MATH 3503 - Winter 2025
Final Exam

April 23, 2025

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1. Transform the given equation into a system of first-order equations.

$$t^2y^{''} + ty^{'} + (t^2 - 0.25)y = 0$$

(1 Marks)

2. Find the solution of the given initial value problem.

$$\mathbf{x}^{'} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}\mathbf{x}; \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(2 Marks)

3. Find the general solution to the system. (You may use any of the methods discussed in class).

$$\mathbf{x}^{'} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

(2 Marks)

4. Find the Laplace Transforms for the following functions using tables of transforms.
   a) $f(t) = 8sin(2t) - cos(3t) + 1$
   b) $f(t) = cos^2(6t)$
   c) $f(t)5(t-3)^2$ (3 Marks)

5. Given $f(t) = t^2e^{2t}$
   a) use the Laplace Transform of derivatives to find $\mathcal{L}\{f(t)\}$. *Hint: Compute* $\mathcal{L}\{f^{''}\}$.
   b) use the derivative of Laplace Transforms to find $\mathcal{L}\{f(t)\}$. *Hint: Let* $g(t) = e^{2t}$, *compute* $G^{''}(s)$
   c) use the definition of the integral transform to find $\mathcal{L}\{f(t)\}$. *Hint:* $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$
   d) suppose that $\mathcal{L}\{g(t)\} = \frac{2}{s-2}$ and $\mathcal{L}\{h(t)\} = \frac{1}{(s-2)^2}$, show that $(g*h)(t) = f(t)$. (4 Marks)

6. Find $\mathcal{L}^{-1}\{F(s)\}$
a) $F(s) = \frac{3}{s^2-4}$
b) $F(s) = \frac{1}{(s-1)^2(s+1)}$
c) $F(s) = \frac{1}{s^2-2s+5}$
d) $F(s) = \frac{8}{s^3(s+2)}$ (4 Marks)

7. Let $f(t)$ be periodic with period two and let

$$f(t) = \begin{cases} 1, & \text{if } -1 \leq t < 0 \\ 0, & \text{if } 0 \leq t < 1 \end{cases} \tag{1}$$

Calculate the coefficients of the Fourier series for f(t). That is, give formulas fo a_0, a_n and b_n so that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi t}{L})$$

for almost all values of t. At what values of t does the Fourier series not converge to f(t)? (3 Marks)

8. Solve the following differential equations by using one of the following techniques: variable-separable, linear (with an integration factor), homogeneous, exact, or Bernoulli. Be sure to include the trivial solution, if it exists. Where initial conditions are given, find the particular solution for the system.
a) $e^y sin(2x)dx + cos(x)(e^{2y} - y)dy = 0$
b) $\frac{dy}{dx} = \frac{y-x}{y+x}$
c) $(5x + 4y)dx + (4x - 8y^3)dy = 0$
d) $(x^4 - y)dx + (x^2y^2 + x)dy = 0$
e) $y^2dx + (x^2 + xy + y^2)dy = 0, y(0) = 1$
f) $(y^2cos(x) - 3x^2y - 2x)dx + (2ysin(x) - x^3 + ln(y))dy = 0, y(0) = e$ (6 Marks)

9. Use the Method of Undetermined Coefficients to solve the following initial value problems.
a) $y^{''} + 9y = cos(3x) + sin(3x), y(0) = 2, y^{'}(0) = 1$
b) $y^{''} - 4y^{'} + 4y = tsin(t), y(0) = 1, y^{'}(0) = 1$
c) $u^{''} + 2u^{'} + u = xe^{-x} + x, u(0) = -2, u^{'}(0) = 1$ (6 Marks)

10. Two springs and two masses ($m_1 = 2kg$ and $m_2 = 1kg$) are attached in a straight line on a horizontal surface, as shown below. The masses are displaced to the right from their equilibrium positions and then released. The mass m_1 is displaced *2m* and the mass m_2 is displaced —textit3m. By applying Newton's Second Law of Motion to each mass, the following two equations describe the coupled oscillating system.

$$\begin{aligned} m_1 x_1^{''} &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2^{''} &= -k_2(x_2 - x_1) \end{aligned} \tag{2}$$

a) Rewrite the equations as a system of first order equations.
b) Express the system in matrix form: $\mathbf{Z}' = \mathbf{AZ} + \mathbf{F}; \mathbf{Z}(t_0) = \mathbf{k}$
c) Solve the initial value problem. Let $k_1 = 4N/m$ and $k_2 = 2N/m$. (4 Marks)

11. Solve the initial value problems.
a) $\mathbf{Y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{Y}; \mathbf{Y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}$
b) $\mathbf{Y}' = \begin{bmatrix} 7 & 15 \\ -3 & 1 \end{bmatrix} \mathbf{Y}; \mathbf{Y}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ (4 Marks)

12. Use the Method of Variation of Parameters to find the solution to the following systems:
a) $\mathbf{Y}' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{Y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
b) $\mathbf{Y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{Y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$ (4 Marks)

13. For each of the following periodic functions f(t):
(i) find the Fourier coefficients of f
(ii) find the Fourier series of f.
(iii) write the first three non-zero harmonic terms of the series
(iiii) plot the periodic function f(t) and the partial sum of the first 10 harmonics of the Fourier Series using software (Geogebra, Python, etc)
a) $f(t) = sin^2(t) for - \pi < t \leq \pi,\ f(t) = f(t + 2\pi)$
b) $f(t) = \begin{cases} 0; & -\pi < t < 0 \\ t^2; & 0 \leq t \leq \pi \end{cases} \quad f(t) = f(t + 2\pi)$
c) $f(t) = \begin{cases} -t; & -4 \leq t < 0 \\ 0; & 0 \leq t < 4 \end{cases} \quad f(t) = f(t + 8)$
d) $f(t) = \begin{cases} t; & 0 \leq t < 2 \\ 2; & 2 \leq t < 4 \\ 6 - t; & 4 \leq t \leq 6 \end{cases} \quad f(t) = f(t + 6)$ (16 Marks)

14. Show whether the following functions are linearly independent or not:
a) $sin(t)$, $cos(t)$
b) $cos(2t)$ ′ $tan(t)$
c) $5t$, $6t^3$
d) e^{3t} , e^t (4 Marks)