Mathematical Finance Winter term 2023/2024

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Mathematical Finance: QF

Exercises (for discussion on Monday, 18.12.2023)

Exercise 1. Consider a market with price process $S = (S^0, S^1)$ and time horizon $N = 1$ given by the following tree.

$\begin{pmatrix} S_0^0 \\ S_0^1 \end{pmatrix}$	$\begin{pmatrix} S_1^0 \\ S_1^1 \end{pmatrix}$	ω	$P(\{\omega\})$
	$\begin{pmatrix} 1 \\ 130 \end{pmatrix}$	ω_1	0.2
$\begin{pmatrix} 1 \\ 100 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 110 \end{pmatrix}$	ω_2	0.3
	$\begin{pmatrix} 1 \\ 80 \end{pmatrix}$	ω_1	0.5

1. Compute the upper and the lower price of a call option X on S^1 with maturity 1 and strike 95.
2. Determine a cheapest superhedge for the call option, i.e. find a self-financing strategy ϕ such that $V_0(\phi) = \pi_U(X)$ and $V_N(\phi) \geq X$.

Exercise 2. (*3 points*)

We consider the CRR model form the lecture with interest rate $\tilde{r} = 0.05$, the initial values $S_0^0 = S_0^1 = 100$ and time-horizon $N = 2$. Moreover, we assume that the one-period relative value increase and decrease of the stock are given by $u = 1.2$ and $d = 0.9$ which occur with probability p and $1 - p$, respectively. In this market we consider a lookback-option with payoff

$$H := \left(\max_{k=0,\dots,N} S_k^1 - 105 \right)^+$$

at time N.

1. Draw a tree containing the price process $S = (S^0, S^1)$ and the contingent claim H.
2. Give reason, why the claim H is attainable.

Exercise 3. (*5 points*)
Consider the market of Exercise 2.

1. Compute the arbitrage-free fair price process S^2 for the contingent claim H.
2. Compute the perfect hedge $\varphi = (\varphi^0, \varphi^1)$ of the lookback option.

Exercise 4. Decide for each of the following markets (represented by the trees) whether they are arbitrage free, and whether they are complete. Explain your answers, also find an arbitrage strategy if there is any. Each line represents one ω, which has positive probability.

(a)

$$\begin{tikzcd}
 & \begin{pmatrix} 0.9 \\ 120 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 100 \end{pmatrix} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \begin{pmatrix} 0.9 \\ 100 \end{pmatrix} \\
 & \begin{pmatrix} 0.9 \\ 90 \end{pmatrix}
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
 & \begin{pmatrix} 4 \\ 120 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 30 \end{pmatrix} \arrow[ur, dash] \arrow[dr, dash] & \\
 & \begin{pmatrix} 8 \\ 240 \end{pmatrix}
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
 & \begin{pmatrix} 3 \\ 27 \\ 3 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 2 \\ 10 \end{pmatrix} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \begin{pmatrix} 4 \\ 48 \\ 0 \end{pmatrix} \\
 & \begin{pmatrix} 3 \\ 27 \\ 15 \end{pmatrix}
\end{tikzcd}$$

(d)

$$\begin{tikzcd}
 & \begin{pmatrix} 1.05 \\ 3.15 \\ 25.2 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 7 \\ 56 \end{pmatrix} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \begin{pmatrix} 1.05 \\ 8.4 \\ 67.2 \end{pmatrix} \\
 & \begin{pmatrix} 1.05 \\ 6.3 \\ 50.4 \end{pmatrix}
\end{tikzcd}$$

Submission of the homework until: Thursday, 14.12.2023, 10.00 a.m. via OLAT.