Project Report: The Dynamics of Tippe Top and Numerical Simulation

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ABSTRACT

In this report, I would like to introduce the dynamics of Tippe Top(TT). A tippe top is a kind of top that when spun, it will spontaneously invert itself to spin on its narrow stem.[6] Because the system have high level of non-linear performance, the dynamics of TT is quite complicated and hard to understand. Preceding works[2, 3] showed the dynamic equations and parameters of TT. And a great numerical stimulation also provided. My work including an introduction of TT and reproduction of their result, further we will carefully talk about the critical behaviors in the system which play the center role in this system but missing in preceding works.

KEYWORDS: Tippe Top, Numerical ODE

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1. Introduction

Tippe Top (TT) is a popular toy which is formed by a half sphere with a small leg. When spun it on the rough surface, TT will begin to precession and nutation. After a period time of nutation, the TT may turn down and spin on its leg and finally lose the balance by perturbation.

Figure 1: The Tippe Top[6]

Although the description of the physical process is clear and simple, obtaining a complete description of the dynamics of inversion has proven to be a difficult problem. Since the equation is nonlinear with 5 degrees of freedom (translation motion on the surface:2, Euler angle:3). Meanwhile, getting the exact friction term is also impossible.

The complexity leads us to the only possible approach to the dynamic of TT, that is numerical simulation. In the main part of the report, we will derive the dynamic equation of TT by minimal assumption, and use 4-order Runge-Kutta methods to solve this ODE equation. The dependence on initial angular velocities and roughness of surfaces will also be discussed.

Further, according to our daily experience, if the initial angle velocity is too small, the TT will spin steadily near the vertical direction, meanwhile, if too big, it probably "fly" from the surface and make the motion hard to describe. This trouble was neglected in the many literatures, in the end of the report we will discuss it.

In summary, the questions we will discuss in the main part of this report were listing following:

1. How to describe the motion of TTs mathematically?
2. How to solve the equation with non-linear and high degree of freedom?
3. The stability of the solution.
4. The dependence of initial condition, especially the angle velocity and the friction factor.

2. The Tippe Top Model-Background

2.1. Euler Angles and Euler Dynamic Equations

The Euler angles are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system.

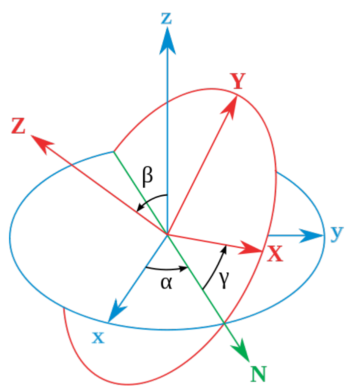

Figure 2: The Euler Angle[5]

The angle velocity can be easily obtained[1] from the Figure 2

$$\omega = \dot{\alpha}\hat{z} + \dot{\beta}\hat{N} + \dot{\gamma}\hat{Z} \tag{2.1}$$

And we projection it onto the rigid body coordinates (X, Y, Z)[1] :

$$\begin{cases} \omega_X = \dot{\beta}\cos\gamma + \dot{\alpha}\sin\beta\sin\gamma \\ \omega_Y = -\dot{\beta}\sin\gamma + \dot{\alpha}\sin\beta\cos\gamma \\ \omega_Z = \dot{\gamma} + \dot{\alpha}\cos\beta \end{cases} \tag{2.2}$$

Further, we choose our rigid body coordinates as the inertial principal axis coordinates (1,2,3) to simplified the inertia tensor as this form[1] :

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{2.3}$$

Further, angle momentum takes:

$$\mathbf{L} = \mathbf{I}\omega \tag{2.4}$$

Under the external force $\mathbf{F}$, the mass center motion equation tell us:

$$m\ddot{\mathbf{r}}_{\mathbf{c}} = \mathbf{F}, \dot{\mathbf{L}} = \mathbf{M} + \mathbf{L} \times \omega \tag{2.5}$$

,where $\mathbf{M}$ is external torque and the $\mathbf{L} \times \omega$ term occur since we choose the rigid body coordinate which is rotating and a non-inertia coordinates. The non-inertia force have to be considered.

Finally, we got[1] :

$$\begin{cases} I_1\dot{\omega}_1 = M_1 + (I_2 - I_3)\,\omega_2\omega_3 \\ I_2\dot{\omega}_2 = M_2 + (I_3 - I_1)\,\omega_3\omega_1 \\ I_3\dot{\omega}_3 = M_3 + (I_1 - I_2)\,\omega_1\omega_2 \end{cases} \tag{2.6}$$

which is well known as Euler Dynamic Equations.

2.2. The Tippe Top

Here we refer the equation in our TT system from[3]

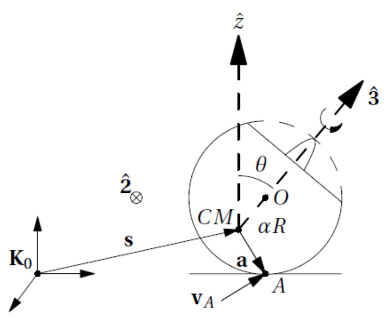

Figure 3: The TT model

In our model, (1,2,3) are the principal axis coordinates, (They form a right hand coordinates). The center of mass CM is shifted from the geometric center O along the axis 3 by αR, and A touches the surface. From CM to point A, there is a vector a which hold the geometrical relation $a = \alpha R\hat{3} - R\hat{z} = R(\sin\theta\hat{1} + (\alpha - \cos\theta)\hat{3})$.

The external force and torque should be considered carefully, the contact constraint force g_n can be determined by Lagrangian Multiples which is a vast work. And the friction, we adopt the form in[2]. $\mathbf{f} = -\mu g_n \mathbf{v_A}$. Where μ has dimension of s/m.

Now, we finally give our equations:

$$\begin{aligned} \mathbf{F} &= (g_n - mg)\hat{z} - \mu g_n \mathbf{v_A} \\ \mathbf{M} &= \mathbf{F} \times \mathbf{a} \end{aligned} \tag{2.7}$$

together with equation(2.2)(2.5)(2.6)(notice $I_1 = I_2$ due to the symmetry), and construct g_n by constraint condition $(\mathbf{s} + \mathbf{a}) \cdot \hat{z} = 0$, we have[3]:

$$\ddot{\theta} = \frac{\sin\theta}{I_1}\left(I_1\dot{\varphi}^2\cos\theta - I_3\omega_3\dot{\varphi} - R\alpha g_n\right) + \frac{R\mu g_n v_x}{I_1}(1 - \alpha\cos\theta) \tag{2.8}$$

$$\ddot{\varphi} = \frac{I_3\dot{\theta}\omega_3 - 2I_1\dot{\theta}\dot{\varphi}\cos\theta - \mu g_n v_y R(\alpha - \cos\theta)}{I_1\sin\theta} \tag{2.9}$$

$$\dot{\omega}_3 = -\frac{\mu g_n v_y R\sin\theta}{I_3} \tag{2.10}$$

$$\dot{v}_x = \frac{R\sin\theta}{I_1}\left(\dot{\varphi}\omega_3\left(I_3(1 - \alpha\cos\theta) - I_1\right) + g_n R\alpha(1 - \alpha\cos\theta) - I_1\alpha\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right)\right) \tag{2.11}$$

$$-\frac{\mu g_n v_x}{mI_1}\left(I_1 + mR^2(1 - \alpha\cos\theta)^2\right) + \dot{\varphi}v_y \tag{2.12}$$

$$\dot{v}_y = -\frac{\mu g_n v_y}{mI_1I_3}\left(I_1I_3 + mR^2I_3(\alpha - \cos\theta)^2 + mR^2I_1\sin^2\theta\right) \tag{2.13}$$

$$+\frac{\omega_3\dot{\theta}R}{I_1}\left(I_3(\alpha - \cos\theta) + I_1\cos\theta\right) - \dot{\varphi}v_x \tag{2.14}$$

,and

$$g_n = \frac{mgI_1 + mR\alpha\left(\cos\theta\left(I_1\dot{\varphi}^2\sin^2\theta + I_1\dot{\theta}^2\right) - I_3\dot{\varphi}\omega_3\sin^2\theta\right)}{I_1 + mR^2\alpha^2\sin^2\theta - mR^2\alpha\sin\theta(1 - \alpha\cos\theta)\mu v_x} \tag{2.15}$$

Here comes a question, if we have a very large ω_3(Which means the third term in numerator is very large), the constrain is not valid. Because to constrain the TT on the surface in this situation, g_n has to be negative. Actually, the desk can NEVER pull our tipper top back! This lead to a float TT which we have foreboded in introduction part. We will discuss it in the end of report.

2.3. Before the Solution

In this subsection, we will discuss the properties which can be obtained from the equation without explicit solution.

When the time scale is enough, we can conclude that the TT may spin steadily without any energy loss. From our intuition, we conclude that in the end of evolution, v_x, v_y, $\dot{v_x}$, $\dot{v_y}$ must vanished. Since if $\mathbf{v} \neq 0$, the friction exists and slows down the TT until stops it.

So (2.14) comes $\frac{\omega_3 \dot{\theta} R}{I_1} \left(I_3(\alpha - \cos\theta) + I_1 \cos\theta \right) = 0$ and leads to $\dot{\theta} = 0$.

Apply $v_x = v_y = \dot{v_x} = \dot{v_y} = \dot{\theta} \equiv 0$ to other equations we conclude:

$$\begin{aligned} I_1 \dot{\varphi}^2 \cos\theta - I_3 \omega_3 \dot{\varphi} - R\alpha g_n = 0 \\ \ddot{\varphi} = 0 \\ \dot{\omega_3} = 0 \\ \dot{\varphi}\omega_3 \left(I_3(1 - \alpha\cos\theta) - I_1 \right) + g_n R\alpha(1 - \alpha\cos\theta) - I_1 \alpha \dot{\varphi}^2 \sin^2\theta = 0 \end{aligned} \tag{2.16}$$

If $\theta \neq \pi$, we have:

$$\omega_3 = \dot{\phi}(\cos\theta - \alpha) \tag{2.17}$$

This relation holds independently on the mass and moment of inertia.

3. Result, Analysis, Discuss

To solve equations (2.8)-(2.15) analytically is impossible, we use 4-order Runge-Kutta Method to take a reliable simulation. Details are included in Appendix

We adopt the parameters in [3] which is listed fellow:

Table 1: Parameters

m	0.02kg
R	0.02m
α	0.3
I_1	2/5mR^2
I_3	131/350mR^2
μ	0.3m/s
g	9.8ms^2

3.1. Result

Figure 4 present our result under special condition $(\theta, \dot{\theta}, \varphi, \dot{\varphi}, \omega, v_x, v_y) = (0.1\,\text{rad}, 0, 0, 0, 155\,\text{rad/s}, 0, 0)$. In this condition, the TT will turn around completely which implies that $\theta = \pi$.

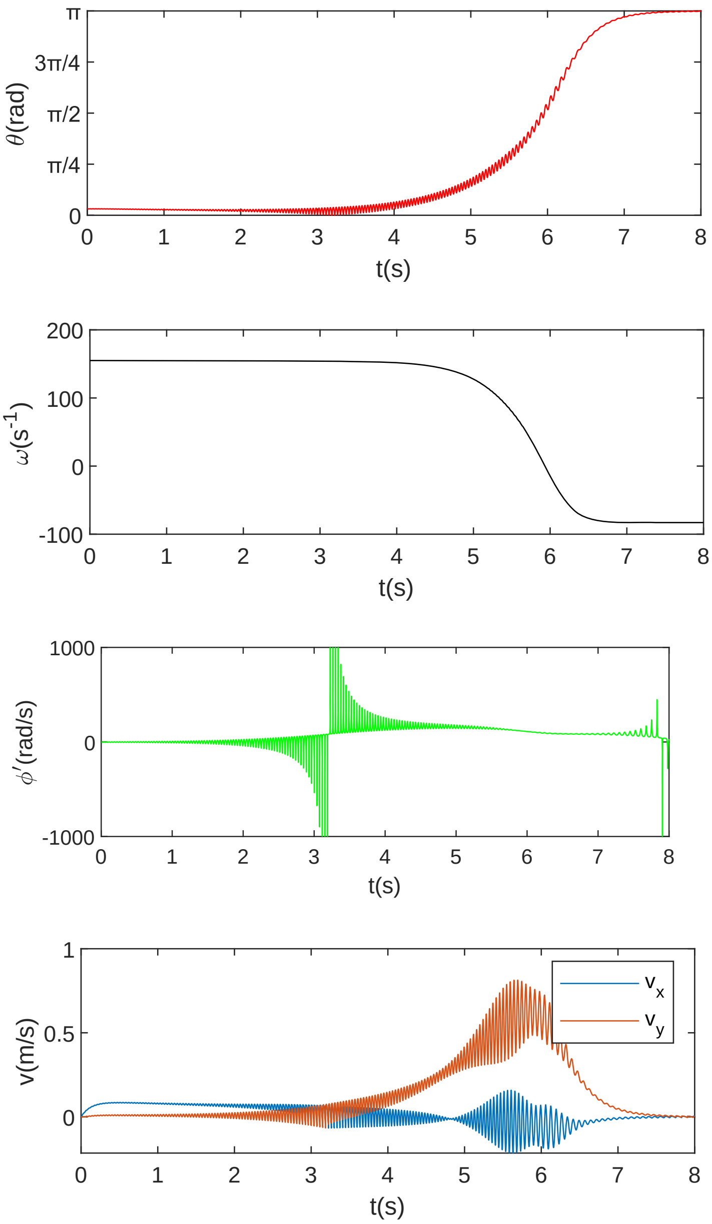

Figure 4: (a)the nutation angleθ with t, it flips over completely around $t = 7.4s$(b)The principle angle velocity around axis 3(c)the precession angle velocity, which reverses at $t = 3.3s$ and $t = 7.9s$.(d)The translation velocity which converges to 0

Figure 5 present another result under condition $(\theta, \dot{\theta}, \varphi, \dot{\varphi}, \omega, v_x, v_y) = (0.1\,\text{rad}, 0, 0, 0, 50\,\text{rad/s}, 0, 0)$.

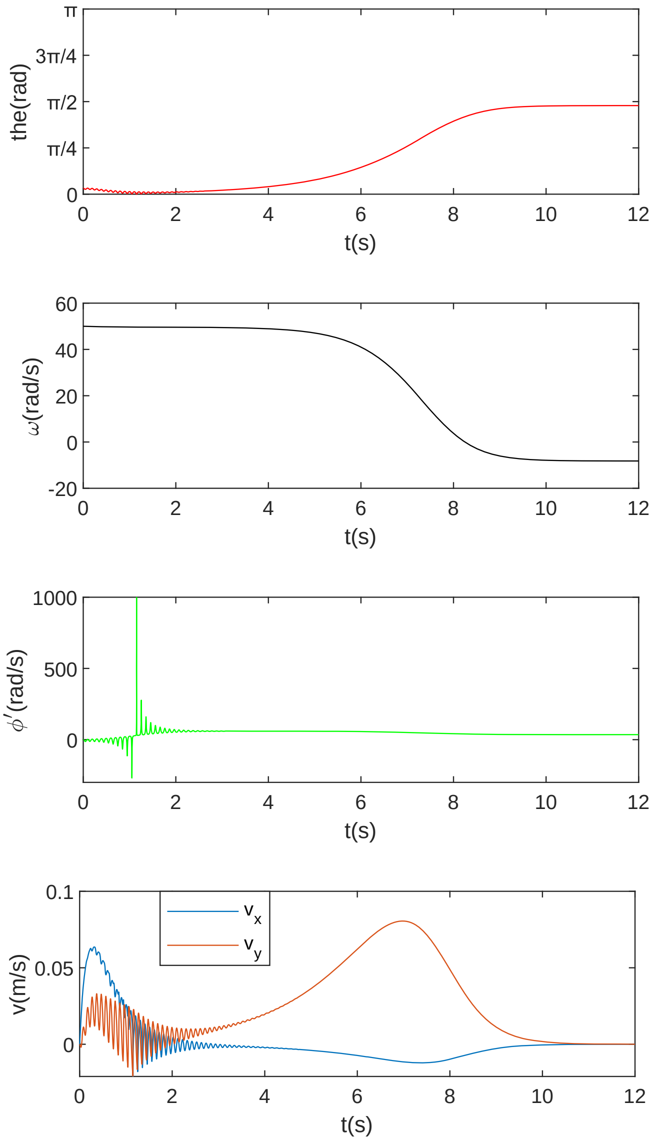

Figure 5: (a)the nutation angle θ with t, it flips over half around $t = 8.9s$(b)The principle angle velocity around axis 3(c)the precession angle velocity, which reverses at $t = 1.1s$ (d)The translation velocity which converges to 0

In the end of the $\theta = 1.503$, $\dot{\varphi} = 35.257$, $\omega = -8.201$, which perfectly correspond to the (2.17) These two figures provide much information:

1. If initial angle velocity is not big enough, the TT can't flip completely.
2. The ω is smooth and behave similarly.

3.2. The dependence on ω_0

Now, we conclude that the initial condition will highly correspond to the flip.

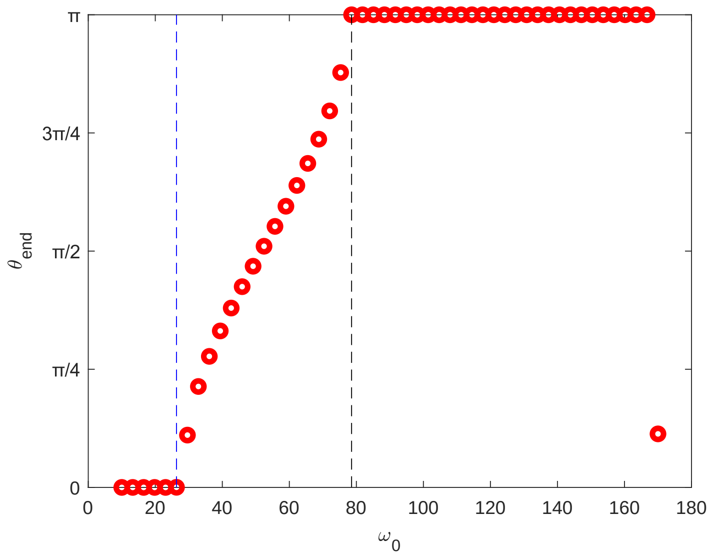

Figure 6: The relation of initial ω and the final θ, the blue dash line and the black dash line is two critical points of θ_{end}

In Figure 6, if the ω is not enough big, the TT even can not nutate and keep stable around $\theta = 0$. When the ω increase, θ will arrive the π and spin on TT's leg. The two critical point is $\omega_{c1} = 26.3$ and $\omega_{c2} = 78.6$. However, when ω increase farther, an abnormal point occur around $\omega_a = 170$. Which should be discussed carefully. And more $t - \theta$ plot at different ω_0 is attached in Appendix.

3.3. The dependence on μ

In our daily experience, in the smooth surface, the tipper top is hard to flip over since there is no torques to reverse the TT. However, if the friction is too large, it also difficult for TT to flip because of obstacle. Numerical simulation correspond to our experience.

More interesting thing is when μ is in the normal range (0.2 1), the flip time is inversely proportional to the μ.(we define the flip time t as minimal t which satisfy $|\theta(t) - \theta_{end}| < 0.01$.)

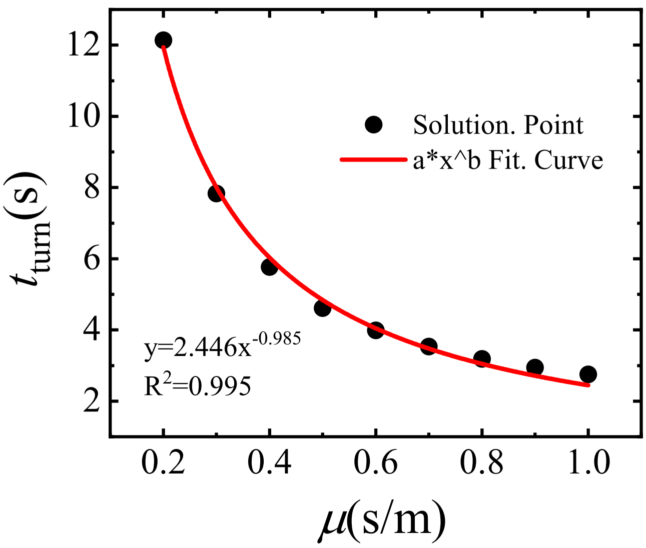

Figure 7: The relation of initial flip time and the fraction factor μ, the red line is the fit curve by ax^b. And left-bottom corner is the function and Pearson R. We conclude that $t \propto \mu^{-0.985}$, which is approach $t \propto \mu^{-1}$

However, thing goes bad uncontrollably if μ is huge.

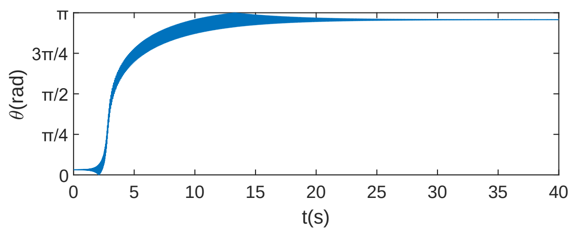

Figure 8: Here we choose the μ as 100(s/m), (In this μ, a wedge with initial velocity 1m/s can just travel for 0.001m. We don't have such a rough surface actually.)

Under the abnormal friction, the behavior of TT is abnormal, it notate around $\theta = \pi$ rapidly and finally stop notate.

In summary, we list our result below:

1. When $\omega_0 < \omega_{c1}$, the TT will spin uprightly.
2. When $\omega_{c1} < \omega_0 < \omega_{c2}$, the TT will keep a constant nutation angle and the final angle velocity holds (2.17)
3. When $\omega_{c2} < \omega_0$, the TT will completely flip over and spin on its leg.

4. When $\omega_0 > 170$, abnormal behavior perform. In the next section we'll point out this is caused by the $g_n < 0$ and is a non-physical situation.
5. In normal range of μ(<1(s/m)), the flip time t and the μ holds $t \propto \mu^{-0.985} \approx \mu^{-1}$.
6. Out of the normal range, the relation mentioned in 5 is lose efficacy.

4. Discussion

From our discussion, our model and solution provide interesting and reliable simulation. However, some special situation was intentionally neglected in the last section. Now, we are going to test it.

In Figure 6, we found that a wrong point came caused by the high angle velocity. So, what have had happened in $\omega_0 > 170$?

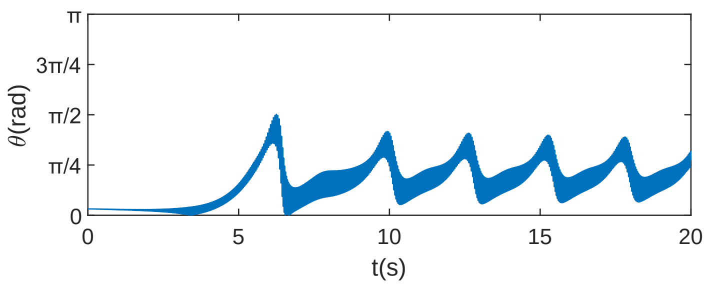

Figure 9: This is the $\theta - t$ relation under the $\omega_0 = 175$ rad/s

The nutation of the TT is quite complex, which mix rapid and smooth nutation. But the after further research we conclude that: this is non-physical solution. Two arguments provided to suppose our idea:

First, we calculate the energy of this TT, the energy function is given:

$$
\begin{aligned}
E =& \frac{1}{2} m R^2 \left[(\alpha - \cos\theta)^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta \right) + \sin^2\theta \left(\dot{\theta}^2 + \omega_3^2 + 2\omega_3 \dot{\varphi} (\alpha - \cos\theta) \right) \right] \\
&+ \frac{1}{2} \left[I_1 \left(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2 \right) + I_3 \omega_3^2 \right] + mgR(1 - \alpha\cos\theta)
\end{aligned}
\tag{4.1}
$$

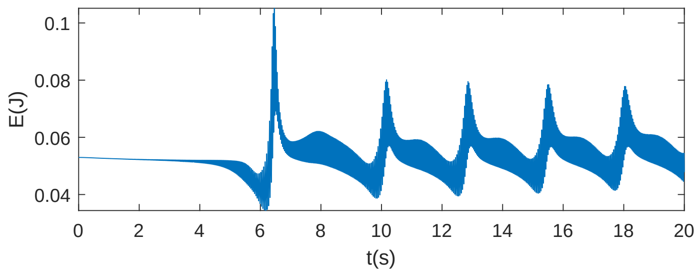

Figure 10: This is the $E - t$ relation under the $\omega_0 = 175\,\text{rad/s}$

This is IMPOSSIBLE because the system is lossing it energy during the motion, the only force can do positive work is the gravity. However, the TT would have felled down for $15R$ to gain such a big energy(0.06J). at t=6s.

Second, the constrain force g_n can never be negative. However, in this system we point that:

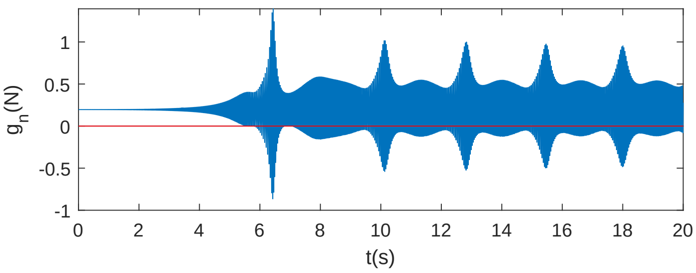

Figure 11: This is the $g_n - t$ relation under the $\omega_0 = 175\,\text{rad/s}$, red line is the 0

So, we can't constrain our Tipper Top on the surface in huge ω. That is, the TT may fly from the surface and make a complex situation which is hard to discuss.

The compromise is "Stop before Fly". At $t = 5.62s$, before the critical time, the motion is:

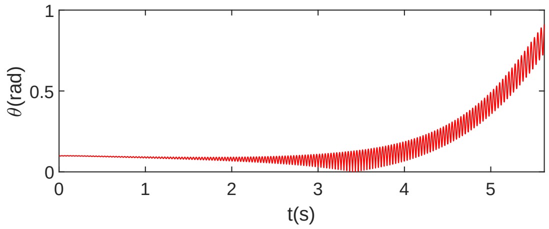

Figure 12: This is the $\theta - t$ relation under the $\omega_0 = 175\,\text{rad/s}$ here we stop before t=5.62s

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5. Critique and Summary

I'll summarize the important concepts for which I gained a better understanding and discuss the numerical or computer techniques I learned. Meanwhile, this work can be improved from some aspect.

First, it's a great review of Classical Mechanics. I use to think this is a boring question to understand the motion of TTs. And in Classical mechanics class we just mention this and do not give a detailed solution. Now, after review the rigid body equation and conception the physical image is much clear.

Second, I also understand the stiff-performance in ode function. In some bad initial condition, the ODE equation is stiff and ode45 function is unbearably slow. Suspecting stiff, I try the ode15s which is a build-in function for stiff ode and get a reliable solution.

Further, some small tricks like using matrix computation rather than "for loop" to improve the speed, adding legend in picture is also valuable.

Also, we can do a better work if further details were included in the report but due to the limitation of time we neglected them. For example $t_{flip} \propto \mu^-1$ is an extraordinary simple and interesting result, the reader can check it by wider range by themselves and the conclusion which shows this dependence is true even for $\mu = 0.01$ may surprise them. I believe that there is a more fundamental ruler guarantees the correctness.

6. Acknowledgment

I would like to thank Prof. Luo for his guidance and suggestion, as well as Yuanye Lin, who discussed questions with me in the night. He also provided acute insight. Especially, Zixuan Yang's encouragement and suggestion are precious and warm.

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Appendix A. Codes

Appendix A.1. The Main Codes

```
clc;
clear;
close all
%% parameter
m=0.02;
R=0.02;
alpha=0.3;
I3=2/5*m*R^2;
I1=131/350*m*R^2;
mu=0.3;
g=9.8;
%% SOLVE
tspan=[0,20];
y0=[0.1;0;0;0;155;0;0];
[t,y]=ode45(@tip,tspan,y0);
the=mod(y(:,1),pi);
dthe=y(:,2);
dphi=y(:,4);
w=y(:,5);
vx=y(:,6);
vy=y(:,7);
%% Result
E=1/2*m*R^2*((alpha-cos(the)).^2.*(dthe.^2+(dphi.*sin(the)).^2)...
+sin(the).^2.*(dthe.^2+w.^2+2*w.*dphi.*(alpha-cos(the))))+...
1/2*(I1*((dphi.*sin(the)).^2+dthe.^2)+I3*w.^2)...
+m*g*R*(1-alpha*cos(the));
gn=(m*g*I1+m*R*alpha*(I1*cos(the).*(dphi.^2.*sin(the).^2...
+dthe.^2)-I3*dphi.*w.*sin(the).^2))...
./(I1+m*R^2*alpha^2*sin(the).^2-m*R^2*alpha ...
*sin(the).*(1-alpha.*cos(the))*mu.*vx);

t(find(abs(the-the(end))<0.01,1))%Find filp time

%% Plot
plot(t,the,'r');
xlabel('t(s)');
ylabel('\theta(rad)')
set(gca,'yTick', 0:pi/4:pi)
set(gca,'ytickLabel',{'0','pi/4','pi/2','3pi/4','pi'})
```

```
set (gcf, 'Position',[0,0,500,150]);
pbaspect([3 1 1]);
```

```
function dtip = tip(t,y)
%tip
%
%1 theta,
%2 dtheta
%3 phi
%4 dphi
%5 ome
%6 vx
%7 vy
%%% theta=[0,pi], phi=[0,2pi]
dtip=zeros(7,1);
%%%parameter
m=0.02;
R=0.02;
alpha=0.3;
I3=2/5*m*R^2;
I1=131/350*m*R^2;
mu=0.3;
g=9.8;
%%
the=mod(y(1),pi);
dthe=y(2);
dphi=y(4);
w=y(5);
vx=y(6);
vy=y(7);
%%
gn=(m*g*I1+m*R*alpha*(I1*cos(the)*(dphi^2*sin(the)^2+dthe^2)...
-I3*dphi*w*sin(the)^2))...
/(I1+m*R^2*alpha^2*sin(the)^2-m*R^2*alpha*sin(the)...
*(1-alpha*cos(the))*mu*vx);
% if gn<0
%      gn=0;
% end

dtip(1)=dthe;
dtip(2)=sin(the)/I1*(I1*dphi^2*cos(the)-I3*w*dphi-R*alpha*gn)...
+R*mu*gn*vx/I1*(1-alpha*cos(the));
dtip(3)=dphi;
dtip(4)=(I3*dthe*w-2*I1*dthe*dphi*cos(the)-mu*gn*vy*R...
```

```
*(alpha-cos(the)))/(I1*sin(the));
dtip(5)=-mu*gn*vy*R*sin(the)/I3;
dtip(6)=R*sin(the)/I1*(dphi*w*(I3*(1-alpha*cos(the))-I1)...
+gn*R*alpha*(1-alpha*cos(the))-I1*alpha*(dthe^2+dphi^2*sin(the)^2))...
-mu*gn*vx/m/I1*(I1+m*R^2*(1-alpha*cos(the))^2)+dphi*vy;
dtip(7)=-mu*gn*vy/m/I1/I3*(I1*I3+m*R^2*I3*(alpha-cos(the))^2...
+m*R^2*I1*sin(the)^2)+...
w*dthe*R/I1*(I3*(alpha-cos(the))+I1*cos(the))-dphi*vx;
end
```

Appendix B. 4-steps Runge-Kutta Method[4]

Runge Kutta method is one of the most popular algorithm to solve ODEs numerically. The accuracy and the speed are both satisfactory in most situation. The idea RUNGE-Kutta methods is to improve the accuracy by calculating intermediate grid points within the interval $[t_n, t_{n+1}]$. And 4-steps RK is the most popular one, for equation $y = f(t, y)$, we have:

$$\begin{aligned}
&Y_1 = y_n \\
&Y_2 = y_n + \tfrac{\Delta t}{2} f\left(Y_1, t_n\right) \\
&Y_3 = y_n + \tfrac{\Delta t}{2} f\left(Y_2, t_n + \tfrac{\Delta t}{2}\right) \\
&Y_4 = y_n + \Delta t f\left(Y_3, t_n + \tfrac{\Delta t}{2}\right) \\
&y_{n+1} = y_n + \tfrac{\Delta t}{6}\left[f\left(Y_1, t_n\right) + 2f\left(Y_2, t_n + \tfrac{\Delta t}{2}\right)\right. \\
&\quad \left. +2f\left(Y_3, t_n + \tfrac{\Delta t}{2}\right) + f\left(Y_4, t_n\right)\right]
\end{aligned} \tag{Appendix B.1}$$

One can proof that this method has an accuracy of order$(\Delta t)^5$.

In MATLAB, there is a build-in function ode45, in our main chapter, we will choose this function to deliver the calculate.

Appendix C. Plots

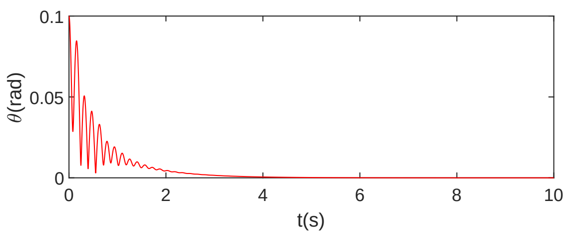

Figure 13: This is the $\theta - t$ relation under the $\omega_0 = 10$ rad/s

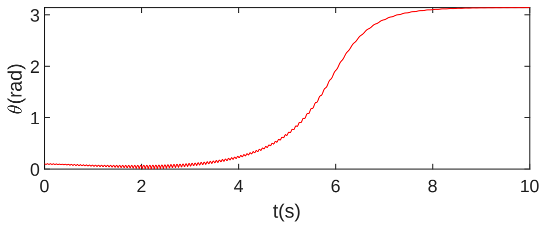

Figure 14: This is the $\theta - t$ relation under the $\omega_0 = 90\,$rad/s

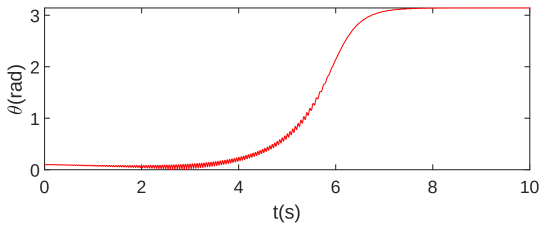

Figure 15: This is the $\theta - t$ relation under the $\omega_0 = 120\,$rad/s

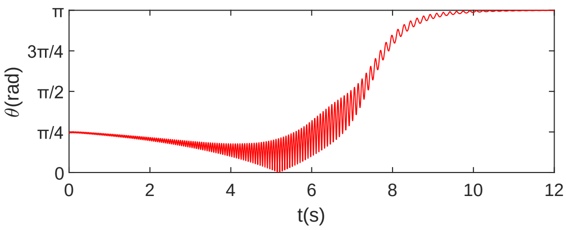

Figure 16: This is the $\theta - t$ relation under the $\omega_0 = 150\,$rad/s and $\theta_0 = pi/4$