Exercise 4

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$$f(v) = N \cdot \exp\left(-\frac{mv^2}{2k_BT}\right) \cdot 4\pi v^2$$

Determine normalization constant:

$$\int_0^\infty f(v)\,dv \stackrel{!}{=} 1 \qquad | \quad v \geq 0$$

$$1 = 4\pi N \int_0^\infty v^2 e^{-\frac{m}{2k_BT}v^2}\,dv \qquad | \; a = \frac{m}{2k_BT}$$

$$\Leftrightarrow \quad N = \frac{1}{4\pi}\left(\int_0^\infty v^2 e^{-av^2}\,dv\right)$$

NR:

$$\int_0^\infty v^2 e^{-av^2}\,dv \qquad \left| \begin{array}{l} u = av^2 \Leftrightarrow v^2 = \frac{u}{a} \Leftrightarrow v = \left(\frac{u}{a}\right)^{1/2} \\ \frac{du}{dv} = 2av \Leftrightarrow dv = \frac{du}{2av} = du\,\frac{1}{2a}\cdot a^{1/2} u^{-1/2} \\ \qquad = du\,\frac{1}{2}(au)^{-1/2} \end{array} \right.$$

$$= \frac{1}{2a^{3/2}} \int_0^\infty u^{1/2} e^{-u}\,du$$

$$= \frac{1}{2a^{3/2}} \int_0^\infty u^{3/2-1} e^{-u}\,du$$

$$= \frac{\Gamma(3/2)}{2a^{3/2}} = \frac{\sqrt{\pi}}{4a^{3/2}} = \frac{\sqrt{\pi}}{4}\left(\frac{2k_BT}{m}\right)^{3/2}$$

$$\Rightarrow \quad N = \frac{1}{4\pi}\,\frac{\sqrt{\pi}}{4a^{3/2}} = \left(\frac{m}{2\pi k_BT}\right)^{1/2}$$

a)

$$f'(v_m) \stackrel{!}{=} 0$$

$$\Leftrightarrow \quad 0 = \frac{d}{dv}\left(N \cdot e^{-av_m^2} \cdot 4\pi v_m^2\right)$$

$$\Leftrightarrow \quad 0 = \frac{d}{dv}\left(v_m^2 e^{-av_m^2}\right)$$

$$\Leftrightarrow \quad 0 = 2v_m e^{-av_m^2} - 2av_m \cdot v_m^2 \cdot e^{-av_m^2}$$

$$\Leftrightarrow \quad 0 = \underbrace{2 v_m e^{-a v_m^2}}_{\neq 0} (1 - 2 a v_m)$$

$$\Leftrightarrow \quad 1 = 2 a v_m$$

$$\Leftrightarrow \quad v_m = \frac{1}{2a}$$

$$\Leftrightarrow \quad a = \frac{1}{2 v_m}$$

$$\Rightarrow \quad N = \frac{1}{(\sqrt{\pi}\, v_m)^3}$$