Tutorial 2: Bayesian Estimation of Linear Regression Models

Exercise 3 - Supplemental Material

Result 1: Likelihood function in terms of OLS quantities

Proof:

$$(\mathbf{y} - \mathbf{X}\beta)'(\mathbf{y} - \mathbf{X}\beta) = \nu\mathbf{s^2} + (\beta - \hat{\beta})'\boldsymbol{\kappa}(\beta - \hat{\beta})$$

First of all, let us rewrite $(y - X\beta)'(y - X\beta)$ by summing and subtracting $X\hat{\beta}$:

$$\begin{aligned}(y - X\beta)'(y - X\beta) &= (y - X\beta + X\hat{\beta} - X\hat{\beta})'(y - X\beta + X\hat{\beta} - X\hat{\beta}) \\ &= ((y - X\hat{\beta}) - (X\beta - X\hat{\beta}))'((y - X\hat{\beta}) - (X\beta - X\hat{\beta})) \\ &= ((y - X\hat{\beta})' - (\beta - \hat{\beta})'X')((y - X\hat{\beta}) - X(\beta - \hat{\beta}))\end{aligned}$$

which implies that $(y - X\beta)'(y - X\beta)$ yields,

$$(y - X\hat{\beta})'(y - X\hat{\beta}) - (y - X\hat{\beta})'X(\beta - \hat{\beta}) - (\beta - \hat{\beta})'X'(y - X\hat{\beta}) + (\beta - \hat{\beta})'X'X(\beta - \hat{\beta}) \quad (1)$$

Note that both central terms are equal to zero since the expression $(y - X\hat{\beta})'X$ yields zero by construction, and thereupon its transpose. The reason for that comes from the exogeneity assumption between the explanatory variables and OLS residuals, which can be proved as follows:

$$\begin{aligned}(y - X\hat{\beta})'X &= (y - X(X'X)^{-1}X'y)'X = (y' - (X(X'X)^{-1}X'y)')X \\ &= (y' - (X'y)'(X(X'X)^{-1})')X = (y' - y'X((X'X)^{-1'}X'))X \\ &= (y' - y'X(X'X)^{-1}X')X = y'X - y'X\underbrace{(X'X)^{-1}X'X}_{=I_K} \\ &= y'X - y'X = 0\end{aligned}$$

Then,

$$(y - X\beta)'(y - X\beta) = \underbrace{(y - X\hat{\beta})'(y - X\hat{\beta})}_{SSR} + (\beta - \hat{\beta})' \underbrace{X'X}_{\boldsymbol{\kappa}} (\beta - \hat{\beta}), \tag{2}$$

where SSR denotes the OLS sum of squared residuals.

Finally, since we define the OLS variance estimator as $s^2 = \frac{1}{\nu}(y - X\hat{\beta})'(y - X\hat{\beta})$ such that

$$\nu s^2 = \nu \, \frac{1}{\nu} \underbrace{(y - X\hat{\beta})'(y - X\hat{\beta})}_{SSR} = SSR,$$

it follows that

$$(y - X\beta)'(y - X\beta) = \nu s^2 + (\beta - \hat{\beta})'\boldsymbol{\kappa}(\beta - \hat{\beta})$$

Result 2: Posterior sum of squared residuals

Solving for $\bar{\nu}\bar{s}^2$ and further simplifying:

$$\begin{aligned}
\bar{\nu}\bar{s}^2 &= \underline{\nu s}^2 + \nu s^2 + \underline{\beta}'\underline{\boldsymbol{\kappa}}\underline{\beta} + \hat{\beta}'\boldsymbol{\kappa}\hat{\beta} - \bar{\beta}'\bar{\boldsymbol{\kappa}}\bar{\beta} \\
&= \underline{\nu s}^2 + \nu s^2 + \underline{\beta}'\underline{\boldsymbol{\kappa}}\underline{\beta} + \hat{\beta}'\boldsymbol{\kappa}\hat{\beta} - (\underline{\boldsymbol{\kappa}}\underline{\beta} + \boldsymbol{\kappa}\hat{\beta})'\bar{\boldsymbol{\kappa}}^{-1}\bar{\boldsymbol{\kappa}}\bar{\boldsymbol{\kappa}}^{-1}(\underline{\boldsymbol{\kappa}}\underline{\beta} + \boldsymbol{\kappa}\hat{\beta}) \\
&= \underline{\nu s}^2 + \nu s^2 + \underline{\beta}'\underline{\boldsymbol{\kappa}}\underline{\beta} + \hat{\beta}'\boldsymbol{\kappa}\hat{\beta} - (\underline{\beta}'\underline{\boldsymbol{\kappa}} + \hat{\beta}'\boldsymbol{\kappa})\bar{\boldsymbol{\kappa}}^{-1}(\underline{\boldsymbol{\kappa}}\underline{\beta} + \boldsymbol{\kappa}\hat{\beta}) \\
&= \underline{\nu s}^2 + \nu s^2 + \underline{\beta}'\underline{\boldsymbol{\kappa}}\underline{\beta} + \hat{\beta}'\boldsymbol{\kappa}\hat{\beta} - \underline{\beta}'\underline{\boldsymbol{\kappa}}\bar{\boldsymbol{\kappa}}^{-1}\underline{\boldsymbol{\kappa}}\underline{\beta} - 2\underline{\beta}'\underline{\boldsymbol{\kappa}}\bar{\boldsymbol{\kappa}}^{-1}\boldsymbol{\kappa}\hat{\beta} - \hat{\beta}'\boldsymbol{\kappa}\bar{\boldsymbol{\kappa}}^{-1}\boldsymbol{\kappa}\hat{\beta} \\
&= \underline{\nu s}^2 + \nu s^2 + \underline{\beta}'(\underline{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}\bar{\boldsymbol{\kappa}}^{-1}\underline{\boldsymbol{\kappa}})\underline{\beta} + \hat{\beta}'(\boldsymbol{\kappa} - \boldsymbol{\kappa}\bar{\boldsymbol{\kappa}}^{-1}\boldsymbol{\kappa})\hat{\beta} - 2\underline{\beta}'\underline{\boldsymbol{\kappa}}\bar{\boldsymbol{\kappa}}^{-1}\boldsymbol{\kappa}\hat{\beta}
\end{aligned}$$

Result 3: Posterior variance-covariance matrix

Prove that

(i) $\underline{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}\bar{\boldsymbol{\kappa}}^{-1}\underline{\boldsymbol{\kappa}} = (\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}$

(ii) $\boldsymbol{\kappa} - \boldsymbol{\kappa}\bar{\boldsymbol{\kappa}}^{-1}\boldsymbol{\kappa} = (\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}$

(iii) $\underline{\boldsymbol{\kappa}}\bar{\boldsymbol{\kappa}}^{-1}\boldsymbol{\kappa} = (\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}$

Start from $\bar{\boldsymbol{\kappa}}^{-1} = (\underline{\boldsymbol{\kappa}} + \boldsymbol{\kappa})^{-1} = [\boldsymbol{\kappa}(\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})\underline{\boldsymbol{\kappa}}]^{-1} = \underline{\boldsymbol{\kappa}}^{-1}(\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}\boldsymbol{\kappa}^{-1}$. Substitute this into (i) which yields $\underline{\boldsymbol{\kappa}} - (\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}\boldsymbol{\kappa}^{-1}\underline{\boldsymbol{\kappa}} = (\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})^{-1}$. Multiply both sides from the right by $(\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})$ which yields $(\underline{\boldsymbol{\kappa}}^{-1} + \boldsymbol{\kappa}^{-1})\underline{\boldsymbol{\kappa}} - \boldsymbol{\kappa}^{-1}\underline{\boldsymbol{\kappa}} = I_k$. Simplifying the left-hand side finally yields $I_k = I_k$ which shows that our claim is correct. Parts (ii) and (iii) can be proved analogously.