Fréchet derivatives under various parameterizations using chain rule

for LEGO module m_parameterization.f90

joeyartech

joeywzhou1986@gmail.com

The University of Texas at Dallas

10/18/2019

Overview

$\boxed{\text{WaveEq lm} \Rightarrow \text{optim x,g}}$

- WaveEq uses lm and produces gkpa, glda, gmu and grho under moduli-density parameterization.
- m_parameterization.f90 converts them to user-specified parameterization, which can be
  - moduli-density (kpa, lda, mu, rho)
  - velocities-density (vp, vs, rho)
  - velocities-impedance (vp, vs, Ip)
  - slowness-density (sp, sps, rho)
- Also we may consider "passive parameters" for hard constraints (e.g. Gardner) between these parameters.

velocities-density

velocities-impedance

slowness-density

PARAMETERIZATION velocities-density
isotropic ACoustic

$$\begin{cases} \kappa = \rho V_P^2 \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\kappa 2\rho V_P \\
\nabla_\rho &= \nabla_\kappa \frac{\partial \kappa}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\kappa V_P^2 + \nabla_{\rho_0}
\end{aligned}$$

$$\begin{cases} V_P = \sqrt{\frac{\kappa}{\rho_0}} \\ \rho = \rho_0 \end{cases}$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{V_P} \frac{\partial V_P}{\partial \kappa} + \nabla_\rho \frac{\partial \rho}{\partial \kappa} \\
&= \nabla_{V_P} \frac{1}{2\rho_0} \sqrt{\frac{\rho_0}{\kappa}} \\
&= \nabla_{V_P} 0.5 \rho_0^{-0.5} \kappa^{-0.5} \\
\nabla_{\rho_0} &= \nabla_{V_P} \frac{\partial V_P}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0} \\
&= \nabla_{V_P} \left(-\frac{\kappa}{2\rho_0^2} \sqrt{\frac{\rho_0}{\kappa}} \right) + \nabla_\rho \\
&= \nabla_{V_P} \left(-0.5 \rho_0^{-1.5} \kappa^{0.5} \right) + \nabla_\rho
\end{aligned}$$

PARAMETERIZATION velocities-density + PASSIVE gardner

isotropic ACoustic

$$\boxed{\rho = c_1 V_P^{c_2}}$$

$$\begin{cases} \kappa = \rho V_P^2 = c_1 V_P^{c_2+2} \\ \rho_0 = \rho = c_1 V_P^{c_2} \end{cases}$$

$$V_P = \left(\frac{\kappa}{c_1}\right)^{\frac{1}{c_2+2}}$$

$$\begin{aligned} \nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\ &= \nabla_\kappa c_1 (c_2+2) V_P^{c_2+1} \\ &\quad + \nabla_{\rho_0} c_1 c_2 V_P^{c_2-1} \\ &= \left(\nabla_\kappa c_1 (c_2+2) V_P + \nabla_{\rho_0} \frac{c_1 c_2}{V_P}\right) V_P^{c_2} \end{aligned}$$

$$\begin{aligned} \nabla_\kappa &= \nabla_{V_P} \frac{\partial V_P}{\partial \kappa} \\ &= \nabla_{V_P} \frac{1}{c_1 (c_2+2)} \left(\frac{\kappa}{c_1}\right)^{-\frac{c_2+1}{c_2+2}} \end{aligned}$$

PARAMETERIZATION velocities-density

isotropic P-SV

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) \\ \mu = \rho V_S^2 \\ \rho_0 = \rho \end{cases}$$

$$\begin{cases} V_P = \sqrt{\frac{\lambda+2\mu}{\rho_0}} \\ V_S = \sqrt{\frac{\mu}{\rho_0}} \\ \rho = \rho_0 \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda 2\rho V_P \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda(-4\rho V_S) + \nabla_\mu 2\rho V_S \\
\nabla_\rho &= \nabla_\lambda \frac{\partial \lambda}{\partial \rho} + \nabla_\mu \frac{\partial \mu}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\lambda(V_P^2 - 2V_S^2) + \nabla_\mu V_S^2 + \nabla_{\rho_0}
\end{aligned}$$

$$\begin{aligned}
\nabla_\lambda &= \nabla_{V_P} \frac{\partial V_P}{\partial \lambda} + \nabla_{V_S} \frac{\partial V_S}{\partial \lambda} + \nabla_\rho \frac{\partial \rho}{\partial \lambda} \\
&= \dots \\
\nabla_\mu &= \nabla_{V_P} \frac{\partial V_P}{\partial \mu} + \nabla_{V_S} \frac{\partial V_S}{\partial \mu} + \nabla_\rho \frac{\partial \rho}{\partial \mu} \\
&= \dots \\
\nabla_{\rho_0} &= \nabla_{V_P} \frac{\partial V_P}{\partial \rho_0} + \nabla_{V_S} \frac{\partial V_S}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0} \\
&= \dots
\end{aligned}$$

PARAMETERIZATION velocities-density + PASSIVE gardner

isotropic P-SV

$$\boxed{\rho = c_1 V_P^{c_2}}$$

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = c_1 V_P^{c_2+2} - 2c_1 V_P^{c_2} V_S^2 \\ \mu = \rho V_S^2 = c_1 V_P^{c_2} V_S^2 \\ \rho_0 = \rho = c_1 V_P^{c_2} \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda c_1 (c_2 + 2) V_P^{c_2+1} + \nabla_\lambda (-2) c_1 c_2 V_P^{c_2-1} V_S^2 \\
&\quad + \nabla_\mu c_1 c_2 V_P^{c_2-1} V_S^2 + \nabla_{\rho_0} c_1 c_2 V_P^{c_2-1} \\
&= \nabla_\lambda \left(c_1 (c_2 + 2) V_P - 2 c_1 c_2 V_P^{-1} V_S^2 \right) V_P^{c_2} + \left(\nabla_\mu V_S^2 + \nabla_{\rho_0} \right) c_1 c_2 V_P^{c_2-1} \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda (-4 c_1 V_P^{c_2} V_S) + \nabla_\mu 2 c_1 V_P^{c_2} V_S \\
&= (\nabla_\lambda (-2) + \nabla_\mu) 2 c_1 V_P^{c_2} V_S
\end{aligned}$$

velocities-density

velocities-impedance

slowness-density

PARAMETERIZATION velocities-impedance

isotropic ACoustic

$$\boxed{I_P = V_P \rho}$$

$$\begin{cases} \kappa = \rho V_P^2 = V_P I_P \\ \rho_0 = \rho = \frac{I_P}{V_P} \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\kappa I_P + \nabla_{\rho_0}(-I_P V_P^{-2}) \\
\nabla_{I_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial I_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial I_P} \\
&= \nabla_\kappa V_P + \nabla_{\rho_0} V_P^{-1}
\end{aligned}$$

$$\begin{cases} V_P = \sqrt{\frac{\kappa}{\rho_0}} \\ I_P = \sqrt{\kappa \rho_0} \end{cases}$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{V_P} \frac{\partial V_P}{\partial \kappa} + \nabla_{I_P} \frac{\partial I_P}{\partial \kappa} \\
&= \nabla_{V_P} \frac{1}{2\rho_0} \sqrt{\frac{\rho_0}{\kappa}} + \nabla_{I_P} \frac{\rho_0}{2\sqrt{\kappa \rho_0}} \\
&= \nabla_{V_P} 0.5 \rho_0^{-0.5} \kappa^{-0.5} + \nabla_{I_P} 0.5 \rho_0^{0.5} \kappa^{-0.5} \\
&= (\nabla_{V_P} + \nabla_{I_P} \rho_0)\, 0.5 \rho_0^{-0.5} \kappa^{-0.5} \\
\nabla_{\rho_0} &= \nabla_{V_P} \frac{\partial V_P}{\partial \rho_0} + \nabla_{I_P} \frac{\partial I_P}{\partial \rho_0} \\
&= \nabla_{V_P} \frac{-\kappa}{2\rho_0^2} \sqrt{\frac{\rho_0}{\kappa}} + \nabla_{I_P} \frac{\kappa}{2\sqrt{\kappa \rho_0}} \\
&= \nabla_{V_P} (-0.5) \rho_0^{-1.5} \kappa^{0.5} + \nabla_{I_P} 0.5 \rho_0^{-0.5} \kappa^{0.5} \\
&= \left(-\nabla_{V_P} \rho_0^{-1} + \nabla_{I_P}\right) 0.5 \rho_0^{-0.5} \kappa^{0.5}
\end{aligned}$$

PARAMETERIZATION velocities-impedance

isotropic P-SV

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = V_P I_P - \frac{2V_S^2 I_P}{V_P} \\ \mu = \rho V_S^2 = \frac{V_S^2 I_P}{V_P} \\ \rho_0 = \rho = \frac{I_P}{V_P} \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda \left(I_P + \frac{2V_S^2 I_P}{V_P^2} \right) + \nabla_\mu \left(-\frac{V_S^2 I_P}{V_P^2} \right) + \nabla_{\rho_0} \left(-\frac{I_P}{V_P^2} \right) \\
&= \nabla_\lambda \left(I_P + \frac{2V_S^2 I_P}{V_P^2} \right) + \nabla_\mu \left(-\frac{V_S^2 I_P}{V_P^2} \right) + \nabla_{\rho_0} \left(-\frac{I_P}{V_P^2} \right) \\
&= \left(\nabla_\lambda (V_P^2 + 2V_S^2) - \nabla_\mu V_S^2 - \nabla_{\rho_0} \right) \frac{I_P}{V_P^2} \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda \frac{-4 V_S I_P}{V_P} + \nabla_\mu \frac{2 V_S I_P}{V_P} \\
&= (-2\nabla_\lambda + \nabla_\mu) \frac{2 V_S I_P}{V_P}
\end{aligned}$$

PARAMETERIZATION velocities-impedance

isotropic P-SV *continued*

$$
\begin{cases}
\lambda = \rho(V_P^2 - 2V_S^2) = V_P I_P - \frac{2V_S^2 I_P}{V_P} \\
\mu = \rho V_S^2 = \frac{V_S^2 I_P}{V_P} \\
\rho_0 = \rho = \frac{I_P}{V_P}
\end{cases}
$$

$$
\begin{aligned}
\nabla_{I_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial I_P} + \nabla_\mu \frac{\partial \mu}{\partial I_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial I_P} \\
&= \nabla_\lambda \left(V_P - \frac{2V_S^2}{V_P} \right) + \nabla_\mu \frac{V_S^2}{V_P} + \nabla_{\rho_0} \frac{1}{V_P} \\
&= \left(\nabla_\lambda (V_P^2 - 2V_S^2) + \nabla_\mu V_S^2 + \nabla_{\rho_0} \right) \frac{1}{V_P}
\end{aligned}
$$

velocities-density

velocities-impedance

slowness-density

PARAMETERIZATION slowness-density

isotropic ACoustic

$$\boxed{S_P = 1/V_P}$$

$$\begin{cases} \kappa = \rho V_P^2 = \rho S_P^{-2} \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{S_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\
&= \nabla_\kappa (-2\rho S_P^{-3}) \\
\nabla_\rho &= \nabla_\kappa \frac{\partial \kappa}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\kappa S_P^{-2} + \nabla_{\rho_0}
\end{aligned}$$

$$\begin{cases} S_P = \sqrt{\frac{\rho_0}{\kappa}} \\ \rho = \rho_0 \end{cases}$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{S_P} \frac{\partial S_P}{\partial \kappa} + \nabla_\rho \frac{\partial \rho}{\partial \kappa} \\
&= \nabla_{S_P} \left(\frac{1}{2} \sqrt{\frac{\kappa}{\rho_0}} \frac{-\rho_0}{\kappa^2} \right) \\
&= \nabla_{S_P} (-0.5) \rho_0^{0.5} \kappa^{-1.5} \\
\nabla_{\rho_0} &= \nabla_{S_P} \frac{\partial S_P}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0} \\
&= \nabla_{S_P} \left(\frac{1}{2} \sqrt{\frac{\kappa}{\rho_0}} \frac{1}{\kappa} \right) + \nabla_\rho \\
&= \nabla_{S_P} 0.5 \rho_0^{-0.5} \kappa^{-0.5} + \nabla_\rho
\end{aligned}$$

PARAMETERIZATION slowness-density + PASSIVE gardner

isotropic ACoustic

$$\boxed{\rho = c_1 S_P^{-c_2}}$$

$$\begin{cases} \kappa = \rho V_P^2 = c_1 S_P^{-c_2-2} \\ \rho_0 = \rho = c_1 S_P^{-c_2} \end{cases}$$

$$\begin{aligned} \nabla_{S_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\ &= \nabla_\kappa c_1(-c_2-2) S_P^{-c_2-3} \\ &\quad + \nabla_{\rho_0}(-c_1 c_2) S_P^{-c_2-1} \\ &= -\left(\nabla_\kappa c_1(c_2+2) + \nabla_{\rho_0} c_1 c_2 S_P^2\right) S_P^{-c_2-3} \end{aligned}$$

$$S_P = \left(\frac{\kappa}{c_1}\right)^{-\frac{1}{c_2+2}}$$

$$\begin{aligned} \nabla_\kappa &= \nabla_{S_P} \frac{\partial S_P}{\partial \kappa} \\ &= \nabla_{S_P} \frac{-1}{c_1(c_2+2)} \left(\frac{\kappa}{c_1}\right)^{-\frac{c_2+3}{c_2+2}} \end{aligned}$$

PARAMETERIZATION slowness-density

isotropic P-SV

$$\boxed{S_{PS} = V_S/V_P = V_S S_P}$$

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = \rho S_P^{-2}(1 - 2S_{PS}^2) \\ \mu = \rho V_S^2 = \rho(S_{PS}/S_P)^2 \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{S_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_P} + \nabla_\mu \frac{\partial \mu}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\
&= \nabla_\lambda \rho(-2) S_P^{-3}(1 - 2S_{PS}^2) + \nabla_\mu \rho S_{PS}^2(-2) S_P^{-3} \\
&= \left(\nabla_\lambda (1 - 2S_{PS}^2) + \nabla_\mu S_{PS}^2\right)(-2)\rho S_P^{-3} \\
\nabla_{S_{PS}} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_{PS}} + \nabla_\mu \frac{\partial \mu}{\partial S_{PS}} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_{PS}} \\
&= \nabla_\lambda \rho S_P^{-2}(-4) S_{PS} + \nabla_\mu 2\rho S_{PS}/S_P^2 \\
&= (-\nabla_\lambda 2 + \nabla_\mu)\, 2\rho S_{PS} S_P^{-2} \\
\nabla_\rho &= \nabla_\lambda \frac{\partial \lambda}{\partial \rho} + \nabla_\mu \frac{\partial \mu}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\lambda S_P^{-2}(1 - 2S_{PS}^2) + \nabla_\mu (S_{PS}/S_P)^2 + \nabla_{\rho_0} \\
&= \left(\nabla_\lambda (1 - 2S_{PS}^2) + \nabla_\mu S_{PS}\right) S_P^{-2} + \nabla_{\rho_0}
\end{aligned}$$

PARAMETERIZATION slowness-density + PASSIVE gardner

isotropic P-SV

$$\boxed{S_{PS} = V_S S_P;\, \rho = c_1 S_P^{-c_2}}$$

$$\begin{cases} \lambda = \rho S_P^{-2}(1 - 2S_{PS}^2) = c_1 S_P^{-c_2-2}(1 - 2S_{PS}^2) \\ \mu = \rho (S_{PS}/S_P)^2 = c_1 S_P^{-c_2-2} S_{PS}^2 \\ \rho_0 = c_1 S_P^{-c_2} \end{cases}$$

$$\begin{aligned}
\nabla_{S_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_P} + \nabla_\mu \frac{\partial \mu}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\
&= \nabla_\lambda c_1(-c_2 - 2) S_P^{-c_2-3}(1 - 2S_{PS}^2) \\
&\quad + \nabla_\mu c_1(-c_2 - 2) S_P^{-c_2-3} S_{PS}^2 + \nabla_{\rho_0} c_1(-c_2) S_P^{-c_2-1} \\
&= \left(\nabla_\lambda (c_2 + 2)(1 - 2S_{PS}^2) + \nabla_\mu (c_2 + 2) S_{PS}^2 + \nabla_{\rho_0} c_2 S_P^2\right)(-c_1) S_P^{-c_2-3} \\
\nabla_{S_{PS}} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_{PS}} + \nabla_\mu \frac{\partial \mu}{\partial S_{PS}} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_{PS}} \\
&= \nabla_\lambda c_1 S_P^{-c_2-2}(-4) S_{PS} + \nabla_\mu c_1 S_P^{-c_2-2} 2 S_{PS} \\
&= (-\nabla_\lambda 2 + \nabla_\mu)\, 2 c_1 S_P^{-c_2-2} S_{PS}
\end{aligned} \tag{1}$$