计算机视觉

COMSW4731_001_2021_3

Computer Vision I: First Principles

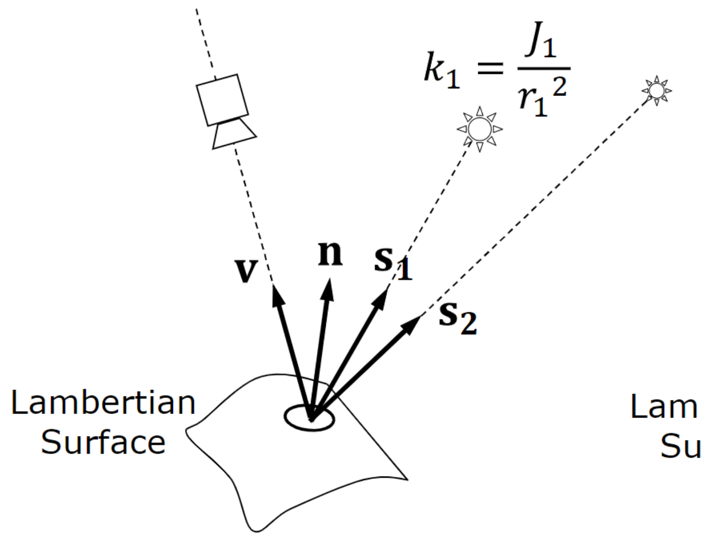
Homework #5

Name Joey Hou

UNI jh4170

侯江天

Problem 1



In the figure above (which is from the slides), $\mathbf{n}$ is the normal vector of the surface, $\mathbf{s}_1$ and $\mathbf{s}_2$ are in the directions of the two light sources, and $\mathbf{v}$ is in the direction of the viewing direction.

Illumination Can Be Viewed As Coming From A Single "Effective" Direction

For

$$I_1 = \frac{\rho}{\pi} k_1 \mathbf{n} \cdot \mathbf{s}_1$$

$$I_2 = \frac{\rho}{\pi} k_2 \mathbf{n} \cdot \mathbf{s}_2$$

The total illumination can be expressed as the sum:

$$I_{\text{total}} = I_1 + I_2 = \frac{\rho}{\pi} k_1 \mathbf{n} \cdot \mathbf{s}_1 + \frac{\rho}{\pi} k_2 \mathbf{n} \cdot \mathbf{s}_2$$

Since the sum is a constant, it can be expressed as the dot product of $\mathbf{n}$ and another vector:

$$I_{\text{total}} = \mathbf{n} \cdot \mathbf{s}$$

By normalizing $\mathbf{s}$ by dividing itself with its magnitude $||\mathbf{s}||$, it can be written in the form of a single effective direction:

$$I_{\text{total}} = ||\mathbf{s}|| \mathbf{n} \cdot \frac{\mathbf{s}}{||\mathbf{s}||} = \frac{\rho}{\pi} k_3 \mathbf{n} \cdot \mathbf{s}_3$$

where

$$\mathbf{s}_3 = \frac{\mathbf{s}}{||\mathbf{s}||}$$

$$k_3 = ||\mathbf{s}|| \frac{\pi}{\rho}$$

Therefore, the given statement of "illumination can be viewed as coming from a single "effective" direction" is shown above.

Equal Intensities

We have

$$I = \frac{\rho}{\pi} k_1 \mathbf{n} \cdot \mathbf{s}_1 = \frac{\rho}{\pi} k_2 \mathbf{n} \cdot \mathbf{s}_2$$

and the total intensity is

$$I_{\text{total}} = \frac{\rho}{\pi} \mathbf{n} \cdot (k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2) = \frac{\rho}{\pi} \hat{k} \mathbf{n} \cdot \hat{\mathbf{s}}$$

First, since the intensities are equal, we have:

$$k_1 = k_2$$

So,

$$\frac{\rho}{\pi} k_1 \mathbf{n} \cdot (\mathbf{s}_1 + \mathbf{s}_2) = \frac{\rho}{\pi} \hat{k} \mathbf{n} \cdot \hat{\mathbf{s}}$$

and

$$k_1 \mathbf{n} \cdot (\mathbf{s}_1 + \mathbf{s}_2) = \hat{k} \mathbf{n} \cdot \hat{\mathbf{s}}$$

For the left side of the equation, the magnitude of the vector $\mathbf{s}_1 + \mathbf{s}_2$ is $||\mathbf{s}_1 + \mathbf{s}_2||$. Hence, we can normalize the vector as:

$$k_1 \mathbf{n} \cdot (\mathbf{s}_1 + \mathbf{s}_2) = k_1 ||\mathbf{s}_1 + \mathbf{s}_2|| \mathbf{n} \cdot \frac{\mathbf{s}_1 + \mathbf{s}_2}{||\mathbf{s}_1 + \mathbf{s}_2||} = \hat{k} \mathbf{n} \cdot \hat{\mathbf{s}}$$

Therefore, $\hat{\mathbf{s}}$ (or $\mathbf{s}_3$) has the following equality with $\mathbf{s}_1$ and $\mathbf{s}_2$:

$$\hat{\mathbf{s}} = \frac{\mathbf{s}_1 + \mathbf{s}_2}{||\mathbf{s}_1 + \mathbf{s}_2||}$$

with $\hat{k}$ (or k_3) be:

$$\hat{k} = k_1 ||\mathbf{s}_1 + \mathbf{s}_2|| = k_2 ||\mathbf{s}_1 + \mathbf{s}_2||$$

Inequal Intensities

We have

$$I_1 = \frac{\rho}{\pi} k_1 \mathbf{n} \cdot \mathbf{s}_1$$

$$I_2 = \frac{\rho}{\pi} k_2 \mathbf{n} \cdot \mathbf{s}_2$$

and the total intensity is

$$I_{\text{total}} = \frac{\rho}{\pi} \mathbf{n} \cdot (k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2) = \frac{\rho}{\pi} \hat{k} \mathbf{n} \cdot \hat{\mathbf{s}}$$

and

$$\mathbf{n} \cdot (k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2) = \hat{k} \mathbf{n} \cdot \hat{\mathbf{s}}$$

For the left side of the equation, the magnitude of the vector $k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2$ is $||k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2||$. Hence, we can normalize the vector as:

$$\mathbf{n} \cdot (k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2) = ||k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2|| \mathbf{n} \cdot \frac{k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2}{||k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2||} = \hat{k} \mathbf{n} \cdot \hat{\mathbf{s}}$$

Therefore, $\hat{\mathbf{s}}$ (or $\mathbf{s}_3$) has the following equality with $\mathbf{s}_1$ and $\mathbf{s}_2$:

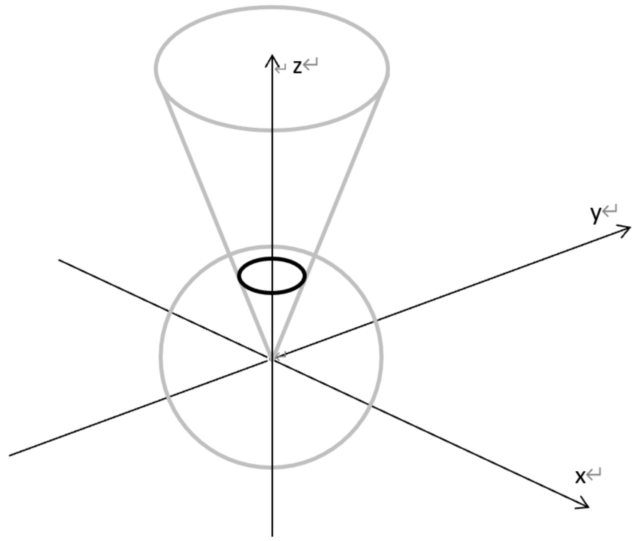
$$\hat{\mathbf{s}} = \frac{k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2}{||k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2||}$$

with $\hat{k}$ (or k_3) be:

$$\hat{k} = ||k_1 \mathbf{s}_1 + k_2 \mathbf{s}_2||$$

Problem 2

(a)



We can set up a coordinate system centered at the center of the Gaussian sphere. Without a loss of generality, we can set the point source on the z-axis, above the Gaussian sphere.

The constant brightness on the Gaussian sphere are generated by light paths from the point source with the same angle θ to the z-axis. That is, all light paths that form the contour of constant brightness form a right cone with the function be in the form of:

$$z^2 = a^2(x^2 + y^2)$$

The Gaussian sphere, centered at the origin point, has the function in the form of:

$$x^2 + y^2 + z^2 = r^2$$

The contour of constant brightness on the sphere is the intersection of these two quadric surfaces. We try to calculate the function of this contour by:

$$x^2 + y^2 + a^2(x^2 + y^2) = r^2$$

$$\Rightarrow (a^2 + 1)(x^2 + y^2) = r^2$$

$$\Rightarrow x^2 + y^2 = \frac{r^2}{a^2 + 1}$$

This is the function of a circle with a radius of $r/\sqrt{a^2 + 1}$.

Therefore, the contours of constant brightness on the Gaussian sphere in the case of a Lambertian surface illuminated by a point source are clrcles (in the above coordinate system, they are centered on z-axis).

(b)

For two different light sources, according to part (a), they will form 2 different circles on the surface of Gaussian sphere.

For two circles C_1 and C_2 on the sphere S, they are on two planes: m and n, respectively. If m and n are not parallel to each other, they will intersect at line l. When m is parallel to n, They share no common points and C_1 doesn't intersect with C_2, too:

$$C_1 \subset m, C_2 \subset n$$

$$m \cap n = \varnothing \Rightarrow C_1 \cap C_2 = \varnothing$$

For the line l, it may or may not intersect with the sphere. The intersection of a line l and a sphere can be no points, one point, or two points:
The function of a line is:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

The function of the Gaussian sphere is:

$$x^2 + y^2 + z^2 = r^2$$

Their intersections are calculated from:

$$y = \frac{b(x - x_0)}{a} + y_0$$

$$z = \frac{c(x - x_0)}{a} + z_0$$

and

$$x^2 + (\frac{b(x - x_0)}{a} + y_0)^2 + (\frac{c(x - x_0)}{a} + z_0)^2 = r^2$$

The above answer shows that the solution of the intersections of a line and a sphere is the solutions of a quadratic equation with two real solutions at most.
Let the set I be the intersection of line l and sphere S:

$$C_1 \subset m, C_2 \subset n, m \cap n = l$$

$$\Rightarrow C_1 \cap C_2 \subset l$$

$$C_1 \subset S, C_2 \subset S, S \cap l = I$$

$$\Rightarrow (C_1 \cap C_2) \cap I \subset I$$

This means that the number of intersections of two different circles on a sphere should be equal or less than 2, the cardinality of I.
Therefore, there could be at most 2 intersections for circles C_1 and C_2 on a sphere (and the example is presented in the hint figure (b)).