Streak artifacts from non-convex metal objects in X-ray tomography

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Radon transform

- Given $f : \mathbb{R}^2 \to \mathbb{R}$ and a line $\ell \in \mathbb{R}^2$, define the Radon transform as

$$\mathscr{R}f(\ell) = \int_\ell f(x)\, dH^1(x).$$

- Can parametrize lines by $(s, \omega) \in \mathbb{R} \times \mathbb{S}^1$ by the relation $\ell = \{x \in \mathbb{R}^2 \,:\, x \cdot \omega = s\}$. Thus we can view the Radon transform mapping to functions on $M := \mathbb{R} \times \mathbb{S}^1$.
- The lines and the Radon transform are invariant under the equivalence relation $(s, \omega) \sim (-s, -\omega)$, so we can also consider the space of lines $\mathcal{L} = M/\sim$ and consider the Radon transform mapping to functions on $\mathcal{L}$.

Beer's Law

- The Radon transform comes up in X-ray tomography: by Beer's Law

$$\frac{dI}{dt} = f(x(t))I(t)$$

  where $f(x)$ is the absorption coefficient.
- Solve ODE to get

$$\frac{I_f}{I_0} = \exp(-\mathscr{R}f(\ell)) \implies \mathscr{R}f(\ell) = -\ln\left(\frac{I_f}{I_0}\right).$$

Filtered Back-projection

- Can define adjoint $\mathscr{R}^*$ by

$$\mathscr{R}^* g(x) = \int_{\mathbb{S}^1} g(x \cdot \omega, \omega)\, d\omega.$$

- Then the filtered back-projection formula gives

$$f = \mathscr{F}\mathscr{R} f, \quad \mathscr{F} := \frac{1}{4\pi} \mathscr{R}^* \mathscr{I}^{-1},$$

where $\mathscr{I}^{-1}$ is the Riesz transform,

$$\mathcal{F}_s \mathscr{I}^{-1} g(\sigma, \omega) = |\sigma| \mathcal{F}_s g(\sigma, \omega).$$

Artifacts

In practice, there are streak artifacts:

Figure: A dental X-ray slice. Picture from
https://www.exxim-cc.com/metal_artifact_reduction.html

Cause: attenuation varying w/energy

- One possible cause is the *beam-hardening effect*: energies of X-rays are not all the same, and some materials (e.g. metals) have attenuation coefficients varying w/energy (so $f = f_E$, E energy).
- Thus the data collected is actually an average:

$$\frac{I_f}{I_0} = \int \exp(-\mathscr{R} f_E)\eta(E)\, dE,$$

  where $\eta(E)\, dE$ is the probability distribution of energies of the X-ray particles.
- Thus the image produced via FBP

$$f_{CT} = \mathscr{F}\left(-\ln\left(\frac{I_f}{I_0}\right)\right)$$

  might not be f.

- If we assume

$$f_E = f_{E_0} + (E - E_0)f$$

(think $f(x) \approx \frac{\partial f_E}{\partial E}(x; E_0)$), and $\eta(E) = \frac{1}{2\epsilon}\chi_{[E_0-\epsilon, E_0+\epsilon]}(E)$, then

$$\begin{aligned} -\ln\left(\frac{I_f}{I_0}\right) &= -\ln\left(\exp(-\mathscr{R}f_{E_0})\int_{E_0-\epsilon}^{E_0+\epsilon}\frac{\exp(-(E-E_0)\mathscr{R}f)}{2\epsilon}\,dE\right) \\ &= \mathscr{R}f_{E_0} - \ln\left(\frac{\sinh(\epsilon\mathscr{R}f)}{\epsilon\mathscr{R}f}\right). \end{aligned}$$

so the image produced is $f_{CT} = f_{E_0} + f_{MA}$, where

$$f_{MA} = \mathscr{F}F(\mathscr{R}f), \quad F(x) = -\ln\left(\frac{\sinh(\epsilon x)}{\epsilon x}\right).$$

General Ideas

See e.g. works of Choi-Park-Seo '14 and Palacios-Uhlmann-Wang '17.
We suppose the attenuation variation f has a geometric structure relating to some shape D (e.g. f is conormal to D).

- Use FIO structure (e.g. canonical relation) of $\mathscr{R}$ to determine the analytic structure (e.g. wavefront set) of $\mathscr{R}f$
- Determine analytic structure of $F(\mathscr{R}f)$ (answer may depend on D)
- Use FIO structure of filtered backprojection to determine analytic structure of the artifact.

Wavefront set

For a distribution u on $\mathbb{R}^n$, its *wavefront set* $WF(u)$ consist of $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash 0)(= T^*\mathbb{R}^n \backslash o)$ where, for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi(x) \neq 0$, we have that $\mathcal{F}(\varphi u)$ does not decrease rapidly in any conical neighborhood of ξ.
This set gives the locations x and the (co)directions ξ of singularities of u.
When applying an operator T, wavefront sets will transfer by the Hörmander-Sato Lemma:

$$WF(Tu) \subset WF'(T) \circ WF(u)$$

where we identify T with its Schwartz kernel (a distribution on the product), and

$$WF'(T) = \{(x, y, \xi, -\eta) : (x, y, \xi, \eta) \in WF(T)\}$$

Conormal/Lagrangian distributions

If $Y \subset \mathbb{R}^n$ is a smooth submanifold, then $u \in L^2$ is (L^2 classically) conormal to Y if, for all finite collections of vector fields $\{V_1, \ldots, V_N\}$ *tangent to* Y, we have $V_1 \ldots V_N u \in L^2$. (E.g. $u = \chi_D$ if $Y = \partial D$.)
In such cases we have $WF(u) \subset N^*Y$. If $Y = \{f = 0\}$ with df non-vanishing on Y, then we can explicitly write $(N^*Y)_x = \text{span } df_x$ for $x \in Y$.
More generally, we can consider *Lagrangian distributions* w.r.t. some Lagrangian submanifold $\Lambda \subset T^*\mathbb{R}^n$ (it can be defined similarly as above, with vector fields replaced by ΨDOs whose symbols vanish on Λ). Then $WF(u) \subset \Lambda$.
In both cases there are explicit ways to parametrize such distributions via oscillatory integrals.

$\mathscr{R}$ as FIO

A Fourier Integral Operator (FIO) is an operator whose Schwartz kernel is a Lagrangian distribution (in the product space)

- The Schwartz kernel of $\mathscr{R}$ can be written as

$$\delta(s - x \cdot \omega) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(s - x\cdot\omega)\lambda}\, d\lambda,$$

  from which we have that $\mathscr{R}$ is an FIO associated to the Lagrangian $N^*(\{s - x \cdot \omega = 0\})$.

- Writing $\omega = (\cos\phi, \sin\phi)$, we have

$$d(s - x\cdot\omega) = ds + (x_1 \sin\phi - x_2 \cos\phi)\, d\phi - (\cos\phi\, dx_1 + \sin\phi\, dx_2)$$

  so

$$\sigma\, ds + \eta\, d\phi - \xi \cdot dx \in N^*(\{s - x\cdot\omega = 0\}) \implies \xi \parallel \omega.$$

- Note for u conormal to a curve $\gamma \subset \mathbb{R}^2$ we have

$$WF(\mathscr{R}u) \subset WF'(\mathscr{R}) \circ WF(u) \subset WF'(\mathscr{R}) \circ N^*(\gamma).$$

- For $(x_0, \xi_0) \in N^*(\gamma)$, note that ξ_0 is conormal to γ at x_0. Furthermore,

$$(s, \omega, \sigma, \eta, x_0, \xi_0) \in WF'(\mathscr{R}) \implies \omega \parallel \xi_0$$

so ω gives a unit conormal to γ at x_0! Furthermore $s = x_0 \cdot \omega$, i.e. the line $\{s = x \cdot \omega\}$ is tangent to γ at x_0.

- Thus

$$S := \pi_{(s,\omega)}(WF'(\mathscr{R}) \circ N^*(\gamma)) = \{(s, \omega) : \{s = x \cdot \omega\} \text{ is tangent to } \gamma\}.$$

If γ is strictly convex, then S is a smooth curve, and $WF'(\mathscr{R}) \circ N^*(\gamma) = N^*(S)$.

FBP wavefront mapping

- The Filtered Backprojection $\mathscr{F}$ is also a FIO, whose canonical relation is the transpose of that of $\mathscr{R}$.
- It is 1-to-2 (for each (x, ξ) there are two $(s, \omega, \sigma, \eta)$ in the relation), though the canonical relation fo $\mathscr{R}$ is 2-to-1 with the corresponding ambiguities.
- Thus, comparing the singularities of $\mathscr{F} F(\mathscr{R} f)$ against $f_{E_0} = \mathscr{F}\mathscr{R} f_{E_0}$ essentially asks:

$$\text{what is } WF(F(\mathscr{R} f)) \backslash WF(\mathscr{R} f_{E_0})?$$

New singularities from multiplication

Suppose for convenience that $f_{E_0} = f = \chi_D$, where D is a union of disjoint connected regions with smooth boundaries.

- Choi-Park-Seo '14 analyzed $F(\mathscr{R}\chi_D)$ by using a Taylor series for F to show

$$WF(F(\mathscr{R}\chi_D)) \backslash WF(\mathscr{R}\chi_D) = WF((\mathscr{R}\chi_D)^2) \backslash WF(\mathscr{R}\chi_D).$$

- Palacios-Uhlmann-Wang '17 used the work to study more quantitatively the situation where D is a union of convex connected domains:
- If D is a single strictly convex region, then $\mathscr{R}\chi_D$ is a bounded function conormal to S, a smooth curve. Then $(\mathscr{R}\chi_D)^2$ is also conormal to S, i.e. no new singularities are generated.

- If D is a union of multiple such regions D_j (so $\chi_D = \sum \chi_{D_j}$), then

$$(\mathscr{R}\chi_D)^2 = \sum (\mathscr{R}\chi_{D_j})^2 + \sum_{i \neq j} (\mathscr{R}\chi_{D_i})(\mathscr{R}\chi_{D_j}).$$

  The first term behaves as before, but the second term in general introduces new singularities.
- In fact, in general $WF((\mathscr{R}\chi_{D_i})(\mathscr{R}\chi_{D_j}))$ can contain any point in $N^*(S_i \cap S_j)$, where S_i, S_j are the curves associated to $\mathscr{R}\chi_{D_i}, \mathscr{R}\chi_{D_j}$. Note $S_i \cap S_j$ is a finite collection of points, and at each point the fiber includes *every* direction (note in N^*S_j the admissible fiber directions at any point are limited to one direction). Such singularities are carried by FBP to the streaking lines.
- Note that $S_i \cap S_j \ni (s, \omega) \iff$ the line $\{s = x \cdot \omega\}$ is tangent to both D_i and D_j; hence in this case

$$\text{sing supp}\,(f_{MA}) \subset \{x \text{ on lines tangent to more than one } D_j\} \cup \partial D.$$

Questions

- Finite iterated regularity?
- Inflection points?

Iterated Regularity

For $k \in \mathbb{N}$ and Hilbert space H of functions on X and a Lie algebra $\mathcal{V}$ of smooth vector fields on X, define

$$I_k H(X; \mathcal{V}) := \{u \in H \,:\, V_1 \ldots V_j u \in H \text{ for all } j \leq k, V_i \in \mathcal{V}\}.$$

Most common example: for S a collection of submanifolds of X, consider $\mathcal{V}(S)$, the collection of vector fields tangent to all submanifolds in S.
More generally, for any conic subset $\Lambda \subset T^*X$, define
$\mathscr{M}(\Lambda) := \{A \in \Psi^1(X) \,:\, \sigma_1(A)|_\Lambda \equiv 0\}$, and define

$$I_k H(X; \mathscr{M}(\Lambda)) := \{u \in H \,:\, A_1 \ldots A_j u \in H \text{ for all } j \leq, A_i \in \mathscr{M}(\Lambda)\}.$$

Note that $I_k H(X; \mathcal{V}(Y)) = I_k H(X; \mathscr{M}(N^*Y))$ if Y is a smooth submanifold. Note as well that $I_\infty L^2(X; \mathcal{V}(Y))$ is the space of "classical" conormal distributions w.r.t. Y.
Define $L^\infty I_k H := L^\infty(X) \cap I_k H$.

Main Theorem

Theorem (Wang-Z. '20)

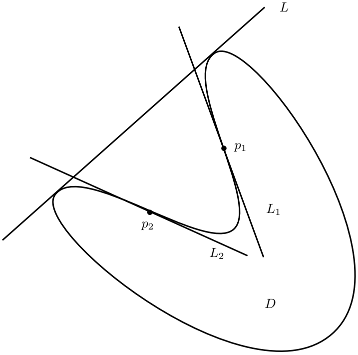
Suppose $D = \cup D_j$ is the union of disjoint simply connected domains with smooth (not necessarily convex) boundary which satisfy some geometric assumptions. If $f \in L^\infty I_k L^2(\mathbb{R}^2; \mathcal{V}(\partial D))$ is compactly supported and $f_{MA} = \mathscr{F} F(\mathscr{R} f)$, then

$$WF(f_{MA}) \subset \left(\bigcup_{\ell \in \mathcal{L}' \cup \mathcal{L}''} N^* \ell \right) \cup N^* \partial D$$

where $\mathcal{L}'$ is the collection of lines tangent to ∂D at more than one point, and $\mathcal{L}''$ is the collection of lines tangent to ∂D at an inflection point.

Our geometric assumptions are:

1. The curvature of the boundary does not vanish on any open set. All points where it does vanish, i.e. inflection points, are simple inflection points.
2. Any straight line is tangent to at most two points on ∂D. If a line is tangent to two points on ∂D, neither of those points are inflection points.



Intertwining Property

Recall if D has smooth and strictly convex boundary that $WF'(\mathscr{R})$ mapped $N^*(\gamma)$ to $N^*(S)$ for a smooth curve S, where $\gamma = \partial D$. We'd like to claim, for compactly supported u,

$$u \in L^\infty I_k L^2(\mathbb{R}^2; \mathcal{V}(\gamma)) \implies \mathscr{R}u \in L^\infty I_k H^{1/2}(M; \mathcal{V}(S)).$$

(Note $\mathscr{R}$ maps L^2_c to $H^{1/2}$.)
Naively, we can hope: for every $\tilde{V} \in \mathcal{V}(S)$, there exists $V \in \mathcal{V}(\gamma)$ such that $\tilde{V}\mathscr{R} = \mathscr{R}V$. Thus when testing for iterated regularity w.r.t. S, we can transfer the test to an iterated regularity test w.r.t. γ.
This doesn't quite work with vector fields, so next we can try: for every $\tilde{A} \in \mathscr{M}(N^*S)$, does there exist $A \in \mathscr{M}(N^*\gamma)$ such that $\tilde{A}\mathscr{R} = \mathscr{R}A$?

Heuristically one can do this by "conjugating" $\tilde{A}$ by $\mathscr{R}$. In fact, Egorov's theorem says that if Q is a two-sided parametrix for $\mathscr{R}$, then $Q\tilde{A}\mathscr{R}$ is a ΨDO on $\mathbb{R}^2$ with symbol $\sigma(\tilde{A}) \circ C$, where we view the canonical relation as a map $C : T^*\mathbb{R}^2 \to T^*M$.

Issue: $\mathscr{R}$, defined as an operator mapping to functions on M, does not admit a two-sided parametrix (the FBP operator $\mathscr{F}$ is only a left inverse). In fact the desired intertwining property is false due to the range symmetry condition on $\mathscr{R}$.

Solution: Work directly on the space of lines $\mathcal{L}$ and redefine $\mathscr{R}$ to map to functions on $\mathcal{L}$. Then it does admit a two-sided parametrix, and we can go through as desired.

(There are also technical issues with properness, but that can be resolved by inserting appropriate cutoffs, which is harmless since the original data is supported in a known compact region.)

Thus for D a single strictly convex object we have

$$\begin{aligned} u \in L^\infty I_k L^2(\mathbb{R}^2; \mathscr{M}(N^*\gamma)) &\implies \mathscr{R}u \in L^\infty I_k H^{1/2}(\mathcal{L}; \mathscr{M}(N^*S)) \\ &\implies \mathscr{R}u \in L^\infty I_k H^{1/2}(\mathcal{L}; \mathcal{V}(S)) \end{aligned}$$

If $D = \cup_{j=1}^N D_j$ is a union of such objects, and S_j the corresponding smooth curves in $\mathcal{L}$, then we have

$$\mathcal{R}u \in \sum L^\infty I_k H^{1/2}(\mathcal{L}; \mathcal{V}(S_j)) \subsetneq L^\infty I_k H^{1/2}(\mathcal{L}; \mathcal{V}(S))$$

where $S = (S_1, \dots, S_N)$.

C^∞-algebra property

The advantage of these spaces is its C^∞-*algebra* property, i.e. it is closed under composition by a C^∞ function:

Theorem (Melrose-Ritter)

For any Lie algebra of vector fields $\mathcal{V}$, we have

$$u \in L^\infty I_k L^2(X;\mathcal{V}), F \in C^\infty(\mathbb{R}) \implies F(u) \in L^\infty I_k L^2(X;\mathcal{V}).$$

First used in studying semilinear wave equation $u_{tt} - \Delta u = F(u)$.
Proof relies on a generalized Gagliardo-Nirenberg inequality (interpolating intermediate L^p norm of intermediate derivatives by high L^p/high derivative and low L^p/low derivative estimates).

Thus we have

$$\mathscr{R}f \in L^\infty I_k L^2(\mathcal{L}; \mathcal{V}(S)) \implies F(\mathscr{R}f) \in L^\infty I_k L^2(\mathcal{L}; \mathcal{V}(S)).$$

Just like in the case of classical conormal distributions, new singularities can be introduced at the intersection points of the submanifolds in S. Given an intersection point $p \in S_i \cap S_j$, we can write

$$I_k L^2(\mathcal{L}; \mathcal{V}(S_i, S_j)) = I_k L^2(\mathcal{L}; \mathcal{V}(S_i, p)) + I_k L^2(\mathcal{L}; \mathcal{V}(S_j, p)).$$

One can show regularity w.r.t. $\mathcal{V}(S_i, p)$ is equivalent to regularity w.r.t $\mathscr{M}(N^*S_i \cup N^*p)$ due to (S_i, p) being *microlocally complete* (cf. Melrose-Ritter), which under $\mathscr{F}$ translates to regularity w.r.t. $N^*\gamma_i \cup N^*L$, where L is the line in $\mathbb{R}^2$ corresponding to $p \in \mathcal{L}$.
Since L intersects γ_i tangentially to order 2 (i.e. not an inflection point), the collection $\{\gamma_i, L\}$ is also microlocally complete. Hence regularity w.r.t. $N^*\gamma_i \cup N^*L$ is equivalent to regularity w.r.t. $\mathcal{V}(\gamma_i, L)$.

Inflection point/cusp

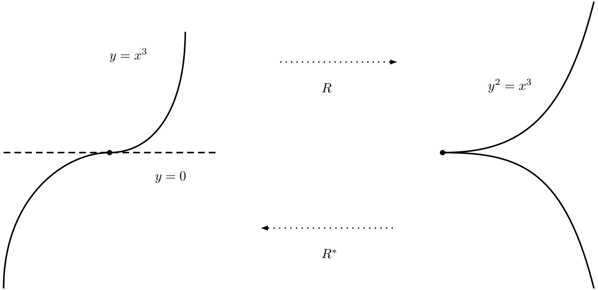
Suppose γ has an inflection point p. Under an orthogonal change of coordinates we have that γ is locally the graph of

$$x_2 = cx_1^3 + O(x_1^4).$$

We suppose $c \neq 0$, i.e. the point is a *simple* inflection point.
In this case $\Lambda = WF'(\mathscr{R}) \circ N^*(\gamma)$ is still a *smooth Lagrangian* submanifold of $T^*\mathcal{L}$; however its projection to the base will be singular: namely it will be a cusp G. Away from the cusp point B we have that Λ coincides with $N^*(G \backslash B)$.

Example: For γ given locally around $(0,0)$ by the graph of $x_2 = x_1^3$, we have S is given near $(0,\pi/2)$ by the set

$$\left\{ \frac{(s/\sin\phi)^2}{2} = \frac{(-\cot\phi)^3}{3} \right\}.$$



Using the ΨDO intertwining property, we can still show

$$u \in I_k L^2(\mathbb{R}^2; \mathcal{V}(\gamma)) \implies \mathscr{R} u \in I_k L^2(\mathcal{L}; \mathscr{M}(\Lambda)) \subsetneq I_k L^2(\mathcal{L}; \mathcal{V}(G)).$$

Note $I_k L^2(\mathcal{L}; \mathcal{V}(G))$ is still a C^∞-algebra, but membership does not say much about the behavior near the cusp (all vector fields in $\mathcal{V}(G)$ must vanish at B).
Moreover, $I_k L^2(\mathcal{L}; \mathscr{M}(\Lambda))$ is not closed under multiplication (essentially by Zworski '94).

Marked Lagrangian Distribution

To remedy this, we use the space of *marked Lagrangian distributions*: Given $\Lambda \subset T^*X$ conic Lagrangian and $\Sigma \subset \Lambda$ smooth and conic, let

$$\mathscr{M}(\Lambda, \Sigma) := \{A \in \Psi^1(X) \, : \, \sigma_1(A)|_\Lambda \equiv 0, H_{\sigma_1(A)} \text{ tangent to } \Sigma\}.$$

Define the iterated regularity spaces $I_k H(X; \mathscr{M}(\Lambda, \Sigma))$ similarly as before. For a cusp G with cusp point B, $\Lambda_G = \overline{N^*(G \backslash B)}$, $\Lambda_B = N^*\{B\}$, $\Sigma = \Lambda_G \cap \Lambda_B$, set

$$J_k(\mathbb{R}^2; G) := L^\infty I_k L^2(\mathbb{R}^2; \mathscr{M}(\Lambda_G, \Sigma)) + L^\infty I_k L^2(\mathbb{R}^2; \mathscr{M}(\Lambda_B, \Sigma)).$$

Then (after localization and identifying a neighborhood of $B \in \mathcal{L}$ with $\mathbb{R}^2$) we have $L^\infty I_k L^2(\mathcal{L}; \mathscr{M}(\Lambda)) \subset J_k(\mathbb{R}^2; G)$, and furthermore (by Melrose '87, also Sá Barreto '91) $J_k(\mathbb{R}^2; G)$ is a C^∞-algebra. (Idea: after performing a non-homogeneous blow-up of the cusp point, one can show the iterated marked Lagrangian regularity is equivalent to iterated regularity w.r.t. vector fields tangent to the blow-up.)
Thus, near the cusp point we have $F(\mathscr{R}f) \in J_k(\mathbb{R}^2; G)$.
Under $\mathscr{F}$, such functions are mapped to $I_k H^{-1/2}(\mathbb{R}^2; \mathscr{V}(\gamma, L))$ where L is the line corresponding to B.

Thus, to summarize the analysis:

- If $\ell \in \mathcal{L}$ does not intersect γ tangentially, or it is tangent to γ at exactly one point (non-inflection), then $\mathscr{R}f \in I_k L^2(\mathcal{L}, \mathcal{V}(S_j))$ for some j in a neighborhood of ℓ. This is preserved under composition with F, so f_{MA} does not produce streak artifacts here.
- If ℓ is tangent to γ at two (non-inflection) points, then locally $\mathscr{R}f \in I_k L^2(\mathcal{L}, \mathcal{V}(S_i)) + I_k L^2(\mathcal{L}, \mathcal{V}(S_j))$. Nonlinear composition puts it into $I_k L^2(\mathcal{L}, \mathcal{V}(S_i, S_j))$, which translates to regularity for f_{MA} in $I_k H^{-1/2}(\mathbb{R}^2, \mathcal{V}(\gamma_i, \ell)) + I_k H^{-1/2}(\mathbb{R}^2, \mathcal{V}(\gamma_j, \ell))$ (i.e. possible additional singularity conormal to ℓ).
- If ℓ is tangent to γ at one inflection point, then locally $\mathscr{R}f \in I_k L^2(\mathcal{L}, \mathscr{M}(\Lambda_G))$ where $\Lambda_G = \overline{N^*(G \setminus B)}$, G a cusp with cusp point B. Nonlinear composition then puts in the marked Lagrangian space J_k, which translates to regularity for f_{MA} in $I_k H^{-1/2}(\mathbb{R}^2, \mathcal{V}(\gamma, \ell))$ (again new singularity conormal to ℓ).

Future Directions

- "Surjectivity": are the C^∞-algebras used to include $F(\mathscr{R}f)$ the smallest possible? If not, then it would be good to get more quantitative description of the new artifacts, esp. in cusp case.
- There is a loss of $1/2$ derivative since in $\mathcal{L}$ we use iterated L^2 spaces, while $\mathscr{R}f$ actually belongs to iterated $H^{1/2}$ space. Would be good to improve this loss.
- Interaction of tangency of multiple inflection points or inflection and non-inflection points.
- Higher order inflection points (corresponding to higher order cusps in $\mathcal{L}$).

Thanks for your attention!