Mathematical Finance Winter term 2023/2024

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Mathematical Finance: QF

Exercises (for discussion on Monday, 06.11.2023)

Exercise 1. Decide whether the following statements are true for all finite sets A, B, C or not. If yes, give an argument why (e.g. Venn diagram), if not, give a counterexample.

1. $A \setminus B = \emptyset \implies A = B$
2. $|A \cup B| = |A| + |B| \implies A \cap B = \emptyset$
3. $B \in \mathfrak{P}(A) \Rightarrow B \not\subset \mathfrak{P}(A)$
4. $A \subset B \implies B^c \subset A^c$
5. $A \cap B \cap C = \emptyset \implies |A \cup B \cup C| = |A| + |B| + |C|$
6. $(A \cap B) \cap (A \setminus B) = \emptyset$
7. $A \cap B \neq A \cup B$
8. $(A \cap B)^c = A^c \cup B^c$

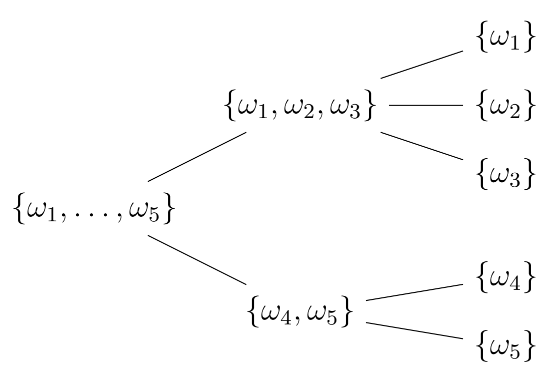
Exercise 2. Consider a probability space $(\Omega, \mathfrak{P}(\Omega), P)$ with $\Omega = \{\omega_1, \dots, \omega_5\}$. The nodes of the trees below represent events, i.e. subsets of the sample space Ω.

1. Suppose that

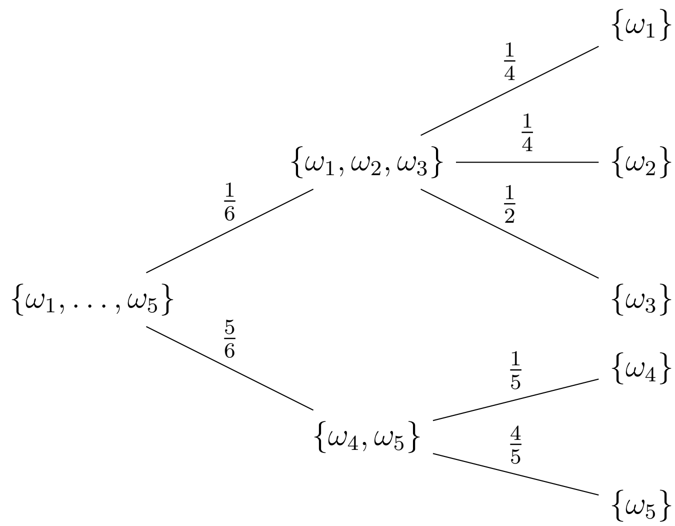
$$P(\{\omega_1\}) = 0.2, \quad P(\{\omega_2\}) = 0.2, \quad P(\{\omega_3\}) = 0.1, \quad P(\{\omega_4\}) = 0.1, \quad P(\{\omega_5\}) = 0.4.$$

   (i) Determine the probabilities of the events (i.e. the nodes) represented in the following tree.

   (ii) Determine for each edge the conditional probability of the event on the right end of the edge, given the event on the left end ($P(\{\omega_1\} \mid \{\omega_1, \omega_2, \omega_3\})$ etc.).



2. Determine $P(\{\omega_1\}), \dots, P(\{\omega_5\})$ if the conditional probabilities of the edges are given as indicated below (i.e. $P(\{\omega_1\} \mid \{\omega_1, \omega_2, \omega_3\}) = \frac{1}{4}$, etc.).



Exercise 3. Many banks offer *reverse convertible bonds*. These are characterized by the maturity T, the underlying asset with prices S_0^1, S_T^1, the nominal amount N, the strike K and the assured interest rate r.
At the beginning the buyer pays the nominal amount. At the end the seller either pays back the nominal amount – in the case $S_T^1 > K$ – or gives $n = \frac{N}{K}$ assets – in the case $S_T^1 \leq K$. In both cases the seller pays the assured interest on the nominal value. This means for the holder the value of S^2 at maturity T is

$$S_T^2 := \begin{cases} Ne^{rT} & \text{if } S_T^1 > K \\ \frac{N}{K} S_T^1 + N(e^{rT} - 1) & \text{if } S_T^1 \leq K. \end{cases}$$

Find a combination of the bond and a call or put option with the same payoff as the reverse convertible bond.

Exercise 4. Consider a market with two securities, 1 and 2. The prices are $S_0^1 = 6, S_0^2 = 11$. Both securities have a term of one year and make a single payout at the end of the year (at $t = 1$), but the payout is random.

1. In this model, there are two outcomes $\Omega = \{\omega_1, \omega_2\}$. Security 1 pays $S_1^1(\omega_1) = 7$ or $S_1^1(\omega_2) = 5$ and security 2 pays $S_1^2(\omega_1) = 14$ or $S_1^2(\omega_2) = 10$.

2. In another model, $\Omega = \{\omega_3, \omega_4\}$. Security 1 pays $S_1^1(\omega_3) = 7$ or $S_1^1(\omega_4) = 5$ and security 2 pays $S_1^2(\omega_3) = 14$ or $S_1^2(\omega_4) = 8$.

For both models determine if there is an arbitrage strategy. If there is one state it explicitly.

Submission of the homework until: Thursday, 02.11.2023, 10.00 a.m. via OLAT.