Example Notes

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Isomorphism

Proposition

Let f be a continuous function, such that for any real numbers a, b, f(a+b) = f(a)f(b). Then, $f(x) \ge 0$ for any $x \in \mathbb{R}$.

Proof

By way of contradiction, suppose f(x) < 0 for some $x \in \mathbb{R}$. Then, f(x) = f(x+0) = f(x)f(0). This implies f(0) = 1. Since f is continuous, f(c) = 0 for some c between x and 0 by the intermediate value theorem. Then, f(x) = f(x-c+c) = f(x-c)f(c) = 0 is a contradiction. Hence, $f(x) \ge 0$ can never be negative.

Limits

Proposition

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences, such that $x_n < y_n$ for any $n \in \mathbb{N}$. Then, $\lim_{n \to \infty} x_n \le \lim_{n \to \infty} y_n$.

Proof

Let $\{x_n\}$, $\{y_n\}$ be sequences, such that $x_n \to x$ and $y_n \to y$. Let $\epsilon > 0$. For any n large enough, we have $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. Then,

$$x_n - y < y_n - y \le \frac{\epsilon}{2}$$

$$x_n - x < y - x + \frac{\epsilon}{2}$$

$$-\frac{\epsilon}{2} < y - x + \frac{\epsilon}{2}$$

$$x - y < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $x - y \le 0$, so $x \le y$.

Happy Number

A happy number n is defined by the process:

- Replaced n with the sum of squares of its digits
- If this process converges to 1, then it is a happy number
- If this process loops in a cycle, then it is not a happy number

Example

```
19
1^{2} + 9^{2} = 82
8^{2} + 2^{2} = 68
6^{2} + 8^{2} = 100
1^{2} + 0^{2} + 0^{2} = 1

2
2^{2} = 4
4^{2} = 16
1^{2} + 6^{2} = 37
3^{2} + 7^{2} = 58
5^{2} + 8^{2} = 89
8^{2} + 9^{2} = 64 + 81 = 145
1^{2} + 4^{2} + 5^{2} = 1 + 16 + 25 = 42
4^{2} + 2^{2} = 16 + 4 = 20
2^{2} + 0^{2} = 4
```