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Home Assignment Econometrics III
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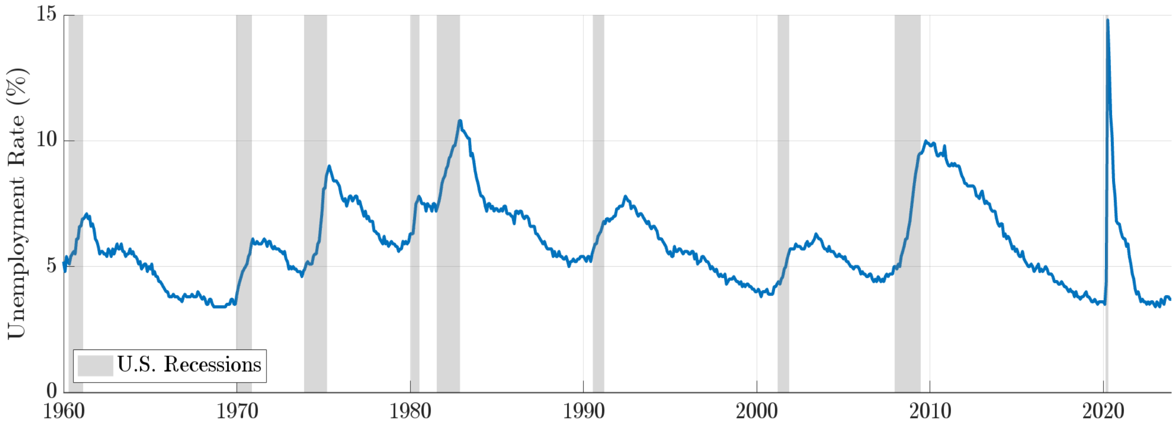
U.S. unemployment dynamics: recession vs. stable periods

The unemployment rate is a key indicator shaping U.S. monetary policy, especially during financial crises when labor markets become more volatile. Understanding how the unemployment rate behaves over time is essential for guiding Federal Reserve actions. Structural factors in U.S. labor markets create asymmetric cycles in the unemployment rate with sudden jumps during economic downturns and more gradual declines during recoveries, as shown below:



This asymmetry also results in heightened volatility (lower precision) during recessions compared to more stable periods. A Bayesian approach, which allows for differing error precision between recession and non-recession periods, offers a first step towards flexibility and provides a clearer picture of how the unemployment rate evolves across economic conditions. To this end, consider the following dynamic linear regression model:

$$\begin{aligned}\text{UNEMP}_t &= \mu + \alpha_1 \text{UNEMP}_{t-1} + \beta_1 \text{INPRO}_{t-1} + \ldots + \beta_q \text{INPRO}_{t-q} + \gamma_1 \text{CPI}_{t-1} + \ldots \\ &\quad + \gamma_q \text{CPI}_{t-q} + \phi_1 \text{BCONF}_{t-1} + \ldots + \phi_q \text{BCONF}_{t-q} + \lambda \text{COVID}_t + \varepsilon_t, \qquad \varepsilon_t \sim N(0, h^{-1}), \quad (1)\end{aligned}$$

where $q = 4$ denotes the distributed lags while the monthly variables are the unemployment rate (UNEMP_t, first-difference), industrial production (INPRO_t, log first-difference), inflation (CPI_t, log first-difference), business confidence indicator (BCONF_t, first-difference), and COVID_t controls for the irregular behavior after the pandemic shock (April to June 2020). Assume that exogeneity between regressors in (1) and ε_t holds. This generalized regression framework can be written in matrix form:

$$y = X\beta + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \Sigma). \qquad (2)$$

Moreover, let us assume two different error precisions h_1 and h_2 for calm and recession periods specified via the precision matrix such that

$$H = \Sigma^{-1} = \begin{bmatrix} h_i & 0 & \ldots & 0 \\ 0 & h_i & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & h_i \end{bmatrix} \quad \text{with} \quad h_i = \begin{cases} h_1 & \text{if } t \text{ is a } \textbf{calm} \text{ period} \\ h_2 & \text{if } t \text{ is a } \textbf{recession} \text{ period} \end{cases} \qquad (3)$$

Finally, suppose prior beliefs concerning the unknown parameters of the model $\theta = (\beta', h_1, h_2)'$ are represented by the independent Normal-Gamma prior densities:

$$\beta \sim \mathcal{N}(\underline{\beta}, \underline{V}) \tag{4}$$

$$h_1 \sim G(\underline{s}_1^{-2}, \underline{\nu}_1) \tag{5}$$

$$h_2 \sim G(\underline{s}_2^{-2}, \underline{\nu}_2) \tag{6}$$

Questions

1. **(1P)** State the joint prior $p(\beta, h_1, h_2)$, the likelihood $p(y|\beta, h_1, h_2)$ and the joint posterior $p(\beta, h_1, h_2|y)$.
   Hint: define $d = \text{diag}(d_1, \ldots, d_T)$ as the diagonal matrix with recession indicators $d_t = 1$, for $t = 1, \ldots, T$, and thus note that (3) can be simplified to

$$H = h_1(I_T - d) + h_2 d \quad \text{with} \quad |H| = h_1^{T_1} h_2^{T_2}, \tag{7}$$

   where $T_2 = \sum_{i=1}^{T} d_t$ is the number of recession periods and $T_1 = T - T_2$ is the number of calm periods in the sample.

2. **(2P)** Find the conditional posteriors $p(\beta|y, h_1, h_2)$, $p(h_1|y, \beta, h_2)$ and $p(h_2|y, \beta, h_1)$ and show in pseudo code how the Gibbs sampler can be used to estimate this model.

3. **(2P)** Write a MATLAB function `indnormgam_posterior.m` that computes the posterior moments of the multiple linear regression model with independent normal-gamma prior following (4)-(6). Specifically, the user should supply data y and X, along with the vector of dummies d, as well as prior parameters $\underline{\beta}$, $\underline{V}$, $\underline{s}_1^{-2}$, $\underline{\nu}_1$, $\underline{s}_2^{-2}$, and $\underline{\nu}_2$. Moreover, let the function's user choose the S_0 burn-in and S_1 MC posterior replications for the Gibbs sampler. From there, the function should compute and hand back posterior parameters, along with posterior means and variances of β, h_1 and h_2.

4. **(2P)** In your Main script, simulate posterior moments of β, h_1 and h_2 using the dataset `US_macro.mat` from 1960 to 2023. For that, choose an informative prior based on OLS estimates (hereby assuming homoskedasticity with $h_1 = h_2$). This means that the prior hyperparameters for β, h_1 and h_2 can be based on the latter OLS quantities. Find reasonable numbers for the S_0 and S_1 iterations. Finally, report the posterior means and standard deviations of β, h_1 and h_2.

5. **(1P)** Check for MCMC convergence either by examining trace plots or CD statistics. How reliable are posterior estimates based on these Markov chains?

6. **(1P)** Compare your posterior estimates for h_1 and h_2 and answer whether there is empirical evidence for lower precision of the unemployment rate in recession periods.

7. **(1P)** Extend your script to compute a one-step-ahead forecast UNEMP_{t+1}^* for 2024M01 based on past values $X_{t-q+1:t}^*$. Plot the predictive density of UNEMP_{t+1}^* and illustrate the conditional posterior mean along with percentiles of your choice.

Q1: 1. **(1P)** State the joint prior $p(\beta, h_1, h_2)$, the likelihood $p(y|\beta, h_1, h_2)$ and the joint posterior $p(\beta, h_1, h_2|y)$.

given

1Step: State joint prior $p(\beta, h_1, h_2)$

$$p(\beta, h_1, h_2) = p(\beta)\, p(h_1)\, p(h_2)$$

$$\propto \underbrace{\exp\left\{-\frac{1}{2}(\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta})\right\}}_{\beta \sim N(\underline{\beta}, \underline{V})} \underbrace{h_1^{\frac{\underline{V}_1-2}{2}} \exp\left\{-\frac{1}{2} h_1 \underline{V}_1 \underline{s}_1^{-2}\right\}}_{h_1 \sim G(\underline{s}_1^{-2}, \underline{V}_1)} \underbrace{h_2^{\frac{\underline{V}_2-2}{2}} \exp\left\{-\frac{1}{2} h_2 \underline{V}_2 \underline{s}_2^{-2}\right\}}_{h_2 \sim G(\underline{s}_2^{-2}, \underline{V}_2)}$$

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given

2Step: State likelihood $p(y|\beta, h_1, h_2)$

Recall: Our model is $y = X\beta + \boxed{\varepsilon}$ with $\varepsilon \sim N(0, \Sigma)$, so the function of parameters for y must be multivariate normal following:

① $E[y] = E[X\beta] + E[\varepsilon] = X\beta$

② $Var[y] = Var[\varepsilon] = \Sigma$

$\Big\}\ y|\beta, H \sim N(X\beta, \Sigma)$

vector y_i

we get:

$$p(y|\beta, h_1, h_2) \propto |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(y-X\beta)'\Sigma^{-1}(y-X\beta)\right\}$$

$$\propto |H|^{\frac{1}{2}} \exp\left\{-\frac{1}{2}(y-X\beta)' H (y-X\beta)\right\}$$

split into two parts

$$\propto h_1^{\frac{T_1}{2}} h_2^{\frac{T_2}{2}} \exp\left\{-\frac{1}{2}(y-X\beta)'\left(h_1(I_T-d)+h_2 d\right)(y-X\beta)\right\}$$

$$\propto h_1^{\frac{T_1}{2}} h_2^{\frac{T_2}{2}} \exp\left\{-\frac{1}{2}\left[\underbrace{(y-X\beta)' h_1 (I_T-d)(y-X\beta)}_{h_1 \text{ part (calm)}} + \underbrace{(y-X\beta)' h_2 d (y-X\beta)}_{h_2 \text{ part (recession)}}\right]\right\}$$

3Step: State the Posterior

$$p(\beta, h_1, h_2 | y) \propto p(\beta) p(h_1) p(h_2)\, p(y | \beta, h_1, h_2)$$

$$\propto \exp\left\{-\frac{1}{2}(\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta})\right\} h_1^{\frac{\underline{\nu}_1-2}{2}} \exp\left\{-\frac{1}{2} h_1 \underline{\nu}_1 \underline{s}_1^{-2}\right\} h_2^{\frac{\underline{\nu}_2-2}{2}} \exp\left\{-\frac{1}{2} h_2 \underline{\nu}_2 \underline{s}_2^{-2}\right\} \cdot$$

$$h_1^{\frac{T_1}{2}} h_2^{\frac{T_2}{2}} \exp\left\{-\frac{1}{2}\left[(y-X\beta)' h_1 (I_T - d)(y-X\beta) + (y-X\beta)' h_2 d (y-X\beta)\right]\right\}$$

simplify & collect terms:

$$\propto h_1^{\frac{T_1+\underline{\nu}_1-2}{2}} h_2^{\frac{T_2+\underline{\nu}_2-2}{2}} \exp\left\{-\frac{1}{2}\left[(\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta}) + h_1 \underline{\nu}_1 \underline{s}_1^{-2} + h_2 \underline{\nu}_2 \underline{s}_2^{-2} + (y-X\beta)' h_1 (I_T - d)(y-X\beta) + (y-X\beta)' h_2 d (y-X\beta)\right]\right\}$$

$\boxed{GR}$ (without case distinction for H)

$$\propto h_1^{\frac{T_1+\underline{\nu}_1-2}{2}} h_2^{\frac{T_2+\underline{\nu}_2-2}{2}} \exp\left\{-\frac{1}{2}\left[(\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta}) + h_1 \underline{\nu}_1 \underline{s}_1^{-2} + h_2 \underline{\nu}_2 \underline{s}_2^{-2} + (y-X\beta)' H (y-X\beta)\right]\right\}$$

⚠ Doesn't have any analytical accessible/explored form. Hence, we must use conditionals and marginals to get results.

Q2:

2. **(2P)** Find the conditional posteriors $p(\beta|y, h_1, h_2)$, $p(h_1|y, \beta, h_2)$ and $p(h_2|y, \beta, h_1)$ and show in pseudo code how the Gibbs sampler can be used to estimate this model.

1Step: Get $p(\beta|y, h_1, h_2)$

→ simply Bayes theorem:

$p(\beta, h_1, h_2) = p(\beta|h_1, h_2, y) \cdot p(h_1 h_2|y)$

Recall: $p(\beta|y, h_1, h_2) = \dfrac{p(\beta, h_1, h_2|y)}{\cancel{p(h_1, h_2|y)}}$ → not depend on β, marginals treated as constants

we get:

$$p(\beta|y, h_1, h_2) \propto p(\beta, h_1, h_2|y)$$

⇒ So we treat our conditional as a function of β leaving h fixed. (Essential later when we want to sample) Hence, we can "ignore" all parts not containing β here.

$$p(\beta|y, h_1, h_2) \propto \exp\left\{-\frac{1}{2}\left[\underbrace{(\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta})}_{A} + \underbrace{(y-X\beta)' H (y-X\beta)}_{B}\right]\right\}$$

⇒ Now expand quadratic forms:

⚠ Don't forget it

$$A: (\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta}) \Rightarrow \boxed{\beta'\underline{V}^{-1}\beta} - \boxed{2\beta'\underline{V}^{-1}\underline{\beta}} + \cancel{\underline{\beta}'\underline{V}^{-1}\underline{\beta}}$$

$$B: (y-X\beta)' H (y-X\beta)' \Rightarrow \boxed{X\beta' H X\beta} - \boxed{2\beta' X' H y} + \cancel{y' H y}$$

int. constants: ⊕

⇒ Put parts together we get:

$\underline{\beta}$ missing!

can be dropped

Eq1: $$p(\beta|y, h_1, h_2) \propto \exp\left\{-\frac{1}{2}\left[\beta'(\underline{V}^{-1} + X'HX)\beta - 2\beta'(\underline{V}^{-1}\underline{\beta} + yHX') + \cancel{\oplus}\right]\right\}$$

⇒ Now lets assume $p(\beta|y, h_1, h_2) \sim \mathcal{N}(\bar{\beta}, \bar{V})$, which is also natural since prior and likelihood are normal for β at last:

$\boxed{Eq2}$: $p(\beta|y,h_1,h_2) \propto \exp\left\{-\frac{1}{2}(\beta-\bar{\beta})'\bar{V}^{-1}(\beta-\bar{\beta})\right\}$

open q.form $\propto \exp\left\{-\frac{1}{2}\left[\beta'\bar{V}^{-1}\beta - 2\beta'\bar{V}^{-1}\bar{\beta} + \cancel{\bar{\beta}'\bar{V}^{-1}\bar{\beta}}\right]\right\}$

int. constant

$\Rightarrow$ Now compare $\boxed{Eq1}$ and $\boxed{Eq2}$ to get posterior parameters. (important for computation)

① $\beta'\bar{V}^{-1}\beta = \beta'(\underline{V}^{-1}+X'HX)\beta \Rightarrow \boxed{\bar{V}^{-1} = \underline{V}^{-1}+X'HX}$

② $2\beta'\bar{V}^{-1}\bar{\beta} = 2\beta'(\underline{V}^{-1}\underline{\beta}+yHX') \Rightarrow \bar{\beta}\bar{V}^{-1} = \underline{V}^{-1}+yHX' \quad | \cdot\bar{V}$ (since Matrix)

$$\boxed{\bar{\beta} = \bar{V}\cdot(\underline{V}^{-1}\underline{\beta}+yHX')}$$

write in front
else could be problem
in MATLAB

Step 2: Get $p(h_1|y,\beta,h_2)$

- Now its basically the same principle, but we have to be careful about the Error precisions. So we have to differentiate between the h_1 and h_2. For h_1 we have

$$p(h_1|y,\beta,h_2) \propto p(h_1,h_2,\beta|y)$$

$$\propto h_1^{\frac{T_1+\underline{\nu}_1-2}{2}} \exp\left\{-\frac{1}{2}\left[h_1\underline{\nu}_1\underline{s}_1^{-2} + (y-X\beta)'h_1(I_T-d)(y-X\beta)\right]\right\}$$

$\boxed{Eq1}$ $$\propto h_1^{\frac{T_1+\underline{\nu}_1-2}{2}} \exp\left\{-\frac{h_1}{2}\left[(y-X\beta)'(I_T-d)(y-X\beta) + \underline{\nu}_1\underline{s}_1^{-2}\right]\right\}$$

$\Rightarrow$ Now compare to $h_1|h_2,y,\beta \sim \text{Gamma}(\bar{s}_1^{-2},\bar{\nu}_1)$ to get posterior moments:

Eq 2 $\quad p(h_1 / h_2, \beta, y) \propto h_1^{\frac{\bar{\nu}_1 - 2}{2}} \exp\left\{-\frac{h_1}{2} \bar{\nu}_1 \bar{S}_1^{-2}\right\}$

We get:

① $\frac{\bar{\nu}_1 - 2}{2} = \frac{T_1 + \underline{\nu}_1 - 2}{2} \Rightarrow$ $\boxed{\bar{\nu}_1 = T_1 + \underline{\nu}_1}$

② $\bar{\nu}_1 \bar{S}_1^{-2} = (y - X\beta)'(I_T - d)(y - X\beta) + \underline{\nu}_1 \underline{S}_1^{-2}$

$$\Rightarrow \boxed{\bar{S}_1^{-2} = \frac{(y - X\beta)'(I_T - d)(y - X\beta) + \underline{\nu}_1 \underline{S}_1^{-2}}{T_1 + \underline{\nu}_1}}$$

Step 3: Get $p(h_2 / h_1, \beta, y)$

$p(h_2 / h_1, \beta, y) \propto p(h_2, h_1, \beta / y)$

Eq 1 $\quad \propto h_2^{\frac{T_2 + \underline{\nu}_1 - 2}{2}} \exp\left\{-\frac{h_2}{2}\left[(y - X\beta)' d (y - X\beta) + \underline{\nu}_2 \underline{S}_2^{-2}\right]\right\}$

$\Rightarrow$ Now compare to $h_1 / h_2, y, \beta \sim \text{Gamma}(\bar{S}_2^{-2}, \bar{\nu}_2)$ to get posterior moments:

Eq 2 $\quad p(h_1 / h_2, \beta, y) \propto h_2^{\frac{\bar{\nu}_2 - 2}{2}} \exp\left\{-\frac{h_2}{2} \bar{\nu}_2 \bar{S}_2^{-2}\right\}$

We get:

① $\frac{\bar{\nu}_2 - 2}{2} = \frac{T_2 + \underline{\nu}_2 - 2}{2} \Rightarrow$ $\boxed{\bar{\nu}_2 = T_2 + \underline{\nu}_2}$

② $\bar{\nu}_2 \bar{S}_2^{-2} = (y - X\beta)' d (y - X\beta) + \underline{\nu}_2 \underline{S}_2^{-2}$

$$\Rightarrow \boxed{\bar{S}_2^{-2} = \frac{(y - X\beta)' d (y - X\beta) + \underline{\nu}_2 \underline{S}_2^{-2}}{T_2 + \underline{\nu}_2}}$$

Step 4: Gibbs sampling - Pseudo code

Recall: Since now we have the conditionals in analytically known forms we can now start with the sampling process.

not same!

① Set prior parameters: $\underline{\beta}, \underline{V}, \underline{s}_1^{-2}, \underline{s}_2^{-2}, \underline{\nu}_1, \underline{\nu}_2$

② Define posterior parameters: $\overline{\beta}, \overline{V}, \overline{s}_1^{-2}, \overline{s}_2^{-2}, \overline{\nu}_1, \overline{\nu}_2$

③ Define no. replications: $S = S0 + S1$ with $S0 =$ burn-in period

Recall: We want to converge

④ Set Initialized: Draw $\beta^{(0)}$ from marginal $p(\beta/y)$ from OLS

⑤ Generate first Draws for error precisions:

- $h_1^{(1)} / \beta^{(0)}$ from $p(h_1 / y, \beta^{(0)}, h_2)$
- $h_2^{(1)} / \beta^{(0)}$ from $p(h_2 / y, \beta^{(0)}, h_1)$

⑥ Generate first draw of $\beta^{(1)} / h_1^{(1)} h_2^{(1)}$ from $p(\beta / y, h_1, h_2)$

⑦ Start iteration with loop counter $i = 2 : S$ Recall: Matlab starts at 1

- Draw $h_1^{(i)} / \beta^{(i-1)}$ from $p(h_1 / y, \beta^{(i-1)}, h_2)$
- Draw $h_2^{(i)} / \beta^{(i-1)}$ from $p(h_2 / y, \beta^{(i-1)}, h_1)$
- Draw $\beta^{(i)} / h_1^{(i)} h_2^{(i)}$ from $p(\beta / y, h_1^{(i)}, h_2^{(i)})$

⑧ Discard burn-in draws $S0$ (Technically, we keep them for inference, but exclude from sample)

⑨ Compute posterior mean $E(\beta|y)$ by sample average $\hat{\beta} = \frac{1}{S1} \sum_{S0+1}^{S1} \beta^{(i)}$

⑩ Compute posterior variance $Var(\beta|y)$ by sample var $\hat{S}^2 = \frac{1}{S1-1} \sum_{S0+1}^{S1} (\beta^{(i)} - \hat{\beta})^2$

⑪ For h_1; h_2 do ⑨ and ⑩ respectively

not per se part of sampling itself