Mathematical Finance: QF

Exercises (for discussion on Monday, 29.01.2024)

Exercise 1. (*8 points*)
Suppose a customer wants to buy a European call option on the stock S^1 with strike 101.75€ and maturity $T = 31.03.2024$. On the market European call options with strikes 100€, 101€, 102€, 103€, 104€, 105€ and maturity $T = 31.03.2024$ are liquidly traded. Their prices are given in the following table:

Strike K_i	100 €	101 €	102 €	103 €	104 €	105 €
Price P_i	7.453 €	6.970 €	6.448 €	5.958 €	5.467 €	5.070 €

Suppose that in a very simple model the price of a European call option on S^1 with strike K and maturity $T = 31.03.2024$ is given by

$$P(K, \vartheta_0, \vartheta_1) = \vartheta_0 + \vartheta_1 K$$

for parameters $\vartheta_0, \vartheta_1 \in \mathbb{R}$.

(a) Find ϑ_0, ϑ_1 such that

$$\sum_{i=1}^{6} (P_i - P(K_i, \vartheta_0, \vartheta_1))^2$$

is minimized.

(b) Use the result from part (a) to determine the approximated fair price of the European call option with strike 103.5€.

(c) Discuss whether for arbitrary strike K the pricing approach used in this exercise may lead to an arbitrage-free market.

Hint: For (a) differentiate with respect to ϑ_0 and ϑ_1 and use the first order criterion to find the minimum.

Exercise 2. Let W be a standard Brownian motion and let $X_t = 4W_t^3$.
Compute the Itō process representation of X, i.e. write it in the form

$$X = X_0 + \ldots \bullet I + \ldots \bullet W \quad \text{or} \quad X = X_0 + \int \ldots ds + \int \ldots dW_s.$$

Exercise 3. *(4 points + 2 bonus points)*
Let $S = (S^0, S^1, S^2, S^3, S^4, S^5)$ denote an arbitrage-free market with time horizon $N \in \mathbb{N}$. We have $S_n^0 = \exp(rn)$ for some $r \geq 0$. Further let S^2 and S^3 denote the value processes for an American call option respectively an American put option with identical strike K and maturity N on the stock S^1 which does not pay dividends. S^4 and S^5 are price processes for the corresponding European call and put options. Prove the following inequalities

$$S_0^1 - K \leq S_0^2 - S_0^3 \leq S_0^1 - K\exp(-rN)$$

which are an analogue to the Put-Call-parity for European options.

Hints:

- For the first inequality consider a stopping time τ^* such that

$$E_Q\left[\frac{(K - S_{\tau^*}^1)^+}{S_{\tau^*}^0}\middle|\mathcal{F}_0\right] = \max_{\tau \in \mathcal{J}_0} E_Q\left[\frac{(K - S_\tau^1)^+}{S_\tau^0}\middle|\mathcal{F}_0\right].$$

  Argue why

$$\max_{\tau \in \mathcal{J}_0} E_Q\left[\frac{(K - S_\tau^1)^+}{S_\tau^0}\middle|\mathcal{F}_0\right] = S_0^3$$

  and apply that $f = f^+ - (-f)^+$ for real valued functions f.

- For the second inequality apply Theorem 6.7 and revisit Sheet 5 Exercise 3.

Submission of the homework until: Thursday, 25.01.2024, 10.00 a.m. via OLAT.