Feedback

- ROBOT-1:
  - Everyone did well at starting up and shutting down the robot
  - It is important to read the RMF and SWP before using the robots
  - Knowing where to find information in the standards is more important than memorizing sections of them

- ROBOT-2:
  - Available on Moodle.
  - To be marked in Week 4, but you can try in week 3 if prefer.

MTRN4230

Robotics

Lecture 3

Coordinate Frames & Homogeneous Transformations

Hoang-Phuong **Phan** – T2 2023

Lecture 2 - Revision

https://kahoot.it/

✓ Sensors & Actuators

- Electric, Hydraulic, Pneumatic
- Micro actuators
- Position, velocity, acceleration
- Tactile sensors

✓ Computer vision

- Colour masking using thresholds
- Perspective transformation

Lecture 2 - Continue

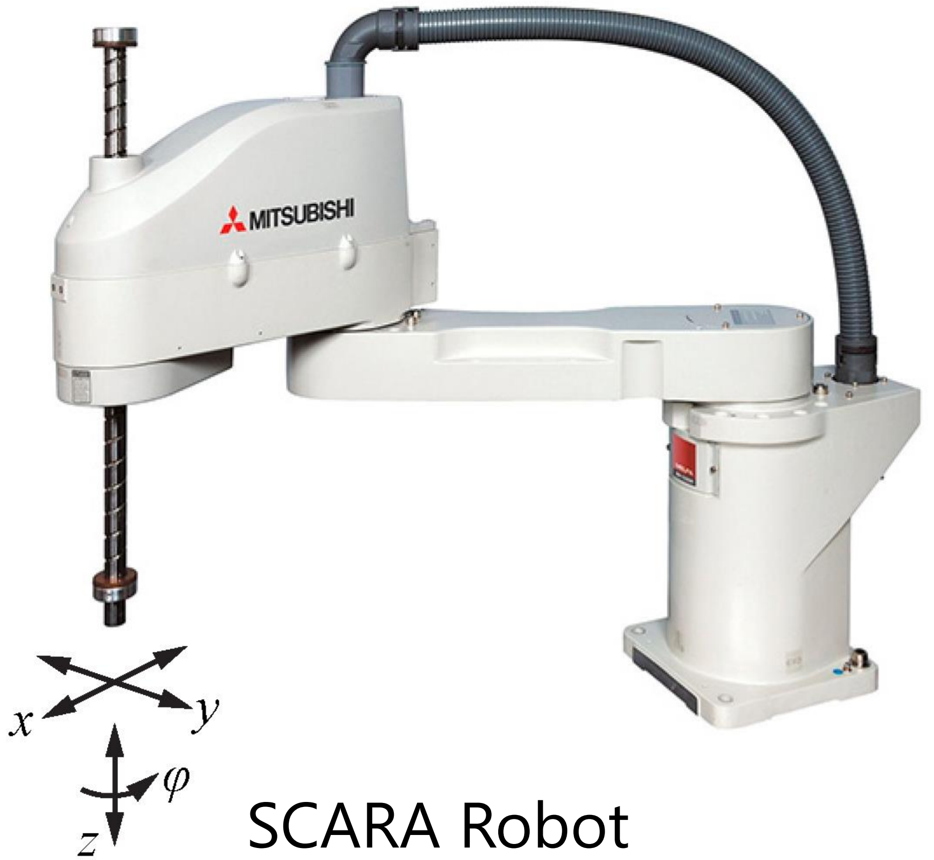
❑ Modelling of 2-link robots

- ❑ Actuators: move the robot
- ❑ Sensors: measure joint angles, detect object

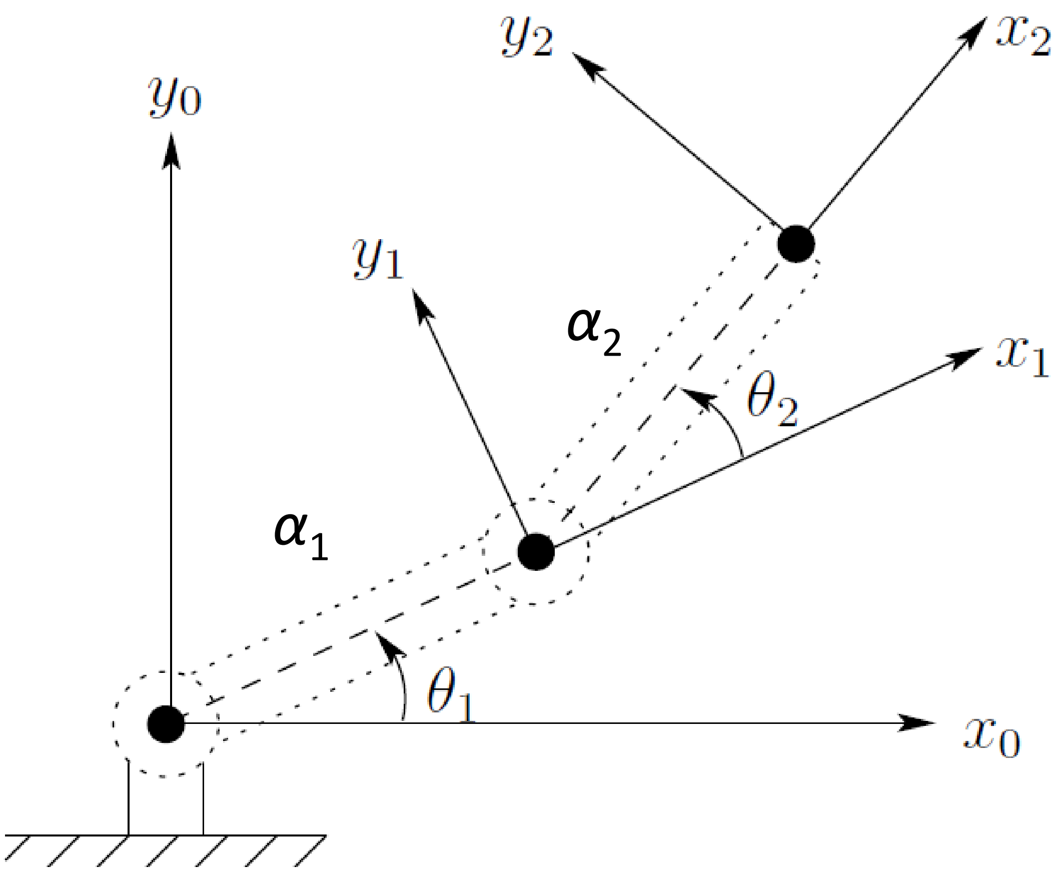
- ❑ How to find the tool position (forward kinematic)
- ❑ How to find the joint angles to reach targeted positions (inverse kinematic)

Forward kinematic equations

- ❑ Only consider the two revolute joints.
- ❑ Observe from the top view



SCARA Robot



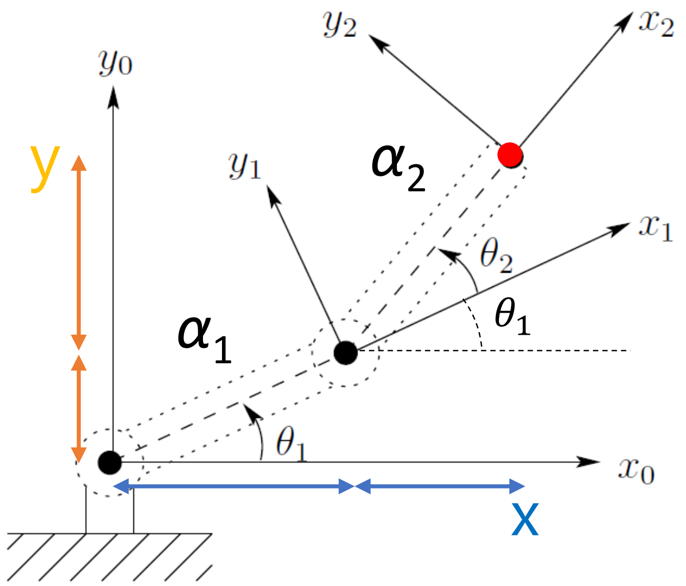
Based on the joint angles (using position sensors), find the position and the orientation of the end effector.

Forward kinematic equations

- **Position** of the tool with respect to a fixed frame (x_0y_0)

$$x = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2)$$

$$y = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)$$

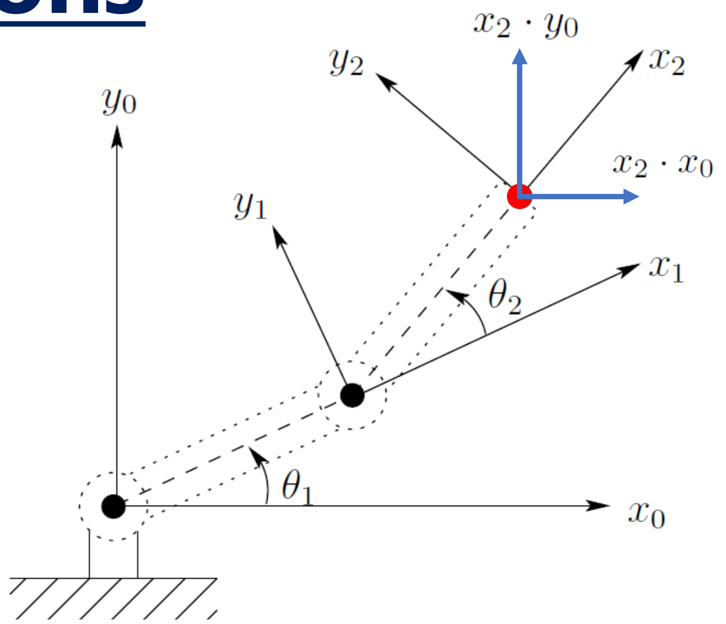


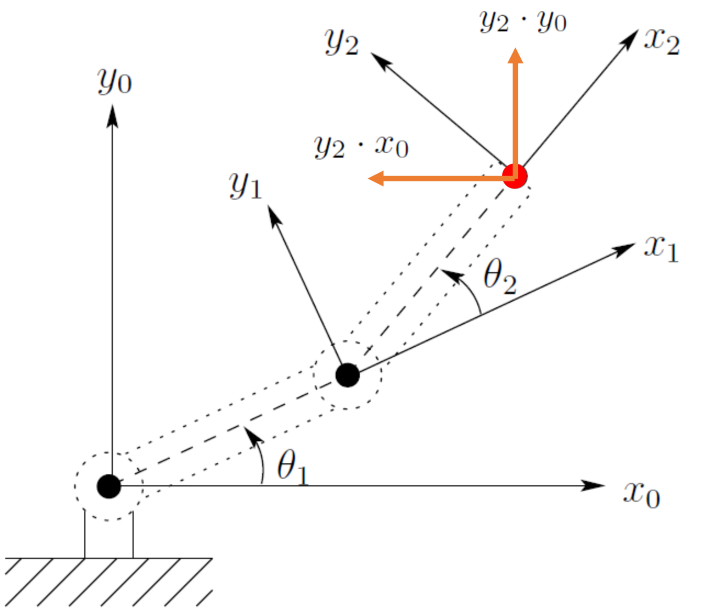
Forward kinematic equations

- The **orientation** of the tool frame (vectors x_2 and y_2)

$$\boxed{\begin{aligned} x_2 \cdot x_0 &= \cos(\theta_1+\theta_2) \\ x_2 \cdot y_0 &= \sin(\theta_1+\theta_2) \end{aligned}}$$

$$\boxed{\begin{aligned} y_2 \cdot x_0 &= -\sin(\theta_1+\theta_2) \\ y_2 \cdot y_0 &= \cos(\theta_1+\theta_2) \end{aligned}}$$





- **Rotation matrix**

$$\begin{bmatrix} \boxed{\begin{matrix} x_2 \cdot x_0 \\ x_2 \cdot y_0 \end{matrix}} & \boxed{\begin{matrix} y_2 \cdot x_0 \\ y_2 \cdot y_0 \end{matrix}} \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

Forward kinematic equations

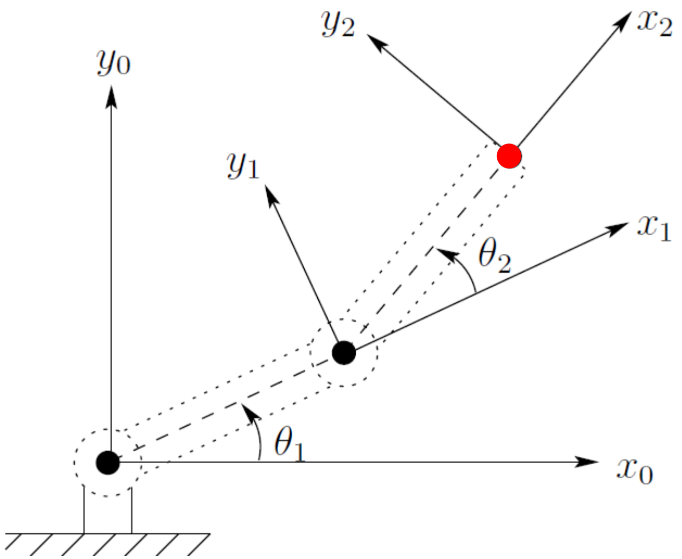
❑ Pose of end-effector

Translation

$$P = \begin{bmatrix} \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) \\ \alpha_1 \sin\theta_1 + \alpha_2\, sin(\theta_1 + \theta_2) \end{bmatrix}$$

Rotation

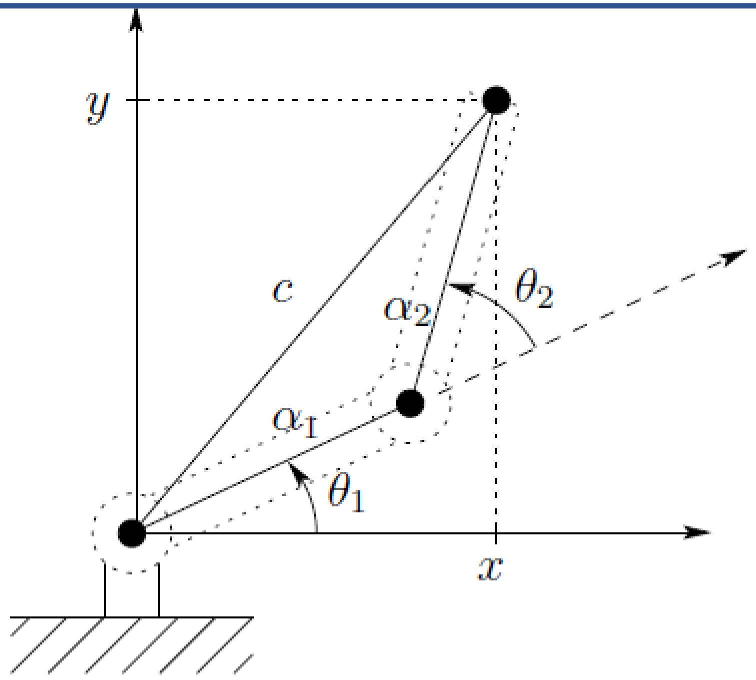
$$R = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$



Represented by a single formular → Homogeneous Transformations (Lecture 3)

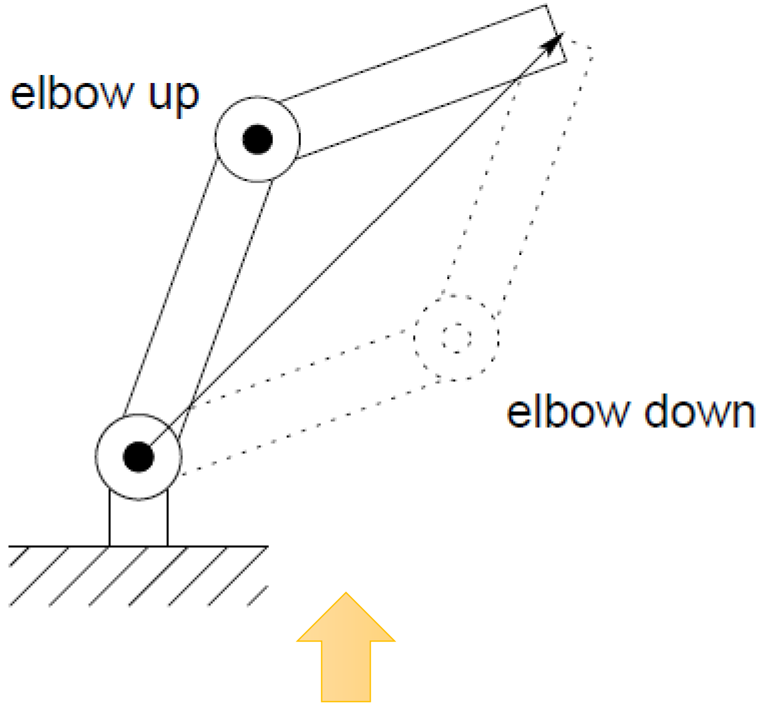
Multiple link robotics → Denavit-Hartenberg Convention (Lecture 4)

Inverse kinematic equations



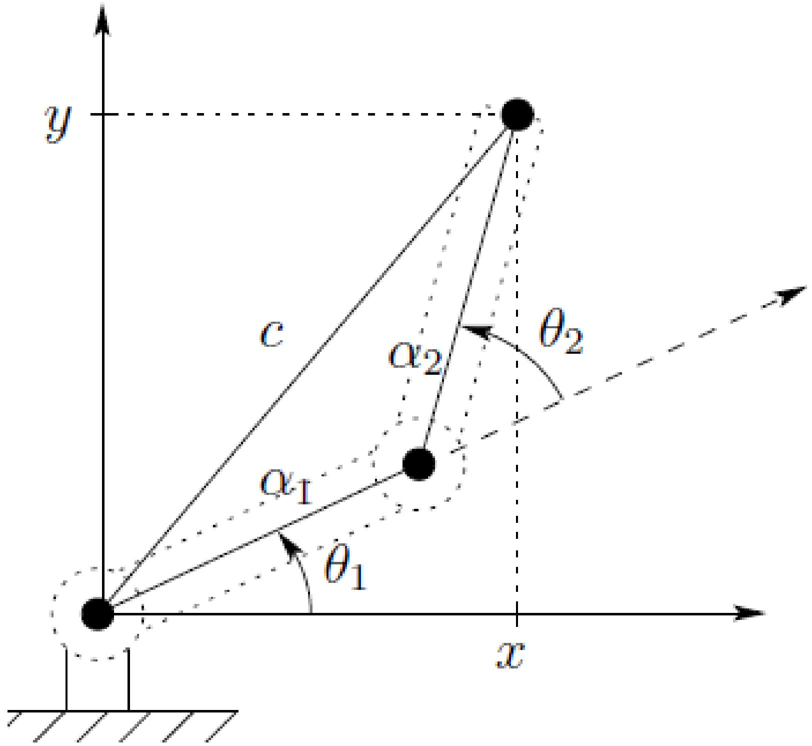
$$\cos\theta_2 = \frac{x^2 + y^2 - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2}$$

$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{\alpha_2 \sin\theta_2}{\alpha_1 + \alpha_2 \cos\theta_2}\right)$$



There are more than one solution (**Redundancy**)

Velocity kinematics



- End-effector velocity

$$\boxed{\begin{aligned} x &= \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) \\ y &= \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2) \end{aligned}}$$

$$\boxed{\begin{aligned} \dot{x} &= -\alpha_1 \sin\theta_1 \cdot \dot{\theta}_1 - \alpha_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{y} &= \alpha_1 \cos\theta_1 \cdot \dot{\theta}_1 + \alpha_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \end{aligned}}$$

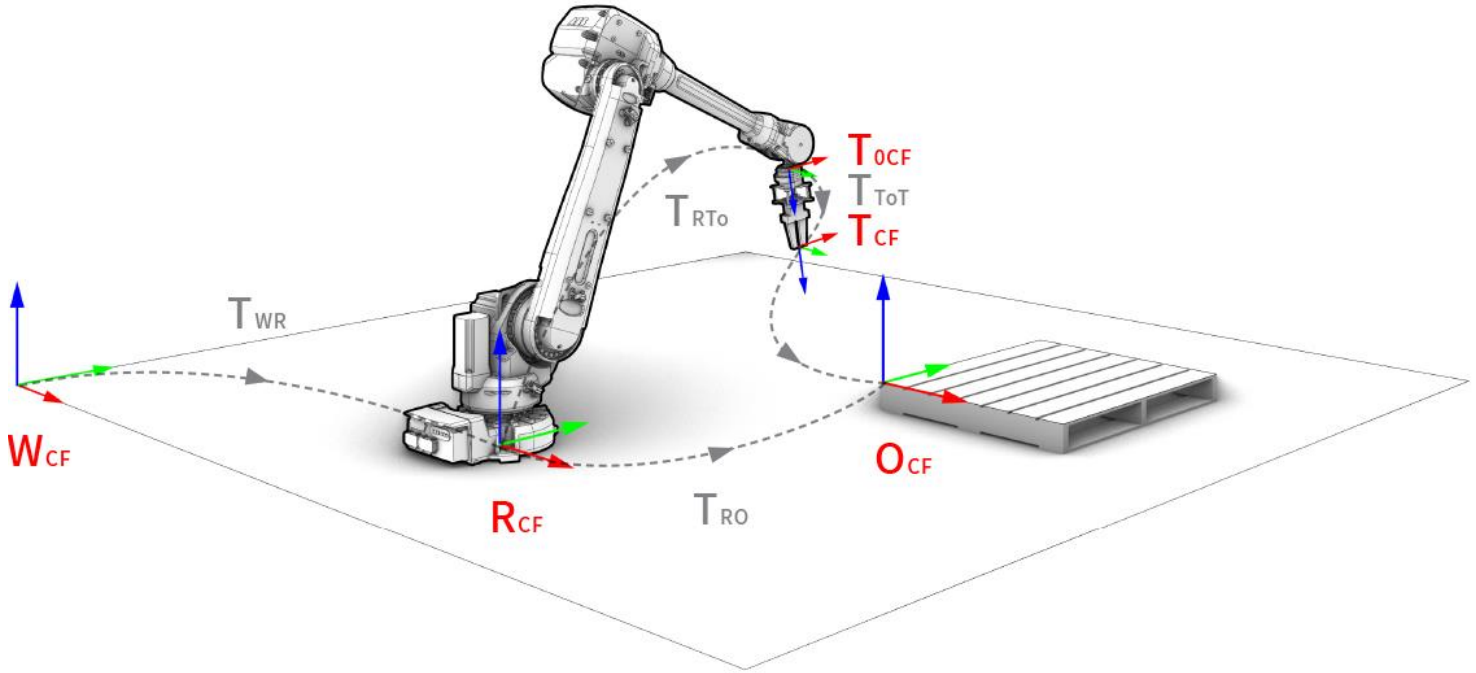
- Matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha_1 \sin\theta_1 - \alpha_2 \sin(\theta_1 + \theta_2) & -\alpha_2 \sin(\theta_1 + \theta_2) \\ \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) & \alpha_2 \cos(\theta_1 + \theta_2) \end{bmatrix}}_{J:\ \text{Jacobian}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Lecture 3 – Learning objectives

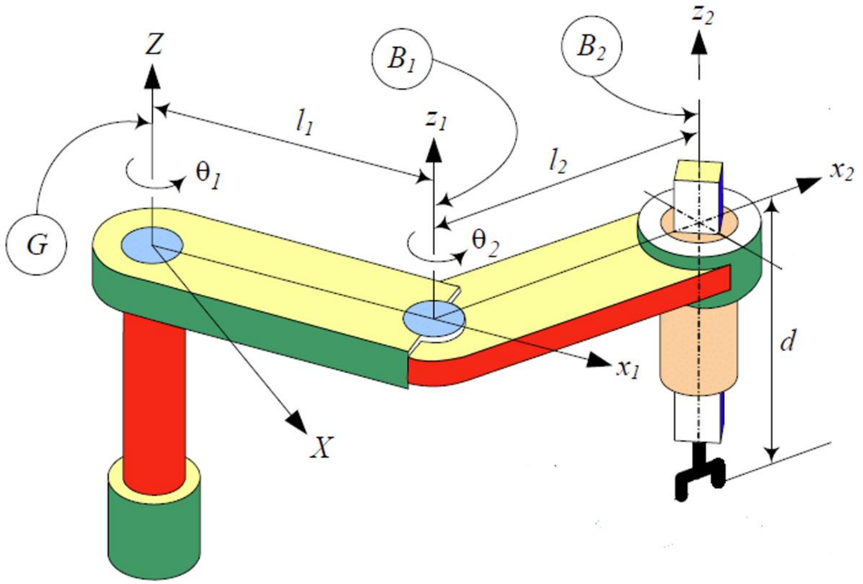
- ❑ Pose of robot frames
- ❑ Transformation operation between frames
- ❑ Rotation matrix (e.g., Euler Angles, Roll-Pitch-Yaw, Axis-Angle)
- ❑ Homogeneous transformation

Coordinate frames



How do we tell the robot where to move without knowing the position and orientation of the end effector ?

Coordinate frames

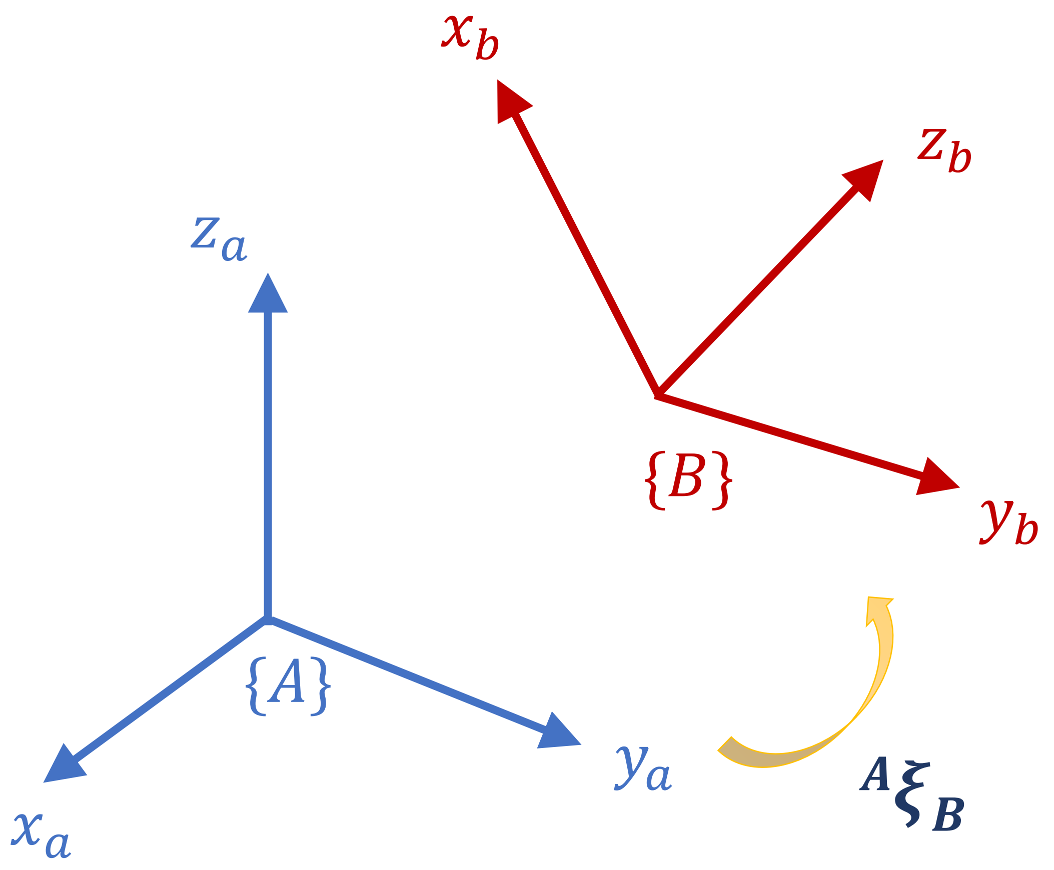


- Motion at each joint move robot arm to its target.
- Difficult to directly calculate the position/orientation of the end-effector.

- Introduce coordinate frames to each joint.
- Specify position/orientation of a joint **with respect to** joint before that.
- Apply a chain-rule to calculate the end-effector position/orientation w.r.t world coordinate frame.

The Pose

- The pose represents the location and orientation of a frame



$^{A}\boldsymbol{\xi}_{B}$

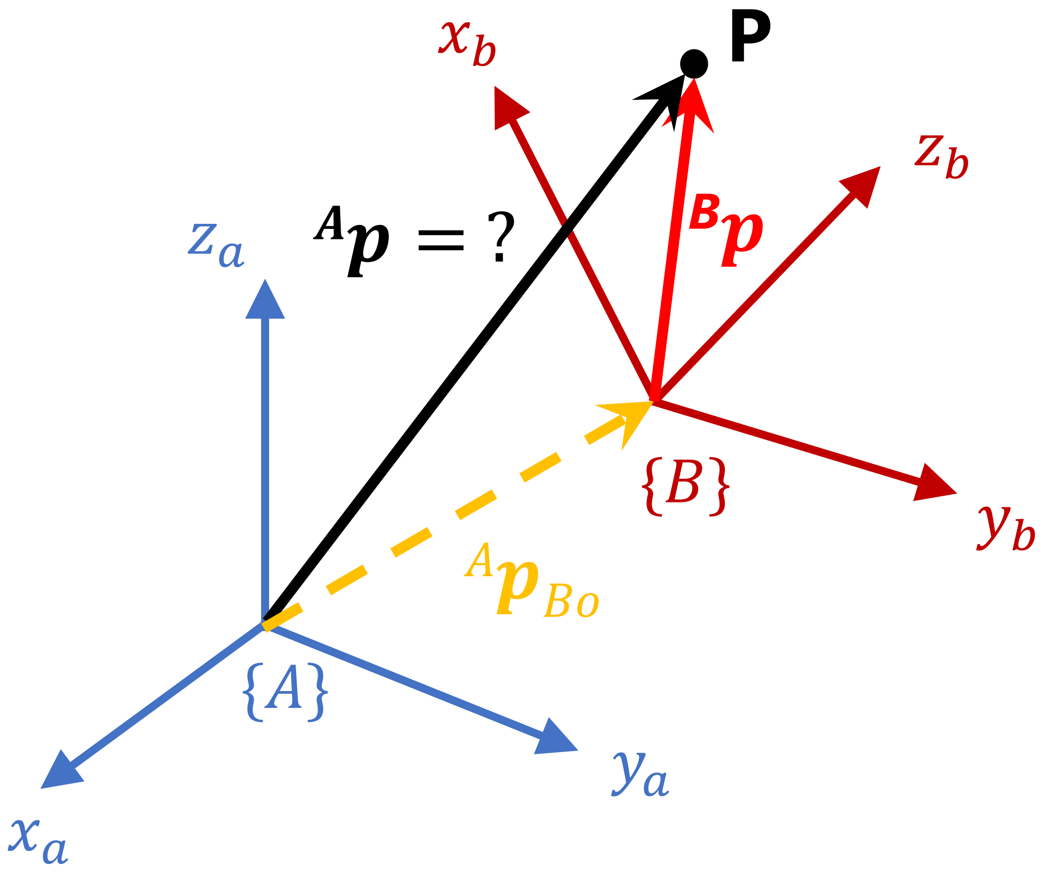
Transformation Operator

$^{A}\boldsymbol{p}_{Bo}$: Translational vector from the origin of $\{A\}$ to the origin of frame $\{B\}$

$^{A}\boldsymbol{R}_{B}$: Rotational matrix expressing the orientation of $\{B\}$ relative to $\{A\}$

Mapping of Frames

- $^{x}\boldsymbol{p}$ - vector of coordinates of point **P** expressed relative to $\{X\}$



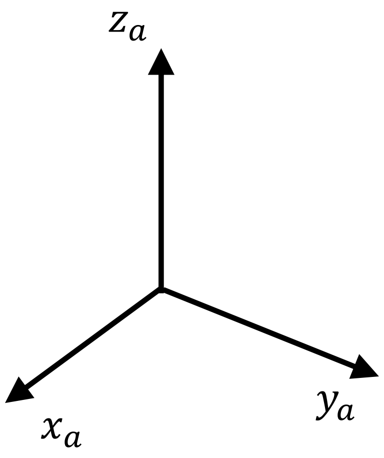
^{A}p: vector of **P** expressed in frame $\{A\}$
^{B}p: vector of **P** expressed in frame {B}

$$\boxed{^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{p}_{Bo}}$$

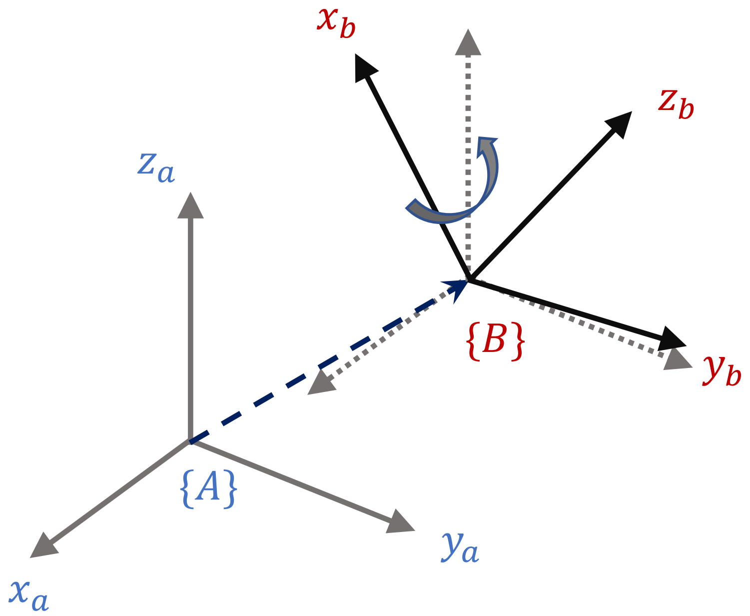
Or,

$$\boxed{^{A}\boldsymbol{p} = {}^{A}\xi_{B}\,{}^{B}\boldsymbol{p}}$$

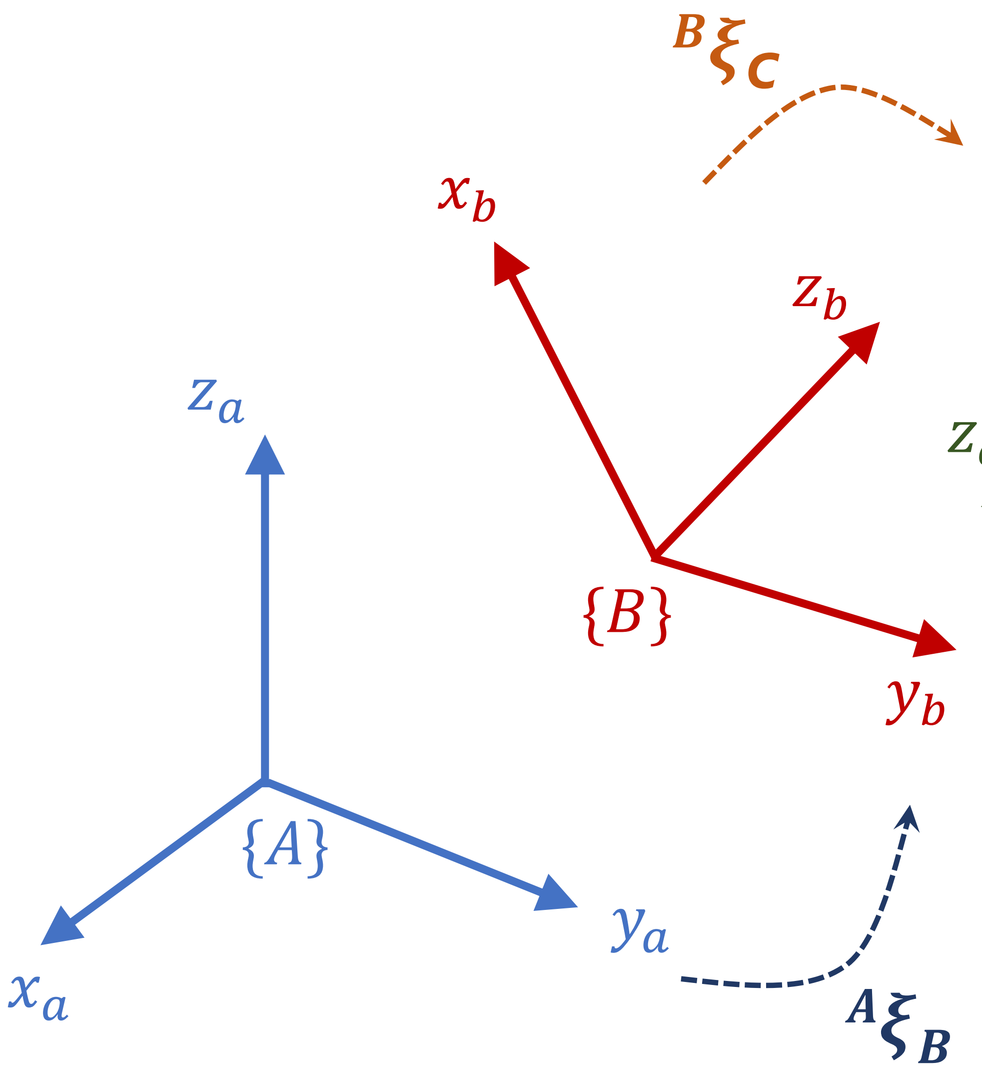
Mapping of Frames



- Always use right-handed coordinate frames.
- Pose contains a translation component (*P*) & a rotational component (*R*).
- When mapping coordinate systems, first translate, then apply rotation.
- Transformation operator (ξ) can take many forms
  - Homogeneous transformation
  - Orthonormal rotation matrixes
  - Quaternions

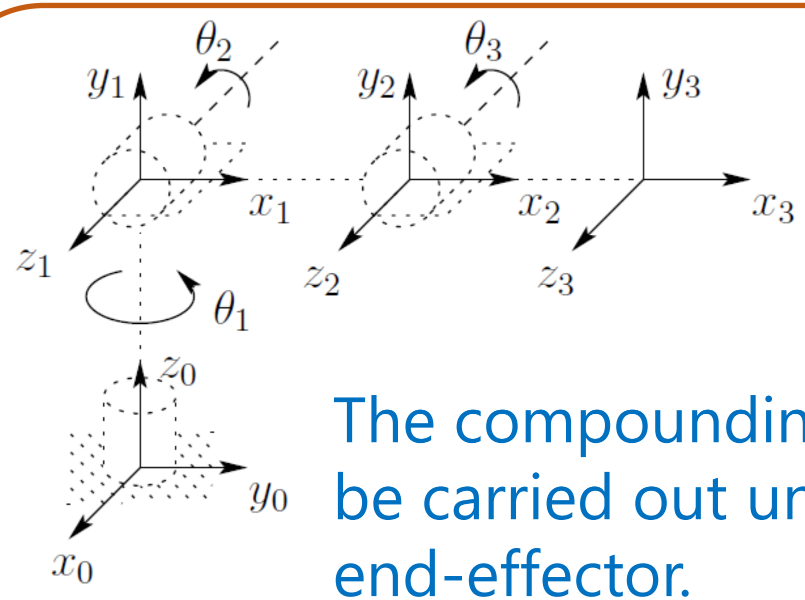


Frame Composition



$$^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C}$$

$\oplus$: compounding operator



The compounding operation can be carried out until we reach the end-effector.

Rules for Frame Composition

$$\xi \oplus 0 = \xi \,;\; \xi \ominus 0 = \xi$$

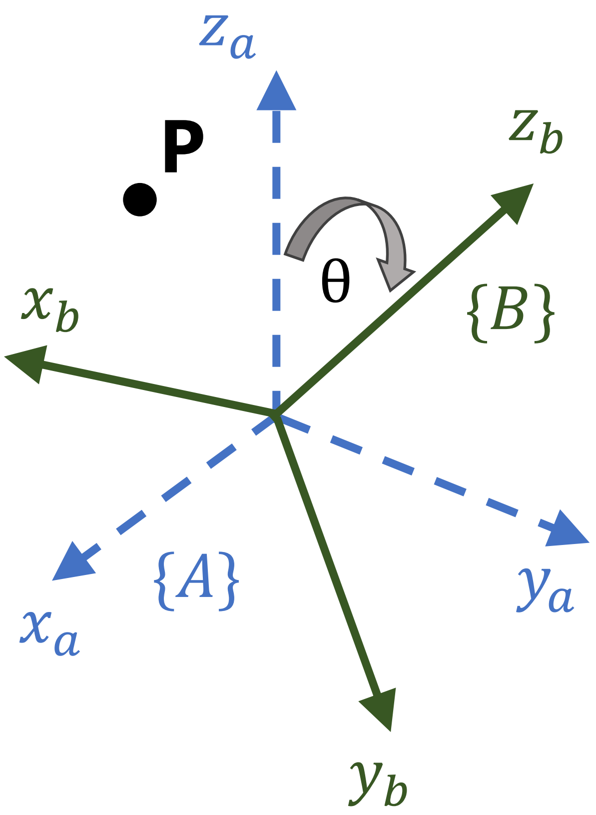
$$\ominus\, {}^{X}\xi_{Y} = {}^{Y}\xi_{X}$$

$$\xi \ominus \xi = 0 \,;\; \ominus \xi \oplus \xi = 0$$

$${}^{X}\xi_{Y} \oplus {}^{Y}\xi_{Z} = {}^{X}\xi_{Z}$$

$$\xi_1 \oplus \xi_2 \neq \xi_2 \oplus \xi_1$$

The Rotation Matrix



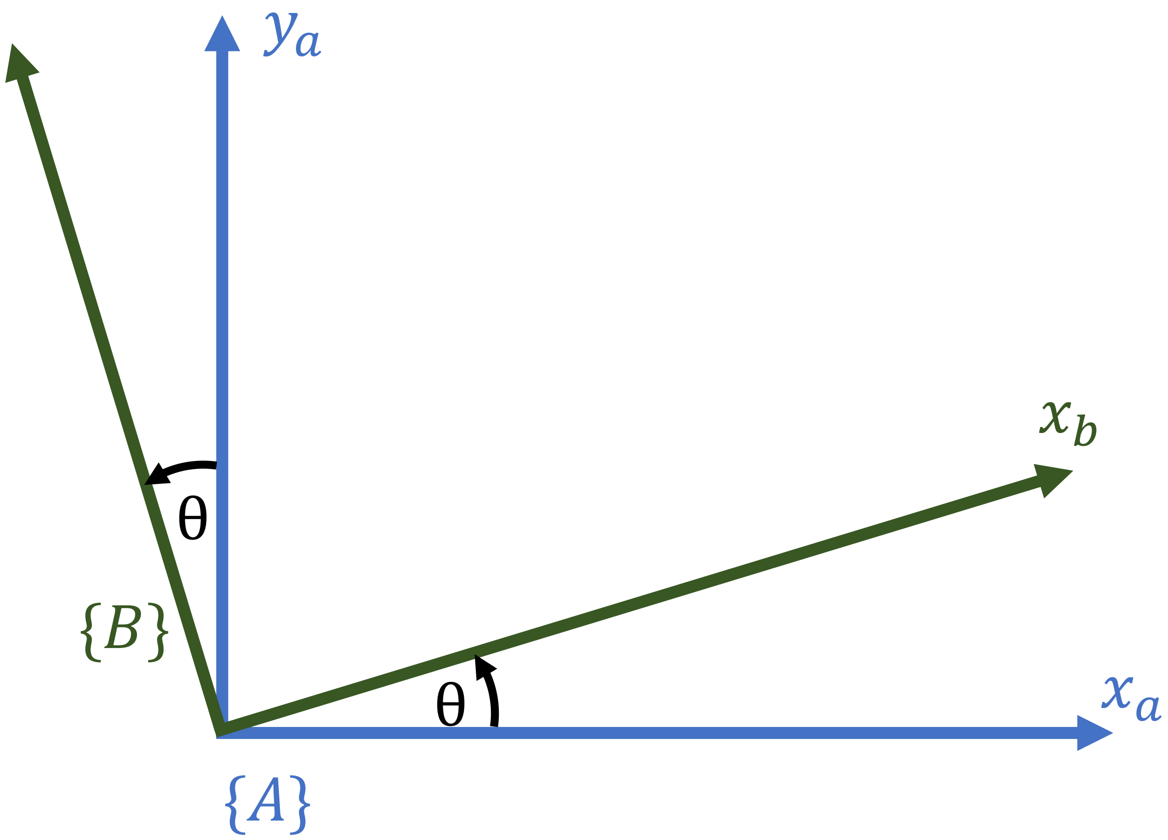
- ${}^{A}R_{B}$ represents the rotation matrix from the coordinates of **point *P*** defined *w.r.t* frame *B* to the coordinates *w.r.t* frame *A*.

$$\boxed{{}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + \cancel{{}^{A}\boldsymbol{p}_{Bo}}}$$

$$\boxed{{}^{A}p = {}^{A}R_{B}\,{}^{B}\boldsymbol{p}}$$

Example 1: Rotation matrix in 2D coordinate

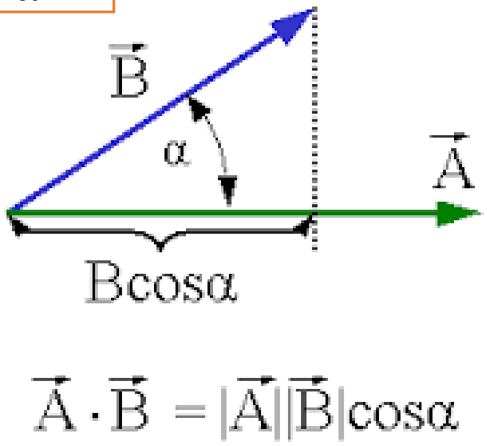
- $[x_a, y_a]$ are unit vectors of the axes x and y in frame {A}; $[x_b, y_b]$ are unit vectors of frame {B}



$$^{A}R_B = \begin{bmatrix} x_b \cdot x_a & y_b \cdot x_a \\ x_b \cdot y_a & y_b \cdot y_a \end{bmatrix}$$

Recall

$$x \cdot y = |x||y|cos\alpha$$

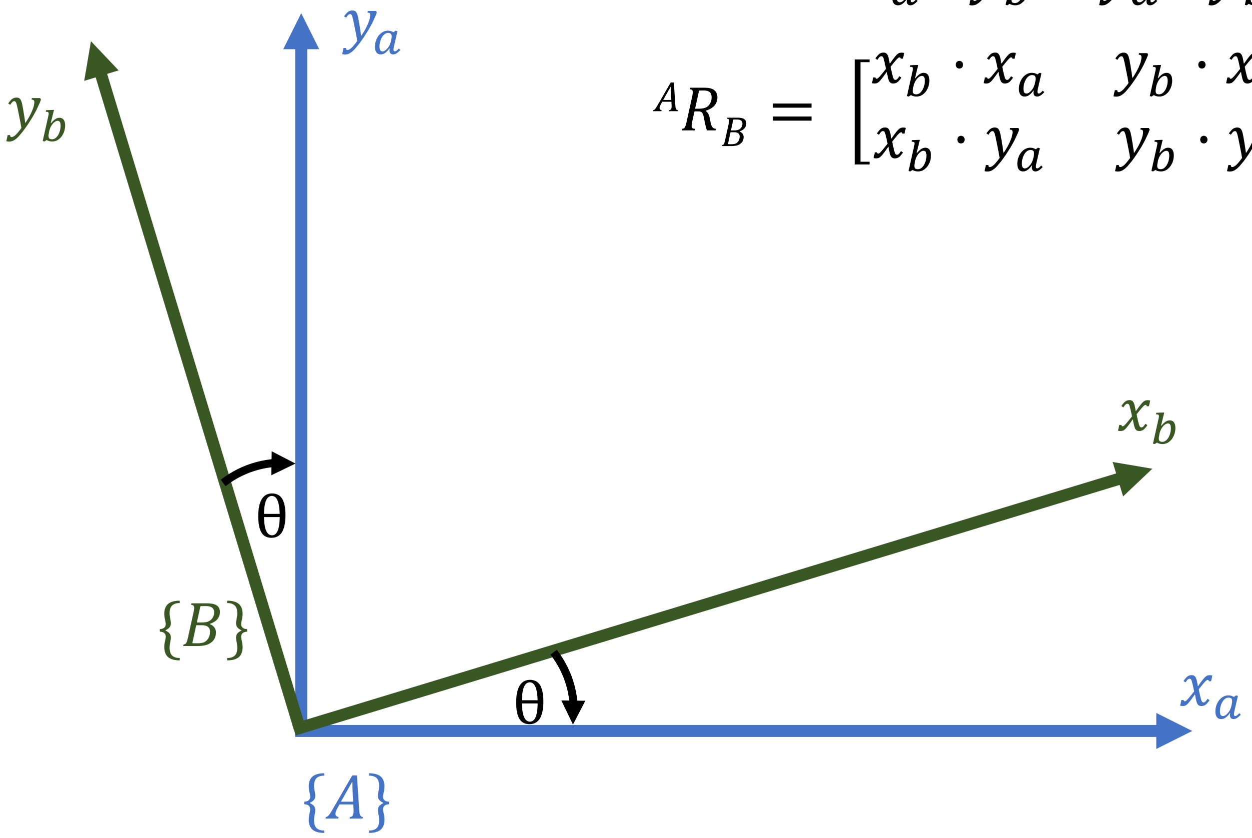


$$|x_a| = |y_a| = |x_b| = |y_b| = 1$$

$$^{A}R_B = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Rotation matrix in 2D coordinate

- The orientation of a rotated coordinate frame [*A*] *w.r.t* frame [*B*]



$$
\left.
\begin{aligned}
{}^{B}R_{A} &= \begin{bmatrix} x_a \cdot x_b & y_a \cdot x_b \\ x_a \cdot y_b & y_a \cdot y_b \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \\
{}^{A}R_{B} &= \begin{bmatrix} x_b \cdot x_a & y_b \cdot x_a \\ x_b \cdot y_a & y_b \cdot y_a \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}
\end{aligned}
\right\} \quad {}^{B}\boldsymbol{R}_{A} = ({}^{A}\boldsymbol{R}_{B})^{T}
$$

Recall

$$\boxed{{}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{R}_{B}{}^{B}\boldsymbol{p}} \Rightarrow \boxed{\begin{cases} {}^{B}\boldsymbol{p} = ({}^{A}\boldsymbol{R}_{B})^{-1}{}^{A}\boldsymbol{p} \\ {}^{B}\boldsymbol{p} = {}^{B}\boldsymbol{R}_{A}{}^{A}\boldsymbol{p} \end{cases}}$$

$$\Downarrow$$

$$\boxed{{}^{B}\boldsymbol{R}_{A} = ({}^{A}\boldsymbol{R}_{B})^{-1} = ({}^{A}\boldsymbol{R}_{B})^{T}}$$

Rotation matrixes are **Orthogonal**

Special Orthogonal Group

- $R^T = R^{-1}$ forms part of the Special Orthogonal group of order n, $SO(n)$.

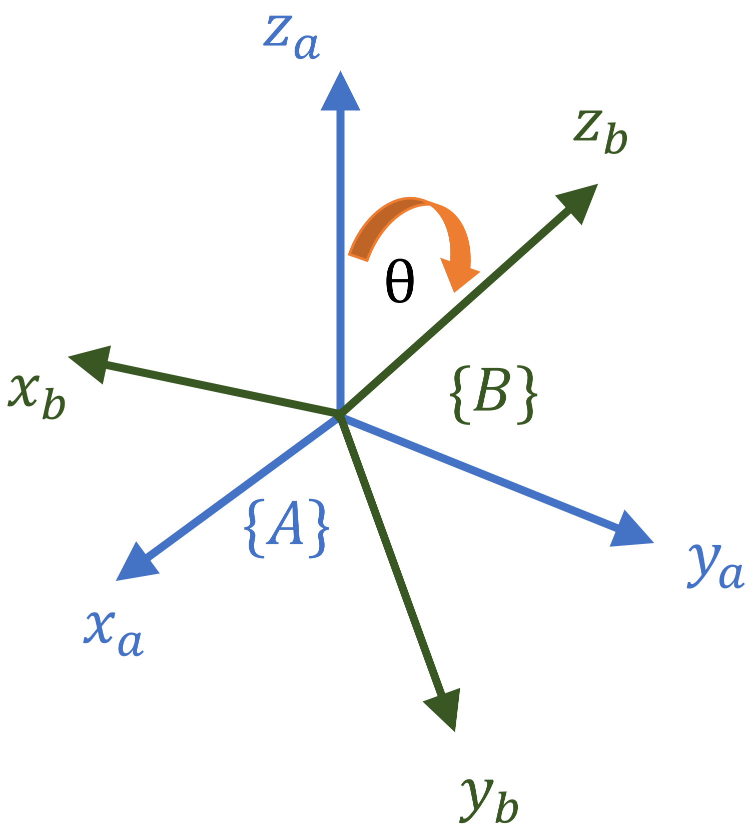
- For example, ${}^{a}R_{b}$ is a 2×2 matrix, which belongs to $SO(2)$.
  For 3D frames, the rotation matrix belong to SO(3)

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

- For any $R \in SO(n)$:
  - The columns (and rows) of R are mutually orthogonal
  - Each column (and row) of R is a unit vector
  - $\det(R) = 1$, hence the length of the vector is unchanged

The 3D Rotation Matrix

- Rotate frame $\{A\}$ by θ around an arbitrary axis



$$^{A}R_{\boldsymbol{B}} = \begin{bmatrix} x_b \cdot x_a & y_b \cdot x_a & z_b \cdot x_a \\ x_b \cdot y_a & y_b \cdot y_a & z_b \cdot y_a \\ x_b \cdot z_a & y_b \cdot z_a & z_b \cdot z_a \end{bmatrix}$$

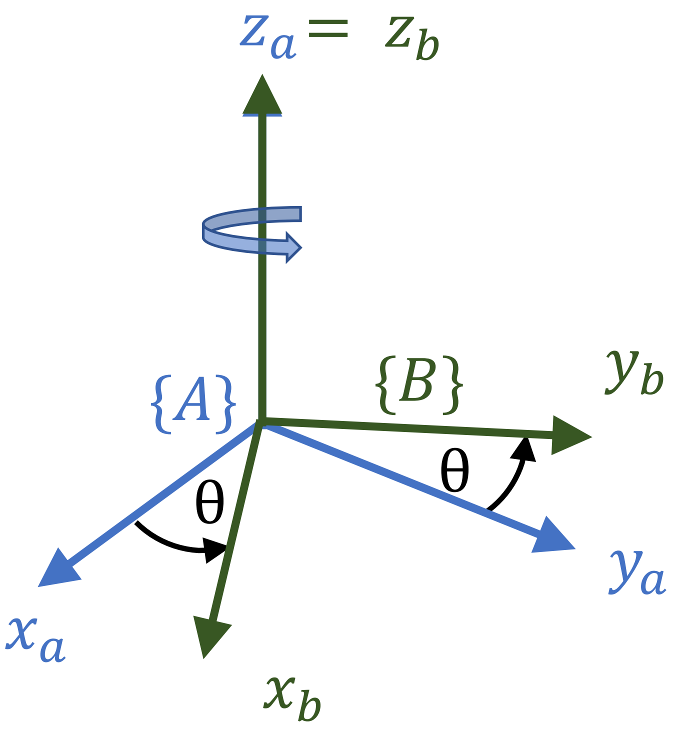
Recall,

$$x \cdot y = |x||y|cos\alpha$$

$^{A}R_{\boldsymbol{B}}$ is a matrix containing cosines, where the values represent angles between respective axes.

Example 2: A rotation about Z, Y, X – axis

❑ About Z – axis



$$^{A}R_{\boldsymbol{B}} = \begin{bmatrix} x_b \cdot x_a & y_b \cdot x_a & z_b \cdot x_a \\ x_b \cdot y_a & y_b \cdot y_a & z_b \cdot y_b \\ x_b \cdot z_a & y_b \cdot z_a & z_b \cdot z_a \end{bmatrix}$$

$$^{A}R_{\boldsymbol{B}} = \begin{bmatrix} cos(\theta) & cos\left(\frac{\pi}{2} + \theta\right) & cos(\pi/2) \\ cos\left(\frac{\pi}{2} - \theta\right) & cos(\theta) & cos(\pi/2) \\ cos(\pi/2) & cos(\pi/2) & cos(0) \end{bmatrix}$$

$$^{A}R_{\boldsymbol{B}} = R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2: A rotation around Z, Y, X –axis

- About X – axis

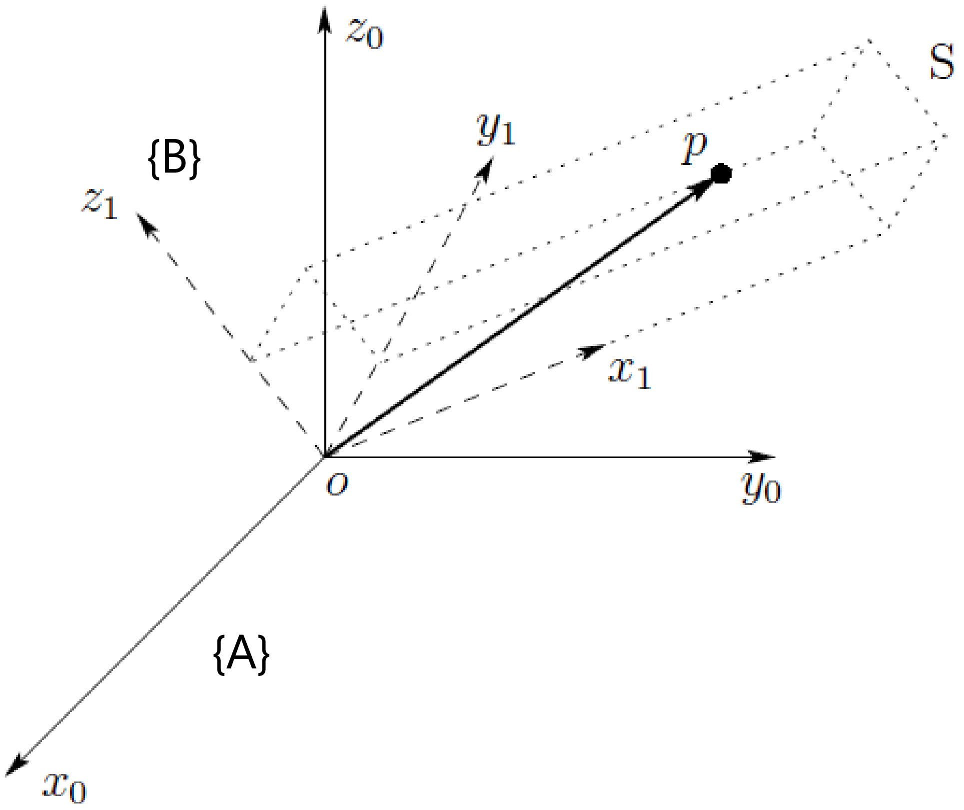
$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

- About Y – axis

$$R_Y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

Transform vectors between coordinate frames

- ${}^{B}\boldsymbol{p}$ - vector of coordinates of vector $\mathbf{P}$ expressed relative to $\{B\}$



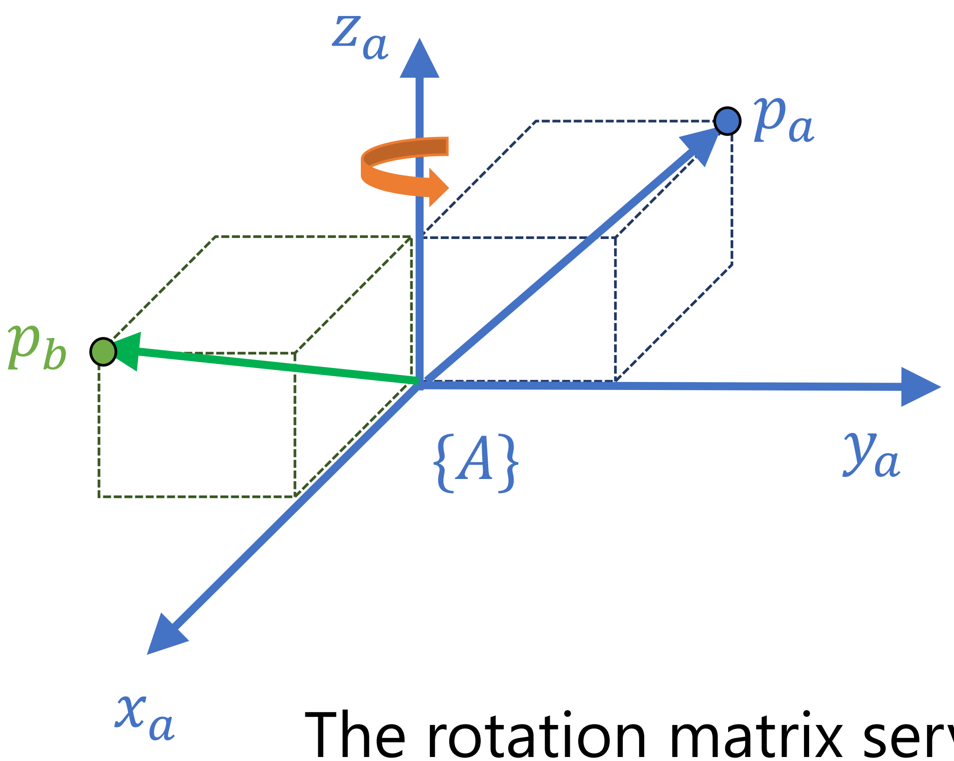
$${}^{B}p = u x_b + v y_b + w z_b = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

${}^{A}\boldsymbol{p}$ - vector of coordinates of the same vector expressed relative to $\{A\}$

$${}^{A}p = {}^{A}R_B \begin{bmatrix} u \\ v \\ w \end{bmatrix} = {}^{A}R_B {}^{B}\boldsymbol{p}$$

Transform a vector in the same coordinate system

- Rotate vector p_a by θ around an arbitrary axis (Rotation matrix: ***R***)



$$\boldsymbol{p}_b = R\boldsymbol{p}_a$$

- For instance, rotate p_a by π about Z-axis

$$R_z = \begin{bmatrix} cos\pi & -sin\pi & 0 \\ sin\pi & cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

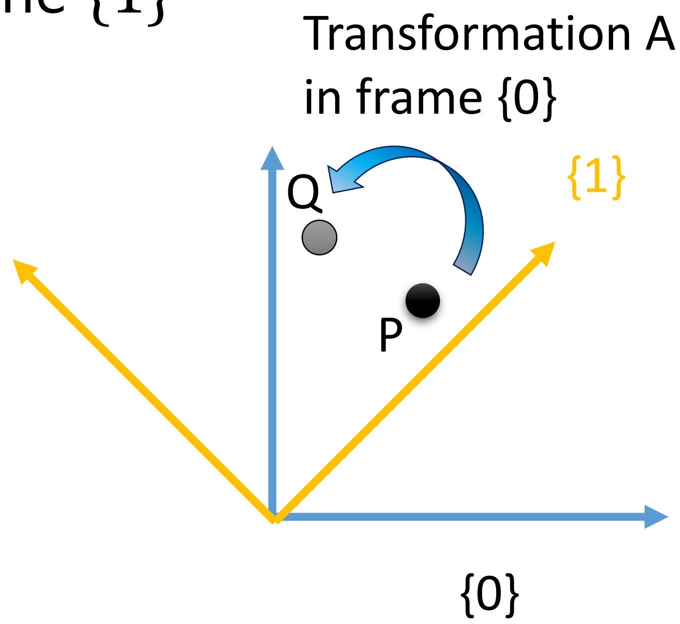
$$^A\boldsymbol{p}_b = R_z{}^A\boldsymbol{p}_a$$

The rotation matrix serves as a transformation operator on an **existing vector *p_a*** and rotating it to **a new vector *p_b*** in the same coordinate system *A*.

Similarity Transformations

- ❑ ${}^{0}R_1$ is the coordinate transformation between frame $\{0\}$ and $\{1\}$
- ❑ A: a transformation defined with respect to Frame $\{0\}$
- ❑ B: the same transformation defined with respect to Frame $\{1\}$

$$\boldsymbol{B} = ?$$



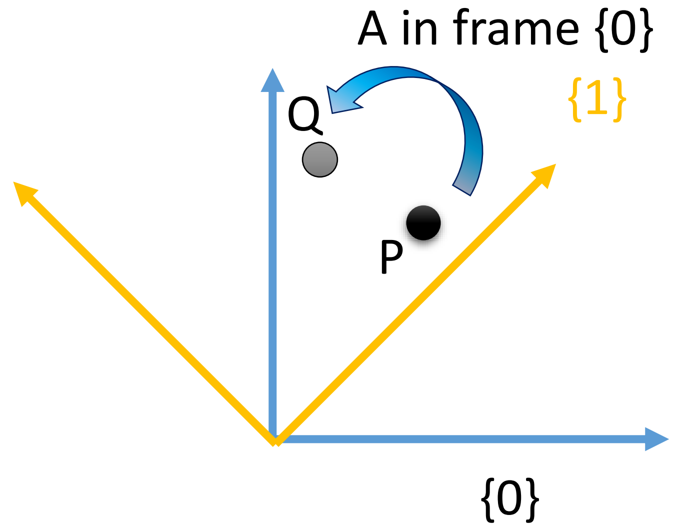
Similarity Transformations

- ❑ The transformation of an arbitrary point ^0P in frame {0} gives vector ^0Q:

$$^0Q = A\ ^0P$$

- ❑ The same points ^0P and ^0Q expressed in frame {1} are ^1P and ^1Q:

$$\begin{cases} ^0P = \ ^oR_1 \ \ ^1P \\ ^0Q = \ ^oR_1 \ \ ^1Q \end{cases} \Rightarrow \begin{cases} ^1P = (^0R_1)^{-1}\ ^0P \\ ^1Q = (^0R_1)^{-1}\ ^0Q \end{cases}$$



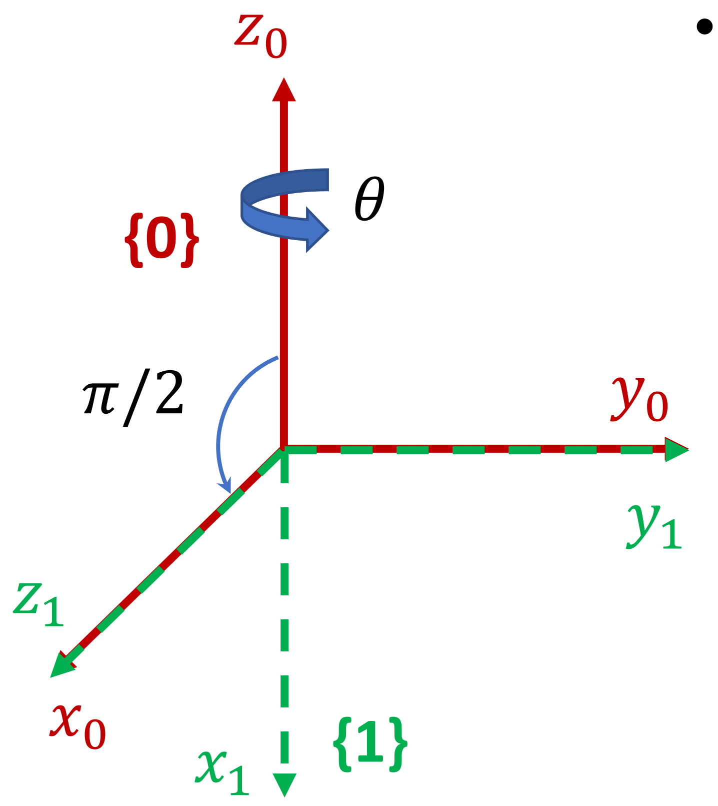
$$^1Q = (^0R_1)^{-1}\ ^0Q = (^0R_1)^{-1}\ A\ ^0P = \underbrace{(^0R_1)^{-1}\ A(^0R_1)}\ ^1P$$

$$^1Q = B\ ^1P \Rightarrow \boxed{\boldsymbol{B = (^0R_1)^{-1} A ^0R_1}}$$

- ❑ Allows us to express the same rotation easily w.r.t different frames
- ❑ Useful for rotation about fixed axes

Example 3: Similarity Transformations

- Frame {1} is obtained by rotating frame {0} about y_0 an angle of $\pi/2$



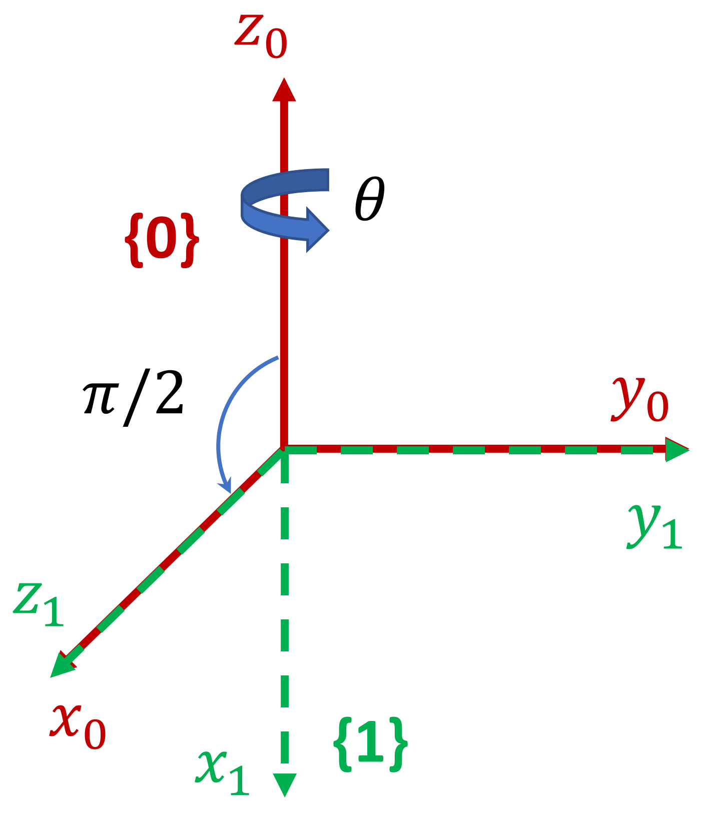
- Transformation A in frame {0} is a rotation by θ about z_o
- What is the same rotation B expressed in frame {1}?

$$ {}^0R_1 = \begin{bmatrix} cos\pi/2 & 0 & sin\pi/2 \\ 0 & 1 & 0 \\ -sin\pi/2 & 0 & cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} $$

Transformation A in frame {0}

$$ A = R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Example 3: Similarity Transformations



- The same transformation B expressed in frame {1}

$$\boldsymbol{B} = ({}^{0}\boldsymbol{R}_{1})^{-1}\boldsymbol{A}\,{}^{0}\boldsymbol{R}_{1}$$

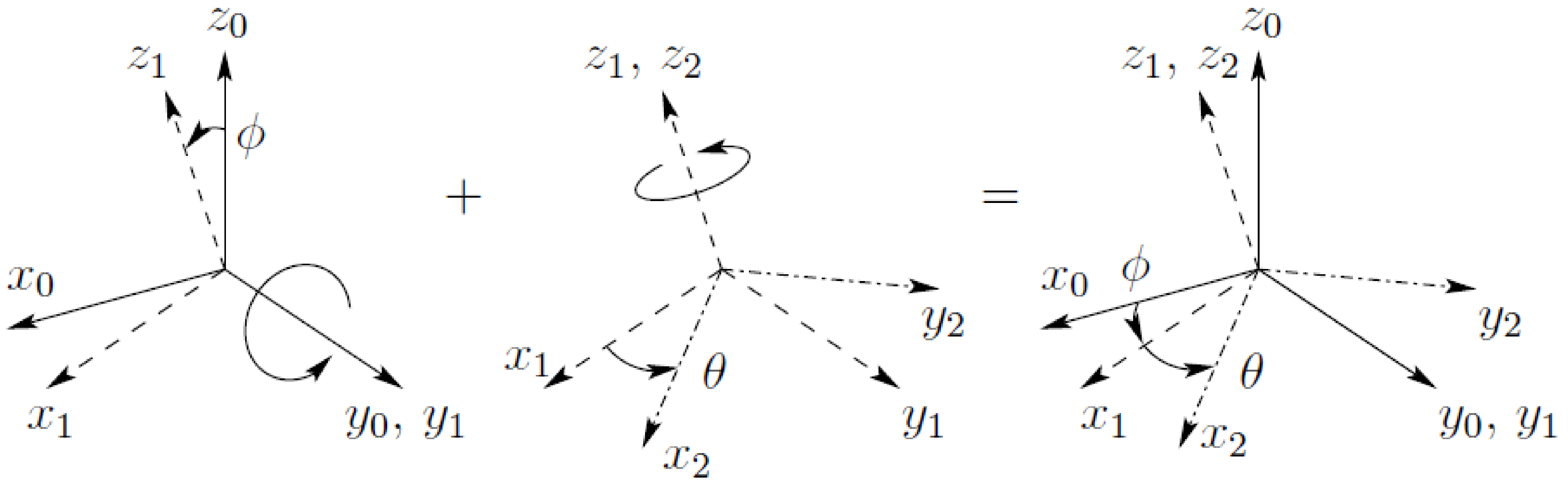
$$= \underbrace{\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right)^{-1}}_{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Special orthogonal matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

Rotation by –θ about x_1

Rotation about Current Axes



Step 1: Rotate about $\mathbf{y_0}$ Step 2: Rotate about $\mathbf{z_1}$ ⟹ $^0\boldsymbol{R}_2$ = ?

$$^0R_2 = R_{y\phi}.R_{z\theta} = {}^0R_1\,{}^1R_2$$

Order of rotation is important!

Quiz 1: Rotation about Current Axes

Step 1: Rotate by α about current **x**

Step 2: Rotate by β about current **y** $\Longrightarrow$ ${}^{0}\boldsymbol{R}_{3}$ **= ?**

Step 3: Rotate by γ about current **z**

Quiz 1: Rotation about Current Axes

Step 1: Rotate by α about current **x**

Step 2: Rotate by β about current **y**

Step 3: Rotate by γ about current **z**

$$R_{x\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha \\ 0 & sin\alpha & cos\alpha \end{bmatrix} \qquad R_{y\beta} = \begin{bmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{bmatrix} \qquad R_{z\gamma} = \begin{bmatrix} cos\gamma & -sin\gamma & 0 \\ sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

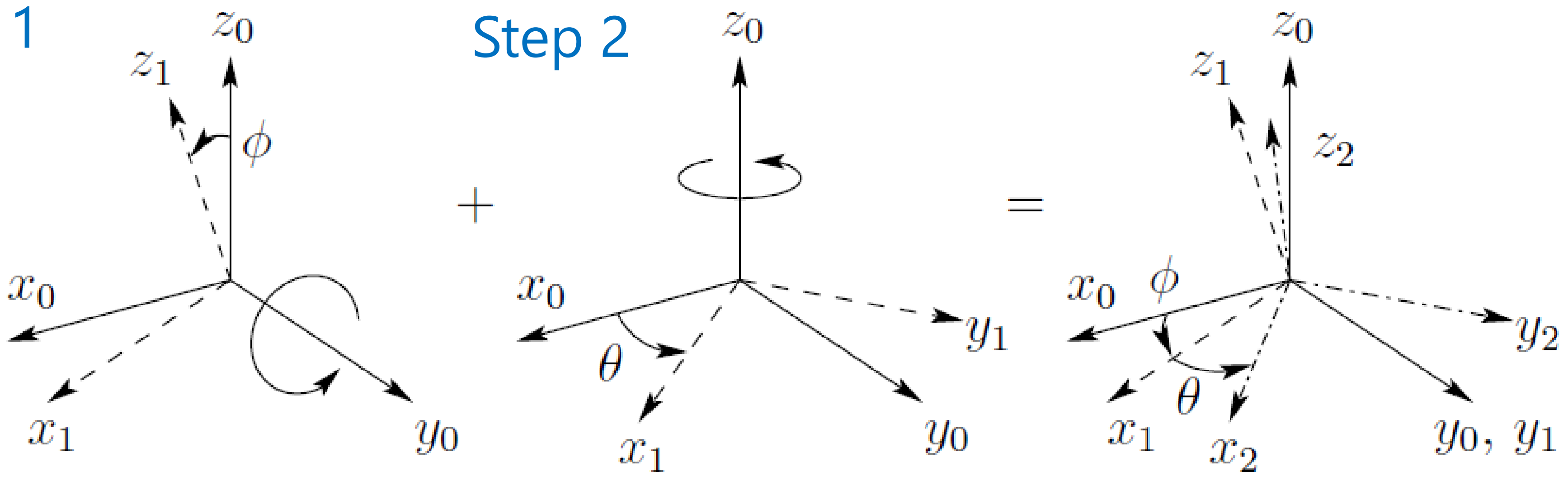
$$^{0}R_{3} = R_{x\alpha}.R_{y\beta}.R_{z\gamma}$$

Step 1 Step 2 Step 3

Rotation about *Fixed* Axes

Step 1

Step 2



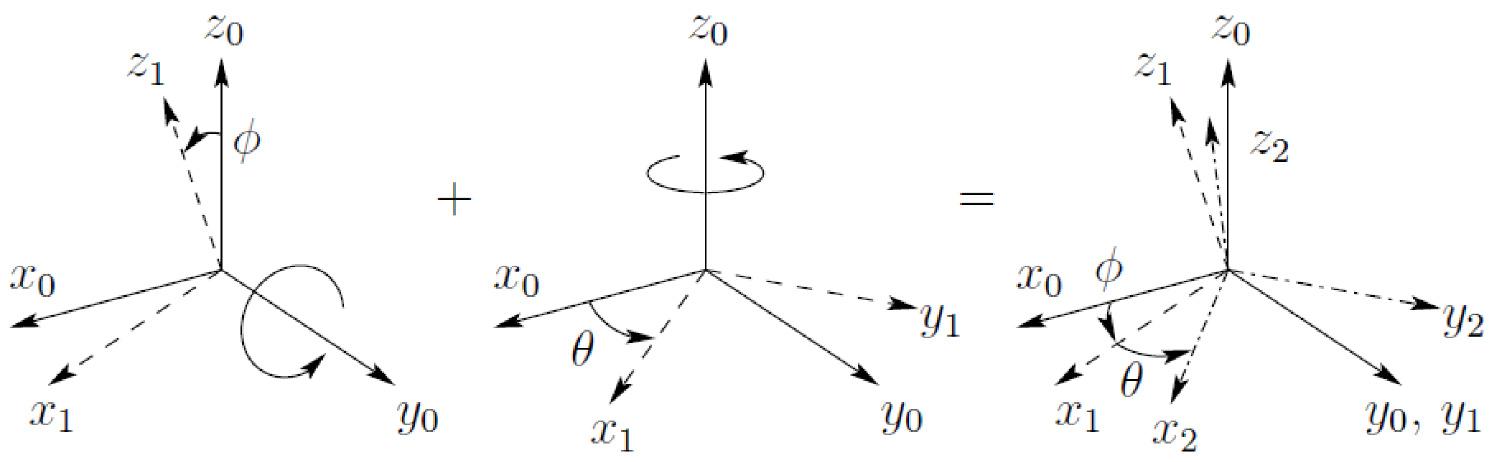
The rotation matrix in step 1: $\quad {}^0R_1 = R_{y\phi}$

The rotation matrix in step 2: $\quad {}^1R_2$ is the similar transformation of rotating by an angle ϴ about z_0 in frame {0}

$$\boxed{{}^0R_2 = {}^0R_1\,{}^1R_2}$$

Rotation by an angle ϴ about z_0 in frame {0}: $R_{z\theta}$

Rotation about *Fixed* Axes



Similarity Transformation expressed in frame {1}

$$\boxed{{}^1R_2 = ({}^0R_1)^{-1}.(\text{Transformation Matrix in Frame}\{0\}).{}^0R_1}$$

$${}^1R_2 = ({}^0R_1)^{-1}.R_{z\theta}.{}^0R_1$$

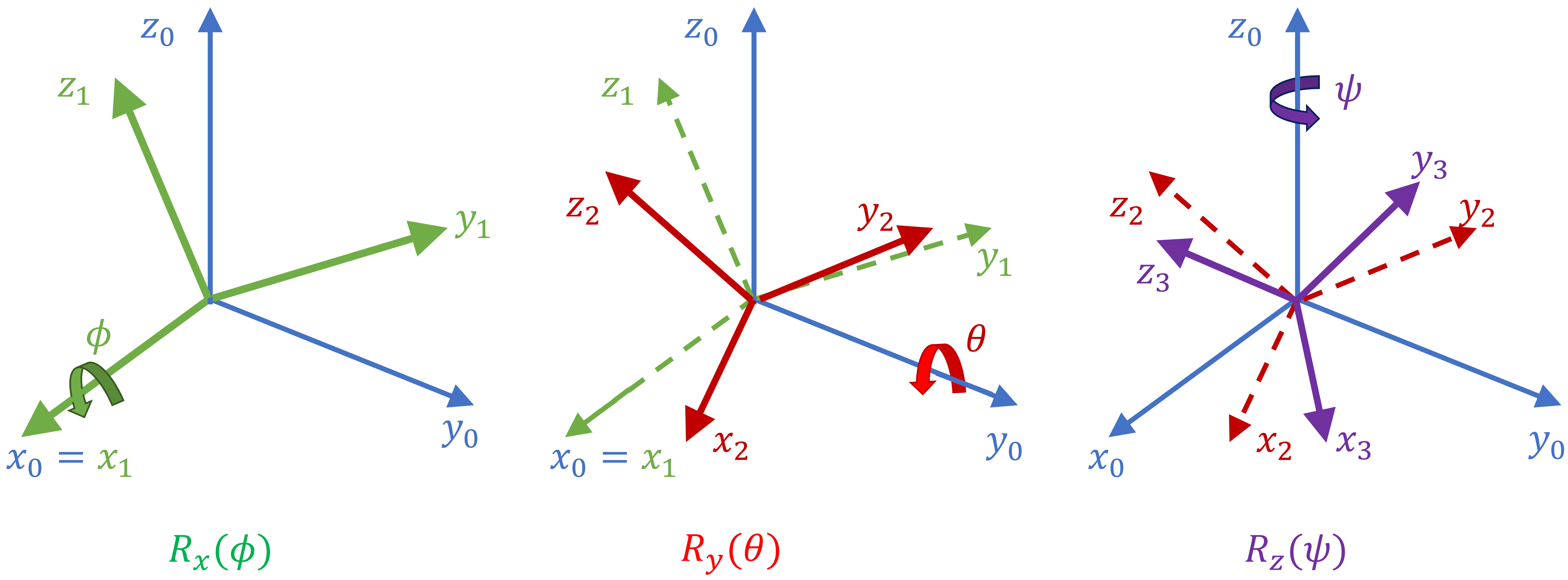
$${}^0R_2 = {}^0R_1.{}^1R_2$$

Rotation 2 Rotation 1

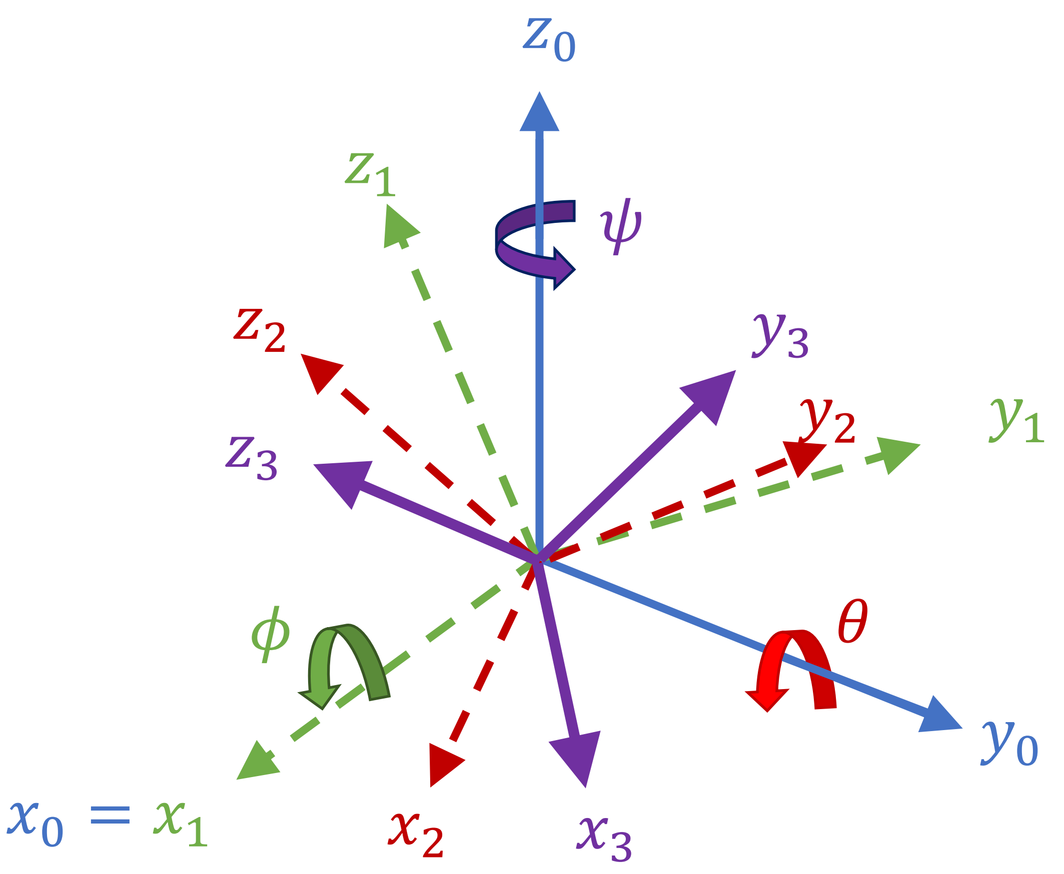
$$\boxed{{}^0R_2 = {}^0R_1.[({}^0R_1)^{-1}.R_{z\theta}.{}^0R_1] = R_{z\theta}.{}^0R_1 = R_{z\theta}.R_{y\phi}}$$

Note that the rotation order **is reversed** from previous case.

Example 4: Rotation about Fixed Axes



Example 4: Rotation about Fixed Axes



Rotation3 Rotation 2 Rotation 1

$$ {}^{0}R_3 = R_z(\psi).\ R_y(\theta).\ R_x(\phi) $$

Quiz 2: Combined Rotations

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

Hint:
If around current axis – post-multiply
If around fixed axis – pre-multiply

$$R = ???$$

Answer

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

If around current axis – post-multiply
If around fixed axis – pre-multiply

$$R = \qquad R_{x,\theta}$$

Answer

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

If around current axis – post-multiply
If around fixed axis – pre-multiply

$$R = \qquad R_{x,\theta} R_{z,\phi}$$

Answer

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

If around current axis – post-multiply
If around fixed axis – pre-multiply

$$R = \quad R_{z,\alpha} R_{x,\theta} R_{z,\phi}$$

Answer

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

If around current axis – post-multiply
If around fixed axis – pre-multiply

$$R = \quad R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

Answer

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

If around current axis – post-multiply
If around fixed axis – pre-multiply

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

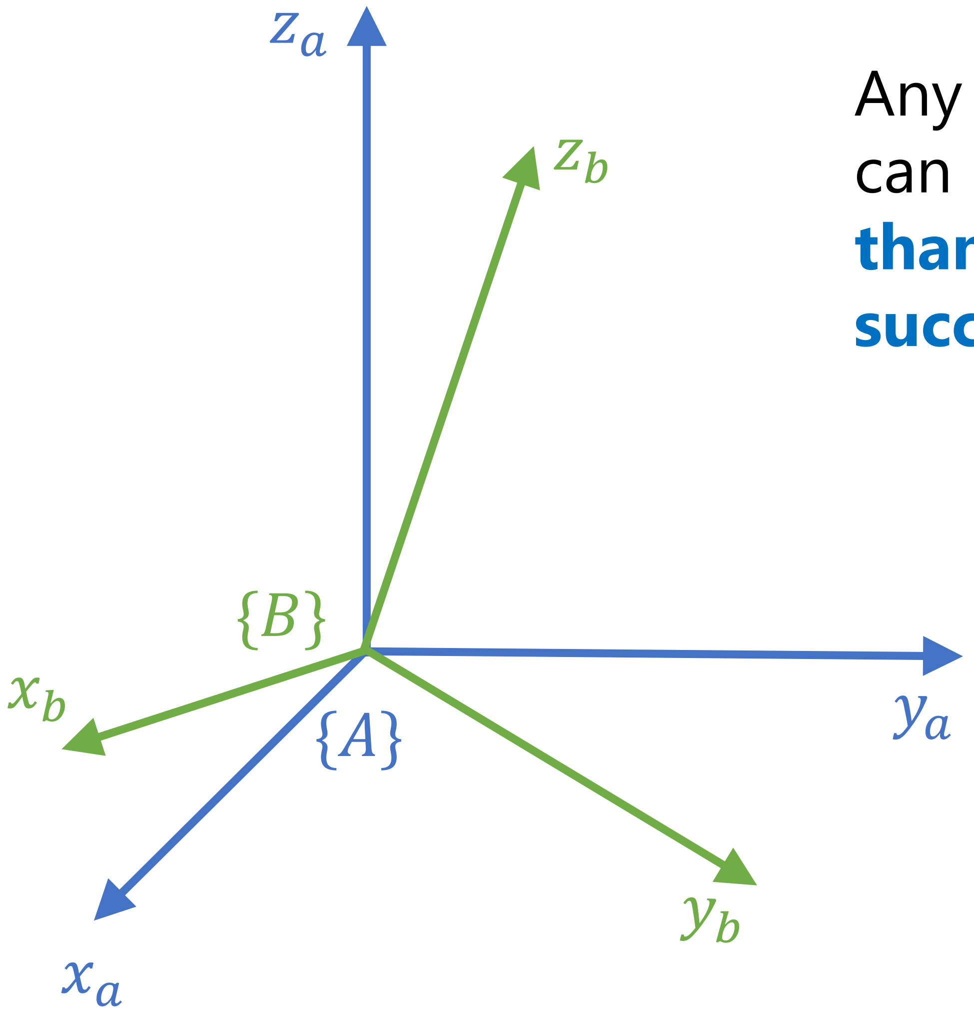
Parameterization of Rotation

- How do we represent a 3D rotation ?

Several conventions exist:

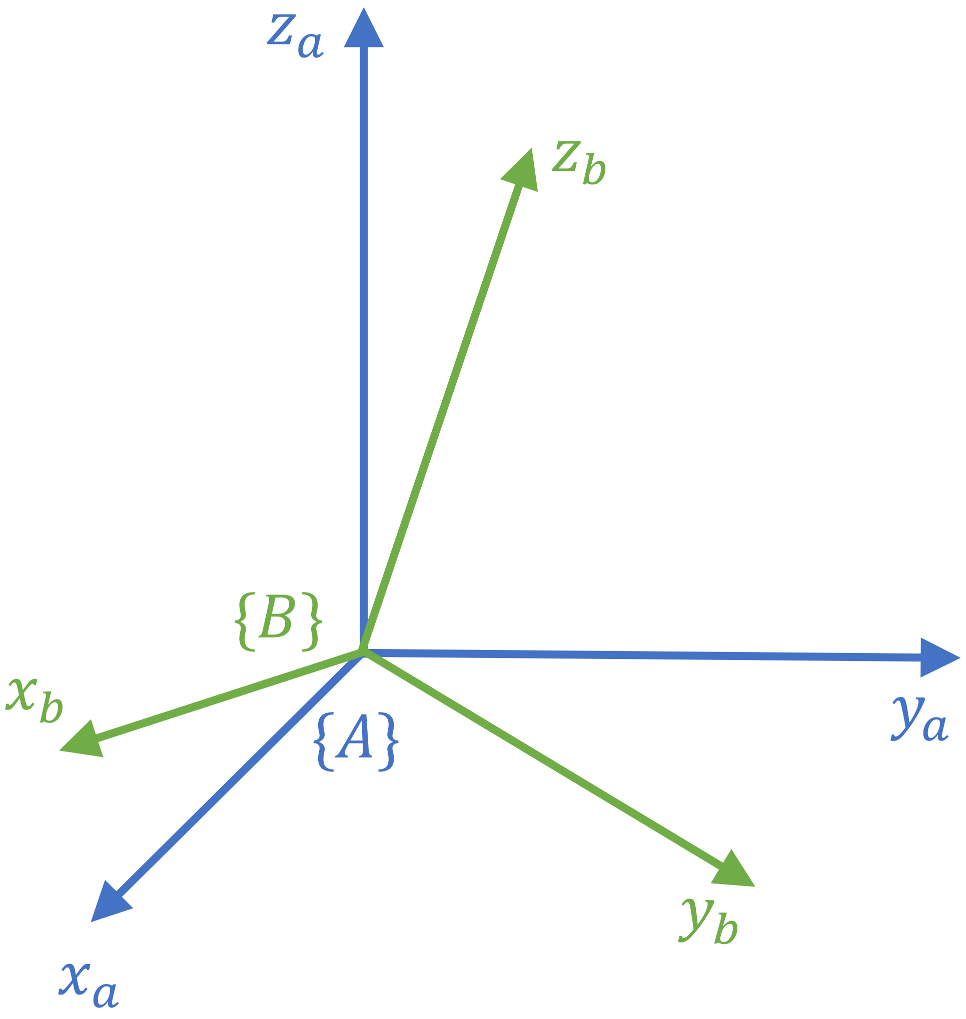
- Euler angle representation
- Roll-pitch-yaw representation
- Axis angle representation
- Quaternions

Euler's Rotational Theorem



Any two independent orthonormal coordinate frames can be related by **a sequence of rotations (not more than 3)** about coordinate axes (e.g., x,y,z), where **no two successive rotations** may be about the **same axis**.

Euler Angles



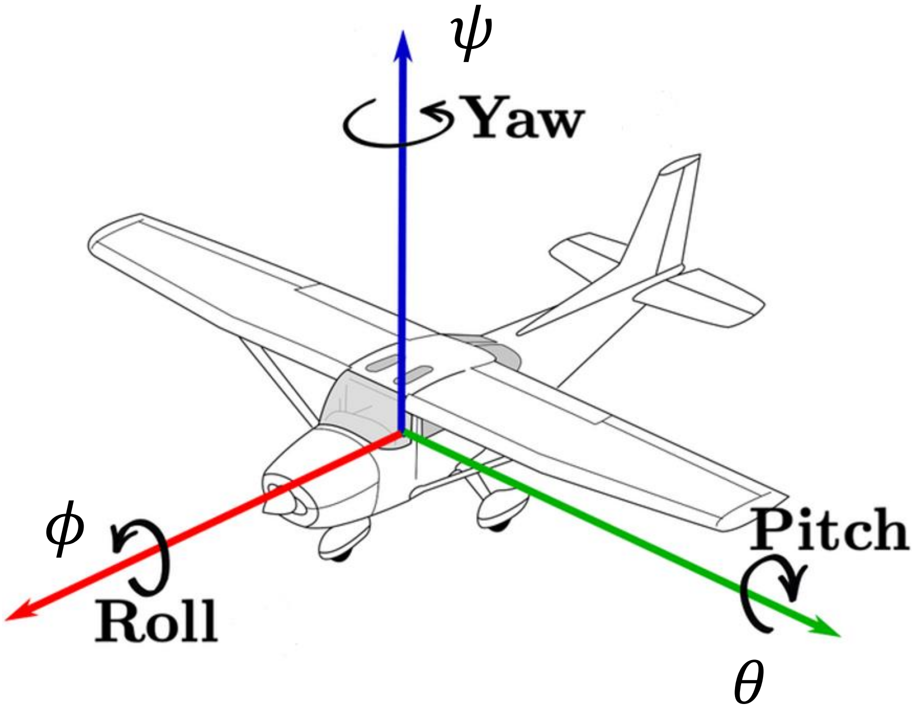
All possible rotation combinations which agree Euler's Rotational Theorem:

$$\begin{matrix} \boldsymbol{XYX} & \boldsymbol{XYZ} & XZY & \boldsymbol{XZX} \\ \boldsymbol{YXY} & YXZ & YZX & \boldsymbol{YZY} \\ ZXY & \boldsymbol{ZXZ} & \boldsymbol{ZYX} & \boldsymbol{ZYZ} \end{matrix}$$

The combinations containing two rotations around the same axis are **Classic Euler Angles.** $\boldsymbol{ZYZ}$ is used in the Robotics toolbox for this course.

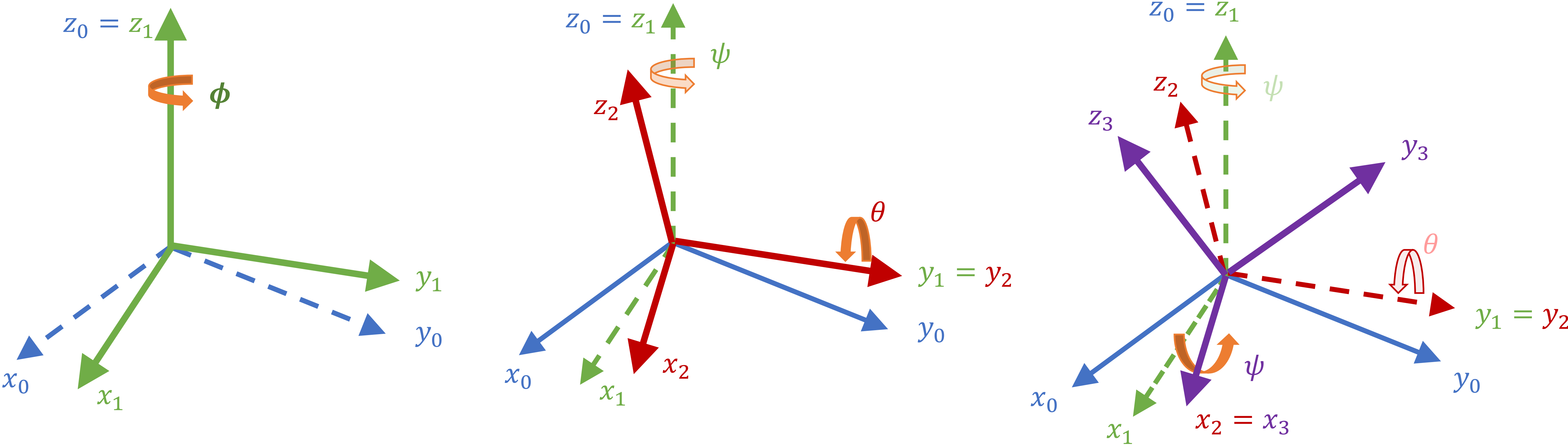
$\boldsymbol{XYZ}$ and $\boldsymbol{ZYX}$ conventions are used in the **Roll-Pitch-Yaw** representation (from Aviation).

Roll-Pitch-Yaw Representation



- The ZYX and XYZ conventions:
  ϕ – Roll ; θ – Pitch ; ψ – Yaw
- Generally, the ZYX convention in current axes in a body-centric coordinate frame is used.
- Roll-Pitch-Yaw can also be expressed in terms of a fixed frame.
- Pay attention to the order (XYZ v.s. ZYX) used in different textbooks !

Example 5: Rotation Around ZYX



$$R_{zyx}(\phi, \theta, \psi) = R_z(\phi).R_y(\theta).R_x(\psi)$$

Example 5: Rotation Around ZYX

$$R_{zyx}(\phi,\theta,\psi) = R_z(\phi).\,R_y(\theta).\,R_x(\psi)$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Example 6: Rotation Around ZYZ (Classic Euler Angles)

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Rotation Around ZYZ (Classic Euler Angles)

❑ Inverse kinematic problem for manipulator

- Given the rotation matrix R_{ZYZ}, what are the rotation angles?

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- If not both r12 and r23 are 0

$$c_\theta = r_{33},\ s_\theta = \pm\sqrt{1 - r_{33}^2}$$

$$\theta = \operatorname{atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right)$$

or

$$\theta = \operatorname{atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right)$$

Rotation Around ZYZ (Classic Euler Angles)

- Inverse kinematic problem for manipulator

  - If $\theta = \mathrm{atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right)$

$$\begin{aligned}\phi &= \mathrm{atan2}(r_{13}, r_{23})\\ \psi &= \mathrm{atan2}(-r_{31}, r_{32})\end{aligned}$$

  - If $\theta = \mathrm{atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right)$

$$\begin{aligned}\phi &= \mathrm{atan2}(-r_{13}, -r_{23})\\ \psi &= \mathrm{atan2}(r_{31}, -r_{32})\end{aligned}$$

Rotation Around ZYZ (Classic Euler Angles)

- Inverse kinematic problem for manipulator

  - If both r_{12} and r_{23} are 0

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

- If $r_{33} = 1$

$$\begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- If $r_{33} = -1$

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- There are infinitely many solutions

Issue of Euler Angle

❑Loss of a DOF with Euler angles that is called Gimbal lock

Consider ZYX rotation

$$R_{zyx}(\phi,\theta,\psi) = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

When $\theta = \frac{\pi}{2}$, $\cos\theta = 0$. Then

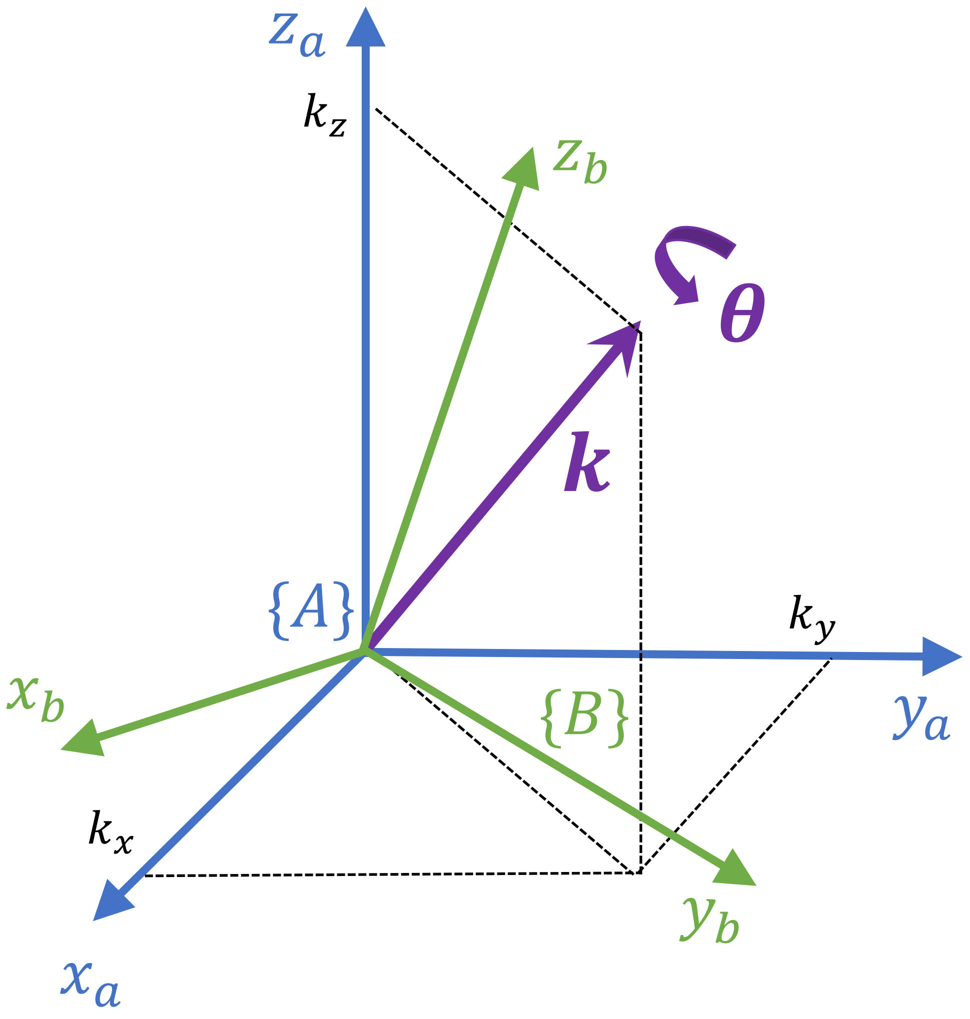
$$R_{ZYX} = \begin{bmatrix} 0 & -s(\Phi+\varphi) & c(\Phi+\varphi) \\ 0 & c(\Phi+\varphi) & -s(\Phi+\varphi) \\ -1 & 0 & 0 \end{bmatrix}$$

Solution: Quaternion-based rotation matrix representation

Change in $\boldsymbol{\Phi}$ and $\boldsymbol{\varphi}$ has the same effect !!

"The gimbal lock problem does not make Euler angles **invalid**, but it makes them unsuited for some practical applications." [https://en.wikipedia.org/wiki/Gimbal_lock]

Axis Angle Representation

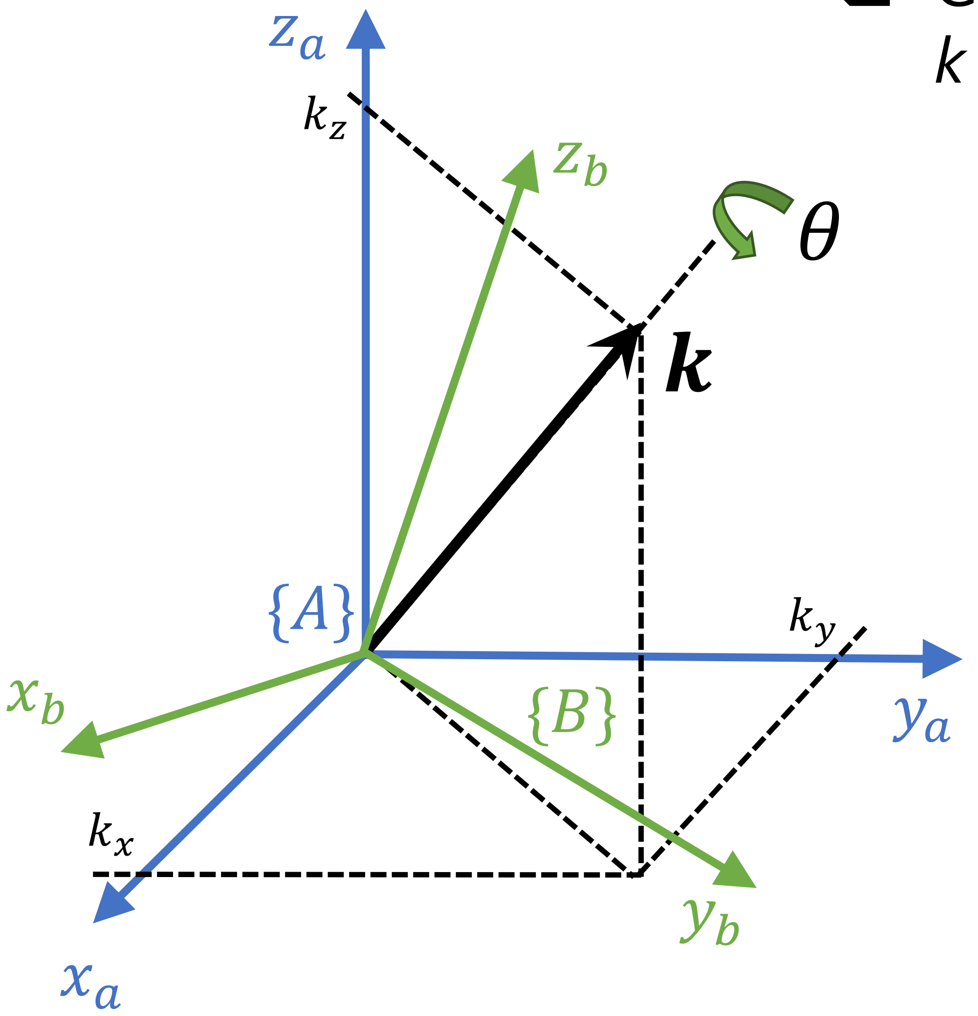


Any two independent orthonormal coordinate frames (e.g., **{A}** and **{B}**) can be related by a **single rotation** about **some axis (*k*)**.

$\boldsymbol{k}$: axis of rotation
$\boldsymbol{\theta}$: angle of rotation about axis $\boldsymbol{k}$

$\boldsymbol{k}$ must be unchanged by the rotation. Therefore, $\boldsymbol{k}$ must be an eigenvector of the rotation matrix R

Axis Angle Representation



- Given the rotation matrix $R \in SO(3)$, the rotation axis k and angle θ are given by:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Example 7: Axis Angle Representation

- Suppose R is generated by a rotation of 60° about x_0, followed by a rotation of 30° about y_1 followed by a rotation of 90° about z_2 (XYZ Euler angles). Find k and θ that represent equivalent rotation.

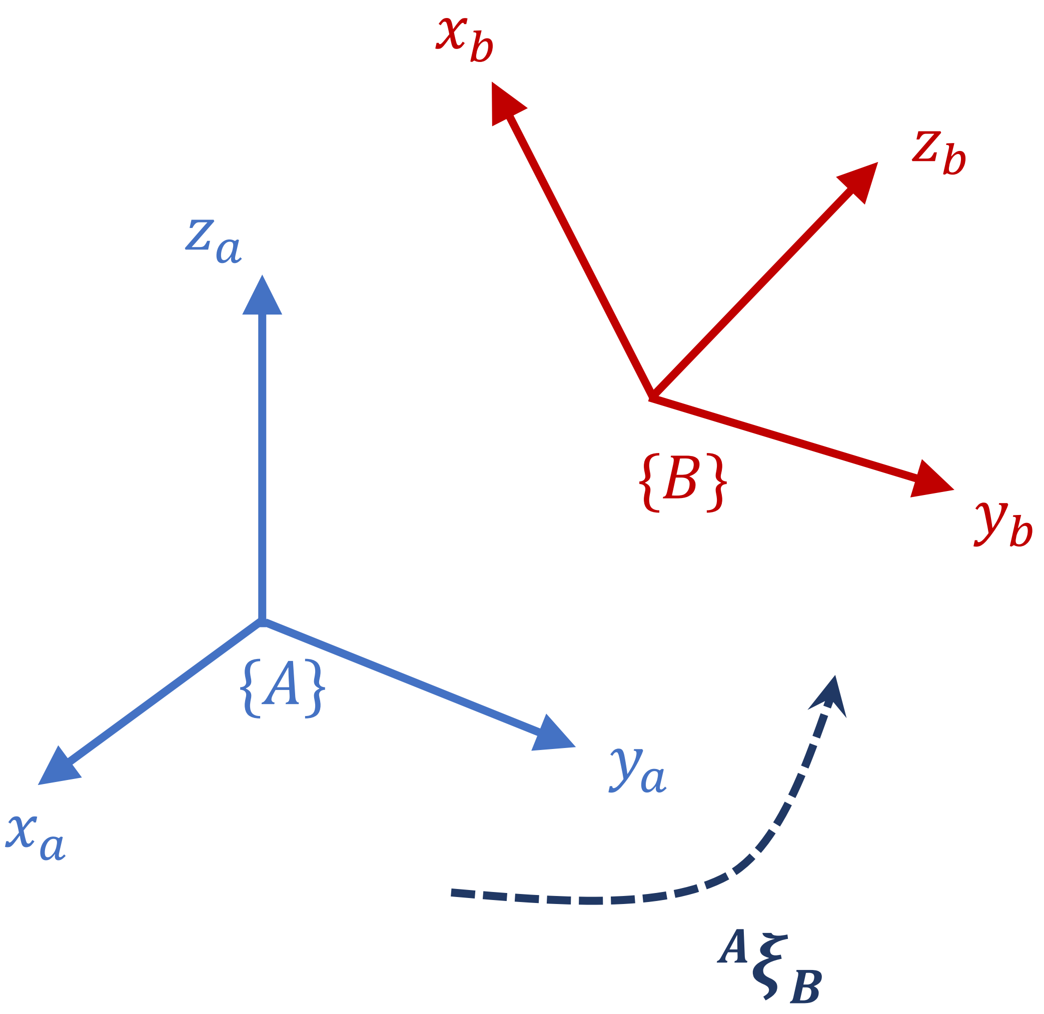
$$R = R_{x,60}R_{y,30}R_{z,90} = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$$

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} \Rightarrow k = \left(\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}} - \frac{1}{2}, \frac{1}{2\sqrt{3}} + \frac{1}{2}\right)^T$$

Homogeneous Transformations

Robot Kinematics



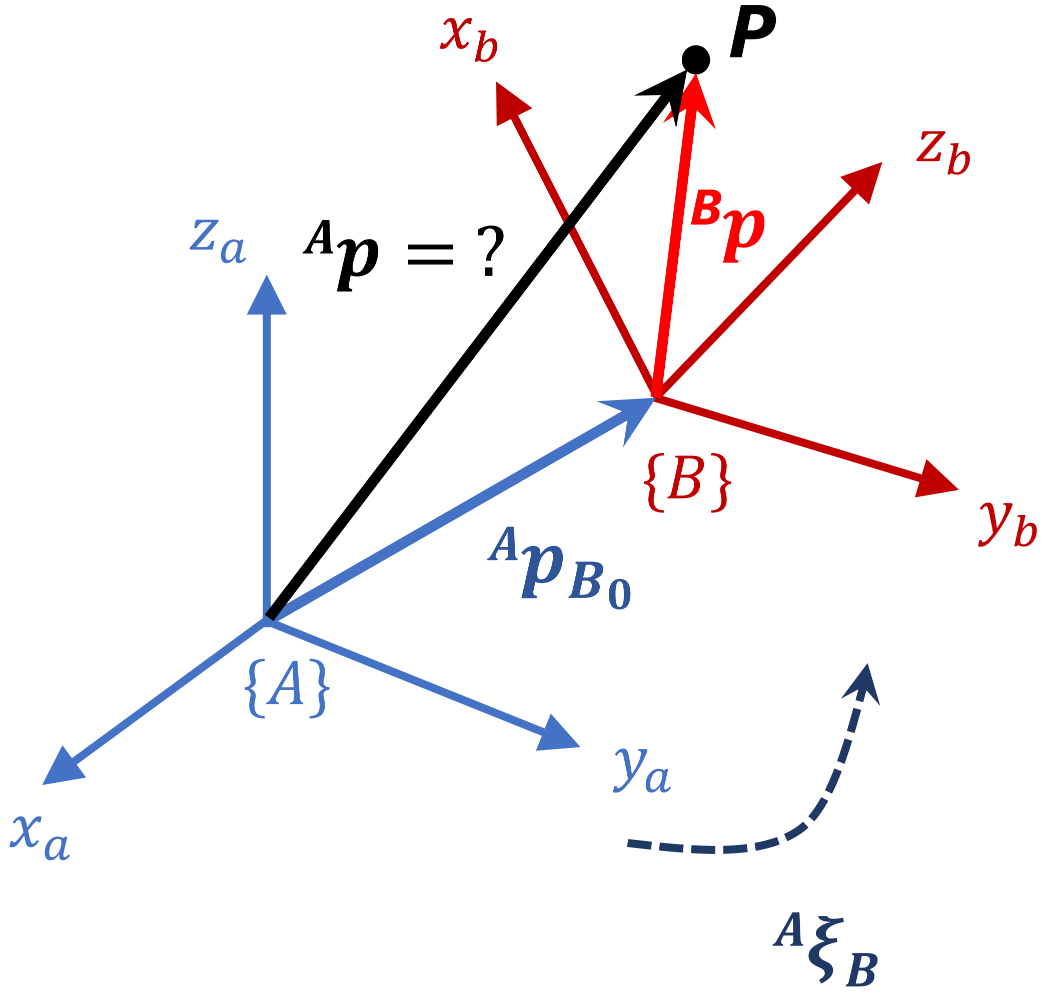
To specify the location of an end-effector we need its Position and orientation – i.e.: a "Frame"

If frame $\{B\}$ represents end effector, its orientation and location w.r.t $\{A\}$ are defined by,

$$ {}^{A}\xi_{B} = \{ {}^{A}R_{B} \, , {}^{A}\boldsymbol{p}_{Bo} \} $$

Orientation: ${}^{A}R_{B}$
Translation: ${}^{A}\boldsymbol{p}_{Bo}$

Mapping of Frames Revisited



Recall,

$${}^{A}\boldsymbol{p} = {}^{A}\xi_{B}\, {}^{B}\boldsymbol{p}$$

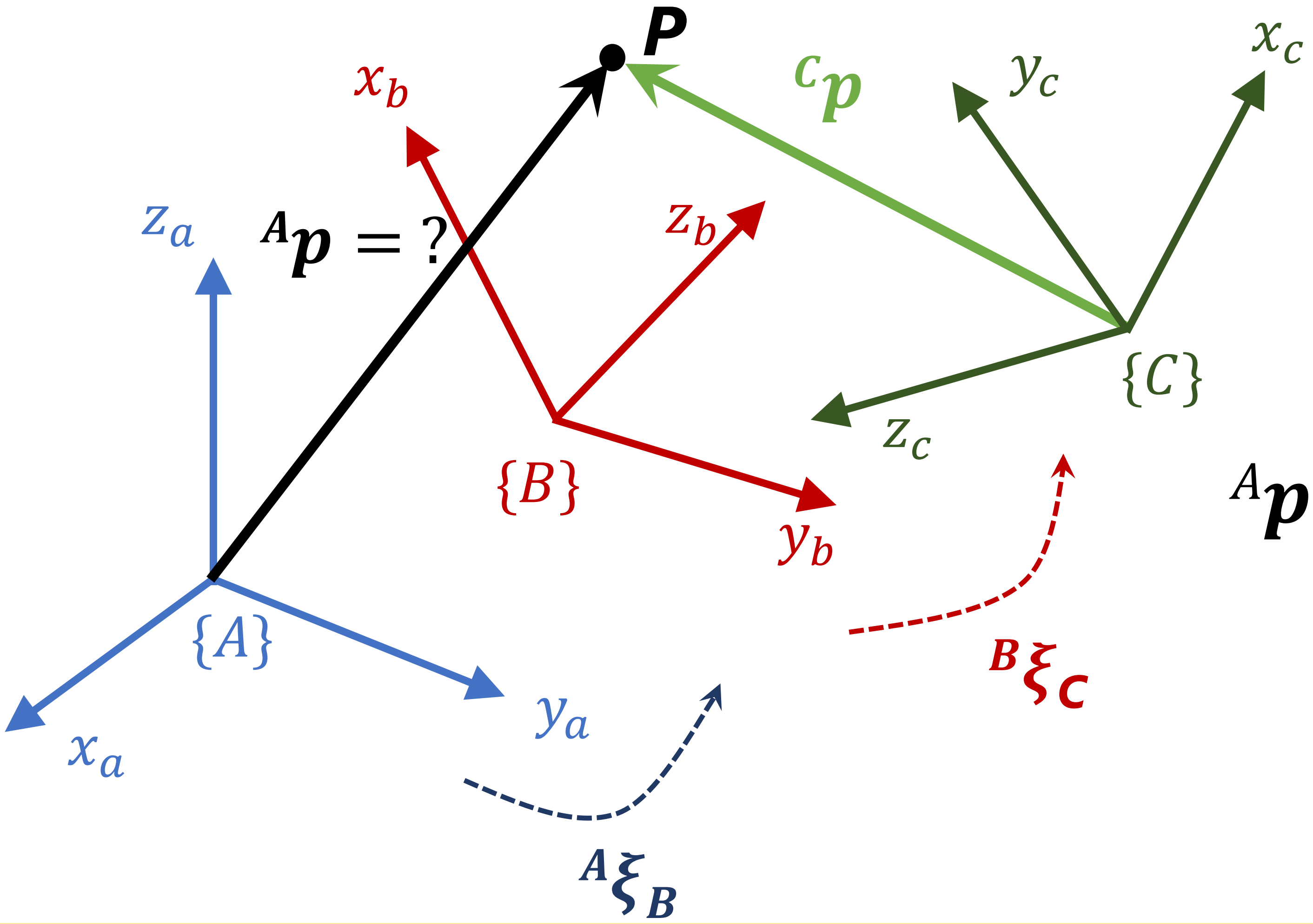
$${}^{A}\boldsymbol{p} = {}^{A}R_{B}\, {}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{p}_{Bo}$$

${}^{A}R_{B}$: **rotational** transformation matrix expressing the orientation of $\{B\}$ relative to $\{A\}$

${}^{A}\boldsymbol{p}_{Bo}$: **translational** transformation vector from the origin of $\{A\}$ to the origin of frame $\{B\}$

Mapping of Frames Revisited

- Two motions (e.g. two-link robots)

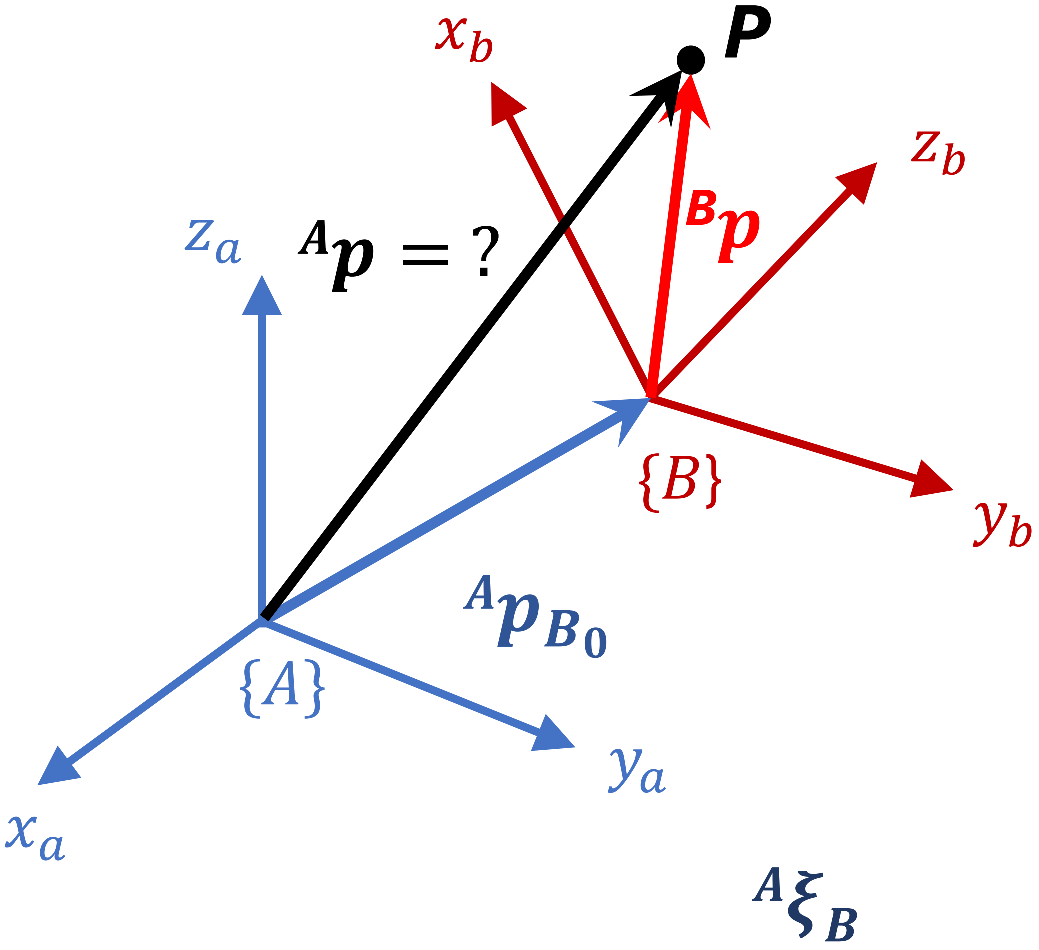


$$^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{p}_{Bo}$$

$$^{B}\boldsymbol{p} = {}^{B}R_{C}\,{}^{C}\boldsymbol{p} + {}^{B}\boldsymbol{p}_{Co}$$

$$^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}R_{C}\,{}^{C}\boldsymbol{p} + {}^{A}R_{B}\,{}^{B}\boldsymbol{p}_{Co} + {}^{A}\boldsymbol{p}_{Bo}$$

Single transformation operator to replace pose ($^{A}\xi_{B}$)



$$^{A}\boldsymbol{p} = {}^{A}R_{B}\, {}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{p}_{Bo}$$

$$^{A}\boldsymbol{p} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{p}_{Bo} \end{bmatrix} \begin{bmatrix} {}^{B}\boldsymbol{p} \\ 1 \end{bmatrix}$$

$$1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\boldsymbol{p} \\ 1 \end{bmatrix}$$

Modify above expression as,

$$\begin{bmatrix} {}^{A}\boldsymbol{p} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{p}_{Bo} \\ 0\ 0\ 0 & 1 \end{bmatrix}}_{^{A}T_{B}} \begin{bmatrix} {}^{B}\boldsymbol{p} \\ 1 \end{bmatrix}$$

Homogeneous Transformation

Homogeneous Transformation Operator (${}^{x}T_{y}$)

$$ {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{p}_{Bo} \\ 0\ 0\ 0 & 1 \end{bmatrix} $$

- ❑ Contains position and orientation information
- ❑ 3 dimensional applications result in a 4 x 4 matrix
- ❑ Can define a frame (i.e., pose)
- ❑ Can map a point defined in frame $\{B\}$, relative to frame $\{A\}$

Rotation-only transformation

$$ {}^{A}T_{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Translation-only transformation

$$ {}^{A}T_{B} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Special Euclidian Group

❑ A rigid body motion comprising of an ordered pair (d, R)

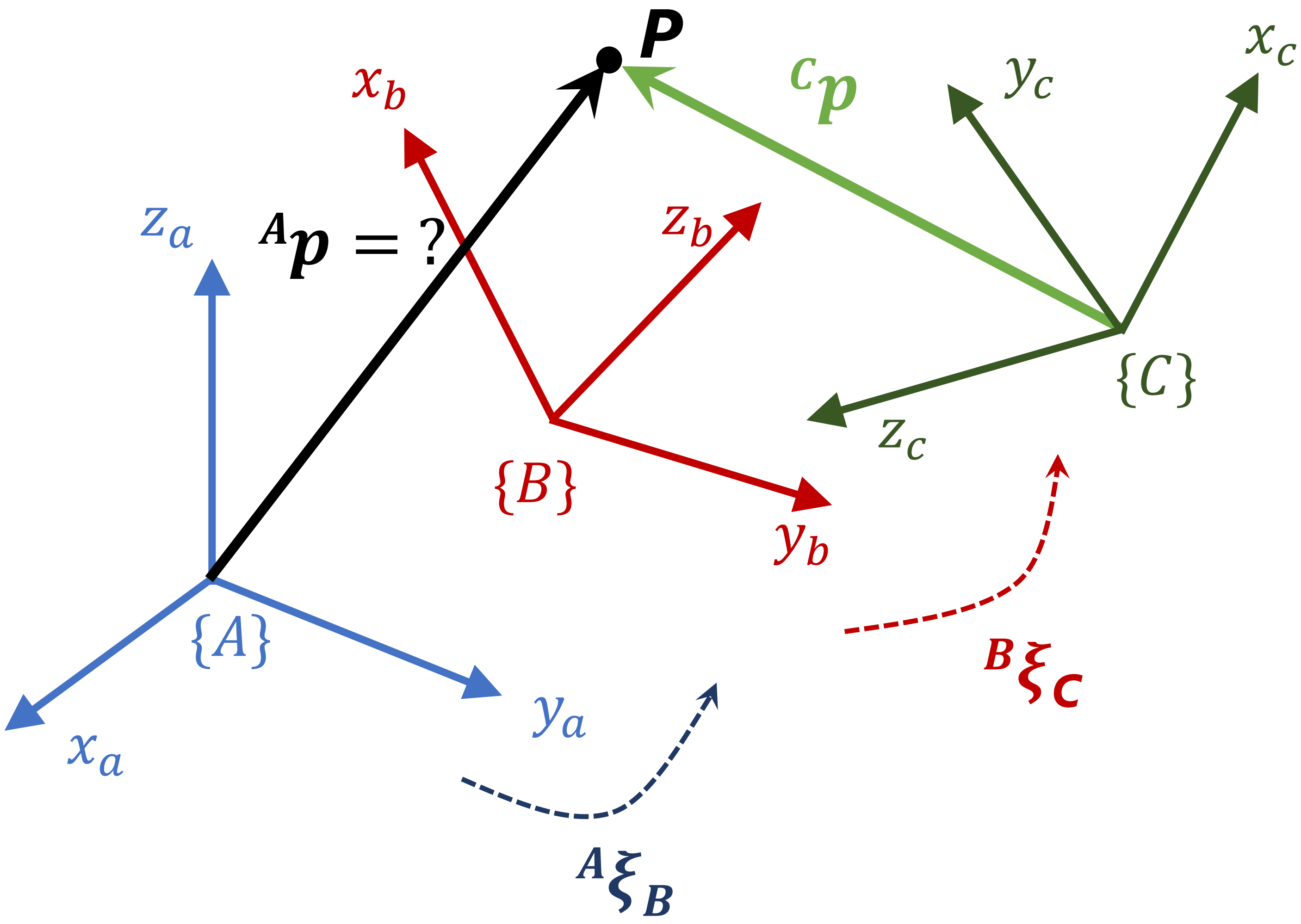
- $\boldsymbol{d} \in \mathbb{R}^3$ is translation operator
- $\boldsymbol{R} \in SO(3)$ is rotation operator

$$T = \left[\begin{array}{c|c} R & d \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

(*d*,*R*) forms the Special Euclidean Group SE(3).

$$\boxed{SE(3) = \mathbb{R}^3 \times SO(3)}$$

Compounding Frames using *T*



Given that we know $^C\boldsymbol{p}$, find $^A\boldsymbol{p}$?

Compound transformation

$$^AT_C = {}^AT_B{}^BT_C$$

Compounding Frames using *T*

$$ {}^{A}T_{C} = \left[\begin{array}{ccc|c} & {}^{A}R_{B} & & {}^{A}\boldsymbol{p}_{Bo} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \times \left[\begin{array}{ccc|c} & {}^{B}R_{C} & & {}^{B}\boldsymbol{p}_{Co} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Rotation Operator Translation Operator

$$ {}^{A}T_{C} = \left[\begin{array}{ccc|c} & {}^{A}R_{B}\,{}^{B}R_{C} & & {}^{A}R_{B}\,{}^{B}\boldsymbol{p}_{Co} + {}^{A}\boldsymbol{p}_{Bo} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

- Multiplication of matrixes is not commutative
- Order of rotations is important

Example 8: Homogeneous Transformation

The homogeneous transformation matrix H that represents a rotation by angle α about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle θ about the current z-axis, is given by:

Answer

$$Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha & 0 \\ 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_{x,b} = \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_{z,d} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot_{z,\theta} = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 8: Homogeneous Transformation

A rotation by angle about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle about the current z-axis

$$H = Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha & -s\alpha & 0 \\ 0 & s\alpha & c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Rules for Robot Arms

- Robot has $n + 1$ links, numbered from 0 to n, base is taken as link 0

- Each joint has a variable q_i:
  - If revolute, $q_i \ = \ \theta_i$ (angle of rotation)
  - If prismatic, $q_i \ = \ di$, (joint displacement)

- ${}^{i-1}T_i$ is the transformation matrix that transforms the coordinates of a point from frame i to frame $i - 1$

Transformation Rules for Robot Arms

- In DH Convention* each homogeneous transformation ${}^{i-1}T_i$ represents the product of four basic homogeneous transformations.

$$ {}^{i-1}T_i = R_{z(i-1)}(\theta_i).\,Q_{z(i-1)}(d_i).\,Q_{xi}(a_i).\,R_{xi}(\alpha_i) $$

angle offset length twist

* Will be discussed in lecture 4

Lecture 3 Summary

- $^{A}\xi_{B}$: Pose of frame B with respect to frame A.
- **Pose** contains rotational **and** translation components
- Rotation matrix can be represented by Classic Euler Angles, Roll-Pitch-Yaw, Axis-Angle conventions
- Homogeneous transformation helps us to represent a point defined in a frame, with respect to another frame.

$$T = \left[\begin{array}{ccc|c} & R & & d \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Lecture 4 – Denavit Hartenberg Convention

- ❑ Familiarise yourself with matrix operations
- ❑ Selection of joint frames using DH method
- ❑ Forward kinematics: Calculate the pose of end-effector using homogeneous transformation

Example 9

A robot is set up 1 meter from a table. The tabletop is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2m above the tabletop with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.