Name / Matriculation number: ______________________________

Exam in **Mathematical Finance**, winter term 2016/2017, Mathematisches Seminar, CAU Kiel

Lecturer: Prof. Dr. Jan Kallsen

Exercise	1	2	3	4	5	6	$\sum$
Points							

Hints

- This exam contains **6 exercises**.
- You have **180 minutes** of time.
- You may give your answers in **German or English**.
- If not stated differently, please **give reasons for your answer**.
- If a certain type of computation is required several times in an exercise, you may **provide your computation for one example** in detail and state only the results for the remaining cases.
- Any numerical solution obtained from your calculations needs to be provided with a precision of four digits after the decimal point.
- Make sure that your handwriting is readable. Non-readable answers will be considered as wrong.

Exercise 1 (2 + 2 + 2 + 2 + 2 = 10 points)

Suppose that $S = (S^0, S^1)$ is an arbitrage-free market with time horizon $N = 10$. The process $S^0 > 0$ models a deterministic bond and S^1 models the price of a risky asset. Moreover, let $\varphi = (\varphi^0, \varphi^1)$ denote a trading strategy and denote by $\hat{V}(\varphi)$ the associated discounted price process. Do we know whether ...

(a) ... there exists a probability measure $\tilde{P}$ such that $\hat{S}^2_n := \hat{V}_0(\varphi) + \varphi \bullet \hat{S}_n$ is a $\tilde{P}$-martingale?

(b) ... the process $\hat{S}^2$ coincides with the discounted price process $\hat{V}(\varphi)$?

(c) ... the upper and lower price of a European put option on S^1 with strike 100 and maturity 10 coincide?

(d) ... the upper and lower price of an option with payoff $S^1_{10} - 100$ at time 10 coincide?

If yes, why? If no, why?

(e) Suppose that both a European call and a European put on S^1 with strike 100 and maturity 10 are traded in the market above. How are the price process of these two options related?

Please give detailed reasons to your answers!

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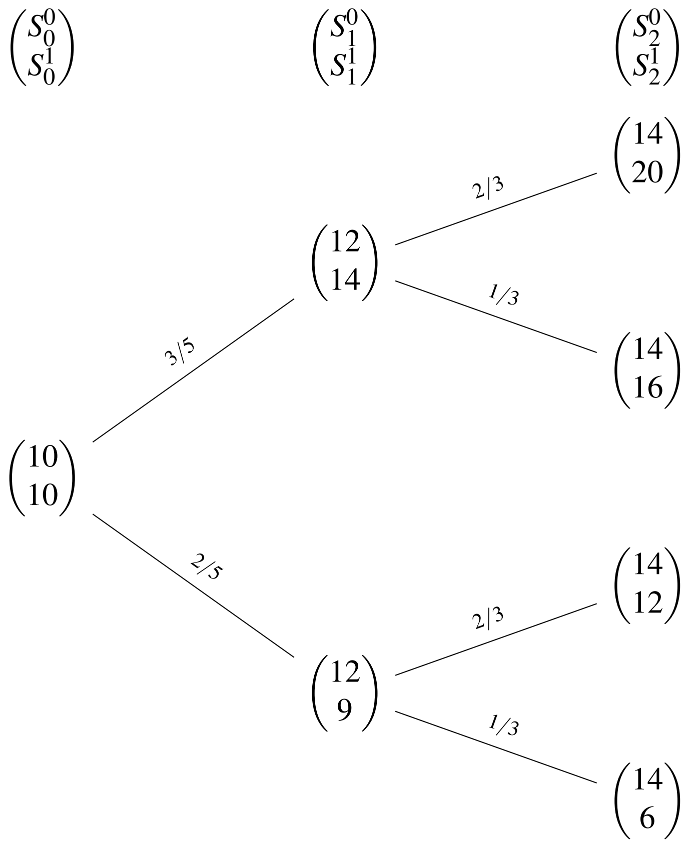
Exercise 2 (2 + 6 + 5 + 2 = 15 points)

We consider a market with time horizon $N = 2$ and price process $S = (S^0, S^1)$ that is given by the following tree. The numbers on the edges denote the P-transition probabilities.



Moreover, we consider a put option on the maximum of S^1 with strike 20 and maturity $N = 2$, i.e. an option with payoff $H := (20 - \max\{S_0^1, S_1^1, S_2^1\})^+$.

Tasks:

(a) Give reasons why there exist a unique fair price process S^2 and a perfect hedging strategy φ for the put.

(b) Determine S^2 and φ, and insert them appropriately in the tree above.

(c) Determine the Doob decomposition of the process S^1 with respect to the measure P.

(d) Is the process S^1 a (sub/super-)martingale with respect to the measure P or neither? Give reason to your answer!

Exercise 3 (4 + 4 = 8 points)

We consider the Ito-process

$$X_t = 1 + X \bullet W_t$$

where W denotes a standard Brownian motion.

Tasks:

(a) Compute the Ito process representation of $Y_t := tX_t$, i.e. write Y_t in the form

$$Y_t = Y_0 + \ldots \bullet I_t + \ldots \bullet W_t.$$

(b) Compute the Ito process representation of $Z_t := X_t \log(X_t)$, i.e. write Z_t in the form

$$Z_t = Z_0 + \ldots \bullet I_t + \ldots \bullet W_t.$$

Exercise 4 (11 + 3 = 14 points)

Let $X_0 := 0$ and suppose that X_1, X_2, X_3 are i.i.d. random variables such that

$$P(X_1 = 1) = 2/3 \quad \text{and} \quad P(X_1 = -1/2) = 1/3.$$

Moreover, denote by $\mathscr{F} := (\mathscr{F}_n)_{n\in\{0,\ldots,3\}}$ the filtration generated by the process $(X_n)_{n\in\{0,\ldots,3\}}$ and let $\mathscr{T}$ denote the set of stopping times with values in $\{0,\ldots,3\}$.

Tasks:

(a) Define $S_0 := 0$ and $S_n := \frac{1}{n}\sum_{k=1}^{n} X_k$ for $n \in \{1,\ldots,3\}$. Calculate

$$\sup_{\tau\in\mathscr{T}} \mathrm{E}\left(S_\tau\right).$$

(b) Determine the optimal stopping times τ_f, τ_s related to the latter stopping problem.

Exercise 5 (5 points)

Consider a market (S^0, S^1, S^2) with numeriare S^0 which evolves according to the following tree.

$$\begin{tikzcd}[column sep=huge]
\begin{pmatrix} S_0^0 \\ S_0^1 \\ S_0^2 \end{pmatrix} & \begin{pmatrix} S_1^0 \\ S_1^1 \\ S_1^2 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 110 \\ 120 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 50 \\ 55 \end{pmatrix} \arrow[ur, dash, "5/7"] \arrow[r, dash, "1/7"] \arrow[dr, dash, "1/7"] & \begin{pmatrix} 2 \\ 100 \\ 110 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 90 \\ 100 \end{pmatrix}
\end{tikzcd}$$

Question: Is the latter market arbitrage-free? Is it complete? Give reason to your answer!

Exercise 6 (5 + 3 = 8 points)

We consider a market (S^0, S^1, D^1) with time horizon $n = 2$ and a predictable process φ^1. The process S^0 models the price of an asset which does not pay any dividends. The process S^1 models the price of an asset which pays dividends. The cumulative dividend process associated to S^1 is given by D^1. Assume that the processes S^0, S^1, D^1 and φ^1 evolve according to the following tree.

$$\varphi^1_0 \begin{pmatrix} S^0_0 \\ S^1_0 \\ D^1_0 \end{pmatrix} \qquad \varphi^1_1 \begin{pmatrix} S^0_1 \\ S^1_1 \\ D^1_1 \end{pmatrix} \qquad \varphi^1_2 \begin{pmatrix} S^0_2 \\ S^1_2 \\ D^1_2 \end{pmatrix}$$

$$\begin{tikzcd}
 & & 15 \begin{pmatrix} 1.2 \\ 1.44 \\ 0.2 \end{pmatrix} \\
 & 10 \begin{pmatrix} 1.1 \\ 1.2 \\ 0.1 \end{pmatrix} \arrow[ur, dash] \arrow[dr, dash] & \\
 & & 15 \begin{pmatrix} 1.2 \\ 0.96 \\ 0.2 \end{pmatrix} \\
5 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \arrow[uur, dash] \arrow[ddr, dash] & & \\
 & & 5 \begin{pmatrix} 1.2 \\ 0.96 \\ 0.2 \end{pmatrix} \\
 & 10 \begin{pmatrix} 1.1 \\ 0.8 \\ 0.1 \end{pmatrix} \arrow[ur, dash] \arrow[dr, dash] & \\
 & & 5 \begin{pmatrix} 1.2 \\ 0.64 \\ 0.2 \end{pmatrix}
\end{tikzcd}$$

Task:

(a) Determine a predictable process φ^0 such that the resulting portfolio $\varphi = (\varphi^0, \varphi^1)$ is self-financing with $V_0(\varphi) = 12$.

(b) Determine the value process $V(\varphi)$ associated to the portfolio φ from task (a).