Research Note; The Chern Number and the Topological Structure of Solid System

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ABSTRACT

In this two weeks I review some literatures about Berry curvature and Chern number including its mathematical and physical means. The quantum Hall effect which relates to Chern number and Berry curvature is also been introduced. As the last research note, this note also give a conclusion of my research project.

KEYWORDS: Chern Number, Hall effect, Berry Curvature,

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1 Mathematical description

To make my understanding more clearer, I think it is necessary to learn something about differential geometry.

1.1 Holonomy and Berry phase

The Berry Phase reflects the holonomy.
A tangent vector on the surface could be various if it travel along a closed curve on surface, that property call holonomy. Figure 1 shows the property.

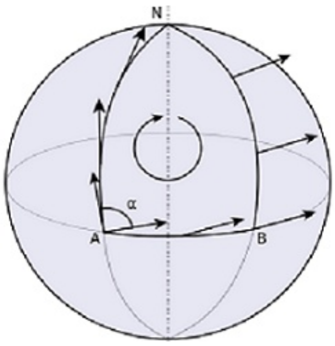

Figure 1: Holonomy of Surface

The Berry Connection naturally have the properties of tangent vector (That's a theory of fiber bundle, the connection or 1-form is just like tangent vector on the surface) so it naturally have the properties of holonomy which leads to the Berry phase. Things look more clear when we have more effective mathematical tools.

1.2 Chern number and Chern-Gauss-Bonet theorem

Gauss-Bonet Theorem told us that:

$$\int KdS = 2\pi\chi \tag{1.1}$$

, where K is Gauss Curvature and χ is Euler characteristic number which related to genus. For a sphere, its Eulers number is 2, for a doughnut, it is 0. The great theorem told us that: there exists a topological invariable showing the properties of a surface which inspire us to find a topological invariables in physical system.

That reminds me of the Berry Curvature and the definition of Chern Number:

$$\int_{BZ} B(k)d\mathbf{k} = 2\pi c \tag{1.2}$$

S. S. Cherns outstanding work shows that the Gauss-Bonet Theorem can be expanded to manifold by replacing Euler characteristic number into Chern number. This theorem is well known as the Chern-Gauss-Bonet theorem. In our band theory, the parameter space is k-space which is highly symmetrical, so we just need to integrate over the first brillouin zone. I dont want to show the mathematical details of it but give some examples.

2 Berry Curvature and Chern Number In Physical system

2.1 Simple Two Level System

The two level system Hamiltonian can be write as (like magnet field and spin):

$$H = h \cdot \sigma \tag{2.1}$$

Think about the rotation of magnet field, the $h = h(\sin\theta\cos\phi, \sin\theta\sin\phi, cos\theta)$. Shows in Figure 2[2]

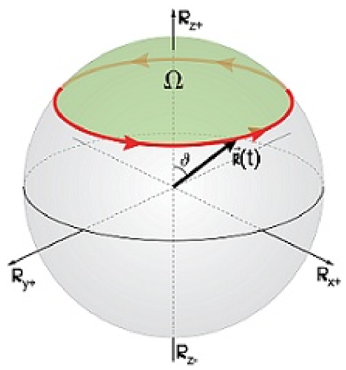

Figure 2: The Adiabatic evolution

The wave-function and berry connection of this system is shows[2]:

$$\begin{aligned}
|u_-\rangle &= (\sin\frac{\theta}{2}\mathrm{e}^{-i\phi}, -\cos\frac{\theta}{2}) \\
|u_+\rangle &= (\cos\frac{\theta}{2}\mathrm{e}^{-i\phi}, \sin\frac{\theta}{2}) \\
A_\theta &= 0 \\
A_\phi &= \sin^2\theta
\end{aligned} \tag{2.2}$$

So, the berry phase can be raised by integrate over the sphere.

$$\gamma(\Omega) = \frac{1}{2}\Omega \tag{2.3}$$

Ω is the solid angle
And the Chern number , shows:

$$c = 1 \tag{2.4}$$

Interesting things raised if our magnet field change its aptitude during the adiabatic evolution.[1]
The Hamiltonian takes this form;

$$H = \begin{Bmatrix} h_z & h_x - ih_y \\ h_x + ih_y & -h_z \end{Bmatrix} \tag{2.5}$$

,which is likely with the Hamiltonian of graphene. The berry curvature now takes:

$$B = \frac{1}{2h^3}\mathbf{h} \tag{2.6}$$

, which remains us of the monopoly.

2.2 The graphene

I have shown it in last note, here I just talk about the deep thought behind it:
The System which is protected by time reversal symmetry (TRS) leads to zero Chern Number.
I find a difficult proof from a literature which use the conception of Topological obstruction. But I got a simple one.

Difficult Whether there is topological obstruction depends on if we can find a global gauge of wave function. If the system without topological obstruction the Chern Number must be zero. I find a not precise explanation. We know that the sphere is an orientable surface. That is to say we can use a "global gauge" (which means all norm vector should point to the outside) to find a continuous norm vector field. But we cant do it on Mobius Strip, and their Euler characteristic number is different. So do the physical system
Another theorem told us that: if the system is protected by TRS, we can find a global gauge of this system and leads to a zero Chern number.

Simple Actually this theorem is not hard to understand in our simple situation, recall the definition of Berry Curvature.

$$B(k.R) = -\Im \sum_{m(\neq n)} \langle \nabla_{\mathbf{R}} \phi_n(k;\mathbf{R}) \,|\, \phi_m(k;\mathbf{R}) \rangle \times \langle \phi_m(k;\mathbf{R}) \,|\, \nabla_{\mathbf{R}} \phi_n(k;\mathbf{R}) \rangle \tag{2.7}$$

Due to the Time reversal symmetry:

$$\phi^*(-k) = \phi(k) \tag{2.8}$$

Replace k to -k, the equation become:

$$B(-k.R) = -\Im \sum_{m(\neq n)} \langle \phi_m(k;\mathbf{R}) \,|\, \nabla_{\mathbf{R}} \phi_n(k;\mathbf{R}) \rangle \times \langle \nabla_{\mathbf{R}} \phi_n(k;\mathbf{R}) \,|\, \phi_m(k;\mathbf{R}) \rangle \tag{2.9}$$

$$= -B(k, R) \tag{2.10}$$

The Berry Curvature is a odd function so the Chern number is Zero naturally.
A figure of last note also reflects this:

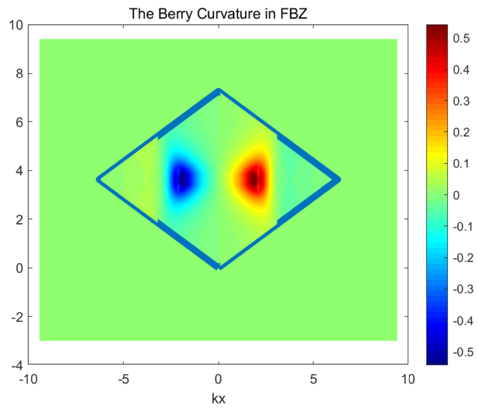

Figure 3: Berry Curvature In Graphene

So, if we want to find a nontrivial Chern Number, breaking of TRS is necessary. The most simple way is: apply a magnet field. Here comes the conception of Hall effect.

3 Quantum Hall Effect[2]

The direct use of Berry Curvature and Chern Number is the explanation of quantum Hall effect. I simply introduce it.
Kiltzing discovered the quantum Hall effect in 1980. They discovered that in strong magnetic field the Hall conductivity is exactly quantized in the unit of $\frac{e^2}{\hbar}$. That can be easily explained by Chern Number.

The solution of time dependent Schödinger Equation is

$$|u_n\rangle - i\hbar \sum_{m(\neq n)} \frac{|u_m\rangle \langle u_m \,|\, \partial u_n/\partial t\rangle}{\varepsilon_n - \varepsilon_m} \tag{3.1}$$

So, the velocity operator takes: $v_n(q) = \dfrac{\partial \varepsilon_n(q)}{\hbar \partial q} - B_n$. B is Berry Curvature.
That leads to

$$\sigma_{xy} = \frac{e^2}{h} \int \frac{B_n(k)}{2\pi} dk = c\frac{e^2}{h} \tag{3.2}$$

4 Further Works and Conclusion

I finish all the works in my proposal. I learned the basic conception of solid state physics, calculate the band structure of Graphene and Kagome lattices. The Berry curvature is also included in my research, from which I learn something about topology and its relation of modern physics.
I'm really like this course and really learn so much. Although my research looks like simple and successful, there are also many questions.
I will do some numerical calculation around my research in later, and find out the spin-effect and the next near atom effect in Tight-Blind Model which I neglected in early. Meanwhile, I will learn something about spin Hall effect and topological insulator later.

References

[1] GROSSO, G., AND G. P. PARRAVICINI (2014): "Chapter 8 - Interacting Electronic-Nuclear Systems and the Adiabatic Principle," in *Solid State Physics (Second Edition)*, ed. by G. Grosso, and G. P. Parravicini, pp. 333 – 390. Academic Press, Amsterdam, second edition edn.

[2] XIAO, D., M.-C. CHANG, AND Q. NIU (2010): "Berry phase effects on electronic properties," *Rev. Mod. Phys.*, 82, 1959–2007.