Double Pendulum

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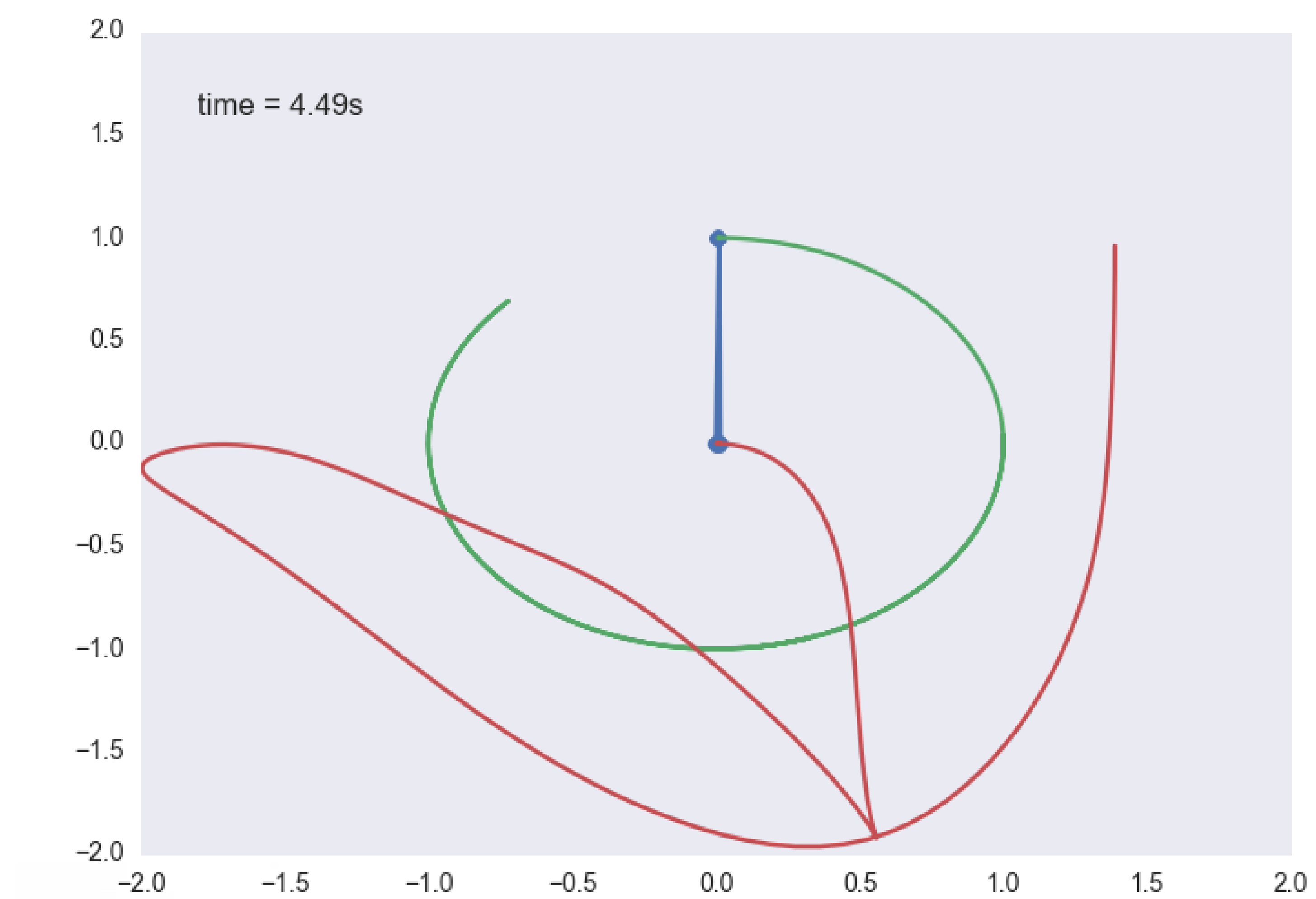
Introduction

The double pendulum is a chaotic system. This means that it is extremely sensitive to the initial conditions. The goal in modeling this system was to find the pair of initial angles that resulted in an equilibrium at the top with the second mass below the first.

In order to limit the scope of the model, it is set to have a 1:1 length ratio, 1:1 mass ratio and zero initial angular velocity for both masses. The initial angles were swept with conservation of energy in mind, such that the starting angle of the second mass was dependant upon the starting angle of the first mass in order to have the exact energy of the equilibrium situation.

Abstract

We modeled the system with massless rods connecting the pendulums and the pivot, and set the pendulums to be identical in terms of length and mass. The only differences were the initial angles of the pendulums, which we swept to find the best values. Our results showed that an initial angle of 92.25 for the first pendulum and 157.1534 for the second pendulum produce our desired result. It is not perfectly stable, since the mathematical error compounds upon itself, but it stays at rest vertically for long enough to satisfy our goal. This error comes from the fact that a chaotic system is very sensitive to initial parameters, and even slightly imperfect ones will result in a buildup of error over time.



$$\theta_1 = 92.25$$

$$\theta_2 = 157.1534$$

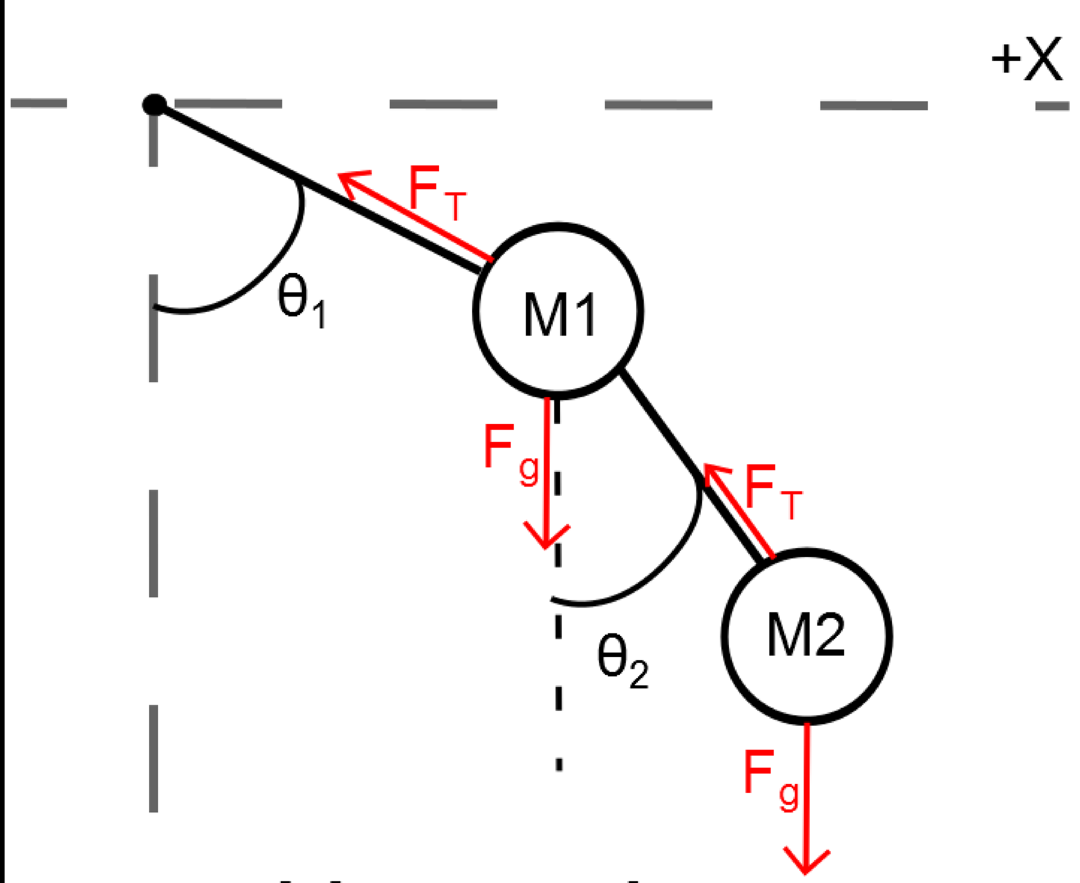
Equations

$$\frac{d\theta_1}{dt} = \omega_1$$

$$\frac{d\omega_1}{dt} = \frac{m_2 l_1 \omega_1^2 sin\Delta\ cos\Delta + m_2 g sin\theta_2 cos\Delta\ + m_2 l_2 \omega_2^2 sin\Delta - (m_1+m_2) g si}{(m_1+m_2) l_1 - m_2 l_1 cos^2\Delta}$$

$$\frac{d\theta_2}{dt} = \omega_2$$

$$\frac{d\omega_2}{dt} = \frac{-m_2 l_2 \omega_2^2 sin\Delta cos\Delta + (m_1+m_2)(g sin\theta_1 cos\Delta - l_1\omega_1^2 sin\Delta - g sin\theta_2}{(m_1+m_2) l_2 - m_2 l_2 cos^2\Delta}$$

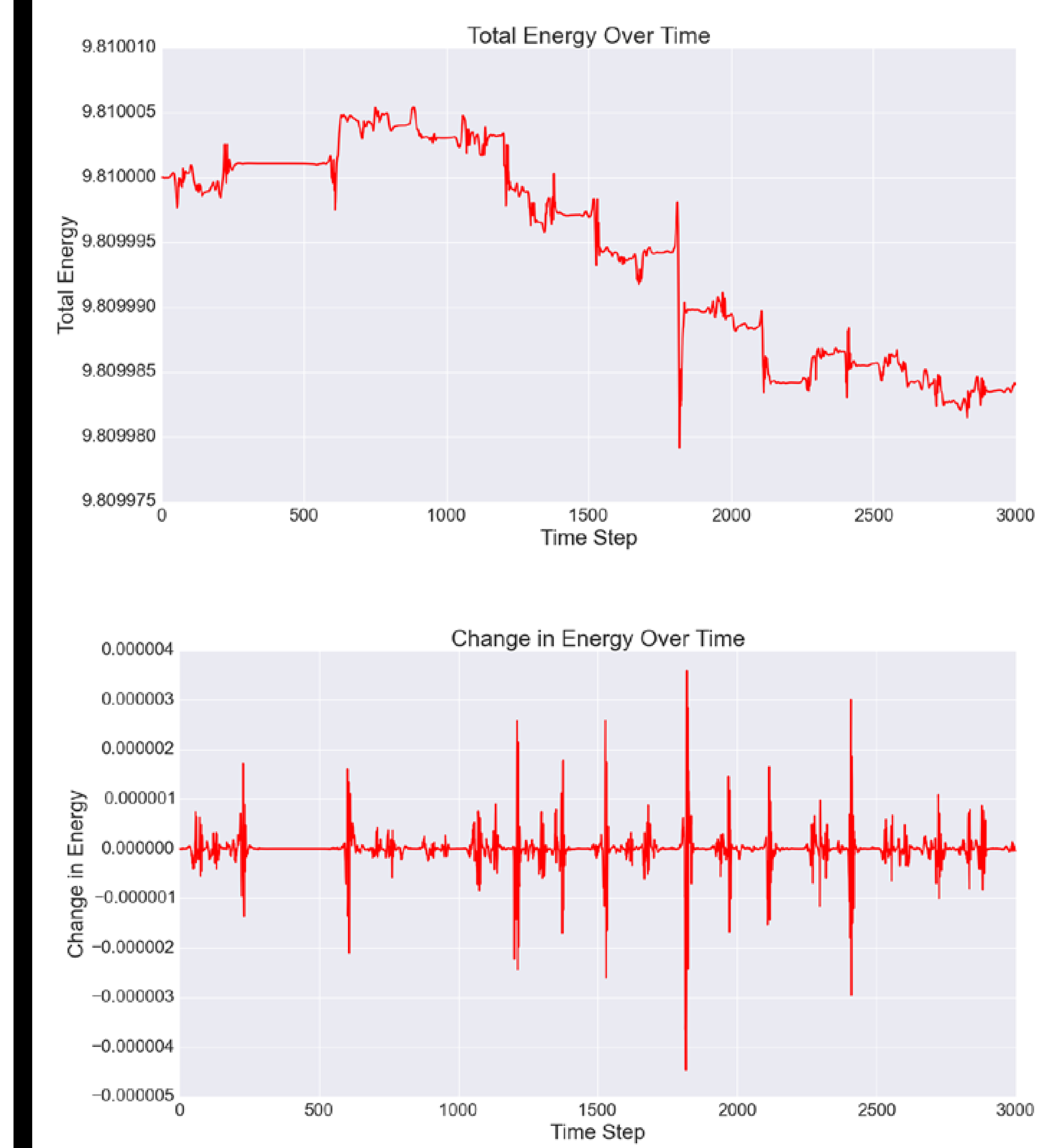


$$U = mgh$$

$$K = {}^1/_2\, mv^2$$

$$L = U + K$$

Validation



As seen in the graph of the total energy and the change in energy over time, the resulting model is quite accurate in terms of energy conservation. Since the sytem being modeled has no external forces or energies, idealy the energy should remain constant throughout. However, due to numerical error, the energy level can vary, and although this varriance is small, a chaotic system such as a double pendulum can have a very different reaction to such a small change in energy levels.

One important thing to note is that although the pendulum does fall after it reaches the equilibrium point, the delta E graph shows a spike in the change of the energy of the system which directly correlates to the point at which the pendulum slowly falls back down. This means that it is numerical error which correlates to the pendulum falling down.

Conclusion

The conclusion that can be drawn from this model is that there are only three possible pairs of initial angles (one pair is starting at the position desired, and the other is the negative of the one demonstated in the animation) for which the double pendulum will swing an then remain at equilibrium as described before. This may seem bizarre but it does make sense to a certain degree. Since there is no initial velocity for either mass, the phase space is limited from the 4D plane to a 2D space.

Before devolving into a lot of math, what this means is that if the model included initial velocites then there would be sets of points correlating to the equilibrium point, but since no initial velocites were included then only a few set of discrete points could be hoped to be found. It turns out that those discrete points happen to be three discrete points.