Lecture 8 - Revision

https://kahoot.it/

✓ Map Based Method

- Reactive planning
- D* Method
- Probabilistic Roadmap method (PRM)

✓ Artificial Potential Field

- Attractive Potential (Conic and Parabolic Well potential)
- Repulsive potential
- Gradient descent

MTRN4230
Robotics

Lecture 9

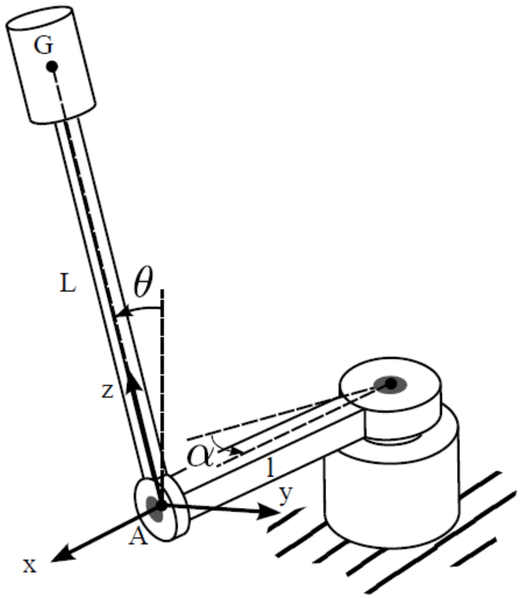
Robotic Dynamics

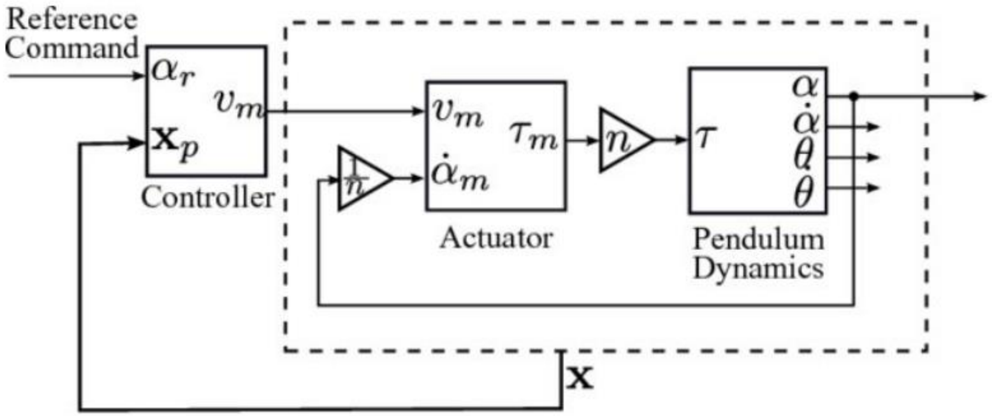
Hoang-Phuong **Phan** – T2 2023

UNSW SYDNEY

Learning Objectives

- ❑ Lagrangian Mechanics
- ❑ Euler-Lagrange Equation
- ❑ Application in N-Link Robots
- ❑ Examples
- ❑ QUIZ 2 format





https://doi.org/10.1145/2744769.2744857

The Need for Dynamics

- To bring a joint or the end-effecter to a desired orientation or position, we need to apply **forces** and **torques** at each joint of the robot arm.
- To determine the forces and torques required at each joint we need to learn about robot dynamics.
- Kinematics help us determine the joint variable set-points
- Dynamics help us build controllers so we can achieve these set-points with the help of **actuators**.

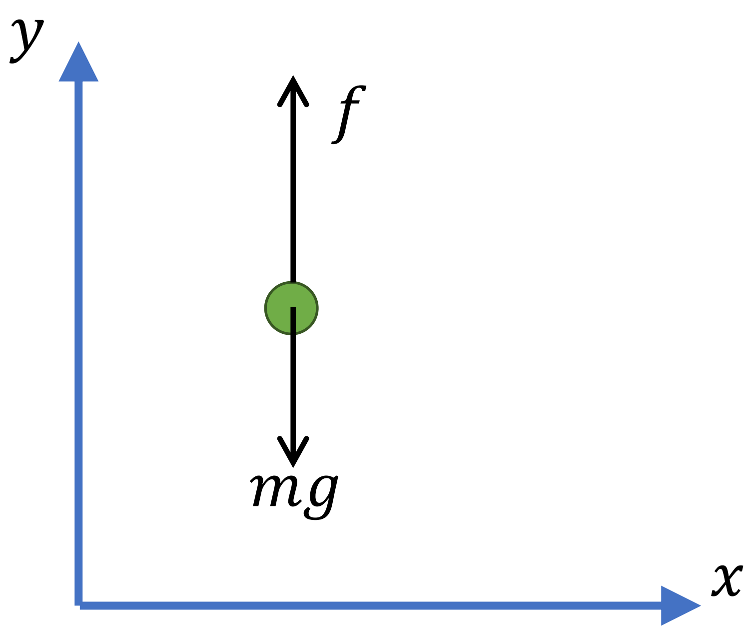
Euler-Lagrange Approach

The Lagrangian

- Lagrangian is the difference between the kinetic and potential energy in a system (Based on the principle of work and energy): $\boldsymbol{L = K - P}$
- Analyses the system as a whole and at the same time
- Based on n generalized coordinates
- No need to calculate constraint forces for each link

Derivation of Euler-Lagrange Equations

- Consider the dynamics of a particle



Kinetic Energy of the system:

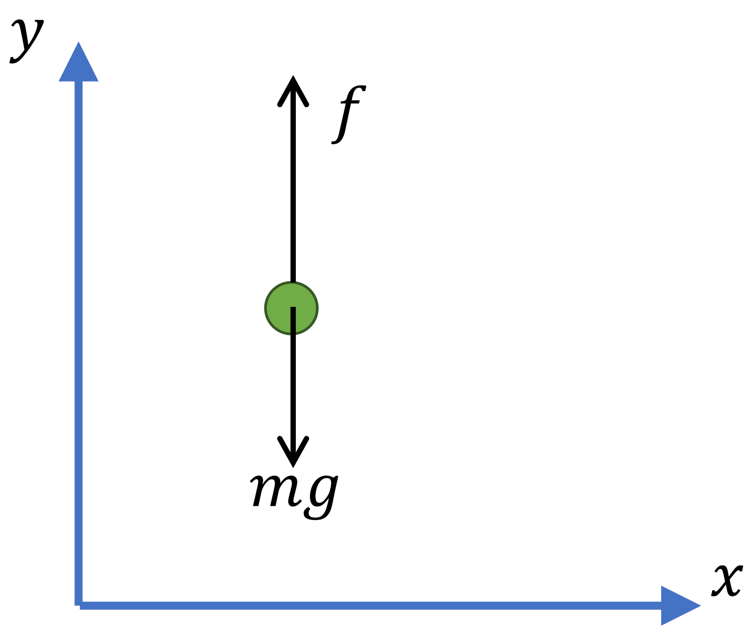
$$\sum F = m.a \quad \text{(Newton's 2}^{\text{nd}}\text{ law)}$$
$$f - mg = m\ddot{y}$$

It can be also seen that,

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\delta}{\delta\dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\delta K}{\delta\dot{y}}$$

Where $K = \frac{1}{2}m\dot{y}^2$ is the kinetic energy of the system.

Derivation of Euler-Lagrange Equations



Potential Energy of the system:

$$P = mgy$$

It can be also seen that,

$$\frac{\delta}{\delta y}(P) = \frac{\delta}{\delta y}(mgy) = mg$$

Where P is the potential energy of the system

Derivation of Euler-Lagrange Equations

- We define the **Lagrangian** $\boldsymbol{L}$ as,

$$\boldsymbol{L = K - P}$$

$$L = K - P = \frac{1}{2}m\dot{y}^2 - mgy$$

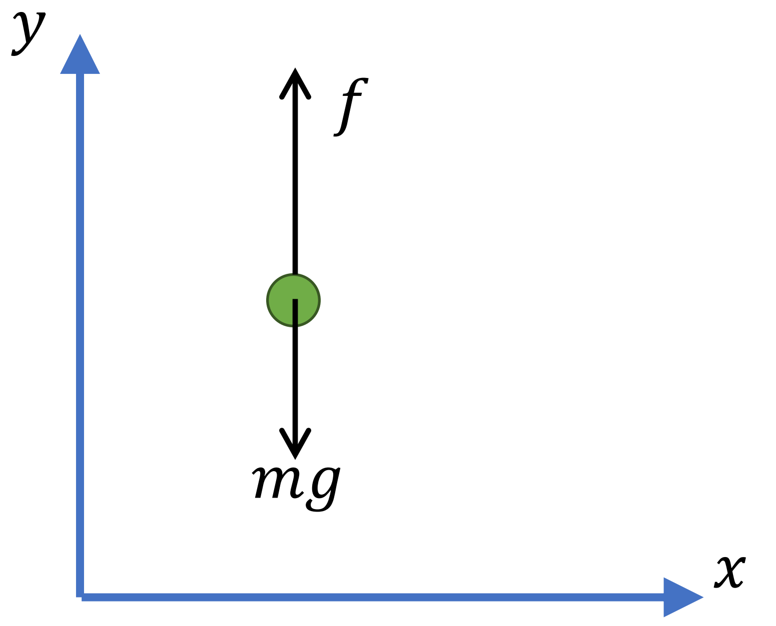
Since,

$$\frac{\delta L}{\delta \dot{y}} = m\dot{y} = \frac{\delta K}{\delta \dot{y}}$$

$$\frac{\delta L}{\delta y} = -mg = \frac{\delta P}{\delta y}$$

Force acting on the system is,

$$\boxed{f = m\ddot{y} + mg = \frac{d}{dt}\frac{\delta L}{\delta \dot{y}} - \frac{\delta L}{\delta y}}$$



The General Case

In the previous example,

f - external force acting on the system

q - **generalised coordinate** (The variable causing change in K and P)

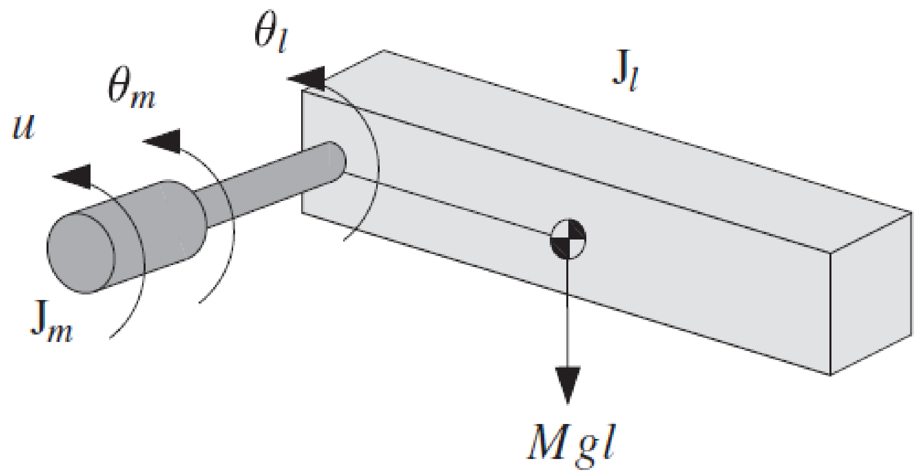
Thus the Euler-Lagrangian equations yield,

$$\boxed{f = \frac{d}{dt}\frac{\delta L}{\delta \dot{q}} - \frac{\delta L}{\delta q}}$$

Also note that the 1-DOF system resulted in a single generalized coordinate.

Single-Link Manipulator

Let θ_l and θ_m denote the angles of the link and motor shaft, respectively. Then $\theta_m = r\theta_l$ where r : 1 is the gear ratio.



- The kinetic energy of the system

$$K = \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_\ell\dot{\theta}_\ell^2$$
$$= \frac{1}{2}(r^2J_m + J_\ell)\dot{\theta}_\ell^2$$

- The potential energy

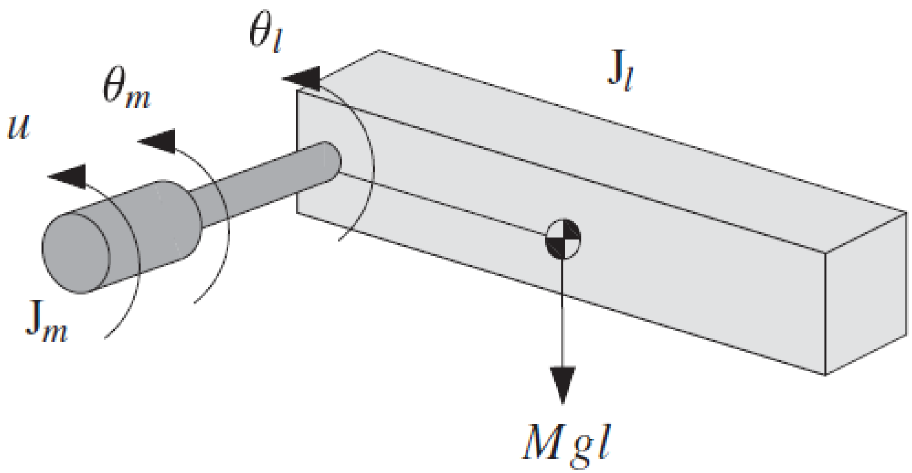
$$P = Mg\ell(1 - \cos\theta_\ell)$$

Defining $J = r^2J_m + J_\ell$

- The Lagrangian $\mathcal{L} = \frac{1}{2}J\dot{\theta}_\ell^2 - Mg\ell(1 - \cos\theta_\ell)$

Single-Link Manipulator

- The Euler-Lagrange equations yields the equation of motion



$$f = \frac{d}{dt}\frac{\delta L}{\delta \dot{q}} - \frac{\delta L}{\delta q}$$

$$\mathcal{L} = \frac{1}{2}J\dot{\theta}_\ell^2 - Mg\ell(1-\cos\theta_\ell)$$

Applied force/torque

$$J\ddot{\theta}_\ell + Mg\ell\sin\theta_\ell = \tau$$

$$\tau = u - B\dot{\theta}_\ell$$

Motor output force System damping force

$$\boxed{J\ddot{\theta}_\ell + B\dot{\theta}_\ell + Mg\ell\sin\theta_\ell = u}$$

Apply PD control to move the joint to the targeted position

$$u = K_P\tilde{q} - K_D\dot{q}$$

Generalised Coordinates for an n-link Robot Manipulator

- For an n-link robot manipulator, the joint variables form a set of generalised coordinates, e.g., $(q_1, q_2, \ldots, q_n)$

- Generalised coordinates should be also independent from one another

- For an n-DOF robot with n generalised coordinates,

$$\boxed{\frac{d}{dt}\frac{\delta L}{\delta \dot{q}_k} - \frac{\delta L}{\delta q_k} = \tau_k; k = 1,2,\ldots,n}$$

Find the Lagrangian of these equations

τ_k- Generalised force (if prismatic) or torque (if revolute) at each joint
$\boldsymbol{q} = (q_1, q_2, q_3 \ldots\ldots, q_n)^T$ - Vector of joint variables

Applying Euler-Lagrange Method to an n-link Manipulator

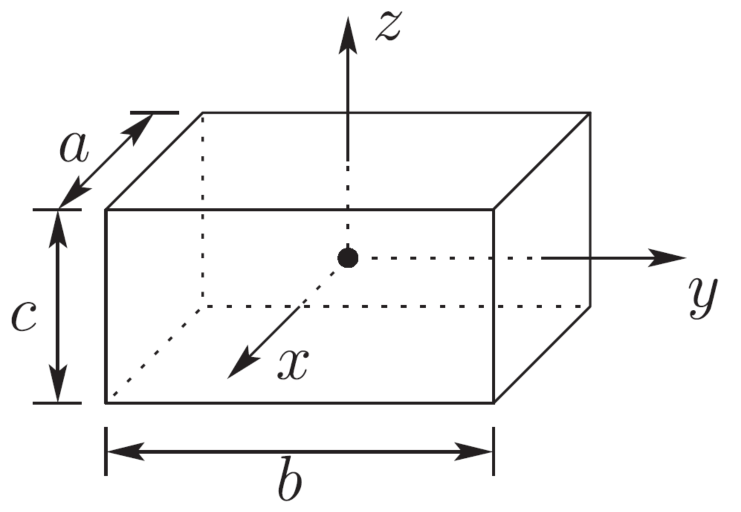
Nomenclature

I_i: Moment of inertia of link i about an axis passing through center of mass of link i
q: The vector of joint variables where q_i is i^{th} generalized coordinate
G: Gravitational vector expressed in the base frame
$J_{\omega i}$: Jacobian sub-matrix w.r.t. the angular velocity of link i
J_{vci}: Jacobian sub-matrix w.r.t. the linear velocity of center of mass of link i
n: Number of joints
v_{ci}: Linear velocity of center of mass of link i
ω_i: Angular velocity of link i
o_{ci}: Vector from origin O_0 to center of mass of link i
τ_k: Generalized force at each joint

Kinetic Energy - K

- Kinetic energy of an object:

$$K = \frac{1}{2} m \boldsymbol{v}_{\boldsymbol{c}}^T \boldsymbol{v}_{\boldsymbol{c}} + \frac{1}{2} \boldsymbol{\omega}^T I \boldsymbol{\omega}$$



where,

- I – 3×3 inertia matrix
- $\boldsymbol{v}_{\boldsymbol{c}}$ – velocity vector of the **center of mass**
- $\boldsymbol{\omega}$ – angular velocity vector

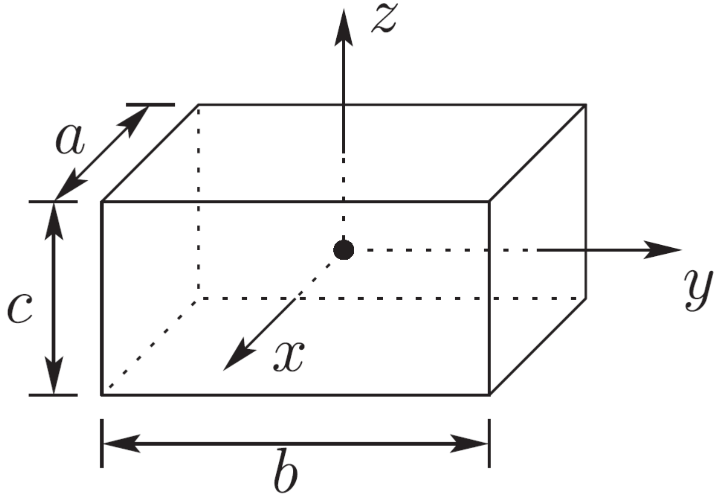
Kinetic Energy - *K*

- I – 3×3 Inertia Matrix (or Inertia Tensor)

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \int\int\int (y^2 + z^2)\rho(x,y,z)dx\ dy\ dz$$
$$I_{yy} = \int\int\int (x^2 + z^2)\rho(x,y,z)dx\ dy\ dz$$
$$I_{zz} = \int\int\int (x^2 + y^2)\rho(x,y,z)dx\ dy\ dz$$

$$I_{xy} = I_{yx} = -\int\int\int xy\rho(x,y,z)dx\ dy\ dz$$
$$I_{xz} = I_{zx} = -\int\int\int xz\rho(x,y,z)dx\ dy\ dz$$
$$I_{yz} = I_{zy} = -\int\int\int yz\rho(x,y,z)dx\ dy\ dz$$



$$I_{xx} = \frac{m}{12}(b^2 + c^2)$$

$$I_{yy} = \frac{m}{12}(a^2 + c^2) \quad ; \quad I_{zz} = \frac{m}{12}(a^2 + b^2)$$

Kinetic Energy for a Single Link

- Kinetic energy of the i[th] link:

$$K_i = \frac{1}{2}\left(m_i \boldsymbol{v}_{\boldsymbol{ci}}^T \boldsymbol{v}_{ci} + \boldsymbol{\omega}_{\boldsymbol{i}}^T I_i \boldsymbol{\omega}_{\boldsymbol{i}}\right)$$

- Apply the Jacobian

$$\boldsymbol{v}_{ci} = J_{vci}(\boldsymbol{q})\,\dot{\boldsymbol{q}}$$

$$\boldsymbol{\omega}_{\boldsymbol{i}} = R_i^T(\boldsymbol{q}) J_{\omega i}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- J_{vci} – Jacobian matrix for velocity of the center of mass of link i
- $J_{\omega i}$ – Jacobian matrix for angular velocity of link i

- $R_i^T(\boldsymbol{q})$ – transformation matrix that transforms the angular velocity vector from the object frame to the inertial frame

Kinetic Energy for a Single Link

We also know,

$$J_{vci}^{j} = \begin{cases} \mathbf{z}_{j-1} \times (\mathbf{o}_{ci} - \mathbf{o}_{j-1}) & \text{if joint } j \text{ is revolute} \\ \mathbf{z}_{j-1} & \text{if joint } j \text{ is prismatic} \end{cases}$$

and,

$$J_{\omega i}^{j} = \begin{cases} \mathbf{z}_{j-1} & \text{if joint } j \text{ is revolute} \\ 0 & \text{if joint } j \text{ is prismatic} \end{cases}$$

Where, $j \leq i$ and $\mathbf{o}_c$ is the position vector of the centre of mass of the ith link.

$$J_{vci} = (J_{vci}^{1}, J_{vci}^{2}, \ldots \ldots . J_{vci}^{i}, 0,0,0,0)$$
$$J_{\omega i} = (J_{\omega i}^{1}, J_{\omega i}^{2}, \ldots \ldots . J_{\omega i}^{i}, 0,0,0,0)$$

See Spong Ch.4, p. 116 (example 4) for the Jacobian J_{vci}^{j}

Kinetic Energy for n-links

- Total kinetic energy becomes,

$$K = \frac{1}{2}\sum_{i=1}^{n}(m_i \boldsymbol{v}_{\boldsymbol{ci}}^T \boldsymbol{v}_{ci} + \boldsymbol{\omega}_{\boldsymbol{i}}^T I_i \boldsymbol{\omega}_{\boldsymbol{i}})$$

$$K = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \underbrace{\sum_{i=1}^{n}\left(m_i J_{vci}(\boldsymbol{q})^T J_{vci}(\boldsymbol{q}) + J_{\omega i}(\boldsymbol{q})^T R_i(\boldsymbol{q}) I_i R_i^T(\boldsymbol{q}) J_{\omega i}(\boldsymbol{q})\right)}\,\dot{\boldsymbol{q}}$$

$D(\boldsymbol{q})$ – inertia matrix with $n \times n$ terms.

$$\boxed{K = \frac{1}{2}\,\dot{\boldsymbol{q}}^T D(\boldsymbol{q})\dot{\boldsymbol{q}}}$$

Potential Energy - P

- P_i is the potential energy stored in link i
- It equals the amount of work required to displace the center of mass of link i from the origin of base frame to position $\boldsymbol{r}_{ci}$

$$P_i = m_i \boldsymbol{g}^T \boldsymbol{r}_{ci}$$

where,

$\boldsymbol{g}^T$ – gravitational vector expressed in base frame $\{0\}$

For n-links,

$$\boxed{P(q) = \sum_{i=1}^{n} m_i \boldsymbol{g}^T \boldsymbol{r}_{ci}}$$

Equations of Motion

- Our aim is to find, $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$

- For this we first consider the Lagrangian,

$$L = K - P$$

$$L = \frac{1}{2}\,\dot{\boldsymbol{q}}^T D(\boldsymbol{q})\dot{\boldsymbol{q}} \quad - P(\boldsymbol{q})$$

We can rewrite Kinetic Energy part as,

$\dot{\boldsymbol{q}}^T D(\boldsymbol{q})\dot{\boldsymbol{q}} \quad = \sum_{i,j}^{n} d_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j$; $\qquad i,j$ – row and column indices of $D(\boldsymbol{q})$

Thus, $$\boxed{L = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j - P(\boldsymbol{q})}$$

Equations of Motion

- Since $P(\boldsymbol{q})$ is not dependent on $\dot{q}$, the **partial derivative w.r.t the k^{th} joint velocity** ($\dot{\boldsymbol{q}}_k$) becomes,

$$L = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j - P(\boldsymbol{q}) \Rightarrow \frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}(\boldsymbol{q})\dot{q}_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}(\boldsymbol{q})\dot{q}_j$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_j \frac{\partial d_{kj}(\boldsymbol{q})}{\partial q_i}\dot{q}_i\dot{q}_j}$$

$$k = 1,2,\ldots\ldots,n$$

Equations of Motion

- Similarly, the **partial derivative w.r.t the k^{th} joint position** (q_k)becomes,

$$L = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(\boldsymbol{q})\dot{q}_k\dot{q}_j - P(\boldsymbol{q})$$

$$\boxed{\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}^{n} \frac{\partial d_{ij}(\boldsymbol{q})}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P(\boldsymbol{q})}{\partial q_k}}$$

Euler-Lagrange Equations

- Using the results from the previous two slides, we can write the Euler-Lagrange equations as,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$

$$\Big(\sum_j d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_j \frac{\partial d_{kj}(\boldsymbol{q})}{\partial q_i}\dot{q}_i\dot{q}_j\Big) - \Big(\frac{1}{2}\sum_{i,j}^{n} \frac{\partial d_{ij}(\boldsymbol{q})}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P(\boldsymbol{q})}{\partial q_k}\Big) = \tau_k$$

$$\boxed{\sum_j d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{ij}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k}$$

Euler-Lagrange Equations

- By interchanging order of summation and using symmetry, we can simplify the Euler-Lagrange as,

$$\sum_{j} d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + g_k(q) = \tau_k; k = 1,2, \dots., n$$

Where,

$$g_k = \frac{\partial P}{\partial q_k}$$

$$c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$$

See Spong Ch.6, p.255 for proof.

Euler-Lagrange Equations

- The system can be further simplified so we can understand the content of each term.

$$\sum_{j} d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + g_k(q) = \tau_k; k = 1,2,\ldots.,n$$

$$\underbrace{M(\boldsymbol{q})\ddot{\boldsymbol{q}}}_{\text{Joint-space inertia matrix}} + \underbrace{C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}}_{\text{Coriolis and centripetal coupling matrix}} + \underbrace{g(\boldsymbol{q})}_{\text{Gravity loading}} = \boldsymbol{\tau};$$

Euler-Lagrange Equations

- Note that,

$$g(\boldsymbol{q}) = (g_1(\boldsymbol{q}), g_2(\boldsymbol{q}), \dots, g_n(\boldsymbol{q}))^T$$

$$C(\boldsymbol{q}, \dot{\boldsymbol{q}})_{kj} = c_{kj} = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i$$

- Here we only considered the effects due to the movement of the links.
- Refer to Corke –Eq. 9.8 for the effects of friction and force due to a wrench applied at the end-effector.

Applications of Euler-Lagrange Equations

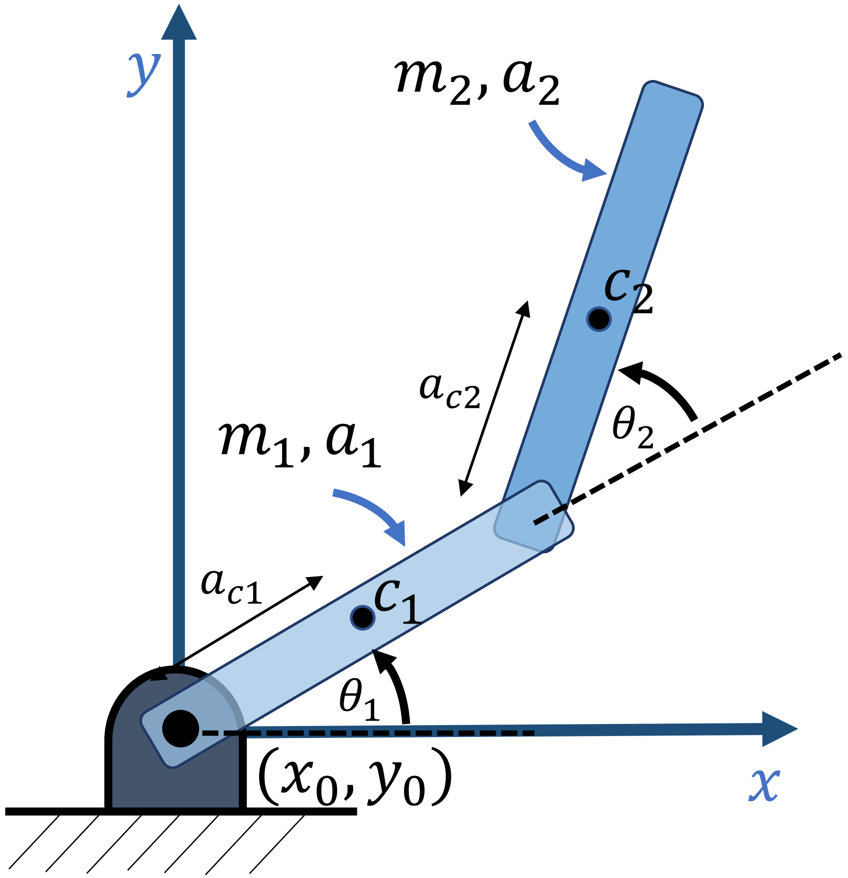
- We now have a differential equation which represents the model of the robot arm.

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} \quad + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \quad + g(\boldsymbol{q}) = \boldsymbol{\tau};$$

- The model is expressed in terms of generalised coordinates (e.g. Joint Variables), so can be directly **applied for motor control at each joint**.

- Solve this equation for specific **initial conditions** and **applied loads** in order to know **how the arm behaves**.

Example: Euler-Lagrange Equations for a Planar Elbow Manipulator



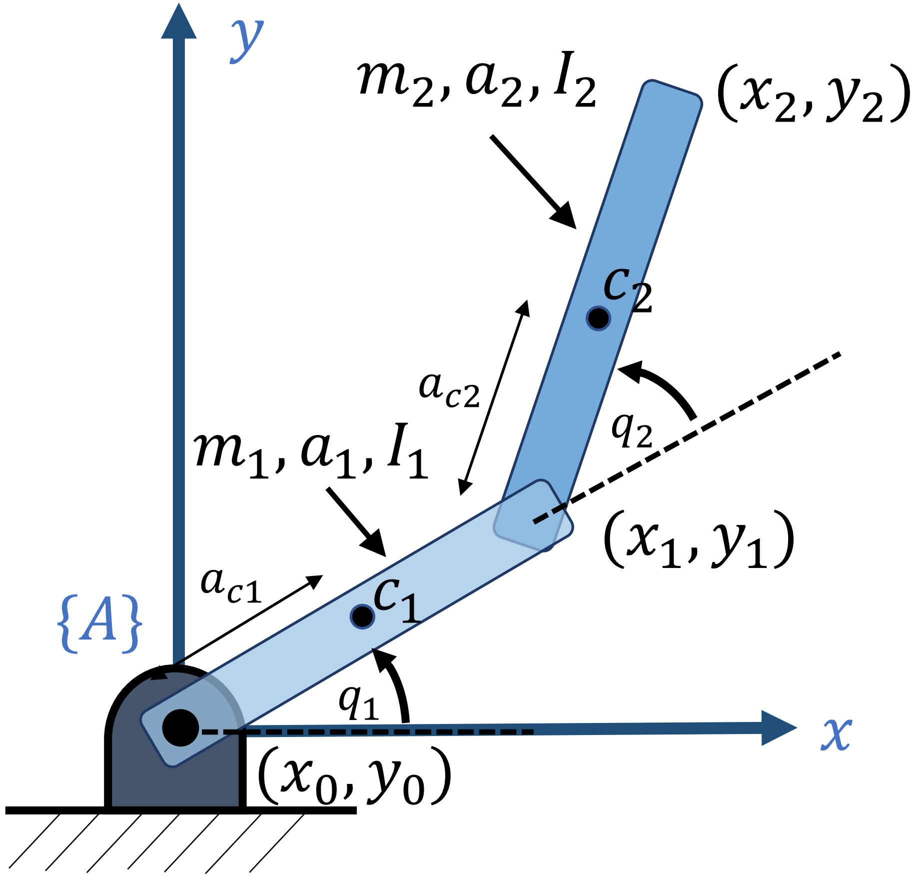
The Step-by-step Approach

$$\sum_{j} d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

$$m_i J_{vci}(\boldsymbol{q})^T J_{vci}(\boldsymbol{q}) + J_{\omega i}(\boldsymbol{q})^T R_i(\boldsymbol{q}) I_i R_i^T(\boldsymbol{q}) J_{\omega i}(\boldsymbol{q})$$

- Step 1: Obtain **DH parameters** and homogeneous transformations
- Step 2: Determine position vectors of **center of gravity** of each link and obtain **gravitational matrix** components
- Step 3: Construct the **Jacobian** matrixes
- Step 4: Find the **kinetic energy**
- Step 5: Construct manipulator matrix $\boldsymbol{D}(\boldsymbol{q})$
- Step 6: Apply the Euler-Lagrange Equation

Step 1: DH table and Homogeneous Transformation



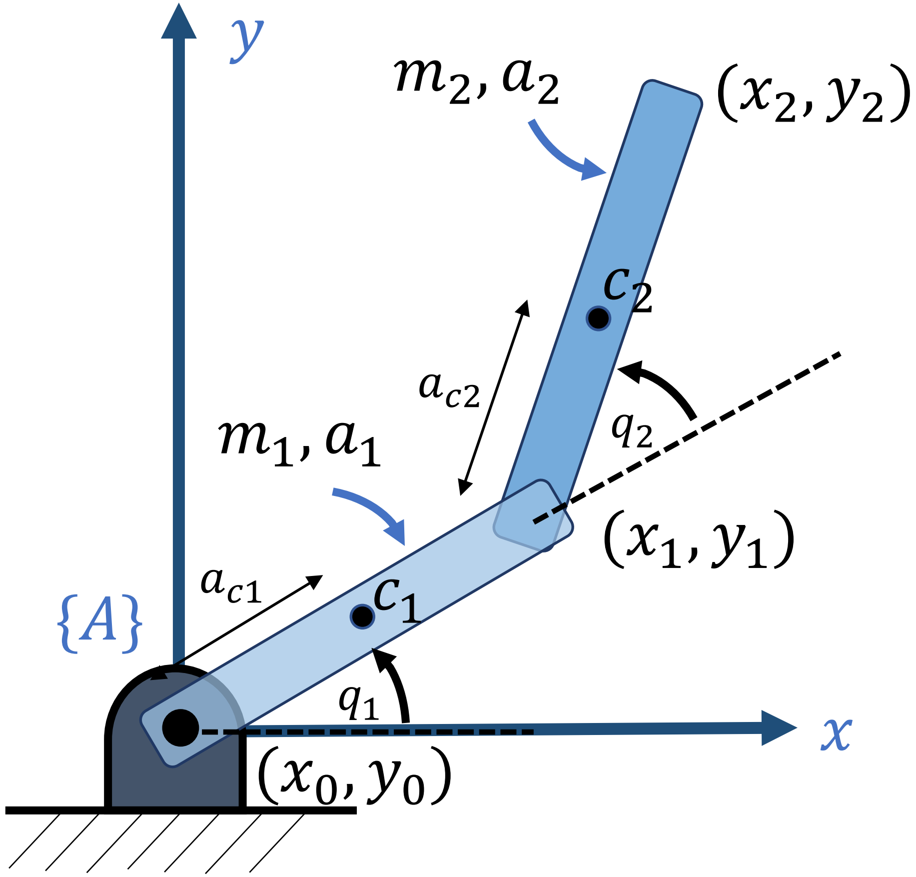
❑ **Step 1:**

i	θ_i	d_i	a_i	α_i
1	q_1	0	a_1	0
2	q_2	0	a_2	0

q_1 and q_2 are the generalized coordinates.

$$x_2 = a_1 cosq_1 + a_2 \cos(q_1 + q_2)$$
$$y_2 = a_1 sinq_1 + a_2 \sin(q_1 + q_2)$$

Step 1: Homogeneous Transformation

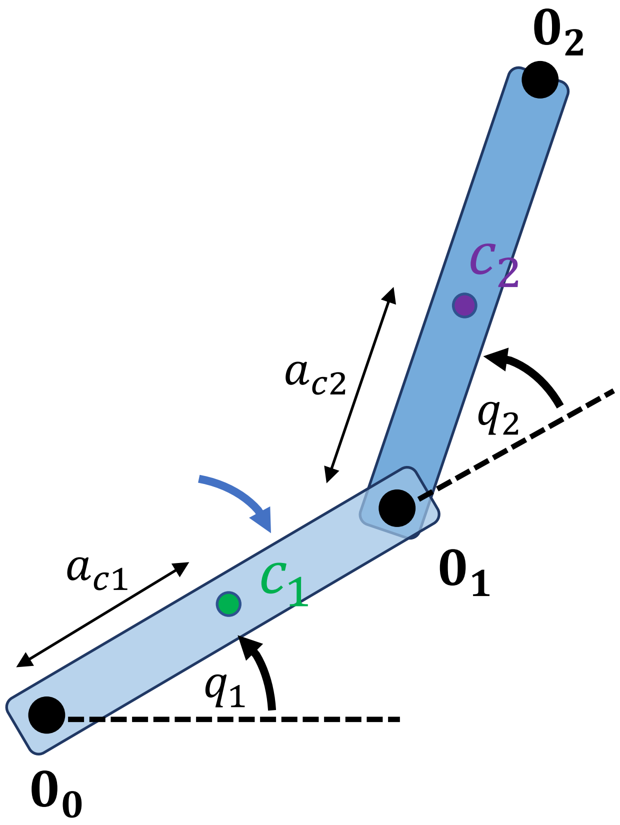


Write down the homogeneous transforms.

$$
{}^{0}\boldsymbol{T}_{1} = \begin{pmatrix} \cos q_1 & -sinq_1 & 0 & a_1 cosq_1 \\ sin\, q_1 & \cos q_1 & 0 & a_1 sinq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{0}\boldsymbol{T}_{2} = \begin{pmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 & a_1 cosq_1 + a_2\cos(q_1+q_2) \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 & a_1 sinq_1 + a_2\sin(q_1+q_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Step 2: Center of Gravity and Potential energy



Using the details of the 4th column of T

$$\mathbf{o}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{o}_1 = \begin{pmatrix} a_1 cosq_1 \\ a_1 sinq_1 \\ 0 \end{pmatrix}; \mathbf{o}_2 = \begin{pmatrix} a_1 cosq_1 + a_2 \cos(q_1 + q_2) \\ a_1 sinq_1 + a_2 \sin(q_1 + q_2) \\ 0 \end{pmatrix}$$

Thus,

$$\mathbf{o}_{c1} = \begin{pmatrix} a_{c1} cosq_1 \\ a_{c1} sinq_1 \\ 0 \end{pmatrix}; \mathbf{o}_{c2} = \begin{pmatrix} a_1 cosq_1 + a_{c2} \cos(q_1 + q_2) \\ a_1 sinq_1 + a_{c2} \sin(q_1 + q_2) \\ 0 \end{pmatrix}$$

And from 3rd column we get,

$$\mathbf{z}_0 = \mathbf{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Step 2: Center of Gravity and Potential energy

$$\mathbf{o}_{c1} = \begin{pmatrix} a_{c1} cos q_1 \\ a_{c1} sin q_1 \\ 0 \end{pmatrix}; \mathbf{o}_{c2} = \begin{pmatrix} a_1 cos q_1 + a_{c2} \cos(q_1 + q_2) \\ a_1 sin q_1 + a_{c2} \sin(q_1 + q_2) \\ 0 \end{pmatrix}$$

Potential energy for link 1,

$$P_1 = m_1 g a_{c1} \sin(q_1)$$

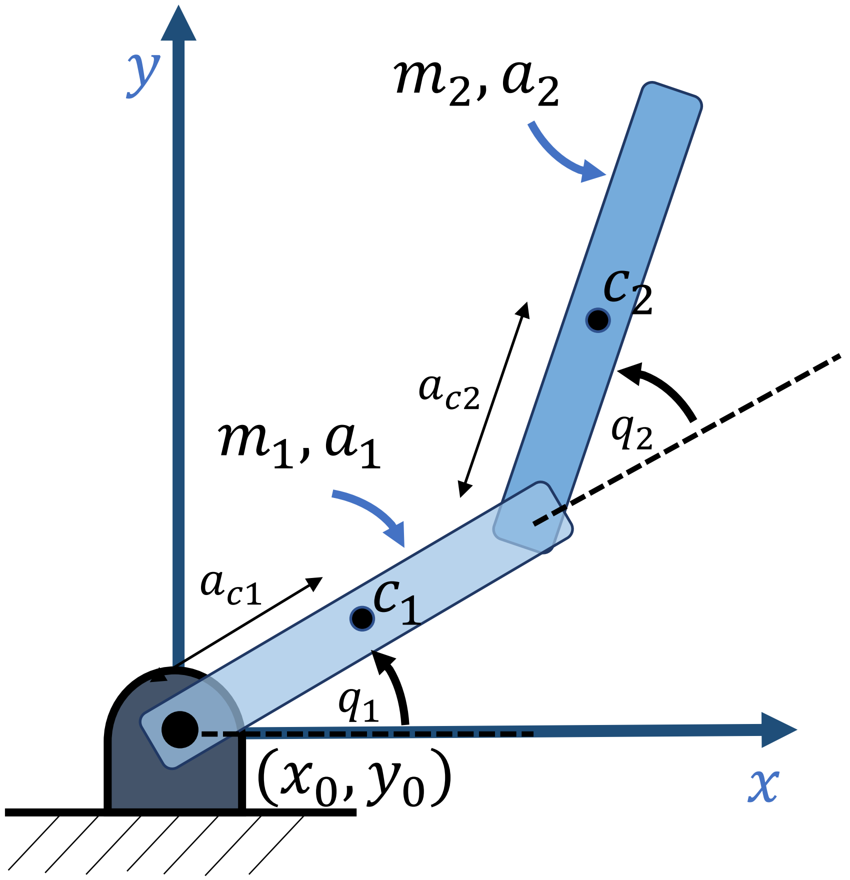
Potential energy for link 2,

$$P_2 = m_2 g(a_1 \sin(q_1) + a_{c2} \sin(q_1 + q_2))$$

Total potential energy,

$$P = P_1 + P_2$$

$$\boxed{\boldsymbol{P = (m_1 a_{c1} + m_2 a_1) g sin(q_1) + m_2 a_{c2} g \, sin(q_1 + q_2)}}$$



Step 3: The Jacobian

- Now we can derive the Jacobian sub-matrices as,

$$J_{vc1} = \begin{pmatrix} z_0 \times (O_{c1} - O_0) & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{pmatrix}$$

$$J_{vc2} = \begin{pmatrix} z_0 \times (O_{c2} - O_0) & z_1 \times (O_{c2} - O_1) \end{pmatrix}$$

$$J_{\omega 1} = \begin{pmatrix} z_0 & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{pmatrix}$$

$$J_{\omega 2} = \begin{pmatrix} z_0 & z_1 \end{pmatrix}$$

See the further reading slide

Step 3: The Jacobian

- Now we can derive the Jacobian sub-matrices as,

$$J_{vc1} = \begin{pmatrix} -a_{c1} sinq_1 & 0 \\ a_{c1} cosq_1 & 0 \\ 0 & 0 \end{pmatrix}; J_{vc2} = \begin{pmatrix} -a_1 sinq_1 - a_{c2} \sin(q_1 + q_2) & -a_{c2} \sin(q_1 + q_2) \\ a_1 cosq_1 + a_{c2} cos(q_1 + q_2) & a_{c2} \cos(q_1 + q_2) \\ 0 & 0 \end{pmatrix}$$

$$J_{\omega 1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}; J_{\omega 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Step 4: Kinetic Energy

- Since rotational kinetic energy is,

$$K_r = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \sum_{i=1}^{2} \left(J_{\omega i}(\boldsymbol{q})^T R_i(\boldsymbol{q}) I_i R_i^T(\boldsymbol{q}) J_{\omega i}(\boldsymbol{q}) \right) \dot{\boldsymbol{q}}$$

$$K_r = \frac{1}{2}\dot{\boldsymbol{q}}^T \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} I_1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} I_2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \dot{\boldsymbol{q}}$$

$$K_r = \frac{1}{2}\dot{\boldsymbol{q}}^T \left\{ I_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + I_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \dot{\boldsymbol{q}} \quad = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{pmatrix} \dot{\boldsymbol{q}}$$

Using $I_{zz} = \frac{m}{12}(a^2 + b^2)$ to calculate I_1 and I_2

Step 4: Kinetic Energy

- The translational kinetic energy is,

$$K_t = \frac{1}{2}\dot{\boldsymbol{q}}^T\{m_1 J_{vc1}(\boldsymbol{q})^T J_{vc1}(\boldsymbol{q}) + m_2 J_{vc2}(\boldsymbol{q})^T J_{vc2}(\boldsymbol{q})\}\dot{\boldsymbol{q}}$$

Thus, total kinetic energy,

$$K = K_t + K_r$$

$$K = \frac{1}{2}\,\dot{\boldsymbol{q}}^T D(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Step 5: Construct the manipulator matrix D(q)

- The inertia matrix $D(\boldsymbol{q})$ becomes

$$D(\boldsymbol{q}) = m_1 J_{vc1}(\boldsymbol{q})^T J_{vc1}(\boldsymbol{q}) + m_2 J_{vc2}(\boldsymbol{q})^T J_{vc2}(\boldsymbol{q}) + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

$$D(\boldsymbol{q}) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

where,

$$d_{11} = m_1 a_{c1}^2 + m_2(a_1^2 + a_{c2}^2 + 2a_1 a_{c2}\cos(q_2)) + I_1 + I_2$$
$$d_{12} = d_{21} = m_2(a_{c2}^2 + a_1 a_{c2}\cos(q_2)) + I_2$$
$$d_{22} = m_2 a_{c2}^2 + I_2$$

Step 6: Apply the Euler-Lagrange Equations

$$\sum_j d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

Where

$$c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$$

$$g_k = \frac{\partial P}{\partial q_k}$$

Where $P = (m_1 a_{c1} + m_2 a_1) g sin(q_1) + m_2 a_{c2} g\, sin(q_1 + q_2)$

Step 6: Apply the Euler-Lagrange Equations

(A) $c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$

$$c_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$

$$c_{121} = c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 a_1 a_{c2}\sin(q_2) = h$$

$$c_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h$$

$$c_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h$$

$$c_{122} = c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0$$

$$c_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0$$

Step 6: Apply the Euler-Lagrange Equations

(B) $g_k = \frac{\partial P}{\partial q_k}$

Thus,

$$g_1 = \frac{\partial P}{\partial q_1} = (m_1 a_{c1} + m_2 a_1) g cos(q_1) + m_2 a_{c2} g \cos(q_1 + q_2)$$

$$g_2 = \frac{\partial P}{\partial q_2} = m_2 a_{c2} g \cos(q_1 + q_2)$$

Step 6: Apply the Euler-Lagrange Equations

Recall

$$\sum_{j} d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + g_k(q) = \tau_k; k = 1,2,\ldots.,n$$

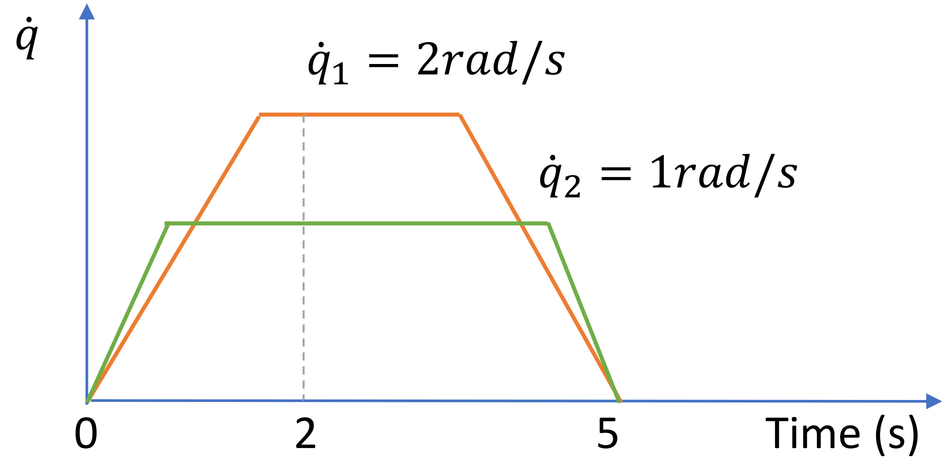
Thus, for the planar elbow manipulator, the final dynamic equations representing the torques at each joint are,

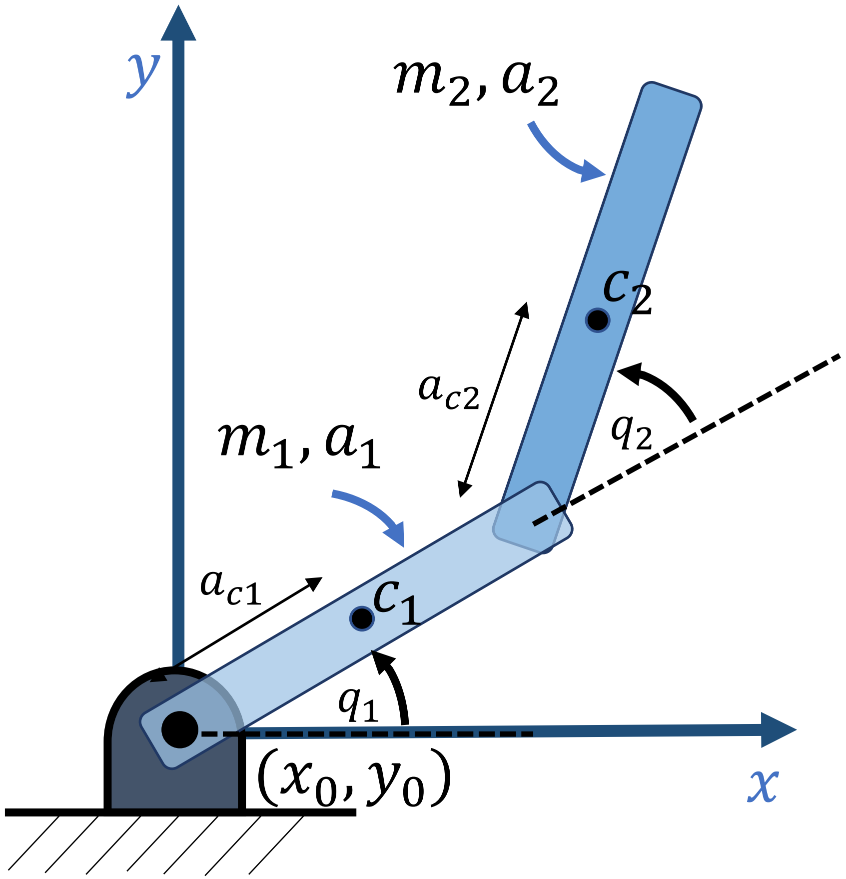
$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 = \tau_1$$
$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 = \tau_2$$

Example: Trapezoidal trajectory

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 = \tau_1$$
$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 = \tau_2$$

Calculate the Coriolis and centripetal coupling torque of τ_1 and τ_2 at t= 2(s)





Example: Trapezoidal trajectory

$$\dot{q}_1 = 2rad/s \qquad \dot{q}_2 = 1rad/s$$

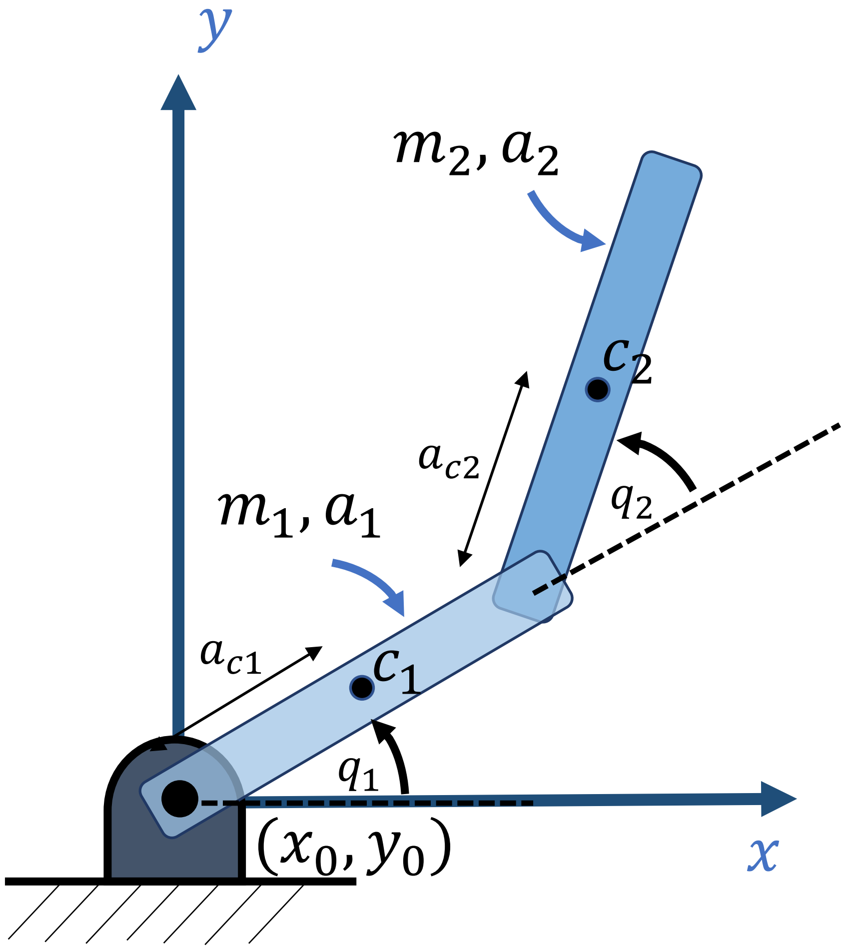
$$\text{Coriolis}(\tau_1) = c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 = 2c_{121} + 2c_{211} + c_{221}$$

$$\text{Coriolis}(\tau_2) = \quad c_{112}\dot{q}_1^2 = 4c_{112}$$

$$c_{121} = c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 a_1 a_{c2}\sin(q_2) = h$$

$$c_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h$$

$$c_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h$$



Summary

- Lagrangian

$$L = K - P$$

- Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{q_k}} - \frac{\delta L}{\delta q_k} = \tau_k; k = 1,2, \dots, n$$

- Euler-Lagrange Equation

$$\sum_j d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + g_k(q) = \tau_k; k = 1,2, \dots., n$$

$$M(\boldsymbol{q})\ddot{\mathbf{q}} \quad + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \quad + g(\boldsymbol{q}) = \boldsymbol{\tau};$$

Next week

- ❑ Quiz 2 (20 Marks)
  - From 16:00 pm, Monday 31 July
  - Quiz contents: Lectures 5 to 9, except Singularity (slides 34 – 44, Lec 5), Acurracy & Repeatability (slides 52 – 56, Lec 7), Grid Based Method (Slides 11-22, Lec 8)

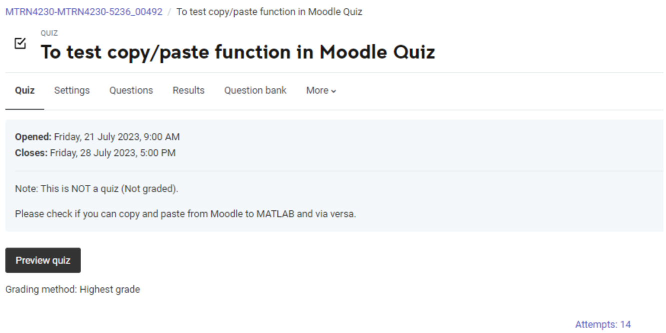
- ❑ Lecture 10: Quiz revision

- ❑ Lab time will be used for Project 2

Next week

- ❑ Quiz 2 preparation: Please practice all examples in the slides (lectures 5 - 9) and watch the tutorial recording (See Week 9 Folder - Moodle)

- ❑ Test copy/past on Moodle

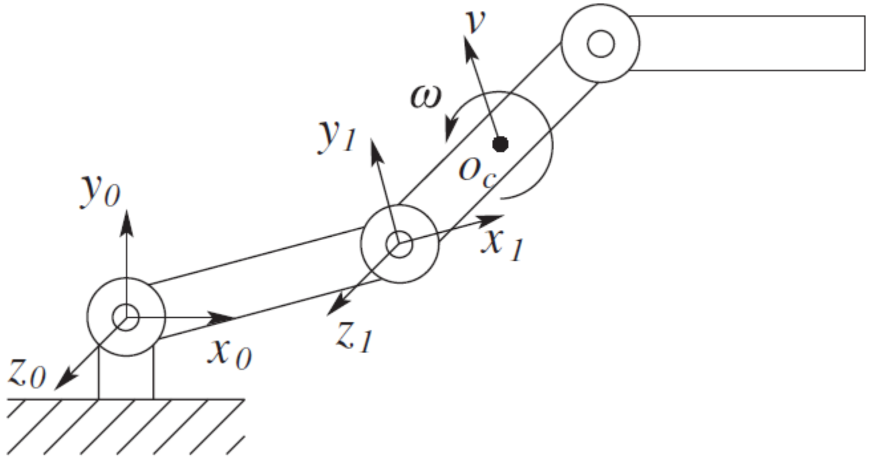


- ❑ Quiz 2 (20 Marks)
  - From 16:00 pm, Monday 31 July (please do not start later than 16:15pm)
  - Time: 95 minutes (extra 5mins for typing answers).
  - There are <u>6 questions</u>, <u>including 2 basic questions </u>(that assess some basic knowledge/equations from lectures 5 – 9. Each question worth 3 marks). The other questions are at the same level as the examples provided in the lecture slides.
  - This is a relatively long Quiz. Please manage your time effectively (15 mins on each question. Some longer questions may take 20 mins. Basic questions may take less time).

Appendix

❑Jacobian for an Arbitrary Point on a Link

Consider a three-link planar manipulator. Suppose we wish to compute the linear velocity v and the angular velocity ω of the center of link 2. In this case we have that



$$J(q) = \begin{bmatrix} z_0 \times (o_c - o_0) & z_1 \times (o_c - o_1) & 0 \\ z_0 & z_1 & 0 \end{bmatrix}$$

which is merely the usual Jacobian with O_c in place of O_n.
Note that in this case the vector O_c must be computed as it is not given directly by the T matrices