2.1–2.2 Sets & Probs

- If S has n elementaries $\Rightarrow$ # event $= 2^n$
- Set Properties: ① Commutitive / ② Associative / ③ Distributive
  ④ $(A\cup B)^c = A^c \cap B^c$, $(A\cap B)^c = A^c \cup B^c$
- Axioms of Probability:
  ① $P(A) \geq 0$ ② $P(S) = 1$
  ③ if $A\cap B = \emptyset$, then $P(A\cup B) = P(A) + P(B)$
- Extensions from Axiom:
  ① $P(A^c) = 1 - P(A)$ ② $0 \leq P(A) \leq 1$ ③ $P(\emptyset) = 0$
  ④ $P(A\cup B) = P(A) + P(B) - P(A\cap B)$
  ⑤ three events: $P(A\cup B\cup C) = P(A) + P(B) + P(C)$
  $- P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$
  ⑥ $P(A\cup B) \leq P(A) + P(B)$
  ⑦ If $A \subseteq B$ then $P(A) \leq P(B)$

2.3 Counting Methods

- Sampling WITH Replacement, WITH Ordering
  \# N elements Total, select k $= n^k$
- NO Replacement, WITH ordering (P)
  \# N total, select out k $= \dfrac{N!}{(N-k)!}$
  ie. $N\times(N-1)\times(N-2)\times\cdots\times(N-k+1)$
  $\rightarrow$ Full Permutation: $N!$
- No Replacement, NO Ordering (C)
  \# N total, select k $= \binom{N}{k} = \dfrac{N!}{k!(N-k)!}$
- When split (n) into multiple (k) groups:
  Ex. n balls in bucket, split into categories:
  n_1个 belong to A^{class}, # $B = n_2$, # $C = n_3$ ⋯ # $K = n_k$
  and $n_1 + n_2 + n_3 + \cdots + n_k = n$
  \# different ways to PUT BALLS IN A ROW
  $= \binom{n}{n_1 n_2 \cdots n_k} = \dfrac{n!}{n_1!\, n_2! \cdots n_k!}$ $\xrightarrow{\text{divided by } 1}$ 2红 3蓝 出现 RBRBB 概率

2.4 Conditional

- Conditional Def: $\boxed{P(A|B) = \dfrac{P(A\cap B)}{P(B)}}$ for $P(B) > 0$
- Th. Total Probability: $P(A) = \sum_{i=1}^{n} P(A|B_i)\, P(B_i)$
- Baye's Rule:
  $P(B_i|A) = \dfrac{P(A\cap B)}{P(A)} = \boxed{\dfrac{P(A|B_i)\cdot P(B_i)}{\sum_{k=1}^{n} P(A|B_k)\cdot P(B_k)}}$

2.5 Independent

- Independent Def: $\boxed{P(A\cap B) = P(A)\,P(B)}$
  means: $P(A|B) = P(A)$, $P(B|A) = P(B)$
- Pairwise Independent: if A, B, C are Pairwise IND:
  $P(A\cap B) = P(A)P(B)$ .. $A \overset{\text{ind}}{\leftrightarrow} B$, $B \overset{\text{ind}}{\leftrightarrow} C$, $A \overset{\text{ind}}{\leftrightarrow} C$
- Jointly Independent:
  based on previouse example, also need:
  $P(A\cap B\cap C) = P(A)P(B)P(C)$

2.6 Sequential Experienments

- Bernoulli: single trial, success or fail
  $P(\text{success}) = p$
- Binomial: n trials, within them, k success
  $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$
- Multinomial: n trials, every trial has M classes, result B_j with probability P_j
  probability that every class happen time $(k_1 k_2 \cdots k_m)$:
  $P(k_1 \cdots k_M) = \dfrac{n!}{k_1!\, k_2! \cdots k_m!} p_1^{k_1} p_2^{k_2} \cdots p_M^{k_m}$
  $\rightarrow$ Ex: 扔飞镖 9次, 落在区域 1,2,3 概率为 0.2, 0.3, 0.5
  每个区域正好落3次的概率: $\dfrac{9!}{3!3!3!}(0.2)^3(0.3)^3(0.5)^3$
- Geometric: repeat trials until first success
  P(first success when m^{th} trial happens):
  $P(m) = (1-p)^{m-1} p$, $m = 1, 2, \cdots$
  $P(\text{超过k次才成功}) = P(m > k) = (1-p)^k$
- Dependent Experiments
  Ex. two-urn experiment
  $P(s_0, s_1, \cdots, s_n) = P(s_0)\, P(s_1|s_0)\, P(s_2|s_1) \cdots P(s_n|s_{n-1})$
- $\sum_{k=0}^{\infty} q^k = \dfrac{1}{1-q}$, $|q| < 1$
- $\sum_{k=1}^{\infty} k q^{k-1} = \dfrac{1}{(1-q)^2}$

3.1–3.2 RandomVariable, Discrete, and pmf

$X : S \to \mathbb{R}, \quad z \mapsto X(z)$

- Probability Mass Function (PMF): $p_X(x) = P(X = x)$
- PMF Props: ① $p_X(x) \geq 0$ ② $\sum_{x\in S_X} p_X(x) = 1$ ③ $P(X\in B) = \sum_{x\in S_B} p_X(x)$

  ①②③ used for pmf valid check

  $p_X(x) = 0$ for all $x \notin S_X$
- Valid RV: 对所有 $B \subseteq \mathbb{R}$，事件 $\{z : X(z) \in B\}$ 都属于 $\mathcal{F}$。
  小x在原本S的事件中存在

3.3 & 3.5 Expectation & Variance, and Important RVs

- Expected Value:
  $E[X] = \mu_X = \boxed{\sum_{x\in S_X} x\, p_X(x)}$
- E Convergence
  pmf 只对规定好的、有限或可数的一组 x 值有意义，在这组之外它就当作不存在（或者算 0）。
  $E[|X|] = \sum_k |x_k|\, p_X(x_k) < \infty$
- Expected RV: if $Z = g(x)$ then we have:
  $E[Z] = E[g(x)] = \sum_k g(x_k)\, p_X(x_k)$
- E Formulas:
  if $Z = a g(X) + b h(X) + c$, then: $\boxed{E[Z] = aE[g(x)] + bE[h(x)] + c}$
  $\rightarrow E[c] = c$
  $\rightarrow E[aX] = aE[X]$
  $\rightarrow E[X+c] = E[X] + c$
  $\rightarrow$ $E[g(X) + h(X)] = E[g(x)] + E[h(x)]$
- Variance: $VAR[X] = \boxed{E[(X-\mu_X)^2] = \sum_x (x-\mu_X)^2 p_X(x)}$
- standard dev: $\boxed{\sigma_X = \sqrt{VAR(X)}}$ $\boxed{VAR[X] = E[X^2] - (E[X])^2}$ $\leftarrow$
- VAR props: $\boxed{VAR[X+c] = VAR[X]}$ $\boxed{VAR[cX] = c^2\, VAR[X]}$
-

分布	pmf	E[X]	Var[X]
Bernoulli(p)	$p^x(1-p)^{1-x}$, x=0,1	p	p(1-p)
Binomial(n,p)	$\binom{n}{k}p^k(1-p)^{n-k}$	np	np(1-p)
Geometric(p)	(1-p)^{k-1}p	1/p	(1-p)/p²
Poisson(λ)	e^{-λ}λ^k/k!	λ	λ
Uniform(1...L)	1/L	(L+1)/2	(L²−1)/12
Zipf(L)	(1/c_L)(1/k)	L/c_L	L(L+1)/(2c_L)−(L/c_L)²

- Poisson
  $P(X=k) = \boxed{e^{-\lambda}\cdot\dfrac{\lambda^k}{k!}}$
  λ: 平均次数
  k: 实际次数

3.4 Conditional RVs

- Def: $\boxed{p_{X|C}(x) = P(X = x \mid C) = \dfrac{P(\{X = x\}\cap C)}{P(C)}}$

  若 C 本身是“关于 X 的事件”，例如 $C = \{a \leq X \leq b\}$，
  则条件 pmf 可直接由 X 的 pmf 算出：

  $$p_{X|C}(x) = \begin{cases} \dfrac{p_X(x)}{P(C)}, & x \in C \\ 0, & x \notin C \end{cases}$$
- Conditional Expectation: $\boxed{E[X|B] = \sum_x x\, p_{X|B}(x)}$
- Conditional VAR: $VAR[X|B] = \boxed{E[(X - E[X|B])^2|B] = \sum_x (x - E[X|B])^2 p_{X|B}(x)}$
- Law of Total Expectation: $\boxed{E[X] = \sum_i E[X|B_i]\, P(B_i)}$ $\quad E[g(X)] = \sum_i E[g(X)|B_i]\, P(B_i)$
- $\sum_{j=0}^{n-1} r^j = \dfrac{1-r^n}{1-r}$, $|r| < 1$