Tutorial 3: Numerical Methods for Bayesian Linear Regression Models

Review the Concepts and Proofs

1. Consider the normal linear regression model. What is the difference between the natural conjugate and the independent normal-gamma prior?
2. What is Monte Carlo integration? Give three examples of often-used integrand functions $g(\theta)$.
3. Explain the probability integral transform.
4. Explain the acceptance-rejection method.
5. Explain importance sampling. Why is it particularly helpful for Monte Carlo integration of the normal linear regression model with inequality constraints?
6. Explain the Gibbs sampler. Why is it particularly helpful for Monte Carlo integration of the normal linear regression model with independent normal-gamma prior?
7. What does Markov chain Monte Carlo (MCMC) mean?
8. Why is it important to check convergence of MCMC algorithms?
9. Why is it, at least in principle, necessary to use a long-run variance estimator when computing the numerical standard error for a MCMC algorithm?
10. What is the Savage-Dickey density ratio?

Exercises

1. Show how you can apply importance sampling to simulate moments of a truncated normal distribution, $x \sim trunc\mathcal{N}(\mu, \sigma^2, a, b)$, $a \le x \le b$, using only normal random numbers.

2. How can you easily simulate draws from the general $t(\mu, \sigma, k)$-distribution for integer values of k if your software offers you only standard normal random numbers? (Hint: think of the definition of the t-distribution.)

3. Suppose you have a posterior distribution of the scalar parameter θ which is logistic with parameters $-\infty < \bar{\alpha} < \infty$ and $0 < \bar{\beta} < \infty$ and cdf

$$F(\theta|\bar{\alpha}, \bar{\beta}) = \left[1 + \exp\left(-\frac{\theta - \bar{\alpha}}{\bar{\beta}}\right)\right]^{-1}.$$

   You want to apply Monte Carlo integration to find (i) the mean of θ, (ii) the variance of θ, and (iii) the expected value of $g(\theta) = \exp(\sqrt{|\theta|} - 1)$ but you have access to random numbers from the uniform distribution and the standard normal distribution only.

   (a) Find the posterior pdf.

   (b) Write a pseudo code (i.e., a step-by-step algorithm) that applies the probability integral transform.

   (c) Consider the acceptance-rejection method. Show that the normal distribution is not a good proposal distribution $h(\theta)$ because the scale factor M is unbounded in this case. (Hint: what happens to $f(\theta)/h(\theta)$ as $\theta \to \infty$?) Next show that the t-distribution should work properly because M is bounded.

   (d) Write a pseudo code that applies the acceptance-rejection method.

   (e) Show in pseudo code how importance sampling can be used. Which importance function do you choose?

4. Consider the normal linear regression model with independent normal-gamma prior, $\beta \sim \mathcal{N}(\underline{\beta}, \underline{V})$ and $h \sim Gamma(\underline{s}^{-2}, \underline{\nu})$. Assume the regression model includes an intercept β_1 and one regressor with slope β_2, i.e., $\beta = (\beta_1, \beta_2)'$.

   (a) Explain how prior knowledge can be used to specify the prior parameters $\underline{\beta}$, $\underline{V}$, $\underline{s}^{-2}$, and $\underline{\nu}$.

(b) State prior and likelihood. Find the posterior pdf.

(c) Find the conditional posterior of β given h and the data.

(d) Find the conditional posterior of h given β and the data.

(e) Show in pseudo code how the acceptance-rejection method can be used to find the posterior mean and variance matrix of β if only standard normal random numbers are available. Find an efficient proposal distribution that minimizes the acceptance rate.

(f) Show in pseudo code how importance sampling can be used to find the posterior mean and variance matrix of β if only standard normal random numbers are available.

(g) Show in pseudo code how Gibbs sampling can be used to find the posterior mean and variance matrix of β if independent normal and gamma random numbers are available.