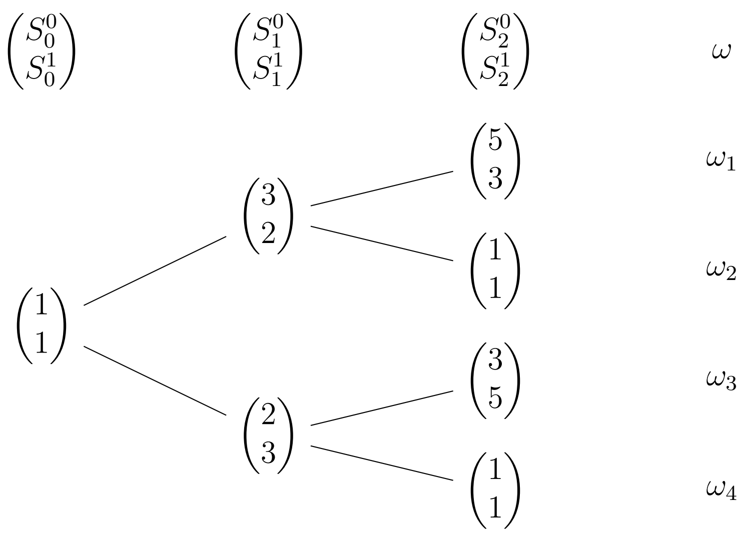
Mathematical Finance Winter term 2023/2024
S. Christensen
P. Le Borne, B. Schroeter, B. Schultz **Sheet QF06**

Mathematical Finance: QF

Exercises (for discussion on Monday, 11.12.2023)

Exercise 1. We consider a price process $S = (S^0, S^1)$ with time horizon $N = 2$. The probability measure P fulfills $P(\{\omega\}) > 0$ for all ω and the filtration is given by the tree.



a) Compute an equivalent martingale measure Q, i.e. find a probability measure Q such that $\hat{S}$ is a Q-martingale.

b) Is the market complete? Explain your answer!

Exercise 2. Given the setting of Exercise 1:

a) Consider the contingent claim $X = \max\{S_2^1 - 1, 0\}$. Find the fair price process for X given by

$$S_n^2 = S_n^0 \mathbb{E}_Q[\hat{X} | \mathcal{F}_n], \quad n = 0, 1, 2.$$

b) Find a self-financing (and predictable) trading strategy $\varphi = (\varphi^0, \varphi^1)$ such that

$$V_2(\varphi) = X.$$

c) Compute $V_0(\varphi)$ and compare it with the initial value S_0^2 of the price process obtained in Part (a).

Exercise 3. Let (S^0, S^1) be the price process in a market with end-time 1. Assume that $S_0^0 = S_0^1 = S_1^0 = 1$ and that

$$S_1^1(\omega) = \begin{cases} x_1, & \text{falls } \omega = \omega_1 \\ x_2, & \text{falls } \omega = \omega_2 \\ x_3, & \text{falls } \omega = \omega_3 \end{cases}$$

with

$$\begin{aligned} p_1 &:= P(\{\omega_1\}) > 0, \\ p_2 &:= P(\{\omega_2\}) > 0, \\ p_3 &:= P(\{\omega_3\}) = 1 - p_1 - p_2 \geq 0 \end{aligned}$$

and $x_1 < x_2 < x_3$.

a) Find specific values for x_1, x_2, x_3 and p_1, p_2, p_3, such that the market admits an arbitrage and give the arbitrage strategy.

b) Find specific values for x_1, x_2, x_3 and p_1, p_2, p_3, such that the market is arbitrage-free and complete.

c) Find specific values for x_1, x_2, x_3 and p_1, p_2, p_3, such that the market is arbitrage-free and not complete.

In all cases explain your choices.

Exercise 4. Consider an inhomogeneous market with time horizon N. S^0 is the riskless asset given by $S_n^0 = 1 + nr$ with a constant $r > 0$. Let the stock S^1 be given by $S_n^1 = \prod_{i=1}^n (1 + \Delta\tilde{X}_i)$ with the $\Delta\tilde{X}_i$, $i = 1, 2, ..., N$ being independent random variables with

$$\begin{aligned} P(\Delta\tilde{X}_i = u_i - 1) = p_i > 0, \\ P(\Delta\tilde{X}_i = d_i - 1) = 1 - p_i > 0, \end{aligned}$$

for real numbers $u_i > d_i > 0$. We consider the filtration is generated by $\mathcal{F}_n := \sigma(\Delta\tilde{X}_1, \ldots, \Delta\tilde{X}_n)$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and set $S_0^0 = S_0^1 = 1$. Show that there is no arbitrage if

$$u_n > \frac{1 + nr}{1 + (n-1)r} > d_n$$

for all $n \in \{1, ..., N\}$.

Submission of the homework until: Thursday, 07.12.2023, 10.00 a.m. via OLAT.