绝对误差 e $\qquad e = x^* - x$ 准确值 近似值

向量范数 性质 ① 非负性 $\|x\| \geq 0$ 且 $x=0 \Leftrightarrow \|x\|=0$

② 齐次性 $\|\alpha x\| = |\alpha|\,\|x\|$

③ 三角不等式 $\|x+y\| \leq \|x\| + \|y\|$

L_p 范数 $\|x\|_p = \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}}, \quad 1 \leq p \leq +\infty$

范数等价 $m \leq \dfrac{R_1(x)}{R_2(x)} \leq M$

矩阵范数 $\|A\| = \sup\limits_{\substack{x \in R^n \\ x \neq 0}} \dfrac{\|Ax\|}{\|x\|}$

性质: ① 非负性 $\|A\| \geq 0$ 且 $\|A\| = 0 \Leftrightarrow A = 0$

② 齐次性 $\|\lambda A\| = |\lambda|\,\|A\|$

③ 三角不等式 $\|A+B\| \leq \|A\| + \|B\|$

④ (A,B同阶) $\|AB\| \leq \|A\|\,\|B\|$

⑤ 相容性 $\|Ax\| \leq \|A\| \cdot \|x\|$ A为n阶阵

常用范数 $\|A\|_1$ 列和最大值

$\|A\|_\infty$ 行和最大值

$\|A\|_2 = \sqrt{\rho(A^TA)}$

谱半径 $\rho(A) = \max\limits_i \{|\lambda_i|\}$ 定理: $\rho(A) \leq \|A\|$

Euclid范数 $\|A\|_E = \left(\sum_i \sum_j |a_{ij}|^2\right)^{\frac{1}{2}}$

不是范数

$\lim\limits_{k\to\infty} A^k = 0 \Leftrightarrow \rho(A) < 1$

条件数 $\mathrm{cond}(A) = \|A\|\,\|A^{-1}\|$

$\mathrm{cond}(A)$ 较大 — 病态矩阵, 小扰动有大失真

误差的事后估计法.

插值

$$f(x)=L_n(x)\approx \frac{x-x_0}{x_0-x_{n+1}}\left(\overset{x_0,\cdots,x_n}{L_n(x)}-\overset{x_1,\cdots,x_{n+1}}{\tilde{L}_n(x)}\right)$$

Lagrange

$$\begin{cases} L_n(x)=\sum_i^n l_i(x)f(x_i) \\ l_i(x_j)=\delta_{ij}=\prod_{\substack{0\le j\le n\\ j\ne i}}\frac{x-x_j}{x_i-x_j}\end{cases}$$

误差 $R_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$ ，$\xi\in[a,b]$

定理 $\sum_i l_i(x)\equiv 1$

差商

$$f[x_0,x_1,\cdots,x_k]=\frac{f[x_1,x_2,\cdots,x_k]-f[x_0,x_1,\cdots x_{k-1}]}{x_k-x_0}$$

性质 ① $f[x_0,\cdots,x_k]=\sum_{i=0}^{k}\dfrac{f(x_i)}{\prod_{j\ne i}(x_i-x_j)}$

② 与节点顺序无关

③ $f(x)$ — m次 $\Rightarrow$ $f[x_0,\cdots,x_{k-1},x]$ — $(m-k)$次

推广 $f[\overbrace{x_0,\cdots x_0}^{n+1个}]=\dfrac{f^{(n)}(x_0)}{n!}$

Newton

$$N_n(x)=f(x_0)+(x-x_0)f[x_0,x_1]+\cdots+(x-x_0)\cdots(x-x_{n-1})f[x_0,\cdots,x_n]$$

→ 差商表对角元

$$\Rightarrow R_n(x)=f[x,x_0,\cdots,x_n]\prod_{i=0}^{n}(x-x_i)$$

$$\| \qquad\qquad\qquad\qquad$$

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)\qquad\Rightarrow f[\underbrace{x,x_0,\cdots x_n}_{n+2个值}]=\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Hermite

$$\begin{cases} H_{2n+1}(x)=\sum_{i=0}^{n}h_i(x)f(x_i)+\sum_{i=0}^{n}g_i(x)f'(x_i) \\ h_i(x)=\left(1-2(x-x_i)\sum_{j\ne i}\frac{1}{x_i-x_j}\right)l_i^2(x) \\ g_i(x)=(x-x_i)l_i^2(x)\end{cases}\Bigg]\rightarrow \begin{matrix} h_i(x_j)=\delta_{ij} \\ h_i'(x_j)=0 \\ g_i(x_j)=0 \\ g_i'(x_j)=\delta_{ij}\end{matrix}$$

或利用推广差商进行 Newton 形式（此时节点有顺序要求）

（可设 $h_i(x)=l_i^2(x)(a_ix+b_i)$ $g_i(x)=c_i(x-x_i)l_i^2(x)$）

$$R(x)=\frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x-x_0)^2\cdots(x-x_n)^2$$

三样条

$$S(x)=S_i(x)=a_ix^3+b_ix^2+c_ix+d \qquad x\in[x_i,x_{i+1}]$$

① $S(x_i)=y_i$，$i=0,1,\cdots n$

$$\begin{cases} S(x_i+0)=S(x_i-0) \\ S'(x_i+0)=S'(x_i-0) \\ S''(x_i+0)=S''(x_i-0) \end{cases} ,i=1,\cdots,n-1$$

M关系式 —— 二阶导数表示

最小二乘法

m关系式 —— 1阶导数表示

线性拟合

$$\begin{cases} p(x)=a+bx \\ Q(a,b)=\sum_{i=1}^{m}(p(x_i)-y_i)^2 \\ \frac{\partial Q}{\partial a}=0 \\ \frac{\partial Q}{\partial b}=0 \end{cases} \Rightarrow \begin{pmatrix} m & \sum_{i=1}^{m}x_i \\ \sum_{i=1}^{m}x_i & \sum_{i=1}^{m}x_i^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{m}y_i \\ \sum_{i=1}^{m}x_iy_i \end{pmatrix}$$

n次拟合

$$\begin{cases} p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n \\ Q(a_0,a_1,\cdots,a_n)=\sum_{i=1}^{m}(p(x_i)-y_i)^2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} m & \sum x_i & \cdots & \sum x_i^n \\ \sum x_i & \sum x_i^2 & \cdots & \sum x_i^{n+1} \\ \vdots & \vdots & & \vdots \\ \sum x_i^n & \sum x_i^{n+1} & \cdots & \sum x_i^{2n} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}=\begin{pmatrix} \sum y_i \\ \sum x_iy_i \\ \vdots \\ \sum x_i^n y_i \end{pmatrix}$$

一般函数拟合 $p(x) = a_0\varphi_0(x) + \cdots + a_n\varphi_n(x)$

$$\begin{pmatrix} \sum \varphi_1^2(x_i) & \sum \varphi_2(x_i)\varphi_1(x_i) & \cdots & \sum \varphi_n(x_i)\varphi_1(x_i) \\ \vdots & \vdots & & \vdots \\ \sum \varphi_1(x_i)\varphi_n(x_i) & \sum \varphi_2(x_i)\varphi_n(x_i) & \cdots & \sum \varphi_n^2(x_i) \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum y_i \varphi_1(x_i) \\ \vdots \\ \sum y_i \varphi_n(x_i) \end{pmatrix}$$

ae^{bx} 拟合 $z = \ln y = \ln a + bx$ $\rightarrow$ 线性拟合

$A + Bx$

矛盾方程组 $Ax = b$

$\Rightarrow A^TAx = A^Tb$ 法方程

数值积分

插值型 $I(f) \doteq \int_a^b L_n(x)dx = \sum_{i=0}^n \left[\int_a^b l_i(x)dx\right] f(x_i)$

$= \sum_i \alpha_i f(x_i)$

误差 $E_n(f) = \int_a^b R_n(x)dx$

n阶插值多项式形式至少有n阶代数精度

$I(x^n) \overset{?}{=} I_n(x^n)$

Newton-Cotes $I(f) \approx \int_a^b L_n(x)dx$ $[a,b]$ n等分

梯形 $I(f) \approx \int_a^b L_1(x)dx = \frac{b-a}{2}[f(a)+f(b)]$

$= T(f)$

一阶代数精度

$E_1(x) = -\frac{f''(\eta)}{12}(b-a)^3$

Simpson $[a,b]$ 2等分.

$$I(f) \approx S(f) = \int_a^b L_2(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

三阶代数精度.

一般 (1) n为奇数 —— n阶代数精度 $E_n(f) = \frac{f^{(n+1)}(\eta)}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x-x_i)\,dx$

(2) n为偶数 —— $(n+1)$阶代数精度 $E_n(f) = \frac{f^{(n+2)}(\eta)}{(n+2)!}\int_a^b x\prod_{i=0}^{n}(x-x_i)\,dx$

复化积分 $I(f) = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,dx$

复化梯形 $T_n(f) = h\left[\frac{1}{2}f(a) + \sum_{i=1}^{n-1} f(a+ih) + \frac{f(b)}{2}\right]$

$$E_n(f) = -\frac{(b-a)^3}{12n^2}f''(\xi)$$

$\left(\exists \xi \in [a,b] \text{ s.t. } \sum_{i=0}^{n-1} f''(\xi_i) = nf''(\xi)\right)$

复化Simpson $S_n(f) = \frac{h}{3}\left[f(a) + 4\sum_{i=0}^{m-1} f(x_{2i+1}) + 2\sum_{i=0}^{m-1} f(x_{2i}) + f(b)\right]$

自动控制误差 $|I(f) - T_{2n}(f)| < \varepsilon \implies |T_{2n}(f) - T_n(f)| < 3\varepsilon$

$$\begin{cases} T_{2n}(f) = \frac{1}{2}\left[T_n(f) + H_n(f)\right] \\ H_n(f) = h_n \sum_{i=0}^{n-1} f(x_{i+\frac{1}{2}}) \\ h_n = \frac{b-a}{n} \end{cases}$$

Romberg

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$$

k: 分点数 $n2^{k-1}$

j: 积分类型.

$|R_{k,k} - R_{k-1,k-1}| < \varepsilon$ 退出循环.

Gauss

Thm $I_n(f)=\sum_{i=1}^{n}\alpha_i f(x_i)$ 代数精度不超过 $2n-1$ 阶

$G_n(f)=\sum_{i=1}^{n}\alpha_i^{(n)} f(x_i^{(n)})$ $x_i^{(n)}$ 为 Legendre 多项式的零点.

$I(f)=\int_{-1}^{1} f(x)\,dx$ 插值 $2n-1$ 阶 代数精度.

一般区间作变量代换 (α_i 查表)

$$\int_a^b f(t)\,dt=\int_{-1}^{1} f\left(\frac{a+b}{2}+\frac{b-a}{2}x\right)\frac{b-a}{2}dx$$

数值微分

精度 $O(h^p)$ p阶精度

1. 插值函数微分

2. 差商 展开得精度

一阶 $D_+[f,h](x_0)=f[x_0,x_0+h]=\dfrac{f(x_0+h)-f(x_0)}{h}$

一阶 $D_-[f,h](x_0)=f[x_0,x_0-h]$

二阶 $D_0[f,h](x_0)=f[x_0-h,x_0+h]$

常微分

$y'=f(x,y)$

Euler法 —— 一阶方法

① 向前

$y'(x_n)\approx\dfrac{y(x_{n+1})-y(x_n)}{h}\Rightarrow y_{n+1}=y_n+hf(x_n,y_n)$ 更简单

② 向后 $y_{n+1}=y_n+hf(x_{n+1},y_{n+1})$ 更稳定、精确.

③ 预估-校正法

$$\begin{cases}\overline{y_{n+1}}=y_n+hf(x_n,y_n) & \text{先显式格式}\\ y_{n+1}=y_n+hf(x_{n+1},\overline{y_{n+1}}) & \text{后隐式格式}\end{cases}$$

精度. $T_{n+1} = O(h^{p+1}) \Rightarrow$ p阶精度

截断误差

基于数值积分. $y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} y'(t)dt.$

① 矩形近似 $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) \to$ Euler后差

$= y_n + hf(x_n, y_n) \to$ Euler前差

② 梯形近似 $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$

预估. 校正法.

线性多步法. $y(x_{n+1}) = y(x_{n-p}) + \int_{x_{n-p}}^{x_{n+1}} y'(x)dx$

① $y'(x) \to$ 插值点不包含 x_{n+1} 显式 $\underbrace{x_n, x_{n-1}, \cdots x_{n-q}}_{(q+1)\text{个点}}$ 阶

② 包含 x_{n+1} 隐式 $x_{n+1}, \cdots, x_{n+1-q}$

Adams — $p=0$

起始点用Euler法. Runge-kutta法算出.

Taylor 类型方法.

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \cdots + \frac{h^m}{m!}y^{(m)}(x_n) + O(h^{m+1})$$

e.g. $m=2$ 时

$$y_{n+1} = y_n + \underline{hf(x_n, y_n) + \frac{h^2}{2}[f_x(x_n, y_n) + f_y(x_n, y_n)f(x_n, y_n)]}$$

Runge-Kutta 法.

用 $c_1 f(x_n, y(x_n)) + c_2 f(x_n + ah, y(x_n) + bhf(x_n, y(x_n))$ 逼近

$$\Rightarrow \begin{cases} y_{n+1} = y_n + h(c_1 k_1 + c_2 k_2) \\ k_1 = f(x_n, y_n) \\ k_2 = f(x_n + ah, y_n + bhk_1) \\ c_1 + c_2 = 1 \\ a = b \end{cases}$$

二阶

$\Rightarrow$ 取 $c_1 = c_2 = \frac{1}{2}$ $a = b = 1$

$$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) \\ k_1 = f(x_n, y_n) \\ k_2 = f(x_n + h, y_n + hk_1) \end{cases}$$

三阶

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3) \\ k_1 = f(x_n, y_n) \\ k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\ k_3 = f(x_n + h, y_n - hk_1 + 2hk_2) \end{cases}$$

四阶

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 = f(x_n, y_n) \\ k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\ k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2) \\ k_4 = f(x_n + h, y_n + hk_3) \end{cases}$$

高阶 $\Rightarrow$ 化为方程组求解.

Def 收敛阶 $\lim_{k\to\infty}\frac{|e_{k+1}|}{|e_k|^n}=M$ n阶收敛

非线性方程 ($x_{k+1}=\varphi(x_k)$ 迭代)

1. 对分法 区间不断减半 (零点存在定理)

$\Rightarrow$ 只能解出区间中的一根

2. 不动点迭代 $\varphi(\hat{x})=\hat{x}$

Thm $\begin{cases} x\in[a,b]\Rightarrow\varphi(x)\in[a,b] \\ |\varphi'(x)|\le L<1 \end{cases}\Rightarrow$ $[a,b]$上有唯一不动点

(令 $\psi=x-\varphi$)

唯一: $|x_1-x_2|<|x_1-x_2|$ 矛盾

误差: $|x_{k+1}-x_k|=|\varphi(x_k)-\varphi(x_{k-1})|\le L|x_k-x_{k-1}|\cdots\cdots\le L^k|x_1-x_0|$

$$|x_{k+p}-x_k|\le|x_{k+p}-x_{k+p-1}|+\cdots+|x_{k+1}-x_k|$$
$$=\frac{L^k(1-L^p)}{1-L}|x_1-x_0|$$
$$\Rightarrow|x^*-x_k|\le\frac{L^k}{1-L}|x_1-x_0|$$

3. Newton 迭代 $x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots$$

$$f(x)=0\Rightarrow f(x_0)+f'(x_0)(x-x_0)\approx0$$

$$\Rightarrow x=x_0-\frac{f(x_0)}{f'(x_0)}\Rightarrow\varphi(x)=x-\frac{f(x)}{f'(x)}$$

① x^*为单根 ($f(x^*)=0$, $f'(x^*)\ne0$)

$\Rightarrow\varphi'(x^*)=0$ $\varphi(x^*)=x^*$

$$e_{k+1}=x_{k+1}-x^*=\varphi(x_k)-\varphi(x^*)\approx\varphi(x^*)+\varphi'(x^*)(x_k-x^*)+\frac{1}{2}\varphi''(x^*)(x_k-x^*)^2-\varphi(x^*)$$

$$=\frac{1}{2}\varphi''(x^*)e_k^2$$ 二阶方法.

② x^*为 p次重根　一阶方法..

修正为 $x_{k+1} = x_k - p\dfrac{f(x_k)}{f'(x_k)}$ 仍为二阶

初值需在零点附近

4. 弦截法

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$ 收敛阶 1.618

5. Newton 解非线性方程组

取线性部分有
$$\begin{cases} J(x_k, y_k)\begin{pmatrix}\Delta x_k \\ \Delta y_k\end{pmatrix} = \begin{pmatrix} -f_1(x_k, y_k) \\ -f_2(x_k, y_k)\end{pmatrix} \\ J(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix} \\ x_{k+1} = x_k + \Delta x_k \quad y_{k+1} = y_k + \Delta y_k \end{cases}$$

直到 $\max(|\Delta x|, |\Delta y|) < \varepsilon$ 为止

线性方程组直接解法

1. 消元法

对角阵 $\sim O(n)$

三角阵 $\sim O(n^2)$　计算量.

Gauss消元 $\sim O(n^3)$

Gauss-Jordan 化为对角阵 $\sim O(n^3)$

Gauss 列主元. 使第 k 步时 $a_{kk} = \max\{a_{kk}, \cdots a_{nk}\}$.

行尺度主元法. $\begin{cases} S_i = i\text{行中最大} \\ \text{使 } \max \dfrac{a_{ik}}{S_i} \text{ 的行为主行} \end{cases}$

Thm. 对角占优阵可直接 Gauss 消元.

2. 分解法. $A = \underset{\text{下三角}}{L}\ \underset{\text{上三角}}{U}$

Doolittle 单位下三角

Courant 单位上三角

LDL^T — 对称正定 　　L 单位下三角, D 对角

Cholesky — 对称正定 　　$A = (L\sqrt{D})(\sqrt{D}L^T) = PP^T$

Thm ① A 的 n 个主子式 $\neq 0$ 　$\exists\ LU$ 分解

② 对角占优阵 　$\exists\ LU$ 分解

③ 实对称正定, $\exists$ 唯一 Cholesky 分解

3. 追赶法. $Ax = f$

$$A = \begin{pmatrix} a_1 & b_1 & & & \\ c_2 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-1} & a_{n-1} & b_{n-1} \\ & & & c_n & a_n \end{pmatrix} = \begin{pmatrix} u_1 & & & \\ c_2 & u_2 & & \\ & \ddots & \ddots & \\ & & c_n & u_n \end{pmatrix} \begin{pmatrix} 1 & v_1 & & \\ & 1 & \ddots & \\ & & \ddots & v_{n-1} \\ & & & 1 \end{pmatrix}$$

令 $c_1 = 0$
$v_n = 0$

$$\begin{cases} u_i = a_i - c_i v_{i-1} \\ v_i = \dfrac{b_i}{u_i} \\ y_i = \dfrac{(f_i - c_i y_{i-1})}{u_i} \end{cases}$$

$$LUx = f \Rightarrow \begin{cases} Ux = y \\ Ly = f \end{cases}$$

$$\Rightarrow x_k = y_k - v_k x_{k+1}$$

线性方程组迭代法

$$x^{(k+1)} = Mx^{(k)} + g$$

$$Nx = Nx - Ax + b \Rightarrow x = \underbrace{N^{-1}(N-A)}_{M}x + \underbrace{N^{-1}b}_{g}$$

Thm. $\rho(M) < 1 \Leftrightarrow$ 迭代收敛

$\|M\| < 1 \Rightarrow$ 迭代收敛

1 Jacobi方法

$$\underbrace{N = D}$$

$$\Rightarrow x^{(k+1)} = \underbrace{D^{-1}(D-A)}x + \underbrace{D^{-1}b}$$

$$M = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \cdots & -\frac{a_{2n}}{a_{22}} \\ \vdots & & \ddots & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{pmatrix}$$

$$g = \begin{pmatrix} \frac{y_1}{a_{11}} \\ \vdots \\ \frac{y_n}{a_{nn}} \end{pmatrix}$$

$$Ax = y$$

Thm A 为严格行/列对角占优阵，则 Jacobi 迭代收敛

2. Gauss-Seidel 迭代 $N = D + L$

$$\begin{cases} x_1^{(k+1)} = b_{12}x_2^{(k)} + \cdots + b_{1n}x_n^{(k)} + g_1 \\ x_2^{(k+1)} = b_{21}x_1^{(k+1)} + b_{23}x_3^{(k)} + \cdots + b_{2n}x_n^{(k)} + g_2 \\ \vdots \end{cases}$$

$$x^{(k+1)} = Sx^{(k)} + f$$

Thm ① S 为严格行/列对角占优 $\Rightarrow$ 收敛

② A 为对称正定阵 $\Rightarrow$ 收敛

松弛迭代 $N = (I + \omega D^{-1}L)$

$$\begin{cases} x_1^{(k+1)} = [b_{12}x_2^{(k)} + \cdots + b_{1n}x_n^{(k)} + g_1]\omega + (1-\omega)x_1^{(k)} \\ x_2^{(k+1)} = [b_{21}x_1^{(k+1)} + b_{23}x_3^{(k)} + \cdots + b_{2n}x_n^{(k)} + g_2]\omega + (1-\omega)x_2^{(k)} \\ \vdots \end{cases}$$

Thm 收敛 $\Rightarrow$ $0 < \omega < 2$

收敛 $\Longleftarrow$ $0 < \omega < 2$, A为对称正定阵

求逆矩阵

$$A\vec{x}_m = \vec{e}_m$$ 解 m个方程组

$$A^{-1} = \sum_m \vec{x}_m$$

特征值 & 特征向量

1. 幂法.

$$x^{(k+1)} = A x^{(k)}$$

① $|\lambda_1| > |\lambda_2| \cdots \geq |\lambda_n|$

$$\begin{cases} \lambda_1 \approx \dfrac{x_i^{(k+1)}}{x_i^{(k)}} \\ V \approx x^{(k)} \end{cases}$$

② $\begin{cases} |\lambda_1| = |\lambda_2| > |\lambda_3| \cdots \geq |\lambda_n| \\ \lambda_1 = -\lambda_2 \end{cases}$

$$\begin{cases} \lambda_1 = \sqrt{\dfrac{x_i^{(k+2)}}{x_i^{(k)}}} \\ V_1 = x^{(k+1)} + \lambda_1 x^{(k)} \\ V_2 = x^{(k+1)} - \lambda_1 x^{(k)} \end{cases}$$

2. 规范运算.

$$\begin{cases} y^{(k)} = \dfrac{x^{(k)}}{\|x^{(k)}\|_\infty} \\ x^{(k+1)} = A y^{(k)} \end{cases}$$

① $\{x^{(k)}\}$ 收敛 $\Rightarrow$ 按模最大只有一个 且 $\lambda_1 > 0$

$$\begin{cases} \lambda_1 \approx \max\limits_i |x_i^{(k+1)}| \\ v_1 \approx y^{(k)} \end{cases}$$

② $\{x^{(2k)}\}$ $\{x^{(2k+1)}\}$ 分别收敛于互为反号的向量.

$\Rightarrow$ 按模最大只有一个, $\lambda_1 < 0$

$$\begin{cases} \lambda_1 \approx -\max_i |x_i^{(k+1)}| \\ v_1 \approx y^{(k)} \end{cases}$$

③ $\{x^{(2k)}\}$ $\{x^{(2k+1)}\}$ 分别收敛于不同向量

$\Rightarrow$ 符号相反 2个模最大

$$\begin{cases} \lambda_1 \approx \sqrt{\dfrac{x_i^{(k+1)}}{y_i^{(k-1)}}} \\ v_1 = x^{(k+1)} + \lambda_1 x^{(k)} \\ v_2 = x^{(k+1)} - \lambda_1 x^{(k)} \end{cases}$$

3. 反幂法. 计算模最小特征值.

A^{-1} 幂法.

$$\begin{cases} \lambda_1' = \dfrac{1}{\lambda_n} \\ v_1' = v_n \end{cases}$$

4. 位移幂法.

A	$\lambda_1, \lambda_2, \cdots, \lambda_n$
$A-\mu I$	$\lambda_1-\mu, \lambda_2-\mu, \cdots, \lambda_n-\mu$

计算 $\max|\lambda_i-\mu|$

$\min|\lambda_i-\mu|$

5. 实对称阵 Jacobi 方法.

Givens 变换

$$B = Q^T(p,q,\theta) A Q(p,q,\theta)$$

$$Q = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \cos\theta & & \sin\theta \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} & -\sin\theta & & \cos\theta & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

$$\begin{cases} s = \dfrac{a_{qq} - a_{pp}}{2a_{pq}} \\ t = \begin{cases} t^2 + 2st - 1 = 0 & \text{接模较小根} \\ 1 & s = 0 \end{cases} \\ \cos\theta = \dfrac{1}{\sqrt{1+t^2}} = c \\ \sin\theta = \dfrac{t}{\sqrt{1+t^2}} = d \end{cases}$$

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