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Abstract

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The properties of one and more space dimensional solitons are discussed on the basis of the works performed mainly in Dubna at the Joint Institute for Nuclear Research, Laboratory of Computing Techniques and Automation. The conception of near integrable systems is formulated. As an example of interaction of Langmuir solitons it is shown that the Hamiltonian system is able to transform continuously its properties from a deep inelastic state up to the complete integrable one, depending on a physical parameter.

The stability and existence problem of many-dimensional solitons is discussed. The results of numerical experiments are given on the dynamics of formation and interaction of many dimensional solitons and pulsons.

There are two large classes of physical phenomena, which may be reduced to the equations possessing soliton and soliton-like solutions. They differ in their problems as well as in the interpretation of the results, although they come together at the classical level.

The problems of studying non-linear wave phenomena in real continuous media are of the first class and field theoretic problems are of the second one.

In the first case the investigation used to go through the following stages: it starts at the classical or quantum discrete level, then with some degree of rigour, the study proceeds to the classical or quasi-classical continuum limit, that defines the form of the resulting equations. As the final result of such a transformation, we have the soliton or quasi-soliton classical solutions (to which through an inverse transformation may, in principle, be given a quantum meaning).

In the second case the models under investigation are constructed with the help of a Lagrangian formalism and allowance for certain requirements come from the general physical conceptions and laws (invariance under the Poincaré group, global or gauge internal symmetries some of which may be broken and so on). In this case classical wave equations appear at the initial stage and their solutions become the basis for constructing "real" quantum objects including that of a soliton type (extended particles)[1]. Here, unlike conventional atomic physics, the quantum soliton properties are determined by classical solutions as $\hbar \to 0$. All this defines the form of the second class of equations.

These directions come together in the intermediate stage of the investigation of the localized (soliton) solutions of the finite energy to the classical wave equations

Here I could mention some opinions about the importance of studying the soliton solution properties occurring in papers of both directions. However, it seems to be purposeless after the aforesaid.

[1] The so-called non-perturbative approach in quantum field theory.

Thus, the investigation of the general properties of the classical solitons (CS) may be carried out, to some extent, forgetting their physical interpretation. As we shall see below there are possible stable solitons to exist in four-dimensional (x, y, z, t) one-field models with a so-called saturable nonlinearity. Undoubtedly, being of interest within the first class, they might be rejected for the second class theories because of renormalizability conditions.

We consider below non-linear phenomena in the systems allowing stochastization, i.e., in non-integrable systems. The behaviour of such systems may be governed by the degree of their proximity to some complete integrable analogs. In this sense the integrable models can be considered as a zero order (non-linear!) approximation to the description of the real physical systems, and further study can be performed *with all discretion* as a perturbation series in this small deviation.

Note, that at present only complete integrable models may be strictly analytically investigated by various methods. But analytical methods are, as a rule, practically helpless (at least, at present) in studying the evolution of non-integrable systems. Therefore, with rare exceptions, all the results on the evolution of ergodic (even one-dimensional) systems were obtained by computer experiments.

2. What I should primarily stress, is that it was a computer which created the Fermi–Pasta–Ulam problem (about 23 years ago) and then discovered solitons. As a result of numerical experiments on the dynamics of KdV nonlinear waves, a concept of the solitons (Zabusky) appeared to be solitary waves which emerged from the collision without changing their shapes and velocities. Somewhat earlier Perring and Skyrme have found via a computer the analogous effects in the framework of the sine-Gordon equation, but for rather different objects.

It is interesting to note, that "two-soliton" solutions (bions) have been obtained a decade earlier (Seeger, Donth and Kochendörfer). Then Ooyama and Saito (1970) have found solitons on "Toda lattice" nearing the FPU problem. Finally, solitons in the framework of the Schrödinger equation with cubic nonlinearity (S3) were discovered by Yajima and Outi in 1971. All references may be found in the review by Scott, Chu and McLaughlin [1].

Thus, the computer creating a new branch in the theory of nonlinear partial differential equations fell behind. A boom time of discovering and studying the completely integrable Hamiltonian systems and the related methods of the inverse scattering problem, Hirota and Bäcklund transformations began. Developing and formalizing these techniques being of an international competition character displayed, that integrable equations may be generated in the unlimited amount. All this gave rise to the view, that the majority (if not all) of the Lagrangian systems are completely integrable.

The first impact on this outlook was done again by a computer. The inelastic interaction of the Langmuir solitons in plasma was discovered at Dubna in 1974 [2]; analogously, for solitons of the "improved" versions of Boussinesq and KdV equations, of the Higgs and Klein–Gordon equations [3]. It turned out, that even a "small" alteration of the equation may render it non-integrable. Moreover, as it was shown, certain particular properties of the integrability disappeared under transformation from the plane (x, t) geometry to the spherically (or cyllindrically) symmetric (r, t) one [4].

The conception of *near* integrable systems, for which, as has been pointed out, integrable equations may serve as somewhat original zero order approximation, appeared. The role of the interaction parameter in an investigation can play the deviation from an appropriate integrable equation. It may be done most easily, if such a deviation may be isolated as a right hand side of the equation with a small parameter [5]. The consequent approximation methods can work in this case. However, we should stress, that such a procedure has to be done with all possible cautions, since acting in this way one could obtain "solutions", which do not subject to the initial equation. A lengthy report on these topics will be delivered by Prof. Fedyanin on this Symposium.

In addition, even in the case of small parameter theory, there are possible soliton-like solutions cardinally different from that to integrable equations [6]. Sometimes it fails to get such a deviation in a pure form: for example, in the case of the Boussinesq equation, we have φ_{xxtt} instead of φ_{xxxx}. The "nearness" of the Higgs and KG3 equations to the sine-Gordon equation is still more non-trivial [3].

It should be emphasized that the numerical experiments are now one of the most powerful tools to investigate Hamiltonian systems, especially to answer if a given system is the complete integrable. The elastic collision of solitons can imply the positive answer (recall KdV). The inelastic interaction of solitons would make searching for the integrability consequences, in particular, many-soliton formulas to be fruitless [7].

Nevertheless, the concept of a nearness between a given system and some integrable one somewhat helps to discover pulsating (bound states) solitons, bions, both in the plane (x, t) [8] geometry and in a spherically symmetric (r, t) one [9], for the KG3 and Higgs equations. If in the plane case one can still find an approximate analytical solution for the pulsons (pulsating solitons), on the contrary, in the (r, t) geometry, the discovery of the pulsons as well as the investigation of their properties are only due to a computer (the conventional analytical methods turned out to be powerless because of the actual nonlinearity and the absence of a small parameter).

The transformation of a "physical" system, describing the interaction of Langmuir and ion-acoustic waves in plasmas, from the strong non-integrable one, when colliding solitons are slow (their fusion is possible) up to the integrable system, at $v \to 1$ [18], have been firstly displayed via a computer in [2]. It is interesting to note here, that the non-linear description of ion-acoustic oscillations (e.g., with the help of the Boussinesq equation) makes the system non-integrable again. The system of two coupled integrable equations is non-integrable! [3]

3. As a result of a great deal of work all over the world, the properties of solitons in the plane (x, t) world have been learned quite well. It was time to proceed to more real and intricate many-dimensional worlds[2]. This transition as should be expected, was non-trivial. Here the stability problem went ahead, when we proceed from one to many space dimensions (unstable solitons were found, in the plane (x, t) world, only in the framework of the KG3 equation). Many papers, especially in plasma theory, were devoted in recent years to studying the stability of plane (x, t) solitons with respect to transverse perturbations [3, 10]. Note, that the solitons of the two well-known integrable equations, the KdV and S3, are in this sense different. The KdV solitons may be stable, that allowed to solve the two space dimensional problem (the Kadometzev–Petviashvili equation); the S3 solitons are unstable that pointed, once again, at the existence of the Langmuir wave collapse in plasmas.

A discouraging theorem was established by Derrick and Hobart 1963–64 that implied there were no stable time-independent soliton-like solutions in more, than one spatial dimension world, in the framework of conventional relativistic non-linear theories (without internal symmetries). Let us define the soliton in many dimensional worlds. As the author knows no integrable systems in non-one-dimensional Minkovski worlds have been obtained as yet (with the exception for the non-relativistic KdV equation and similar ones). Our definition therefore, practically coincides with that of quasi-particle solution: the soliton is a solution of certain nonlinear equation, which has finite energy, momentum, "charge," "correct" asymptotics at infinity and safety interaction factor.

4. The Derrick–Hobart theorem states, in fact, that in the stationary case the Hamiltonian of a given system, as a functional of field φ, $H[\varphi]$ determines a surface, which cannot be a valley. To stabilize the system additional constraints (integrals of motion) have to be. There are possible two ways:

(a) Constructing the system with non-trivial topology Faddeev, Polyakov, t'Hooft and many others: there is, in addition, a very excellent review by (S. Coleman [19]).

(b) Considering theories with internal global or gauge symmetries. Later we do not deal with the first approach.

In the second case the soliton provides the Hamiltonian with a conditional extremum that may lead to the soliton stability (Zastavenko, Lee and Makhankov). Let us show this in the example of the global $U(1)$ symmetry, $\psi \to \psi\, e^{i\theta}$. Then we have an additional conservation law

$$\frac{dQ}{dt} = 0, \quad Q = -2\int (\psi_t^* \psi - \psi_t \psi^*)\, d^D x \tag{1}$$

here and later D is the number of Euclidean coordinates.

The solution time-dependence of the form $\psi_s = \varphi\, e^{-i\omega t}$, minimizes the energy functional, H. Hereat, the stability condition of soliton solution becomes (Vakhitov and Kolokolov, Friedberg, Lee and Sirlin, Makhankov [3])

$$\begin{aligned} &\frac{dQ}{d\omega} < 0, \quad (Q\text{-theorem}) \text{ relativistic case} \\ &\frac{dS}{d\omega} < 0, \quad S = \int \varphi^2\, d^D x, \quad \omega < 0 \quad \text{non-relativistic case} \end{aligned} \tag{2}$$

In [11] (see also [3], p. 100) it was shown, that in the systems with so-called saturable non-linearity, there exist stable Q-solitons in a certain region of ω. In the framework of the one-field φ_D^4 theory, there are no stable solutions at $D \geqslant 2$.

Note, that in all the cases considered, we dealt with the complex field (1) that implies the exact conservation law taking place.

One can search for a soliton-like solution, in the case of real scalar field theory, in the following form

[2] Note, that there appeared nearly 350 papers regarding solitons only in 1977.

$$\chi = \operatorname{Re}\varphi\, e^{-i\omega t} \quad \text{(pulson)} \tag{3}$$

The approximate conservation law

$$\frac{dS}{dt} \simeq 0 \tag{4}$$

may arise, under certain conditions, with the exponential accuracy (Bogolubsky and Makhankov [8] Manakov [12]).

5. Consider the coordinate dependence, $\varphi(r)$, of solitons. We restrict ourselves by spherically symmetric solutions, in D-dimensional Euclidean space or in $(D+1)$-dimensional Minkowski world (in the latter case the mass term is renormalized), to the following equation of quite a general type with polynomial Lagrangian.

$$\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{d\varphi}{dr}\right) - x^2\varphi + g^2\varphi^{p-1} - a\varphi^{q-1} = 0 \tag{5}$$

Having used the "virial theorem" and integrating eq. (5) with weight $\varphi_r r^{D-1}$ over r, we come to the necessary conditions of soliton-like solution existence [13]

$$\left(1 - p\frac{D-2}{2D}\right)U_p > \left(1 - q\frac{D-2}{2D}\right)U_q \tag{6}$$

$$(p-2)U_p > (q-2)U_q \tag{7}$$

Here

$$U_p = \frac{g^2}{p}\int_0^\infty \varphi^p r^{D-1}\,dr, \quad U_q = \frac{a}{q}\int_0^\infty \varphi^q r^{D-1}\,dr$$

Note that for the instanton solutions ($x^2 = a = 0$), the condition (6) coincides with the renormability condition of the theory. A similar situation takes place for vector fields $A_\mu = f(x)x_\mu/|x|$.

For example, when $p = 4$, $q = 6$, $D = 3$, we have a set of solutions φ_n (n numbers nodes of the field function) stable in a certain area of parameters ω, g^2, a.

The existence theorem for $D = 3$ and quite a general nonlinearity was proved in [14].

At the end of this section we note, that static properties of solitons were studied for more complex scalar and vector gauge theories such as SU(2) × SU(2), SU(3) × SU(3), as well as including colour spinor (quark) fields, i.e., SU(3) × SU(3) × SU(3) [15] (one of the authors of these investigations, Prof. Lee, has spoken on them).

6. To justify somewhat the title of this report let us proceed again to a computer. The dynamics of solitons in many-dimensional worlds is, so far, almost completely an eparchy of the computer. With its help pulsons–quasi-bions in spherical three-dimensional space have been discovered, and their properties investigated [4]. The *ss*-solutions of a soliton type were studied for the following equations describing real scalar fields[3]

$$\varphi_{tt} - \Delta_{rr}\varphi - \varphi + \varphi^3 = 0 \quad \text{GLH} \tag{8.1}$$

$$\varphi_{tt} - \Delta_{rr}\varphi + \sin\varphi = 0 \quad \text{SG (integrable in } x, t \text{ world)} \tag{8.2}$$

$$\varphi_{tt} - \Delta_{rr}\varphi + \varphi - \varphi^3 = 0 \quad \text{KG3} \tag{8.3}$$

The constants of motion are respectively

$$E = 4\pi\int \mathcal{H} r^2\,dr, \quad \mathcal{H} = \tfrac{1}{2}[\varphi_t^2 + \varphi_r^2 + V_i(\varphi)] \tag{9}$$

where

$$V_1(\varphi) = \tfrac{1}{2}(\varphi^2 - 1)^2, \quad V_2(\varphi) = 2(1 - \cos\varphi),$$
$$V_3(\varphi) = \varphi^2\left(1 - \frac{\varphi^2}{2}\right) \tag{10}$$

As has been pointed out, stable Q-solitons ($dQ/d\omega < 0$) for the KG3 equation do not exist, when $D = 3$. The computer experiments carried out at Dubna exhibit the same situation for pulsating solutions of type (3) to eq. (8.3). The KG3-pulsons as well as the KG3 Q-solitons decay by two instability modes – dissipative and singular [3].

Solutions to eqs. (8.1) and (8.2) have non-trivial asymptotics (the vacuum values) at infinity ($r \to \infty$)

$$\varphi_{\mathrm{GLH}}|_{r\to\infty} \to \pm 1,$$

$$\varphi_{\mathrm{SG}}|_{r\to\infty} \to \pm 2\pi n$$

to which minima of potentials (10) correspond (function $V_3(\varphi)$ has only one relative minimum $V_3(0) = 0$). Fixation of one of these vacua as a boundary condition may lead to the quasi-stable (i.e., weakly radiating linear waves) pulsons to occur. Their lifetime is of the order of 10^3 oscillation periods (adiabatic invariant (4)!). Note, that the decay time of a KG3-pulson is of the order of several (3–4) oscillations (for details see [3]).

It should be stressed as well, that no nonradiating objects similar to the bions in plane (x, t) world, were found for the SG equation.

We chose the following boundary conditions

$$\varphi^v_{\mathrm{GLH}} = -1, \quad \varphi^v_{\mathrm{SG}} = 0$$

and obtained long-lived pulsons (with a bell-shape coordinate dependence) under quite an arbitrary initial conditions, only if [8]

$$\varphi(0,0) > \mathrm{const}_1 > 1 \quad \text{GLH}$$

$$\varphi(0,0) > \mathrm{const}_2 > 2\pi \quad \text{SG}$$

Somewhat later analogous objects were found by the group from ITEP (Moscow) [16].

Quasi-stable pulsons of greater and greater field amplitudes in the origin, $\varphi(0, 0) > 2\pi n$ might be expected to exist for many-vacuum sine-Gordon theory. Such objects have been found (although they have no analog in the plane (x, t) world); their amplitudes lie in the region $\varphi(0, t) \in (3\pi, 4\pi)$. Then in the region $(2\pi, 3\pi)$, these pulsons become unstable. As a result, we have a cascade process: irradiating slightly a heavy pulson diminishes its amplitude inside the interval $(3\pi, 4\pi)$, rapidly goes through the interval $(2\pi, 3\pi)$, and then, in the area $(\pi, 2\pi)$, we have again a weakly radiating pulson, which becomes unstable in $(0, \pi)$. It is easy to notice, that such a pulson behaviour is intimately connected to a potential shape $V(\varphi)$ (Bogolubsky).

To shorten computer run time the initial conditions were taken in the quasi-bion form

$$\varphi(r, 0) = 4x\,\mathrm{arctg}\,[\mathrm{tg}\,\alpha\,\mathrm{sech}\,(\sin\alpha r)]$$

with changing parameters α and x.

We have just considered the dynamical properties of solitons, which relate their formation and instability. But a real dynamics, in which particular properties of solitons may be exhibited, indeed, is their interaction.

Let us now proceed to discussing the results known to the author, on collisions of two space-dimensional cylindrically symmetric ((CS) in each rest frame) solitons. We mention here

[3] According to the Derrick–Hobart theorem there exist no stable time-independent solitons in the framework of such theories.

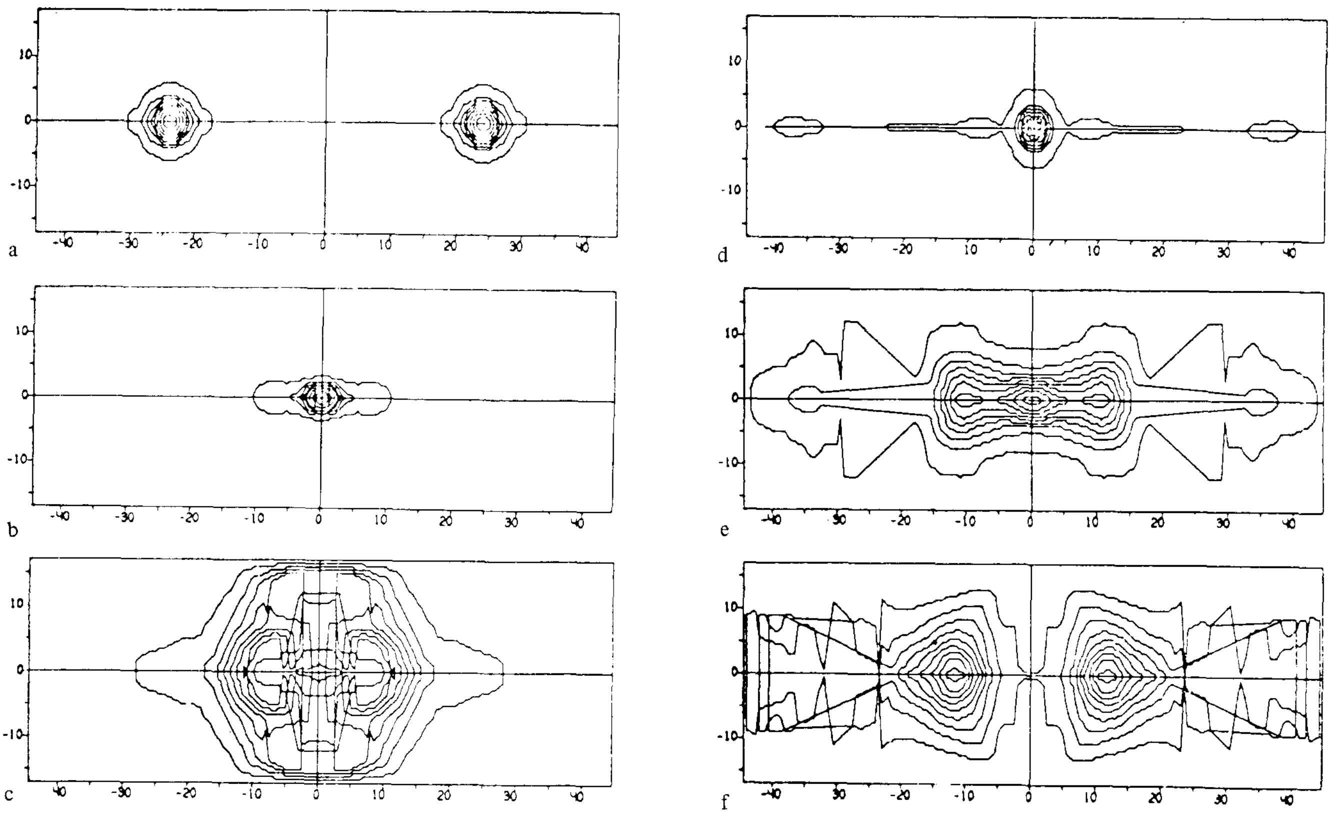

Fig. 1. Bound state ("resonance") of two Q solitons at $v = 0.2$, $\mu = 0.95$. Figures with isohypses correspond to plane cross sections of Fig. 2. $H = \mathcal{H}/\mathcal{H}_{\max}$, $\mathcal{H}_{\max} = 2.28 = \mathcal{H}\ (t = 120)$.

only two numerical experiments carried out at Dubna, concerning the interaction of (a) unstable KG3-pulsons [17] and (b) Q-solitons of the KG equation with the saturable nonlinearity of $|\psi|^2(1 + |\psi|^2)^{-1}$ type. It should be noticed, that there are already very impressive movies by Tappert on interaction of CS solitons in the framework of the Schrödinger equation with exponential nonlinearity (saturable)

$$i\psi_t + \Delta_{rr}\psi + \frac{1}{\alpha}\psi(1 - \exp\{-\alpha|\psi|^2\}) = 0,$$

which models the behaviour of Langmuir wave packets, in the vicinity of a stationary state.

To proceed further we would note that all the pulsons of the KG3, GLH and SG equations turned out to be stable against small angle perturbations $\delta\varphi(\vartheta)$ [17]. In the first series of computations (Bogolubsky, Shvachka and the author), the head-on collision of two unstable KG3-pulsons of type

$$\varphi(x, y, t) = A\varphi(u_0\sqrt{\gamma_i^2(x - v_it)^2 + y^2}) \times \cos(\sqrt{1 - u_0^2}\gamma_i(t - v_ix))$$

have been studied. Here v_i is a velocity of ith pulson in light velocity units, $v_1 = -v_2 = v = 0.2, 03, 04, 06$, $\gamma_i^{-2} = 1 - v_i^2$ and $\varphi(r)$ is a soluton of the following boundary problem

$$\varphi_{rr} + \frac{1}{r}\varphi_r - \varphi + \varphi^3 = 0, \quad \varphi_r(0) = \varphi(\infty) = 0$$

The collision time was taken less than the instability rate. The behaviour of pulsons resembles that of the one-dimensional 1s-solitons discussed above [2]. When the velocity of pulsons, v is greater than a critical one $v_{cr} \simeq 0.3$, pulsons emerge from collision and, only then, decay by dissipative or singular modes. As $v \leqslant 0.3$, pulsons coalesce into one collapsing afterwards. A significant fact is that the number of unstable "quasi-particles" is conserved, at $v > v_{cr}$, where they live approximately for the same time as they do in free states. This takes place, despite the pulson collision bringing about a great, although self-consistent, perturbation of the order of unity for each of them. Here, in two space-dimentional world we observe, apparently, the exhibition of soliton-like (in Zabusky sense) properties by unstable pulsons.

The second series of calculations (Kummer, Shvachka and the author) was undertaken to study head-on collisions of stable Q-solitons in the model

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \psi - \frac{|\psi|^2}{1 + |\psi|^2} = 0$$

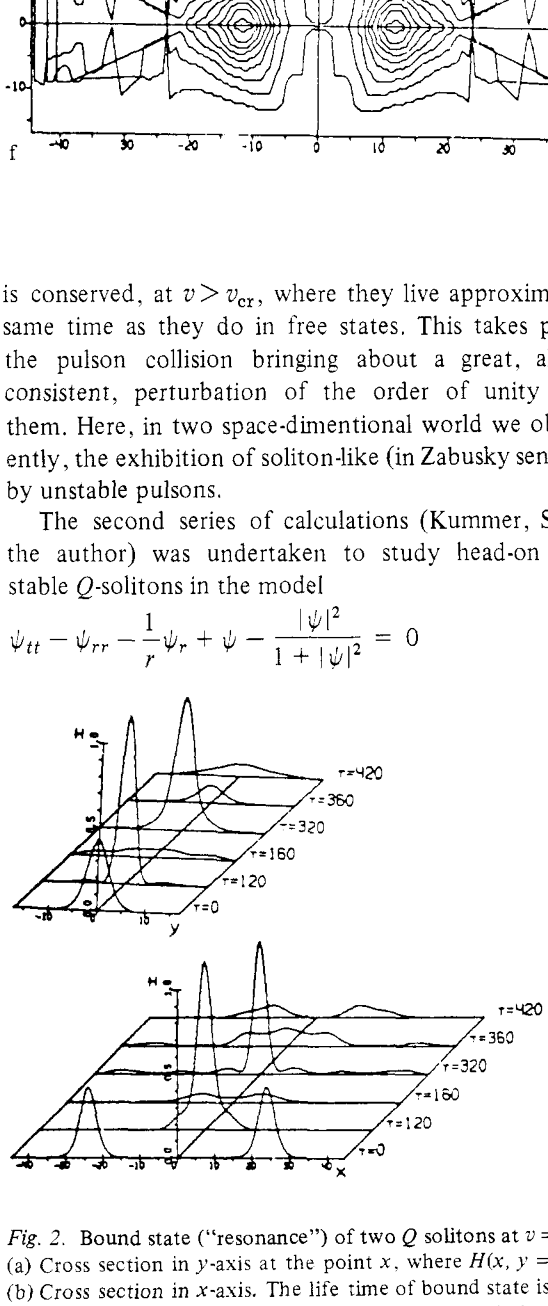

Fig. 2. Bound state ("resonance") of two Q solitons at $v = 0.2$, $\mu = 0.95$; (a) Cross section in y-axis at the point x, where $H(x, y = 0)$ is maximal. (b) Cross section in x-axis. The life time of bound state is approximately 2×10^2, it means that the system oscillates several times (≈ 4) before decaying.

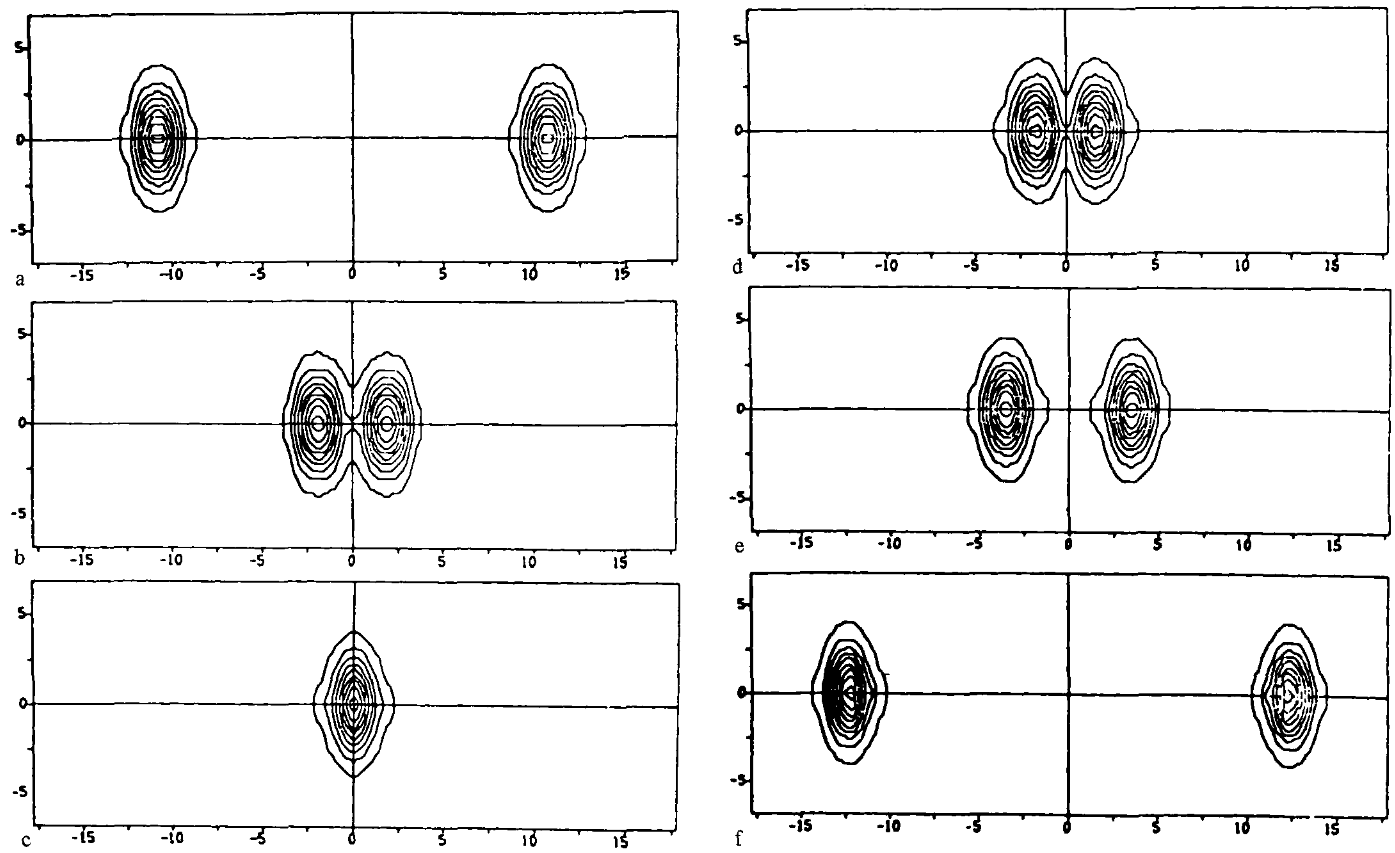

Fig. 3. Elastic interaction of Q-solitons at $v = 0.9$, $\mu = 0.7$. Figures with isohypses correspond to plane cross-section of Fig. 4. $H = \mathcal{H}/\mathcal{H}_{\max}$, $\mathcal{H}_{\max} = 69.4 = \mathcal{H}\ (t = 12)$.

The behaviour of Q-solitons in these collisions is very similar to the pure soliton one (when $v \to 1$): an interaction turns out to be nearly elastic, and a certain shift in their positions have been observed. In the region of small $v \simeq 0.2$, we have observed an oscillating bound state of two solitons, apparently, slight radiating. The results of two such experiments are depicted in Figs. 1–4.

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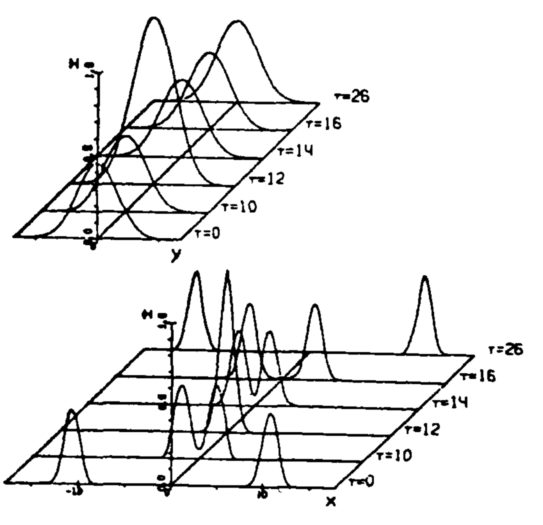

Fig. 4. Elastic interaction of Q solitons at $v = 0.9$, $\mu = 0.7$.

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Note in proof: The detailed results on the investigation of stable Q-soliton interactions have been submitted to Physica Scripta in October 1978.