MATH 3503 - Winter 2025

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Table of contents

Preface

These notes are for the Winter 2025 term of the course MATH 3503 at the University of New Brunswick. These notes will cover more advanced calculus topics for engineers.

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Chapter 1

Introduction

This is a book created from markdown and executable code.

See @knuth84 for additional discussion of literate programming.

Chapter 2

First Order ODEs

2.1 Introduction

First order Ordinary Differential Equations (ODEs) are the types of equations that you were exposed to in your first calculus class. These are derivative functions, made up of first derivatives of one independent variable. In the most general case, these look like

$\frac{dy}{dx} = f(x, y)$

In the special case where f(x,y) is actually only a function of x (i.e., f(x,y) = f(x)), then the most general form of a solution looks like

$y(x) = \int f(x)dx$

In this section, we will look at various techniques that can be used to solve other forms of first order ODEs. In all of the following cases, we may or may not have an initial condition. When we do, this initial condition can be used to determine any constants of integration that pop up during calculating the solution.

2.2 Seperable ODEs

One of the simplest forms of first order ODE is the seperable form. This happens when you can write the differential equation in the form of

$y^{'} = f(x)g(y)$

When you have an equation that looks like this, you can rearrange it to look like

$\frac{dy}{dx} = f(x)g(y)$
$\frac{1}{g(y)}dy = f(x)dx$

Solving the equation becomes as simple as integrating both sides of the rearranged equation.

$\int \frac{1}{g(y)} dy = \int f(x) dx$

2.3 Linear Equations and Integrating Factors

The next step in complexity for first order ODEs is if we have an extra function of x and y. The general form in this case looks like

$y^{'} + p(x)y = f(x)$

This type of equation is considered linear because there are no higher powers of y or $y^{'}$. In order to solve these types of equations, we need to use an integrating factor, defined by

$I(x) = e^{\int p(x)dx}$

This extra factor can now get used in solving the differential equation by plugging it into

$y = \frac{1}{I(x)} \int I(x) * f(x) dx$

2.4 Exact/Homogeneous

Exact equations are special first order ODEs that are derived from a potential function. These are common types of equations that occur in physics and engineering. In these types of cases, you have a potential field that defines the behaviour described by the differential equation. In the general case, we may have the potential defined by

$F(x, y)$

In this general case, we can look at the total derivative of F(x,y).

$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$

This should look familiar from the MATH 2513 class, when you covered partial differential equations. We can use shorter nomenclature and rewrite this as

$dF = M(x, y)dx + N(x, y)dy$

If we assume that the potential function is a constant function, then we know that $dF = 0$, and we can rewrite our equation as

$M(x, y)dx + N(x, y)dy = 0$

This nows looks like something that we may run into when working with first order ODEs. Specifically, this is a homogeneous ODE. If we can write our problem equation in this form, we can work backwards to discover what the solution is. The first step is to integrate either M or N. You usually pick whichever one is simpler to work with. In the M case, we end up with

$F = \int M dx + C(y)$

The important part is the extra term of C(y). You then take this solution for F(x,y) and take the derivative with respect to y. You then set this equal to N, in order to solve for what C(y) is. You can also start by integrating N first, and you end up with an extra term of C(x).

$F = \int N dy + C(x)$

We can now take the derivative with respect to x and set it equal M. In this way, we can solve for C(x).

In some cases, an equation of the form

$M dx + N dy = 0$

actually isn't exact. If we aren't sure, we can test for exactness by checking to see if we have

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

This comes from the fact that both M and N are partial derivatives of the potential function F. If you remember from MATH 2513, we have

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$

So, if we fail this test, what can we do? In these cases, we can multiply by an integrating factor to get it into an exact form. There is a derivation of how to get the following equation, but we can use

$P(x) = \frac{M_y - N_x}{N}$

to find a function. This becomes our integrating factor, that we can substitute into

$e^{\int P(x) dx}(M dx + N dy) = 0$

This new equation is now exact, and we can use the above technique to solve for a solution.

2.5 Bernoulli

There are several classes of equations where we can use a change of variable to help make the differential equation simpler to solve. One of these types are Bernoulli equations. These have a general form of

$y' + p(x)y = q(x)y^n$

In this case, we can start by dividing out both sides of the equation by y^n. This gives us

$y^{-n}y^{'} + p(x)y^{1-n} = q(x)$

We can now do a change of variables to

$u = y^{1-n}$

with the first derivative being

$u^{'} = (1-n)y^{-n}y^{'}$

Substituting this back into the original differential equation, we end up with

$\frac{u^{'}}{(1-n)y^{-n}} + p(x)u = q(x)$

The resulting new differential equation will then need to be solved using one of the other techniques you learned. In some cases, it may be seperable, in some cases you may need to use an integrating factor. Once you have a solution for u, you can then substitute it back into the change of variable you did above.

Chapter 3

Second Order ODEs

Second order ODEs are differential equations where we may have second derivatives of y with respect to x. The general form looks like

$A(x)y^{''} + B(x)y^{'} + C(x)y = F(x)$

Usually, we divide through so that the $y^{''}$ term is all by itself. If we do this, we end up with

$y^{''} + p(x)y^{'} + q(x)y = f(x)$

3.1 Homogeneous with Constant Coefficients

The first class of second order ODEs we will look at are homogeneous equations. This is the special case of setting $f(x) = 0$. We will also further constrain the equations for this section by only looking at constant coefficients. This special case looks like

$y^{''} + by^{'} + cy = 0$

In the most general case, there should be two linearly independent solutions $(y_1 and y_2)$ that combine to make the general solution. This general solution looks like

$y = C_1 y_1 + C_2 y_2$

You can check for linear independence by using the Wronskian. If we have linear independence, then we have

$\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix} \neq 0$

In order to find these two solutions to the differential equation, we start by setting up the characteristic equation that is associated with the differential equation. This characteristic equation is given by

$ar^2 + br + c = 0$

We can then find out what the two values for r are. Because this is just a quadratic equation, it is simply

$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

There are three cases when solving for r.

- Case 1 - r_1 and r_2 are real and unique
- Case 2 - r_1 and r_2 are equal
- Case 3 - r_1 and r_2 are complex

3.1.1 Case 1

If we have case 1, we can easily construct our solution. In this case, it would look like

$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

3.1.2 Case 2

If we have case 2, we can't simply use the same method as for case 1. We would end up with two equal equations. In this case, we can separate y_1 and y_2 by multiplying one of them by x. This would give us the solution

$y = C_1 e^{rx} + C_2 x e^{rx}$

3.1.3 Case 3

Case 3 is a new one if you have never considered the imaginary case for the quadratic equation. In this case, the solution to the characteristic equation is

$r_{1,2} = \alpha \pm \beta i$

In this case, we use the real and imaginary parts of r to make up our solution for y. The general form becomes

$y = C_1 e^{\alpha x} cos(\beta x) + C_2 e^{\alpha x} sin(\beta x)$

3.2 Initial Value Problem

In all of the above cases, we would have constants of integration that would show up in the solutions. In the first order ODE case, a single initial value is enough to define the constant of integration that appears. In the second order case, we will need two initial values. These take the form of

$y(x_0) = y_0$
$y^{'}(x_0) = y_0^{'}$

These can be plugged into the general solution for y to find a particular solution.

3.3 Non-homogeneous Differential Equations with Constant Coefficients

Up to now, we have been working with homogeneous equations where f(x)=0. In many physical situations, we end up with a non-homogeneous differential equation. The f(x) term is often called a driving function or forcing function.

In order to solve these types of equations, we start by assuming that the equation is homogeneous. In this way, we can find a complimentary solution, y_c that solves the equation when we pretend that f(x)=0. The second step is to come up with a second solution, y_p, called the particular solution. We will look at two options of finding this particular solution.

3.3.1 Undetermined Coefficients

The first method is to use the method of undetermined coefficients. The first step is to find y_c. This is done by assuming that the ODE is actually homogeneous, or of the form

$ay^{''} + by^{'} + cy = 0$

In this case, we can set up a characteristic equation that is associated with the homogeneous version of our ODE.

$ar^2 + br + c = 0$

We now need to solve for the values of r, using this basic quadratic equation. The solution is given by

$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

This gives us three cases for possible values of r.

3.3.1.1 Case 1

In this case, both values of r are real and unique. For this case, we have a solution of y given by

$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

3.3.1.2 Case 2

In this case, r is real, but both values are equal. In this case, we need to adjust one of the possible terms for y so that everything stays linearly independent. We then get

$y = C_1 e^{rx} + C_2 x e^{rx}$

3.3.1.3 Case 3

In this case, we end up with complex values for r, of the form

$r = \alpha \pm \beta i$

This leads to solutions for y as given by

$y = C_1 e^{\alpha x} cos(\beta x) + C_2 e^{\alpha x} sin(\beta x)$

Once we have this solution for the homogeneous version, we need to find the extra part to account for the right hand side of the equation, f(x). This extra part is called the driving function or forcing function. We actually make a guess based on the form of f(x). The table below covers the standard types of f(x) that we tend to run into.

$f(x)$	y_p
$ax + b$	$Ax + B$
$ax^2 + bx + c$	$Ax^2 + Bx + C$
$asin(bx)$	$Asin(bx) + Bcos(bx)$
$acos(bx)$	$Acos(bx) + Bsin(bx)$
ae^{bx}	Ae^{bx}
$(ax + b)e^{cx}$	$(Ax + B)e^{cx}$
$ae^{bx} sin(cx)$	$Ae^{bx} sin(cx) + Be^{bx} cos(cx)$
$(ax^2 + bx + c)e^{dx} cos(\pi x)$	$(Ax^2 + Bx + C)e^{dx} cos(\pi x) +$ $(Dx^2 + Ex + F)e^{dx} sin(\pi x)$
$ax + e^{bx}$	$Ax + B + Ce^{bx}$
$a + e^{bx}$	$A + Be^{bx}$
$ae^{bx} + csin(dx)$	$Ae^{bx} + Csin(dx) + Dcos(dx)$
$ax^2 + bcos(cx)$	$Ax^2 + Bx + C + Dcos(cx) + Esin(cx)$

In order to find the coefficients, you would take the first and second derivatives of your guess and plug it into the origianl equation in question. Setting this equal to f(x), you can then compare terms from the left hand side to the equivalent terms on the right hand side to find each of the coefficients. This then gives you the y_p part of the general solution.

NOTE - You need to make sure that y_c and y_p are linearly independent. If they aren't, you can multiply y_p by some power of x in order to make the solutions linearly independent.

3.3.2 Variation of Parameters

In order to use variation of parameters, you start the same way as above to find the y_c solution. Once you have that, you will have two partial solutions y_1 and y_2, since we have

$y_c = C_1 y_1 + C_2 y_2$

The second solution is given by

$y_p = u_1 y_1 + u_2 y_2$

In order to find u_1 and u_2, we use the following formulas:

$$u_1^{'} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2^{'} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}}$$

and

$$u_2^{'} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1^{'} & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}}$$

Once you integrate these to get u_1 and u_2, you can plug them back into y_p.