Lecture 1 9.9.

隐函数 $S: \{F(x,y,z)=c\}$.

$\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0 \quad (x_0, y_0, z_0) \in S$.

$\exists\, U \ni (x_0, y_0) \quad U \subseteq \mathbb{R}^2 \quad g: U \to \mathbb{R}$

s.t. $F(x, y, g(x,y)) \equiv C \quad \forall (x,y) \in U$

local $\to$ global.

Topological space

X topological space $\quad \mathcal{T}$ (topology) collection of subsets (open sets)

① $\phi, X \in \mathcal{T}$

② finite $\cap \quad \bigcap_{i=1}^{n} A_i \in \mathcal{T}$

③ contable $\cup \quad \bigcup_{i=1} A_i \in \mathcal{T}$

Basis: $\mathcal{B}$ is a basis of $(X, \mathcal{T})$

if $\mathcal{B}$ generates $\mathcal{T}$

$\forall\, U \in \mathcal{T} \quad U = \bigcup_i A_i, \bigcap_i^n A_i$

secondary countable. $(X, \mathcal{T})$ if $\exists \mathcal{B}$. is countable.

Hausdoff: X is call Hausdoff

$\forall p, q \in X \quad \exists\, U^{open} \ni p. \quad \exists\, V^{open} \ni q \quad U \cap V = \phi$

Example:

$\mathbb{R}^n \quad \{B_r(p) \mid p \in Q^R, r \in Q\}$

$B_r(p) = \{q \in \mathbb{R}^n \mid |q-p| < r\}$?

Contnus map: $\to$ function.

$f: X \to Y$ is contiuns if $f^{-1}(V)$ is open in $X \quad \forall\, V$ open in Y

Homeomorphism: $f: X \to Y$ is a homeophism
同胚
$\begin{cases} 1 \text{ to } 1 \text{ on to.} \\ f, f^{-1} \text{ both continus} \end{cases}$

Manifolds:

M is topological space is called n-d manifold if

① Hausdoff

② 2-countable.

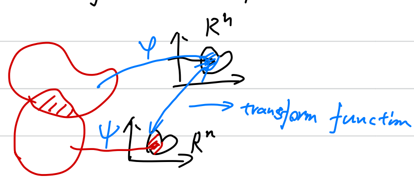
③ $\forall p \in M, \exists$ chart $(U, \varphi), p \in U$

U is open

$\varphi: U \to \varphi(U)^{open} \subseteq \mathbb{R}^n$ is a homeomophism

smooth

moreover if $\forall\, U \cap V \neq \phi$



red $\to$ blue $\varphi \circ \psi^{-1}$ is smooth

Smooth structure: maximal ~~choice~~ collections of smooth compartable charts. $\mathcal{A}$ is atlas cover the manifolds and $\forall (U,\varphi) \notin \mathcal{A}$ $(V,\psi) \in \mathcal{A}$, (U,φ) is not computtable

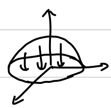
Rmk: $\exists$ more than one smooth structure

$\exists$ TM with No smooth structure.

Example:

$S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \in \mathbb{R}^{n+1}$

$U_i^+ = \{\vec{x} \in S^n \mid x_i > 0\}$ $\quad \varphi_i^+(\vec{x}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_{n+1})$



$U_i^- = \{\vec{x} \in S^n \mid x_i < 0\}$ $\quad \varphi_i^- = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_{n+1})$

$\varphi_j^+ \circ (\varphi_i^-)^{-1}(w_1, \ldots, w_n)$

$\downarrow (\varphi_i^-)^{-1}$

$(w_1, \ldots, w_{i-1}, \sqrt{1-\prod w_i^2}, w_i, \ldots, w_n)$

$\downarrow \varphi_j^+$

$w_1, \ldots w_{i-1}, -\sqrt{1-\prod w}, w_i, \ldots, w_{j-1}, w_j \ldots)$

2. $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$

$(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$

$\Leftrightarrow (x_1, \ldots, x_n) = \lambda(y_1, \ldots, y_n)$, $\lambda \neq 0$

$[x_1, \ldots, x_{n+1}]$

$U_i = \{[x_1, \ldots, x_{n+1}] \mid x_i \neq 0\}$

$\varphi_i: \mathbb{RP}^n \to \mathbb{R}^n$ $\quad \varphi_i([x_1, \ldots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i}\right)$

check:

1. U_i is open, $\bigcup_{i=1}^{n+1} U_i = \mathbb{RP}^n$, Tr
2. φ_i is homeomorphism. 1 to 1 and onto.

Manifolds with boundary:

① Hausdorff

② 2-countable. (U,φ)

③ $\forall p \in M$, $\exists$ local charts s.t. $\varphi: U \to \varphi(U)$ is a homeomophism.

where $\varphi(U)$ is either ~~an open set in $\mathbb{R}^n$~~ or $\mathbb{R}^n_+$

Lecture 2. 9.11 / All smooth manifolds

1. Smooth function

Defn: $f: M \to \mathbb{R}$ is smooth if

$\forall p \in M$, $\exists (U,\varphi)$, $p \in U$, s.t. $f \circ \varphi^{-1}$ is smooth

$f \circ \varphi^{-1}$ is coordinate represent of f

Prop. if f is smooth, $\forall (U,\psi)$, $f \circ \psi^{-1}$ is smooth

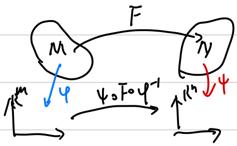
$f \circ \varphi^{-1}$ is smooth, $\underline{f \circ \varphi^{-1}} \circ \underline{\varphi \circ \psi^{-1}}$, compatable because M is smooth.

S^2 $\{x^2+y^2+z^2=1\}$ $\quad f(x,y,z) = z$

(U_x^+, φ_x^+) $\quad \varphi_x^+(x,y,z) = (y,z)$ $\quad f \circ \varphi_x^{-1} = z$

(U_z^+, φ_z^+) $\quad \varphi_z^+(x,y,z) = (x,y)$ $\quad f \circ \varphi_z^{-1} = \sqrt{1-x^2-y^2}$

Smooth map:



$F: M \to N$ is smooth map if $\forall p \in M$,

$\exists (U,\varphi)$, (V,ψ) $\quad p\in U$, $f(p)\in V$

s.t. $\psi\circ F\circ\varphi^{-1}$ is smooth

diffeomorphism: F is $\begin{cases} 1 \text{ to } 1 \text{ onto (bijection)} \\ F, F^{-1} \text{ is smooth.} \end{cases}$

Tangent space and vector

tangent vector views as linear maps on smooth function near p

$C_p^\infty = \{ f \mid \exists U \ni p,\ f(U) \text{ is smooth}\}$

Defn: a linear map $v: C_p^\infty \to R$ satisfy:

$v(fg) = f(p)v(g) + g(p)v(f)$ $\quad$ ① $v(\text{constant}) = 0$

$v \in T_pM$

(U,φ)
W.O.L.G. $\varphi(p) = 0$. $\quad C_p^\infty = \{f, \dots\}$ $\quad C_0^\infty = \{F \mid F = f\circ\varphi^{-1},\ f\in C_p^\infty\}$

define $(\partial x_i)_0 : C_0^\infty \to R$

$$(\partial x_i)_0 F = \frac{\partial F}{\partial x_i}\Big|_{\vec{x}=\vec{0}}$$

Prop: $T_pM \cong \mathrm{Span}\{(\partial x_i)_0\}$

proof: define $(\partial x_i)_p : C_p^\infty \to R$ by

$$(\partial x_i)_p f = \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}\Big|_p$$

obviously $(\partial x_i)_p \leftrightarrow (\partial x_i)_0$

claim: $v = \sum_i v_i (\partial x_i)_p$. $\quad$ Lemma.

① $v(F\circ\varphi)$ $\quad \underline{\forall F: \varphi(U)\to R,\ F(x) = F(0) + \sum_i h_i(x)\, x_i}$

$$v(F\circ\varphi) = \underbrace{v(F(0)\circ\varphi)}_{\text{constant.}} + \sum_i v[h_i(x)\circ\varphi)\cdot(x_i\circ\varphi)]$$

$$= \sum_i [v(h_i(x)\circ\varphi)\cdot(x_i\circ\varphi)(p) + v(x_i\circ\varphi)(h_i(x)\circ\varphi)(p)]$$

$$= \sum_i v(x_i\circ\varphi)\, h_i(0)$$

$$v = \sum_i v_i (\partial x_i)_p$$

$$v(f) = \sum_i v_i \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}\Big|_p = \sum_i v_i \frac{\partial F}{\partial x_i}\Big|_0 = \sum_i v_i h_i(0)$$

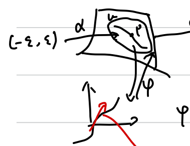
So $v_i = v(x_i\circ\varphi)$

$F(x) = F(0) + \sum_i h_i(x)\, x_i$

$\frac{\partial F}{\partial x_i} = h_i(x) + \sum_j \frac{\partial h_j(x)}{\partial x_i} x_j$

$\frac{\partial F}{\partial x_i}\Big|_0 = h_i(0)$

Tangent vector of curve



$\alpha: (-\varepsilon,\varepsilon)\to M$ $\quad$ W.L.O.G. $\alpha(0) = p$. chart.

$\alpha((-\varepsilon,\varepsilon)) \subset U \ni p$ $\quad (U,\varphi)$ $\quad$ w.p. $\varphi(p) = 0$

$\varphi\circ\alpha$ is a curve on R^n

$v: C_p^\infty \to R$

$$v(f) \overset{\text{define}}{=} \frac{d}{dt}[f\circ\alpha(t)]\Big|_0 \qquad v\in T_pM$$

$$= \frac{d}{dt}(f\circ\varphi^{-1}\circ\varphi\circ\alpha)$$

$$= \sum_i \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi\circ\alpha)_i'(t)$$

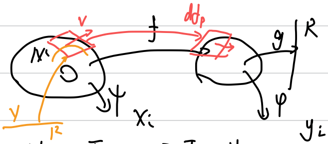
$$= \left[\sum_i (\varphi\circ\alpha)_i'(t)(\partial x_i)_p\right] f$$

Tangent vector of i.

$$v = \sum_i (\varphi\circ\alpha)_i'(t)\, \partial x_{i_p}$$

$$v \overset{\text{def}}{=} \alpha'(t)$$

The differential of Smooth Maps



$(df)_p : T_pM \to T_{f(p)}N$

for $g \in C^\infty(N)$ equalance: $(f\circ\gamma)'(0)(h) = v(h\circ f)$

$df_p(v)(g) = v(g\circ f)$

or: $df_p(v) = (f\circ\gamma)'(0)$ $\gamma: R \to M$, $(-\varepsilon,\varepsilon)$, $\gamma'(0) = v$, $\gamma(0) = p$

$df_p(v)(g) = v(g\circ f)$

Check: $df_p(\alpha_1 v_1 + \alpha_2 v_2)(g)$

$= (\alpha_1 v_1 + \alpha_2 v_2)(g\circ f)$

$= \alpha_1 v_1(g\circ f) + \alpha_2 v_2(g\circ f)$

$= (\alpha_1\, df_p(v_1) + \alpha_2\, df_p(v_2))\, g$

$df_p(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1\, df_p(v_1) + \alpha_2\, df_p(v_2)$ ✓

df_p is linear!

$f(\ )$

$df_p(\partial x_i)g = \partial x_i(g\circ f)$

$= \frac{\partial(g\circ f\circ\psi^{-1})}{\partial x_i}\Big|_0$ y_i

$= \frac{\partial(g\circ\varphi^{-1}\circ\varphi\circ f\circ\psi^{-1})}{\partial x_i}\Big|_0$

$= \sum_j \frac{\partial(g\circ\varphi^{-1})}{\partial y_j}\frac{\partial y_j}{\partial x_i}$ $y_i = (\varphi\circ f\circ\psi$

So: $df_p(\partial x_i) = \sum_j \frac{\partial y_j}{\partial x_i}\partial y_j$

So $df_p(\gamma'(0)) = df_p\left(\sum_i x_i'(t)\,\partial x_i\right)$

$= \sum_j\sum_i \frac{\partial y_j}{\partial x_i} x_i'(t)\,\partial y_j$

$$\underbrace{\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial y_n}{\partial x_1} & & \frac{\partial y_n}{\partial x_m}\end{pmatrix}_{n\times m}}_{\text{Jacobi of } f} \begin{pmatrix}\partial x_1 \\ \vdots \\ \partial x_m\end{pmatrix}_{m\times n} = \begin{pmatrix}\partial y_1 \\ \vdots \\ \partial y_n\end{pmatrix}_{n\times 1}$$

Rank:

definition: We define rank of f at p is the rank of $(df)_p$ donate: $\mathrm{rank}_f(p)$

if $\forall p \in M$, $df_p = r$ donate $\mathrm{rank}_f = r$

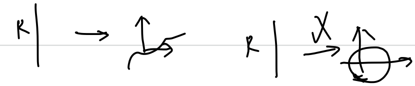
$f: M \to N$

Submersions: $\mathrm{rank}_f = \dim N$ (surjective) $(n<m)$

Immersions: $\mathrm{rank}_f = \dim M$ (injective) $(m<n)$

Embeddings: ① f is homeomorphism from M to $f(M)$

② : immersion

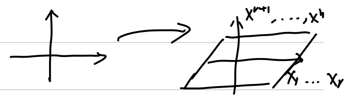


Rank Theorem

M^m, N^n $f: M \to N$ smooth, rank $f = k$.

$\forall p \in M$, $\exists (U,\psi) \ni p$, $(V,\varphi) \ni F(q)$

$F := \psi\circ f\circ\varphi^{-1}$ $F(x_1,\dots,x_m) = (x^1,\dots,x^r,0,\dots,0)$



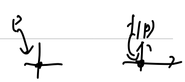
$(\tilde U,\tilde\varphi)$, $(\tilde V,\tilde\psi)$ of p. f(p)

$\tilde\varphi(\tilde u) = (x_1,\dots,x_k; x_{k+1},\dots,x_m) = (X_1, X_2)$

$\tilde\psi(\tilde V) = (y_1,\dots,y_k; y_{k+1},\dots,y_K) = (Y_1, Y_2)$

$\tilde F := \tilde\psi\circ f\circ\tilde\varphi^{-1} = (Y_1(X_1,X_2), Y_2(X_1,X_2))$

$\tilde F(0,0) = (0,0)$ ✓



since $(df)_p$ is rank r.

$(df)_{k\times k}$ is not singular:

or $DF_{k\times k}$ is not singula $\varphi(U) \to R^m$

consider: $G(X_1, X_2) = (Y_1(X_1,X_2), X_2)$

$DG = \begin{pmatrix} \frac{\partial Y_1}{\partial X_1}\Big|_{k\times k} & \frac{\partial Y_1}{\partial X_2} \\ 0 & I \end{pmatrix}$ is not singular.

$F\circ G^{-1}(W_1, W_2)$

$G:(\bar X_1, \bar X_2)\to(W_1, W_2)$

$F\ (\bar X_1, \bar X_2)\to(Y_1, Y_2)$

$(W_1, W_2)\xrightarrow{G^{-1}}(Y_1, Y_2)\xrightarrow{F}(W_1, \tilde Y_2(W_1, W_2))$

$F\circ G^{-1}(w_1, w_2) = (W_1, \tilde Y_2(W_1, W_2))$

$\tilde Y_2(w_1, w_2) = Y_2(\bar X_1(w_1, w_2), \bar X_2(w_2))$

$D(F\circ G^{-1}) = \begin{pmatrix} 1 & 0 \\ \frac{\partial Y_2}{\partial W_1} & \frac{\partial Y_2}{\partial W_2} \end{pmatrix}$

and $F\circ G^{-1}$ rank k

so $\frac{\partial Y_2}{\partial W_2} = 0$

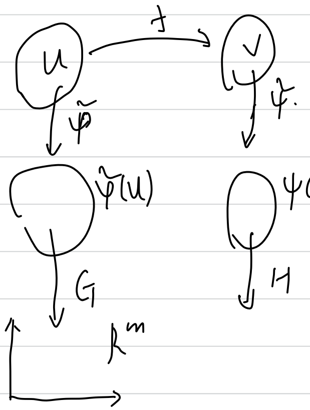
$F\circ G^{-1}(W_1, W_2) = (W_1, \tilde Y_2(W_1))$

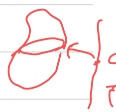
defn $H: V\to R^m$

$H(Y_1, Y_2) = (Y_1, Y_2 - \tilde Y_2(Y_1))$

$DH = \begin{pmatrix} 1 & -\frac{\partial \tilde Y_2}{\partial Y_1} \\ 0 & 1 \end{pmatrix}$ det > 0 smooth

$H\circ F\circ G^{-1}(W_1, W_2) = (W_1, 0)$



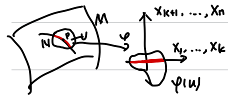


cite

Submanifolds:

$N\subset M^n$ is subset of SM   M

if $\forall p\in N$, $\exists (U,\varphi)\ni p$, $\varphi(U\cap N) = \{\dots, x_k, x_{k+1}=\dots=x_n=0\}$



Then, $(U\cap N, \varphi|_{U\cap N})$ given rise to a local chart of N at p. and U is submanifolds of M.

(equal)

Defn: Submanifolds

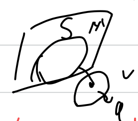
N^k is called (regular submanifolds of M if

$N\hookrightarrow M$ is an Embedding (see John. Lee Thm. 5.8)

$S = \Phi^{-1}(p)$, for $p\in N$

Thm: $\Phi: M^m\to N^n$ smooth map of constant rank k; Then, each level set of Φ Smooth submanifolds with dim. $m-k$

level set



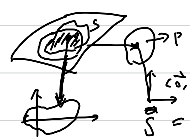
Proof: $\forall q\in N$, $\Phi^{-1}(q) = S$

special case : Φ immersion dim = 0

$\forall p\in S$ $\exists (U,\varphi)\ni p$ (V,ψ) of $\Phi(p)$

s.t. $\psi\circ\Phi\circ\varphi^{-1}(x_1,\dots,x_m) = (x_1,\dots,x_k,0,\dots,0)$ $\forall (x_1,\dots,x_m)$

so: $\varphi(S\cap U) = \{0,\dots,0,x_{k+1},\dots,x_n\}$ $\leftarrow \varphi(S\cap U)$



$F^{-1}(0,\dots,0)$

$S = \Phi^{-1}\circ\psi^{-1}(\vec 0)$

$\varphi\circ\Phi^{-1}\circ\psi^{-1}(\vec 0) = (0,\dots,0,x_{k+1},\dots,x_n)$

[Constant Rank Level set]

Defn: $f: M^m \to N^n$ map.

$p \in M$ is regular pts if $(df)_p$ is surjective ($\text{rank}_f(p) = n$)

$p \in M$ is critical pt if $\not\nearrow$

$q \in N$ is regular value if $\forall p \in f^{-1}(q)$ is regular pts

critical value if . . .

$f^{-1}(q)$ is call regular level set

Thm. $\Phi: M^m \to N^n$ (not necessary to be CR)

regular level set is submanifolds.

Proof: $\tilde{M} = \{p \mid p \text{ is regular pt}\}$

$\tilde{M}$ is open in M ?

obversly: $\Phi^{-1}(q) \subseteq M$

$\Phi|_{\tilde{M}}$ is constant rank

$\Phi^{-1}(q) \hookrightarrow \tilde{M}$ according to last thm.

and $\tilde{M} \hookrightarrow M$?

So $\Phi^{-1}(q) \hookrightarrow M$

Example:

1. $M_n = \{n \times n$ matrics of $\mathbb{R}\}$ is a smooth manifolds of $\dim(n^2)$

$[a_{ij}] \simeq \mathbb{R}^{n^2}$ ✓

2. $GL_n = \{A \in M_n \mid \det(A) \neq 0\}$ is a smooth ... of $\dim n^2$

3. $S_n = \{A \in GL_n \mid A = A^T\}$ dim $\frac{n(n+1)}{2}$

4.? $O_n = \{AA^T = Id\}$ dim?

$\Phi: GL_n \to S_n$

$A \to AA^T$ Proof: Id is regular value.

Vector Fields

1. define: $X: M \to TM$

$p \to X_p$

$(U, (x^i)) \ni p$

: $X_p = \sum_i X^i(p) \frac{\partial}{\partial x^i}\Big|_p$ $X^i: U \to \mathbb{R}$

$\mathfrak{X}_M = \{$collection of all smooth functions$\}$

X as a map: $C^\infty \to C^\infty$ by define

$Xf = \sum \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} X^i$

or $(Xf)(p) = X_p f$

Lie bracket

$[X, Y]: C^\infty \to C^\infty$

$f \mapsto X(Y(f)) - Y(X(f))$

Proper: ① $[\lambda_1 X_1 + \lambda_2 X_2, Y] = \lambda_1 [X_1, Y] + \lambda_2 [X_2, Y]$

② $[X, Y] = -[Y, X]$

③ $[X, [Y, Z]] + \cdots\cdots$

Intergral curve of vector Fields:

$X \in \mathfrak{X}(M)$ $\alpha: (-\varepsilon, \varepsilon) \to M$ is intergral curve if

if $\alpha'(t) = X_{\alpha(t)}$

under (U, φ) $\alpha([-\varepsilon, \varepsilon]) \subseteq U$

$X_{\alpha(t)} = \sum_i X^i(\alpha(t)) \frac{\partial}{\partial x^i}$

$\alpha'(t) = \sum_i \dot{\alpha}^i \frac{\partial}{\partial x^i}$

$\dot{\alpha}^i = X^i(\alpha^1, \ldots, \alpha^n)$

Recall Thm: Local existance and unique.

(1) $\forall (\alpha^1(0), \dots, \alpha^n(0)), \exists \varepsilon > 0$, $\exists!$ solution define on $(-\varepsilon, \varepsilon)$

(2) if $\alpha^i(t) = \beta^i(t)$ at some point t_0, they are same.

(3) $\forall \vec{x_0} = (\alpha^1(0), \dots, \alpha^n(0))$, $\exists U \ni \vec{x_0}$, s.t. $\exists \varepsilon > 0$ $\alpha((-\varepsilon, \varepsilon)) \subset U$

Defn: Vector Fields is complete if integral curve is defn on R

anti-example: $X = x^2 \partial x$ on R $\quad c_X(p)(t)$ is

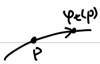
$\alpha'(t) = \alpha^2(t)$

$\alpha = -\frac{1}{t}$ $\quad t = 0$ not defn

if X is complete on M

we define $\varphi_t : M \to M$ by

$\varphi_t(p) = \Phi(t, p)$ $\quad \Phi(t,p)$ is intergral curve start at p



$$\begin{cases} \Phi'(0, p) = X_p \\ \Phi(0, p) = p. \end{cases}$$

$\varphi_{-t} = \varphi_t^{-1}$

$$\begin{cases} \varphi \text{ is smooth} \\ \varphi \text{ is one to one and on to.} \end{cases}$$

Thm: If M is closed $\begin{cases} \text{compact} \\ \text{no boundary.} \end{cases}$

$\forall X \in \mathfrak{X}(M)$, X is complete.

Proof: $\forall y \in U_p$, we can find $(-\varepsilon_p, \varepsilon_p)$ $c_X(p)(t)$

$\{U_p\}^{open}$ is open covering of M

$\varepsilon = \sup\{\varepsilon_p\}$

$\exists (-\varepsilon, \varepsilon)$, $\forall p \in M$, $c_p(t)$ defined here.

Partition of Unity

1. bump-function (cut-off function)

support of f: $\operatorname{supp}(f) = \overline{\{p \mid f(p) \neq 0\}}$

Properities of bump function

$\exists f : R^n \to R$ Smooth

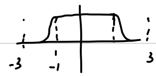
① $0 \le f \le 1$ $\qquad B_r = \{x \mid |x| < r\}$.

② $f|_{B_1} \equiv 1$

③ $\operatorname{supp}(f) \subset B_3$

Proof: $f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0 \end{cases}$

$h(t) = \dfrac{f(3-t)}{f(3-t) + f(t-1)}$



Lemma:

X is TS,

$\{U_\alpha\}$ is covering of X, $\bigcup_\alpha U_\alpha = X$

$\{V_\beta\}$ is refinement of $\{U_\alpha\}$ if $\begin{cases} \bigcup_\beta V_\beta = X \\ \forall \beta, \exists \alpha, \ V_\beta \subset U_\alpha \end{cases}$

$\{U_\alpha\}$ Local finity $\forall p \in X$, $\exists V^{open} \ni p$, s.t. $\underbrace{V \cap U_\alpha \neq \emptyset}_{\text{finite } \alpha}$

example: $U_n = (0, 2^{-n})$

$\bigcup_{n=-\infty}^{\infty} = (0, +\infty)$

$\exists n$, $x < 2^{-n}$.

X is para compact

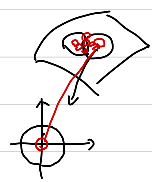
Lemma: $\{V_\alpha\}$ is open covering of M (Manifolds)

$\exists$ at most countable $\{V_\beta\}$ a refinement of $\{U_\alpha\}$ and is locally finite.

Let: (U_β, φ_β) is charts of M satisfy:

$\varphi_\beta(U_\beta) = B_3$

$\{\varphi_\beta^{-1}(B_1)\}_{\beta=1}^{\infty} = M$



Secondary

Defn:

$\{f_i\}_{i=1}^{\infty}$ $f_i \in C^{\infty}(M)$ is call partition of unity of

① $\forall i$ $0 \le f_i \le 1$

② $\{\mathrm{Supp}(f_i)\}_{i=1}^{\infty}$ is locally finite covering of M

③ $\sum_{i=1}^{\infty} f_i(x) = 1$

Defn: if $\{\mathrm{Supp}(f_i)\}_{i=1}^{\infty}$ is refinement of U_α then $\{f_i\}$ is said to be subordinate of $\{U_\alpha\}$

Theorem:

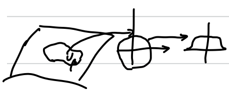
M sm, $\{U_\alpha\}$ open covering of M.

$\exists$ partition of Unit $\{f_i\}_{i=1}^{\infty}$ subordinate to $\{U_\alpha\}$

by lemma: $\exists \{U_\beta\}_{\beta=1}^{\infty}$ locally finite refinement of U_α

and $\{U_\beta, V_\beta\}$ $\varphi_\beta(U_\beta) = B_3$ $\{\varphi_\beta^{-1}(B_1)\}_{\beta=1}^{\infty}$ is open covering (also refinement of $\{U_\beta\}$)

$$\tilde{f}_\beta(x) = \begin{cases} h \circ \varphi_\beta & x \in U_\beta \\ 0 & x \notin U_\beta \end{cases}$$



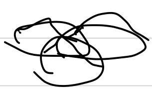
$F = \sum_{\beta=1}^{\infty} \tilde{f}_\beta$

① F is smooth $\forall p$, only finite β, $p \in U_\beta$

② $F > 0$ $\{\varphi_\beta^{-1}(B_1)\}_{\beta=1}^{\infty} = X$ $\Downarrow$ $\forall p$, $\exists \beta$ $\tilde{f}_\beta(p) = 1 > 0$

and $\tilde{f}_\beta \ge 0$, so $F > 0$

$f_\beta = \dfrac{\tilde{f}_\beta}{F}$



Exterior Algebra.

V n-dim vector space $\quad V=\mathbb{R}^3$

$\{e_1,\dots,e_n\}$ is a basic

we defi $\quad a\in\Lambda^2V = a_1 x\wedge y + a_2 y\wedge z + a_3 x\wedge z \quad \dim(\Lambda^k V)=\binom{n}{k}$

$\Lambda^k V := \operatorname{span}\{e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k} \mid i_1<i_2<\cdots<i_k\}$

$\Lambda^* V := \bigoplus_{k=0}^{n}\Lambda^k V \quad \rightarrow$ external algebra of V. multiply was defined by $\wedge$

satisfy: $e_i\wedge e_j = -e_j\wedge e_i \quad (e_i\wedge e_i = 0)$

Algebra: Vector space + Ring

$\forall u,v\in\Lambda^* V \qquad u\wedge v\in\Lambda^* V.$

$\lambda(u\wedge v) = (\lambda u)\wedge v = u\wedge(\lambda v) \qquad u\wedge v = (-1)^{kl}\, v\wedge u$

Dual vector space:

V^* linear map for V to $R \quad \dim(V^*)=n$

$\{e_1^*,\dots,e_n^*\}$ and $e_i^*(e_j)=\delta_{ij}$

k-alternating form $\qquad$ at one point

$w: \underbrace{V\times V\times\cdots\times V}_{k} \to R$

1. multi linear
   $w(aX_1+bY_1, X_2,\dots,X_n) = a w(X_1,X_2,\dots,X_n) + b w(Y_1,\dots,Y_n)$
2. $w(X_{\sigma(1)},\dots,X_{\sigma(n)}) = \operatorname{sgn}(\sigma)\, w(X_1,\dots,X_n)$

同构

exterior algebra on V^*

$\Lambda^* V^* = \bigoplus_{k=0}^{n}\Lambda^k V^* \quad$ (k-covector

$w\in\Lambda^k V^* \quad$ assume $w = \alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_n.$

where $\alpha_i = \sum_{j=1}^{k}\lambda_{ij} e_j^* \in \Lambda^1 V^* \quad$ (or dual vector)

Define: $w(V_1,\dots,V_k) = \frac{1}{k!}\det(\alpha_i(V_j))$

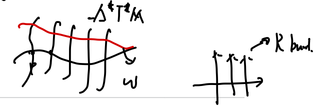
we can easily check w is k-alternating form

We define the alternating tensor field of V is $\Omega^k M$ (under Local chart)

$w\in\Omega = \sum_I w_I\, dx^I \qquad w_I$ is a continuous function on V.

and k is the order.

ω: a section of $\Lambda^* T^* M$



Differential forms on $\mathbb{R}^n$

$f\in C^\infty(U) \qquad df = \sum_i \frac{\partial f}{\partial x_i}dx_i$

$f\in\Omega^k(M) \qquad \Omega^k(M) = \operatorname{span}\{dx_{i_1}\wedge dx_{i_2}\cdots dx_{i_k} \mid i_1<i_2<\cdots<i_k\}$

Exterior differention

$d: \Omega^k(M)\to\Omega^{k+1}(M)$

$\omega \to d\omega$

$d\Big(\sum_{|I|=k} f_I dx_I\Big) = \sum_I (df_I)\wedge dx_I$

$d\circ d = 0.$: $\quad df_I = \sum_i\Big(\frac{\partial f_I}{\partial x_i}dx_i\Big)$

$d\omega = \sum_I\sum_i \frac{\partial f_I}{\partial x_i} dx_i\wedge dx_I$

$d^2\omega = \sum_I\sum_{i,j}\frac{\partial^2 f_I}{\partial x_i\partial x_j} dx_j\wedge dx_i\wedge dx_I$

$= \sum_I\sum_{i<j}\Big(\frac{\partial^2 f_I}{\partial x_i\partial x_j}\Big)(dx_j\wedge dx_i + dx_i\wedge dx_j) = 0$

Differential forms on M, w is call k-form (section of $\Lambda^k\dots$ bundle, $\forall$), $\Omega^k(M) = \{w\}$ collection

i) $w_p\in\Lambda^k T_pM \qquad \Omega^k(M)$ is section of $\Lambda^k T_pM$ bundle.

ii) (U,φ) Local chart $\qquad w = \sum_I f_I dx^I \qquad$ (on different (U,φ)?)

w is also a smooth tensor field on M $\rightarrow$ 不一样 $\quad$ 上面 dx_i 是 $\mathbb{R}^n$ 中的基

Note: $\quad T_pM = \operatorname{span}\{\partial x_1,\dots,\partial x_n\}$

covector $T_p^*M \quad \operatorname{span}\{dx_1,\dots,dx_n\} \qquad dx_i: T_pM\to R$

$= \Lambda^1 T_pM$

下面 dx_i 是 T_p^*M 的基，形式上一样是因为我们自动略去了 φ

Satisfy: $dx_i(\partial x_j) = \delta_{ij}$

at M

Defn 2:

$w: \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k} \to C^\infty(M).$

with: ① $w(X_{i_1},\dots,X_{i_k}) = \operatorname{sgn}(\sigma)\, w(X_1,\dots,X_k)$

② multi linear.

Operators

1. $d: \Omega^k(M) \to \Omega^{k+1}(M)$ $\quad i_X: \Omega^k(M) \to \Omega^{k-1}(M)$

$\omega \to d\omega$

$d\omega(X_1, \dots, X_N) =$ $\quad (i_X\omega)(X_1, \dots, X^{k-1}) = \omega(X, X_1, \dots, X^{k-1})$

$\frac{1}{k+1}\{\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \hat{X_i}, \dots X_{k+1}))$

$+\sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X_i}, \dots, \hat{X_j}, \dots, X_{k+1})$

or $\omega \mapsto d\omega$

chart $\downarrow$

$\sum f_I dx^I \to \sum df_I \wedge dx^I$

2. Push forward on vector field

$f: M \to N$ smooth function

$\Downarrow$ induce a push forward of X vector field

$f_* X \in \mathfrak{X}(N)$

$(f_*X)_{f(p)} = df_p(X_p)$ $\quad f^{-1}_* = (f_*)^{-1}$ can we?

$M \xrightarrow{f} N$, $\quad N \xrightarrow{f^{-1}} M$

$(f^{-1})_* = (f_*)^{-1}$ $\quad T_{f(p)}N \to T_pM$

$(f^{-1})_*X = [(f^{-1})_*X]_p = d(f^{-1})_{f(p)} X_{f(p)}$

$= (df_p)^{-1} X_{f(p)}$ delte

$(f_*[(f^{-1})_*X])_{f(p)}$

$= (df_p)([(f^{-1})_*X]_p)$

$= X_{f(p)} \; df_p^{-1}$

$f_* \circ (f^{-1})_* = 1$

$(f^{-1})_* = (f_*)^{-1}$

3 Pull back on cotvectors fields

$f: \omega \in \Omega^1(N)$ $\quad f: M \to N$ $\quad \omega \in \Omega^1(N)$

$(f^*\omega)(X) = \omega(f_*X)$ $\quad f^*\omega \in \Omega^1(M)$

$(f^*\omega)_p(X_p) = \omega((f_*X)_{f(p)})$

$= \omega(df_p(X_p))$

Push forward and Pull back on Tensor fields

$\Gamma(T^{(k,l)}TM)$ $\quad \Gamma(T^{(k,0)}TM) = \mathfrak{X}^k(M)$

$f_*: \Gamma(T^{(k,l)}TM) \to \Gamma(T^{(k,l)}TN)$

$(f_*A)(X_1, \dots, X_l, \omega_1, \dots, \omega_k) = A(f_*^{-1}X_1, \dots, f_*^{-1}X_l, f^*\omega_1, \dots, f^*\omega_k)$

$(f^*A)(X_1, \dots, X_l, \omega_1, \dots, \omega_k) = A(f_*X_1, \dots, f_*X_l, f^{*-1}\omega_1, \dots, f^{*-1}\omega_k)$

$f_*(f^*A)(X_1, \dots, X_l, \omega_1, \dots, \omega_k) = A(f_*^{*}X_1, \dots, f_*\omega_1, \dots, f_*\omega_k)$

$= (f^*A)(f_*X_1, \dots, f_*X_l, f^{*-1}\omega_1, \dots, f^{*-1}\omega_k)$

$= A(X_1, \dots, \omega_1, \omega_2, \dots \omega_k)$

$f_* = f^{*-1}$

$f: M \to N$

$(f_*)^{-1} = (f^{-1})_*$?

5: Lie derivative: $\mathcal{L}_X$ $\quad X \in \mathfrak{X}(M)$ and complete

$\theta_t: M \to M$

$\theta_t^*: \Gamma(T^{(k,l)}TM) \to \Gamma(T^{(k,l)}TM)$

$\theta_{t*}:$ $\quad \vdots$ $\quad (F^*_*)^{-1} = F^{-1}_*$?

Define: $\mathcal{L}_X A = \frac{d}{dt}(\theta_t^* A)$

for $Y \in \mathfrak{X}(M)$

$\mathcal{L}_X Y = \frac{d}{dt}(\theta_t^* Y) = \frac{d}{dt}(\theta_{t*}^{-1} Y) = \frac{d}{dt}(\theta_{-t*} Y)$

$= \lim_{t \to 0} \frac{\theta_{-t*}Y - Y}{t}$ $\quad T_pM \leftarrow \mathfrak{X}(M) \ni Y$

$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{(\theta_{-t*}Y)_p - Y_p}{t}$

$= \frac{(d\theta_{-t})_{\theta_t(p)} Y_{\theta_t(p)} - Y_p}{t}$

$(\mathcal{L}_X Y)_p f = \frac{((d\theta_{-t})_{\theta_t(p)} Y_{\theta_t(p)})f - Y_p f}{t}$ $\quad df_p(v)(g) = v(g \circ f)$

$= \frac{Y_{\theta_t(p)}(f \circ \theta_{-t}) - Y_p f}{t}$

$= \frac{Y_{\theta_t(p)}(f \circ \theta_{-t}) - Y_{\theta_t(p)}(f) + Y_{\theta_t(p)} f - Y_p f}{t}$

$= Y_{\theta_t(p)}\left(\frac{f \circ \theta_{-t} - f}{t}\right) + \frac{Y_{\theta_t(p)} f - Y_p f}{t}$

$\downarrow$

Xf

$= -Y_p(Xf) + X_p(Yf)$

$= [X, Y]_p f$

$\mathcal{L}_X Y = [X, Y]$

$(\mathcal{L}_X \omega) = \frac{d}{dt}(\theta_t^* \omega)$

$(\mathcal{L}_X\omega)(X_1,\dots,X_k) = \frac{(\theta_t^*\omega)(X_1,\dots,X_k) - \omega(X_1,\dots,X_k)}{t}$

$= \frac{\omega(\theta_{t*}X_1,\dots,\theta_{t*}X_k) - \omega(X_1,\dots,X_k)}{t}$

on $\omega\in\Omega^k(M)$

$\mathcal{L}_X = i_X \circ d + d\circ i_X$ Cartan.

on $\Omega^k(M)$

$\mathcal{L}_X\circ i_Y - i_Y\circ \mathcal{L}_X = i_{[X,Y]}$ Excersise

Summary:
$$\begin{cases} i_Y: \Omega^k\to\Omega^{k-1} \\ d:\Omega^k\to\Omega^{k+1} \\ \mathcal{L}_X: \Omega^k\to\Omega^k \\ f^*: \Omega^k(N)\to\Omega^k(M) \\ f_*: \mathfrak{X}(M)\to\mathfrak{X}(N) \end{cases} \Rightarrow \begin{matrix} T^kTM\to T^kTM \\ T^kTN\to T^kTM \\ T^kTM\to T^kTN \end{matrix}$$

Integral of ω :

M^n $\quad\omega\in\Omega^n(M)$ in (U,φ), (U,ψ)

$\psi\downarrow$
$$\begin{cases} \omega|_\varphi = f(x_1,\dots,x_n)\,dx_1\wedge dx_2\wedge\cdots\wedge dx_n \\ \omega|_\psi = g(y_1,\dots,y_n)\,dy_1\wedge\cdots\wedge dy_n \end{cases}$$

DF of $\mathbb{R}^n$ DF of M DF of $\mathbb{R}^n$

define: $\int_U \omega = \int_{\varphi(U)} \omega|_\varphi$

$\omega = \int_D f(x_1,\dots,x_n)\,dx_1\wedge\cdots\wedge dx_n$

$= \int_D f(x_1,\dots,x_n)\,dx_1dx_2\cdots dx_n$

Is that well defined???

$g(y_1,\dots,y_n)\,dy_1\wedge dy_2\wedge\cdots\wedge dy_n$

$y_1 = y_1(x_1,\dots,x_n)$

$\wedge\; dy_1 = \sum_i \frac{\partial y_1}{\partial x_i}dx_i \wedge \sum_i \frac{\partial y_2}{\partial x_i}dx_i\wedge\cdots$

$\Rightarrow \det\left(\frac{\partial y_i}{\partial x_i}\right)$

So $g\begin{pmatrix} y_1(x_1\cdots x_n) \\ \vdots \\ y_n(x_1\cdots x_n)\end{pmatrix}\det\left(\frac{\partial y_i}{\partial x_i}\right) = f(x_1,\dots,x_n)$

well defined:

$\left|\det\left(\frac{\partial y_i}{\partial x_j}\right)\right| = \det\left(\frac{\partial y_i}{\partial x_j}\right) \Rightarrow \det>0.$

paracompact is need! no, every manifolds is paracompact

M is call orientation if $\exists(U_\alpha,\varphi_\alpha)$

$\bigcup_\alpha U_\alpha = M$ and $\det(\varphi_\alpha\circ\varphi_\beta^{-1})>0$

for orientation compatible covering $\{(U_\alpha,\varphi_\alpha)\}$ is local finity.

$\Downarrow$

$\exists$ partition of unity:
$$\begin{cases} \mathrm{supp}\{f_\alpha\}\subseteq U_\alpha \quad \text{refine} \\ \sum_\alpha f_\alpha \equiv 1 \Rightarrow \bigcup_\alpha \mathrm{supp}\{f_\alpha\} = M \\ 0\le f_\alpha\le 1 \end{cases}$$

If supp(ω) is compact

$\mathrm{supp}(\omega) = \overline{\{p\in M, \omega_p\ne 0\}}$

we define

$\int_M\omega = \underbrace{\sum_\alpha}_{\text{only finite sum is non-zero}} \int_{U_\alpha} f_\alpha\omega$

$= \sum_\alpha \int_{\mathrm{supp}(f_\alpha)\cap\mathrm{supp}(\omega)} f_\alpha \omega$

finite compact

= finite

if $\{g_\beta\}_\beta$ is also a P.o.U

$\sum_\beta\int_{U_\beta} g_\beta\omega = \sum_\alpha\sum_\beta\int_{U_\beta\cap U_\alpha} g_\beta f_\alpha\omega$

‖ because $\sum_\beta\sum_\alpha\int g_\beta f_\alpha\omega$ is finite!

$\sum_\alpha\int_{U_\alpha} f_\alpha\omega = \sum_\beta\sum_\alpha\int_{U_\beta} f_\alpha\omega\, g_\beta$

Stokes

$\omega\in\Omega^{n-1}(M)$ $\quad d\omega\in\Omega^n(M)$ and supp(ω) is compact,

then: $\int_M d\omega = \int_{\partial M}\omega$

Remark : ∂M is orientable, and we can define +

if V is vector space with $\dim(V)=n$

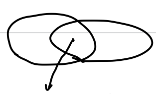
we call span$\{e_1,\dots,e_n\}$ ~ span$\{e_1',\dots,e_n'\}$ iff $\det(a_{ij})>0$

$V/\sim = \{[+1],[-1]\}$

M orientable $\to \exists(U_\alpha,\varphi_\alpha)$ $\bigcup U_\alpha = M$ and

$\det(\varphi_\alpha\circ\varphi_\beta^{-1})>0$ $\Updownarrow$

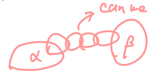
at every point p $\{\partial_1,\dots,\partial_n\}_p$ is on same orientable



$p\{\partial_1,\dots,\partial_n\}$

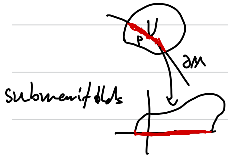
proof $\forall U_\alpha, U_\beta$, $\{dx_1^\alpha,\dots,dx_n^\alpha\}\sim\{dx_1^\beta,\dots,dx_n^\beta\}$

can we?



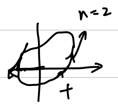
define induced orientation of ∂M as

$\forall p \in \partial M \quad \exists (U,\varphi)$ p ori



∂M $\quad \varphi_U$ is open in $\mathbb{R}^n_+$?

$T_p \partial M$ positive orientation define : $\{(-1)^n \partial_1, \dots, \partial_{n-1}\}$



for $\mathbb{R}$

$\partial x, \partial y$

Carry our proof on $\mathbb{R}^n$ or H^{n+1}

$$\omega = \sum_{i=1}^{n} f_i(x)\, dx_1 \wedge \cdots \widehat{dx_i}, \dots, dx_n$$

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots dx_n$$

$$\int_{\mathbb{R}^n} d\omega = \sum_{i=1}^{n} \int (-1)^{i-1} \int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i \; d\mathbb{R}^{n-1}$$

since $\operatorname{supp}\{f_i(x)\}$ is compact.

$$\frac{\partial f_i}{\partial x_i} dx_i = f_i \Big|_{-\infty}^{\infty} \quad \text{since } f_i \text{ support compact}$$

$$\int_{\partial \mathbb{R}^n} \omega = 0. \quad \checkmark$$

if $\displaystyle \int_{H^n} d\omega = \sum_{i=1}^{n} \int_{H^n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \dots dx_n$

only when $i = n$

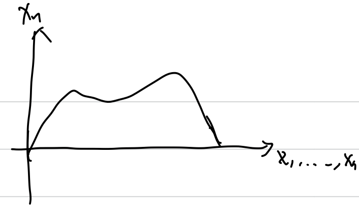
$$= (-1)^{n-1} \int_0^{\infty} \frac{\partial f_n}{\partial x_n} dx_n \cdot \int_{\mathbb{R}^{n-1}} dx_1 \cdots dx_{n-1}$$

$$= \quad f_n(x_1, \dots, x_{n-1}, \infty) - f_n(x_1, \dots, x_{n-1}, 0)$$

$$= (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x_1, \dots, x_{n-1}, 0)\, dx_1, \dots, dx_n$$

$$\int_{\partial H_n} \omega = (-1) \sum_{i=1}^{n} \int f_i(x_1, \dots, x_{n-1}, 0)\, dx_1\, dx_2 \cdots \widehat{dx_i}, \dots, dx_n$$

$i \neq n \implies \int_0^0 dx_n = 0$



$$\int_{\partial H_n} \omega = (-1)^n \int_n (\dots, 0)$$

$$dx_1 \dots dx_{n-1}$$

So; our result is right for $\omega \in \Lambda^{n-1}(H^n)$ and $\omega \in \Lambda^k(\mathbb{R}^{n-1})$ / compact.

if ω support on single chart (U, φ)

$$\int_M d\omega = \underbrace{\int_{H^n} (\varphi^{-1})^* d\omega}_{\text{compact.}} = \int_{H^n} d((\varphi^{-1})^* \omega)$$

$$= \int_{\partial H^n} (\varphi^{-1})^* \omega$$

$$= \int_{\partial M} \omega \qquad (\varphi^{-1})(\partial H^n) = \partial M$$

and

$$\omega = \sum_\alpha f_\alpha \omega \qquad \int_{\partial M} \sum_\alpha f_\alpha \omega = \sum_\alpha \int_{\partial M} f_\alpha \omega = \sum_\alpha \int_M d(f_\alpha \omega)$$

$$\sum_\alpha d(f_\alpha \omega) = \sum_\alpha f_\alpha d\omega + \sum_\alpha d f_\alpha \wedge d\omega$$

$$= d\omega + d\Big(\sum_\alpha f_\alpha\Big) \wedge d\omega$$

$$= d\omega$$

$$\text{So} = \int_M d\omega$$

de Rham Cohomology:

$d: \Omega^k(M) \to \Omega^{k+1}(M)$

$Z^k := \ker(d: \Omega^k(M) \to \Omega^{k+1}(M))$ $\Rightarrow$ closed form

$B^k := \ker(d: \Omega^{k-1}(M) \to \Omega^k(M))$ $\Rightarrow$ exact form

Obviously $B^k \subset Z^k$ since $d^2 = 0$

define: $H^k_{dR}(M) := Z^k / B^k$ $\leftarrow$ M的第k个 dR上

In general:

chain complex: $0 \xrightarrow{d} A^0 \xrightarrow{d} A^1 \to \cdots \to A^n \to 0$

if $d^2 = 0$

$$H^k = \frac{\ker(d \; k \to k+1)}{\operatorname{Im}(d: k-1 \to k)}$$

de Rham:

$[\omega] \in H^k(M)$ $\quad d\omega = 0$

$[\omega] = [\omega']$ iff $\omega - \omega' = df$ $\Rightarrow \omega$ cohomology to ω'

Ex $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ $\quad M = \mathbb{R}^2 \setminus \{0\}$

Induced Cohomology maps: (Functional)

$F: M \to N$ $\quad F^*: \Omega^k(N) \to \Omega^k(M)$ $\Rightarrow F^*$ $H^k_{dR}(N) \to H^k_{dR}(M)$

① if ω is ~~exact~~ closed in N

$d(F^*\omega) = F^*(d\omega) = 0$

② if ω is exact in N

$F^*(d\eta) = d(F^*\eta)$

$\Rightarrow$ $F^*(Z^k(N)) \subseteq Z^k(M)$, $F^*(B^k(N)) \subseteq B^k(M)$

define $F^*: H^k_{dR}(N) \to H^k_{dR}(M)$

$F^*[\omega] = [F^*\omega]$

if $[\omega'] = [\omega]$ $\quad \omega' = \omega + d\eta$

$[F^*\omega'] = [F^*\omega + F^* d\eta] = [F^*\omega + d(F^*\eta)]$

$= [F^*\omega]$

well defined.

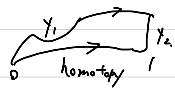
Homotopy and Foundamental Group.

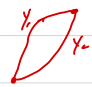
X, Y TSs $\quad F_0, F_1: X \to Y$ continuous

a homotopy from F_0 to F_1 is a continuous map

$H: X \times I \to Y$

$$\begin{cases} H(x,0) = F_0 \\ H(x,1) = F_1 \end{cases}$$





homotopic relative $\{0,1\}$

$\Downarrow$

path-homotopic

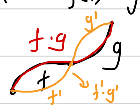
path-homotopic $f_0 \sim f_1$ if $I \times I$

$$\begin{cases} H(s,0) = f_0(s) \\ H(s,1) = f_1(s) \\ H(0,t) = f_0(0) = f_1(0) \\ H(1,t) = f_0(1) = f_1(1) \end{cases}$$

$f_1 \sim f_2$ equivalence relation so

$[f] \Rightarrow$ path class

$f, g \quad I \to X \qquad f(1) = g(0)$



product: $f \cdot g(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$

$\Downarrow$

$f' \cdot g' \sim f \cdot g$

$[f] \cdot [g] = [f \cdot g]$ $\Rightarrow$ a group

2. Homotopy equivalence:

$F: X \to Y$ smooth maps, if $\exists\, G: Y \to X$ conti...

and $G \circ F \simeq Id_X$ $\quad F \circ G \simeq Id_Y$ $\Rightarrow F$ is homotopy equalvelence

and X, Y homotopy equalvalent.

$S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$

$\iota: S^{n-1} \hookrightarrow \mathbb{R}^n \qquad t \circ \iota = Id_{S^{n-1}}$

$t: r(x) \to \frac{x}{|x|} \qquad \iota \circ t = \frac{x}{|x|} \qquad tx + (1-t)\frac{x}{|x|}$

① de Rham is homotopy invariant.

If

$M \to N$ is homotopy equivalent, $H^k_{dR}(M) \cong H^k_{dR}(N)$

1. Star shape: $\iota : \{c\} \hookrightarrow U$. $f : U \to \{c\}$. $f \circ \iota = Id_c$, $\iota \circ f \simeq$

$H(x,t) = c + t(x-c)$

$H^k_{dR}(U) \simeq H^k_{dR}(\{c\})$

$H^k_{dR}(U) = 0$ for $k \geq 1$

② $H^0_{dR}(M) \cong \mathbb{R}$ for connected M.

$$\begin{cases} Z^0 = f & df = 0 \Rightarrow f = C \\ B^0 = \{0\} & Z^0/B^0 \cong \mathbb{R} \end{cases}$$

③ $H^1_{dR}(M) \to \mathrm{Hom}(\pi_1(M,q), \mathbb{R})$ Fundamental group based and q

$[\omega]$

$\Phi[\omega] : \pi_1(M,q) \to \mathbb{R}$

$\Phi[\omega][\gamma] = \int_\gamma \omega$ not enough tools.

well defined and injective (Surject)

$[\omega']$ $[\omega + d\omega]$ $\int_\gamma d\omega = \int_{\partial\gamma} \omega = 0$

$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$ [homework]

Injective

If: $0 = \int_{\varphi} \omega$

$= \int_\alpha \omega + \int_\sigma \omega - \int_\alpha \omega$

$= \int_\sigma \omega$ So: $\int_\gamma \omega = 0$ $\forall \gamma$ is closed.

$\Downarrow$ $\omega = dg = [0]$

The MV Thm:

$M = U^{open} \cup V^{open}$ and $U \cap V \neq \phi$

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U\cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

i, j, k, l inclusion map

$$\begin{tikzcd} & \Omega^k(U) \arrow[dr, "i^*"] & \\ \Omega^k(M) \arrow[ur, "k^*"] \arrow[dr, "l^*"'] & & \Omega^k(U\cap V) \\ & \Omega^k(V) \arrow[ur, "j^*"'] & \end{tikzcd}$$

$\Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U\cap V)$

$(k^* \oplus l^*)(\omega) = (\omega|_U, \omega|_V)$

$(i^* - j^*)(\omega, \eta) = \omega|_{U\cap V} - \eta|_{U\cap V}$

$\ker(F^p) = \mathrm{Im}\, F_{p-1}^{p-1}$

$\boxed{\text{Exact sequence}}$: $V^{p-1} \xrightarrow{F^{p-1}} V^p \xrightarrow{F^p} V^{p+1} \cdots$

MV state: $\exists \delta_k : \Omega^k(U\cap V) \to \Omega^{k+1}(M)$

and sequence is exact

$\xrightarrow{\delta_{k-1}} \Omega^k(M) \to \quad \to \quad \xrightarrow{\delta_k} \Omega^{k+1}(M)$

Short exact:

$0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0 \Rightarrow$ ψ is injective. $\ker(\varphi) = \mathrm{Im}(\psi)$ φ is surjective

$\ker(\psi) = 0$ $\mathrm{Im}(\varphi) = C$

Zigzag lemma: (and commutative)

If: $0 \to A^p \xrightarrow{F} B^p \xrightarrow{G} C^p \to 0$ short exact

$\forall p$, $\exists \delta : H^k(C) \to H^{p+1}(A)$ s.t.

$\to H^k(A) \xrightarrow{F^*} H^k(B) \xrightarrow{G^*} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \to \cdots$

is exact

$$\begin{tikzcd} 0 \arrow[r] & A^k \arrow[r, "F"] \arrow[d, "d"] & B^k \arrow[r, "G"] \arrow[d, "d"] & C^k \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A^{k+1} \arrow[r, "F"] \arrow[d] & B^{k+1} \arrow[r, "G"] \arrow[d, "d"] & C^{k+1} \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A^{k+2} \arrow[r, "F"] & B^{k+2} \arrow[r, "G"] & C^{k+2} \arrow[r] & 0 \end{tikzcd}$$

F, G, d is commute

$\ker(G) = \mathrm{Im}(F)$

since G is surjective.

$$\begin{array}{ccccccccc}
0 \to & A^k & \xrightarrow{F} & B^k & \xrightarrow{G} & C^k & \to 0 \\
& \downarrow d & & \downarrow d & & \downarrow d & \\
0 \to & A^{k+1} & \xrightarrow{F} & B^{k+1} & \xrightarrow{G} & C^{k+1} & \to 0 \\
& \downarrow & & \downarrow d & & \downarrow d & \\
0 \to & A^{k+2} & \xrightarrow{F} & B^{k+2} & \xrightarrow{G} & C^{k+2} & \to 0
\end{array}$$

b^k, $Gb^k = c^k$, $[c^k]$, $d a^{k+1}$

F, G, d is commute

$dc^k = 0$ by define

G is surject

$\exists\, b^k \quad Gb^k = c^k$

since $G(d\, db^k \in$

$\exists\, a^{k+1}$

$= db^k$.

$I(Gb^k) = 0$

$(G) = \mathrm{Im}(F)$

$F(d a^{k+1}) = d(F$

$= 0$

So $da^{k+1} = 0$

$\delta: c^k \to a^{k+1}$

define $\delta: [c^k] \to [a^{k+1}]$

Proof MV.

①

$$0 \to \Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U\cap V) \to 0$$

short

③

① $k^* \oplus l^*$ is injective:

if $(\omega|_U, \omega|_V) = (0,0)$

$\Downarrow$

$\omega = 0$

② $(i^* - j^*)(\omega|_U, \omega|_V) = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$

a. $\mathrm{Im}(k^* \oplus l^*) \subseteq \mathrm{Ker}(i^* - j^*)$

$0 = (i^* - j^*)(y, y') = y|_{U\cap V} - y'|_{U\cap V}$

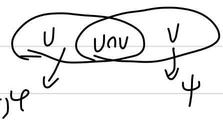
So $y = y'$ on $U \cap V$

So: $\sigma = \begin{cases} y & \text{on } U \\ y' & \text{on } V \end{cases}$ σ is smooth every where

and $(y, y') = (k^* \oplus l^*)\sigma$

$\forall \omega \in \Omega^k(U\cap V)\ \exists\, \omega_1 \in U,\ \omega_2 \in V$

$\omega = \omega_1 - \omega_2$



$\omega_1 = \begin{cases} \psi\omega & U\cap V \\ 0 & U \setminus \mathrm{supp}\,\psi \end{cases}$

$\omega_2 = \begin{cases} -\varphi\omega & U\cap V \\ 0 & V \setminus \mathrm{supp}\,\varphi \end{cases}$

$\omega_1 - \omega_2 = (\psi + \varphi)\omega = \omega.$

3. Degree theory:

M, N: compact, orientable, connected, smooth ...

of dimensions n

1) $\forall \omega \in \Omega^n(N)$

$$\int_M F^*\omega = k\int_N \omega$$

2) If q is regular value

$k = \sum_{p \in F^{-1}(q)} \mathrm{sgn}(p)$

$\mathrm{sgn}(p) = \begin{cases} + & dF_p \text{ preserve oriente} \quad \det(dF_p) > 0. \\ - & \det(dF_p) < 0. \end{cases}$

Riemannian metric:

Vector space:

Inner product: $V\times V\to R$

$u,v \to (u,v)$

① bilinear $(\lambda_1 u_1+\lambda_2 u_2, v) = \lambda_1(u_1,v)+\lambda_2(u_2,v)$

② Symmetric $(u,v)=(v,u)$

③ positive difinite: $(u,u)\ge 0$ and $(u,u)=0$ iff $u=0$

M, SM, g is call Riemannian metric on M if it asign to each T_pM and inner product which is various

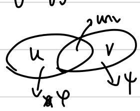
smooth: $g: \mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty$

$g_p: T_pM\times T_pM\to R$ $\quad g_p$

local chart of p $(U,\varphi(x_1,\dots,x_n))$

$g_{ij}=g(\partial x_i,\partial x_j)$ $\quad\{\partial x_1,\dots,\partial x_j\}$

$g=g_{ij}dx^i dx^j$ $\quad\{dx^1,\dots,dx^j\}$ dual basis



$g_{ij}=g(\partial x_i,\partial x_j)$

$\hat g_{ij}=g(\partial y_i,\partial y_j)=g\left(\left(\frac{\partial x_k}{\partial y_i}\right)\partial x_k, \left(\frac{\partial x_\ell}{\partial y_j}\right)\partial x_\ell\right)$

$=\left(\frac{\partial x_k}{\partial y_i}\right)g_{k\ell}\left(\frac{\partial x_\ell}{\partial y_j}\right)$

$\hat g_{ij}=g_{k\ell}\left(\frac{\partial x_k}{\partial y_i}\right)\left(\frac{\partial x_\ell}{\partial y_j}\right)$

Hodge:

Orthogonal basis: $\{e_i,\dots,e_j\}$ on T_pM (GS-orth)

$g_{ij}=\delta_{ij}$

$\{e_i^*,\dots,e_j^*\}$ dual basis and set it to orth by def

⇓

g^{ij} independent of $\{e_i,\dots,e_j\}$

$\Lambda^kT_p^*M=\operatorname{span}\{e^*_{i_1}\wedge\dots\wedge e^*_{i_k}\mid i_1<\dots<i_k\}$ see orthonormal.

$\langle a,b\rangle_p=\sum_I a^I b^I$

$a=\sum_I a^I e^{*I}$

$b=\sum_I b^I e^{*I}$

$\langle \alpha_1^*\wedge\dots\wedge\alpha_k^*, \beta_1^*\wedge\dots\wedge\beta_k^*\rangle$

$=\det(\langle\alpha_i^*,\alpha_j^*\rangle)$

④ orthon

Hodge *

$*: \Lambda^kT_p^*M\to\Lambda^{n-k}T_p^*M$

$e_I^*\to e_J^*\,sgn(I,J)$

$I\cup J=\{1,\dots,n\}$

$sgn(I,J)=sgn(i_1,\dots,i_k,j_1,\dots,j_{n-k})$

⇓

varing p

$*:\Omega^k(M)\to\Omega^{n-k}(M)$

$*1\in\Omega^n(M):=V_m$ volume form on M

2 Lagrnge:

Frame work: on each p ⇒ we find an orthonormal basis: $V_m=\theta_1\wedge\dots\wedge\theta_n$

or on local chart:

$V_m=\sqrt{\det(g_{ij})}\,dx_1\wedge dx_2\wedge\dots\wedge dx_n$

① $*(f\omega+g\eta)=f*\omega+g*\eta$

② $**\omega=(-1)^{k(n-k)}\omega$

③ $\omega\wedge*\eta=\eta\wedge*\omega=\langle\omega,\eta\rangle V_m$

④ $*$ $\quad\langle\omega,\eta\rangle$

⑤ $\langle*\omega,*\eta\rangle=\langle\omega,\eta\rangle$

② $\omega_p=\sum_I f_I e_I$ we just proof $\omega_p=e_I$ for linearity

$*e_I=sgn(I,J)e_J$ $\quad **e_I=\overset{k\ n-k}{sgn(I,J)}\ \overset{n-k\ k}{sgn(J,I)}$

$*e_J=sgn(J,I)e_I$ $\quad sgn(J,I)=(-1)^{k(n-k)}$

1 3 4 / 2 5

sgn(13425)

25134

③ $\omega_p=\sum_I f_I e_I$ $\quad I=(e_{i_1},\dots,e_{i_k})$

$\eta_p=\sum_{I'} g_{I'}e_{I'}$

$\omega_p\wedge*\eta_p = e_I\wedge e_{J'}\neq 0$ iff $(I,J)=\{1,\dots,n\}$

So $\omega_p\wedge*\eta_p=\underbrace{\sum_I f_I g_I}_{\langle\omega_p,\eta_p\rangle}\ \underbrace{e_I\wedge e_J\ sgn(I,J)}_{e_1\wedge\dots\wedge e_n \to V_m}$

$\langle *w, *\eta \rangle$

$w = \sum_I f_I e_I$

$\eta = \sum_{I'} g_{I'} e_{I'}$

$*w = \sum_I f_I e_J \quad sgn(I,J)$

$*\eta = \sum_{I'} g_{I'} e_{J'}$

$\langle *w, *\eta \rangle = \sum_I f_I g_I \quad \checkmark$

Inner product on $\Omega^k(M)$

$w, \eta \in \Omega^k(M) \quad M \cdots\cdots$

$(w,\eta) := \int_M \langle w, \eta \rangle V_m$

$= \int_M w \wedge *\eta$

$\delta : \Omega^k \to \Omega^{k-1} \quad w \in \Omega^{k-1}(M) \quad \eta \in \Omega^k(M)$

$(dw, \eta) = (w, \delta\eta)$

$w \in \Omega^k(M)$

$= \int_M dw \wedge *\eta \qquad d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$

$= \int_M d(w \wedge *\eta) - \int_M (-1)^{k-1} w \wedge d(*\eta)$

$= (-1)^k \int_M w \wedge d(*\eta)$

$= (-1)^k \int_M w \wedge ** \, d(*\eta) \quad (-1)^{(n-k+1)(k-1)}$

$= \int_M w \wedge *(*d*\eta) \, (-1)^{(n-k+1)(k-1)+k} \qquad n(k+1)+1$

$(n-k+1)(k-1)+k$

$= n(k-1) - (k-1)^2 + k.$

$n(k-1) - k + 1 + k.$

$= n(k-1)+1 = n(k+1)+1$

$\delta = (-1)^{n(k+1)+1} * d *$

$\Delta = d\delta + \delta d \; : \; \Omega^k \to \Omega^k.$

$\Delta : \quad \Omega^1(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$

$w = f\,dx^1$

$dw = \frac{\partial f}{\partial x_i} dx^i \wedge dx^1$

$= -\sum_{2}^{n} \frac{\partial f}{\partial x^i} dx^1 \wedge dx^i \qquad i = 2 \sim n$

$*dw = -\sum_2^n \frac{\partial f}{\partial x^i} dx^2 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n \quad (-1)^{i-2}$ i

$d*dw = (-1)^{i+1} d \sum_2^n \frac{\partial f}{\partial x^i} dx^2 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n$

$= (-1)^{i+1} \frac{\partial^2 f}{\partial x^1 \partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$

$+ (-1)^{i+1} \sum_{i=2}^n \frac{\partial^2 f}{\partial x^{i2}} dx^i \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$

$= \sum_{i=2}^n \frac{\partial^2 f}{\partial x^{i2}} dx^2 \wedge \overset{i-1}{\cdots} \wedge dx^n$

$*d*dw = (-1)^{n} \sum_{i=2}^n \frac{\partial^2 f}{\partial x^{i2}} dx^1 + (-1)^{n-1} \sum_{i=2}^n \frac{\partial^2 f}{\partial x^i} dx^1$

$d*d*w = \frac{\partial^2 f}{\partial x_1^2} dx^1 + \sum_i \frac{\partial^2 f}{\partial x_1 \partial x_i} dx^i$

$d\delta + \delta d \quad w$

$\delta \quad \Omega^1 \to \Omega^0 \quad (-1)^{2n+1} d*d* + (-1)^{3n+1} *d*d.$

$- \frac{\partial^2 f}{\partial x_1^2} dx^1 \quad - \quad \sum_i \frac{\partial^2 f}{\partial x_1 \partial x^i} dx^i$

$- \sum_i \frac{\partial^2 f}{\partial x^{i2}} dx^1$

① $\Delta * = * \Delta$

② $(\Delta w, \eta) = (\omega, \Delta \eta)$

③ $\Delta w = 0$ iff $dw = 0$ $\delta w = 0$

$(\Delta w, \eta) = (d\delta w + \delta d w, \eta)$

$= (\delta w, \delta \eta) + (dw, d\eta)$

$= (w, \delta \eta)$ ✓

$0 = (\Delta w, w) = (\delta w, \delta w) + (dw, dw) = 0$

$\delta w = 0$ and $dw = 0$.

Hodge $H^k(M) = \{w \mid \Delta w = 0\}$

$H^k(M) \cong H^k_{DR}(M)$

$f: w \Leftrightarrow [w]$

① Injection

$fw = [0]$ then $w = 0$

if $w = d\eta$ then $w = 0$

$(w, w) = (d\eta, w) = (\eta, \delta w) = 0$

$w = 0$

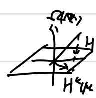
② $\forall w$, $dw = 0$ $\exists! w'$ $w' - w'' = d\eta$

on $\Delta w' = 0$

Choosing a basis of $H^k(M)$ $\beta_1 \dots \beta_r$

$H: \Omega^k(M) \to H^k(M)$

$Hw = \sum_{i=1}^{r} (w, \beta_i)\beta_i$



Hodge decomposed:

$\forall w \in \Omega^k(M)$

$w = Hw + d\eta + \delta\theta$

since $dA^{k-1} \perp \delta A^{k+1} \perp H^k$

proof: if: $\Delta w = 0$

$(w, d\eta) = (\delta w, \eta) = 0$

$(w, \delta\theta) = (dw, \theta) = 0$

$(\delta\eta, d\theta) = (\eta, d^2\theta) = 0$.

$A^k = H^k \oplus dA^{k-1} \oplus \delta A^{k+1}$

if so $w = Hw + d\eta + \delta\theta$ uniquely

since: $dw = 0 \Rightarrow d\delta\theta = 0$

$0 = (\theta, d\delta\theta) = (\delta\theta, \delta\theta) \Rightarrow \delta\theta = 0$

$\Rightarrow w = Hw + d\eta$ ✓

Riemannian Geometry:

(M, g)

① Affine Connection:

$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

$\nabla(X, Y) \xrightarrow{\text{记为}} \nabla_X Y$

1) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$ C^∞-linear

2) $\nabla_X (fY) = f \nabla_X Y + X(f) Y$

3) $\nabla_X (\lambda_1 Y_1 + \lambda_2 Y_2) = \lambda_1 \nabla_X Y_1 + \lambda_2 \nabla_X Y_2$

Thm: (Levi-Civita) (M, g) Riemannian M

$\exists! \nabla$

1) $X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z)$

2) $\nabla_X Y - \nabla_Y X = [X, Y]$

LC-Connection

Proof:

$X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z)$

$Y(Z, X) = (\nabla_Y Z, X) + (Z, \nabla_Y X)$

$Z(X, Y) = (\nabla_Z X, Y) + (X, \nabla_Z Y)$

① + ② - ③

$$([Y, Z], X) + ([X, Z], Y) + 2(\nabla_X Y, Z)$$
$$- (\nabla_X Y, Z)$$
$$+ (Z, \nabla_X X)$$

$$(\nabla_X Y, Z) = \frac{1}{2}\big(X(Y, Z) + Y(Z, X) - Z(X, Y)$$
$$- (X, [Y, Z]) - (Y, [X, Z])$$
$$- (Z, [Y, X]) \big)$$

取 Z 为 basis $\Rightarrow \nabla_Y X$

$\partial_i, \partial_j, \partial_k$ local chart

$(\partial_k, \nabla_{\partial_j} \partial_i) = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k})$

$\nabla_{\partial_j} \partial_i = \Gamma^l_{ij} \partial_l$

$g_{kl} \Gamma^l_{ij} = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k})$

$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (g_{ik,j} + g_{ik,i} - g_{ij,k})$

$(g^{kl})^{-1} = g_{lk}.$