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Abstract

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We discuss the stability of three-dimensional planar Langmuir solitons which are solutions of both the correct Zakharov equations and the simplified, curl-free equations which many authors have considered. We resolve the differences between various results obtained by different authors, show that the simplified equations used by them are unlikely to be a good approximation, and estimate the growth rates for the case of the exact equations.

1. Introduction

The stability of Langmuir solitons is a problem which has a certain amount of interest in connection with a possible soliton theory of strong Langmuir turbulence. In one dimension a Langmuir soliton is stable, but scaling of the energy shows that solitons in two or three dimensions are unstable (see, e.g., Gibbons et al., [1]). Consider now a plasma which is pumped by a plane, electrostatic wave. If the turbulent energy level is sufficiently high, the modulational growth rate for longitudinal perturbations is much greater than for transverse perturbations [2, 3]. We may therefore expect that planar Langmuir solitons will form. The question now arises as to their stability. As one-dimensional solitons are stable, it seems likely that planar solitons in three dimensions are unstable to transverse perturbations. At that point, however, the literature becomes confused and it is the aim of the present paper to sort out the situation.

In the next section we write down the basic equations and discuss their perturbations. In Section 3 we discuss a special case of the basic equations which has been studied by most other authors with different and conflicting results. We resolve these differences. In the last section we consider the general equations and estimate the growth rates for that case. We show that the special case studied by other authors is unlikely to be a good approximation.

2. Basic equations and perturbed solitons

Our basic equations are the Zakharov (1972) equations

$$\mathrm{i}\frac{\partial \boldsymbol{E}}{\partial t} + \nabla(\nabla . \boldsymbol{E}) - \alpha[\nabla \wedge [\nabla \wedge \boldsymbol{E}]] = n\boldsymbol{E} \qquad (1)$$

$$\frac{\partial^2 n}{\partial t^2} - \nabla^2 n = \nabla^2 |\boldsymbol{E}|^2 \qquad (2)$$

where $\boldsymbol{E}$ is the slowly varying Langmuir electric field amplitude, n the density variation, α is given by the equation (m_e: electron mass; k_B: Boltzmann constant; T_e: electron temperature)

$$\alpha = m_e c^2/3k_B T_e, \qquad (3)$$

and $\boldsymbol{E}$, n, x, and t are all suitably normalized (we use the same normalization as in Gibbons et al., [1] or Thornhill and ter Haar, [3]. As T_e for typical plasmas ranges from 10^4 to 10^8 K, α is always a large parameter, a fact which is of the utmost importance for the discussion in the present paper.

We shall now consider the following planar soliton solution of eq. (1) and (2):

$$\boldsymbol{E}_0 = E_0\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}} aA \operatorname{sech} A\xi \exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t). \qquad (4)$$

$$n_0 = -2A^2 \operatorname{sech}^2 A\xi, \qquad (5)$$

where $\hat{\boldsymbol{x}}$ is the unit vector in the x-direction, and

$$\xi = x - Vt, \quad V = 2k_0, \quad a^2 = 2(1-V^2),$$
$$A^2 = k_0^2 - \omega_0 \qquad (6)$$

There is, of course, no y- or z-dependence in expression (4). We now consider perturbations to the planar soliton of the form

$$\boldsymbol{E} = E_0\hat{\boldsymbol{x}} + \delta\boldsymbol{E}, \quad n = n_0 + \delta n \qquad (7)$$

We shall discuss two special cases:

(I) planar geometry, in which case we have

$$\begin{aligned} \delta E_x &= F_x(\xi, t)\exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t)\cos(k_\perp y) \\ \delta E_y &= F_y(\xi, t)\exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t)\sin(k_\perp y) \\ \delta n &= n(\xi, t)\cos(k_\perp y) \end{aligned} \qquad (8)$$

where $k_\perp$ is the wavenumber of the transverse perturbation; and

(II) cylindrical geometry (axial symmetry), when

$$\begin{aligned} \delta E_x &= F_x(\xi, t)\exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t) J_0(k_\perp r) \\ \delta E_r &= F_r(\xi, t)\exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t) J_1(k_\perp r) \\ \delta n &= n(\xi, t) J_0(k_\perp r) \end{aligned} \qquad (9)$$

where J_0 and J_1 are Bessel functions of zeroth and first order, respectively. For cylindrical geometry we have assumed axial symmetry of fields and density. Substituting the expressions (7) and (9) into (1) and linearising the equations (assuming small perturbations) we obtain

$$\mathrm{i}\frac{\partial}{\partial t}\delta E_x + \frac{\partial^2}{\partial x^2}\delta E_x + \alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\delta E_x\right) - (\alpha - 1)\frac{\partial}{\partial x}V_r = n_0\delta E_x + \delta n E_0 \qquad (10)$$

$$\mathrm{i}\frac{\partial}{\partial t}\delta E_r + \alpha\frac{\partial^2}{\partial x^2}\delta E_r + \frac{\partial}{\partial r}V_r - (\alpha - 1)\frac{\partial^2}{\partial x \partial r}\delta E_x = n_0 E_r \qquad (11)$$

where

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$$V_r = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) \tag{12}$$

We have here considered the cylindrical geometry; the planar case (I) can be similarly handled.

If the variables are changed from x, t to ξ, t and the explicit form of δE given in (9) is substituted, we obtain

$$\mathrm{i}\frac{\partial F_x}{\partial t} + (\omega_0 - k_0^2)F_x + \frac{\partial^2 F_x}{\partial \xi^2} - \alpha k_\perp^2 F_x$$

$$-(\alpha-1)\left(\frac{\partial}{\partial \xi} + \mathrm{i}k_0\right)k_\perp F_r = n(\xi, t)E_0 + n_0 F_x, \tag{13}$$

$$\mathrm{i}\frac{\partial F_r}{\partial t} + (\omega_0 - \alpha k_0^2)F_r + \alpha\frac{\partial^2 F_r}{\partial \xi^2} + 2\mathrm{i}k_0(\alpha-1)\frac{\partial F_r}{\partial \xi}$$

$$+(\alpha-1)k_\perp\left(\frac{\partial}{\partial \xi} + \mathrm{i}k_0\right)F_x - k_\perp^2 F_r = n_0 F_r \tag{14}$$

Various properties of the Bessel functions J_0 and J_1 have been used to obtain these equations. Suppose now that the perturbations have a time dependence of the form $\mathrm{e}^{\gamma t}$ with γ real and that

$$F_x = f_x + \mathrm{i}g_x, \quad F_r = f_r + \mathrm{i}g_r \tag{15}$$

where the f's and g's are real. A final change of variable from ξ to $z = A\xi$ gives the equations

$$Lf_x - an \operatorname{sech} z = \Gamma g_x + \alpha\kappa^2 f_x + \beta\kappa\left\{\frac{\mathrm{d}f_r}{\mathrm{d}z} - \bar{k}_0 g_r\right\} \tag{16}$$

$$Lg_x = -\Gamma f_x + \alpha\kappa^2 g_x + \beta\kappa\left\{\frac{\mathrm{d}g_r}{\mathrm{d}z} + \bar{k}_0 f_r\right\} \tag{17}$$

$$Kg_r = -\Gamma f_r + \kappa^2 g_r - \beta\kappa\left(\frac{\mathrm{d}g_x}{\mathrm{d}z} + \bar{k}_0 f_x\right) - 2\bar{k}_0\beta\frac{\mathrm{d}f_r}{\mathrm{d}z} \tag{18}$$

$$Kf_r = \Gamma g_r + \kappa^2 f_r - \beta\kappa\left(\frac{\mathrm{d}f_x}{\mathrm{d}z} - \bar{k}_0 g_x\right) + 2\bar{k}_0\beta\frac{\mathrm{d}g_r}{\mathrm{d}z} \tag{19}$$

where the operators L and K are defined by

$$L = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2\operatorname{sech}^2 z - 1 \tag{20}$$

$$K = \alpha\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2\operatorname{sech}^2 z - 1 + \beta\bar{k}_0^2 \tag{21}$$

and the parameters have been changed to

$$\beta = \alpha - 1, \quad \bar{k}_0 = k_0/A, \quad \kappa = k_\perp/A, \quad \Gamma = \gamma/A^2 \tag{22}$$

The same equations (16) to (19) result, if we consider soliton stability for the planar geometry, with F_y replacing F_r.

An important property of eqs. (16) to (19) is immediately apparent: the case when the parameter $\alpha = 1$ (and hence $\beta = 0$) is unusual. When $\beta = 0$, eqs. (16), (17) and (18), (19) are no longer coupled and the radial and longitudinal field components become independent. From a physical point of view this is of considerable importance, as stabilizing or destabilizing effects of the transverse component of field cannot affect the longitudinal component. However, setting $\alpha = 1$ gives just the equations which have been considered by previous authors; we have then $\nabla(\nabla \cdot E) - \alpha\nabla \times \nabla \times E = \nabla^2 E$ in eq. (1).

To close the system of eqs. (16) to (19) we use the sound eq. (2) which in the new variables may be written (see Schmidt (1975))

$$\left(A\Gamma - V\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 n + \left(\kappa^2 - \frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)n = -2a\left(\kappa^2 - \frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)(f_x \operatorname{sech} z) \tag{23}$$

Note that this equation determines n in terms of f_x only and does not involve f_r or g_r. In principle, this makes it possible to have different growth rates Γ in the longitudinal and transverse directions, with the time development of n being determined by the longitudinal field growth rate.

3. The curl-free equations

Several authors have considered eqs. (1) and (2) with $\alpha = 1$; they reach different and conflicting conclusions about soliton instability. Both Zakharov and Rubenchik [4] and Degtyarev, Zakharov and Rudakov [5] have found a real value for Γ which grows linearly with κ for small κ. Schmidt [9] has also found a two-dimensional soliton to be unstable but obtains $\Gamma \propto \sqrt{\kappa}$. Pereira, Sudan and Denavit [6] also obtain $\Gamma \sim \kappa$ and present some interesting computer calculations. Infeld and Rowlands [7] however, find soliton stability and claim to have found inconsistencies in previous authors' work. In this section we resolve the differences.

Consider eqs. (16) to (19) with α set equal to 1:

$$Lf_x - an \operatorname{sech} z - \kappa^2 f_x = \Gamma g_x, \quad Lg_x - \kappa^2 gx = -\Gamma f_x \tag{24}$$

$$Lg_r - \kappa^2 g_r = -\Gamma f_r, \quad Lf_r - \kappa^2 f_r = \Gamma g_r \tag{25}$$

Equations (25) combine to give

$$\frac{\mathrm{d}^2 F_r}{\mathrm{d}z^2} + (2\operatorname{sech}^2 z - 1 - \kappa^2 + \mathrm{i}\Gamma)F_r = 0 \tag{26}$$

It should be noted at this point that in deriving (26) we started from the assumption that Γ in eqs. (16)–(19) was real. However, by considering eqs. (13) and (14) and setting $\alpha = 1$ directly, it may be seen that (26) holds for complex as well as real values of Γ.

Substituting $\theta = \tanh z$ transforms this equation to

$$(1-\theta^2)\frac{\mathrm{d}^2 F_r}{\mathrm{d}\theta^2} - 2\theta\frac{\mathrm{d}F_r}{\mathrm{d}\theta} + \left(2 - \frac{1+\kappa^2 - \mathrm{i}\Gamma}{1-\theta^2}\right)F_r = 0 \tag{27}$$

which is Legendre's associated equation with solutions $p_\nu^\mu(\theta)$ and $Q_\nu^\mu(\theta)$ where

$$\nu(\nu+1) = 2 \quad \text{and} \quad \mu^2 = 1 + \kappa^2 - \mathrm{i}\Gamma \tag{28}$$

Since we require that $|F_r| \to 0$ as $z \to \pm\infty$, it is necessary that [8]

$$\mu = \text{integer} \tag{29}$$

Thus Γ is purely imaginary and the transverse field component oscillates in time but does not grow. (Note that Γ may take imaginary values, despite the assumption of eq. (15), because (26) is an equation for the complex function F_r). Since eq. (24) is not coupled with (25), we see that the growth rate in (24) is different and may be real. To consider these equations further, a tractable form for n must be obtained from (23). In the static regime, eqs. (1) and (2) reduce to the non-linear Schrödinger equation (NLS) since in that case $n = -|E|^2$ and stability of NLS solitons to transverse perturbations has been considered by Zakharov and Rubenchik [4]; further detail was given by Degtyarev, Zakharov and Rudakov [5]. The static regime is valid for $A\Gamma \ll 1$ (see eq. (23)). The opposite condition, $A\Gamma \gg 1$, has been considered by Schmidt [9]. We shall now consider these two limiting cases separately.

3.1. $A\Gamma \ll 1$

For $A\Gamma \ll 1$, $V^2 \ll 1$ and $\kappa \ll 1$, eq. (23) gives

$$n = -4/af_x \operatorname{sech} z \tag{30}$$

and the perturbation eq. (24) become

$$L_1 f_x = -\Gamma g_x, \tag{31}$$

$$L_0 g_x = \Gamma f_x, \tag{32}$$

where

$$L_1 = -\frac{d^2}{dz^2} + (1 - 6\operatorname{sech}^2 z) + \kappa^2 \quad \text{and} \quad L_0 = -L + \kappa^2 \tag{33}$$

Both L_0 and L_1 are self-adjoint and their properties have been investigated by Zakharov and Rubenchik [4]. L may be written in the form

$$Lf = \frac{1}{\operatorname{sech} z}\frac{d}{dz}\left(\operatorname{sech}^2 z \frac{d}{dz}\left(\frac{f}{\operatorname{sech} - z}\right)\right), \tag{34}$$

which, since sech $z \neq 0$, shows that $-L$ is a non-negative operator. Hence L_0 is positive definite and the inverse L_0^{-1} exists. From (31) and (32) we obtain

$$L_1 f = -\Gamma^2 L_0^{-1} f \tag{35}$$

Let λ_0 be the minimum eigenvalue of L_0 so that

$$\lambda_0^{-1} = \max_{\text{all } f} \langle f L_0^{-1} f\rangle / \|f\|^2 \tag{36}$$

where $\langle\ \rangle$ denotes integration over z from $-\infty$ to $+\infty$ (see Stakgold [10]). The maximum in this equation is actually attained by setting $f = \operatorname{sech} z$ which is an eigenfunction of L_0. Since we have assumed Γ real, there is an instability of the soliton only if L_1 has at least one negative eigenvalue. Call λ_1 the smallest eigenvalue of L_1. Then we find from (35) that

$$-\Gamma^2 \geqslant \lambda_0\lambda_1 \quad \text{or} \quad \Gamma^2 \leqslant \lambda_0|\lambda_1| \tag{37}$$

which gives an upper bound on Γ. By considering L_0 and L_1 expressed in a form similar to eq. (27), it is possible to show that the operators L_0 and L_1 have the eignevalues $\lambda^{(0)}$ and $\lambda^{(1)}$, respectively, which are given by the relations

$$\lambda^{(0)} = \kappa^2, \quad \lambda^{(1)} = -3 + \kappa^2 + 4n - n^2, \quad n = 0, 1, 2 \tag{38}$$

(see Morse and Feshbach [11]. Thus λ_1 is negative and we obtain the growth rate

$$\Gamma^2 \sim \kappa^2(3 - \kappa^2) \tag{39}$$

Note that this growth rate is independent of soliton speed V. Using a different approach, Degtyarev, Zakharov and Rudakov [5] study the instability in a little more detail and obtain

$$\Gamma^2 = \frac{\kappa^2(4 - \kappa^2)}{1 + \mu k_\perp^2}, \tag{40}$$

where μ is a constant. In both cases we have $\Gamma \sim \kappa$ for small κ.

3.2. $A\Gamma \gg 1$

The analysis for this case has been given by Schmidt [9]. From the sound equation we obtain

$$n = -\frac{2a}{A^2\Gamma^2}\frac{d^2}{dz^2}(f_x \operatorname{sech} z) \tag{41}$$

Equation (24) gives

$$Lf_x - \kappa^2 fx + \frac{2a^2}{A^2\Gamma^2}\frac{d^2}{dz^2}(f_x \operatorname{sech} z)\operatorname{sech} z = \Gamma g_x \tag{42}$$

The properties of the operator on the left-hand side are not easily obtained; matters are further complicated by the presence of the factor $(A^2\Gamma^2)^{-1}$. However, an approximate growth rate may be found by taking the product of eq. (42) and the second of equation (24) with sech z and integrating over all z. After requiring that $A\Gamma \gg 1$, $A \gg 1$, but $\kappa \ll 1$ and $\Gamma \ll 1$, it is possible to obtain

$$\Gamma^2 = \tfrac{1}{2}[-\kappa^4 + \sqrt{(\kappa^8 + 16\kappa^2 q/A^2)}] \tag{43}$$

where $q = \sqrt{[\frac{8}{15}(1 - V^2)]}$. For κ small this gives $\Gamma \sim \sqrt{\kappa}$. Expression (43) holds for $A\Gamma \gg 1$, which leads to the following restrictions on κ:

$$\frac{1}{2A\sqrt{q}} < \kappa < 2\sqrt{q} \tag{44}$$

The analysis given in Section 2.1 above was subject to the restriction $A\Gamma \ll 1$, $\kappa \ll 1$ and also $V^2 \ll 1$. It is not possible to give as careful an analysis of (24) and (25) when the restriction on V^2 is relaxed. For V^2 no longer small, the equations (24) and (25) remain strongly coupled and some other approach to soliton stability must be tried (see below from eq. (49) onwards). The opposite case, Section 3.2, is physically rather obscure as it applies to the situation where $A\Gamma \gg 1$ but $\Gamma \ll 1$. This suggests that the analysis given by Schmidt will only apply to extremely large amplitude solitons which the Zakharov equations, for physical reasons, cannot describe; it is likely that there is no region of validity for the result (43). However, for the smaller values of κ allowed by (44), it is possible that (43) might describe the growth.

In Figure 1 we compare the results of Schmidt (eq. (43)) with those found in Section 3.1 for the case $V = 0$. The two dispersion relations are seen not to be in disagreement: their regions of validity do not overlap. The interpolated curve matches the two curves I and II across the interval about $\kappa \approx 1/\sqrt{3A}$ where each curve is only approximate.

In a recent paper, Infeld and Rowlands [7] have studied eqns. (23) and (24) by using a multiple-scaling technique (an excellent review of scaling techniques and perturbation methods is given in Nayfeh [12]). In contrast to the results just given, they find that the Langmuir soliton described by the Zakharov, eqs. (1) and (2) with $\alpha = 1$ is *stable* to transverse perturbations;

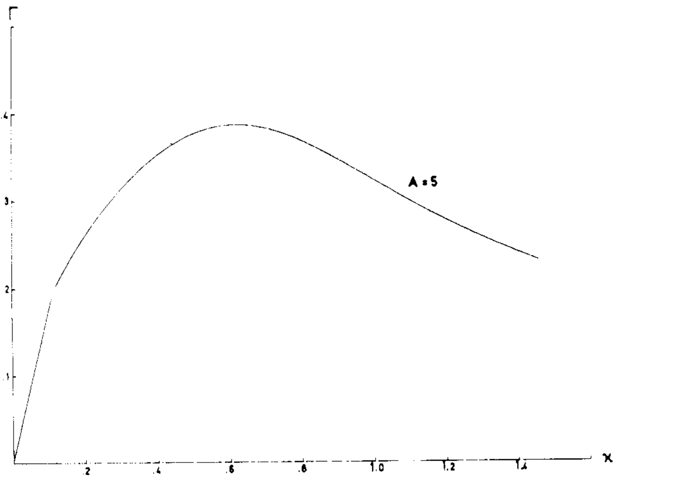

Fig. 1. Soliton instability growth rate ($\alpha = 1$). Curve I: $\Gamma \sim \kappa\sqrt{3}$ (Case a). Curve II: Γ given by eq. (43) (Case b).

they also attempt to point out errors in previously published work. What in fact they showed was that the *particular* perturbation considered in their work is stable. This particular perturbation is of the type

$$\delta E = \epsilon F_x(A\xi) \exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t) \cos k_\perp y \tag{45}$$

where ϵ is a small parameter (for purposes of calculation it is convenient to retain ϵ explicitly). The perturbation function F_x may be expanded as a series in $k_\perp$, if we take $k_\perp$ to be the small parameter ϵ. In terms of κ we have

$$F_x = F_0 + \kappa F_1 + \kappa^2 F_2 + \ldots \tag{46}$$

To lowest order in κ we find from (24) that

$$F_0 = \text{const.}\, x \frac{\mathrm{d}}{\mathrm{d}z} \operatorname{sech} z + \mathrm{i}\, \text{const.}\, x \operatorname{sech} z \tag{47}$$

This perturbation is equivalent in *effect* to an infinitesimal translation of the soliton $E_0(z)$, as we have

$$\begin{aligned} E_0(z+\kappa) &= E_0(z) + \kappa \frac{\partial}{\partial z} E_0(z) + 0(\kappa^2) \\ &= E_0(z) + \kappa \left\{ \frac{\mathrm{d}}{\mathrm{d}z} \operatorname{sech} z + \mathrm{i}\bar{k}_0 \operatorname{sech} z \right\} \\ &\quad \times A \exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t) + 0(\kappa^2) \end{aligned} \tag{48}$$

Note that the coefficient A is constant in this calculation.

In a similar way we find that the density perturbation n is also a simple translation. Both n and F_x are a very particular type of perturbation and it is not surprising that the Langmuir soliton is stable against small translations. To confirm this argument we note that it is possible to apply multiple scaling techniques (following Infeld and Rowlands) to the non-linear Schrödinger equation; the NLS soliton is predicted by such an analysis to be stable against transverse perturbations. However the analyses given by Zakharov [13], Zakharov and Rubenchik [4] and Yajima [14] of the NLS clearly show the NLS soliton to be *unstable* to transverse perturbation. In particular, Yajima shows the soliton to be stable to perturbations for which the coefficient A is held constant, but that if A is taken to be a slowly varying function of time and transverse position then the soliton is unstable. Motivated by this result we return to the problem of Langmuir soliton stability.

Consider a rather general form of perturbed soliton written in the following way

$$E = \mu A f(\xi) \exp(\mathrm{i}\theta + \mathrm{i}/2V(x - x_0)) \tag{49}$$

and

$$n = -2A^2 g(\xi) \tag{50}$$

where A and V are now slowly varying functions of time and transverse position and are specified by

$$\begin{aligned} A &= A_0 + a(y,t) \\ V &= V_0 + v(y,t) \\ \mu &= \sqrt{(1 - V^2)} \\ \xi &= A(x - x_0) \\ \frac{\partial x_0}{\partial t} &= V, \quad \frac{\partial \theta}{\partial t} = A^2 + V^2/4 \end{aligned} \tag{51}$$

A_0 and V_0 are the values of width and speed for the unperturbed soliton. Note particularly the form of the exponential factor in E and the corresponding change in θ. The argument of the exponential has been written in this way so that we have the relation

$$\frac{\partial}{\partial t}\left(\theta + \frac{V}{2}(x - x_0)\right) = A^2 - V^2/4 + \frac{1}{2}\frac{\partial v}{\partial t}(x - x_0) \tag{52}$$

In a recent paper, Hojo [15] has considered a perturbed Langmuir soliton written in a very similar way to eqs. (49)–(51) but without the important correction (52). To show this difference, consider the unperturbed soliton given in eqs. (4)–(6) and look at the complex phase factor

$$\exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t) = \exp(\mathrm{i}V/2x + \mathrm{i}(A^2 - V^2/4)t)$$

The important change made in (51) results from writing this phase factor in the form

$$\exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t) = \exp(\mathrm{i}(A^2 + V^2/4)t + \mathrm{i}V/2(x - Vt))$$

We see that the phase is then written in terms of time t and position relative to the centre of the soliton $(x - Vt)$. When a *perturbed* soliton is considered, the phase must in the same way be written in terms of time t and position relative to the centre of the *perturbed* soliton. If the position is measured relative to the motion of an unperturbed soliton, as is implicit in Hojo's calculation (see the eqs. (7)–(9) in his paper), then secularities appear in the differential equations for the perturbation functions $a(y,t)$ and $v(y,t)$. The equations which he obtains are then in error and the conclusion that Langmuir solitons are stable to transverse perturbations is also incorrect. We shall give some details of the calculation based on (49)–(51) both to show the correct equations and also to demonstrate clearly that Langmuir solitons are unstable.

We make the further substitution in eqs. (49)–(51)

$$f = f_0 + R + \mathrm{i}I, \quad g = g_0 + g_1$$

where f_0 and g_0 correspond to the unperturbed soliton. R, I and g_1 are real functions of ξ. Substituting into equation (1) and linearizing the equations assuming R, I and g_1 are small, we obtain the following system of coupled equations:

$$\begin{aligned} A_0^3(R'' - R + 2Rg_0 + 2f_0 g_1) &= -\alpha \frac{\partial^2 a}{\partial y^2}\frac{\partial}{\partial \xi}(\xi f_0) \\ &\quad + \tfrac{1}{2}\xi f_0 \frac{\partial v}{\partial t} + \alpha A_0^2 f_0' \frac{\partial^2 x_0}{\partial y^2} + \alpha \frac{V_0 A_0}{1 - V_0^2}\frac{\partial^2 v}{\partial y^2} f_0 \end{aligned} \tag{53}$$

$$\begin{aligned} A_0^3(I'' - I + 2Ig_0) &= -\frac{\partial a}{\partial t}\frac{\partial}{\partial \xi}(\xi f_0) + \frac{V_0 A_0}{1 - V_0^2}\frac{\partial v}{\partial t} f_0 \\ &\quad - \alpha A_0 f_0 \left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\xi}{2A_0}\frac{\partial^2 v}{\partial y^2} - \frac{V_0}{2}\frac{\partial^2 x_0}{\partial y^2}\right) \end{aligned} \tag{54}$$

$$\begin{aligned} A_0^3(1 - V_0^2)\frac{\partial^2}{\partial \xi^2}(g_1 - f_0 R) &= \left(\frac{\partial^2 a}{\partial t^2} - V_0^2 \frac{\partial^2 a}{\partial y^2}\right)(2g_0 + \xi g_0') \\ &\quad - 2A_0 V_0 \frac{\partial a}{\partial t}\left(2g_0' + \frac{\partial}{\partial \xi}(\xi g_0')\right) - A_0^2 g_0' \frac{\partial v}{\partial t} \\ &\quad - 2V_0 A_0 \frac{\partial^2 v}{\partial y^2} g_0 + V_0^2 A_0^2 \frac{\partial^2 x_0}{\partial y^2} g_0' \end{aligned} \tag{55}$$

The notation indicates derivatives with respect to ξ. Following the method used by Yajima [14], or Hojo [15], these three coupled equations may be reduced to just two equations for $a(y,t)$ and $v(y,t)$. For example, if we multiply both sides of eq. (54) by $f_0(\xi)$ and integrate over ξ from $-\infty$ to $+\infty$ we obtain

$$\frac{\partial a}{\partial t} + 2\alpha A_0\left(\frac{\partial^2 \theta}{\partial y^2} - V_0/2\,\frac{\partial^2 x_0}{\partial y^2}\right) - \frac{2V_0A_0}{1-V_0^2}\frac{\partial v}{\partial t} = 0$$

Differentiating with respect to t and using (51) gives the equation

$$\frac{\partial^2 a}{\partial t^2} + 4\alpha A_0^2\,\frac{\partial^2 a}{\partial y^2} - \frac{2A_0V_0}{1-V_0^2}\frac{\partial^2 v}{\partial t^2} = 0 \tag{56}$$

By a similar procedure involving eqs. (53) and (55) we obtain a second equation

$$24V_0A_0\frac{\partial^2 a}{\partial t^2} + (3 - 3V_0^2 + 8A_0^2)\frac{\partial^2 v}{\partial t^2} - 4A_0^2(\alpha(1-V_0^2) + 2V_0^2)\frac{\partial^2 v}{\partial y^2} = 0 \tag{57}$$

The coupled system (56), (57) is very similar to that given by Hojo [15] if we set $\alpha = 1$. However, he has an additional term in (56) of the form $t(\partial^3 v/\partial y^2\,\partial t)$ which arises from an error in the assumed form of the perturbed soliton; it is this additional term that leads him to predict soliton stability. Consider now the system (57), (58) and assume that

$$a = \bar{A}\exp(\omega t)\cos(k_\perp y)$$
$$v = \bar{V}\exp(\omega t)\cos(k_\perp y)$$

The condition for non-zero values of $\bar{A}$, $\bar{V}$ is then given by

$$(\mu_1 + \lambda_2)\left(\frac{\omega}{k}\right)^4 + \left(\frac{\omega}{k}\right)^2(\mu_2 - \lambda_1\mu_1) - \lambda_1\mu_2 = 0 \tag{58}$$

where

$$\lambda_1 = 4\alpha A_0^2,\quad \lambda_2 = \frac{2V_0A_0}{1-V_0^2}$$

$$\mu_1 = \frac{3 - 3V_0^2 + 8A_0^2}{24V_0A_0},\quad \mu_2 = \frac{A_0}{6V_0}(\alpha(1-V_0^2) + 2V_0^2)$$

Equation (58) always has a real solution for ω/k and hence a Langmuir soliton is always unstable to transverse perturbations. However, the growth rate is dependent on the soliton speed as well as the amplitude. We may write

$$\omega = 2A_0\sqrt{\alpha}k \times C(A_0, V_0)$$

where $C(A_0, V_0)$ is independent of k. $C(A_0, V_0)$ is plotted in Fig. 2 as a function of V_0 for various values of A_0 and with α set equal to 1. Figure 3 shows $C(A_0, V_0)$ for the case $\alpha = 10$. When α is greater than 1 we see that the instability growth rate is larger but not qualitatively different from the case $\alpha = 1$ studied by previous authors. It is important to notice that the curl-term in (1) here plays a small role only because we have assumed that the perturbed field has just one component. More generally, the soliton might be perturbed with components of field transverse to the one-dimensional soliton field; in such a problem the curl term will be much more significant. Further, the curl-free assumption used by many authors will break down when the length scale of the perturbation becomes comparable to the soliton width ($k_\perp \sim 1$). These problems will be considered further in the next section. We now compare the growth rate $\omega = 2A_0kC(A_0, V_0)$ (for $\alpha = 1$) with previous author's work. Zakharov and Rubenchik [4] consider transverse stability of nonlinear Schrödinger solitons and obtained a growth rate $2A_0k$, in agreement with the value of a stationary Langmuir soliton. Degtyarev, Zakharov and Rudakov [5] considered a stationary Langmuir soliton of the curl-free Zakharov equations and obtained the result (40), which is in reasonable agreement with (58).

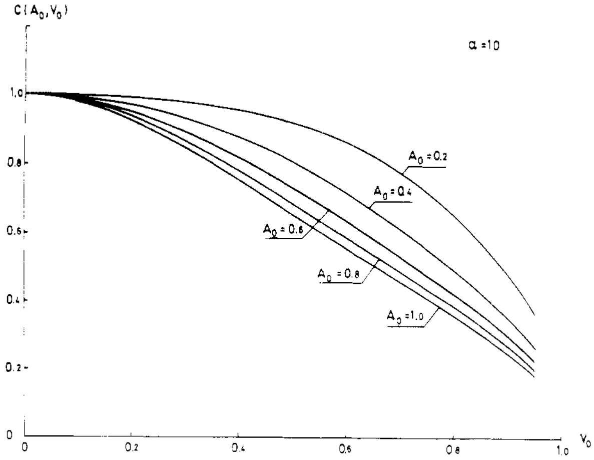

Fig. 3. The coefficient C as a function of $V(\alpha = 10)$.

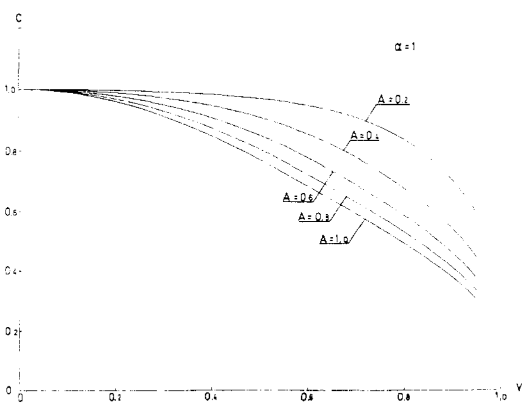

Fig. 2. The coefficient C as a function of $V(\alpha = 1)$.

Most of the published calculations have considered Langmuir collapse and are not relevant to our discussion. However, the stability of two-dimensional planar solitons has been considered by Denavit, Pereira and Sudan [16] and Pereira, Sudan and Denavit [6]; any analytic results on soliton stability in two dimensions should be compared with their work. In the study of Pereira, Sudan and Denavit [6] the curl-free equations were considered with initial condition

$$E(x, 0) = \sqrt{2}A\,\mathrm{sech}\,Ax\exp(ik_0x),\, n(x, 0)$$
$$n(x, 0) = -2A^2\,\mathrm{sech}^2 Ax(1 + 0.2\cos k_\perp y),$$

that is, the initial condition is a soliton with a transversely perturbed density. The curl-free Zakharov equations were solved using a Fourier representation and imposing periodic boundary conditions with periodicity lengths L_x and $L_y = 2\pi/k_\perp$. For typical values of the parameters ($A = 2$) the growth rate of a perturbed stationary soliton was found to have the following key features:

(a) up to $k_\perp \sim 1$, the growth rate γ was approximately constant at $\gamma = 0.7$–0.8;

(b) for $k_\perp \gtrsim 1$, γ decreased sharply to zero at $k_\perp \approx 1.3$.

(c) by varying A it was found that $\gamma \propto A^2$.

These results are in disagreement with the theory presented a moment ago which suggests that $\gamma \sim 2Ak_\perp$. The reason for this appears to be in the method of computer solution of the curl-free Zakharov equations: the periodic boundary conditions play an important role. Abdulloev et al. [17] have already considered the problem of boundary conditions in the context of one-dimensional soliton interactions. They point out that if we consider, say, a perturbed soliton emitting sound waves, the periodicity condition requires that the ion-sound waves re-enter the system and interact with the soliton. This comment applies to the later stages of the transverse instability given by Pereira, Sudan and Denavit [6]; (see, for example, their Fig. 10) but not to the initial stage where γ is found to be constant. The explanation of constant growth rate that they give is spurious; see Appendix A.

The most likely cause of the rather unusual results quoted above can be seen from the initial condition considered numerically by Pereira, Sudan and Denavit: the perturbation of the ion-density is 20% of the unperturbed soliton state. Nonlinearities are likely to be of dominant importance immediately and will completely mask any linear growth rate. The numerical experiment should be repeated with a perturbation several orders of magnitude smaller if a comparison with linear growth rates is desired; we expect that such a calculation would give the linear growth rate given above.

4. Growth rate of soliton instability for the full equations

We now return to eqs. (16) to (19) and consider the case where $\alpha \neq 1$, but κ is small. These equations are too complicated to solve exactly but it is possible to obtain estimates of the growth rate and also to demonstrate the importance of the terms multiplied by the (large) parameter α. First consider (18) and (19) supposing that $\kappa \ll 1$. Then we have the equations

$$Kf_r = \Gamma_r g_r + 2\bar{k}_0\beta \frac{dg_r}{dz} \tag{60}$$

$$Kg_r = -\Gamma_r f_r - 2\bar{k}_0\beta \frac{df_r}{dz} \tag{61}$$

In this case of cylindrical geometry, for $\kappa \ll 1$ the radial field has become decoupled from the longitudinal field and the growth rate Γ_r for the radial field is different from that in the longitudinal direction. Equations (60) and (61) are most conveniently expressed in terms of $F_r = f_r + ig_r$:

$$KF_r + 2\bar{k}_0\beta i \frac{dF_r}{dz} = -i\Gamma_r F_r, \tag{62}$$

which for large α may be written approximately as

$$\left(\frac{d}{dz} + i\bar{k}_0 \frac{\beta}{\alpha}\right)^2 F_r = -\frac{1}{\alpha}[i\Gamma_r + 2\,\mathrm{sech}^2 z - 1 - \bar{k}_0^2]F_r \tag{63}$$

Put $F_r = g(z)\exp[-i\bar{k}_0(\beta/\alpha)z]$ so that eq. (63) becomes

$$\frac{d^2g}{dg^2} + \frac{1}{\alpha}[i\Gamma_r - (1 - k_0^2) + 2\,\mathrm{sech}^2 z]\,g = 0 \tag{64}$$

which is the same form as eq. (26). Thus the solution for $g(z)$ is just

$$g(z) = \text{constant} \times P_\nu^\mu(\tanh z) \tag{65}$$

where

$$\nu(\nu+1) = \frac{2}{\alpha} \quad \text{and} \quad \mu^2\alpha = 1 + \bar{k}_0^2 - i\Gamma_r \tag{66}$$

Since we require that $g(z) \to 0$, as $z \to \pm\infty$, the growth rate Γ_r must be purely imaginary (since $\mu + \nu =$ zero). This calculation is accurate to order κ^0; once terms of order κ and higher are included the radial field eqs. (18) and (19) become linked to the longitudinal. Now return to eqs. (16) and (17) and substitute the approximate form (65) for the radial field. To make calculations simple we restrict attention first to the static regime in which

$$n = -\frac{4}{a} f_x\,\mathrm{sech}\,z \tag{67}$$

Then eqs. (16) and (17) may be written as

$$L_1 f_x = -\Gamma g_x - \beta\kappa\omega_R \tag{68}$$

$$L_0 g_x = \Gamma f_x - \beta\kappa\omega_I \tag{69}$$

(compare with (31) and (32)), where we have used the notation

$$\omega_R + i\omega_i = \frac{dF_r}{dz} + i\bar{k}_0 F_r \tag{70}$$

while ω_R, ω_I are real. In eq. (70) we substitute the approximate radial field calculated above. The growth rate Γ is the growth rate in the longitudinal direction and will be shown to be real. Substituting for F_r we obtain approximately

$$\frac{dF_r}{dz} + i\bar{k}_0 F_r \approx \frac{dg}{dz}\exp(i\bar{k}_0 z) \tag{71}$$

since $\alpha \ll 1$. Now operate on (69) with L_1 and substitute from (68) to obtain

$$L_1 L_0 g_x = -\Gamma^2 g_x - \beta\kappa\Gamma\omega_R - \beta\kappa L_1\omega_I \tag{72}$$

Following the procedure for (36) and (37) we obtain

$$\frac{\langle g_x L_1 L_0 g_x\rangle}{\langle g_x^2\rangle} = -\Gamma^2 - \beta\kappa\Gamma\frac{\langle g_x\omega_R\rangle}{\langle g_x^2\rangle} - \beta\kappa\frac{\langle g_x L_1\omega_I\rangle}{\langle g_x^2\rangle} \tag{73}$$

This equation gives approximately

$$\Gamma^2 + \beta\kappa\Gamma\tilde{\omega}_R + \beta\kappa\tilde{\omega}_I + \lambda_0\lambda_1 = 0 \tag{74}$$

where λ_0 and λ_1 are the eigenvalues given by eq. (38) and $\tilde{\omega}_R$, $\tilde{\omega}_I$ are defined by

$$\tilde{\omega}_R = \langle g_x\omega_R\rangle/\|g_x\|^2 \tag{75}$$

$$\tilde{\omega}_I = \langle g_x L_1\omega_I\rangle/\|g_x\|^2 \tag{76}$$

To consider the approximate relation (74) further we must evaluate (75) and (76). They need to be evaluated only to lowest order in κ so we take g_x to be the solution of (see (69)) $L_0 g_x = 0$, that is, $g_x = \mathrm{sech}\,z$. Also we set

$$\lambda_0\lambda_1 \approx -3\kappa^2 \tag{77}$$

From eq. (74) we have

$$\Gamma \approx \tfrac{1}{2}\{-\beta\kappa\tilde{\omega}_R + \sqrt{(\beta^2\kappa^2\tilde{\omega}_R^2 + 12\kappa^2 - 4\beta\kappa\tilde{\omega}_I)}\}. \tag{78}$$

The exact values of the quantities $\tilde{\omega}_R$ and $\tilde{\omega}_I$ are not important; however their signs are. We evaluate $\tilde{\omega}_R$ and $\tilde{\omega}_I$ approximately in Appendix B, and their values are found to be

$$\tilde{\omega}_R = \mathrm{Re}\left\{\pi(\mu + i\bar{k}_0)\sec\frac{\pi}{2}(\mu + i\bar{k}_0)/\Gamma(-\mu)\right\} \tag{79}$$

$$\tilde{\omega}_I = -\mathrm{Im}\left\{\pi(\mu + i\bar{k}_0)[1 - (\mu + i\bar{k}_0)^2]\sec\frac{\pi}{2} \times (\mu + i\bar{k}_0)/6\Gamma(-\mu)\right\} \tag{80}$$

Note that $\tilde{\omega}_I$ is negative, so that Γ is real and there is a real, physical instability. The dependence of Γ on κ is slightly more complicated to see. Consider first the values of κ for which $\beta\kappa \ll 1$. Since β is large we have

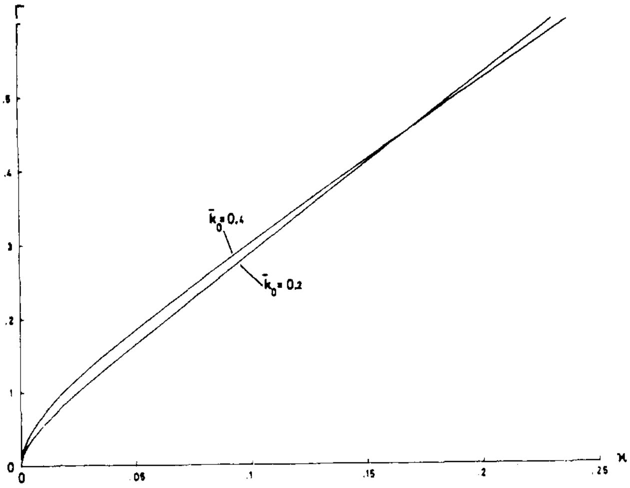

Fig. 4. Growth rate as a function of κ for two different values of $\bar{k}_0$ ($\alpha = 10$).

$$\Gamma \sim \sqrt{(\beta\kappa)}\sqrt{(-4\omega_I)} \tag{81}$$

For larger values of κ the growth rate is very close to a linear function of κ. See Fig. 4.

The exact value of Γ given in (81) is probably not to be relied on exactly, as the calculation has been very approximate. However, the importance of the term $\alpha\nabla\times\nabla\times \boldsymbol{E}$ in (1) is clear; inclusion of the term produces a very different growth rate from that calculated for the curl-free equations in Section 3. Note that the eq. (78) reduces to $\Gamma \sim \sqrt{3\kappa}$ (in agreement with (39)) when we set $\alpha = 1$ that is, $\beta = 0$. It is of interest to observe that the soliton in both cylindrical and planar geometry, when described by the *full* equations (1) and (2), is unstable to the very simple perturbations given in (8) and (9); a more complicated perturbation of the type (49) is not needed to demonstrate instability.

The growth rate given in (81) was obtained using the static approximation and hence can only apply to very slowly moving solitons. It would be desirable to analyze the system (16)–(19) without the restriction $V^2 \ll 1$. It was shown in Section 3 that when $\alpha = 1$ and just one component of field is considered, that the growth rate decreases as the solitons move faster. It seems likely that this will also be true if the full system (16)–(19) is analyzed more carefully, but more work is needed on this point.

The instability with growth rate given in (78) arises in the following way. To lowest order in κ we find that the transverse component of field just oscillates in time. Because the transverse and longitudinal fields are coupled through the form $\alpha[\nabla\times[\nabla\times \boldsymbol{E}]]$, the longitudinal component is driven by the oscillating transverse component and grows with a real growth rate. A more complete analysis of eqs. (16) to (19) should give a real transverse growth rate as well as a real longitudinal Γ. There has been one computer calculation which is relevant to this discussion: Zakharov, Mastryukov and Synakh [18] solved eqs. (1) and (2) numerically with $\alpha = 10$ and using cylindrical geometry. The result of their calculation is given in Fig. 5. The collapse is very anisotropic and the ratio of characteristic lengths l_z and l_r in the longitudinal and radial directions respectively is

$$l_z/l_r \sim \tfrac{1}{6} \tag{82}$$

This difference in collapse rates suggests that the transverse growth rate of soliton perturbations is considerably less than the longitudinal growth rate.

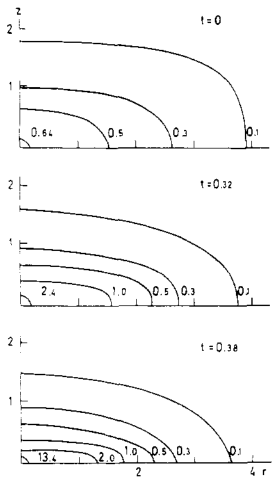

Fig. 5. Collapse of initial disturbance in axisymmetric case; collapse described by eqs. (1) and (2) with $\alpha = 10$. Lines are constant values of $|E|^2$. After V. E. Zakharov, A. F. Mastryukov and V. S. Synakh 1975 Fiz. Plasmy **1**, 614 [18].

Acknowledgements

We express our thanks to Dr. J. Gibbons and Dr. S. G. Thornhill for many helpful discussions and to the Rhodes Trust for the award of a Rhodes Scholarship to one of us (M.J.W.) during the tenure of which this research was carried out.

Appendix A

In this appendix we consider the analysis of a particular perturbation initial value problem given by Pereira, Sudan and Denavit ([6], Appendix). Since they claim to obtain a soliton instability growth rate independent of the perturbation wavenumber $k_\perp$ and hence in agreement with their computer results, it is important to see if their analysis is correct. If it is not, then their numerical results contradict all theoretical analyses. This would lend support to the view expressed in the main text of the present paper that the numerical results are determined by the application of unsuitable initial conditions. We now show that their analysis is indeed incorrect.

Consider the initial problem where the electric field $E(x, 0)$ is that of soliton but the density is perturbed away from the corresponding soliton values:

$$n(x, 0) = -|E(x, 0)|^2\{1 + 2\epsilon\cos k_\perp y\} \tag{A.1}$$

The initial tendency of the system is for ions to rush in and fill the excess density dip; this movement will be described by the sound equation with negligible driving term:

$$n_{tt} - \nabla^2 n \approx 0 \tag{A.2}$$

Hence we have the ion-density perturbation $N_1(x, y, t)$ given by

$$N_1(x, y, t) = \tfrac{1}{2}[N_1(x-t, y, 0) + N_1(x+t, y, 0)] \tag{A.3}$$

A simple expansion around $x = 0$ then gives

$$N_1(0, y, t) = N_1(0, y, 0) + t^2\left\{\frac{\partial^2}{\partial x^2}N_1(x, y, 0)|_{x=0}\right\} + 0(t^4) \tag{A.4}$$

This expression shows that for small times we have $N_1 \propto t^2$ and this result is confirmed by the computer results of Pereira, Sudan and Denavit. However, it is important to recognize that (A.4) holds only for times when the ions are moving almost freely to fill in the excess density dip in (A.1). Such a result would be obtained for a small departure from *any* density dip in which the gas pressure is balanced by the electric field pressure. During the early times for which (A.4) is valid, there will also be an increase in $|E|^2$ until n and $-|E|^2$ are in balance. Both of these processes have nothing to do with the stability of a soliton to transverse perturbations. Pereira, Sudan and Denavit consider the development of the electric field from an initial, stationary ($v = 0$) soliton by writing E in the form

$$E(x, y, t) = E_s(x) \exp(iS(x, y, t) + iA^2 t) \tag{A.5}$$

where $E_s(x)$ is a one-dimensional soliton profile. Equation (A.5) is substituted into the curl-free basic equations ($\alpha = 1$) and then linearized to obtain

$$\frac{\partial S}{\partial t} = i\frac{\partial^2 S}{\partial x^2} - N_1 \tag{A.6}$$

To obtain this equation, Pereira, Sudan and Denavit have *neglected y derivatives* and considered the neighbourhood about $x = 0$. The initial conditions are (A.1) and the requirement that $\partial n/\partial t = 0$ at $t = 0$.

Consider now the development of the ion-density perturbation N_1. The linearized sound equations gives

$$\frac{\partial^2 N_1}{\partial t^2} - \frac{\partial^2 N_1}{\partial x^2} = \frac{\partial^2}{\partial x^2}\{|E_s(x)|^2\, 2\,\mathrm{Im}\,(S(x, y, t))\} \tag{A.7}$$

For small times t we have $S \approx 0$ and N_1 is given by (A.4). From (A.6) we obtain

$$S_1 = -\int_0^t N_1(x, y, \tau)\, d\tau \tag{A.8}$$

where S_1 is an approximation to S. Since S_1 is real it makes no contribution in (A.7). Thus the perturbation N_1 may be used in calculating the second approximation S_2 to S. After a little calculation we obtain for S near $x = 0$

$$S(x, y, t) \simeq S_1(x, y, t) + \tfrac{1}{2}\{2N_1(0, y, 0) - [N_1(x - t, y, 0) + N_1(x + t, y, 0)]\}. \tag{A.9}$$

From (A.5) we obtain the field energy near $x = 0$ as

$$|E|^2 = |E_s(x)|^2 \exp(-\{2N_1(0, y, 0) - [N_1(x - t, y, 0) + N_1(x + t, y, 0)]\}) \tag{A.10}$$

The argument of the exponential is approximately $2t^2\, \partial^2 N_1/\partial t^2$, but for N_1 we have $\partial^2 N_1/\partial t^2 \simeq \partial^2 N_1/\partial x^2$. Thus equation (A.10) becomes

$$|E|^2 \simeq |E_s(x)|^2 \left\{1 + 2t^2\left(\frac{\partial^2 N_1}{\partial x^2}\right)\right\}. \tag{A.11}$$

An expansion of N_1 about $x = 0$ from (A.1) gives

$$|E|^2 \simeq |E_s(x)|^2\{1 + 8\epsilon A^4 t^2 \cos k_\perp y\} \tag{A.12}$$

where an explicit form for $E_s(x)$ has been used.

Pereira, Sudan and Denavit conclude from (A.12) that the instability growth rate γ is then given by

$$\gamma^2 = 8\epsilon A^2$$

Note that this is not a linear growth rate but comes from an exponential growth. To arrive at (A.11) it has been necessary to use

$$\frac{\partial^2 N_1}{\partial t^2} \simeq \frac{\partial^2 N_1}{\partial x^2}$$

at several points and so the equation refers to the free propagation of the perturbation. It does *not* describe a soliton instability. Perhaps the clearest way to check this is to substitute a form of N_1 which contains no y-dependence at all. The arguments of Pereira, Sudan and Denavit would then give a soliton unstable in one dimension; this is known to be incorrect.

Appendix B

In this Appendix we evaluate the expressions given in eqs. (74) and (75). For $\kappa \ll 1$ we have $g_x \simeq \mathrm{sech}\, z$ so $\|g_x\|^2 = 2$. The expressions for $\tilde{\omega}_R$ and $\tilde{\omega}_I$ are then

$$\tilde{\omega}_R = \mathrm{Re}\,\tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{sech}\, z\, \frac{d}{dz} P_\nu^\mu(\tanh z) \exp(i\bar{k}_0 z)\, dz$$

$$= \mathrm{Re}\int_{-1}^{+} \sqrt{(1-\theta^2)}\,\frac{dP_\nu^\mu(\theta)}{d\theta}\left(\frac{1+\theta}{1-\theta}\right)^{i\bar{k}_0/2} d\theta \tag{B.1}$$

and

$$\tilde{\omega}_I = \mathrm{Im}\,\tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{sech}\, z L_1\left\{\frac{d}{dz}P_\nu^\mu(\tanh z)\exp(i\bar{k}_0 z)\right\} dz \tag{B.2}$$

Since L_1 is a self-adjoint operator and $L_1\, \mathrm{sech}\, z = -4\,\mathrm{sech}^2 z$ we have from (B.2)

$$\tilde{\omega}_I = -\mathrm{Im}\, 2\int_{-\infty}^{\infty} \mathrm{sech}^3 \frac{d}{dz} P_\nu^\mu(\tanh z)\exp(i\bar{k}_0 z)\, dz$$

$$= -\int_{-1}^{+1} (1-\theta^2)^{3/2}\frac{d}{d\theta}P_\nu^\mu(\theta)\left(\frac{1+\theta}{1-\theta}\right)^{i\bar{k}_0/2} d\theta \tag{B.3}$$

To evaluate (B.1) and (B.3) we use the following property of the associated Legendre functions:

$$\sqrt{(1-\theta^2)}\frac{d}{d\theta}P_\nu^\mu(\theta) = \tfrac{1}{2}(\nu+\mu)(\nu-\mu+1)P_\nu^{\mu+1}(\theta) + \tfrac{1}{2}P_\nu^{\mu+1}(\theta) \tag{B.4}$$

For the problem under consideration we have $\nu(\nu+1) = 2/\alpha$ where α is large. Thus

$$\nu \simeq 2/\alpha \quad \text{or} \quad -(1 + 2/\alpha) \tag{B.5}$$

The associated Legendre function $P_\nu^\mu(\theta)$ may be defined by the expression (see Erdelyi [8], Vol. I, p. 122)

$$P_\nu^\mu(\theta) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+\theta}{1-\theta}\right)^{\mu/2} F(-\nu, \nu+1, 1-\mu; \tfrac{1}{2}(1-\theta))$$

where F is the hypergeometric function. For $\nu \ll 1$ we have the approximate form

$$P_\nu^\mu(\theta) \approx \frac{1}{\Gamma(1-\mu)}\left(\frac{1+\theta}{1-\theta}\right)^{\mu/2} \tag{B.6}$$

By substituting (B.4) into (B.1) and (B.3), we see that it is necessary to evaluate integrals of the type

$$\int_{-1}^{+1} P_\nu^\mu(\theta)\left(\frac{1+\theta}{1-\theta}\right)^{\lambda/2} d\theta$$

and

$$\int_{-1}^{+1} (1-\theta^2)P_\nu^\mu(\theta)\left(\frac{1+\theta}{1-\theta}\right)^{\lambda/2} d\theta$$

From (B.6) these integrals may be written (approximately) in the general form

$$\int_{-1}^{+1} (1+\theta)^p (1-\theta)^q\, d\theta \tag{B.7}$$

where p, q are in general complex. Making the change of variable from θ to $t = \frac{1}{2}(1+\theta)$ we see that (B.7) becomes

$$2^{p+q+1}\int_0^1 t^p(1-t)^q\,\mathrm{d}t = 2^{p+q+1}B(1+p, 1+q) \tag{B.8}$$

where B is the beta function and takes the value

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{B.9}$$

With this information we consider now the evaluation of an integral of the form

$$\begin{aligned}\int_{-1} P_\nu^\mu(\theta)\left(\frac{1+\theta}{1-\theta}\right)^{\lambda/2}\mathrm{d}\theta &\simeq \frac{1}{\Gamma(1-\mu)}\int_{-1}^{+1}\left(\frac{1+\theta}{1-\theta}\right)^{\lambda/2+\mu/2}\mathrm{d}\theta\\ &= \frac{2}{\Gamma(1-\mu)}\int_0^1 t^{\lambda/2+\mu/2}(1-t)^{-\lambda/2-\mu/2}\,\mathrm{d}t\\ &= \frac{2}{\Gamma(1-\mu)}B\left(1+\frac{\lambda+\mu}{2}, 1-\frac{\lambda+\mu}{2}\right)\\ &= \frac{2}{\Gamma(1-\mu)}\Gamma\left(1+\frac{\lambda+\mu}{2}\right)\Gamma\left(1-\frac{\lambda+\mu}{2}\right)\\ &= \frac{\lambda+\mu}{\Gamma(1-\mu)}\Gamma\left(\frac{\lambda+\mu}{2}\right)\Gamma\left(1-\frac{\lambda+\mu}{2}\right)\end{aligned} \tag{B.10}$$

The gamma function has the property that

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec}\pi z \tag{B.11}$$

Substituting (B.11) into (B.10) we obtain

$$\begin{aligned}I(\nu,\mu,\lambda) &\equiv \int_{-1}^{+1} P_\nu^\mu(\theta)\left(\frac{1+\theta}{1-\theta}\right)^{\lambda/2}\mathrm{d}\theta\\ &\simeq \frac{\pi(\lambda+\mu)}{\Gamma(1-\mu)}\operatorname{cosec}\frac{\pi}{2}(\lambda+\mu)\end{aligned} \tag{B.12}$$

We can now evaluate $\tilde{\omega}_\mathrm{R}$. Substituting (B.4) into (B.1) and writing the result in terms of $I(\nu,\mu,\lambda)$ defined in (B.12) we obtain

$$\begin{aligned}\tilde{\omega}_\mathrm{R} = \mathrm{Re}\,\{&\tfrac{1}{2}(\nu+\mu)(\nu-\mu+1)I(\nu,\mu-1,\mathrm{i}\bar{k}_0)\\ &+\tfrac{1}{2}I(\nu,\mu+1,\mathrm{i}\bar{k}_0)\}\end{aligned} \tag{B.13}$$

For $\nu \ll 1$ we may evaluate the terms inside the bracket by using (B.12) to obtain

$$\tilde{\omega}_\mathrm{R} = \mathrm{Re}\left\{\frac{\pi(\mu+\mathrm{i}\bar{k}_0)}{\Gamma(-\mu)}\sec\frac{\pi}{2}(\mu+\mathrm{i}\bar{k}_0)\right\}. \tag{B.14}$$

Expanding the secant function and using the property that μ is real, we obtain finally

$$\begin{aligned}\tilde{\omega}_\mathrm{R} &\simeq \frac{\pi}{\Gamma(-\mu)}\\ &\times\left(\frac{\mu\cos\mu(\pi/2)\cosh\bar{k}_0(\pi/2)-\bar{k}_0\sin\mu(\pi/2)\sinh\bar{k}_0(\pi/2)}{\cos^2\mu(\pi/2)\cosh^2\bar{k}_0(\pi/2)+\sin^2\mu(\pi/2)\sinh^2 k_0(\pi/2)}\right)\end{aligned} \tag{B.15}$$

We now turn to the evaluation of $\tilde{\omega}_\mathrm{I}$ given in (B.3). Substituting (B.4) into (B.3) gives

$$\begin{aligned}\tilde{\omega}_\mathrm{I} = -\mathrm{Im}\,\{&\tfrac{1}{2}(\nu+\mu)(\nu-\mu+1)K(\nu,\mu-1,\mathrm{i}\bar{k}_0)\\ &+\tfrac{1}{2}K(\nu,\mu+1,\mathrm{i}\bar{k}_0)\}\end{aligned} \tag{B.16}$$

where the function K is defined by

$$K(\nu,\mu,\lambda) \equiv \int_{-1}^{+1}(1-\theta^2)P_\nu^\mu(\theta)\left(\frac{1+\theta}{1-\theta}\right)^{\lambda/2}\mathrm{d}\theta$$

The quantity K may be evaluated approximately by using (B.6) and subsequent relations to obtain

$$K(\mu,\nu,\lambda) \simeq \frac{4}{\Gamma(1-\mu)}B\left(2-\frac{\lambda+\mu}{2}, 2+\frac{\lambda+\mu}{2}\right) \tag{B.17}$$

$$= \frac{\pi(\lambda+\mu)[4-(\lambda+\mu)^2]\operatorname{cosec}(\pi/2)(\lambda+\mu)}{6\Gamma(1-\mu)} \tag{B.18}$$

Substituting (B.18) into (B.16) and carrying out some algebraic manipulation gives

$$\begin{aligned}\tilde{\omega}_\mathrm{I} &\simeq -\mathrm{Im}\,\frac{\pi(\mu+\mathrm{i}\bar{k}_0)[1-(\mu+\mathrm{i}k_0)^2]\sec(\pi/2)(\mu+\mathrm{i}\bar{k}_0)}{6\Gamma(-\mu)}\\ &= -\frac{\pi}{6\Gamma(-\mu)}\\ &\times\left\{\frac{(\mu+3\mu\bar{k}_0^2-\mu^3)\sin(\pi\mu/2)\sinh(\pi\bar{k}_0/2)+(\bar{k}_0-3\mu^2\bar{k}_0+\bar{k}_0^3)\cos(\pi\mu/2)\cosh(\pi\bar{k}_0/2)}{\cos^2(\pi\mu/2)\cosh^2(\pi\bar{k}_0/2)+\sin^2(\pi\mu/2)\sinh^2(\bar{k}_0\pi/2)}\right\}\end{aligned} \tag{B.19}$$

From (B.19 and B.15) we obtain the ratio

$$\begin{aligned}-\frac{\tilde{\omega}_\mathrm{I}}{\tilde{\omega}_\mathrm{R}} &= \frac{1}{6}\\ &\times\left(\frac{(\mu+3\mu\bar{k}_0^2-\mu^3)\sin(\pi\mu/2)\sinh(k_0\pi/2)+(\bar{k}_0-3\mu^2\bar{k}_0+\bar{k}_0^3)\cos(\mu\pi/2)\cosh(\bar{k}_0\pi/2)}{\mu\cos(\pi\mu/2)\cosh(\pi\bar{k}_0/2)-\bar{k}_0\sin(\pi\mu/2)\sinh(\bar{k}_0\pi/2)}\right)\end{aligned} \tag{B.20}$$

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