Mathematical Finance Winter term 2023/2024
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Mathematical Finance: QF

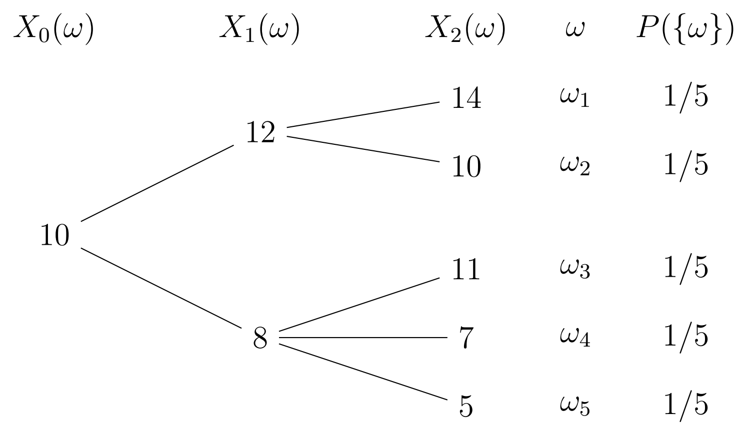
Exercises (for discussion on Monday, 20.11.2023)

Exercise 1. We roll a fair die 6 times. The outcome of each roll is a random variable $\xi_1, \xi_2, \dots, \xi_6$. The stochastic process $X = (X_1, X_2, \dots, X_6)$ is given by the sum over the ξ_i, $X_n = \sum_{i=1}^n \xi_i$. Let $(\mathcal{F}_1, \dots, \mathcal{F}_6)$ be the filtration generated by X.

1. Use Definition 3.5. to prove that $\tau := 6 - 4 \cdot \mathbb{1}_{\{X_2 \leq 7\}}$ is a stopping time.
2. Decide whether the following times are stopping times ('yes' or 'no' are sufficient):
   (a) The minimum of 'the first time X exceeds the number 12' and '$t = 6$'
   (b) The roll with the most eyes, i.e. the time with the maximal value of $\Delta X_n = \xi_n$ (if there are different possibilities take the first time)
   (c) The last time X is even
3. We consider the stochastic process $X = (X_1, X_2, \dots, X_6)$ from (2).
   (a) Find a filtration $(\mathcal{G}_1, \dots, \mathcal{G}_6)$, such that all times in (2) are stopping times with respect to $(\mathcal{G}_1, \dots, \mathcal{G}_6)$
   (b) Find a filtration $(\mathcal{G}'_1, \dots, \mathcal{G}'_6)$, such that none of the times in (2) are stopping times with respect to $(\mathcal{G}'_1, \dots, \mathcal{G}'_6)$

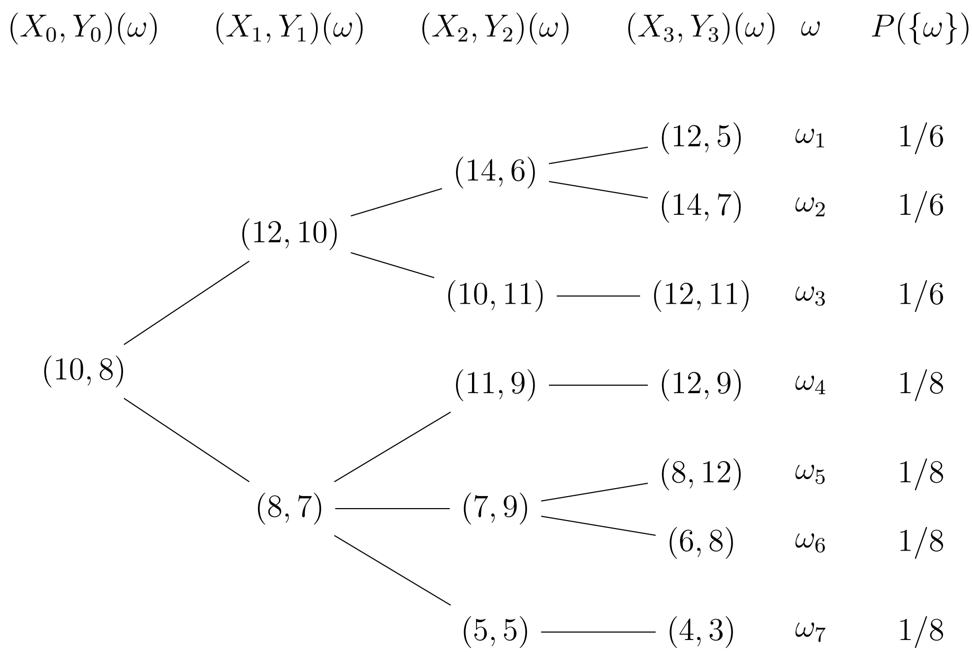
Exercise 2. You are on a TV show. As a prize you are presented with two envelopes containing money. The amount of money X^1 in the first envelope is distributed in the following way: $P(X^1 = 1) = 0.4$, $P(X^1 = 2) = 0.3$, $P(X^1 = 3) = 0.2$, $P(X^1 = 4) = 0.1$. The amount of money in the second envelope X^2 is independent of X^1 with distribution $P(X^2 = 1) = 0.4$, $P(X^2 = 3) = 0.3$, $P(X^2 = 4) = 0.2$, $P(X^2 = 5) = 0.1$.
You open the first envelope and it contains $n \in \{1, 2, 3, 4\}$ euros. Now you can decide to keep the money, or to take the other envelope. For what values of n is it optimal to take the other envelope for a price of n? Calculate the expected profit using the optimal strategy.
Hint: Consider illustrating this by a tree.

Exercise 3. We consider a stochastic process $X = (X_0, X_1, X_2)$ on the probability space $(\Omega, \mathfrak{P}(\Omega), P)$ with $\Omega = \{\omega_1, \dots, \omega_5\}$. The process X and the probability measure P are given by the following tree:



Determine the Doob decomposition of X. To this end, determine for each $n = 0, 1, 2$ and each node in the tree (i.e. each event) on level n the value of A_n and M_n on the respective event (The filtration is given by the tree).

Exercise 4. We consider two stochastic processes $X = (X_0, X_1, X_2)$ and $Y = (Y_0, Y_1, Y_2)$ on the probability space $(\Omega, \mathfrak{P}(\Omega), P)$ with $\Omega = \{\omega_1, \ldots, \omega_7\}$. The processes X, Y, the probability measure P and the filtration $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ are given by the following tree:



Compute

1. $X_- \bullet Y$ at each node in the tree above (recall $(X_-)_n := X_{n-1}$),
2. $[X, X]$ at each node in the tree above,
3. $[X, Y]$ at each node in the tree above.

Submission of the homework until: Thursday, 16.11.2023, 10.00 a.m. via OLAT.