2.1-2.2 Sets & Probs

- If S has n elementaries $\Rightarrow$ # event $=2^n$
- Set properties: ① Commutitive / ② Associative 结合律 / ③ Distributive
  ④ $(A\cup B)^c = A^c\cap B^c$, $(A\cap B)^c = A^c\cup B^c$
- Axioms of Probability:
  ① $P(A)\geq 0$ ② $P(S)=1$
  ③ if $A\cap B=\emptyset$, then $P(A\cup B)=P(A)+P(B)$
- Extensions from Axiom:
  ① $P(A^c)=1-P(A)$ ② $0\leq P(A)\leq 1$ ③ $P(\emptyset)=0$
  ④ $P(A\cup B) = P(A)+P(B)-P(A\cap B)$
  ⑤ three events: $P(A\cup B\cup C) = P(A)+P(B)+P(C) - P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$
  ⑥ $P(A\cup B)\leq P(A)+P(B)$ ⑦ If $A\subseteq B$ then $P(A)\leq P(B)$

2.3 Counting Methods

- Sampling WITH Replacement, WITH Ordering
  # N elements total, select k $= n^k$
- NO Replacement, WITH ordering (P)
  # N total, select out k $= \frac{N!}{(N-k)!}$
  ie. $N\times(N-1)\times(N-2)\times\cdots\times(N-k+1)$
  $\longrightarrow$ Full Permutation: $N!$
- No Replacement, NO Ordering (C)
  # N total, select k $= \binom{N}{k} = \frac{N!}{k!(N-k)!}$
- When split $_{(n)}$ into multiple $_{(k)}$ groups:
  Ex. n balls in bucket, split into categories:
  n_1 belong to A^{class}, # $B=n_2$, # $C=n_3$ … # $k=n_k$
  and $n_1+n_2+n_3+\cdots+n_k=n$
  # different ways to PUT BALLS IN A ROW
  $=\binom{n}{n_1 n_2\cdots n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$ divided by $\xrightarrow{1}$ 2红3蓝出现RBRBB概率

2.4 Conditional

- Conditional Def: $\boxed{P(A|B)=\frac{P(A\cap B)}{P(B)}}$ for $P(B)>0$
- Th. Total Probability: $\boxed{P(A)=\sum_{i=1}^{n}P(A|B_i)P(B_i)}$
- Baye's Rule:
  $P(B_i|A)=\frac{P(A\cap B)}{P(A)} = \boxed{\frac{P(A|B_i)\cdot P(B_i)}{\sum_{k=1}^{n}P(A|B_k)\cdot P(B_k)}}$

2.5 Independent

- Independent Def: $\boxed{P(A\cap B) = P(A)P(B)}$
  means: $P(A|B)=P(A)$, $P(B|A)=P(B)$
- Pairwise Independent: if A,B,C are Pairwise IND:
  $P(A\cap B)=P(A)P(B)$… $A\overset{ind}{\Leftrightarrow}B$, $B\overset{ind}{\Leftrightarrow}C$, $A\overset{ind}{\Leftrightarrow}C$
- Jointly Independent:
  based on previous example, also need:
  $P(A\cap B\cap C)=P(A)P(B)P(C)$

2.6 Sequential Experiments

- Bernoulli: single trial, success or fail
  $P(\text{success})=P$
- Binomial: n trials, within them, k success
  $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$
- Multinomial: n trials, every trial has M classes, result B_j with probability P_j
  probability that every class happen time $(k_1 k_2\cdots k_M)$:
  $P(k_1\cdots k_M) = \frac{n!}{k_1!k_2!\cdots k_M!}P_1^{k_1}P_2^{k_2}\cdots P_M^{k_M}$
  $\rightarrow$ Ex: 扔飞镖 9次，落在区域 1,2,3 概率为 0.2, 0.3, 0.5
  每个区域正好落3次的概率: $\frac{9!}{3!3!3!}(0.2)^3(0.3)^3(0.5)^3$
- Geometric: repeat trials until first success
  P(first success when m^{th} trial happens):
  $P(m)=(1-p)^{m-1}p$, $m=1,2,\cdots$
  P(超过k次才成功) $=P(m>k) = (1-p)^k$
- Dependent Experiments
  Ex. two-urn experiment
  $P(S_0,S_1,\cdots,S_n)=P(S_0)P(S_1|S_0)P(S_2|S_1)\cdots P(S_n|S_{n-1})$
  - $\sum_{k=0}^{\infty}q^k=\frac{1}{1-q}$, $|q|<1$
  - $\sum_{k=1}^{\infty}kq^{k-1}=\frac{1}{(1-q)^2}$

3.1-3.2 RandomVariable, Discrete, and pmf

$X:S\to\mathbb{R}$, $z\mapsto X(z)$

- Probability Mass Function (PMF): $p_X(x)=P(X=x)$ ①②③ used for pmf valid check
- PMF Props: ① $p_X(x)\geq 0$ ② $\sum_{x\in S_X}p_X(x)=1$ ③ $P(X\in B)=\sum_{x\in S_B}p_X(x)$
- Valid RV: 对所有 $B\subseteq\mathbb{R}$，事件 $\{z: X(z)\in B\}$ 都属于 $\mathcal{F}$。 小X在原本S的事件中存在
  $P_X(x)=0$ for all $x\notin S_X$

3.3&3.5 Expectation & Variance, and Important RVs

- Expected Value:
  $E[X]=M_X=\boxed{\sum_{x\in S_X}x\,p_X(x)}$
- E Convergence: $E[|X|]=\sum_k|x_k|p_X(x_k)<\infty$
  pmf只对规定好的、有限或可数的一组 x 值有意义，在这组之外它就当作不存在（或者算 0）。
- Expected RV: if $Z=g(x)$ then we have:
  $E[Z]=E[g(x)]=\sum_k g(x_k)p_X(x_k)$
- E Formulas:
  if $Z=ag(x)+bh(x)+C$, then: $\boxed{E[Z]=aE[g(x)]+bE[h(x)]+C}$
  $\rightarrow E[c]=C$
  $\rightarrow E[aX]=aE[X]$ $\rightarrow$ $\boxed{E[g(x)+h(x)] = E[g(x)]+E[h(x)]}$
  $\rightarrow E[X+c]=E[X]+C$
- Variance: $VAR[X]=\boxed{E[(X-M_X)^2]=\sum_x(x-M_X)^2p_X(x)}$
- standard dev: $\boxed{\sigma_X=\sqrt{VAR(X)}}$ $\boxed{VAR[X]=E[X^2]-(E[X])^2}$
- VAR props: $\boxed{VAR[X+C]=VAR[X]}$ $\boxed{VAR[cX]=c^2VAR[X]}$

分布	pmf	E[X]	Var[X]
Bernoulli(p)	$p^x(1-p)^{1-x}$, x=0,1	p	p(1-p)
Binomial(n,p)	$\binom{n}{k}p^k(1-p)^{n-k}$	np	np(1-p)
Geometric(p)	(1-p)^{k-1}p	1/p	(1-p)/p²
Poisson(λ)	e^{-λ}λ^k/k!	λ	λ
Uniform(1...L)	1/L	(L+1)/2	(L²−1)/12
Zipf(L)	(1/c_L)(1/k)	L/c_L	L(L+1)/(2c_L)−(L/c_L)²

- Poisson
  $P(X=k) = \boxed{e^{-\lambda}\cdot\frac{\lambda^k}{k!}}$
  λ: 平均次数
  k: 实际次数

3.4 Conditional RVs

若 C 本身是"关于 X 的事件"，例如 $C=\{a\leq X\leq b\}$，则条件 pmf 可直接由 X 的 pmf 算出：

$p_{X|C}(x)=\begin{cases}\frac{p_X(x)}{P(C)}, & x\in C\\ 0, & x\notin C\end{cases}$

- Def: $\boxed{p_{X|C}(x)=P(X=x\mid C)=\frac{P(\{X=x\}\cap C)}{P(C)}}$
- Conditional Expectation: $\boxed{E[X|B]=\sum_x x\,p_{X|B}(x)}$
- Conditional VAR: $VAR[X|B]=\boxed{E[(X-E[X|B])^2|B]=\sum_x(x-E[X|B])^2p_{X|B}(x)}$
- Law of Total Expectation: $\boxed{E[X]=\sum_i E[X|B_i]\,P(B_i)}$ $E[g(X)]=\sum_i E[g(X)|B_i]\,P(B_i)$
- $\sum_{j=0}^{n-1}r^j=\frac{1-r^n}{1-r}$, $|r|<1$

4.1 CDF

$\boxed{F_X(x)=P(X\leq x)}$

- Props: $\boxed{0\leq F_X(x)\leq 1}$, $\boxed{\lim_{x\to-\infty}F_X(x)=0,\ \lim_{x\to\infty}F_X(x)=1}$
  $\boxed{\text{if } a<b \text{ then } F_X(a)\leq F_X(b)}$ $\boxed{F_X(b)=\lim_{x\to b^+}F_X(x)}$ right cont'
  $\boxed{P(a<X\leq b)=F_X(b)-F_X(a)}$
- to pmf: $P_X(x_k)=P(X=x_k)=F_X(x_k)-F_X(x_k^-)$

4.2 PDF

- $\boxed{f(x)=\frac{d}{dx}F_X(x)}$
- $P(a<x<b)=\int_a^b f_X(x)\,dx$
- $F_X(x)=\int_{-\infty}^{x}f_X(t)\,dt$
- Props: $\boxed{f_X(x)\geq 0}$ $\boxed{\int_{-\infty}^{\infty}f_X(x)\,dx=1}$

分布	支持集	pdf $f_X(x)$	cdf $F_X(x)$	$E[X]$	$VAR(X)$
Uniform $U[a,b]$	$a\leq x\leq b$	$\frac{1}{b-a}$	$\begin{cases}0, & x<a\\ \frac{x-a}{b-a}, & a\leq x\leq b\\ 1, & x>b\end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential $\text{Exp}(\lambda)$	$x\geq 0$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gaussian / Normal $\mathcal{N}(m,\sigma^2)$	$(-\infty,\infty)$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}$	$\Phi\left(\frac{x-m}{\sigma}\right)$	m	σ^2
Standard Normal $\mathcal{N}(0,1)$	$(-\infty,\infty)$	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$	$\Phi(x)$	0	1

Ex. Chain Rule: $\frac{d}{dx}\sin(x^2)=\cos(x^2)\cdot 2x$

4.3 Expected Value of X

- $\boxed{E[X]=\int_{-\infty}^{\infty}x\,f(x)\,dx}$ Exists only when $\int_{-\infty}^{\infty}|x|f(x)\,dx<\infty$
- $E[X]=\sum_k x_k P(X=x_k)$ for discrete (D)
- if $f_X(m+x)=f_X(m-x)$ then $E[X]=m$
- If $X\geq 0$:
  $E[X]=\int_0^{\infty}(1-F_X(t))\,dt$; $=\sum_{0}^{\infty}P(X>k)$
- props: $E\left[\sum_{k=1}^{n}g_k(X)\right]=\sum_{k=1}^{n}E[g_k(X)]$
- Moments: $E[X^n]=\int_{-\infty}^{\infty}x^n f_X(x)\,dx$

4.4 Imp Continuous RV

- Q-function $Q(x)=\boxed{\frac{1}{\sqrt{2\pi}}\int_x^{\infty}e^{-t^2/2}dt}=\boxed{1-\Phi(x)}$
- $\boxed{Q(-x)=1-Q(x)}$
- Normal CDF: $\boxed{\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt}$
- G$\Leftrightarrow$Normal transform: $\boxed{Z=\frac{X-m}{\sigma}}$

表示把任意高斯随机变量 $X\sim\mathcal{N}(m,\sigma^2)$ 标准化，变成标准正态随

- Exp Memoriless: $\boxed{P(X>t+h\mid X>t)=P(X>h)}$

5.8 example:

$Z=X+Y$

$F_Z(z)=P(Z\leq z)=P(X+Y\leq z)=\iint_{x+y\leq z}f_{X,Y}(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{z-x}f_{X,Y}(x,y)\,dy\,dx$

$f_Z(z)=\frac{d}{dz}F_Z(z)=\int_{-\infty}^{\infty}\frac{d}{dz}\left(\int_{-\infty}^{z-x}f_{X,Y}(x,y)\,dy\right)dx=\int_{-\infty}^{\infty}f_{X,Y}(x,z-x)\,dx$

$X\perp Y\Rightarrow f_{X,Y}(x,y)=f_X(x)f_Y(y)\Rightarrow\boxed{f_Z(z)=\int_{-\infty}^{\infty}f_X(x)f_Y(z-x)\,dx}$ (convolution)

4.5 Functions of RV

已知 X 的 pdf / cdf，给定关系 $Y=g(X)$，求 Y 的 pdf / cdf。

- rule: $P(Y\in C)=P(g(X)\in C)=P(X\in B)$
- 首选方法: $\boxed{F_Y(y)=P(Y\le y)=P(g(X)\le y)}$
- 若 $g(x)$ 严格单调，$y=g(x)$：

$$\boxed{f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|}\quad (x=g^{-1}(y))$$

  - 线性变换专用公式（必须秒写）

$$Y=aX+b\ \Rightarrow\ \boxed{f_Y(y)=\frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)}$$

  - 附带结论（Gaussian 专用）：

$$X\sim\mathcal{N}(m,\sigma^2)\Rightarrow Y\sim\mathcal{N}(am+b,a^2\sigma^2)$$

- 多解情形的 pdf 总公式（难点核心）

若 $y=g(x)$ 在 $x_1,\dots,x_n$ 处成立：

$$\boxed{f_Y(y)=\sum_k\frac{f_X(x_k)}{\left|\frac{dy}{dx}\Big|_{x=x_k}\right|}}$$

- Ex. $Y=X^2$, need to get: $P(Y\le y)$

if $y<0$, $P(Y\le 0)=0$, $\therefore F_Y(y)=0=f(y)$

else: $X^2\le y\iff-\sqrt{y}\le X\le\sqrt{y}$

$F_Y(y)=P(-\sqrt{y}\le X\le\sqrt{y})=F_X(\sqrt{y})-F_X(-\sqrt{y}),\quad y\ge 0$ CDF

for pdf of $y>0$: $\frac{d}{dy}F_X(\sqrt{y})=f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}}$; $\frac{d}{dy}F_X(-\sqrt{y})=f_X(-\sqrt{y})\cdot\left(-\frac{1}{2\sqrt{y}}\right)$

$\therefore\ f_Y(y)=\frac{f_X(\sqrt{y})+f_X(-\sqrt{y})}{2\sqrt{y}},\quad y>0$

4.6 Markov & Chebyshev

- Markov: $\boxed{P(X\ge a)\le\frac{E[X]}{a}}$
- Chebyshev: $\boxed{P(|X-M|\ge a)\le\frac{\sigma^2}{a^2}}$

4.7 Transformation Methods

- Characteristic function:

$$\boxed{\phi_X(v)=E[e^{jvX}]=\int_{-\infty}^{\infty}f_X(x)\,e^{jvx}\,dx}$$

discrete: $\phi_X(v)=\sum_k p_X(k)e^{jvk}$

- Reverse: $p_X(k)=\frac{1}{2\pi}\int_0^{2\pi}\phi_X(v)e^{-jvk}dv$
- Moment: $\boxed{E[X^n]=\frac{1}{j^n}\frac{d^n}{dv^n}\phi_X(v)\Big|_{v=0}}$

5.1 Two RVs

Example 5.1：抽学生名字 → (身高,体重)

- X(z)=学生身高
- Y(z)=学生体重
- 如事件 {H ≤ 183, W ≤ 82} 对应一个二维区域。

- Example format:

$B=\{X\in A_1,\ Y\in A_2\}$

5.2 Pairs of discrete RVs

- Joint pmf:

$$\boxed{p_{X,Y}(x,y)=P(X=x,\ Y=y)}$$

- $P(B)=\sum_{(x,y)\in B}p_{X,Y}(x,y)$
- Marginal pmf:

$$\boxed{p_X(x)=\sum_y p_{X,Y}(x,y)}\quad\boxed{p_Y(y)=\sum_x p_{X,Y}(x,y)}$$

5.3 Joint CDF of two RVs

- Joint CDF:

$$\boxed{F_{X,Y}(x,y)=P(X\le x,\ Y\le y)}$$

- Props:

1) If $x_1\le x_2$, $y_1\le y_2$ then $F_{X,Y}(x_1,y_1)\le F_{X,Y}(x_2,y_2)$

2) $\boxed{F_{X,Y}(x,-\infty)=0,\quad F_{X,Y}(-\infty,y)=0}$ $\boxed{F_{X,Y}(\infty,\infty)=1}$

3) $\boxed{F_X(x)=F_{X,Y}(x,\infty)\qquad F_Y(y)=F_{X,Y}(\infty,y)}$

4) $\boxed{\lim_{x\to a^+}F_{X,Y}(x,y)=F_{X,Y}(a,y)\qquad \lim_{y\to b^+}F_{X,Y}(x,y)=F_{X,Y}(x,b)}$

5) $$\boxed{\begin{aligned}P(x_1<X\le x_2,\ y_1<Y\le y_2)&=F_{X,Y}(x_2,y_2)-F_{X,Y}(x_2,y_1)\\&\quad-F_{X,Y}(x_1,y_2)+F_{X,Y}(x_1,y_1)\end{aligned}}$$

5.4 Joint PDF of two Cont RVs

如果 (X,Y) 是“联合连续”（jointly continuous），则它们所有事件的概率都可以通过一个二元 pdf f_{X,Y}(x,y) 来积分求得。

- $\boxed{P[(X,Y)\in B]=\iint_B f_{X,Y}(x,y)\,dx\,dy}$
- $\boxed{F_{X,Y}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}f_{X,Y}(u,v)\,dv\,du}$ $\boxed{f_{X,Y}(x,y)=\frac{\partial^2}{\partial x\,\partial y}F_{X,Y}(x,y)}$
- $\boxed{f_X(x)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)\,dy\qquad f_Y(y)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)\,dx}$

4️⃣ 协方差的直觉含义

- $\mathrm{COV}(X,Y)>0$: 大的 X 往往伴随大的 Y
- $\mathrm{COV}(X,Y)<0$: 大的 X 往往伴随小的 Y
- $\mathrm{COV}(X,Y)\approx 0$: 线性相关性弱

5.5 Independent RVs

两个随机变量 X 与 Y 独立，当且仅当：
任意关于 X 的事件与任意关于 Y 的事件都独立。
其等价条件是：
联合分布 = 边缘分布的乘积。

- $\boxed{f_{X,Y}(x,y)=f_X(x)\cdot f_Y(y)}$ $\boxed{P_{X,Y}(x,y)=P_X(x)\cdot P_Y(y)}$
  - $\boxed{F_{X,Y}(x,y)=F_X(x)\cdot F_Y(y)}$

5.6 Joint Moments & E of 2RVs

- Expected Value:

$$\boxed{E[g(X,Y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\,f_{X,Y}(x,y)\,dx\,dy}$$

$E[g(X,Y)]=\sum_i\sum_j g(x_i,y_j)\,p_{X,Y}(x_i,y_j)$ discrete

- If X, Y Independent: $(g(X,Y)=g_1(X)\,g_2(Y))$

$$\boxed{E[g_1(X)g_2(Y)]=E[g_1(X)]\,E[g_2(Y)]}$$

- Joint Moments: the (j,k)-th: $\boxed{E[X^jY^k]}$

orthogonal if $E[XY]=0$

- Covariance:

$$\boxed{\mathrm{COV}(X,Y)=E[(X-E[X])(Y-E[Y])]}$$

$$\boxed{\mathrm{COV}(X,Y)=E[XY]-E[X]E[Y]}$$

- If X, Y Independent:

$E[XY]=E[X]E[Y]\Rightarrow\boxed{\mathrm{COV}(X,Y)=0}$

- ↳ CANNOT REVERT!
- Correlation Coefficient: $\boxed{\rho_{X,Y}=\frac{\mathrm{COV}(X,Y)}{\sigma_X\sigma_Y}}$

$\sigma_X=\sqrt{\mathrm{VAR}(X)},\quad\sigma_Y=\sqrt{\mathrm{VAR}(Y)}$ $\quad -1\le\rho_{X,Y}\le 1$

5.7 Cond' prob & Conditional E

- 离散情型: $\boxed{p_{Y|X}(y\mid x)=\frac{p_{X,Y}(x,y)}{p_X(x)}}$

$p_{X,Y}(x,y)=p_{Y|X}(y\mid x)\,p_X(x)$

$P(Y\in A)=\sum_x P(Y\in A\mid X=x)\,p_X(x)$

- If X, Y independent: $\boxed{p_{Y|X}(y\mid x)=p_Y(y)}$
- 连续情型: conditional pdf: $\boxed{f_{Y|X}(y\mid x)=\frac{f_{X,Y}(x,y)}{f_X(x)}}$

conditional cdf: $\boxed{F_{Y|X}(y\mid x)=\int_{-\infty}^{y}f_{Y|X}(t\mid x)\,dt}$

total prob's continuous version: $\boxed{P(Y\in A)=\int_{-\infty}^{\infty}P(Y\in A\mid X=x)\,f_X(x)\,dx}$

- cond' cont' E: $\boxed{E[Y\mid X=x]=\int_{-\infty}^{\infty}y\,f_{Y|X}(y\mid x)\,dy}$ Σ if discont'

cond' E is a function: $\boxed{E[Y\mid X=x]=g(x)}$ 于是我们可以定义随机变量：$\boxed{E[Y\mid X]}$

- Law of total E: $\boxed{E[Y]=E[E[Y\mid X]]}$

if 例 $Y=aX+b+N,\quad N\perp X,\ E[N]=0$
then $\boxed{E[Y\mid X]=aX+b}$

- 若 X,Y 独立: $\boxed{E[Y\mid X]=E[Y]}$

5.8 Functions of 2 RVs

5.8 研究 由 (X,Y) 生成的新随机变量（一个或两个）如何求分布。

1. 一个函数 $Z=g(X,Y)$
2. 两个函数 $(Z_1,Z_2)=(g_1(X,Y),g_2(X,Y))$
3. 线性变换 的 pdf（Jacobian）

1️⃣ 万能思路（最稳）

$$\boxed{F_Z(z)=P(Z\le z)=P(g(X,Y)\le z)}$$

把不等式变成 (x,y) 平面上的区域 R_z，对联合 pdf 积分：

$$F_Z(z)=\iint_{R_z}f_{X,Y}(x,y)\,dx\,dy$$

再对 z 求导得 $f_Z(z)$。

三、条件法（降维神器）

若已知 $Y=y$，则 $Z=g(X,y)$ 是一维问题，可用 4.5 的方法先求

$f_{Z|Y}(z|y)$

再“平均回去”：

$$\boxed{f_Z(z)=\int f_{Z|Y}(z|y)\,f_Y(y)\,dy}$$

直觉：先在每个切片里算，再对切片加权平均。

2.两个函数

1️⃣ CDF 法（通用）

$$\boxed{F_{Z_1,Z_2}(z_1,z_2)=P(g_1(X,Y)\le z_1,\ g_2(X,Y)\le z_2)}$$

把条件翻译成 (x,y) 平面区域，对 $f_{X,Y}$ 积分。

$\begin{cases}V=aX+bY\\W=cX+eY\end{cases}$ $\quad A^{-1}=\frac{1}{\det A}\begin{pmatrix}e&-b\\-c&a\end{pmatrix}$ $\quad$ ae−bc

- Jacobian: set: $\begin{pmatrix}V\\W\end{pmatrix}=A\begin{pmatrix}X\\Y\end{pmatrix},\quad A=\begin{pmatrix}a&b\\c&e\end{pmatrix},\ \det A\ne 0$

$$\boxed{f_{V,W}(v,w)=\underbrace{f_{X,Y}\left(A^{-1}(v,w)\right)}_{f_{X,Y}(x,y)}\cdot\frac{1}{|\det A|}}$$

7.1 Sum of RVs

$S_n=X_1+X_2+\cdots+X_n\quad E[S_n]=E[X_1]+\cdots+E[X_n]$

- VAR — two RVs: $Z=X+Y$; $\mathrm{VAR}(Z)=E[(Z-\bar{E_Z})^2]$ $\boxed{\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)+2\,\mathrm{Cov}(X,Y)}$
- VAR — n variables: $\mathrm{Var}(S_n)=\sum_{k=1}^{n}\mathrm{Var}(X_k)+\sum_{j\ne k}\mathrm{Cov}(X_j,X_k)$ if 相互独立: $\boxed{\mathrm{Var}(S_n)=\sum_{k=1}^{n}\mathrm{Var}(X_k)}$
- VAR — if $X_k\overset{\text{iid}}{\sim}(\mu,\sigma^2)$ $\boxed{E[S_n]=n\mu,}$ $\boxed{\mathrm{Var}(S_n)=n\sigma^2}$
- 已知 $X_1,\dots,X_n$ 独立，如何求 S_n 的 pdf?

1️⃣ 两个变量：卷积（你在 5.8 已经见过）

若 $Z=X+Y$，且 X,Y 独立：

$$\boxed{f_Z(z)=\int_{-\infty}^{\infty}f_X(x)\,f_Y(z-x)\,dx}$$

- 2️⃣ 多个变量：特征函数是王者 $\mathcal{L}_X(v)=E[e^{jvX}]$

$\boxed{\mathcal{L}_{X+Y}(v)=\mathcal{L}_X(v)\,\mathcal{L}_Y(v)}$ 2个 多个: $\mathcal{L}_{S_n}(v)=\prod_{k=1}^{n}\mathcal{L}_{X_k}(v)$

7.2 Sample Mean & Law of Large Num

$X_1,X_2,\dots,X_n$ iid, $\quad E[X]=\mu,\quad \mathrm{Var}(X)=\sigma^2$

- 样本均值 Sample Mean $\quad M_n=\frac{1}{n}\sum_{k=1}^{n}X_k$ $\quad\frac{1}{n}S_n$

$M_n=\frac{1}{n}(X_1+X_2+\cdots+X_n)$

- 均值与方差： $\boxed{E[M_n]=\mu}$ $\boxed{\mathrm{Var}(M_n)=\frac{\sigma^2}{n}}$
- chebichev 在这里:

$$\boxed{P(|M_n-\mu|\ge\varepsilon)\le\frac{\sigma^2}{n\varepsilon^2}}$$

- WLLN: $\boxed{\lim_{n\to\infty}P(|M_n-\mu|\le\varepsilon)=1}$ $\boxed{P(|M_n-\mu|>\varepsilon)\xrightarrow[n\to\infty]{}0}$
- SLLN: $\boxed{P\left(\lim_{n\to\infty}M_n=\mu\right)=1}$

7.3 Central Limit Theorem

- $\boxed{Z_n=\frac{S_n-n\mu}{\sigma\sqrt{n}}=\frac{\sqrt{n}(M_n-\mu)}{\sigma}}$ $\quad E[Z_n]=0$, $\mathrm{Var}(Z_n)=1$
- CLT: $\boxed{Z_n\xrightarrow{d}\mathcal{N}(0,1)}$

equal writing: $\boxed{S_n\approx\mathcal{N}(n\mu,\ n\sigma^2)}$ $\boxed{M_n\approx\mathcal{N}\left(\mu,\ \frac{\sigma^2}{n}\right)}$

- 概率近似模板: $\boxed{P(S_n\le a)\approx\Phi\left(\frac{a-n\mu}{\sigma\sqrt{n}}\right)}$

$$\boxed{P(M_n\le m)\approx\Phi\left(\frac{\sqrt{n}(m-\mu)}{\sigma}\right)}$$

Z_n 是：
“样本均值偏离真实均值多少个‘标准差单位’”。