Tutorial 2: Bayesian Estimation of Linear Regression Models

Review the Concepts and Proofs

1. What is the normal-gamma distribution? How can it be constructed from a conditional and a marginal distribution?
2. How are precision and variance of a normal distribution related?
3. What is an improper prior?
4. Characterize the general multivariate t distribution. How can probabilities of its marginal distributions be computed?
5. What is meant by Bayesian model averaging?

Exercises

1. Let $y = (y_1, \dots, y_N)'$ be a random sample from a normal distribution with unknown mean μ and known precision h,

$$p(y_i|\mu, h) = (2\pi)^{-\frac{1}{2}} h^{\frac{1}{2}} \exp\left[-\frac{1}{2}h(y_i - \mu)^2\right].$$

Suppose prior beliefs concerning μ are represented by a normal distribution with mean $\underline{\mu}$ and precision $\underline{\kappa}$:

$$p(\mu|\underline{\mu}, \underline{\kappa}) = (2\pi)^{-\frac{1}{2}} \underline{\kappa}^{\frac{1}{2}} \exp\left[-\frac{1}{2}\underline{\kappa}(\mu - \underline{\mu})^2\right].$$

(a) Find the posterior distribution and $\mathrm{E}(\theta|y)$.

(b) Suppose a researcher has in mind a previous sample $x = (x_1, \dots, x_M)'$ from the same distribution when specifying her prior. This previous sample had mean $\bar{x}$ (reported in the literature). How should she specify her prior and how interpret the Bayesian point estimator $\mathrm{E}(\theta|y)$? What can she do when she does not fully trust in the validity of the previous sample mean?

2. Let $y = (y_1, \ldots, y_N)'$ be a random sample from a normal distribution with unknown mean μ and precision h,

$$p(y_i|\mu, h) = (2\pi)^{-\frac{1}{2}} h^{\frac{1}{2}} \exp\left[-\frac{1}{2}h(y_i - \mu)^2\right].$$

Suppose prior beliefs concerning μ are, conditional on h, represented by a normal distribution with mean $\underline{\mu}$ and precision $h\underline{\kappa}$, $\mu \sim \mathcal{N}(\underline{\mu}, (h\underline{\kappa})^{-1})$, with density

$$p(\mu|\underline{\mu}, h\underline{\kappa}) = (2\pi)^{-\frac{1}{2}} (h\underline{\kappa})^{\frac{1}{2}} \exp\left[-\frac{1}{2}h\underline{\kappa}(\mu - \underline{\mu})^2\right],$$

and prior beliefs concerning h are represented by a gamma distribution with parameters $\underline{s}^{-2}$ and $\underline{\nu}$, $h \sim \text{Gamma}(\underline{s}^{-2}, \underline{\nu})$, with density

$$p(h|\underline{s}^{-2}, \underline{\nu}) = \left(\frac{2}{\underline{s}^2\underline{\nu}}\right)^{\frac{\underline{\nu}}{2}} \Gamma\left(\frac{\underline{\nu}}{2}\right)^{-1} h^{\frac{\underline{\nu}-2}{2}} \exp\left[-\frac{1}{2}h\underline{\nu}\underline{s}^2\right].$$

   (a) Find the posterior distribution.

   (b) Find and interpret $\mathrm{E}(\mu|y)$ and $\mathrm{E}(h|y)$.

3. Consider the multiple regression model with k regressors

$$y = X\beta + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, h^{-1}).$$

Let (y_i, x_i), $i = 1, \ldots, N$, be a random sample and assume X is exogenous. Suppose prior beliefs concerning β and h are represented by a multivariate normal-gamma distribution with parameters $\underline{\beta}$, $\underline{V}$, $\underline{s}^2$, and $\underline{\nu}$.

   (a) Find the posterior distribution.

   (b) Find $\mathrm{E}(\mu|y)$, $\mathrm{Var}(\mu|y)$, $\mathrm{E}(h|y)$ and $\mathrm{Var}(h|y)$.