Bayesian Econometrics
Tutorial 02 - Bayesian Estimation of Linear Regression Models

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Winter Term 2024/25

Review the Concepts and Proofs

- 1. What is the normal-gamma distribution? How can it be constructed from a conditional and a marginal distribution?
- 2. How are precision and variance of a normal distribution related?
- 3. What is an improper prior?
- 4. Characterize the general multivariate t distribution. How can probabilities of its marginal distributions be computed?
- 5. What is meant by Bayesian model averaging?

Exercise 1

Let $y = (y_1, \dots, y_N)'$ be a random sample from a normal distribution with unknown mean μ and known precision h,

$$p(y_i|\mu, h) = (2\pi)^{-\frac{1}{2}} h^{\frac{1}{2}} \exp\left[-\frac{1}{2}h(y_i - \mu)^2\right].$$

Suppose prior beliefs concerning μ are represented by a normal distribution with mean $\underline{\mu}$ and precision $\underline{\kappa}$:

$$p(\mu|\underline{\mu}, \underline{\kappa}) = (2\pi)^{-\frac{1}{2}} \underline{\kappa}^{\frac{1}{2}} \exp\left[-\frac{1}{2}\underline{\kappa}(\mu - \underline{\mu})^2\right].$$

- (a) **Find the posterior distribution and** $\mathrm{E}(\mu|y)$**.**
- (b) **Suppose a researcher has in mind a previous sample $x = (x_1, \dots, x_M)'$ from the same distribution when specifying her prior. This previous sample had mean $\bar{x}$ (reported in the literature). How should she specify her prior and interpret the Bayesian point estimator** $\mathrm{E}(\theta|y)$**? What can she do when she does not fully trust in the validity of the previous sample mean?**

Exercise 2

Let $y = (y_1, \ldots, y_N)'$ be a random sample from a normal distribution with unknown mean μ and precision h,

$$p(y_i|\mu, h) = (2\pi)^{-\frac{1}{2}} h^{\frac{1}{2}} \exp\left[-\frac{1}{2}h(y_i - \mu)^2\right].$$

Suppose prior beliefs concerning μ are, conditional on h, represented by a normal distribution with mean $\underline{\mu}$ and precision $h\underline{\kappa}$, $\mu \sim \mathcal{N}(\underline{\mu}, (h\underline{\kappa})^{-1})$, with density

$$p(\mu|\underline{\mu}, h\underline{\kappa}) = (2\pi)^{-\frac{1}{2}} (h\underline{\kappa})^{\frac{1}{2}} \exp\left[-\frac{1}{2}h\underline{\kappa}(\mu - \underline{\mu})^2\right],$$

and prior beliefs concerning h are represented by a gamma distribution with parameters $\underline{s}^{-2}$ and $\underline{\nu}$, $h \sim$ Gamma$(\underline{s}^{-2}, \underline{\nu})$, with density

$$p(h|\underline{s}^{-2}, \underline{\nu}) = \left(\frac{2}{\underline{s}^2\underline{\nu}}\right)^{\frac{\underline{\nu}}{2}} \Gamma\left(\frac{\underline{\nu}}{2}\right)^{-1} h^{\frac{\underline{\nu}-2}{2}} \exp\left[-\frac{1}{2}h\underline{\nu}\underline{s}^2\right].$$

- (a) **Find the posterior distribution.** (b) **Find and interpret** $\mathrm{E}(\mu|y)$ **and** $\mathrm{E}(h|y)$**.**

Exercise 3

Consider the multiple regression model with k regressors

$$y = X\beta + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, h^{-1}I).$$

Let (y_i, x_i), $i = 1, \ldots, N$, be a random sample and assume X is exogenous. Suppose prior beliefs concerning β and h are represented by a multivariate normal-gamma distribution with parameters $\underline{\beta}$, $\underline{V}$, $\underline{s}^2$, and $\underline{\nu}$.

- (a) **Find the posterior distribution.**
- (b) **Find** $\mathrm{E}(\beta|y)$**,** $\mathrm{Var}(\beta|y)$**,** $\mathrm{E}(h|y)$ **and** $\mathrm{Var}(h|y)$**.**

Exercise 3 (b): Solution

Recall that $\theta|y \sim NG(\bar{\beta}, \bar{\boldsymbol{\kappa}}^{-1}, \bar{s}^{-2}, \bar{\nu})$, such that

$$\begin{aligned} \beta|h &\sim N(\bar{\beta}, h\bar{\boldsymbol{\kappa}}) \\ h &\sim G(\bar{s}^{-2}, \bar{\nu}) \end{aligned}$$

Hence, starting by the precision parameter h, we have that

$$\mathrm{E}(h|y) = \bar{s}^{-2} = \frac{\underline{\nu} + N}{\underline{\nu}\underline{s}^2 + \nu s^2 + (\hat{\beta} - \underline{\beta})'(\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}(\hat{\beta} - \underline{\beta})}$$

and

$$\mathrm{Var}(h|y) = \frac{2\,(\bar{s}^{-2})^2}{\bar{\nu}} = \frac{2}{\bar{s}^4\,\bar{\nu}} = \frac{2(\underline{\nu} + N)}{[\underline{\nu}\underline{s}^2 + \nu s^2 + (\hat{\beta} - \underline{\beta})'(\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}(\hat{\beta} - \underline{\beta})]^2}$$

Exercise 3 (b): Solution

Hint: $f_G(y|\mu,\nu) = c_G^{-1}\, y^{\frac{\nu-2}{2}} \exp(-\frac{y\nu}{2\mu})$, where $c_G^{-1} = (\frac{\nu}{2\mu})^{\frac{\nu}{2}}\, \Gamma(\frac{\nu}{2})^{-1}$

For the posterior point estimators of β, let us apply the conditional-marginal factorization:

$$
\begin{aligned}
p(\beta) &= \int p(\beta, h|y)dh = \int \underbrace{p(\beta|h,y)}_{\text{conditional normal}} \quad \underbrace{p(h)}_{\text{marginal gamma}} \quad dh \\
&= \bar{\chi} \int h^{\frac{k+\bar{\nu}-2}{2}} \exp\left[-\frac{h}{2}\left\{(\beta-\bar{\beta})'\bar{\boldsymbol{\kappa}}(\beta-\bar{\beta}) + \bar{\nu}\bar{s}^2\right\}\right] dh \\
&= \bar{\chi} \int \underbrace{h^{\frac{\nu^*-2}{2}} \exp\left[-\frac{h}{2}\frac{\nu^*}{\mu^*}\right]}_{\text{Gamma kernel with } G(\mu^*,\nu^*)} dh
\end{aligned}
$$

with $\nu^* = k + \bar{\nu}$ and $\mu^* = \nu^*[(\beta-\bar{\beta})'\bar{\boldsymbol{\kappa}}(\beta-\bar{\beta}) + \bar{\nu}\bar{s}^2]^{-1}$, and where the integrating constant of the joint posterior is given by

$$
\bar{\chi} = \frac{|\bar{\boldsymbol{\kappa}}|^{\frac{1}{2}}}{(\frac{2\bar{s}^{-2}}{\bar{\nu}})^{\frac{\bar{\nu}}{2}}\,\Gamma(\frac{\bar{\nu}}{2})(2\pi)^{\frac{k}{2}}}
$$

Exercise 3 (b): Solution

From there recall that $\int G(\mu^*, \nu^*) dh = 1$ and define the Gamma integrating constant as $c_G = \left(\frac{2\mu^*}{\nu^*}\right)^{\frac{\nu^*}{2}} \Gamma\left(\frac{\nu^*}{2}\right)$.

Then

$$p(\beta) = \bar{\chi}\, c_G \underbrace{\int c_G^{-1}\, h^{\frac{\nu^*-2}{2}} \exp\left[-\frac{h}{2}\frac{\nu^*}{\mu^*}\right] dh}_{=1}$$

$$= \bar{\chi}\, c_G = \left[\frac{|\bar{\boldsymbol{\kappa}}|^{\frac{1}{2}}}{\left(\frac{2\bar{s}^{-2}}{\bar{\nu}}\right)^{\frac{\bar{\nu}}{2}} \Gamma\left(\frac{\bar{\nu}}{2}\right)(2\pi)^{\frac{k}{2}}}\right] \left[\left(\frac{2\mu^*}{\nu^*}\right)^{\frac{\nu^*}{2}} \Gamma\left(\frac{\nu^*}{2}\right)\right]$$

$$= \ldots$$

$$= \frac{\bar{\nu}^{\frac{\bar{\nu}}{2}} \Gamma\left(\frac{\bar{\nu}+k}{2}\right)}{\pi^{\frac{k}{2}} \Gamma\left(\frac{\bar{\nu}}{2}\right)} |\bar{s}^2 \bar{\boldsymbol{\kappa}}^{-1}|^{-\frac{1}{2}} \left[\bar{\nu} + (\beta - \bar{\beta})'(\bar{s}^2 \bar{\boldsymbol{\kappa}}^{-1})^{-1}(\beta - \bar{\beta})\right]^{-\frac{\bar{\nu}+k}{2}}$$

This is the pdf of a multivariate t distribution (see Koop's textbook page 328).

Exercise 3 (b): Solution

Hence, $p(\beta) \sim t(\bar{\beta}, \bar{s}^2\bar{\boldsymbol{\kappa}}^{-1}, \bar{\nu})$ such that

$$E(\beta|y) = \bar{\beta} = (\underline{\boldsymbol{\kappa}} + \boldsymbol{\kappa})^{-1}(\underline{\boldsymbol{\kappa}}\underline{\beta} + \boldsymbol{\kappa}\hat{\beta})$$

and

$$\begin{aligned} \mathrm{Var}(\beta|y) &= \frac{\bar{\nu}}{\bar{\nu}-2}\bar{s}^2\bar{\boldsymbol{\kappa}}^{-1} \\ &= \frac{\underline{\nu}\underline{s}^2 + \nu s^2 + (\hat{\beta} - \underline{\beta})'(\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}(\hat{\beta} - \underline{\beta})}{\underline{\nu} + N - 2}(\underline{\boldsymbol{\kappa}} + \boldsymbol{\kappa})^{-1} \end{aligned}$$