The Travel Time Tomography Inverse Problem for Transversely Isotropic Elastic Media

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The Elastic Wave Equation

- Consider the linear elastic wave equation

$$u_{tt} - Eu = 0, \quad u : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3, \quad (Eu)_i = \rho^{-1} \sum_{jkl} \partial_j (c_{ijkl} \partial_l u_k).$$

- The tensor c_{ijkl} satisfies the symmetries $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$.
- The principal symbol of the elastic wave operator is a matrix given by $-\tau^2 \mathrm{Id} + \sigma(-E)(x, \xi)$, with

$$\sigma(-E)(x, \xi) = \left(\sum_{j,l} \frac{c_{ijkl}(x)}{\rho(x)} \xi_j \xi_l \right)_{ik}$$

symmetric and positive definite for all (x, ξ).

Elastic Waves

- The wavefront set of u must be contained in the set of $(t, x, \tau, \xi) \in T^*(\mathbb{R}_t \times \mathbb{R}^3_x)$ where $-\tau^2 \mathrm{Id} + \sigma(-E)(x, \xi)$ is not invertible, i.e. the points where $\tau^2 = G_j(x, \xi)$ where $G_j(x, \xi)$ ($j = 1, 2, 3$) are the eigenvalues of $\sigma(-E)(x, \xi)$.
- A classical propagation of singularities result states that within $\{\tau^2 = G_j(x, \xi)\}$ the singularities of u will be invariant under the Hamilton flow of $\tau^2 - G_j(x, \xi)$ (assuming constant multiplicity of the eigenvalues).
- Note that if G is the dual metric function of some metric g, then the Hamilton flow of $\frac{1}{2}(\tau^2 - G)$ restricted to $\{\tau = 1\}$ is exactly the geodesic flow with respect to g, and the singularities would propagate in the same manner as the singularities for the scalar wave equation $u_{tt} - \Delta_g u = 0$.
- Thus, the Hamiltonian dynamics with respect to the Hamiltonian $\tau^2 - G_j$ describe the dynamics of the *elastic waves*, with G_j called the wave speeds.

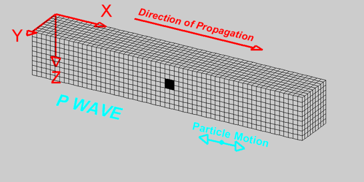

Figure 1: P wave propagation

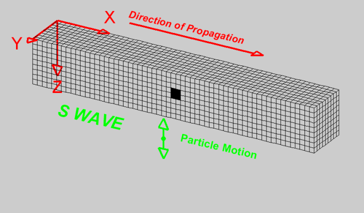

Figure 2: S wave propagation

Animation courtesy of L. Braile, Purdue University: `https://web.ics.purdue.edu/~braile/edumod/waves/WaveDemo.htm`

The Inverse Problem

We study the travel time tomography problem for transversely isotropic elasticity: given a bounded domain $\Omega \subset \mathbb{R}^3$, suppose we know the travel times of the elastic waves between any two points on the boundary.

Problem

Does knowledge of the travel times of the wave speeds determine the elasticity tensor?

(i.e. the injectivity problem)

In general, too many independent components to consider: focus first on simpler (i.e. more symmetric) cases.

Isotropic Elasticity

- The elasticity tensor a priori has up to 81 independent components (4-tensor in 3-d); however inherent symmetries reduce this to 21 components.
- In the case of (fully) *isotropic* elasticity this is reduced to just 2, often denoted by the Lamé parameters λ and μ.
- The wave speeds (eigenvalues) can be explicitly computed:

$$G_P(x,\xi) = \frac{\lambda(x) + 2\mu(x)}{\rho(x)}|\xi|^2$$

$$G_S(x,\xi) = \frac{\mu(x)}{\rho(x)}|\xi|^2 \qquad \text{(multiplicity 2)}$$

- Note that the eigenvalues are (dual) Riemannian metric functions on $T^*\mathbb{R}^3$: thus Hamilton trajectories are geodesics, and knowledge of travel times is the same as knowledge of geodesic lengths w.r.t this metric.
- Hence injectivity of travel time data for isotropic elasticity would follow from boundary rigidity results in Riemannian geometry.
- Can use the work of Stefanov-Uhlmann-Vasy on boundary rigidity ('17) to prove injectivity for isotropic elasticity, assuming geometric conditions such as the "convex foliation" condition.
- More generally, one can use the Dirichlet-to-Neumann (DN) map to recover the Lamé parameters.

Transversely Isotropic Elasticity

- Here there is a pointwise axis of isotropy around which the material behaves isotropically.
- This models rock formations in the Earth (and other planets), as well as the brain.

Figure 3: An example of transversely isotropic elasticity. The normal vector to the rock layers provides the axis of isotropy. Picture from Wikipedia.

- We denote the axis of isotropy by $\bar{\xi}$ (viewed as a covector field). There are additionally 5 independent components ("material parameters"), denoted a_{11}, a_{13}, a_{33}, a_{55}, a_{66} (Voigt notation).
- The wave speeds are labeled G_{qP}, G_{qSH}, G_{qSV}, with

$$\begin{aligned} G_{qSH}(x,\xi) &= a_{66}|\xi'|^2 + a_{55}\xi_3^2 \\ G_{\pm}(x,\xi) &= (a_{11}+a_{55})|\xi'|^2 + (a_{33}+a_{55})\xi_3^2 \\ &\qquad \pm \sqrt{((a_{11}-a_{55})|\xi'|^2 + (a_{33}-a_{55})\xi_3^2)^2 - 4E^2|\xi'|^2\xi_3^2} \end{aligned}$$

where[1] $E^2 = (a_{11}-a_{55})(a_{33}-a_{55}) - (a_{13}+a_{55})^2$, $+$ refers to the qP wave speed and $-$ refers to the qSV wave speed, and $|\xi'|^2 = \xi_1^2 + \xi_2^2$. The coordinates are taken so that $\bar{\xi}(x)$ aligns with the dx_3 axis.

[1] Despite the notation, we don't necessarily have $E^2 \geq 0$.

Previous results in T.I. Elasticity

- Mazzucato-Rachele '07 showed that the DN map determined the travel times assuming the "disjoint mode" assumption, and also showed that the qSH travel times determined two material parameters and the axis of isotropy whenever boundary rigidity assumptions hold.
- de Hoop-Uhlmann-Vasy '19 also showed recovery of two material parameters (a_{55} and a_{66}) and the axis, assuming the convex foliation condition as well as the assumption that $\ker \overline{\xi}$ is an integrable hyperplane distribution (i.e. $\overline{\xi}$ is a smooth multiple of a closed form). They also showed recovery of the remaining parameters assuming that a_{33} and E^2 are known functions of a_{11}.

From now we assume that the axis is known and satisfies the assumption above, and that a_{55} and a_{66} are known as well.
We also assume $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, and that the boundary is strictly convex with respect to either qP or qSV Hamiltonian dynamics. Assume as well that G_{qP} and G_{qSV} are strictly convex in the fiber variable, and that the Hamiltonian dynamics do not have "conjugate points".
For ν one of the parameters a_{11}, a_{33}, or E^2, suppose $\{a_\nu\}$ and $\{\tilde{a}_\nu\}$ are two sets of parameters, and let $r_\nu = \tilde{a}_\nu - a_\nu$, with r_ν supported in Ω.

Results

With these assumptions:

Theorem (Z. '19)

Suppose for $\nu = a_{11}$, a_{33}, or E^2 that the other parameters are known. Then we have

$$\|\nabla r_\nu\|_{L^2} \leq C\|r_\nu\|_{H^{1/2}}$$

if $\nu = a_{11}$ or a_{33} and the qP travel times agree, or if $\nu = E^2$ and either the qP or the qSV travel times agree. In particular, if r_ν is known to be supported in a set of sufficiently small width, then $r_\nu \equiv 0$, i.e. we can recover a_ν among parameters differing in sets of small width.

Remark: A priori assumptions on the support of r_ν are natural in time-lapse monitoring problems, where one wishes to keep track of elastic changes in a relatively small "reservoir" region, outside of which the elasticity can be assumed to remain constant.

With the same assumptions as before:

Theorem (Z. '19)

Suppose a_{11} is known. If the qP and the qSH travel times agree, then we have

$$\|(\nabla r_{33}, \nabla r_{E^2})\|_{L^2} \leq C\|(r_{33}, r_{E^2})\|_{H^{1/2}}.$$

In particular, if (r_{33}, r_{E^2}) is known to be supported in a set of sufficiently small width, then $r_{33}, r_{E^2} \equiv 0$, i.e. we can recover a_{33} and E^2 among parameters differing in sets of small width. Similarly if a_{33} is known and we try to recover (a_{11}, E^2).

With the same assumptions as before:

Theorem (Z. '19)

Suppose there is a known functional relationship $a_{33} = f(a_{11}, E^2)$ *with* $\frac{\partial f}{\partial a_{11}} \geq 0$. *If the qP and the qSH travel times agree, then we have*

$$\|(\nabla r_{11}, \nabla r_{E^2})\|_{L^2} \leq C\|(r_{11}, r_{E^2})\|_{H^{1/2}}.$$

Similar results hold if we have a functional relationship $E^2 = f(a_{11}, a_{33})$ *with* $\left|\frac{\partial f}{\partial a_{33}}\right| > 0$, *or* $a_{11} = f(a_{33}, E^2)$ *with the derivatives* $\frac{\partial f}{\partial a_{33}}$ *and* $\frac{\partial f}{\partial E^2}$ *constant and* $\frac{\partial f}{\partial a_{33}} \neq 0$.

In particular, if the differences are known to be supported in a set of sufficiently small width, then $r_\nu \equiv 0$, *i.e. we can recover* a_ν *among parameters differing in sets of small width.*

Travel time vs. lens data

We will work with the *lens* data: in addition to the travel times between any pair of points on the boundary, we suppose we also know the incoming and exiting (co)vectors of the corresponding Hamilton trajectory.

Lemma

If Ω has strictly convex smooth boundary, and every pair of points on the boundary are connected by a unique Hamilton trajectory, and the material parameters' values (and hence the Hamiltonian) is known at the boundary, then the travel time data determines the lens data.

Proof sketch: the exiting covector on the trajectory from x_0 to x_1 equals $d\tau_{x_0}|_{x_1}$ where $\tau_{x_0} = \operatorname{dist}(x_0, \cdot)$. (Generalization of argument first presented in Michel '81 on boundary rigidity.)

Stefanov-Uhlmann Pseudolinearization

Used to solve problems in boundary rigidity.
Given two vector fields V and $\tilde{V}$, and given their corresponding flows $Z(t,z)$ and $\tilde{Z}(t,z)$, we have

$$\tilde{Z}(t,z) - Z(t,z) = \int_0^t \frac{\partial \tilde{Z}}{\partial z}(t-s, Z(s,z)) \cdot (\tilde{V} - V)|_{Z(s,z)}\, ds.$$

The proof follows from an application of the fundamental theorem of calculus to the function $s \mapsto \tilde{Z}(t-s, Z(s,z))$.

We apply the result to V and $\tilde{V}$ corresponding to the Hamilton vector fields of G, where G is either the qP or qSV wave speed. Note

$$\tilde{V} - V = -\partial_x(\tilde{G} - G) \cdot \partial_\xi + \partial_\xi(\tilde{G} - G) \cdot \partial_x$$

and also that we can write

$$(\tilde{G} - G) = \sum_\nu E^\nu r_\nu, \quad E^\nu(x, \xi) = \int_0^1 \frac{\partial G}{\partial \nu}(a_{11} + sr_{11}, a_{33} + sr_{33}, E^2 + sr_{E^2}; \xi)\, ds$$

(so e.g. for r_ν small we have $E^\nu(x, \xi) \approx \frac{\partial G}{\partial \nu}(a_\nu(x); \xi)$).

$$\tilde{V} - V = -\left(\sum_\nu E^\nu \nabla r_\nu + \partial_x E^\nu r_\nu\right) \cdot \partial_\xi + \left(\sum_\nu \partial_\xi E^\nu r_\nu\right) \cdot \partial_x.$$

Substituting this into the Stefanov-Uhlmann pseudolinearization formula and keeping the bottom three rows (i.e. the rows corresponding to $\frac{\partial \tilde{\Xi}}{\partial(x,\xi)}$) thus gives

$$\vec{0}_3 = \sum_\nu I^\nu[\nabla r_\nu](x,\xi) + \tilde{I}^\nu[r_\nu](x,\xi)$$

for all (x,ξ), where

$$I^\nu[f_1,f_2,f_3](x,\xi) = -\int_{\mathbb{R}} E^\nu(X(t),\Xi(t)) \frac{\partial \tilde{\Xi}}{\partial \xi}(\tau(X(t),\Xi(t)),(X(t),\Xi(t))) \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} (X(t))\, dt$$

and

$$\tilde{I}^\nu[f](x,\xi) = \int_{\mathbb{R}} \left(-\partial_x E^\nu \frac{\partial \tilde{\Xi}}{\partial \xi}(\tau(\cdot),\cdot) + \partial_\xi E^\nu \frac{\partial \tilde{\Xi}}{\partial x}(\tau(\cdot),\cdot) \right) (X(t),\Xi(t)) f(X(t))\, dt$$

with $(X(t),\Xi(t)) = (X(t,x,\xi),\Xi(t,x,\xi))$ (i.e. the trajectory corresponding passing through (x,ξ)). In other words, we have constructed operators I^ν and $\tilde{I}^\nu$, *which depend on the unknown parameters a_ν and $\tilde{a}_\nu$*, for which the differences r_ν satisfy a linear equation.

Formal adjoint

We compose with a formal adjoint L to have the operators map from functions on $\mathbb{R}^3$ back to functions on $\mathbb{R}^3$. This L has the form

$$Lv(x) = \int_{\mathbb{S}^2} \chi(x,\omega) B(x,\omega) v(x,\xi(\omega))\, d\omega,$$

where χ is smooth, B is smooth and matrix-valued, and $\xi(\omega)$ denotes the inverse of the Hamilton map $\xi \mapsto \partial_\xi G$ (so that the Hamilton bicharacteristic with momentum $\xi(\omega)$ has x-tangent vector ω).
We will take our χ to vanish outside a small neighborhood of the equatorial sphere $\overline{\xi} \cap \mathbb{S}^2$ and to be identically one in a smaller neighborhood, and choose B to make the composed operators principally scalar.

Write $N_\pm^\nu = L \circ I_\pm^\nu$ (ν for the parameter in question, $\pm$ for the qP/qSV wave speed).

Stability Estimates

We thus have

$$\vec{0}_3 = \sum_{\nu} N^{\nu}_{\pm}[\nabla r_{\nu}] + \tilde{N}^{\nu}_{\pm}[r_{\nu}].$$

For the one parameter recovery problem, it suffices to show a stability estimate of the following form:

Theorem

For $u \in H^1(\mathbb{R}^3)$ with compact support, we have

$$\|\nabla u\|_{L^2} \leq C\left(\|f\|_{H^2} + \|u\|_{H^{1/2}}\right)$$

where $f = N^{\nu}_{\pm}\nabla u + \tilde{N}^{\nu}_{\pm} u$.

Stability Estimates

We thus have

$$\vec{0}_3 = \sum_{\nu} N_{\pm}^{\nu}[\nabla r_\nu] + \tilde{N}_{\pm}^{\nu}[r_\nu].$$

Similarly for the two parameters recovery problem (say recovering a_{11} and E^2 knowing a_{33}), it suffices to show a stability estimate of the following:

Theorem

For $u_1, u_2 \in H^1(\mathbb{R}^3)$ with compact support, we have

$$\|\nabla(u_1, u_2)\|_{L^2} \leq C\left(\|f\|_{H^2} + \|(u_1, u_2)\|_{H^{1/2}}\right)$$

where $f = \begin{pmatrix} N_{+}^{11} & N_{+}^{E^2} \\ N_{-}^{11} & N_{-}^{E^2} \end{pmatrix} \begin{pmatrix} \nabla u_1 \\ \nabla u_2 \end{pmatrix} + \begin{pmatrix} \tilde{N}_{+}^{11} & \tilde{N}_{+}^{E^2} \\ \tilde{N}_{-}^{11} & \tilde{N}_{-}^{E^2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$

Similarly for the functional relationship recovery problem.

Symbol calculation

To show the estimates, we show $N^{\nu}_{\pm}$ and $\tilde{N}^{\nu}_{\pm}$ are (matrix-valued) ΨDOs of order -1, and we compute (qualitatively) some of their symbols:

Proposition

For $\nu = 11, 33, E^2$, we have that $N^{\nu}_{\pm}$ and $\tilde{N}^{\nu}_{\pm}$ are matrix-valued ΨDOs of order -1, with $N^{\nu}_{\pm}$ having scalar-valued (i.e. multiples of the identity) principal symbols. Furthermore N^{11}_{+} is elliptic, while for $N = N^{33}_{+}, N^{E^2}_{\pm}$ we can write their left-reduced symbols in the form

$$\sigma_L(N)(x,\xi) = |\xi|^{-3}(p_2(x,\xi) + ip_1(x,\xi) + P_1(x,\xi) + P_0(x,\xi))$$

where $p_i, P_i \in S^i(T^*\mathbb{R}^3)$, p_2 is nonnegative and vanishes only on $\Sigma = span\ \overline{\xi}$, where it vanishes nondegenerately quadratically, p_1 is real and nonzero on Σ (assuming certain geological assumptions), and P_1 vanishes on Σ. Finally, all operators $\tilde{N}^{\nu}_{\pm}$ except $\tilde{N}^{11}_{\pm}$ have (matrix-valued) principal symbols which vanish on Σ.

To show the statement, we can use oscillatory testing to write the symbol as an oscillatory integral. Then by stationary phase we have

$$\sigma(N^{\nu}_{\pm})(x,\xi) = a_{-1}|\xi|^{-1} + a_{-2}|\xi|^{-2} + \dots$$

where

$$a_{-1} = 2\pi \int_{\xi^{\perp}\cap\mathbb{S}^2} \chi(x,\omega) E^{\nu}_{\pm}(x,\xi(\omega))\, d\mathbb{S}^1(\omega).$$

For E^{11}_{-}, $E^{33}_{\pm}$, and $E^{E^2}_{\pm}$, we have $E^{\nu}_{\pm}(x,\xi) = f^{\nu}_{\pm}(x,\xi)\cdot(\xi\cdot\overline{\xi}(x))^2$ with $f^{\nu}_{\pm}$ having a definite sign, and $\xi(\omega)\cdot\overline{\xi} = 0 \iff \overline{\xi}$ annihilates ω.
Thus the integral, taken over some circle $\mathbb{S}^1$, is nonzero unless that circle is $\overline{\xi}^{\perp}\cap\mathbb{S}^2$; hence the principal symbol is nonvanishing except along Σ, where it vanishes quadratically.
The term a_{-2} can be computed via stationary phase as well. Under reasonable geological assumptions this term is never vanishing near Σ.

Parabolic parametrices: Boutet de Monvel's calculus

A prototypical example of an operator with symbol of the form $p_2 + ip_1$ where $p_2 \in S^2$ vanishes quadratically on a fiber-dimension 1 subset and p_1 is real-valued and is elliptic on that subset is the heat operator: $\partial_t - \Delta$ has symbol $|\xi|^2 + i\tau$ which is of the above form. In this case its inverse will satisfy $(1/2, 0)$ symbol estimates:

$$\left| D^{\beta}_{(\xi,\tau)} D^{\alpha}_{x} \left(\frac{1}{|\xi|^2 + i\tau} \right) \right| \leq C|(\xi,\tau)|^{-1-|\beta|/2}.$$

This is enough to prove parabolic regularity results.
We use a symbol calculus first developed by Boutet de Monvel '74 which generalizes the above observation.

To do so, choose local coordinates such that $\Sigma = \text{span } dx_n = \{\xi' = 0\} \subset T^*\mathbb{R}^n$ where $\xi' = (\xi_1, \ldots, \xi_{n-1})$, and let $d_\Sigma = \left(\frac{|\xi'|^2}{|\xi|^2} + \frac{1}{|\xi|}\right)^{1/2}$.

Definition

Let $m, k \in \mathbb{R}$. The space $S^{m,k}(T^*M, \Sigma)$ is the set of all $a \in C^\infty(T^*M; \mathbb{C})$ satisfying the property that whenever $W^\alpha = W^{\alpha_1} \ldots W^{\alpha_{|\alpha|}}$ is a product of vector fields on $T^*M \backslash o$ homogeneous of degree 0, and $V^\beta = V^{\beta_1} \ldots V^{\beta_{|\beta|}}$ is a product of vector fields hom. of degree 0 *and* tangent to Σ, that we have the local estimate

$$|W^\alpha V^\beta a| \leq C|\xi|^m d_\Sigma^{k-|\alpha|}.$$

Roughly speaking $S^{m,k}$ are symbols of order m whose principal part vanishes to order k on Σ, with the subprincipal symbols of order less than $k/2$ lower also vanishing as well.

Some properties of this symbol calculus:

- For $k \leq 0$, we have $S^{m,k}(T^*M, \Sigma) \subset S^{m-k/2}_{1/2}(T^*M)$.
- If $a \in S^m_{1,0}(T^*M)$ and $\sigma_m(a)$ vanishes to order k on Σ $(k = 0, 1, 2)$, then $a \in S^{m,k}(T^*M, \Sigma)$.
- If $a \in S^{m,k}(T^*M, \Sigma)$ and $|a| \geq c|\xi|^m d_\Sigma^k$, then $a^{-1} \in S^{-m,-k}(T^*M, \Sigma)$.
- The symbol class is invariant under diffeomorphisms.

We can quantize the symbol calculus to an operator calculus $\Psi^{m,k}(M, \Sigma)$. Some properties:

- The operator class is also invariant under diffeomorphisms, and one can associate a principal symbol to operators in $\Psi^{m,k}(M, \Sigma)$.
- The calculus is closed under composition, with the principal symbol of the composition the product of the principal symbols.

Proof of Stability Estimate (one parameter)

For example, suppose a_{11} and E^2 is known, and we aim to recover a_{33} from the qP travel times. Recall we want to prove

$$\|\nabla u\|_{L^2} \leq C\left(\|f\|_{H^2} + \|u\|_{H^{1/2}}\right)$$

where $f = N_+^{33}\nabla u + \tilde{N}_+^{33} u$.

Proof of Stability Estimate (one parameter)

From symbol calculations, we have $N_+^{33} \in \Psi^{-1,2}(\mathbb{R}^3, \Sigma)$ and $\tilde{N}_+^{33} \in \Psi^{-1,1}(\mathbb{R}^3, \Sigma)$. Moreover, N_+^{33} is "elliptic" in this symbol class. Thus $q = \sigma(N_+^{33})^{-1} \in S^{1,-2}(T^*\mathbb{R}^3, \Sigma)$ quantizes to $Q \in \Psi^{1,-2}(\mathbb{R}^3, \Sigma)$ with $Q \circ N_+^{33} = I + R$, $R \in \Psi^{-1,-1}(\mathbb{R}^3, \Sigma) \otimes \mathrm{Mat}_{3\times 3}(\mathbb{C})$. Applying Q to both sides of $f = N_+^{33}\nabla u + \tilde{N}_+^{33} u$ gives

$$Qf = \nabla u + R[\nabla u] + (Q \circ \tilde{N}_+^{33})u.$$

Note $Q \in \Psi^2_{1/2}(\mathbb{R}^3)$, $R \in \Psi^{-1/2}_{1/2}(\mathbb{R}^3) \otimes \mathrm{Mat}_{3\times 3}(\mathbb{C})$, and $Q \circ \tilde{N}_+^{33} \in \Psi^{0,-1}(\mathbb{R}^3, \Sigma) \otimes \mathrm{Mat}_{3\times 3}(\mathbb{C}) \subset \Psi^{1/2}_{1/2}(\mathbb{R}^3) \otimes \mathrm{Mat}_{3\times 3}(\mathbb{C})$, so

$$\begin{aligned}
\|\nabla u\|_{L^2(\mathbb{R}^n)} &\le \|Qf\|_{L^2(\mathbb{R}^n)} + \|R[\nabla u]\|_{L^2(\mathbb{R}^n)} + \|(Q \circ \tilde{N})u\|_{L^2(\mathbb{R}^n)} \\
&\le C(\|f\|_{H^2(\mathbb{R}^n)} + \|\nabla u\|_{H^{-1/2}(\mathbb{R}^n)} + \|u\|_{H^{1/2}(\mathbb{R}^n)}) \\
&\le C(\|f\|_{H^2(\mathbb{R}^n)} + \|u\|_{H^{1/2}(\mathbb{R}^n)}).
\end{aligned}$$

Future directions

- The argument is made globally, and to have injectivity we need global assumption that the differences a priori have small width of support.
- Would like to make "local" argument in the style of de Hoop-Uhlmann-Vasy '19, using the scattering calculus
  - One challenge is that the corresponding scattering operator will have subprincipal symbol vanishing at the boundary. Thus we can't directly apply this calculus.
  - Note that the Boutet de Monvel calculus essentially "blows up Σ at fiber infinity parabolically"; possible adaptation is to perform further blowup at the spatial boundary.

Thanks for your attention!