that these are indeed subspaces of V, that $U \cap W$ is contained in both U and W, and that $U + W$ contains both U and W.

Theorem 1.4.16. *Let V be a vector space over $\mathbb{F}$, and let U and W be finite dimensional subspaces of V. Then $U+W$ is finite dimensional and*

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Proof. Exercise. □

Remark 1.4.17. Although we can have intersections of vector subspaces, it may not be the case that we have unions of vector subspaces. For example, suppose we have $V = \mathbb{R}^2$, and we let

$$U = \{(x,0) : x \in \mathbb{R}\} \quad \text{and} \quad W = \{(0,y) : y \in \mathbb{R}\}$$

i.e. U is the set of all points on the x-axis, and W is the set of all points on the y-axis. Their union $U \cup W$ is the set of all points on the x-axis or all points on the y-axis.

To see that $U \cup W$ is not a subspace of $\mathbb{R}^2$, we claim that it is not closed under vector addition. Consider the points $(3,0),(0,4) \in U \cup W$. But then the point

$$(3,4) = (3,0) + (0,4) \notin U \cup W$$

Therefore, as $(3,4) \notin U \cup W$, $U \cup W$ is not closed under vector addition, as claimed.

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