定解问题适定性　　　　解存在，唯一，稳定

（调和方程的混合问题不稳定）

一阶线性　　$a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+cu=f$

① $\frac{\partial u}{\partial y}+k(x,y)u=f(x,y)$

$$\Rightarrow u=e^{-\int_{y_0}^{y}k(x,t)dt}\left[\int_{y_0}^{y}e^{\int_{y_0}^{s}k(x,t)dt}f(x,s)ds+g(x)\right]$$

② 特征线　$\frac{dx}{a}=\frac{dy}{b}$　$\Rightarrow$　$\xi=\varphi(x,y)$　取线性无关 $\eta=\psi(x,y)$

作变量代换 $\Rightarrow$ $\bigcirc\frac{\partial u}{\partial \eta}+cu=f$ $\Rightarrow$ ①

③ 达朗贝尔公式　$u(t,x)=\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, & t>0, \quad -\infty<x<+\infty\\ u|_{t=0}=\varphi(x), & \frac{\partial u}{\partial t}|_{t=0}=\psi(x)\end{cases}$$

半直线问题　$u(t,0)=0$ $\Rightarrow$ $\varphi.\psi$ 奇延拓

$\frac{\partial u}{\partial t}(t,0)=0$ $\Rightarrow$ $\varphi.\psi$ 偶延拓

中心对称球面波　$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\Delta_3 u\\ u|_{t=0}=\varphi(r), \quad \frac{\partial u}{\partial t}|_{t=0}=\psi(r)\end{cases}$

$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}\right)\xlongequal{v=ru}\frac{\partial^2 v}{\partial t^2}=a^2\frac{\partial^2 v}{\partial r^2}$$

齐次：

二阶线性　$a_{11}\frac{\partial^2 u}{\partial x^2}+2a_{12}\frac{\partial^2 u}{\partial x\partial y}+a_{22}\frac{\partial^2 u}{\partial y^2}+b_1\frac{\partial u}{\partial x}+b_2\frac{\partial u}{\partial y}+cu=f(x,y)$

① 特征线：$a_{11}(dy)^2-2a_{12}dxdy+a_{22}(dx)^2=0$

判别式 $\Delta = a_{12}^2 - a_{11}a_{22}$

1) $\Delta > 0$ 双曲型

$\Rightarrow \begin{cases} \varphi(x,y)=h_1 \\ \psi(x,y)=h_2 \end{cases} \Rightarrow \dfrac{\partial^2 u}{\partial\xi\partial\eta} + \left(\square\dfrac{\partial u}{\partial\xi} + \square\dfrac{\partial u}{\partial\eta} + \square u\right) = 0$

2) $\Delta = 0$ 抛物型

$\Rightarrow \varphi(x,y) = h$ 再取 $\psi(x,y)$ 使 $J = \dfrac{\partial(\varphi,\psi)}{\partial(x,y)} \neq 0$

$\Rightarrow \dfrac{\partial^2 u}{\partial\eta^2} + \left(\square\dfrac{\partial u}{\partial\xi} + \square\dfrac{\partial u}{\partial\eta} + \square u\right) = 0$

3) $\Delta < 0$ 椭圆型

$\Rightarrow \varphi(x,y) \pm i\psi(x,y) = h \Rightarrow \begin{cases} \xi = \varphi(x,y) \\ \eta = \psi(x,y) \end{cases}$

$\Rightarrow \dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2} + \left(\square\dfrac{\partial u}{\partial\xi} + \square\dfrac{\partial u}{\partial\eta} + \square u\right) = 0$

② 叠加原理 & 齐次化原理

$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2} + f(t,x) \\ u|_{t=0} = 0 \quad u_t|_{t=0} = 0 \end{cases} \qquad u = \int_0^t w(t,x;\tau)\,d\tau$

$\Rightarrow \begin{cases} \dfrac{\partial^2 w}{\partial t^2} = a^2\dfrac{\partial^2 w}{\partial x^2}, & t > \tau \\ w|_{t=\tau} = 0, \quad \dfrac{\partial w}{\partial t}\Big|_{t=\tau} = f(\tau, x) \end{cases}$

Euler 方程 $r^2R'' + rR' - n^2R = 0 \xRightarrow{t=\ln r} \dfrac{d^2R}{dt^2} - n^2R = 0$

极坐标 Laplace

分离变量：
$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} = 0, & r < a \\ u|_{r=a} = f(\theta) \end{cases}$

$\Rightarrow u(r,\theta) = \dfrac{C_0}{2} + \dfrac{D_0}{2}\ln r + \sum_{n=1}^{+\infty}(C_n r^n + D_n r^{-n})(A_n\cos n\theta + B_n\sin n\theta)$

$k>0,\ q\geq 0\quad \rho>0$

S-L定理 $$[k(x)X'(x)]'-q(x)X(x)+\lambda\rho(x)X(x)=0$$

($b_0X''+b_1X'+b_2X+\lambda X=0$ 两边同乘 $\frac{1}{b_0}e^{\int\frac{b_1}{b_0}dx}$ 得到)

①非负性 $\begin{cases}\lambda_n>0\\ \lambda_0=0\end{cases}$ $\Longleftrightarrow$ $q(x)\equiv 0$ 无Ⅰ.Ⅲ类边界条件

② 可数性 对每个固有值，只有一个线性独立固有函数

③ 正交性 不同固有值对应的固有函数相互加权正交

$$\int_a^b X_n(x)X_m(x)\rho(x)dx=0\quad(n\neq m)$$

④ 展开 $$\begin{cases}f(x)=\sum_{n=1}^{+\infty}C_nX_n(x)\\ C_n=\dfrac{\int_a^b f(x)X_n(x)\rho(x)dx}{\|X_n(x)\|^2}\end{cases}$$

⑤ 附加条件：

1) $k(a)/k(b)\neq 0$，则可在此端点附加一.二.三类边界条件

2) 如再有 $k(a)=k(b)$ 则可附加周期性边界条件

3) $k(a)=0/k(b)=0$，可附加自然边界条件 (此点称为方程的奇点) 此时 $q(x)$在此点至多为一级极点

例题

非齐次

$$\begin{cases}\dfrac{\partial^2u}{\partial t^2}-a^2\dfrac{\partial^2u}{\partial x^2}=f(t,x),\quad t>0,\quad 0<x<l\\ u|_{x=0}=u|_{x=l}=0\\ u|_{t=0}=\varphi(x),\quad \dfrac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)\end{cases}$$

冲量原理 $u=u_1+u_2$

$$\Rightarrow\begin{cases}\dfrac{\partial^2u_1}{\partial t^2}-a^2\dfrac{\partial^2u_1}{\partial x^2}=f(t,x)\\ u_1|_{x=0}=u_1|_{x=l}=0\\ u_1|_{t=0}=0,\quad \dfrac{\partial u_1}{\partial t}\Big|_{t=0}=0\end{cases}\qquad \begin{cases}\dfrac{\partial^2u_2}{\partial t^2}-a^2\dfrac{\partial^2u_2}{\partial x^2}=0\\ u_2|_{x=0}=u_2|_{x=l}=0\\ u_2|_{t=0}=\varphi(x),\quad \dfrac{\partial u_2}{\partial t}\Big|_{t=0}=\psi(x)\end{cases}$$

$\Downarrow$

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = 0 \\ w|_{x=0} = w|_{x=l} = 0 \\ w|_{t=\tau} = 0, \quad \frac{\partial w}{\partial t}\Big|_{t=\tau} = f(\tau, x) \quad \longrightarrow t' = t-\tau \\ u_1 = \int_0^t w(t, x; \tau)\, d\tau \end{cases}$$

Fourier展开法

$$\begin{cases} u(t,x) = \underbrace{\sum_{n=1}^{+\infty} T_n(t) \sin\frac{n\pi}{l} x}_{展开} \\ f(t,x) = \\ \varphi(x) = \\ \psi(x) = \end{cases}$$

特解 $T_n^* = \int_0^t \frac{T_{n_1}(\tau) T_{n_2}(t) - T_{n_1}(t) T_{n_2}(\tau)}{W(\tau)} f_n(\tau) d\tau$

$$\Rightarrow \begin{cases} T_n'' + \left(\frac{an\pi}{l}\right)^2 T_n = f_n \\ T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n \end{cases} \Rightarrow \text{Laplace 变换法求解}$$

一般情况

(转化为齐次边界条件)

$$\begin{cases} L_t u + L_x u = f(t,x), \quad t>0, \quad a<x<b \\ \left(\alpha_1 u - \beta_1 \frac{\partial u}{\partial x}\right)\Big|_{x=a} = g_1(t) \qquad \left(\alpha_2 u - \beta_2 \frac{\partial u}{\partial x}\right)\Big|_{x=b} = g_2(t) \\ u|_{t=0} = \varphi(x) \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$$

令 $\begin{cases} u = v(t,x) + w(t,x) \\ \text{可取} \quad v(t,x) = A(t)x + B(t) \end{cases}$

$$\Rightarrow \begin{cases} (\alpha_1 a - \beta_1) A(t) + \alpha_1 B(t) = g_1(t) \\ (\alpha_2 b - \beta_2) A(t) + \alpha_2 B(t) = g_2(t) \end{cases} \Rightarrow v(t,x)$$

$$\Rightarrow \begin{cases} L_t w + L_x w = f(t,x) - L_t v - L_x v \\ \left(\alpha_1 u - \beta_1 \frac{\partial u}{\partial x}\right)\Big|_{x=a} = 0 \qquad \left(\alpha_2 u - \beta_2 \frac{\partial u}{\partial x}\right)\Big|_{x=b} = 0 \\ w|_{t=0} = \varphi(x) - v|_{t=0} \qquad \frac{\partial w}{\partial t}\Big|_{t=0} = \psi(x) - \frac{\partial v}{\partial t}\Big|_{t=0} \end{cases}$$

二元 Fourier 展开

$$\begin{cases} f(x,y) = \sum_{n,m=1}^{+\infty} C_{nm} X_n(x) Y_{nm}(y) \\ C_{nm} = \dfrac{\int_0^a \int_0^b f(x,y) X_n(x) Y_{nm}(y) \rho(x) \sigma(y) \, dy \, dx}{\|X_n(x)\|^2 \, \|Y_{nm}(y)\|^2} \end{cases}$$

幂级数解

$$\begin{cases} w''(z) + p(z) w'(z) + q(z) w = 0 \\ w(z_0) = a_0, \quad w'(z_0) = a_1 \end{cases}$$

Thm $p(z) \cdot q(z)$ 在 $|z-z_0|<R$ 内解析 $\Longrightarrow$ $|z-z_0|<R$ 内存在唯一解且解析

Def (正则奇点) z_0 为 $p(z)$ 至多一级极点，$q(z)$ 至多二级极点

Thm z_0 为正则奇点，则在 $0<|z-z_0|<R$ 上有

$$\begin{cases} w_1(z) = (z-z_0)^{\rho_1} \sum_{n=0}^{+\infty} a_n (z-z_0)^n \\ w_2(z) = \alpha w_1(z) \ln(z-z_0) + (z-z_0)^{\rho_2} \sum_{n=0}^{+\infty} b_n (z-z_0)^n \end{cases}$$

z_0 的指标

Thm z_0 为非正则奇点（$p \cdot q$ 分别超过一级、二级），有广义幂级数解

$$\sum_{n=-\infty}^{n=+\infty}$$

Legendre 函数

$$\Delta_3 u + k^2 u = 0$$

球坐标分离变量展开

θ 的方程 令 $x=\cos\theta$ $\Rightarrow$ $[(1-x^2)y']' + \left(\lambda - \dfrac{m^2}{1-x^2}\right) y = 0$

$m=0$ 时 $[(1-x^2)y']' + \lambda y = 0$ $\quad |y(\pm 1)| < +\infty$

$\Rightarrow$ $y(x) = C P_n(x) + D Q_n(x)$

第一类 第二类

$|H(0)|, |H(\pi)| < \infty$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$\frac{1}{\sin\theta} [\sin\theta H']' + \left(\lambda - \frac{\mu}{\sin^2\theta}\right) H = 0$$

$$(1-2xt+t^2)^{-\frac{1}{2}} = \begin{cases} \sum_{n=0}^{+\infty} P_n(x) t^n, & |t|<1 \\ \frac{1}{t}\sum_{n=0}^{+\infty} P_n(x) \left(\frac{1}{t}\right)^n, & |t|>1 \end{cases}$$

$P_n(-x) = (-1)^n P_n(x)$

$P_n(1) = 1$ $(-1,1)$ 内有且仅有 n 个单零点

递推 $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1) P_n(x)$

$$P_n(0) = \begin{cases} 0, & n=2m+1 \geq 1 \\ \frac{(-1)^m (2m-1)!!}{(2m)!!}, & n=2m \geq 2 \\ 1, & n=0 \end{cases} \qquad \frac{(-1)^{\frac{n}{2}}(n-1)!!}{n!!}$$

$$\int_0^1 x^m P_n(x)\,dx = \frac{m}{m+n+1}\int_0^1 x^{m-1} P_{n-1}(x)\,dx$$

固有值问题 $\begin{cases} [(1-x^2)y']' + \lambda y = 0 \\ |y(\pm 1)| < +\infty \end{cases}$ $\quad \lambda_n = n(n+1)$, $P_n(x)$

a, b 端 $k(x)$ 至多一级零点，$q(x)$ 至多一级极点 $\Rightarrow$ 正则奇点

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

轴对称、$\Delta_3 u = 0$ $\Rightarrow$ $u(r,\theta) = \sum_{n=0}^{+\infty} [C_n r^n + D_n r^{-(n+1)}] P_n(\cos\theta)$

拌随 Legendre $\begin{cases} [(1-x^2)y']' + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \\ |y(\pm 1)| < +\infty \end{cases}$

$\Rightarrow \begin{cases} \lambda_n = n(n+1) \\ P_n^m(x) \end{cases}$ $\quad P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x)$

$$\|P_n^m(x)\|^2 = \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}$$

权函数 $\rho(x) = 1$

一般情况：

$$\Rightarrow u(r,\theta,\varphi)=\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\left(A_n r^n+B_n\frac{1}{r^{n+1}}\right)P_n^m(\cos\theta)(C_{nm}\cos m\varphi+D_{nm}\sin m\varphi)$$

球谐函数 $Y_{nm}(\theta,\varphi)=\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix}P_n^m(\cos\theta)$

权函数 $P(\theta)=\sin\theta$

$$\begin{cases}\|Y_{nm}\|^2=\dfrac{2\pi}{2n+1}\cdot\dfrac{(n+m)!}{(n-m)!}, & m\geq 1\\ \|Y_{n0}\|^2=\dfrac{4\pi}{2n+1}\end{cases}$$

$$\begin{cases}f(\theta,\varphi)=\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\left[C_{nm}Y_{nm}^{(1)}(\theta,\varphi)+D_{nm}Y_{nm}^{(2)}(\theta,\varphi)\right]\\ C_{nm}=\dfrac{1}{N_{nm}^2}\int_0^{2\pi}\int_0^{\pi}f(\theta,\varphi)\cos m\varphi\, P_n^m(\cos\theta)\sin\theta\, d\theta d\varphi\\ D_{nm}=\dfrac{1}{N_{nm}^2}\int_0^{2\pi}\int_0^{\pi}f(\theta,\varphi)\sin m\varphi\, P_n^m(\cos\theta)\sin\theta\, d\theta d\varphi\end{cases}$$

$$\Rightarrow u(r,\theta,\varphi)=\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\left(A_n r^n+B_n\frac{1}{r^{n+1}}\right)P_n^m(\cos\theta)(C_{nm}\cos m\varphi+D_{nm}\sin m\varphi)$$

Laplace 方程第 II 边值问题解不一定存在

Bessel 函数

Helmholtz 方程 柱坐标下

$$\begin{cases}(rR')'+\left(\lambda r-\dfrac{\nu^2}{r}\right)R=0, & 0<r<a\\ |R(0)|<+\infty, & \alpha R(a)+\beta R'(a)=0\end{cases}$$

$x=\omega r$

$\lambda=\omega^2$

$$\Rightarrow x^2y''+xy'+(x^2-\nu^2)y=0$$

$$\Rightarrow R_\nu(r)=CJ_\nu(\omega r)+DN_\nu(\omega r)$$

权函数 $\rho(r)=r$

$$\begin{cases} f(r) = \sum_{n=0/1}^{+\infty} C_n J_\nu(\omega_n r) \\ C_n = \frac{1}{N_{\nu n}^2} \int_0^a r f(r) J_\nu(\omega_n r)\,dr \end{cases}$$

a端为第二类边界条件 $n=0$

$J_\nu(\omega_0 r) = 1$

$$N_{\nu n}^2 = \int_0^a r J_\nu^2(\omega_n r)\,dr = \begin{cases} \frac{a^2}{2} J_{\nu+1}^2(\omega_n a) & \text{a端 I类边界} \\ \frac{1}{2}\left[a^2 - \frac{\nu^2}{\omega_n^2}\right] J_\nu^2(\omega_n a) & \text{II类边界} \\ \frac{1}{2}\left[a^2 - \frac{\nu^2}{\omega_n^2} + \frac{a^2\alpha^2}{\beta^2\omega_n^2}\right] J_\nu^2(\omega_n a) & \text{III类边界} \end{cases}$$

递推

$$\begin{cases} (x^\nu J_\nu)' = x^\nu J_{\nu-1} \\ (x^{-\nu} J_\nu)' = -x^{-\nu} J_{\nu+1} \\ 2\nu J_\nu = x(J_{\nu-1} + J_{\nu+1}) \end{cases}$$

特别 $J_0' = -J_1$

母函数

$$e^{\frac{x}{2}(\zeta - \zeta^{-1})} = \sum_{n=-\infty}^{+\infty} J_n(x) \zeta^n$$

$$= \sum_{l=0}^{+\infty} \frac{1}{l!} \left(\frac{x}{2}\zeta\right)^l \cdot \sum_{k=0}^{+\infty} \frac{1}{k!} \left(-\frac{x}{2}\zeta^{-1}\right)^k$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}\right] \zeta^n$$

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x\sin\theta - n\theta)\,d\theta$$

零点

$$\lim_{x\to 0^+} J_\nu(x) = \begin{cases} 1, & \nu = 0 \\ 0, & \nu > 0 \end{cases}$$

$$\lim_{x\to 0^+} N_\nu(x) = -\infty, \quad \nu \geq 0$$

$$\lim_{x\to +\infty} J_\nu(x) = \lim_{x\to +\infty} N_\nu(x) = 0$$

$$J_\nu(-x) = (-1)^\nu J_\nu(x)$$

$$J_{-\nu}(x) = (-1)^\nu J_\nu(x)$$

Fourier 变换

$$\begin{cases}\bar{f}(\lambda)=\int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}dx \\ f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\bar{f}(\lambda)e^{i\lambda x}d\lambda\end{cases}$$

$$F[f'(x)] = i\lambda F[f(x)]$$

$$F[f(x)e^{i\lambda_0 x}] = \bar{f}(\lambda-\lambda_0)$$

$$F[f(x-x_0)] = \bar{f}(\lambda)e^{-ix_0\lambda}$$

$$F[f(\alpha x)] = \frac{1}{|\alpha|}\bar{f}\left(\frac{\lambda}{\alpha}\right)$$

$$F\left[\int_{-\infty}^{x} f(\xi)d\xi\right] = \frac{1}{i\lambda}\bar{f}(\lambda)$$

$$F[f(x)*g(x)] = \bar{f}(\lambda)\bar{g}(\lambda)$$

$$F^{-1}[e^{-a\lambda^2}] = \frac{1}{\sqrt{\pi}\sqrt{4a}}e^{-\frac{x^2}{4a}}$$

Fourier 正弦, 余弦变换

$$\begin{cases}\bar{f}_s = F_s[f(x)] = \int_0^{+\infty} f(x)\sin\lambda x\,dx & f(x)\text{ 奇延拓} \\ \bar{f}_c = F_c[f(x)] = \int_0^{+\infty} f(x)\cos\lambda x\,dx & f(x)\text{ 偶延拓}\end{cases}$$

$$\begin{cases}F_s^{-1}[\bar{f}_s(\lambda)] = \frac{2}{\pi}\int_0^{+\infty}\bar{f}_s(\lambda)\sin\lambda x\,d\lambda \\ F_c^{-1}[\bar{f}_c(\lambda)] = \frac{2}{\pi}\int_0^{+\infty}\bar{f}_c(\lambda)\cos\lambda x\,d\lambda\end{cases}$$

$u|_{x=0} = u_0 \Rightarrow$ 正弦变换

$u_t|_{x=0} = u_0 \Rightarrow$ 余弦变换

高维 Fourier

$$\begin{cases}\bar{f}(\lambda,\mu,\nu) = \iiint_{-\infty}^{+\infty} f(x,y,z)e^{-i(\lambda x+\mu y+\nu z)}dx\,dy\,dz \\ f(x,y,z)=\frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty}\bar{f}(\lambda,\mu,\nu)e^{i(\lambda x+\mu y+\nu z)}d\lambda\,d\mu\,d\nu\end{cases}$$

Laplace 变换 $\left(f(t) = f(t) H(t) \right)$

$$\begin{cases} \bar{f}(p) = \int_0^{+\infty} f(t) e^{-pt} dt \\ f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \bar{f}(p) e^{pt} dp \end{cases} \quad \left(\begin{matrix} \mathrm{Re}\,p \to \infty \text{时} \\ \bar{f}(p) \to 0 \end{matrix} \right)$$

$$L[f^{(n)}(t)] = p^n \bar{f}(p) - p^{n-1} f(+0) - \cdots\cdots - f^{(n-1)}(+0)$$

$$L[f(t-\tau)] = L[f(t-\tau)H(t-\tau)] = e^{-p\tau} \bar{f}(p)$$

$$L[f(t) e^{p_0 t}] = \bar{f}(p-p_0)$$

$$L[(-t)^n f(t)] = \bar{f}^{(n)}(p)$$

$$L[\int_0^t f(\tau) d\tau] = \frac{\bar{f}(p)}{p}$$

（关于 t 二阶 且关于 t 的 初始条件为齐次时用）

$$\int_{-\infty}^{+\infty} \delta(x) dx = \int_0^{+\infty} \delta(x) dx = 1$$

$$\delta(x) * f(x) = f(x)$$

$$\delta^{(n)}(x) * f(x) = f^{(n)}(x) \qquad L(f(x) * g(x)) = (Lf) * g = f * Lg$$

$$\begin{cases} F[\delta(x)] = 1 \\ F[1] = 2\pi \delta(\lambda) \\ F[e^{iax}] = 2\pi \delta(\lambda - a) \\ F[x] = 2\pi i \delta'(\lambda) \end{cases} \quad \Rightarrow F[\cos, \sin]$$

$$\delta(x-\xi) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x-\xi) \cos nx \, dx = \frac{1}{\pi} \cos n\xi, \quad b_n = \frac{1}{\pi} \sin n\xi$$

$C_0^\infty(R)$ 无穷次连续可导而且在一个有界的闭区间外三0

高维 δ

$$\delta(x,y,z)=\delta(x)\delta(y)\delta(z)$$

$$F^{-1}[f]=\frac{1}{(2\pi)^3}\iiint_{R^3} f(\lambda,\mu,\nu)\,e^{i(\lambda x+\mu y+\nu z)}\,d\lambda\, d\mu\, d\nu$$

Green 公式

$$\oiint_S u\frac{\partial v}{\partial n}dS=\iiint_V u\Delta v\, dV+\iiint_V \nabla u\cdot\nabla v\, dV$$

$$\oiint_S \left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS=\iiint_V (u\Delta v-v\Delta u)\,dV$$

Helmholtz 算子

$$\Rightarrow \oiint_S \left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS=\iiint_V \left(u(\Delta\pm k^2)v-v(\Delta\pm k^2)u\right)dV$$

基本解

$\Delta_2 u+k^2u=0$ $\qquad U(x,y)=\frac{1}{2\pi}\ln(kr)$

$\Delta_2 u=0$ $\qquad V(x,y)=-\frac{1}{2\pi}\ln\frac{1}{r}=\frac{1}{2\pi}\ln r$

$\Delta_3 u+k^2u=0$ $\qquad U(x,y,z)=\frac{-e^{ikr}}{4\pi r}$

$\Delta_3 u=0$ $\qquad U(x,y,z)=-\frac{1}{4\pi r}$

$Lu=f(M)$ 型

$$\begin{cases} LU(M-M_0)=\delta(M-M_0)\\ u=U(M)*f(M)\end{cases}$$

场位方程边值问题

$$\begin{cases}\Delta u=-f(M)\\ u|_{\partial V}=\varphi(M)\end{cases}\Rightarrow \begin{cases}\Delta u_1=-f(M)\\ u_1|_{\partial V}=0\end{cases}\quad \begin{cases}\Delta u_2=0\\ u_2|_{\partial V}=\varphi(M)\end{cases}$$

$$\Rightarrow u(M)=\int_V f(M_0)G(M;M_0)\,dM_0-\oint_{\partial V}\varphi(M_0)\frac{\partial G}{\partial n_0}dS_0$$

Green 函数

$$\begin{cases} \Delta G(M;M_0) = -\delta(M-M_0) \\ G|_{M\in\partial V} = 0 \end{cases}$$

倒易性 $G(M_2;M_1) = G(M_1;M_2)$

① 上半空间

$$G(M;M_0) = \frac{1}{4\pi r(M,M_0)} - \frac{1}{4\pi r(M,M_1)}$$

② 球域内:

$M_0 = (\xi,\eta,\zeta)$ $\quad M_1 = \frac{R^2}{\rho^2}(\xi,\eta,\zeta)$

$+\varepsilon$ $\quad -\frac{R}{\rho}\varepsilon$

$$G(M;M_0) = \frac{1}{4\pi}\left[\frac{1}{r(M,M_0)} - \frac{R}{\rho\, r(M,M_1)}\right]$$

② 圆域内

$M_0 = (\xi,\eta)$ $\quad M_1 = \frac{R^2}{\rho^2}(\xi,\eta)$

$$G(M;M_0) = \frac{1}{2\pi}\left[\ln\frac{1}{r(M,M_0)} - \ln\frac{1}{r(M,M_1)} - \ln\frac{R}{\rho}\right]$$

③ 保形变换 $\quad$ 未知域 $G(z,z_0)$

$f(z) \longrightarrow$ 已知域 $G'(z,z_0)$

则 $G(z,z_0) = G'(f(z), f(z_0))$

④ Fourier 方法

$u_t = Lu$ 型.

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + f(t,M), t>0 \\ u|_{t=0} = \varphi(M) \end{cases} \Rightarrow \begin{cases} \frac{\partial U}{\partial t} = LU, t>0 \\ U|_{t=0} = \delta(M) \end{cases}$$

$$\Rightarrow u(t,M) = U(t,M) * \varphi(M) + \int_0^t U(t-\tau, M) * f(\tau, M) d\tau$$

proof: $\begin{cases} \frac{\partial u_1}{\partial t} = Lu_1 \\ u_1|_{t=0} = \varphi(M) \end{cases}$ $\begin{cases} \frac{\partial u_2}{\partial t} = Lu_2 + f(t,M) \\ u_2|_{t=0} = 0 \end{cases}$

$\Downarrow$ 冲量原理

$$\begin{cases} \frac{\partial w}{\partial t} = Lw \\ w|_{t=\tau} = f(\tau, M) \\ u_2 = \int_0^t w(t,x;\tau) d\tau \end{cases}$$

$u_{tt} = Lu$ 型

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = Lu + f(t,M) \\ u|_{t=0} = \varphi(M), \ \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(M) \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 U}{\partial t^2} = LU \\ U|_{t=0} = 0, \ \frac{\partial U}{\partial t}\Big|_{t=0} = \delta(M) \end{cases}$$

$$\Rightarrow u(t,M) = U(t,M) * \psi(M) + \frac{\partial}{\partial t}[U(t,M) * \varphi(M)] + \int_0^t U(t-\tau, M) * f(\tau, M) d\tau$$

方程总结：

弦振动 $$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(t,x)$$ $a^2 = \frac{T}{\rho}$

$f(t,x) = \frac{g(t,x)}{\rho}$

端点受力 $$\left.\frac{\partial u}{\partial n}\right|_{x=x_1} = -\left.\frac{F_1(t) - ku}{T_1}\right|_{x=x_1}$$

热传导 $$\frac{\partial u}{\partial t} = a^2 \Delta u + f(t,x,y,z)$$ $a^2 = \frac{k}{c\rho}$ 热传导系数

$f(t,x,y,z) = \frac{g(t,x,y,z)}{c\rho}$

边界条件

$$\left. k\frac{\partial u}{\partial n}\right|_{\partial V} = \left. \underbrace{h}_{\text{热交换系数}}(\theta - u)\right|_{\partial V}$$