Partial Global Recovery in the Elastic Travel Time Tomography Problem for Transversely Isotropic Media

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Transversely Isotropic Elasticity and Elastic Waves

We consider the travel time tomography problem for transversely isotropic elastic media. The context is the linear elastic wave equation

$$u_{tt} - Eu = 0, \quad u : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3, \quad (Eu)_i = \rho^{-1} \sum_{jkl} \partial_j (c_{ijkl} \partial_l u_k).$$

The principal symbol of the elastic wave operator is a matrix given by $-\tau^2 \mathrm{Id} + \sigma(-E)(x, \xi)$, with $\sigma(-E)(x, \xi)$ symmetric and positive definite for all (x, ξ). Thus by elliptic regularity the wavefront set of u must be contained in the set of $(t, x, \tau, \xi) \in T^*(\mathbb{R}_t \times \mathbb{R}^3_x)$ where $-\tau^2 \mathrm{Id} + \sigma(-E)(x, \xi)$ is not invertible, i.e. the points where $\tau^2 = G_j(x, \xi)$ where $G_j(x, \xi)$ $(j = 1, 2, 3)$ are the eigenvalues of $\sigma(-E)(x, \xi)$. Furthermore, a classical propagation of singularities result states that within $\{\tau^2 = G_j(x, \xi)\}$ the singularities of u will be invariant under the Hamilton flow of $\tau^2 - G_j(x, \xi)$ (assuming constant multiplicity of the eigenvalues). Note that if G is the dual metric function of some metric g, then the Hamilton flow of $\frac{1}{2}(\tau^2 - G)$ is exactly the geodesic flow with respect to g, and the singularities would propagate in the same manner as the singularities for the scalar wave equation $u_{tt} - \Delta_g u = 0$. Thus, the Hamiltonian dynamics with respect to the Hamiltonian $\tau^2 - G_j$ describe the dynamics of the *elastic waves*, with G_j called the wave speeds.

The elasticity tensor a priori has 21 independent components. In the case of *transversely isotropic* elasticity there is an *axis of isotropy* (denoted $\overline{\xi}(x)$) around which the material behaves isotropically, and furthermore the independence is cut down to 5 independent components ("material parameters"), labeled a_{11}, a_{33}, a_{55}, a_{66}, and E^2 (with $E^2 = (a_{11} - a_{55})(a_{33} - a_{55}) - (a_{13} + a_{55})^2$ in some notations). Then the wave speeds are given by $G_{qSH}(x, \xi) = a_{66}(x)|\xi'|^2 + a_{55}(x)\xi_3^2$ and

$$G_\pm(x, \xi) = (a_{11} + a_{55})|\xi'|^2 + (a_{33} + a_{55})\xi_3^2 \pm \sqrt{((a_{11} - a_{55})|\xi'|^2 + (a_{33} - a_{55})\xi_3^2)^2 - 4E^2|\xi'|^2\xi_3^2}$$

where $+$ refers to the qP wave speed and $-$ refers to the qSV wave speed, and $|\xi'|^2 = \xi_1^2 + \xi_2^2$. The coordinates are taken so that $\overline{\xi}(x)$ aligns with the dx_3 axis.

Figure 1: An example of transversely isotropic elasticity. The normal vector to the rock layers provides the axis of isotropy. Picture from Wikipedia.

The Inverse Problem

We study the travel time tomography problem for transversely isotropic elasticity: does knowledge of the travel times of the wave speeds determine the material parameters?

In [1], the authors showed that the axis of isotropy $\overline{\xi}$ and the parameters a_{55} and a_{66} can be recovered from the qSH wave speed, assuming that the kernel of the axis of isotropy $\overline{\xi}$ is an integrable hyperplane distribution, i.e. $\overline{\xi}$ is a smooth multiple of some closed 1-form df, as well as geometric conditions such as a "convex foliation" condition. We thus assume that a_{55}, a_{66}, and the axis are known, and we focus on recovering a_{11}, a_{33}, and E^2 from the qP and qSV wave speeds.

Pseudolinearization formula

We will make use of the *Stefanov-Uhlmann pseudolinearization formula*, which has been used to solve problems in boundary rigidity. The formula says the following: given two vector fields V and $\tilde{V}$, and given their corresponding flows $X(t, x)$ and $\tilde{X}(t, x)$, we have

$$\tilde{X}(t, x) - X(t, x) = \int_0^t \frac{\partial \tilde{X}}{\partial x}(t - s, X(s, x)) \cdot (\tilde{V} - V)|_{X(s,x)} \, ds.$$

We apply the result to V and $\tilde{V}$ corresponding to the Hamilton flow of G, where G is either the qP or qSV wave speed. If (a_{11}, a_{33}, E^2) and $(\tilde{a_{11}}, \tilde{a_{33}}, \tilde{E^2})$ are two sets of coefficients giving the same travel time data, and we let $r_\nu = \tilde{a_\nu} - a_\nu$ for ν one of the parameters, then we have

$$\vec{0} = \sum_\nu I^\nu[\nabla r_\nu](x, \xi) + \tilde{I}^\nu[r_\nu](x, \xi)$$

for all (x, ξ), where $I^\nu : C_c^\infty(\mathbb{R}^3; \mathbb{C}^3) \to C^\infty(T^*\mathbb{R}^3; \mathbb{C}^3)$ and $\tilde{I}^\nu : C_c^\infty(\mathbb{R}^3; \mathbb{C}) \to C^\infty(T^*\mathbb{R}^3; \mathbb{C}^3)$ are certain matrix-weighted ray transforms (we identify a point in $T^*\mathbb{R}^3$ with the Hamilton bicharacteristic through that point). If we can analyze the operators I^ν and $\tilde{I}^\nu$, then we may be able to use the pseudolinearization formula to show that $r_\nu \equiv 0$ for each ν, i.e. the two sets of parameters were actually the same.

We compose with a formal adjoint L to have the operators map from functions on $\mathbb{R}^3$ back to functions on $\mathbb{R}^3$. This L has the form $Lv(x) = \int_{\mathbb{S}^2} \chi(x, \omega) B(x, \omega) v(x, \xi(\omega)) \, d\omega$, where χ is smooth and $\xi(\omega)$ denotes the inverse of the Hamilton map $\xi \mapsto \partial_\xi G$ (so that the Hamilton bicharacteristic with momentum $\xi(\omega)$ has x-tangent vector ω), and B is a matrix to make the composed operators principally scalar. We will take our χ to vanish outside a small neighborhood of the equatorial sphere $\overline{\xi} \cap \mathbb{S}^2$ and to be identically one in a smaller neighborhood. Write $N_\pm^\nu = L \circ I_\pm^\nu$ (ν for the parameter in question, $\pm$ for the qP/qSV wave speed).

Symbol estimates

It turns out all operators in question are -1 order (classical) pseudodifferential operators (ΨDO). Furthermore N_+^{11} is elliptic, while for $N = N_+^{33}, N_\pm^{E^2}$ we can write their left-reduced symbols in the form

$$\sigma_L(N)(x, \xi) = |\xi|^{-3}(p_2(x, \xi) + ip_1(x, \xi) + P_1(x, \xi) + P_0(x, \xi))$$

where $p_i, P_i \in S^i(\mathbb{R}^3)$, p_2 is nonnegative and vanishes only on $\Sigma = \operatorname{span} \overline{\xi}$, where it vanishes nondegenerately, p_1 is real and nonzero on Σ, and P_1 vanishes on Σ. To show this, use oscillatory testing to write the symbol of the operator as an oscillatory integral of the form

$$\sigma_L(N)(x, \zeta) = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} e^{i(X(t, x, \xi(\omega)) - x) \cdot \zeta} C(x, t, \omega) \, dt \, d\omega$$

for some matrix-valued function $C(x, t, \omega)$, and use the stationary phase asymptotic expansion up to order $j = 1$ terms.

These operators are thus not elliptic; however the expression $p_2 + ip_1$ is of "parabolic type": for example, the heat operator $\partial_t - \Delta$ has symbol $|\xi|^2 + i\tau$ which is of the above form. In this case its inverse will satisfy $(1/2, 0)$ symbol estimates:

$$\left| D_x^\alpha D_\xi^\beta \left(\frac{1}{p_2 + ip_1} \right) \right| \le C_{\alpha,\beta} |\xi|^{-1 - \frac{|\beta|}{2}} \quad \text{for } |\xi| \ge 1.$$

Stability and Injectivity Results

From symbol estimates, especially those for the "parabolic type" operators, we can "recover" two of the three parameters (assuming the other is known) as follows: for suitable χ, for a_{11} and E^2, if we let $f = \begin{pmatrix} N_+^{11} & N_+^{E^2} \\ N_-^{11} & N_-^{E^2} \end{pmatrix} \begin{pmatrix} \nabla r_{11} \\ \nabla r_{E^2} \end{pmatrix} + \begin{pmatrix} \tilde{N}_+^{11} & \tilde{N}_+^{E^2} \\ \tilde{N}_-^{11} & \tilde{N}_-^{E^2} \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{E^2} \end{pmatrix}$, then we have the stability estimate

$$\|\nabla(r_{11}, r_{E^2})\|_{L^2} \le C(\|f\|_{H^2} + \|(r_{11}, r_{E^2})\|_{H^{1/2}}).$$

In particular, if a_{33} is known (so then $f = 0$), and if we can a priori control $\|(r_{11}, r_{E^2})\|_{H^{1/2}}$ by $\|\nabla(r_{11}, r_{E^2})\|_{L^2}$ (e.g. by Poincaré's inequality if the differences are known to be supported on sets of small width), then we can conclude $\nabla(r_{11}, r_{E^2}) \equiv 0 \implies r_{11}, r_{E^2} \equiv 0$, i.e. we have injectivity. A similar result holds for a_{33} and E^2 if we know a_{11}.

We can also consider the case where there is a known functional relationship between the coefficients (e.g. a_{11} is a known function of a_{33} and E^2). Then in all three possible functional relationship cases, we obtain similar stability/injectivity results as above.

References

[1] M. V. de Hoop, G. Uhlmann, and A. Vasy, "Recovery of material parameters in transversely isotropic media," 2019. arXiv: `1902.09394 [math.AP]`.

[2] Y. Zou, "Partial global recovery in the elastic travel time tomography problem for transversely isotropic media," In progress.