ON AN ANALOGUE OF THE MEAN VALUE THEOREM IN $\mathbb{R}^2$

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The goal of this note is to prove the following statement:

Proposition. *Let $\vec{x} : [a,b] \to \mathbb{R}^2$ be a C^1 regular curve with no self-intersections (in particular $\vec{x}(b) \neq \vec{x}(a)$). Then there exists $c \in (a,b)$ such that $\vec{x}\,'(c)$ points in the same direction as $\vec{x}(b) - \vec{x}(a)$, i.e. there exist $c \in (a,b)$ and $\alpha > 0$ such that*

$$\vec{x}\,'(c) = \alpha(\vec{x}(b) - \vec{x}(a)).$$

Note that if we allow $\vec{x}$ to have self-intersections, but still require that $\vec{x}(b) \neq \vec{x}(a)$, then we can obtain a weaker conclusion that there exists c such that $\vec{x}\,'(c)$ is parallel to $\vec{x}(b) - \vec{x}(a)$, but not necessarily pointing in the *same* direction as $(\vec{x}(b) - \vec{x}(a))$ (i.e. $\vec{x}\,'(c)$ is a possibly negative multiple of $\vec{x}(b) - \vec{x}(a)$). This follows by applying Rolle's Theorem to the function $f(t) = \vec{v} \cdot \vec{x}(t)$, where $\vec{v} \in \mathbb{R}^2 \backslash \{\vec{0}\}$ is orthogonal to $\vec{x}(b) - \vec{x}(a)$.

One can view this statement as a generalization of the usual Mean Value Theorem in 1-dimension, which can be rephrased as saying that for any two points on the graph of a C^1 function, there is some intermediate point on the graph whose tangent line is parallel to the secant line between the original two points.

Moreover, if we parametrize the graph of a function $f : \mathbb{R} \to \mathbb{R}$ by $(t, f(t))$, then in this case the tangent vector at a point where the tangent line is parallel to the secant line also points in the same direction as the secant vector, essentially because the tangent vector in this case always has a component pointing in the positive x-direction. For general curves into $\mathbb{R}^2$, this is no longer the case, and if we allow for self-intersections it is possible that any tangent vector is parallel to the secant vector must actually point in the opposite direction of the secant vector (an explicit example is given by $\vec{x}(t) = (t^2 - 1, t^3 - t)$, noting that $\vec{x}(t) - \vec{x}(-t)$ points in the positive y-direction for sufficiently large t, whereas $\vec{x}\,'(t) = (2t, 3t^2 - 1)$ points vertically only when $t = 0$, where it points in the negative y-direction).

Proof. We review some facts about cones and convex sets.

For a set E of vectors in $\mathbb{R}^2 \backslash \{0\}$, let $LC(E)$ by the linear cone generated by E, i.e.

$$LC(E) = \{t\vec{v} \,:\, t > 0, \vec{v} \in E\}$$

and let $CC(E)$ be the convex cone generated by E, i.e.

$$CC(E) = \left\{ \sum_{i=1}^{n} t_i \vec{v_i} \,:\, t_1, \ldots, t_n > 0, \vec{v_1}, \ldots, \vec{v_n} \in E \right\}.$$

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Note that by definition $CC(E)$ is convex, i.e. closed under finite convex combinations, and moreover it is the convex hull of $LC(E)$.

Lemma 1. *Suppose $G \subset \mathbb{R}^2$ is convex, and $\vec{w} : [0,1] \to G$ is continuous. Then $\int_0^1 \vec{w}(t)\,dt \in G$ as well.*

To show this, we use the so-called *convex separation* result:

Lemma 2. *Let G be convex and compact. Suppose $\vec{v_0}$ is a vector in $\mathbb{R}^2$ such that $\vec{v_0} \notin G$. Then there exists $\vec{u_0} \in \mathbb{R}^2$ such that*

$$\vec{u_0} \cdot \vec{v_0} < \min_{\vec{v} \in G} \vec{u_0} \cdot \vec{v}.$$

Proof. Note that there exists $\vec{v}^* \in G$ which minimizes the distance $\|\vec{v} - \vec{v_0}\|$ over all $\vec{v} \in G$ since the distance is continuous and G is compact. Let $\vec{u_0} = \vec{v}^* - \vec{v_0}$. Then $\vec{u_0} \neq \vec{0}$ since $\vec{v_0} \notin G$. Moreover, if $c^* = \vec{u_0} \cdot \vec{v}^*$, then I claim that $\min_{\vec{v} \in G} \vec{u_0} \cdot \vec{v} = c^*$. Indeed, if instead we had $\vec{u_0} \cdot \vec{v_1} < c^*$ for some $\vec{v_1} \in G$, then if we let

$$f(t) = \|(1-t)\vec{v}^* + t\vec{v_1} - \vec{v_0}\|^2 = \|\vec{v}^* - \vec{v_0} + t(\vec{v_1} - \vec{v}^*)\|^2$$

we have

$$\begin{aligned} f(t) &= (\vec{v}^* - \vec{v_0} + t(\vec{v_1} - \vec{v}^*)) \cdot (\vec{v}^* - \vec{v_0} + t(\vec{v_1} - \vec{v}^*)) \\ &= \|\vec{v}^* - \vec{v_0}\|^2 + 2t(\vec{v}^* - \vec{v_0}) \cdot (\vec{v_1} - \vec{v}^*) + t^2\|\vec{v_1} - \vec{v}^*\|^2 \\ &= \|\vec{v}^* - \vec{v_0}\|^2 + 2t(\vec{u_0} \cdot \vec{v_1} - \vec{u_0} \cdot \vec{v}^*) + t^2\|\vec{v_1} - \vec{v}^*\|^2. \end{aligned}$$

In particular,

$$f'(0) = 2(\vec{u_0} \cdot \vec{v_1} - \vec{u_0} \cdot \vec{v}^*) = 2(\vec{u_0} \cdot \vec{v_1} - c^*) < 0,$$

which would mean that f would *not* attain its minimum value on $[0,1]$ at $t = 0$. But that means there would exist $t_0 \in (0,1)$ such that

$$\|((1-t_0)\vec{v}^* + t_0\vec{v_1}) - \vec{v_0}\|^2 = f(t_0) < f(0) = \|\vec{v}^* - \vec{v_0}\|^2,$$

and $(1-t_0)\vec{v}^* + t_0\vec{v_1} \in G$, thus contradicting the assumption that $\vec{v}^*$ minimizes the distance to v_0 in G. Thus, we have $\vec{u_0} \cdot \vec{v} \geq c^*$ for all $\vec{v} \in G$. Since

$$\vec{u_0} \cdot \vec{v_0} = \vec{u_0} \cdot (\vec{v}^* - (\vec{v}^* - \vec{v_0})) = \vec{u_0} \cdot \vec{v}^* - \vec{u_0} \cdot \vec{u_0} = c^* - \|\vec{u_0}\|^2 < c^*,$$

it follows that $\vec{u_0} \cdot \vec{v_0} < \min_{\vec{v} \in G} \vec{u_0} \cdot \vec{v}$, as desired. □

Proof of Lemma 1. By replacing G with the convex hull of the image of $\vec{w}$ (which is compact), we may assume without loss of generality that G is compact. Note for any $\vec{u_0} \in \mathbb{R}^2$ we have $\vec{u_0} \cdot \int_0^1 \vec{w(t)}\,dt = \int_0^1 \vec{u_0} \cdot \vec{w}(t)\,dt$, and moreover

$$\min_{s \in [0,1]} \vec{u_0} \cdot \vec{w}(s) \leq \vec{u_0} \cdot \vec{w}(t)$$

for any $t \in [0,1]$. Integrating over $t \in [0,1]$ and noting the left-hand side is just a constant, we have

$$\min_{s \in [0,1]} \vec{u_0} \cdot \vec{w}(s) \leq \int_0^1 \vec{u_0} \cdot \vec{w}(t)\,dt = \vec{u_0} \cdot \left(\int_0^1 \vec{w}(t)\,dt\right).$$

This shows that we must have $\int_0^1 \vec{w}(t)\,dt \in G$, since otherwise the convex separation result would furnish an example of a vector $\vec{u_0} \in \mathbb{R}^2$ for which the above inequality would not hold. $\square$

Lemma 3. *Suppose $E = \{\vec{v}(s)\}_{s\in[c,d]}$ is the trace of some continuous path $\vec{v}$ on a compact interval $[c,d]$, and suppose there exists $\vec{v_0} \in \mathbb{R}^2$ with the property that $\vec{v_0}\cdot\vec{v} > 0$ for all $\vec{v} \in E$. Then $LC(E)$ is convex, so in particular we have*

$$CC(E) = LC(E).$$

Proof. Note that $\vec{v_0} \neq \vec{0}$, so let $\vec{v_0} = r\begin{pmatrix}\cos(\theta_0)\\ \sin(\theta_0)\end{pmatrix}$ for some $r > 0$ and $\theta_0 \in \mathbb{R}$. Since the entirety of the trace of $\vec{v}$ is contained in the half-plane $\{\vec{v_0}\cdot\vec{v} > 0\}$, there exists a continuous angle function $\theta(s)$ such that

$$\vec{v}(s) = r(s)\begin{pmatrix}\cos(\theta(s))\\ \sin(\theta(s))\end{pmatrix}$$

for all $s \in [c,d]$, where $r(s) = \|\vec{v}(s)\|$, and θ satisfies $\theta_0 - \pi/2 < \theta(s) < \theta_0 + \pi/2$. Let θ_{min} and θ_{max} be the minimum and maximum values of θ on $[c,d]$, which exist since θ is continuous and $[c,d]$ is compact. By the Intermediate Value Theorem we have

$$\theta([c,d]) = [\theta_{min}, \theta_{max}],$$

and hence

$$\begin{aligned} LC(E) = \{t\vec{v}(s) \,:\, t > 0, s \in [c,d]\} &= \left\{t'\begin{pmatrix}\cos(\theta(s))\\ \sin(\theta(s))\end{pmatrix} \,:\, t' > 0, s \in [c,d]\right\} \\ &= \left\{t'\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} \,:\, t' > 0, \theta \in [\theta_{min}, \theta_{max}]\right\}. \end{aligned}$$

Since $\theta_0 - \pi/2 < \theta_{min} \leq \theta_{max} < \theta_0 + \pi/2$, we have $\theta_{max} - \theta_{min} < \pi$, so $LC(E)$ is convex, as desired. $\square$

We now have enough to prove the proposition. Suppose $\vec{x}$ has no self-intersections. Let $\Delta = \{(s,t) \,:\, a \leq s \leq t \leq b\}$, and let $\Gamma : \Delta \to \mathbb{R}^2$ be the secant function associated to $\vec{x}$ defined by

$$\Gamma(s,t) = \begin{cases} \frac{\vec{x}(t)-\vec{x}(s)}{t-s} & s < t \\ \vec{x}\,'(t) & s = t \end{cases}.$$

Then Γ is never equal to $\vec{0}$ by assumption of $\vec{x}$ being regular and having no self-intersections. Moreover, from the Fundamental Theorem of Calculus, we have

$$\Gamma(s,t) = \int_0^1 \vec{x}\,'(s + y(t-s))\,dy. \tag{1}$$

(For $s = t$ this is clear, and for $s < t$ this follows noting that the integrand is the derivative of $y \mapsto \frac{1}{t-s}(\vec{x}(s + y(t-s)))$.) Thus, we have that Γ is continuous on Δ. If we let $\tilde{\Gamma}(s,t) = \frac{\Gamma(s,t)}{\|\Gamma(s,t)\|}$ be the associated unit secant function associated to Γ, then

$\tilde{\Gamma}$ is well-defined and continuous as a map $\tilde{\Gamma} : \Delta \to S^1$, since $\|\Gamma(s,t)\| \neq 0$ for any $(s,t) \in \Delta$. Note that $\tilde{\Gamma}(t,t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$.

Since $\tilde{\Gamma}$ is continuous on Δ, and Δ is compact, it follows that $\tilde{\Gamma}$ is uniformly continuous on Δ. Thus, we can find $\delta_0 > 0$ such that for $(s,t), (s',t') \in \Delta$ we have

$$|(s',t') - (s,t)| \leq \delta_0 \implies \|\tilde{\Gamma}(s',t') - \tilde{\Gamma}(s,t)\| < 1.$$

Note that for unit vectors $\vec{v}$, $\vec{w}$ we have

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\vec{v} \cdot \vec{w} = 2(1 - \vec{v} \cdot \vec{w})$$

we have

$$\|\vec{v} - \vec{w}\| < 1 \iff 1 - \vec{v} \cdot \vec{w} < 1/2 \iff \vec{v} \cdot \vec{w} > 1/2,$$

so in particular we have $\tilde{\Gamma}(s',t') \cdot \tilde{\Gamma}(s,t) > 1/2$ if $|(s',t') - (s,t)| \leq \delta_0$.

For $0 \leq \delta \leq b - a$, let

$$\Delta_\delta = \{(s,t) \in \Delta \, : \, t - s = \delta\}.$$

Note that Δ_0 consists of points on the diagonal $\{(t,t) \, : \, a \leq t \leq b\}$, Δ_{b-a} consists of the single point (a,b), and $\Delta = \cup_{\delta \in [0,b-a]} \Delta_\delta$. Let

$$C_\delta = LC(\tilde{\Gamma}(\Delta_\delta)).$$

Note that if ℓ is a line segment in Δ_δ with the property that there exists $(s,t) \in \Delta$ such that $|(s',t') - (s,t)| \leq \delta_0$ for all $(s',t') \in \ell$, then

$$CC(\tilde{\Gamma}(\ell)) = LC(\tilde{\Gamma}(\ell)) \subset LC(\tilde{\Gamma}(\Delta_\delta)) = C_\delta.$$

Indeed, by the uniform continuity assumption we have that $|(s',t') - (s,t)| \leq \delta_0 \implies \tilde{\Gamma}(s,t) \cdot \tilde{\Gamma}(s',t') > 1/2$ for all $(s',t') \in \ell$, so $\tilde{\Gamma}(\ell)$ satisfies the assumptions of the above lemma, and hence $CC(\tilde{(\Gamma(\ell))}) = LC(\tilde{(\Gamma(\ell))}) \subset C_\delta$.

We now show the following statement:

$$\text{if } 0 \leq \delta \leq b - a, \text{ then } C_\delta \subseteq C_0.$$

We first show this when $\delta \leq \delta_0$. It suffices to show that $\Gamma(s,t) \in C_0$ for all $(s,t) \in \Delta_\delta$ since $\Gamma(s,t)$ is a positive multiple of $\tilde{\Gamma}(s,t)$, and in fact we claim that

$$\Gamma(s,t) \in CC(\Gamma(\ell_{s,t})) = CC(\tilde{\Gamma}(\ell_{s,t})),$$

where $\ell_{s,t}$ is the line segment $\ell_{s,t} = \{(s',s') \, : \, s \leq s' \leq t\}$. Indeed, this follows from Equation (1), as we have

$$\Gamma(s,t) = \int_0^1 \vec{x}'(s + y(t-s)) \, dy = \int_0^1 \Gamma(s + y(t-s), s + y(t-s)) \, dy.$$

The path $(s + x(t-s), s + x(t-s))$ traces out precisely the line segment $\ell_{t,s}$, and hence the integral above lies in $CC(\Gamma(\ell_{s,t})) = CC(\tilde{\Gamma}(\ell_{s,t}))$. Since all points in $\ell_{s,t}$ are at most a distance δ from (s,t), and hence at most distance δ_0 from (s,t), it follows that $CC(\tilde{\Gamma}(\ell_{s,t})) \subset C_0$, as observed above. Hence, if $(s,t) \in \Delta_\delta$ with $\delta \leq \delta_0$, then $\Gamma(s,t) \in C_0$, and hence $C_\delta \subseteq C_0$.

We now show that if $\delta, \delta' > 0$ satisfy $0 < \delta - \delta' \le \delta_0$, then

$$C_\delta \subset C_{\delta'} + C_0.$$

If this is shown, then partitioning $[0, b-a]$ as $0 = \delta_1 < \cdots < \delta_n = b - a$ with $\delta_{i+1} - \delta_i < \delta_0$ for $1 \le i \le n-1$, we see by induction that $C_{\delta_i} \subseteq C_0$ for all $1 \le i \le n-1$, and hence $C_\delta \subseteq C_0$ for all $\delta \in [0, b-a]$. To show this claim, we simply rewrite

$$\begin{aligned}(t-s)\Gamma(s,t) = \vec{x}(t) - \vec{x}(s) &= (\vec{x}(t) - \vec{x}(t - (\delta - \delta'))) + (\vec{x}(t - (\delta - \delta')) - \vec{x}(s)) \\ &= (\delta - \delta')\Gamma(t - (\delta - \delta'), t) + ((t-s) - (\delta - \delta'))\Gamma(s, t - (\delta - \delta')),\end{aligned}$$

and note that if $(s,t) \in \Delta_\delta$, then $((t-s) - (\delta - \delta')) = \delta - (\delta - \delta') = \delta'$. It follows that

$$((t-s) - (\delta - \delta'))\Gamma(s, t - (\delta - \delta')) \in LC(\Gamma(\Delta_{\delta'})) = C_{\delta'},$$

and since $(t - (\delta - \delta'), t) \in \Delta_{\delta - \delta'}$ and $\delta - \delta' \le \delta_0$, we have

$$(\delta - \delta')\Gamma(t - (\delta - \delta'), t) \in LC(\Gamma(\Delta_{\delta - \delta'})) = C_{\delta - \delta'} \subseteq C_0.$$

Hence, we have $\Gamma(s,t) \in C_{\delta'} + C_0$ for any $(s,t) \in \Delta_\delta$, so $C_\delta \subseteq C_{\delta'} + C_0$, as claimed. The induction then gives $C_\delta \subseteq C_0$ for all $0 \le \delta \le b - a$, as desired.

In particular, C_0 contains C_{b-a}, and since Δ_{b-a} is just the single point (a,b), we have

$$C_{b-a} = LC(\Gamma(\Delta_{b-a})) = LC(\Gamma(a,b)) = LC(\vec{x}(b) - \vec{x}(a)).$$

Thus, $\vec{x}(b) - \vec{x}(a) \in C_0$, and since

$$C_0 = LC(\Gamma(\Delta_0)) = \{t\vec{x}'(s) \,:\, t > 0, a \le s \le b\},$$

it follows that there exists some $c \in [a,b]$ and $t > 0$ such that $t\vec{x}'(c) = \vec{x}(b) - \vec{x}(a)$, i.e. $\vec{x}'(c)$ is a positive multiple of $\vec{x}(b) - \vec{x}(a)$, as desired.

$\square$