COMSW4731_001_2021_3

Computer Vision I: First Principles

Homework #2

计算机视觉

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Problem 1

The given matrix is:

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$$

Eigenvalues & Extreme Values of Moment of Inertia

$$A - \lambda I_2 = \begin{bmatrix} a-\lambda & \frac{b}{2} \\ \frac{b}{2} & c-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = (a-\lambda)(c-\lambda) - \frac{b^2}{4} = (ac - \frac{b^2}{4}) - (a+c)\lambda + \lambda^2 = 0$$

and

$$\lambda_{1,2} = \frac{(a+c) \pm \sqrt{(a+c)^2 - (4ac - b^2)}}{(2ac - \frac{b^2}{2})} = \frac{(a+c) \pm \sqrt{(a-c)^2 + b^2}}{2ac - \frac{b^2}{2}}$$

According to the lecture slides, we have:

$$E = a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta$$

$$= -\frac{1}{2}a(1 - 2\sin^2\theta) - \frac{1}{2}b(2\sin\theta\cos\theta) + \frac{1}{2}c(2\cos^2\theta - 1) + \frac{a}{2} + \frac{c}{2}$$

$$= -\frac{1}{2}a\cos 2\theta - \frac{1}{2}b\sin 2\theta + \frac{1}{2}c\cos 2\theta + \frac{a+c}{2}$$

If E is at its extreme values, we have

$$\tan 2\theta = \frac{b}{a-c}$$

and since

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

if let $\tan\theta = x$, we have:

$$\frac{b}{a-c} = \frac{2x}{1-x^2}$$

and

$$b - bx^2 = (2a-2c)x \Rightarrow bx^2 + (2a-2c)x - b = 0$$

The solutions are

$$x_{1,2} = \frac{(2c-2a) \pm \sqrt{(2a-2c)^2 + 4b^2}}{2b} = \frac{(c-a) \pm \sqrt{(a-c)^2 + b^2}}{b}$$

Since

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}, \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

We can rewrite the equation of E as:

$$E = -\frac{1}{2}a\cos 2\theta - \frac{1}{2}b\sin 2\theta + \frac{1}{2}c\cos 2\theta + \frac{a+c}{2}$$

$$= -\frac{1}{2}a\frac{1-\tan^2\theta}{1+\tan^2\theta} - \frac{1}{2}b\frac{2\tan\theta}{1+\tan^2\theta} + \frac{1}{2}c\frac{1-\tan^2\theta}{1+\tan^2\theta} + \frac{a+c}{2}$$

$$= -\frac{a}{2}\frac{1-x^2}{1+x^2} - \frac{b}{2}\frac{2x}{1+x^2} + \frac{c}{2}\frac{1-x^2}{1+x^2} + \frac{a+c}{2}$$

This is an equation with x solved before. Since there are two values of x, we can plug in two values and simplify.

By calculation, we have:

$$E_{1,2} = \frac{(a+c) \pm \sqrt{(a-c)^2 + b^2}}{2ac - \frac{b^2}{2}}$$

This is same as the eigenvalues we've solved before. Therefore, we have shown that the extreme values of E are the eigenvalues of the given matrix.

E is real

The matrix A above is symmetric ($A = A^T$). Because the eigenvalues of a symmetric matrix are real, we thus know that $\lambda_{1,2} = E_{1,2}$ are real, too. It is then proofed that E is real.

E is non-negative

Since for $E_{1,2}$:

$$E_1 = \frac{(a+c) + \sqrt{(a-c)^2 + b^2}}{2ac - \frac{b^2}{2}} > \frac{(a+c) - \sqrt{(a-c)^2 + b^2}}{2ac - \frac{b^2}{2}} = E_2$$

We consider E_2 in this case.

If $E_2 < 0$, we have either $(a+c) + \sqrt{(a-c)^2 + b^2} < 0$ or $2ac - \frac{b^2}{2} < 0$ (and only one of them is true at the same time). Suppose $(a+c) + \sqrt{(a-c)^2 + b^2} < 0$ and we have

$$(a+c) < \sqrt{(a-c)^2 + b^2}$$

and since $a, c > 0$ because of their definition, we have $a + c > 0$ and

$$(a+c)^2 < (a-c)^2 + b^2$$

$$2ac < -2ac + b^2$$

and

$$4ac < b^2$$

However, we've already suppose that in this case, $2ac - \frac{b^2}{2} > 0$. That is,

$$4ac > b^2$$

This creates a contradiction. Therefore, E_2 cannot be negative, which makes $E_{1,2} \geq 0$.

4ac$\geq b^2$

The matrix A is real ($a, \frac{b}{2}, c$ are all real numbers because they are defined as integrals of real functions over real planes). A real symmetric matrix has negative eigenvalues if and only if it is not positive semi-definite. We have proved that the eigenvalues of A are non-negative, so by applying Sylvester's criterion, we know that A is positive semi-definite with the following property:

A has a non-negative determinant.

Since $\det(A) = ac - \frac{b^2}{4} \geq 0$, it is proved that

$$4ac - b^2 \geq 0$$

When is E zero?

Since for $E_{1,2}$:

$$E_1 = \frac{(a+c) + \sqrt{(a-c)^2 + b^2}}{2ac - \frac{b^2}{2}} > \frac{(a+c) - \sqrt{(a-c)^2 + b^2}}{2ac - \frac{b^2}{2}} = E_2$$

We consider E_2 in this case.

When

$$(a+c) - \sqrt{(a-c)^2 + b^2} = 0$$

E is zero. The above equation can be simplified as:

$$(a+c)^2 = (a-c)^2 + b^2$$

$$4ac = b^2$$

Therefore, when

$$4ac = b^2$$

, $E = 0$.