Stat_1112_HW5

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Stat_1112_HW5

MA2011 Statistics Homework Assignment 5

Due date: 2023/05/04

Student ID: ______________________

Name: ______________________

1. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Given an $\alpha \in (0,1)$.
(i) Using the joint distribution of $\bar{X}$ and S^2 to construct a $100(1-\alpha)\%$ confidence interval for μ.
(ii) Using the exact distribution of the MLE $\hat{\sigma}^2$ to construct an exact $100(1-\alpha)\%$ confidence interval for σ^2.
(iii) Using the asymptotic distribution of $\hat{\sigma}^2$ to construct an approximate $100(1-\alpha)\%$ confidence interval for σ^2.

(i) Note that $\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

Therefore, $\mathbb{P}\left(-t_{n-1,\frac{\alpha}{2}} \le \dfrac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{n-1,1-\frac{\alpha}{2}}\right) = 1-\alpha$,

where $t_{n-1,q}$ is the qth quantile for t_{n-1},

and $t_{n-1,q} = -t_{n-1,1-q}$, for $q \in (0,1)$.

$\Rightarrow$ $100(1-\alpha)\%$ C.I. for μ : $\bar{X} \pm \dfrac{S}{\sqrt{n}} t_{n-1,1-\frac{\alpha}{2}}$

(ii) Note that $\dfrac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$.

Therefore, $\mathbb{P}\left(\chi^2_{n-1,\frac{\alpha}{2}} \le \dfrac{n\hat{\sigma}^2}{\sigma^2} \le \chi^2_{n-1,1-\frac{\alpha}{2}}\right) = 1-\alpha$,

where $\chi^2_{n-1,q}$ is the qth quantile for χ^2_{n-1}, $q \in (0,1)$.

$\Rightarrow$ $100(1-\alpha)\%$ C.I. for σ^2 : $\left(\dfrac{n\hat{\sigma}^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \dfrac{n\hat{\sigma}^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$.

(iii) Note that $\sqrt{n}(\hat{\sigma}^2-\sigma^2) \xrightarrow{d} N(0, 2\sigma^4)$

Therefore, $\mathbb{P}\left(z_{\frac{\alpha}{2}} \le \dfrac{\hat{\sigma}^2-\sigma^2}{\sqrt{2\sigma^4/n}} \le z_{1-\frac{\alpha}{2}}\right) \approx 1-\alpha$,

where z_q is the qth quantile for $N(0,1)$,

and $z_q = -z_{1-q}$, $q \in (0,1)$.

$\Rightarrow \mathbb{P}\left(-z_{1-\frac{\alpha}{2}} \le \dfrac{\hat{\sigma}^2-\sigma^2}{\sqrt{2\hat{\sigma}^4/n}} \le z_{1-\frac{\alpha}{2}}\right) \approx 1-\alpha$

$\Rightarrow$ approximate $100(1-\alpha)\%$ C.I. for σ^2 :

$\hat{\sigma}^2 \pm \sqrt{\frac{2}{n}}\,\hat{\sigma}^2 \cdot z_{1-\frac{\alpha}{2}} = \hat{\sigma}^2\left(1 \pm \sqrt{\frac{2}{n}}\, z_{1-\frac{\alpha}{2}}\right)$

2. Suppose that we observe $\hat{\mu} = 2.5$ and $\hat{\sigma}^2 = 4$. Using R to compute the following estimates: $(\alpha = 0.05)$
(a) Suppose $n = 16$, repeat #1(i,ii,iii).
(b) Suppose $n = 100$, repeat #1(i,ii,iii).

(a) (i) By R, $t_{15,0.975} \approx 2.131$, $(n-1)S^2 = n\hat{\sigma}^2 \Rightarrow \dfrac{S}{\sqrt{n}} = \sqrt{\dfrac{\hat{\sigma}^2}{n-1}}$

95% C.I. for μ : $2.5 \pm \sqrt{\frac{4}{15}} \cdot t_{15,0.975} \approx (1.399, 3.601)$

(ii) By R, $\chi^2_{15,0.025} \approx 6.262$, $\chi^2_{15,0.975} \approx 27.488$

95% C.I. for σ^2 : $\left(\dfrac{16 \cdot 4}{\chi^2_{15,0.975}}, \dfrac{16 \cdot 4}{\chi^2_{15,0.025}}\right) \approx (2.328, 10.220)$

(iii) By R, $z_{0.975} \approx 1.960$.

app. 95% C.I. for σ^2 : $4\left(1 \pm \sqrt{\frac{2}{16}}\, z_{1-\frac{\alpha}{2}}\right) \approx (1.228, 6.772)$

(b) (i) $t_{99,0.975} \approx 1.984$.

95% C.I. for μ : $2.5 \pm \sqrt{\frac{4}{99}} \cdot t_{99,0.975} \approx (2.101, 2.899)$

(ii) $\chi^2_{99,0.025} \approx 73.361$, $\chi^2_{99,0.975} \approx 128.422$

95% C.I. for σ^2 : $\left(\dfrac{100 \cdot 4}{\chi^2_{99,0.995}}, \dfrac{100 \cdot 4}{\chi^2_{99,0.025}}\right) \approx (3.115, 5.452)$

(iii) app. 95% C.I. for σ^2 : $4\left(1 \pm \sqrt{2/100} \cdot z_{1-\frac{\alpha}{2}}\right) \approx (2.891, 5.109)$

(Over Please)

3. Suppose that X is a discrete random variable with $P(X = 1) = \theta$ and $P(X = \cancel{2}\ 0) = 1 - \theta$, where $\theta \in (0,1)$ is unknown. Let $X_1, \ldots, X_{10}$ be iid copies of X.
(a) Find the MLE $\hat{\theta}$ and its asymptotic distribution.
(b) Given 10 observations: $(1,1,0,0,1,0,0,0,1,0)$. Construct an approximate 90% confidence interval for θ by ~~(b)~~ (a).

(a) The pmf of X : $f(x|\theta) = \theta^x (1-\theta)^{1-x}$, $x = 0, 1$.

Log likelihood : $\ell(\theta) = \sum_{i=1}^{10} \log f(X_i|\theta) = \sum_{i=1}^{10} \left(X_i \log\theta + (1-X_i)\log(1-\theta)\right) = 10\left(\bar{X}\log\theta + (1-\bar{X})\log(1-\theta)\right)$

where $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$.

$\dfrac{d}{d\theta}\ell(\theta) = 0 \Rightarrow \dfrac{\bar{X}}{\theta} - \dfrac{1-\bar{X}}{1-\theta} = 0 \Rightarrow$ The MLE $\hat{\theta} = \bar{X}$.

$I(\theta) = \mathbb{E}\left(\frac{-d^2}{d\theta^2} \log f(X|\theta)\right) = \mathbb{E}\left(\frac{-d}{d\theta}\left(\frac{X}{\theta} - \frac{1-X}{1-\theta}\right)\right) = \mathbb{E}\left(\frac{X}{\theta^2} + \frac{1-X}{(1-\theta)^2}\right)$

$= \dfrac{\mathbb{E}(X)}{\theta^2} + \dfrac{1-\mathbb{E}(X)}{(1-\theta)^2} = \dfrac{\theta}{\theta^2} + \dfrac{1-\theta}{(1-\theta)^2} = \dfrac{1}{\theta(1-\theta)}$.

$\Rightarrow \sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N\left(0, \frac{1}{I(\theta)}\right) = N(0, \theta(1-\theta))$

(b) An approximate 90% C.I. for θ is $(\hat{\theta} = 0.4, n = 10)$

$\hat{\theta} \pm \sqrt{\dfrac{\hat{\theta}(1-\hat{\theta})}{n}} \cdot z_{0.95} \approx (0.145, 0.655)$

4. Suppose that $Y_1, \ldots, Y_{10} \overset{iid}{\sim} Poisson(\lambda)$, where $\lambda > 0$ is unknown.
(a) Find the MLE $\hat{\lambda}$ and its asymptotic distribution.
(b) Given 10 observations: $(1,1,0,0,1,3,0,0,1,0)$. Construct an approximate 90% confidence interval for λ by (b).

(a) The pmf for Y_i : $f(y|\lambda) = \frac{1}{y!}\lambda^y e^{-\lambda}$, $y = 0, 1, 2, \ldots$

Log likelihood: $\ell(\lambda) = \sum_{i=1}^{10} \log f(Y_i|\lambda) = \sum_{i=1}^{10} \left(-\log(Y_i!) + Y_i \log\lambda - \lambda\right)$

$\dfrac{d}{d\lambda}\ell(\lambda) = 0 \Rightarrow \sum_{i=1}^{10}\left(\dfrac{Y_i}{\lambda} - 1\right) = 0 \Rightarrow$ The MLE $\hat{\lambda} = \bar{Y} = \frac{1}{10}\sum_{i=1}^{10} Y_i$.

$I(\lambda) = \mathbb{E}\left(\frac{-d^2}{d\lambda^2}\log f(Y|\lambda)\right) = \mathbb{E}\left(\frac{-d}{d\lambda}\left(\frac{Y}{\lambda} - 1\right)\right) = \dfrac{\mathbb{E}(Y)}{\lambda^2} = \dfrac{1}{\lambda}$

$\Rightarrow \sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{d} N\left(0, \frac{1}{I(\lambda)}\right) = N(0, \lambda)$

(b) Using the observation, $\hat{\lambda} = \frac{7}{10} = 0.7$, and $n = 10$.

An approximate 90% C.I. for λ is

$\hat{\lambda} \pm \sqrt{\dfrac{\hat{\lambda}}{n}} \cdot z_{0.95} \approx (0.265, 1.135)$

(The End)