Mathematical Finance Winter term 2023/2024

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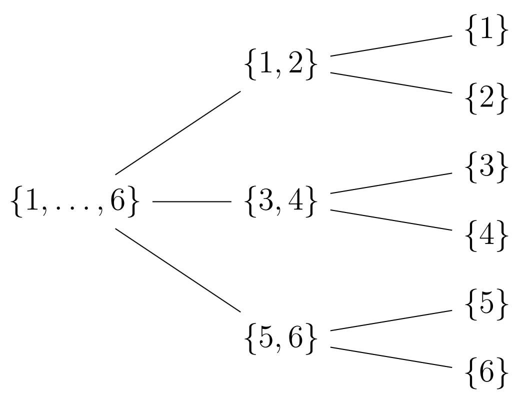
Mathematical Finance: QF

Exercises (for discussion on Monday, 13.11.2023)

Exercise 1. Let X and Y be random variables representing the outcomes of two independently thrown fair dice.

1. Compute $E(X+Y \mid \{X=4\})$, i.e. the conditional expectation of the sum of both scores given that the first die shows 4.
2. Compute $E(X \mid \{X+Y=9\})$, i.e. the conditional expectation of the score of the first die given that the sum of both scores is 9.

Exercise 2. Let $(\Omega, \mathfrak{P}(\Omega), P)$ be a probability space with $\Omega = \{1, \ldots, 6\}$ and $P(\{1\}) = 0.3$, $P(\{2\}) = 0.1$, $P(\{3\}) = 0.1$, $P(\{4\}) = 0.2$, $P(\{5\}) = 0.2$, $P(\{6\}) = 0.1$.
Furthermore, consider the σ-algebra $\mathcal{F} = \sigma(F_1, F_2, F_3)$ with $F_1 = \{1,2\}$, $F_2 = \{3,4\}$, $F_3 = \{5,6\}$ and the random variable $X : \Omega \to \mathbb{R}$ with $X(\omega) = \omega$. We can illustrate the filtration $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \mathcal{F}$ and $\mathcal{F}_2 := \mathfrak{P}(\Omega)$ in the following tree:



Find the expectation of X and the conditional expectations of X given the events in the tree.

Exercise 3. Let $(\Omega, \mathfrak{P}(\Omega), P)$ be a probability space with $\Omega = \{\omega_1, \ldots, \omega_7\}$. We consider the stochastic process $X = (X_0, \ldots, X_4)$ given by the following table.

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$	$X_4(\omega)$
ω_1	1	2	2	3	3.5
ω_2	1	2	2	1	1
ω_3	1	0.5	0.5	0.5	3
ω_4	1	0.5	0.5	0.5	0.5
ω_5	1	1	2	2	3
ω_6	1	1	0.5	2	3
ω_7	1	1	0.5	1	1

1. Represent the filtration $\mathcal{F}_0, \ldots, \mathcal{F}_4$ generated by the process X in a tree.
2. Determine for $n = 0, \ldots, 4$ and each node on level n the values of X_n, X_{n-} and ΔX_n on the respective event.
3. Suppose that the probability measure P is given by the following table.

ω	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$P(\{\omega\})$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$

   Is the process X a martingale, a submartingale, a supermartingale, or none of these? Prove your answer!

Exercise 4. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras to a set Ω.

1. Is $\mathcal{A} \cup \mathcal{B}$ always a σ-algebra? Prove your answer!
2. Is $\mathcal{A} \cap \mathcal{B}$ always a σ-algebra? Prove your answer!

Submission of the homework until: Thursday, 09.11.2023, 10.00 a.m. via OLAT.