Fréchet derivatives under various parameterizations using chain rule

for LEGO module m_parameterization.f90

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Overview

$\boxed{\text{WaveEq lm} \Rightarrow \text{optim x,g}}$

- WaveEq uses lm and produces gkpa, glda, gmu and grho under moduli-density parameterization.
- m_parameterization.f90 converts them to user-specified parameterization, which can be
  - moduli-density (kpa, lda, mu, rho)
  - velocities-density (vp, vs, rho)
  - velocities-impedance (vp, vs, Ip)
  - slowness-density (sp, sps, rho)
- Also we may consider "passive parameters" for hard constraints (e.g. Gardner) between these parameters.

PARAMETERIZATION velocities-density

isotropic ACoustic

$$\begin{cases} \kappa = \rho V_P^2 \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\kappa 2\rho V_P \\
\nabla_\rho &= \nabla_\kappa \frac{\partial \kappa}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\kappa V_P^2 + \nabla_{\rho_0}
\end{aligned}$$

$$\begin{cases} V_P = \sqrt{\frac{\kappa}{\rho_0}} \\ \rho = \rho_0 \end{cases}$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{V_P} \frac{\partial V_P}{\partial \kappa} + \nabla_\rho \frac{\partial \rho}{\partial \kappa} \\
&= \nabla_{V_P} \frac{1}{2\rho_0} \sqrt{\frac{\rho_0}{\kappa}} \\
&= \nabla_{V_P} 0.5 \rho_0^{-0.5} \kappa^{-0.5} \\
\nabla_{\rho_0} &= \nabla_{V_P} \frac{\partial V_P}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0} \\
&= \nabla_{V_P} \left(-\frac{\kappa}{2\rho_0^2} \sqrt{\frac{\rho_0}{\kappa}} \right) + \nabla_\rho \\
&= \nabla_{V_P} \left(-0.5 \rho_0^{-1.5} \kappa^{0.5} \right) + \nabla_\rho
\end{aligned}$$

PARAMETERIZATION velocities-density + PASSIVE gardner

isotropic ACoustic

$$\boxed{\rho = c_1 V_P^{c_2}}$$

$$\begin{cases} \kappa = \rho V_P^2 = c_1 V_P^{c_2+2} \\ \rho_0 = \rho = c_1 V_P^{c_2} \end{cases}$$

$$V_P = \left(\frac{\kappa}{c_1}\right)^{\frac{1}{c_2+2}}$$

$$\begin{aligned} \nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\ &= \nabla_\kappa c_1 (c_2+2) V_P^{c_2+1} \\ &\quad + \nabla_{\rho_0} c_1 c_2 V_P^{c_2-1} \\ &= \left(\nabla_\kappa c_1 (c_2+2) V_P + \nabla_{\rho_0} \frac{c_1 c_2}{V_P} \right) V_P^{c_2} \end{aligned}$$

$$\begin{aligned} \nabla_\kappa &= \nabla_{V_P} \frac{\partial V_P}{\partial \kappa} \\ &= \nabla_{V_P} \frac{1}{c_1 (c_2+2)} \left(\frac{\kappa}{c_1}\right)^{\frac{c_2+1}{c_2+2}} \end{aligned}$$

PARAMETERIZATION velocities-density

isotropic P-SV

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) \\ \mu = \rho V_S^2 \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda 2\rho V_P \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda(-4\rho V_S) + \nabla_\mu 2\rho V_S \\
\nabla_\rho &= \nabla_\lambda \frac{\partial \lambda}{\partial \rho} + \nabla_\mu \frac{\partial \mu}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\lambda(V_P^2 - 2V_S^2) + \nabla_\mu V_S^2 + \nabla_{\rho_0}
\end{aligned}$$

$$\begin{cases} V_P = \sqrt{\frac{\lambda+2\mu}{\rho_0}} \\ V_S = \sqrt{\frac{\mu}{\rho_0}} \\ \rho = \rho_0 \end{cases}$$

$$\begin{aligned}
\nabla_\lambda &= \nabla_{V_P} \frac{\partial V_P}{\partial \lambda} + \nabla_{V_S} \frac{\partial V_S}{\partial \lambda} + \nabla_\rho \frac{\partial \rho}{\partial \lambda} \\
&= \dots \\
\nabla_\mu &= \nabla_{V_P} \frac{\partial V_P}{\partial \mu} + \nabla_{V_S} \frac{\partial V_S}{\partial \mu} + \nabla_\rho \frac{\partial \rho}{\partial \mu} \\
&= \dots \\
\nabla_{\rho_0} &= \nabla_{V_P} \frac{\partial V_P}{\partial \rho_0} + \nabla_{V_S} \frac{\partial V_S}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0} \\
&= \dots
\end{aligned}$$

PARAMETERIZATION velocities-density + PASSIVE gardner

isotropic P-SV

$$\boxed{\rho = c_1 V_P^{c_2}}$$

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = c_1 V_P^{c_2+2} - 2c_1 V_P^{c_2} V_S^2 \\ \mu = \rho V_S^2 = c_1 V_P^{c_2} V_S^2 \\ \rho_0 = \rho = c_1 V_P^{c_2} \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda c_1 (c_2 + 2) V_P^{c_2+1} + \nabla_\lambda (-2) c_1 c_2 V_P^{c_2-1} V_S^2 \\
&\quad + \nabla_\mu c_1 c_2 V_P^{c_2-1} V_S^2 + \nabla_{\rho_0} c_1 c_2 V_P^{c_2-1} \\
&= \nabla_\lambda \left(c_1 (c_2+2) V_P - 2c_1 c_2 V_P^{-1} V_S^2 \right) V_P^{c_2} + \left(\nabla_\mu V_S^2 + \nabla_{\rho_0} \right) c_1 c_2 V_P^{c_2-1} \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda (-4c_1 V_P^{c_2} V_S) + \nabla_\mu 2c_1 V_P^{c_2} V_S \\
&= (\nabla_\lambda (-2) + \nabla_\mu) 2c_1 V_P^{c_2} V_S
\end{aligned}$$