Machinery for a Local Transversely Isotropic Elasticity Inverse Problem

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The Inverse Problem

- Consider an elastic material (modeled as a manifold (M, g)) with transverse isotropy.
  - ▶ It will have 5 elastic parameters + a direction of isotropy (possibly varying in the interior).
  - ▶ There are 3 kinds of associated waves (P, qSH, qSV).
- Suppose we know the travel times of these waves between points on the boundary. Can we uniquely recover the elastic parameters in the interior?
- In the case of full isotropy (2 elastic parameters), the answer is yes (Stefanov-Uhlmann-Vasy '17), assuming nice geometry of the material (convex foliation condition).

The isotropic recovery argument

- In isotropic case, the p and s waves propagate along geodesics with speeds c_p and c_s.
- To argue for uniqueness of p wave speed, suppose there are two wave speeds c_p and $\tilde{c}_p$ giving the same travel time data. Construct an operator $I : C^\infty(M) \to C^\infty(T^*M)$ such that $I[c_p - \tilde{c}_p] \equiv 0$ (Stefanov-Uhlmann pseudolinearization formula).
- Construct normal operator $I^*I : C^\infty(M) \to C^\infty(M)$, and attempt to invert locally: i.e. cut M by an artificial convex boundary to consider $M \cap \{x \geq -c\}$, and show that I^*I belongs to the scattering algebra $\Psi_{sc}(M \cap \{x \geq -c\})$.
- Show the operator is elliptic if c is small enough, and hence invertible up to compact error (that can also be absorbed if c is small enough).

Unfortunately, the proof doesn't quite carry over for the transversely isotropic setting:

- The normal operator $I^*I \in \Psi_{sc}(M \cap \{x \geq -c\})$ is no longer elliptic (principal symbol vanishes quadratically along the transverse isotropy axis).

However, in some cases non-elliptic operators can be inverted:

- Guillemin and Uhlmann ('78) constructed, for manifolds *without boundary* M, the paired Lagrangian calculus $I(\Lambda_0, \Lambda_1)$ associated to a pair of intersecting Lagrangian submanifolds Λ_0, Λ_1 of T^*M.
  - Can be used to invert ΨDOs of the form $\sum P_i^2 + \sum A_i P_i + B$, where $P_i \in \Psi^1(M)$, $A_i, B \in \Psi^0(M)$, assuming that $\Sigma := \cap \sigma(P_i)^{-1}(\{0\})$ is coisotropic and a condition on the coisotropic foliation of Σ. These operators are "degenerately elliptic."
  - Composes nicely with ΨDOs and FIOs, and sometimes also with other PLD's, and has a symbol map that is compatible with composition.

Goal: Constructing analogous calculus for scattering ΨDO

Scattering calculus $\Psi_{sc}(X)$ is (an) analogue of the ΨDO calculus on manifolds *with boundary* X. Thus the goal now is to construct the analogue of the PLD calculus on manifolds with boundary.

- Ideally the calculus constructed will compose well with scattering ΨDO, and have a symbol map describing this composition, so that the parametrix construction can be easily described in terms of symbols.
- There will be additional technical considerations: e.g. a parametrix for a sc-ΨDO will be a distribution on $X \times X$, so one will be careful how the Lagrangians of $X \times X$ interact with the boundary/corner.

The non-elliptic operator in the transversely isotropic problem is of "degenerately elliptic" form, so constructing such a calculus should hopefully invert this operator, up to controllable errors.