

# An Approximate Exponentiated Weibull Joint Envelope-Phase Distribution

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# Outline

- 1 Introduction
  - Atmospheric Turbulence Channel Statistics
- 2 Joint Envelope-Phase Probability Density Function
- 3 Conclusions

# Introduction

- Free-space optical (FSO) systems
  - Short range, outdoor and indoor wireless
  - Large bandwidth, high average capacity
- Major challenge: atmospheric turbulence
  - Scintillation
  - Beam Wander
  - Beam Spreading
- Improvement in performance
  - Enlarge the aperture in the receiver
- Statistical characterization of the laser beam
  - Log-normal, Gamma-Gamma
  - Exponentiated Weibull

# Exponentiated Weibull Statistics

- Barrios proposed the following model for the irradiance at the receiver:

$$I^p \triangleq \sum_{n=1}^m w_n I_n^p$$

which was then approximated as

$$I \approx \lim_{p \rightarrow \infty} \left( \sum_{n=1}^m w_n I_n^p \right)^{1/p} = \max \{I_1, I_2, \dots, I_m\}$$

- $I_n \sim \text{Weibull}(\alpha)$  iid  $\Rightarrow I \sim \text{Exp. Weibull}(\alpha, m)$
- Suitable to fit weak, moderate, and strong atmospheric turbulence
- However, no phase information was taken into account

# Phase Distribution

- To assess the coherent detection of optical signals subject to atmospheric turbulence
- To compute bit error rate for communications systems using phase modulation schemes
- To determine optimum receiver aperture diameter
- Belmonte assumed Gaussian random phase fluctuations<sup>1</sup>
- Perlot assumed Beta distributed phase and also that phase and amplitude were independent<sup>2</sup>

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<sup>1</sup>A. Belmonte and J. M. Kahn, "Performance of synchronous optical receivers using atmospheric compensation techniques", Opt. Express, vol. 16, no. 18, pp. 14 151-14 162, Sep. 2008

<sup>2</sup>N. Perlot, "Turbulence-induced fading probability in coherent optical communication through the atmosphere," Appl. Opt. 46, 7218-7226 (2007)

# Joint Envelope-Phase Density Function

*Proposition:* Let  $R \in \mathbb{R}^+$  and  $\Theta \in [0, \frac{4\pi}{\alpha})$ ,  $\alpha > 0$ , be random variables (rvs) representing, respectively, the envelope and the phase of the exponentiated Weibull signal with shape parameters  $\alpha > 0$  and  $\mu > 0$ , and scale parameter  $\Omega > 0$ . Hence, the joint probability density function (pdf) of the random vector  $(R, \Theta)$ , denoted as  $p_{R,\Theta}(r, \theta)$ , is given by

$$p_{R,\Theta}(r, \theta) = \frac{(\alpha\mu)^2}{4\pi\Omega} \left(\frac{r}{\Omega}\right)^{\alpha-1} \exp\left(-\left(\frac{r}{\Omega}\right)^\alpha\right) \times \\ \left[ \operatorname{erf}\left(\left(\frac{r}{\Omega}\right)^{\alpha/2} \left| \cos\left(\frac{\alpha\theta}{2}\right) \right| \right) \times \operatorname{erf}\left(\left(\frac{r}{\Omega}\right)^{\alpha/2} \left| \sin\left(\frac{\alpha\theta}{2}\right) \right| \right) \right]^{\mu-1}.$$

# Joint Envelope-Phase Density Function

*Proof:*

- $R \in \mathbb{R}^+, \Theta \in [0, \frac{4\pi}{\alpha}), \alpha \in \mathbb{R}^+$
- $Z = X + jY \triangleq R^{\alpha/2} \exp(j\frac{\alpha\Theta}{2})$
- $X = R^{\alpha/2} \cos(\frac{\alpha\Theta}{2}); Y = R^{\alpha/2} \sin(\frac{\alpha\Theta}{2})$
- $X^2 \triangleq \max_{1 \leq i \leq m} X_i^2$  and  $Y^2 \triangleq \max_{1 \leq i \leq m} Y_i^2, m \in \mathbb{N}$ ,

in which  $X_i$  and  $Y_i$  are assumed to be independent Gaussian distributed rvs with zero mean and variance  $\frac{\Omega^\alpha}{2}$ , representing the scattered fields in the in-phase and quadrature components, respectively

# Joint Envelope-Phase Density Function

It suffices to find the pdf of  $X$ . First, define  $W \triangleq X^2$ . Since  $W$  is the maximum of a sequence of Chi-square rvs, it follows that

$$p_W(w) = \frac{m}{\sqrt{\Omega^\alpha \pi w}} \exp\left(-\frac{w}{\Omega^\alpha}\right) \left[\operatorname{erf}\left(\sqrt{\frac{w}{\Omega^\alpha}}\right)\right]^{m-1}, w > 0$$

Note that  $|X| = \sqrt{W}$ , and the pdf of  $|X|$  is

$$p_{|X|}(x) = \frac{2m}{\sqrt{\Omega^\alpha \pi}} \exp\left(-\frac{x^2}{\Omega^\alpha}\right) \left[\operatorname{erf}\left(\frac{x}{\Omega^{\alpha/2}}\right)\right]^{m-1}, x > 0$$



# Joint Envelope-Phase Density Function

Motivated by the fact that for  $m = 1$ ,  $X$  reduces to a Gaussian rv, it is feasible to postulate that positive and negative values of  $X$  are equally likely. In this case

$$p_X(x) = \frac{m}{\sqrt{\Omega^\alpha \pi}} \exp\left(-\frac{x^2}{\Omega^\alpha}\right) \left[\operatorname{erf}\left(\frac{|x|}{\Omega^{\alpha/2}}\right)\right]^{m-1}, \quad x \in \mathbb{R}.$$

Hence, using the assumption that  $X$  and  $Y$  are independent, it follows that

$$p_{R,\Theta}(r, \theta) = |J| p_X\left(r^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\theta}{2}\right)\right) p_Y\left(r^{\frac{\alpha}{2}} \sin\left(\frac{\alpha\theta}{2}\right)\right),$$

in which  $|J| = \frac{\alpha^2 r^{\alpha-1}}{4}$  is the determinant of the Jacobian. ■

# Joint Envelope-Phase Density Function

Although  $p_{R,\Theta}$  was derived for  $m \in \mathbb{N}$ , there is no mathematical constraints in using  $m \in \mathbb{R}^+$ , therefore,  $m$  is replaced by  $\mu \in \mathbb{R}^+$

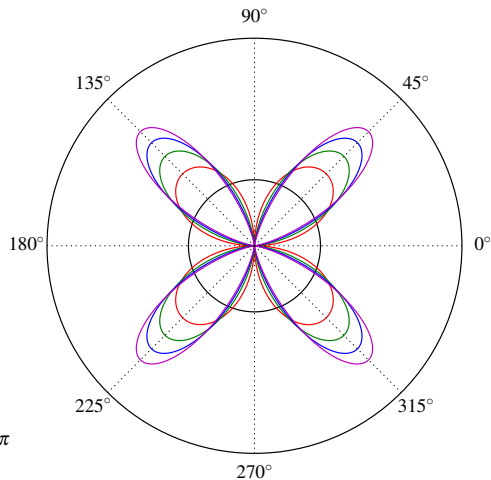
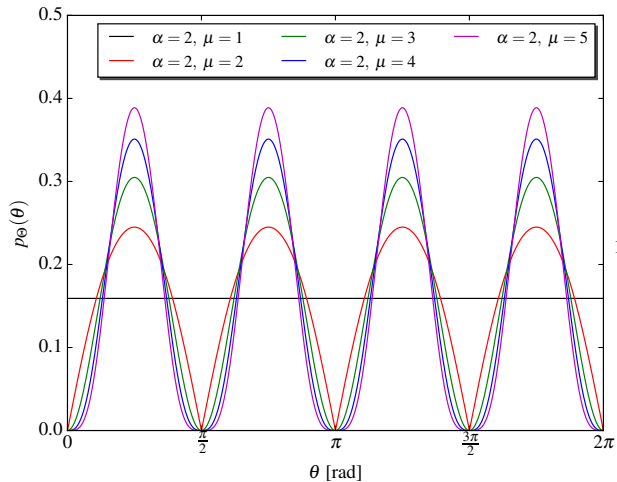
The phase density function may be computed as

$$p_{\Theta}(\theta) = \int_0^{+\infty} p_{R,\Theta}(r, \theta) \, dr$$

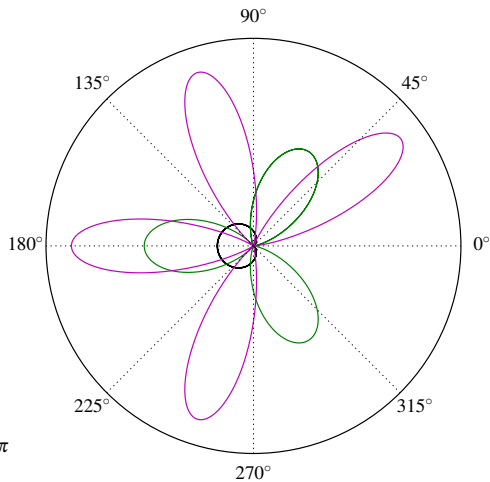
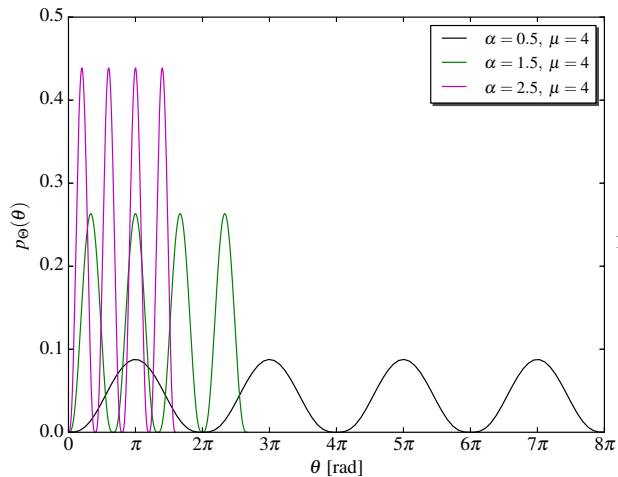
## Remarks

- As expected, in case  $\mu = 1$ ,  $p_{R,\Theta}$  reduces to the joint Weibull distribution
- For  $\mu = 1$  and  $\alpha = 2$ ,  $p_{R,\Theta}$  specializes to the joint Rayleigh distribution
- Apart from the case in which  $\mu = 1$ , the envelope and the phase are not independent rvs
- Furthermore, the phase  $\Theta$  is uniform if and only if  $\mu = 1$

# Phase Density Function



# Phase Density Function



## Conclusions and Future Works

- A novel, approximate, and closed-form expression for the exponentiated Weibull joint envelope-phase distribution has been derived
- The exponentiated Weibull model fading finds applications in free-space optical communications channels subject to a variety of atmospheric turbulence conditions
- The proposed joint distribution may be applied to determine the performance, reliability, and high order statistics of such communications channels in many scenarios
- Further investigations may be conducted to derive an analytical expression for the marginal phase density function and to validate it with experimental data

