Lecture 4 - Revision

✓ DH convention

i	θ_i	d_i	a_i	α_i
1	25	0	3	0
2	40	0	4	0
3	0	0	0	0

✓ RVC Toolbox

```
L(1) = Link('revolute', 'd', d1, 'a', a1, 'alpha', alpha1, 'offset', 0);
L(2) = Link('revolute', 'd', d2, 'a', a2, 'alpha', alpha2, 'offset', 0);
L(3) = Link('revolute', 'd', d3, 'a', a3, 'alpha', alpha3, 'offset', 0);
```

```
robot= SerialLink(L, 'name', ' three link');
```

```
Matrix= robot.fkine([theta]);
```

MTRN4230

Robotics

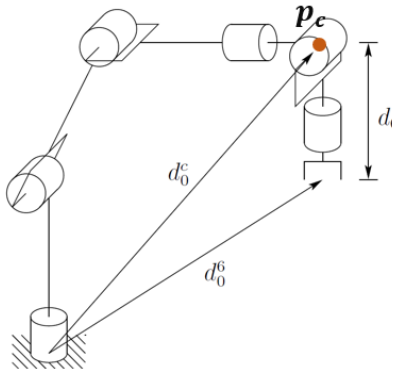
Lecture 5

Inverse Kinematics & The Jacobian

Hoang-Phuong **Phan** – T2 2023

Learning Objectives

- Inverse kinematics



- The Jacobian

$$J_i = \begin{pmatrix} {}^0\mathbf{z}_{i-1} \times ({}^0\mathbf{o}_n - {}^0\mathbf{o}_{i-1}) \\ \mathbf{z}_{i-1} \end{pmatrix} \text{ if joint } i \text{ is revolute;}$$

Or

$$J_i = \begin{pmatrix} {}^0\mathbf{z}_{i-1} \\ 0 \end{pmatrix} \text{ if joint } i \text{ is prismatic.}$$

- Quiz 1 revision

Inverse Kinematics

- Forward kinematics: Given joint variables $(q_1, q_2, \ldots . q_n)$, find the orientation and position of the end effector:

$$^0T_n = \begin{pmatrix} R & \boldsymbol{p} \\ 0 & 1 \end{pmatrix}$$

- Inverse kinematics: Given the homogeneous transformation matrix,

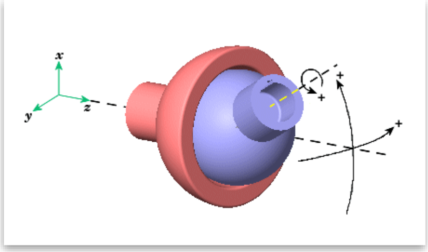
$$^0T_n = \begin{pmatrix} R & \boldsymbol{p} \\ 0 & 1 \end{pmatrix},$$

Can we find unique or multiple solutions for the robot's joint variables $q_1, q_2, \ldots . q_n$ associated with joints $1,2, \ldots ., n$ respectively ?

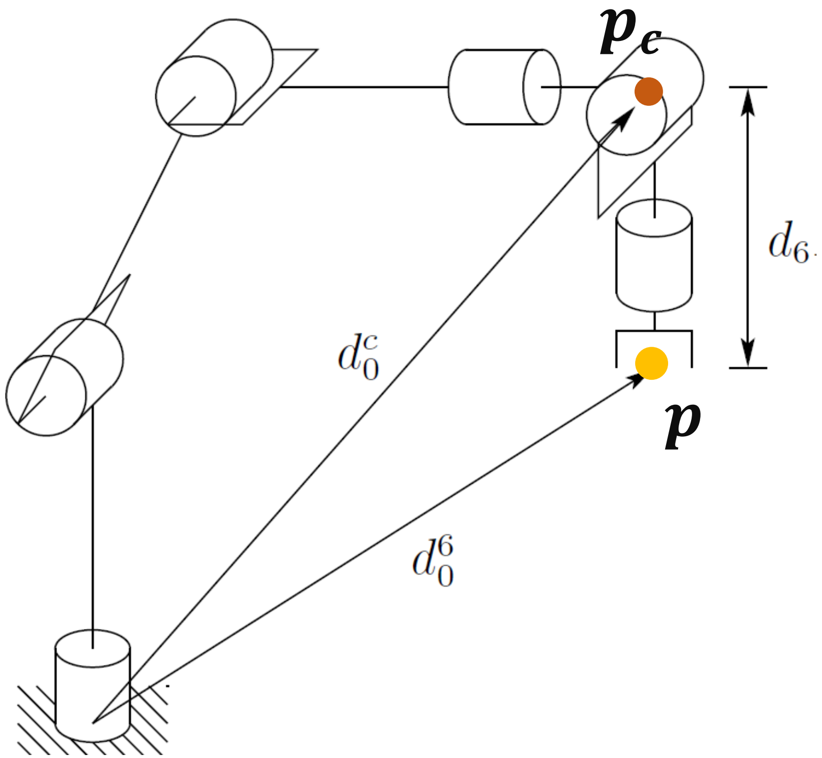
Inverse Kinematics

- Three methods
  - Kinematic decoupling
  - Algebraic approach (Lecture 2 – 2 link robot)
  - Numerical method
    - Consider the inverse kinematic problem as an optimisation problem
    - Minimise the error between the forward kinematic solution and the desired pose.

Kinematic Decoupling



- Requirements:
  - Manipulator has six joints
  - The last three joint origins intersect at a single point (e.g., spherical wrist)



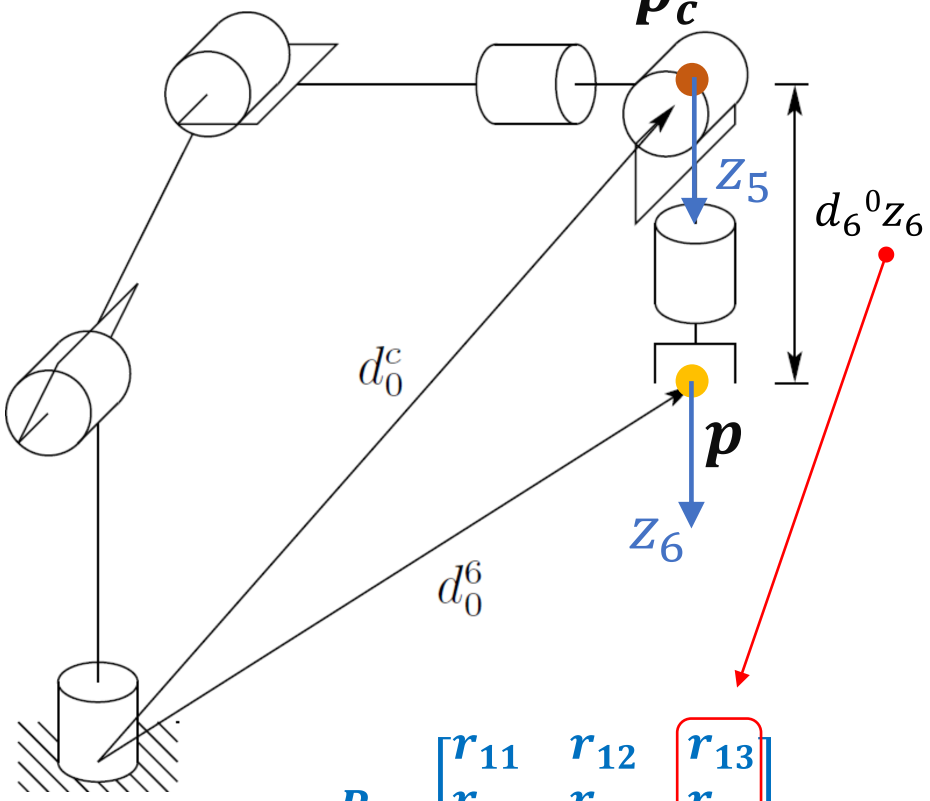
- Separate into two simpler problems
  - First solve (q_1, q_2, q_3) for the position of intersection of the 3 wrist axes (wrist center).
  - Then solve (q_4, q_5, q_6) for the required orientation of the wrist.

Kinematic Decoupling

- D-H Convention of 6 joints

$$^0T_n(q_1, q_2, \dots, q_6) = {^0T_1}{^1T_2}\dots{^5T_6} = \begin{pmatrix} R & \boldsymbol{p} \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{gathered} {^0\boldsymbol{R}_6}(q_1, q_2, \dots, q_6) = {^0\boldsymbol{R}_3}{^3\boldsymbol{R}_6} \\ {^0\boldsymbol{p}_6}(q_1, q_2, \dots, q_6) = \boldsymbol{p} \end{gathered}}$$



$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- The position of the wrist center is a function of q_1, q_2 and q_3
- The position of the tool frame: $\boldsymbol{p} = \boldsymbol{p_c} + \boldsymbol{d_6}{^0\boldsymbol{z_6}}$
- Therefore, the position of the coordinate of the wrist centre $\boldsymbol{p}_c$ is:

$$\boxed{\boldsymbol{p_c} = \boldsymbol{p} - \boldsymbol{d_6}{^0\boldsymbol{z_6}} = \begin{pmatrix} \boldsymbol{x_c} \\ \boldsymbol{y_c} \\ \boldsymbol{z_c} \end{pmatrix}}$$

Kinematic Decoupling

- Step 1: find q_1, q_2, q_3 such that $\boldsymbol{p}_c = \begin{pmatrix} \boldsymbol{x_c} \\ \boldsymbol{y_c} \\ \boldsymbol{z_c} \end{pmatrix}$
- Step 2: use q_1, q_2, q_3 to evaluate ${}^0R_3$

$$ {}^0T_3(q_1, q_2, q_3) = {}^0T_1\,{}^1T_2\,{}^2T_3 = \begin{pmatrix} {}^0R_3 & {}^0P_3 \\ 0 & 1 \end{pmatrix} $$

- Step3: The orientation of the wrist center is a function of q_4, q_5 and q_6.
  Find a set of Euler angles corresponding to the rotation matrix:

  - ${}^3R_6 = ({}^0R_3)^{-1}\,R = ({}^0R_3)^T\,R$

Numerical methods

- Consider the inverse kinematic problem as an optimisation problem
- Minimise the error between the forward kinematic solution and the desired pose.
- The solution can be found by using one of many function minimisation algorithms, and an initial guess at the joint coordinates

For instance, position error

$$\varepsilon_p = \sqrt{(x - x_d)^2 + (y - y_d)^2 + (z - z_d)^2}$$

Orientation error

$$\varepsilon_r = \frac{\|R - R_d\|}{\|R_d\|}$$

https://www.mdpi.com/2218-6581/12/2/38

The Jacobian

$$\boxed{\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{bmatrix} \boldsymbol{J_v} \\ \boldsymbol{J_\omega} \end{bmatrix} \dot{q}}$$

Pose of robot end-effector

Consider an n-link manipulator with joint variables $q_1, \ldots, q_n$

$$ {}^0T_n(\mathbf{q}) = \begin{pmatrix} {}^0R_n(\mathbf{q}) & {}^0\mathbf{o}_n(\mathbf{q}) \\ 0 & 1 \end{pmatrix} $$

End effector position and orientation expressed in a vector:

$$ d = \begin{pmatrix} x \\ y \\ z \\ \theta \\ \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{q}) \\ f_2(\mathbf{q}) \\ f_3(\mathbf{q}) \\ f_4(\mathbf{q}) \\ f_5(\mathbf{q}) \\ f_6(\mathbf{q}) \end{pmatrix} $$

The Jacobian

$$\dot{d} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \frac{\delta f_1}{\delta q_1} & \frac{\delta f_1}{\delta q_2} & \cdots & \frac{\delta f_1}{\delta q_n} \\ \frac{\delta f_2}{\delta q_1} & \frac{\delta f_2}{\delta q_2} & \cdots & \frac{\delta f_2}{\delta q_n} \\ \frac{\delta f_3}{\delta q_1} & \frac{\delta f_3}{\delta q_2} & \cdots & \frac{\delta f_3}{\delta q_n} \\ \frac{\delta f_4}{\delta q_1} & \frac{\delta f_4}{\delta q_2} & \cdots & \frac{\delta f_4}{\delta q_n} \\ \frac{\delta f_5}{\delta q_1} & \frac{\delta f_5}{\delta q_2} & & \frac{\delta f_5}{\delta q_n} \\ \frac{\delta f_6}{\delta q_1} & \frac{\delta f_6}{\delta q_2} & & \frac{\delta f_6}{\delta q_n} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{q}_n \end{pmatrix}$$

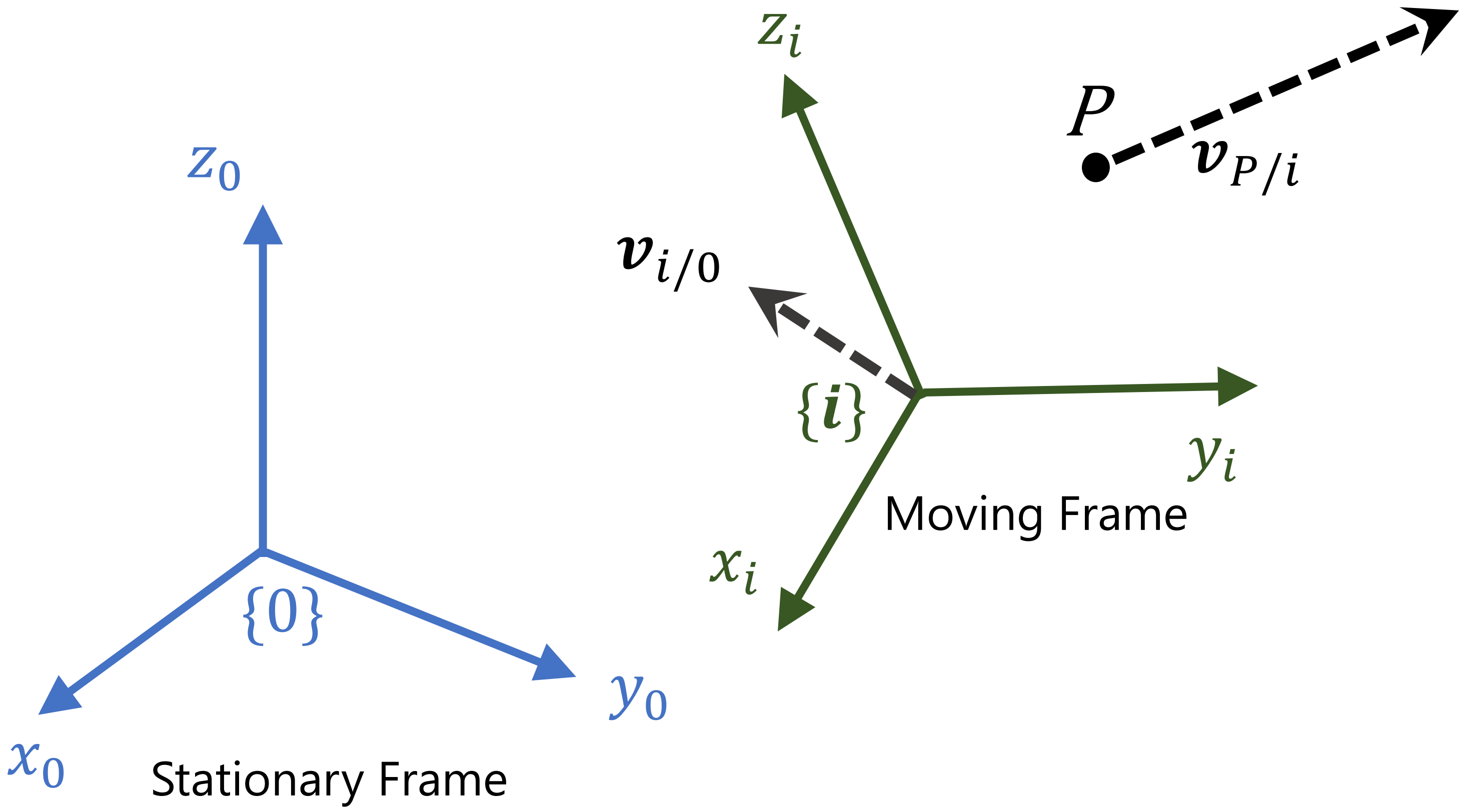
$$\boxed{\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{bmatrix} \boldsymbol{J_v} \\ \boldsymbol{J_\omega} \end{bmatrix} \dot{q}}$$

Find $\boldsymbol{J_v}$ and $\boldsymbol{J_\omega}$ from DH parameters

- $J(\boldsymbol{q})$ is the **Jacobian** where, $J_{ij} = \frac{\delta f_i}{\delta q_j}$
- The Jacobian is a $6 \times n$ matrix

Find J_v and J_ω

Linear Velocity due to Pure Translation

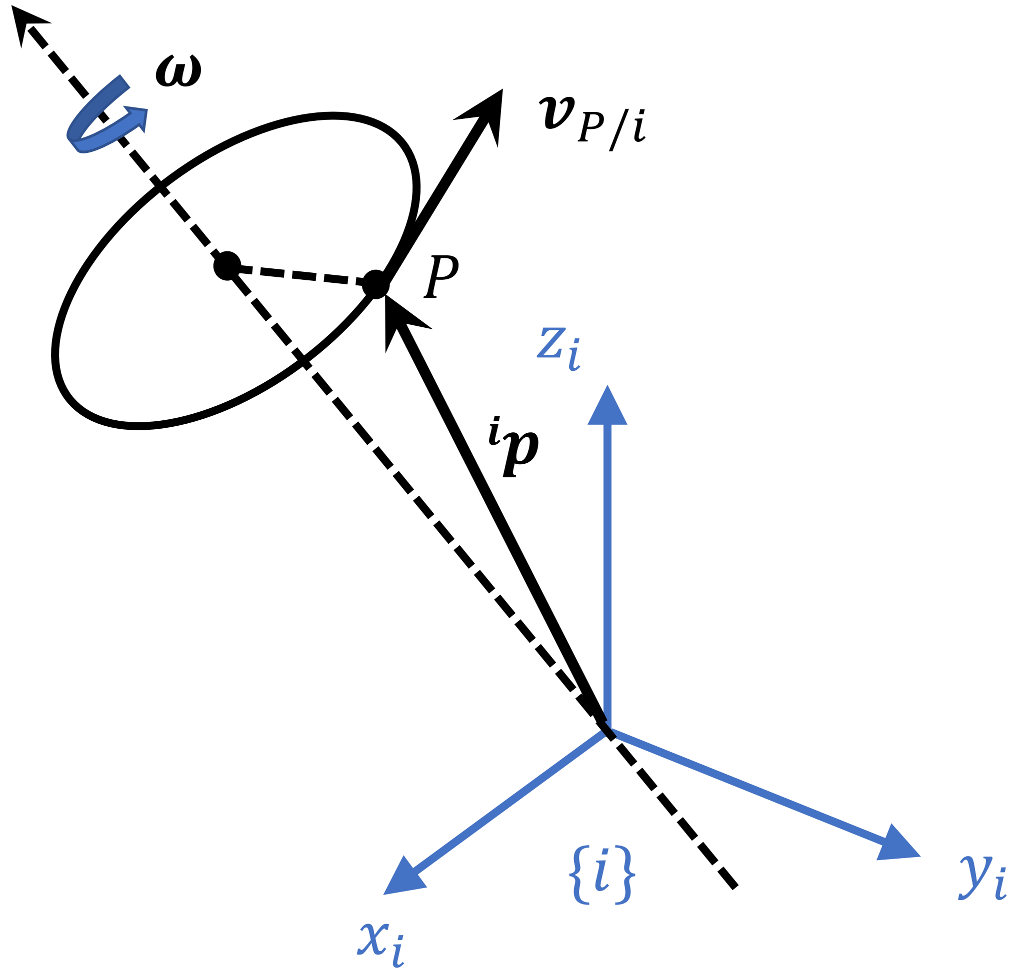


$$\boxed{\boldsymbol{v}_{P/0} = \boldsymbol{v}_{i/0} + \boldsymbol{v}_{P/i}}$$

- $\boldsymbol{v}_{i/0}$: Linear velocity of frame $\{i\}$ with respect to frame $\{0\}$ (i.e., ${}^0\boldsymbol{v}_i$)
- $\boldsymbol{v}_{P/i}$ is the relative velocity of P with respect to the origin of frame {i}

Find J_v and J_ω

❑ Angular Motion



The linear velocity of point P expressed in frame {i}

$$\boxed{\boldsymbol{v}_{P/i} = \boldsymbol{\omega} \times {}^{i}\boldsymbol{p}}$$

$\boldsymbol{v}_{P/i}$ is perpendicular to the plane formed by $\boldsymbol{\omega}$ and ${}^{i}\boldsymbol{p}$

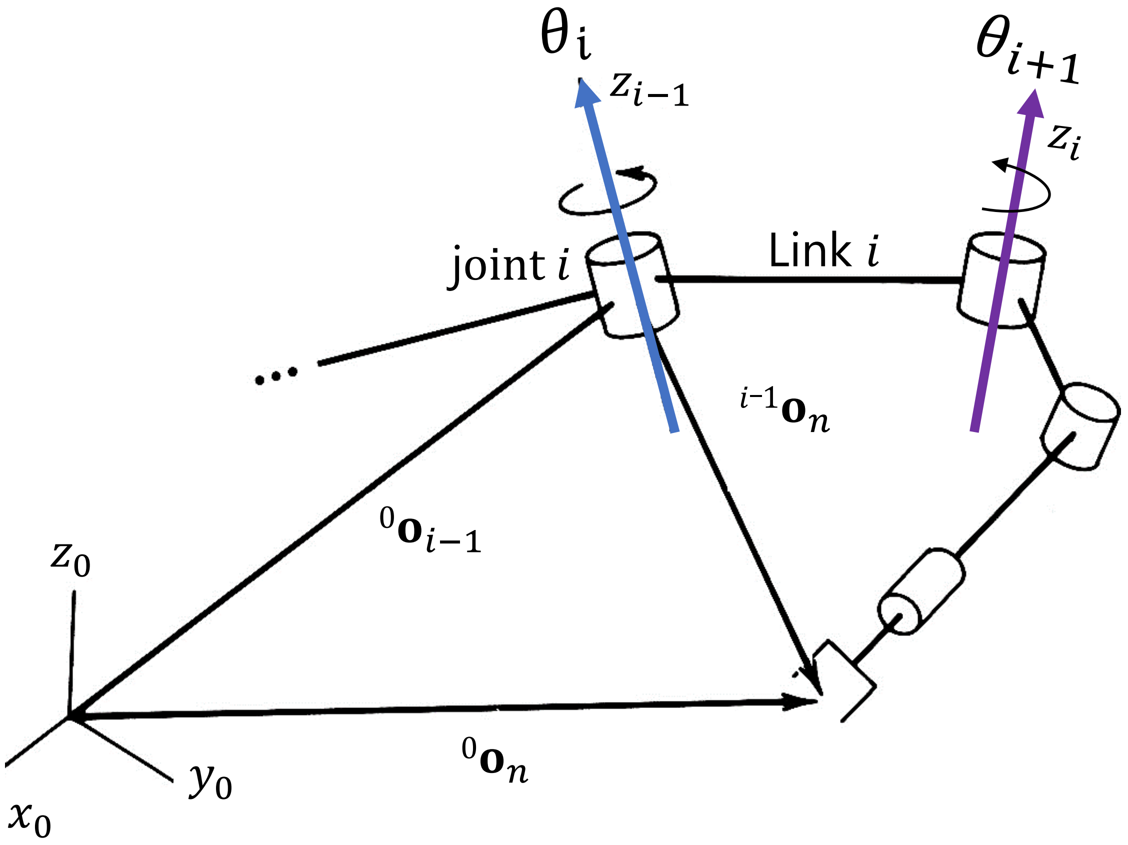
Step 1 – Find J_ω so that $\boldsymbol{\omega} = J_\omega \dot{q}$

- Joint coordinate vector $\mathbf{q} = (q_1, q_2, \ldots, q_n)^T$

- Joint coordinate $q_i = \begin{cases} \theta_i \text{ if revolute} \\ d_i \text{ if primatic} \end{cases}$

- Define a joint operator ρ_i

$$\rho_i = \begin{cases} 1 \text{ if i is revolute} \\ 0 \text{ if i is prismatic} \end{cases}$$



- Link i rotates by an angle $q_i = \theta_i$ around axis z_{i-1}, then the translation motion of end effector is normal to (z_{i-1} and ${}^{i-1}\mathbf{o}_n$) plane

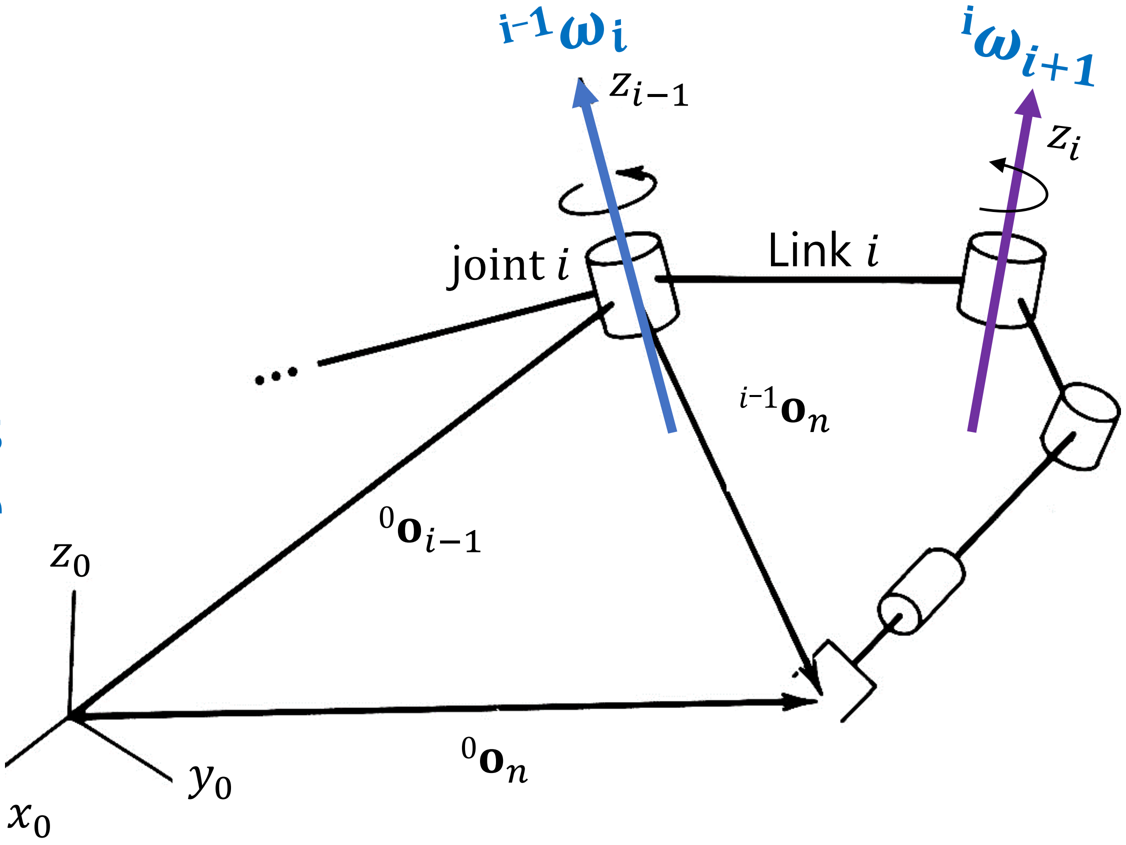
Step 1 – Find J_ω so that $\omega = J_\omega \dot{q}$

❑ Angular velocity

- ${}^{i-1}\boldsymbol{\omega}_i = \dot{q}_i \boldsymbol{k}$ ($\boldsymbol{k}$ is the unit vector in z_{i-1} direction)
- We can vectorially add angular velocities provided they are expressed relative to the common base frame:

$$\boldsymbol{\omega}_n = \rho_1 \dot{q}_1 \boldsymbol{k} + \rho_2 \dot{q}_2 \boldsymbol{k} + \ \ldots + \rho_n \dot{q}_n \boldsymbol{k}$$
$$= \sum \rho_i \dot{q}_i {}^0\mathbf{z}_{i-1}$$

- ${}^0\mathbf{z}_{i-1}$ denotes the unit vector $\boldsymbol{k}$ of frame $\{i-1\}$ expressed w.r.t. frame {0}: ${}^0\mathbf{z}_{i-1} = {}^0R_{i-1}\, \boldsymbol{k}$



Step 1 – Find J_ω so that $\omega = J_\omega \dot{q}$

$$\boldsymbol{\omega}_n = \sum_1^n \rho_i \dot{q}_i {}^0\mathbf{z}_{i-1} = \sum_1^n \rho_i {}^0\mathbf{z}_{i-1} \dot{q}_i$$

$$\omega_n = \left(\rho_1 {}^0z_0 \quad \rho_2 {}^0z_1 \quad \cdots \quad \rho_{n-1} {}^0z_n \quad \rho_{n-1} {}^0z_n\right) \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_{n-1} \\ \dot{q}_n \end{pmatrix}$$

$$\omega_n = J_\omega \dot{\mathrm{q}}$$

$$J_\omega = \left(\boldsymbol{\rho}_1 {}^0\mathbf{z}_0 \quad \boldsymbol{\rho}_2 {}^0\mathbf{z}_1 \quad \ldots \quad \boldsymbol{\rho}_n {}^0\mathbf{z}_{n-1}\right)$$

Step 2 – Find J_v so that $v = J_v \dot{q}$

❑ Linear velocity

Recall $\,^0T_n(q_1, q_2, \dots, q_n) = \,^0T_n(\mathbf{q}) = \begin{pmatrix} ^0R_n(\mathbf{q}) & ^0\mathbf{o}_n(\mathbf{q}) \\ 0 & 1 \end{pmatrix}$

The linear velocity of the end-effector is $\,^0\boldsymbol{v}_n = \,^0\dot{\mathbf{o}}_n$

$$\boldsymbol{v} = \,^0\dot{\mathbf{o}}_n = \sum_{1}^{n} \underbrace{\frac{\delta\,^0o_n}{\delta q_i}}_{i\text{ th column of } J_v} \dot{\mathbf{q}}_i$$

Find $\frac{\delta\,^0\mathrm{o}_n}{\delta q_i}$ as a function of DH parameters

Step 2 – Find J_v so that $v = J_v\dot{q}$

Case 1: Joint i is prismatic

$$^0\mathbf{o}_n = {}^0\mathbf{o}_{i-1} + {}^0R_{i-1}\,{}^{i\text{-}1}\mathbf{o}_n$$

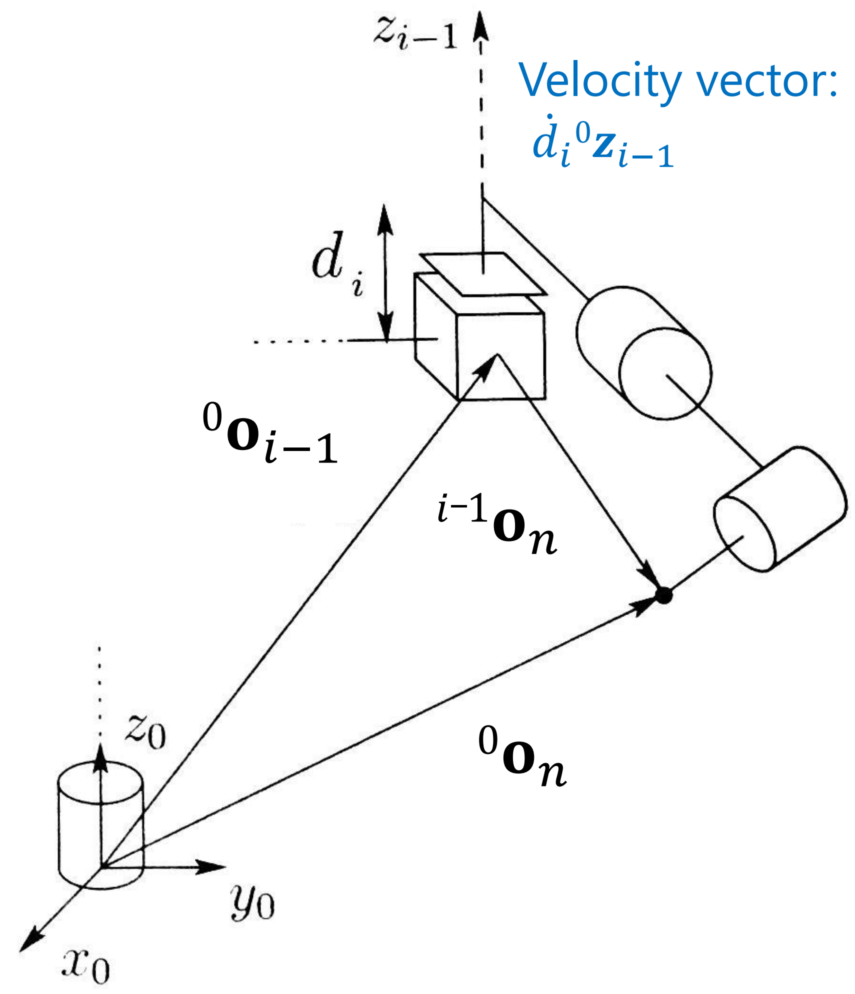
Method: Fix all joints except joint i, In this case both $^0\mathbf{o}_{i-1}$ and $^0R_{i-1}$ are constants. Then,

$$^0\dot{\mathbf{o}}_n = {}^0R_{i-1}\,{}^{i\text{-}1}\dot{\mathbf{o}}_n$$

Velocity of end effector becomes the velocity of prismatic joint.

$$^{i\text{-}1}\dot{\mathbf{o}}_n = {}^{i\text{-}1}\dot{\mathbf{o}}_i$$

$$\boxed{^0\dot{\mathbf{o}}_n = {}^0R_{i-1}\,{}^{i\text{-}1}\dot{\mathbf{o}}_i}$$



Step 2 – Find J_v so that $v = J_v \dot{q}$

Case 1: Joint i is prismatic

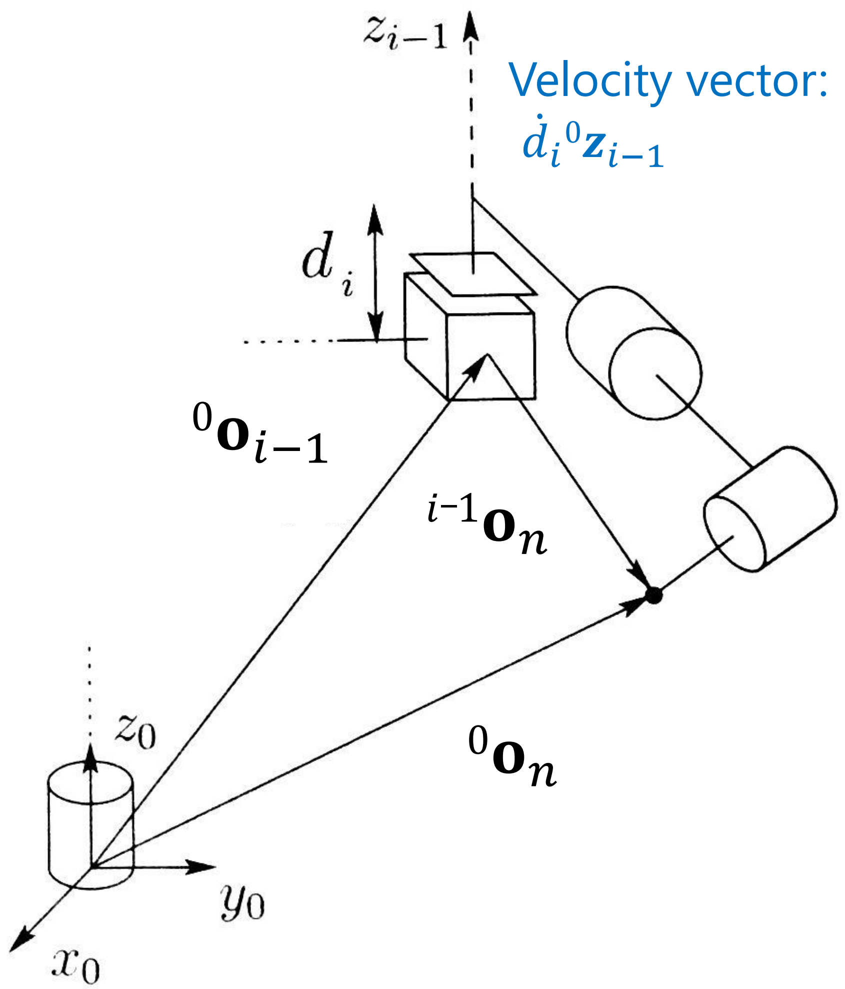
$$^0\dot{\mathbf{o}}_n = {}^0R_{i-1}\,{}^{i\text{-}1}\dot{\mathbf{o}}_i$$

$$^0\dot{\mathbf{o}}_n = {}^0R_{i-1}\dot{d}_i\boldsymbol{k}$$

$\boldsymbol{k}$ is the unit vector in z_{i-1} direction.

$$^0\dot{\mathbf{o}}_n = \dot{d}_i\,{}^0R_{i-1}\boldsymbol{k}$$
$$= \dot{q}_i\,{}^0\mathbf{z}_{i-1} = {}^0\mathbf{z}_{i-1}\,\dot{q}_i$$

$^0\mathbf{z}_{i-1}$ is the **i^{th} column of J_v**



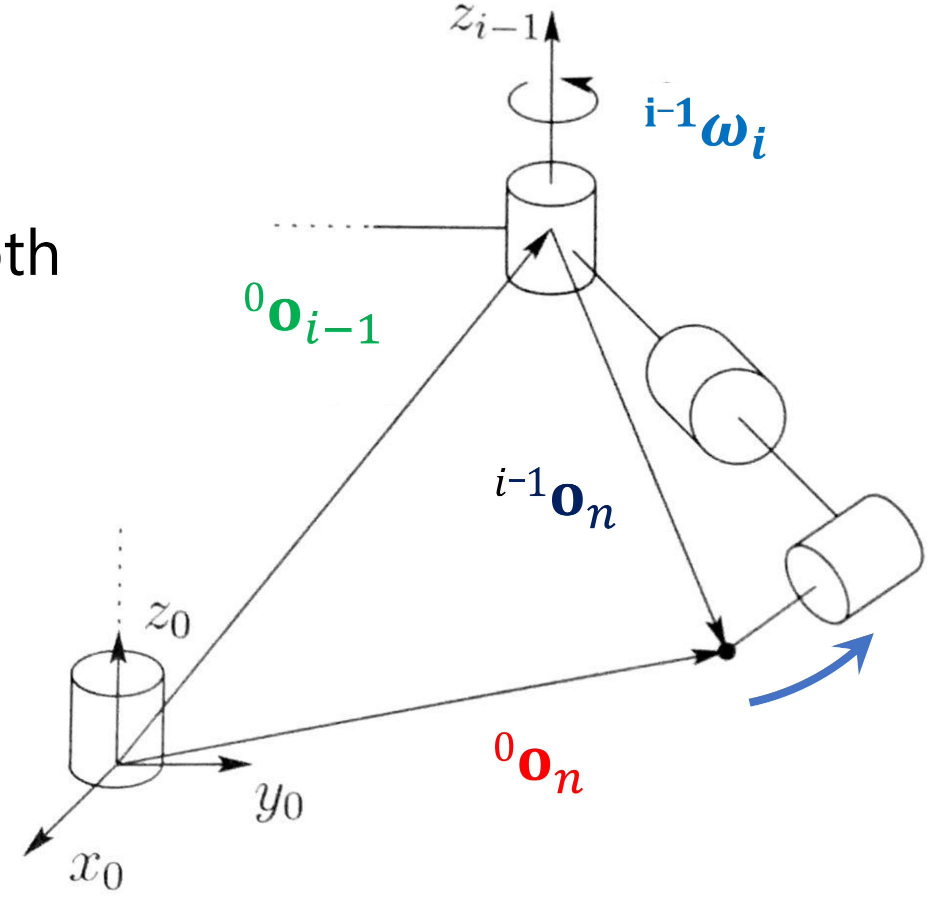
Step 2 – Find J_v so that $v = J_v\dot{q}$

❑ Case 2: Joint i is revolute

$$ {}^{0}\mathbf{o}_n - {}^{0}\mathbf{o}_{i-1} = {}^{0}R_{i-1}\,{}^{i\text{-}1}\mathbf{o}_n $$

Method: Fix all joints and actuate joint i. In this case both ${}^{0}\mathbf{o}_{i-1}$ and ${}^{0}R_{i-1}$ are constants. Then,

$$ {}^{0}\dot{\mathbf{o}}_n = {}^{0}R_{i-1}\,{}^{i\text{-}1}\dot{\mathbf{o}}_n $$



Step 2 – Find J_v so that $v = J_v \dot{q}$

- Motion of link i is a rotation q_i (or θ_i) around z_{i-1}:

$$^{i\text{-}1}\dot{\mathbf{o}}_n = \dot{q}_i \boldsymbol{k} \times {}^{i\text{-}1}\mathbf{o}_n$$

- Since, $^0\dot{\mathbf{o}}_n = {}^0R_{i-1}{}^{i\text{-}1}\dot{\mathbf{o}}_n$

$$^0\dot{\mathbf{o}}_n = {}^0R_{i-1}(\dot{q}_i \boldsymbol{k} \times {}^{i\text{-}1}\mathbf{o}_n)$$

$\dot{q}$ is scalar $\Rightarrow$

$$^0\dot{\mathbf{o}}_n = \dot{q}_i {}^0R_{i-1}(\boldsymbol{k} \times {}^{i\text{-}1}\mathbf{o}_n)$$

$R.(a \times b) = (Ra) \times (Rb)$ $\Rightarrow$

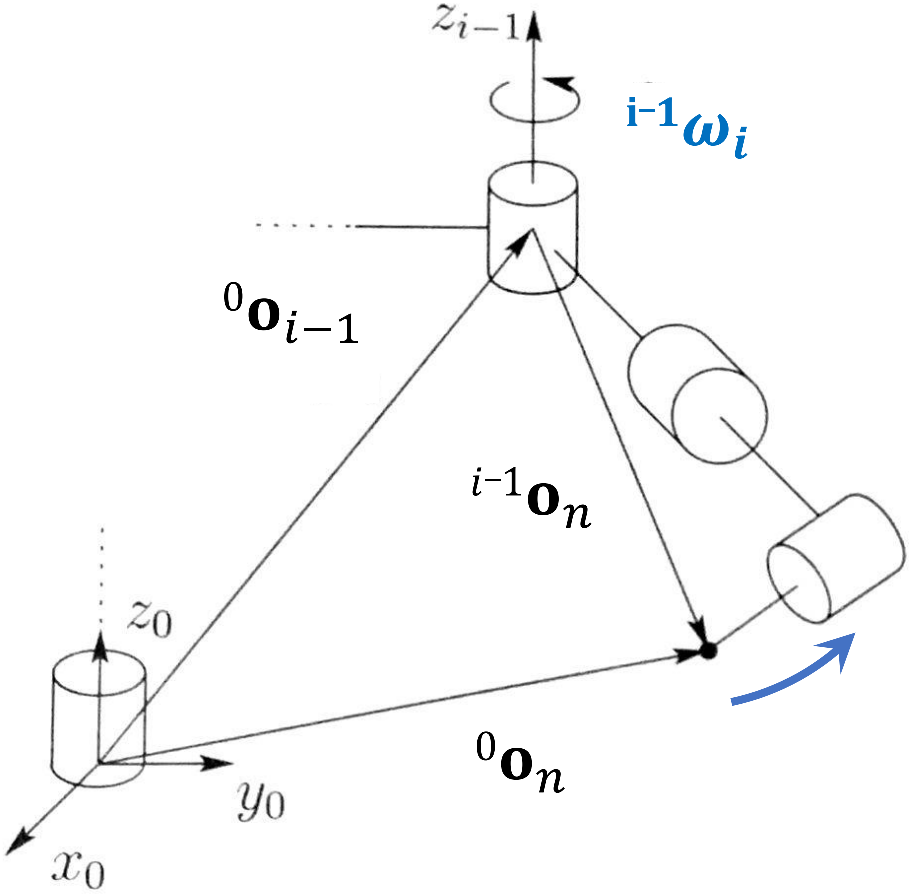
$$^0\dot{\mathbf{o}}_n = \dot{q}_i({}^0R_{i-1}\boldsymbol{k}) \times ({}^0R_{i-1}{}^{i\text{-}1}\mathbf{o}_n)$$

$$\Downarrow$$

$$^0\dot{\mathbf{o}}_n = \dot{q}_i {}^0\mathbf{z}_{i-1} \times {}^0R_{i-1}{}^{i\text{-}1}\mathbf{o}_n$$

$^0R_{i-1}{}^{i\text{-}1}\mathbf{o}_n = {}^0\mathbf{o}_n - {}^0\mathbf{o}_{i-1}$ $\Rightarrow$

$$\boxed{^0\dot{\mathbf{o}}_n = \dot{q}_i {}^0\mathbf{z}_{i-1} \times ({}^0\mathbf{o}_n - {}^0\mathbf{o}_{i-1})}$$



$^0\boldsymbol{z}_{i-1} \times ({}^0\mathbf{o}_n - {}^0\mathbf{o}_{i-1})$ is the i^{th} column of J_v

Step 3 – Combine J_v and J_ω to form the Jacobian

- Combining the upper and lower halves of the Jacobian gives us,

$$J = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix} = \begin{pmatrix} J_1 & J_2 & \cdots & J_n \end{pmatrix}$$, where the i^{th} column is given by,

$$\boxed{J_i = \begin{pmatrix} {}^0\mathbf{z}_{i-1} \times ({}^0\mathbf{o}_n - {}^0\mathbf{o}_{i-1}) \\ {}^0\mathbf{z}_{i-1} \end{pmatrix} \text{ if joint } i \text{ is revolute;}}$$

Or

$$\boxed{J_i = \begin{pmatrix} {}^0\mathbf{z}_{i-1} \\ 0 \end{pmatrix} \text{ if joint } i \text{ is prismatic.}}$$

KEY EQUATIONS of Today's Lecture

Source of Data for the Jacobian

- All quantities needed are available from forward kinematics (using D-H).

$$
{}^{0}T_i = \begin{pmatrix} {}^{0}R_i & {}^{0}\mathbf{o}_i \\ 0 & 1 \end{pmatrix} = \begin{array}{cccc} & & {}^{0}\boldsymbol{z}_i & {}^{0}\boldsymbol{O}_i \\ \end{array} \begin{pmatrix} r_{1,1} & r_{1,2} & \boxed{r_{1,3}} & \boxed{o_{1,4}} \\ r_{2,1} & r_{2,2} & \boxed{r_{2,3}} & \boxed{o_{2,4}} \\ r_{3,1} & r_{3,2} & \boxed{r_{3,3}} & \boxed{o_{3,4}} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

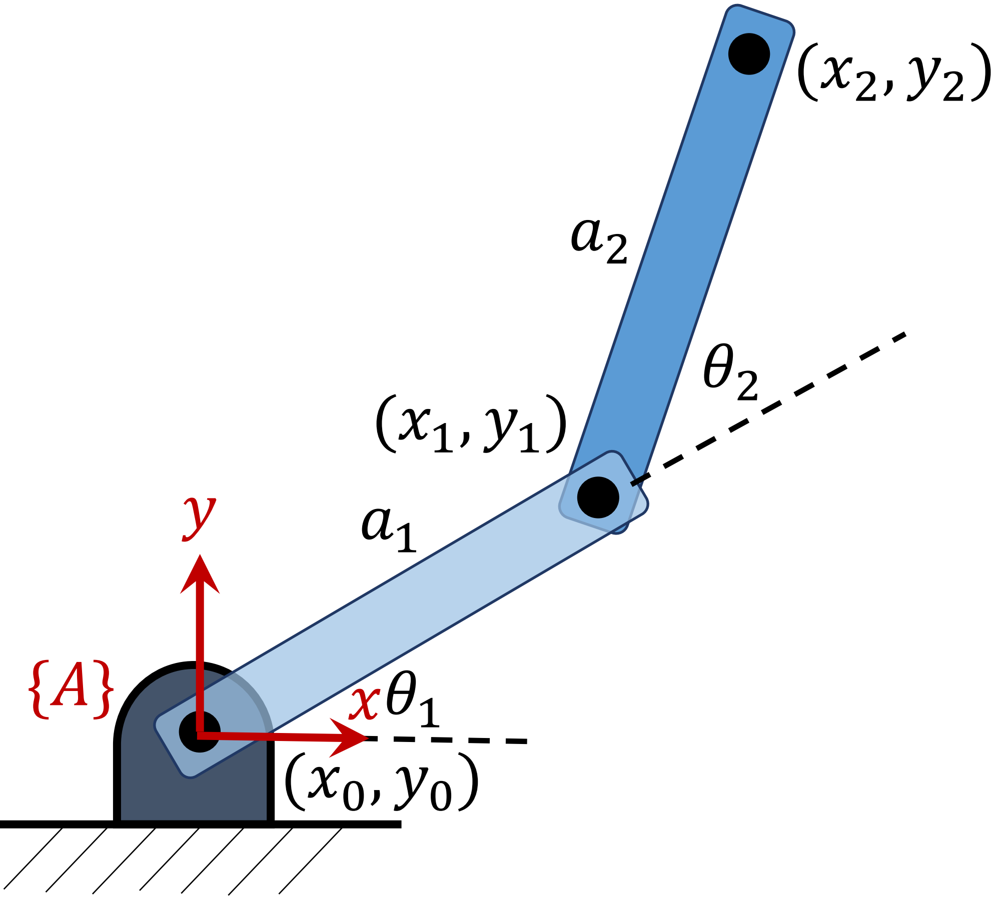
- Joint-to-joint transformation matrices (${}^{i\text{-}1}T_i$) help us evaluate ${}^{0}T_i$.

$$
{}^{i\text{-}1}T_i = \begin{pmatrix} cos\theta_i & -sin\theta_i cos\alpha_i & sin\theta_i sin\alpha_i & a_i cos\theta_i \\ sin\theta_i & cos\theta_i cos\alpha_i & -cos\theta_i sin\alpha_i & a_i sin\theta_i \\ 0 & sin\alpha_i & cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Applications of the Jacobian

- ❑ Planning and execution of smooth trajectories
- ❑ Determination of singular configurations
- ❑ Derivation of the dynamic equations of motion
- ❑ Transformation of forces and torques from end effector to joints

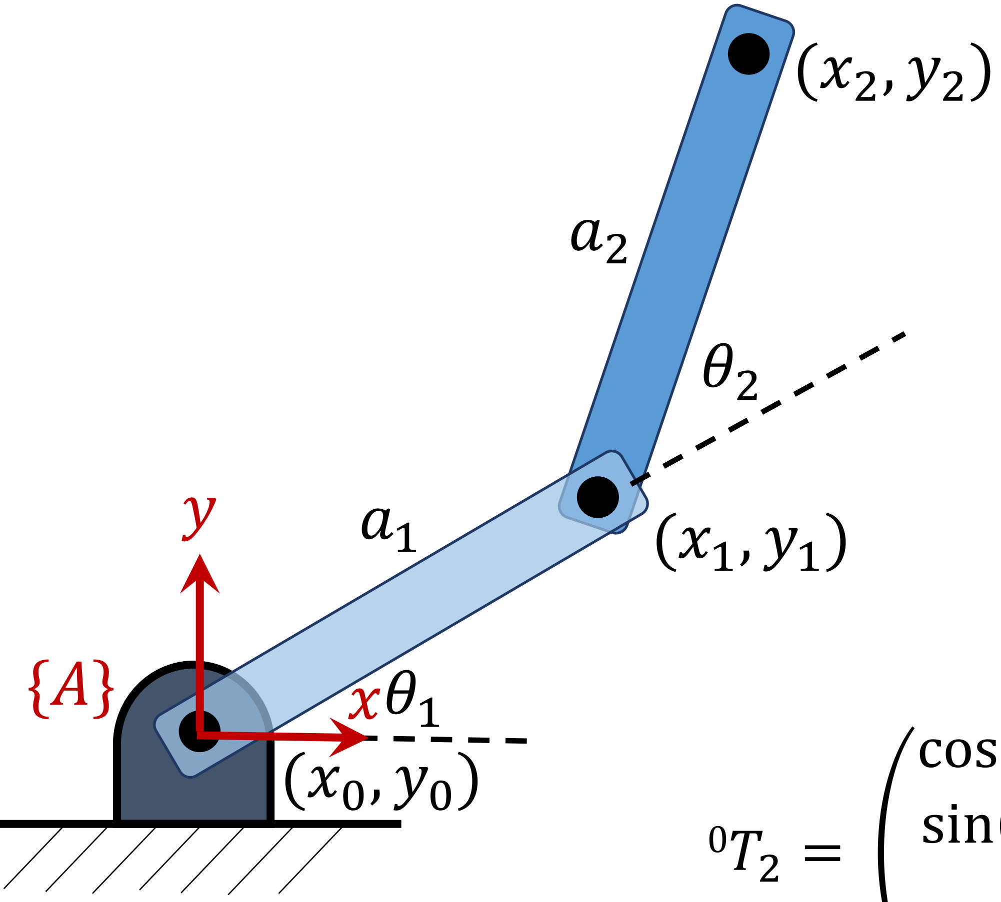
Example 1: Calculate Jacobian



i	θ_i	d_i	a_i	α_i
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0

θ_1 and θ_2 are variable parameters (q_i) for the two revolute joints.

Example 1: Calculate Jacobian



Recall $^0T_2 = {}^0T_1\,{}^1T_2$

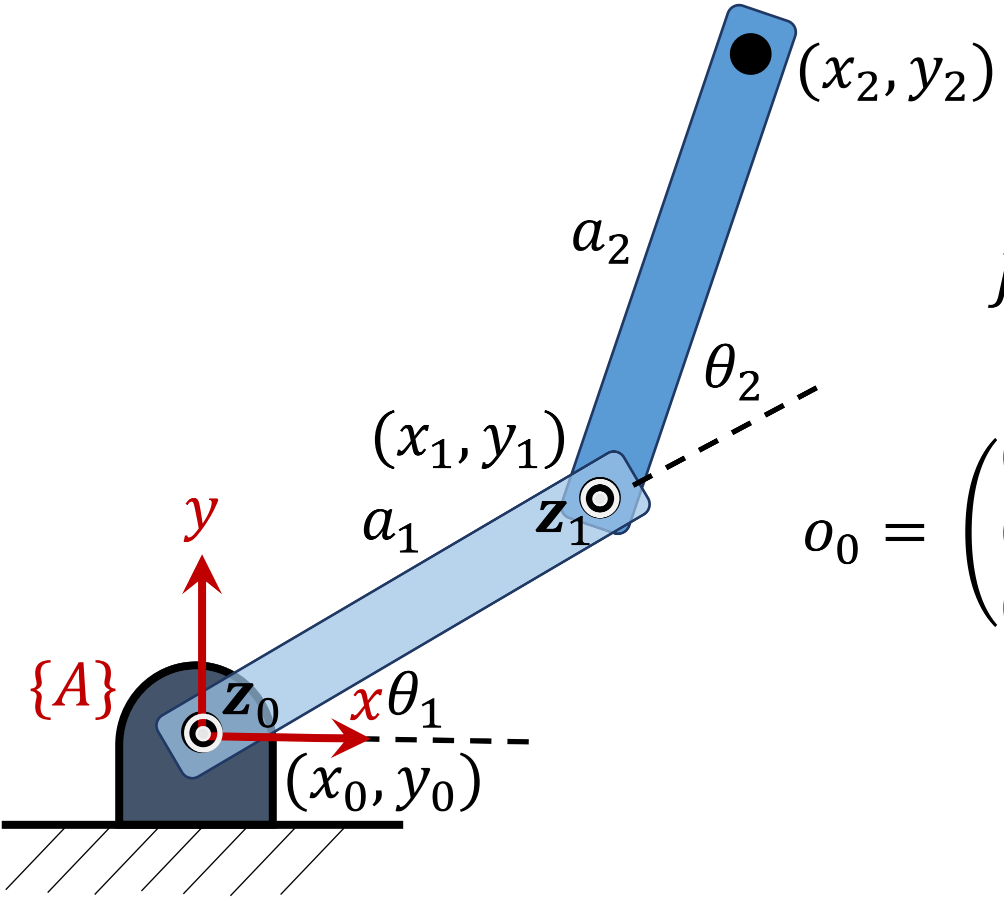
$$
\begin{array}{cccc} & & \boldsymbol{z_1} & \boldsymbol{o_1} \\ \end{array}
$$

$$
{}^0T_1 = \begin{pmatrix} cos\theta_1 & -sin\theta_1 & \boxed{0} & a_1 cos\theta_1 \\ sin\theta_1 & cos\theta_1 & 0 & a_1 sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{array}{cccc} & & \boldsymbol{z_2} & \boldsymbol{o_2} \\ \end{array}
$$

$$
{}^0T_2 = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_1 cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_1 sin\theta_1 + a_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Example 1: Calculate Jacobian

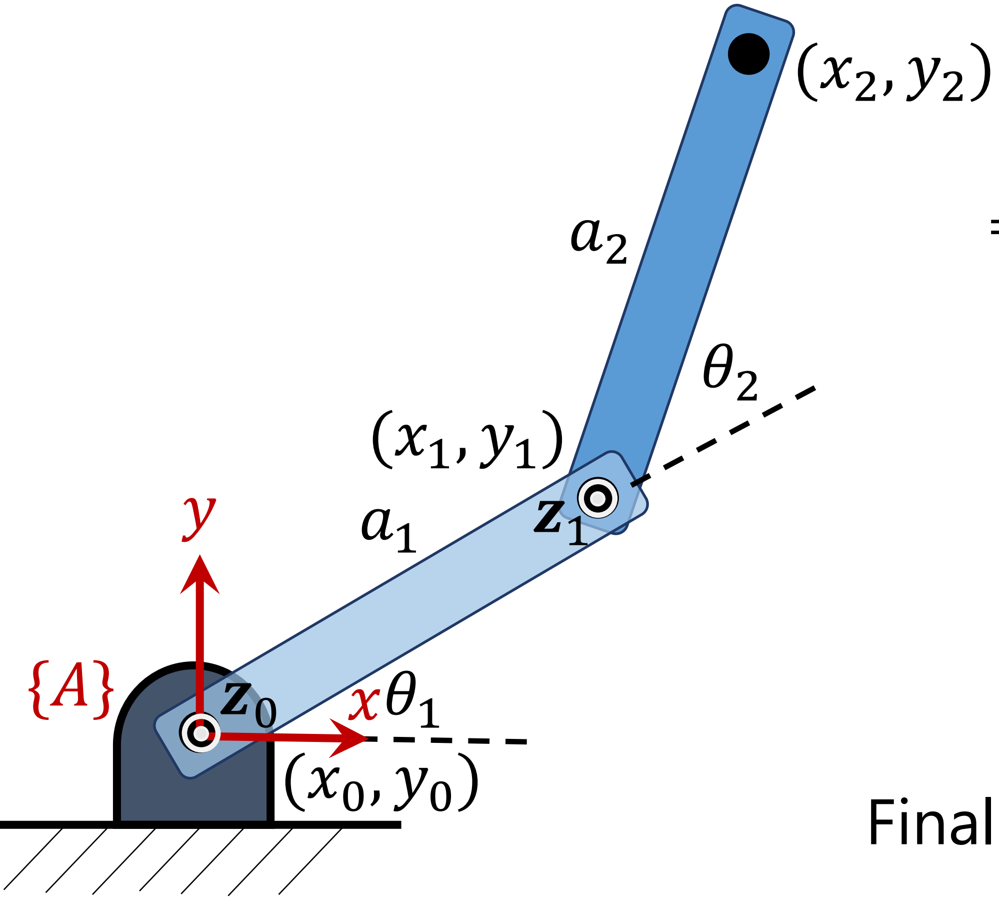


$$J = \begin{pmatrix} J_1 & J_2 \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0 \times (\mathbf{o}_2 - \mathbf{o}_0) & \mathbf{z}_1 \times (\mathbf{o}_2 - \mathbf{o}_1) \\ \mathbf{z}_0 & \mathbf{z}_1 \end{pmatrix}$$

$$o_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; o_1 = \begin{pmatrix} a_1 cos\theta_1 \\ a_1 sin\theta_1 \\ 0 \end{pmatrix}; o_2 = \begin{pmatrix} a_1 cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix}$$

$$\mathbf{z}_0 = \mathbf{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example 1: Calculate Jacobian



$$J = (J_1 \quad J_2) = \begin{pmatrix} \mathbf{z}_0 \times (\mathbf{o}_2 - \mathbf{o}_0) & \mathbf{z}_1 \times (\mathbf{o}_2 - \mathbf{o}_1) \\ \mathbf{z}_0 & \mathbf{z}_1 \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} a_1 cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} a_2 \cos(\theta_1 + \theta_2) \\ a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

Finally, $J = \begin{pmatrix} -a_1 sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$

The Inverse Jacobian

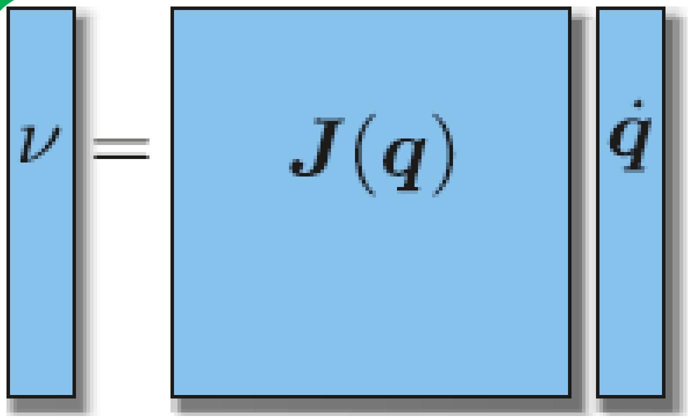
We know that,

$$^{0}\dot{\boldsymbol{d}}_{n} = \begin{pmatrix} ^{0}\boldsymbol{v}_{n} \\ ^{0}\boldsymbol{\omega}_{n} \end{pmatrix} = {}^{0}J_{n}\dot{\boldsymbol{q}}$$
$$^{0}J_{n}^{-1}\ {}^{0}\dot{\boldsymbol{d}}_{n} = {}^{0}J_{n}^{-1}\ {}^{0}J_{n}\dot{\boldsymbol{q}}$$

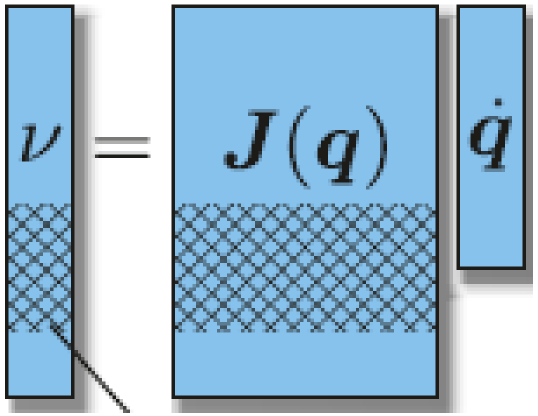
$$\boxed{\dot{\boldsymbol{q}} = {}^{0}J_{n}^{-1}\ {}^{0}\dot{\boldsymbol{d}}_{n}}$$

This involves solving a set of linear equations.

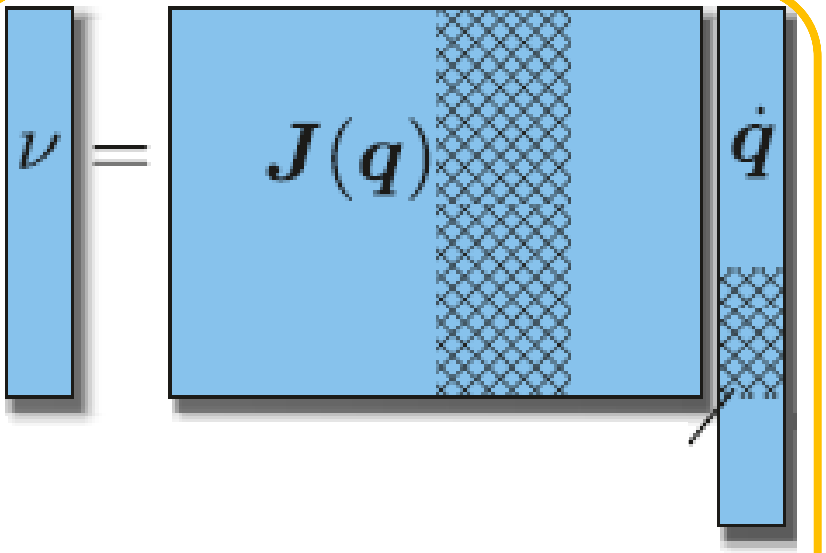
Over and Under-actuated Manipulators



- A robot with 6 joints allows 6 DOF movement
- Can inverse the Jacobian



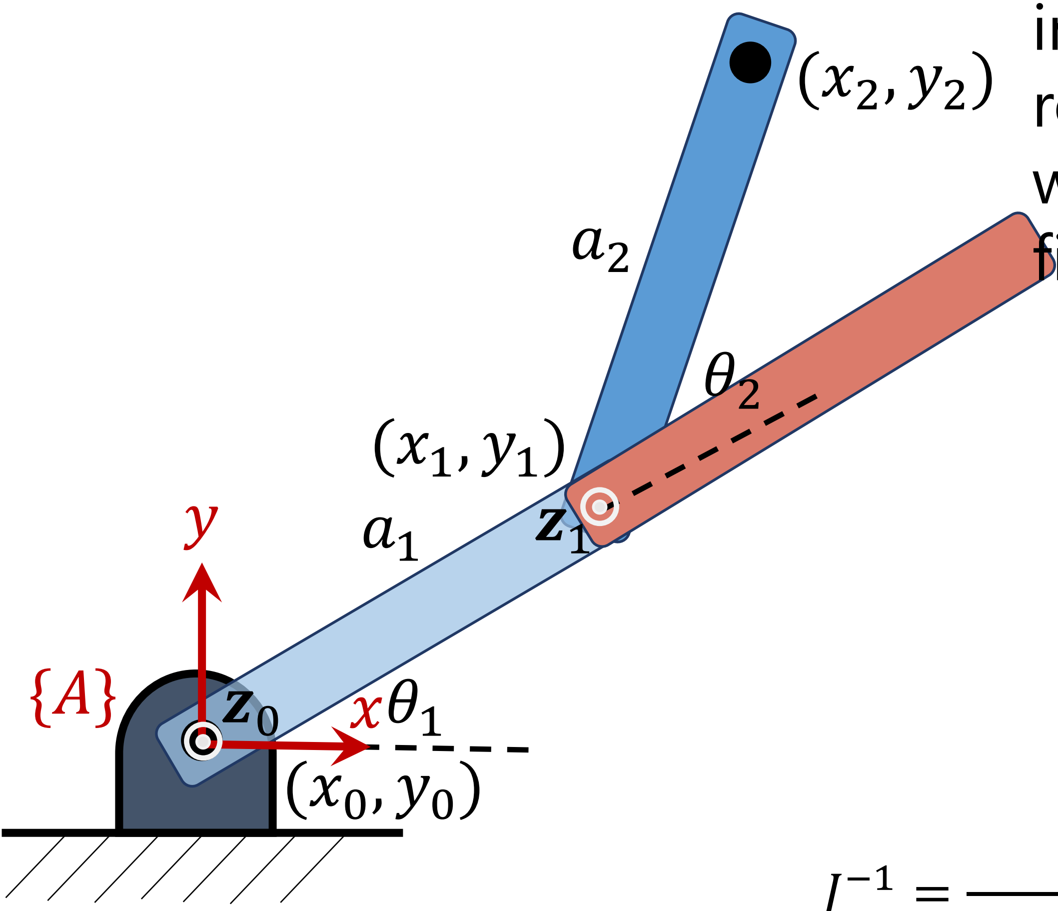
- **Under-actuated**: < 6 joints
- Some degrees of freedom are uncontrolled
- Square it up by deleting some rows of the Jacobian



- **Over actuated**: >6 joints
- Need to lock some by deleting some columns
- J(q) becomes square

For further details, see section 8.4 of Corke (2017).

Example 2: Under-Actuated Manipulator



In our 2-link example, we cannot control movement in the z-direction, or any of the three end effector rotations independent of the (x,y) position. Therefore, we delete the bottom 4 rows of the Jacobian before finding the inverse.

$$J = \begin{pmatrix} -a_1 sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$J^{-1} = \frac{1}{a_1 a_2 \sin\theta_2} \begin{pmatrix} a_2 \cos(\theta_1 + \theta_2) & a_2 \sin(\theta_1 + \theta_2) \\ -a_1 cos\theta_1 - a_2 \cos(\theta_1 + \theta_2) & -a_1 sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

$\theta_2 = 0$ creates a singularity for this robot

Singularities

What are singularities?

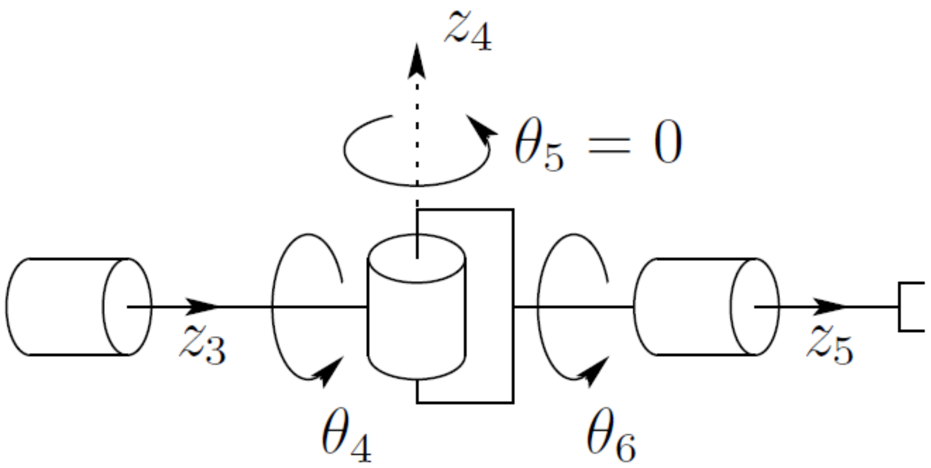
- A robot configuration from which certain motions become unattainable. A certain configuration $\mathbf{q}$ is said to be singular if $\det\left(J(\mathbf{q})\right) = 0.$ What happen next: The robot may move very fast or lose some DOFs

- Near singularities there will not exist a unique solution to the inverse kinematics problem. There may be no solution or an infinite number of solutions

- Singularities usually correspond to points on the boundary of the manipulator workspace (maximum reach of the manipulator)

Example of singularity positions

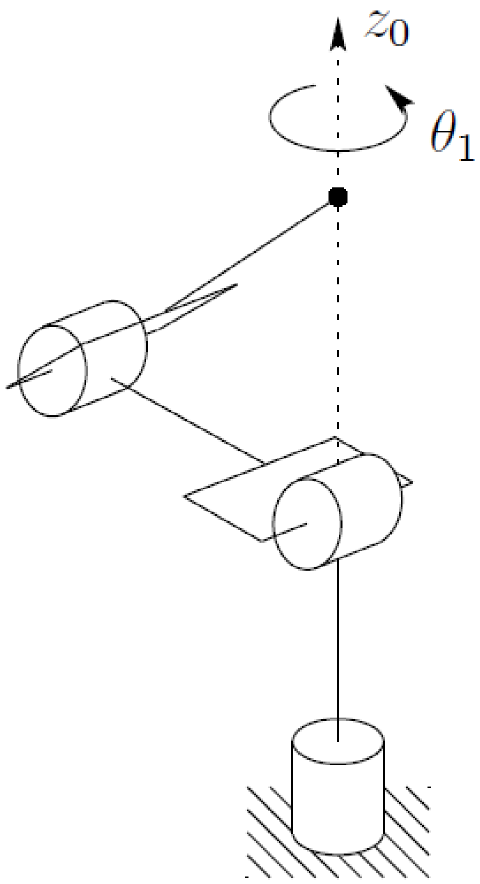
- Wrist singularity
  - Lose 1 DOF
  - Equal and opposite rotation about Z_3 and Z_5 results in no net motion of the end-effector.



Z_3 and Z_5 are linearly dependent

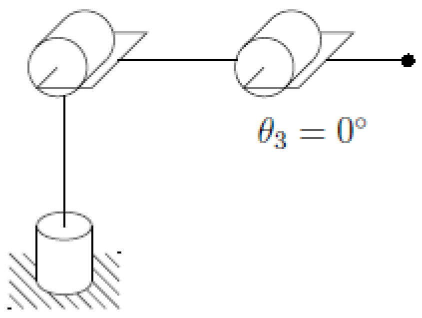
Example of singularity positions

- Shoulder singularity
  - First three joints used for position control
  - When the wrist center intersects the axis of the base rotation (Z_0), any rotation about the Z_0 does not change the position of the wrist center.



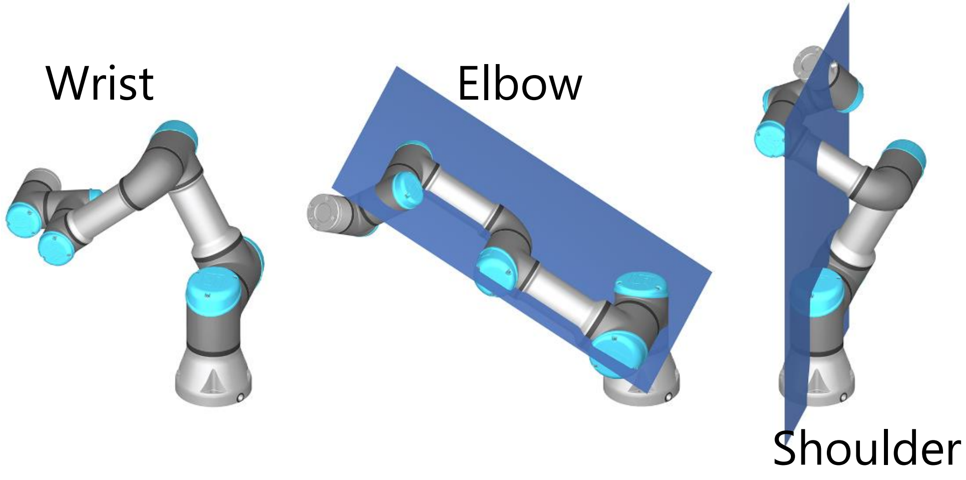
Example of singularity positions

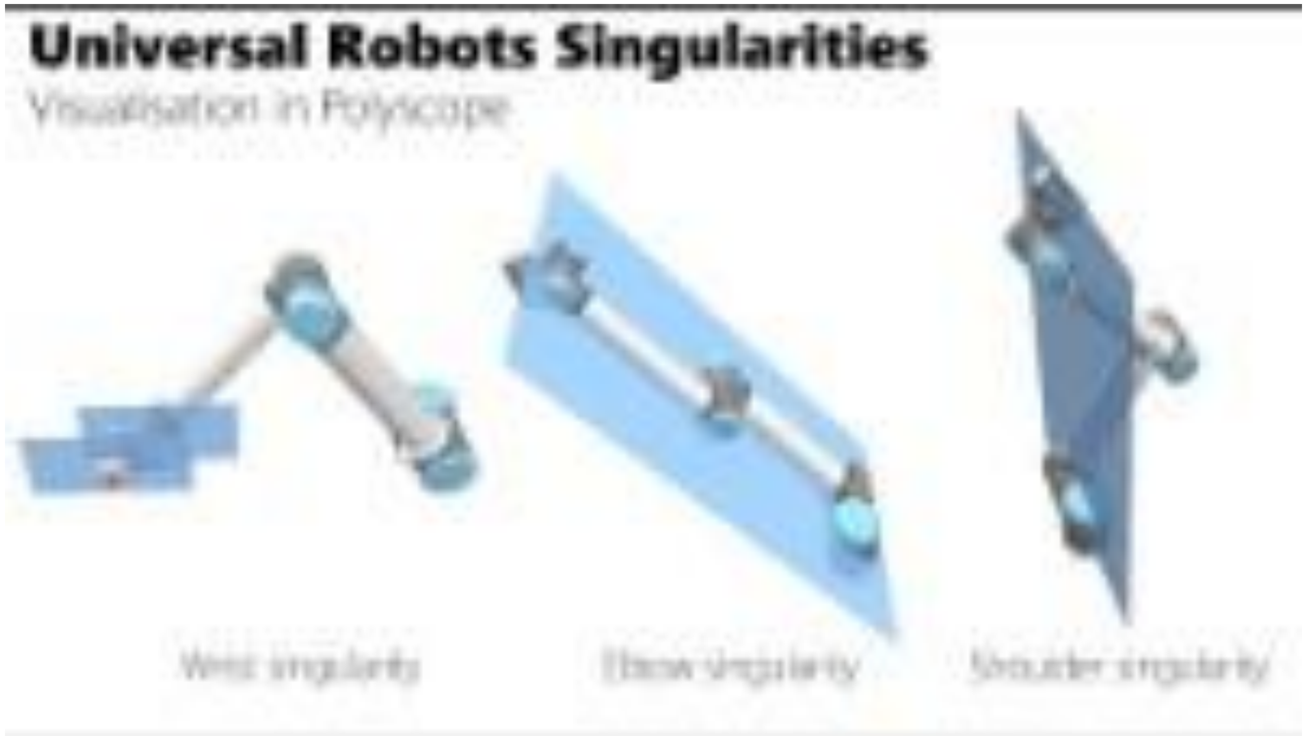
- Elbow singularity
  - Centre of joint 2, centre of join 3, and wrist centre are in the same line
  - The robot reach the boundary of the manipulator workspace



Elbow

Example: UR Series Singularities





- Wrist singularity: Joints 4 and 6 are parallel
- Shoulder singularity: Centre of the wrist aligns with joint 1
- Elbow singularity: Centre of joint 2, centre of join 3, and wrist centre are in the same line

Example: UR Series Singularities

https://www.youtube.com/watch?v=6Wmw4lUHlX8&ab_channel=CoRoETS

Decoupling of Singularities

- Consider the Jacobian mapping:

$$\dot{\mathbf{d}} \quad = J(\mathbf{q})\dot{\mathbf{q}}$$

  - A certain configuration $\mathbf{q}$ is said to be singular if $\det\big(J(\mathbf{q})\big) = 0$
  - Sometimes it helps to divide the Jacobian into sub-matrices to identify the locations of singularities.

Singularity of 6-DOF Manipulator with a Wrist

- Divide the Jacobian into sub-matrices to identify the locations of singularities

  - 6-DOF manipulator with a 3-DOF arm and a 3-DOF wrist:

$$J = (J_p | J_o) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} Linear\ arm & Linear\ wrist \\ Angular\ arm & Angular\ wrist \end{pmatrix}$$

  - Since the last 3 joints are revolute,

$$J_o = \begin{pmatrix} \mathbf{z}_3 \times (\mathbf{o}_6 - \mathbf{o}_3) & \mathbf{z}_4 \times (\mathbf{o}_6 - \mathbf{o}_4) & \mathbf{z}_5 \times (\mathbf{o}_6 - \mathbf{o}_5) \\ \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{pmatrix}$$

Singular Configurations

$$J_o = \begin{pmatrix} \mathbf{z}_3 \times (\mathbf{o}_6 - \mathbf{o}_3) & \mathbf{z}_4 \times (\mathbf{o}_6 - \mathbf{o}_4) & \mathbf{z}_5 \times (\mathbf{o}_6 - \mathbf{o}_5) \\ \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{pmatrix}$$

- ❑ We can choose a coordinate frame such that $\mathbf{o}_3 = \mathbf{o}_4 = \mathbf{o}_5 = \mathbf{o}_6$

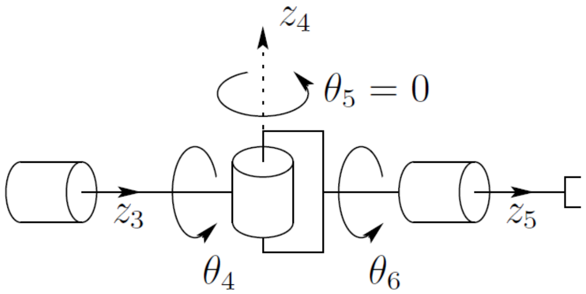
The J_0 becomes,

$$J_o = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{pmatrix}$$

Hence,

$$J = \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix}$$

$$\boxed{\det(J) = \det(J_{11}) \times \det(J_{22}) = 0}$$



Wrist singularity Z_3 and Z_5 are linearly dependent

Singular Configurations

- The set of singular configurations of the manipulator is,

the union of

the set of arm configurations satisfying,

$$\det(J_{11}) = 0$$

AND

the set of wrist configurations satisfying

$$\det(J_{22}) = 0$$

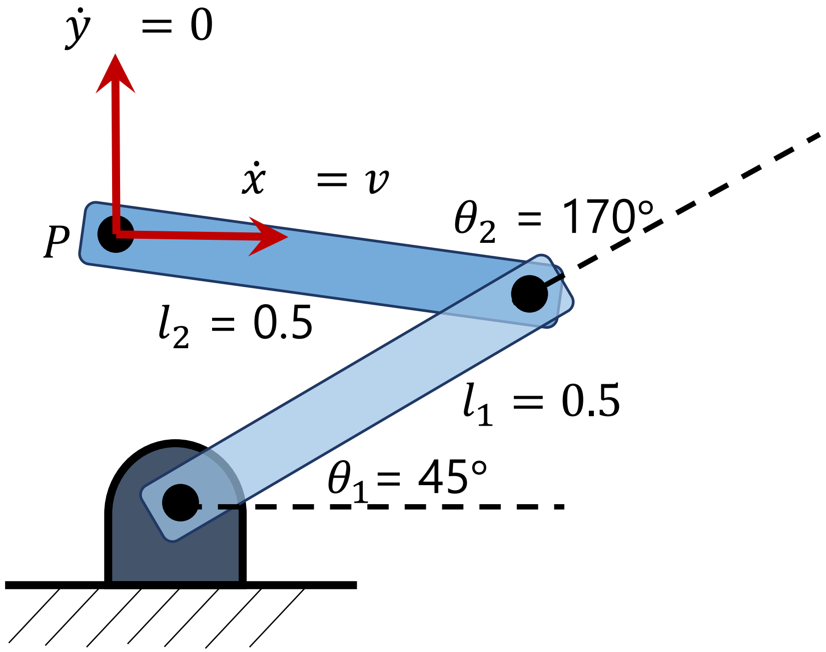
$$J_{22} = (z_3 \quad z_4 \quad z_5)$$

E.g., if z_3 and z_5 are parallel (Wrist singularity)

IMPORTANT: This form of the Jacobian is to be used only to determine the singularities and not any relationship between the velocity of the end-effector and joint velocities

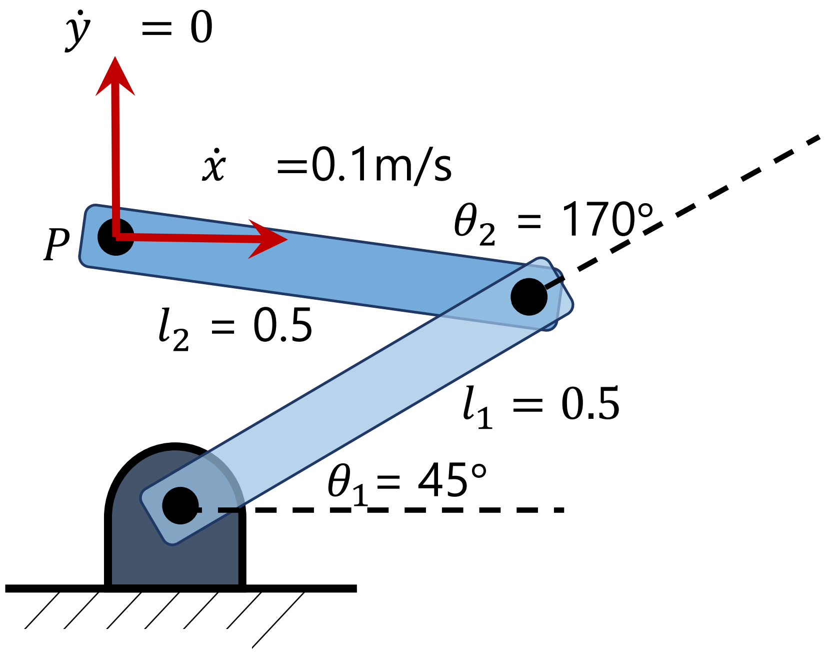
Example 3: Calculate Join Velocities

The end effector P is moving at a velocity v in the positive x-direction. Calculate the angular velocities (in terms of v) that each revolute axis has to turn at, that will give rise to the correct motion of the end-effector.



Example 3: Calculate Join Velocities

Answer



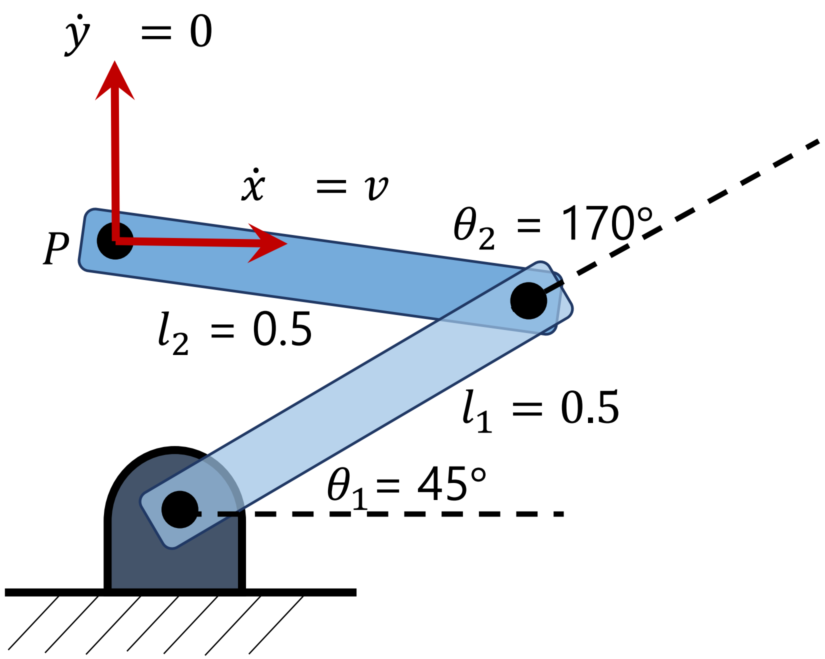

```
L(1) = Link([0 0 0.5 0]);  % a1 = 0.5  (lecture 4 revision)
L(2) = Link([0 0 0.5 0]);  % a2=0.5
two_link = SerialLink(L, 'name', 'two link');

qi = deg2rad([45,170]); % angle theta1 and theta2

% Calculate transformation matrix
T = two_link.fkine(qi)

% Calculate Jacobian at configuration qi
J = two_link.jacob0(qi)

%Take subset of Jacobian for just x-y position
Jxy = J(1:2, :)

q_dot = inv(Jxy) * [0.1 0]'  % angular velocity
```

```
T =
   -0.8192    0.5736         0   -0.05602
   -0.5736   -0.8192         0    0.06677
         0         0         1          0
         0         0         0          1
```

```
J =
   -0.0668    0.2868
   -0.0560   -0.4096
         0         0
         0         0
         0         0
    1.0000    1.0000
```

Example 4: Calculate End-Effector Velocities

For the robot in Example 3, using the forward Jacobian, calculate the fastest horizontal speed possible at the point P, if it is known that the motors at the joints cannot rotate faster than 20rad/s.



Answer

"Joints cannot rotate faster than 20rad/s" means joint velocity is within the range of -20rad/s and 20rad/s.

[Vx Vy]'= Jxy*[q1_dot q2_dot]'

From example 1

```
J =

   -0.0668    0.2868
   -0.0560   -0.4096
         0         0
         0         0
         0         0
    1.0000    1.0000
```

Therefore,

Vx = -0.0668 * q1_dot + 0.2868 * q2_dot

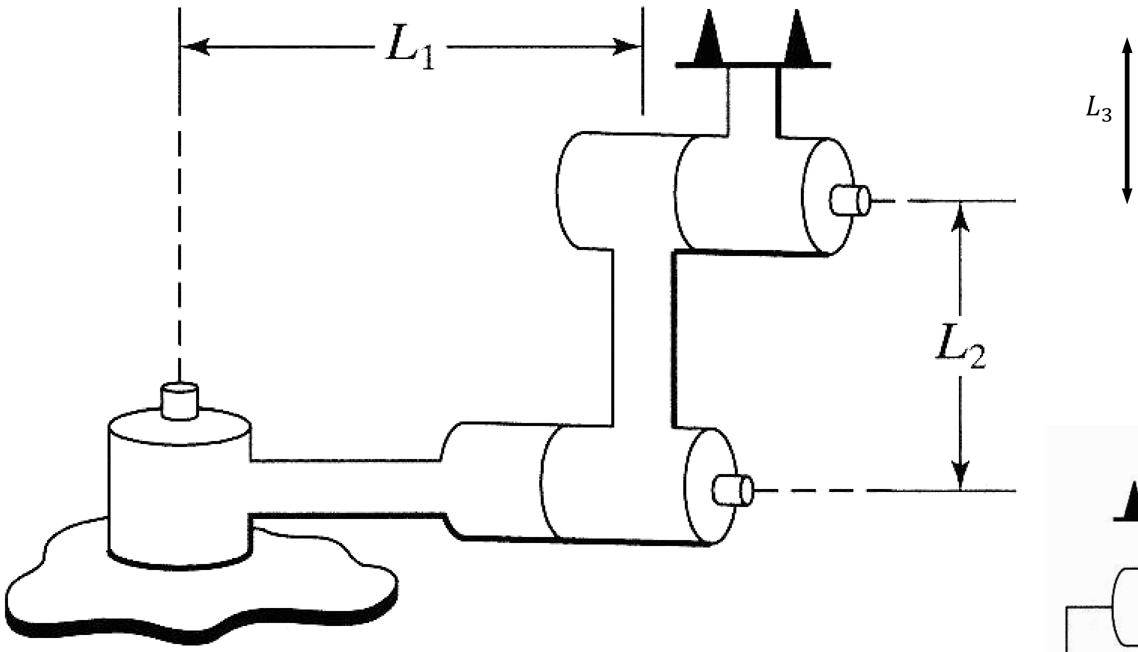
Max(Vx) = - 0.0668 * (-20rad/s) + 0.2868 * 20rad/s
= 7.072 (m/s)

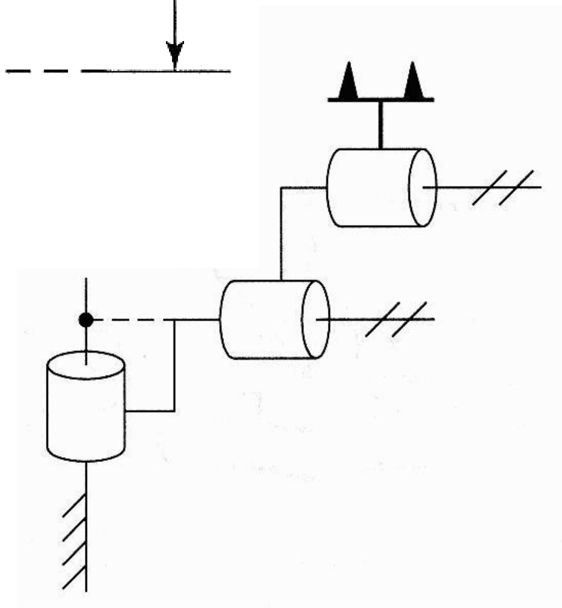
Example 5: Calculate the Jacobian

Using the steps outlined in class (This example is very useful for your Project 1)

- Establish the D-H table for this robot
- Derive the forward kinematic transformation operator ${}^{0}T_{3}$ for this robot
- Derive its Jacobian

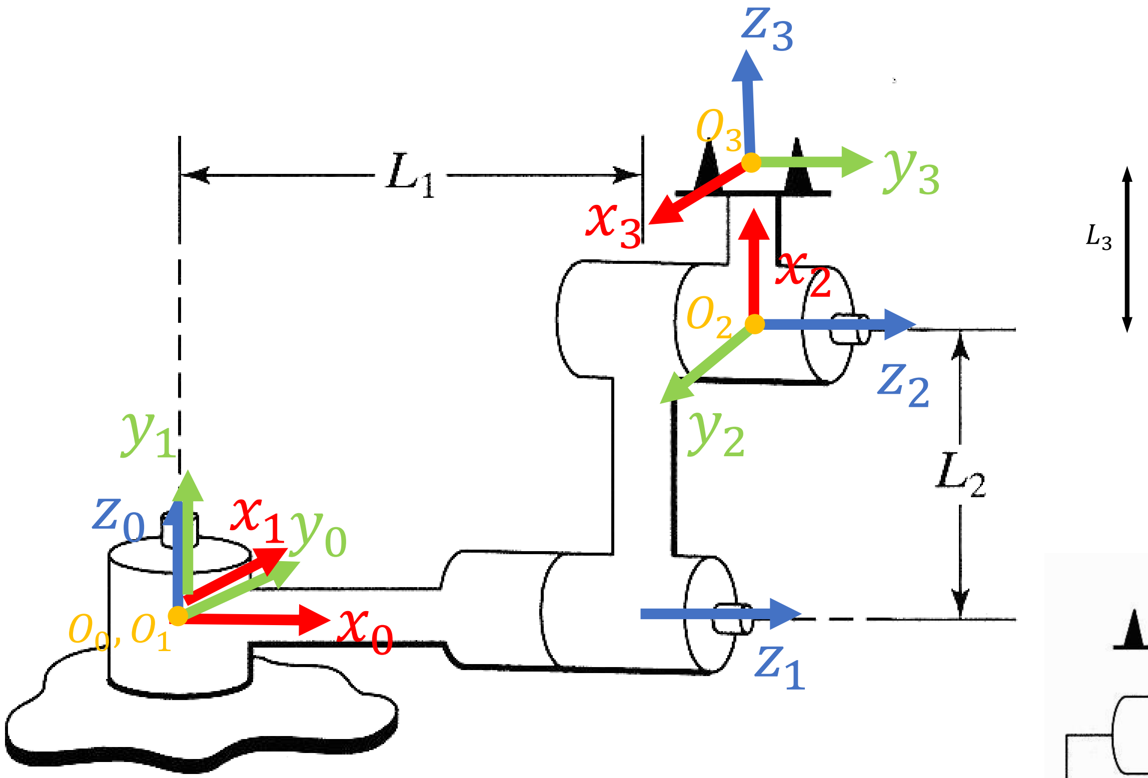
(use Matlab)

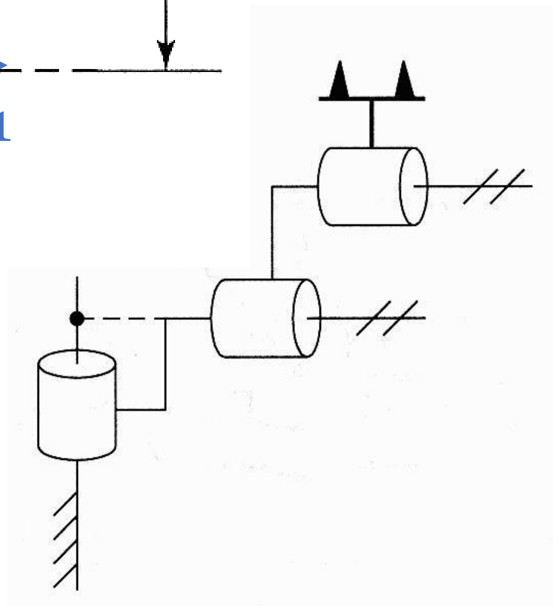




Example Question 5: Calculate the Jacobian

i	θ_i	d_i	a_i	α_i
1	θ_1+90°	0	0	90°
2	θ_2+90°	L_1	L_2	0°
3	θ_3 + 90°	L_3	0	90°





Summary – Lecture 5

- Inverse kinematics
  - Kinematic Decoupling

- The Jacobian
  - Relates joint velocities to end effector pose velocities
  - DH convention allows this to be computed in a straightforward manner

- Singularities
  - Occur when the determinant of the Jacobian is zero
  - Should be avoided where possible, particularly noticeable when attempting linear moves

Next lecture – Robot trajectory (Lecturer: James Stevens)

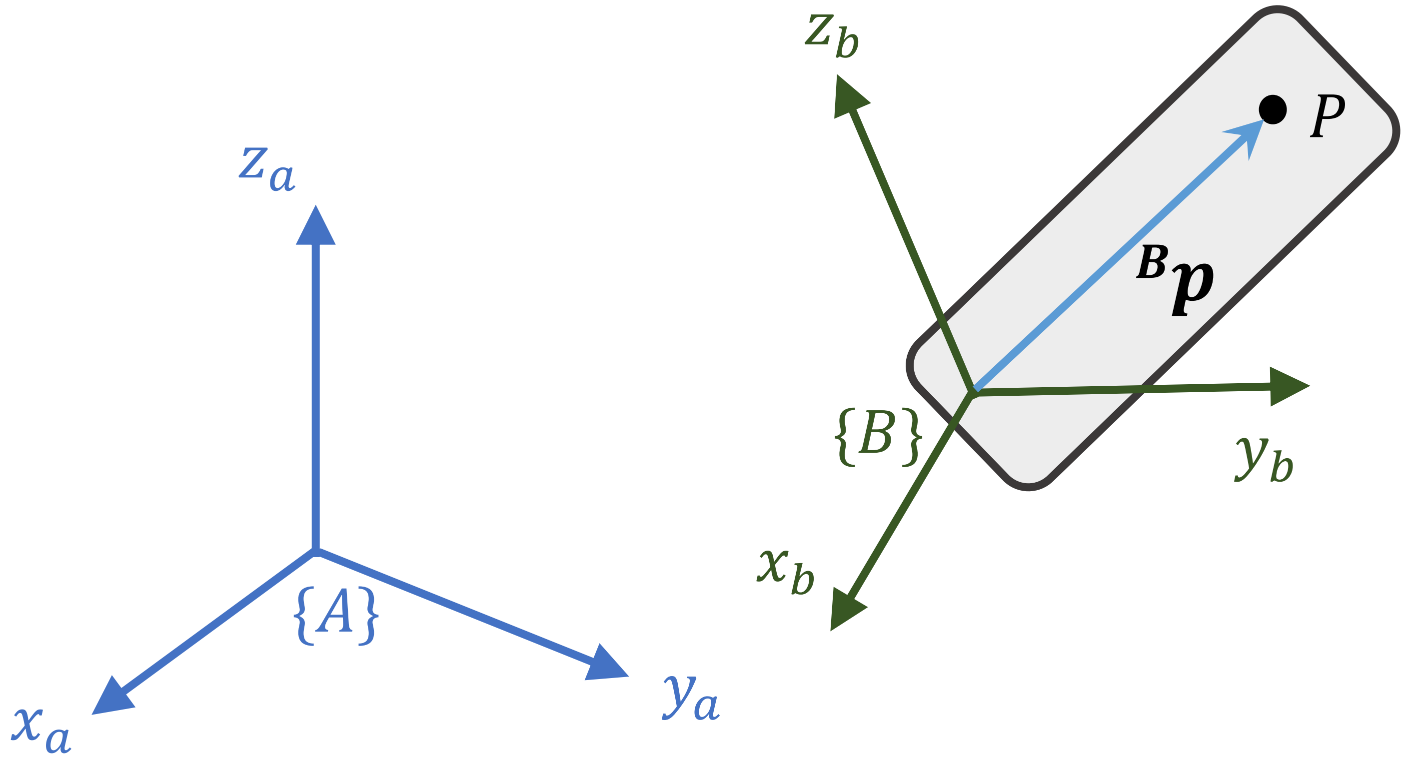
- Trajectory/path generation
- Joint space trajectory design examples
- Accuracy vs repeatability

Appendix

Appendix

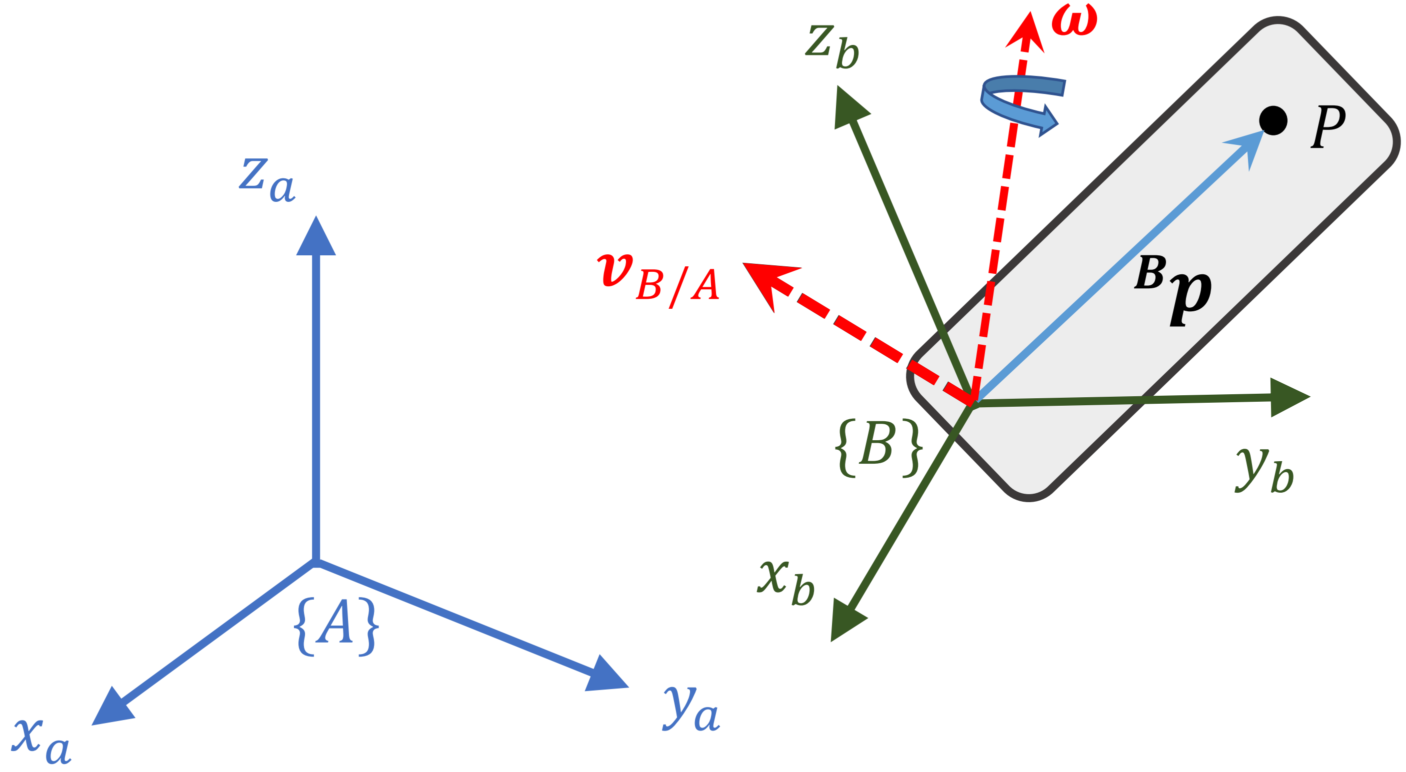
❑ Simultaneous Linear & Angular Motion

- Consider point P on a rigid body.
- Frame $\{B\}$ is attached to the body.
- ${}^{B}\boldsymbol{p}$ is not subjected to change on a rigid body.



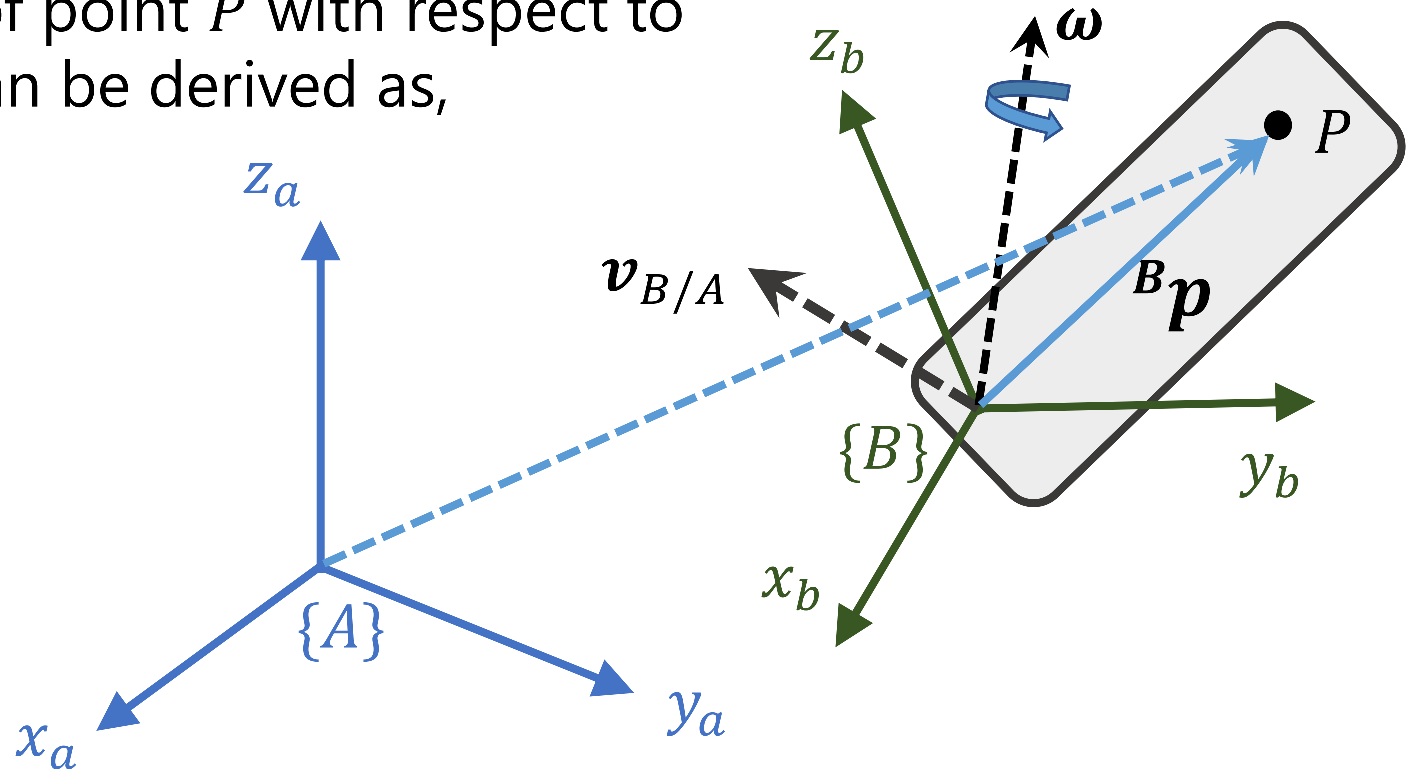
Appendix

- Suppose the rigid body is moving at a linear velocity of $\boldsymbol{v}$ with respect to $\{A\}$ while rotating around a fixed axis.



Appendix

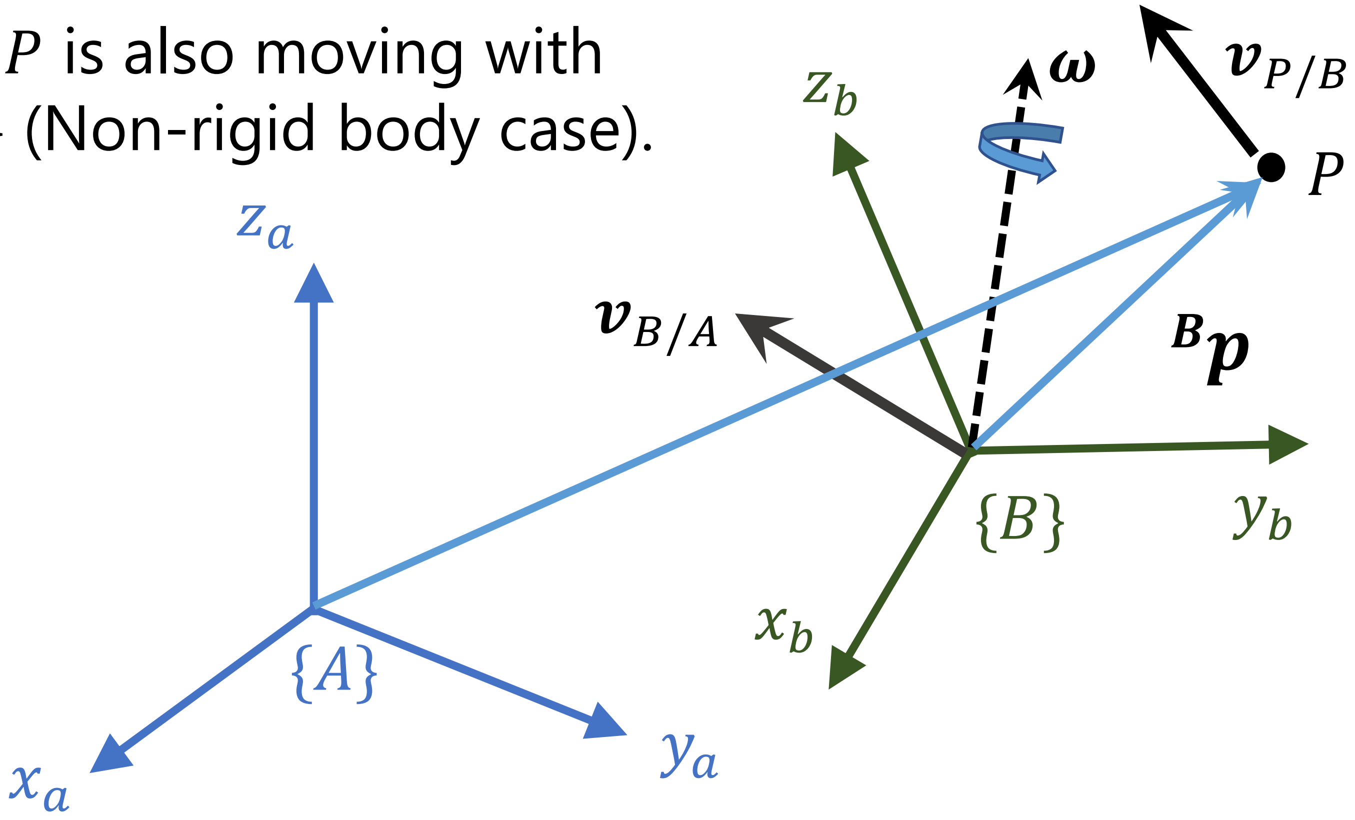
- The linear velocity of point P with respect to frame $\{A\}$ ($\boldsymbol{v}_{P/A}$), can be derived as,



$$\boxed{\boldsymbol{v}_{P/A} = \boldsymbol{v}_{B/A} + \boldsymbol{\omega} \times {}^{A}\boldsymbol{p} = \boldsymbol{v}_{B/A} + \boldsymbol{\omega} \times {}^{A}R_B\,{}^{B}\boldsymbol{p}}$$

Appendix

- Now suppose point P is also moving with respect to frame $\{B\}$ (Non-rigid body case).



$$\boxed{\boldsymbol{v}_{P/A} = \boldsymbol{v}_{B/A} + \omega \times {}^{A}R_{B}{}^{B}\boldsymbol{p} + {}^{A}R_{B}\boldsymbol{v}_{P/B}}$$