Example Notes

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Contents

Isomorphism

Proposition

Let f be a continuous function, such that for any real numbers a, b, $f(a+b) = f(a)f(b)$. Then, $f(x) \geq 0$ for any $x \in \mathbb{R}$.

Proof

By way of contradiction, suppose $f(x) < 0$ for some $x \in \mathbb{R}$. Then, $f(x) = f(x+0) = f(x)f(0)$. This implies $f(0) = 1$. Since f is continuous, $f(c) = 0$ for some c between x and 0 by the intermediate value theorem. Then, $f(x) = f(x-c+c) = f(x-c)f(c) = 0$ is a contradiction. Hence, $f(x) \geq 0$ can never be negative.

Limits

Proposition

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences, such that $x_n < y_n$ for any $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Proof

Let $\{x_n\}$, $\{y_n\}$ be sequences, such that $x_n \to x$ and $y_n \to y$. Let $\epsilon > 0$. For any n large enough, we have $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. Then,

$$\begin{aligned} x_n - y < y_n - y \leq \frac{\epsilon}{2} \\ x_n - x < y - x + \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} < y - x + \frac{\epsilon}{2} \\ x - y < \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $x - y \leq 0$, so $x \leq y$.

Happy Number

A happy number `n` is defined by the process:

- Replaced `n` with the sum of squares of its digits
- If this process converges to 1, then it is a happy number
- If this process loops in a cycle, then it is not a happy number

Example

```
19
1^2 + 9^2 = 82
8^2 + 2^2 = 68
6^2 + 8^2 = 100
1^2 + 0^2 + 0^2 = 1

2
2^2 = 4
4^2 = 16
1^2 + 6^2 = 37
3^2 + 7^2 = 58
5^2 + 8^2 = 89
8^2 + 9^2 = 64 + 81 = 145
1^2 + 4^2 + 5^2 = 1 + 16 + 25 = 42
4^2 + 2^2 = 16 + 4 = 20
2^2 + 0^2 = 4
```