# An Approximate Exponentiated Weibull Joint Envelope-Phase Distribution

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# Outline

# Introduction

- Free-space optical (FSO) systems
  - Short range, outdoor and indoor wireless
  - Large bandwidth, high average capacity
- Major challenge: atmospheric turbulence
  - Scintillation
  - Beam Wander
  - Beam Spreading
- Improvement in performance
  - Enlarge the aperture in the receiver
- Statistical characterization of the laser beam
  - Log-normal, Gamma-Gamma
  - Exponentiated Weibull

# Exponentiated Weibull Statistics

- Barrios proposed the following model for the irradiance at the receiver:

$$I^p \triangleq \sum_{n=1}^{m} w_n I_n^p$$

which was then approximated as

$$I \approx \lim_{p\to\infty}\left(\sum_{n=1}^{m} w_n I_n^p\right)^{1/p} = \max\{I_1, I_2, \ldots, I_m\}$$

- $I_n \sim$ Weibull ($\alpha$) iid $\Rightarrow I \sim$ Exp. Weibull($\alpha, m$)
- Suitable to fit weak, moderate, and strong atmospheric turbulence
- However, no phase information was taken into account

# Phase Distribution

- To asses the coherent detection of optical signals subject to atmospheric turbulence
- To compute bit error rate for communications systems using phase modulation schemes
- To determine optimum receiver aperture diameter
- Belmonte assumed Gaussian random phase fluctuations[1]
- Perlot assumed Beta distributed phase and also that phase and amplitude were independent[2]

[1] A. Belmonte and J. M. Kahn, "Performance of synchronous optical receivers using atmospheric compensation techniques", Opt. Express, vol. 16, no. 18, pp. 14 151-14 162, Sep. 2008

[2] N. Perlot, "Turbulence-induced fading probability in coherent optical communication through the atmosphere," Appl. Opt. 46, 7218-7226 (2007)

# Joint Envelope-Phase Density Function

*Proposition*: Let $R \in \mathbb{R}^{+}$ and $\Theta \in \left[0, \frac{4\pi}{\alpha}\right)$, $\alpha > 0$, be random variables (rvs) representing, respectively, the envelope and the phase of the exponentiated Weibull signal with shape parameters $\alpha > 0$ and $\mu > 0$, and scale parameter $\Omega > 0$. Hence, the joint probability density function (pdf) of the random vector $(R, \Theta)$, denoted as $p_{R,\Theta}(r,\theta)$, is given by

$$
\begin{aligned}
p_{R,\Theta}(r,\theta) = & \frac{(\alpha\mu)^2}{4\pi\Omega}\left(\frac{r}{\Omega}\right)^{\alpha-1} \exp\left(-\left(\frac{r}{\Omega}\right)^{\alpha}\right) \times \\
& \left[\operatorname{erf}\left(\left(\frac{r}{\Omega}\right)^{\alpha/2}\left|\cos\left(\frac{\alpha\theta}{2}\right)\right|\right) \times \right. \\
& \left.\operatorname{erf}\left(\left(\frac{r}{\Omega}\right)^{\alpha/2}\left|\sin\left(\frac{\alpha\theta}{2}\right)\right|\right)\right]^{\mu-1}.
\end{aligned}
$$

# Joint Envelope-Phase Density Function

*Proof:*

- $R \in \mathbb{R}^+$, $\Theta \in \left[0, \frac{4\pi}{\alpha}\right)$, $\alpha \in \mathbb{R}^+$
- $Z = X + jY \triangleq R^{\alpha/2} \exp\left(j\frac{\alpha\Theta}{2}\right)$
- $X = R^{\alpha/2} \cos\left(\frac{\alpha\Theta}{2}\right); \;\; Y = R^{\alpha/2} \sin\left(\frac{\alpha\Theta}{2}\right)$
- $X^2 \triangleq \max\limits_{1 \leq i \leq m} X_i^2$ and $Y^2 \triangleq \max\limits_{1 \leq i \leq m} Y_i^2$, $m \in \mathbb{N}$,

in which $X_i$ and $Y_i$ are assumed to be independent Gaussian distributed rvs with zero mean and variance $\frac{\Omega^\alpha}{2}$, representing the scattered fields in the in-phase and quadrature components, respectively

# Joint Envelope-Phase Density Function

It suffices to find the pdf of $X$. First, define $W \triangleq X^2$. Since $W$ is the maximum of a sequence of Chi-square rvs, it follows that

$$p_W(w) = \frac{m}{\sqrt{\Omega^\alpha \pi w}} \exp\left(-\frac{w}{\Omega^\alpha}\right)\left[\operatorname{erf}\left(\sqrt{\frac{w}{\Omega^\alpha}}\right)\right]^{m-1}, w > 0$$

Note that $|X| = \sqrt{W}$, and the pdf of $|X|$ is

$$p_{|X|}(x) = \frac{2m}{\sqrt{\Omega^\alpha \pi}} \exp\left(-\frac{x^2}{\Omega^\alpha}\right)\left[\operatorname{erf}\left(\frac{x}{\Omega^{\alpha/2}}\right)\right]^{m-1}, x > 0$$

# Joint Envelope-Phase Density Function

Motivated by the fact that for $m = 1$, $X$ reduces to a Gaussian rv, it is feasible to postulate that positive and negative values of $X$ are equally likely. In this case

$$p_X(x) = \frac{m}{\sqrt{\Omega^\alpha \pi}} \exp\left(-\frac{x^2}{\Omega^\alpha}\right)\left[\operatorname{erf}\left(\frac{|x|}{\Omega^{\alpha/2}}\right)\right]^{m-1}, \; x \in \mathbb{R}.$$

Hence, using the assumption that $X$ and $Y$ are independent, it follows that

$$p_{R,\Theta}(r,\theta) = |J| p_X\left(r^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\theta}{2}\right)\right) p_Y\left(r^{\frac{\alpha}{2}} \sin\left(\frac{\alpha\theta}{2}\right)\right),$$

in which $|J| = \frac{\alpha^2 r^{\alpha-1}}{4}$ is the determinant of the Jacobian. ■

# Joint Evenlope-Phase Density Function

Although $p_{R,\Theta}$ was derived for $m \in \mathbb{N}$, there is no mathematical constraints in using $m \in \mathbb{R}^+$, therefore, $m$ is replaced by $\mu \in \mathbb{R}^+$

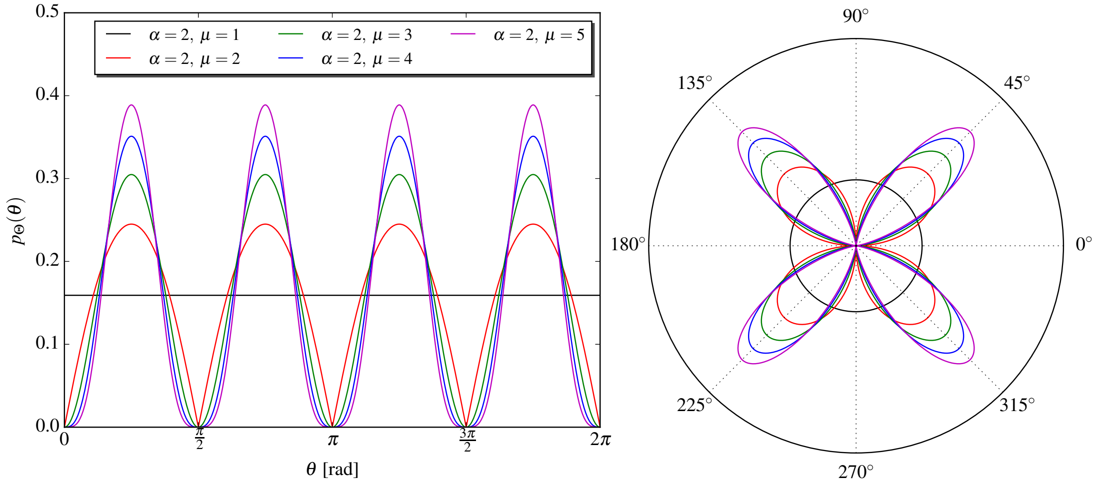
The phase density function may be computed as

$$p_\Theta(\theta) = \int_0^{+\infty} p_{R,\Theta}(r,\theta)\,\mathrm{d}r$$

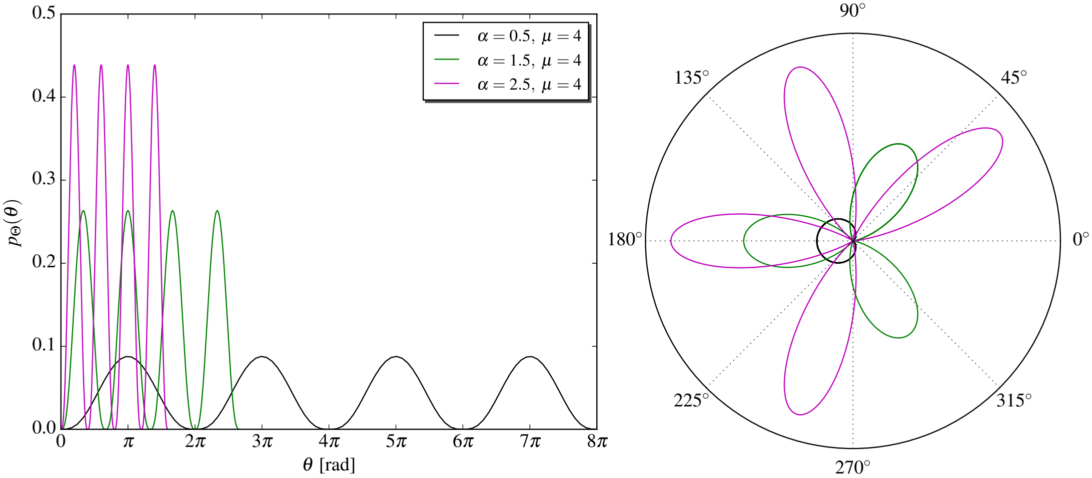
# Remarks

- As expected, in case $\mu = 1$, $p_{R,\Theta}$ reduces to the joint Weibull distribution
- For $\mu = 1$ and $\alpha = 2$, $p_{R,\Theta}$ specializes to the joint Rayleigh distribution
- Apart from the case in which $\mu = 1$, the envelope and the phase are not independent rvs
- Futhermore, the phase $\Theta$ is uniform if and only if $\mu = 1$

# Phase Density Function



# Phase Density Function



# Conclusions and Future Works

- A novel, approximate, and closed-form expression for the exponentiated Weibull joint envelope-phase distribution has been derived
- The exponentiated Weibull model fading finds applications in free-space optical communications channels subject to a variety of atmospheric turbulence conditions
- The proposed joint distribution may be applied to determine the performance, reliability, and high order statistics of such communications channels in many scenarios
- Further investigations may be conducted to derive an analytical expression for the marginal phase density function and to validate it with experimental data

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