Example Notes

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Contents

Isomorphism

Proposition

Let f be a continuous function, such that for any real numbers a, b, $f(a+b) = f(a)f(b)$. Then, $f(x) \geq 0$ for any $x \in \mathbb{R}$.

Proof

By way of contradiction, suppose $f(x) < 0$ for some $x \in \mathbb{R}$. Then, $f(x) = f(x+0) = f(x)f(0)$. This implies $f(0) = 1$. Since f is continuous, $f(c) = 0$ for some c between x and 0 by the intermediate value theorem. Then, $f(x) = f(x-c+c) = f(x-c)f(c) = 0$ is a contradiction. Hence, $f(x) \geq 0$ can never be negative.

Limits

Proposition

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences, such that $x_n < y_n$ for any $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Proof

Let $\{x_n\}$, $\{y_n\}$ be sequences, such that $x_n \to x$ and $y_n \to y$. Let $\epsilon > 0$. For any n large enough, we have $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. Then,

$$\begin{aligned} x_n - y < y_n - y &\leq \frac{\epsilon}{2} \\ x_n - x &< y - x + \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} &< y - x + \frac{\epsilon}{2} \\ x - y &< \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $x - y \leq 0$, so $x \leq y$.

Happy Number

A happy number `n` is defined by the process:

- Replaced `n` with the sum of squares of its digits
- If this process converges to 1, then it is a happy number
- If this process loops in a cycle, then it is not a happy number

Example

```
19
1^2 + 9^2 = 82
8^2 + 2^2 = 68
6^2 + 8^2 = 100
1^2 + 0^2 + 0^2 = 1

2
2^2 = 4
4^2 = 16
1^2 + 6^2 = 37
3^2 + 7^2 = 58
5^2 + 8^2 = 89
8^2 + 9^2 = 64 + 81 = 145
1^2 + 4^2 + 5^2 = 1 + 16 + 25 = 42
4^2 + 2^2 = 16 + 4 = 20
```

```
2^2 + 0^2 = 4
```

The following function verifies if a number is happy:

```
def is_happy(num: int):
    visited = Set()

    def _is_happy(num: int):
        if num == 1:
            return True
        if num in visited:
            return False
        return _is_happy(sum(int(c)**2 for c in str(num)))

    return _is_happy(num)
```

Eulerian Graphs

Proposition

A nontrivial connected graph is Eulerian iff every vertex has even degree.

Proof

Let G be a nontrivial connected graph whose vertices all have even degree. Let C be a maximal length trail in G.

Claim: C is closed. If not, it has an endpoint u with odd degree in C. Since u has even degree in G, there exists an edge $uv \in E(G) \setminus E(C)$. But, $C + uv$ is a longer trail $\perp$. So, trail of maximum length must be closed.

If $E(C) = E(G)$, then we're done. BWOC, suppose not. Since G is connected, there exists some $xy \in G$, where $x \in V(C)$. Consider $G - E(C)$. Since C is a circuit, all vertices have an even degree in C, so they have even degree in $G - E(C)$. Consider a maximal length trail D starting at $xy \in G - E(C)$. As before, D is closed. Splice D into G at x: $xCxDxCx$. This is longer than C $\perp$.