Bayesian Econometrics
PC Tutorial 01

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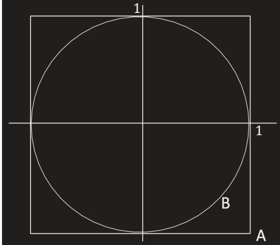
Useful MATLAB programs, codes and links

- Webpage of Gary Koop's textbook.
- Bayesian Econometric Methods textbook's page: Posterior simulation via MCMC methods, latent variable models, time series models, BVARs.
- The BEAR toolbox: user-friendly toolbox created by ECB's team. BVARs, Mixed-frequency BVARs, FAVARs, stochastic volatility.
- Codes on the monograph "Koop & Korobilis (2010). Bayesian Multivariate Time Series Methods for Empirical Macroeconomics".
- Codes on the monograph "Blake, Mumtaz (2017). Applied Bayesian econometrics for central bankers".
- Gary Koop's page: Bayesian VARs (BVARs), TVP-VARs, factor models.
- Dimitri Korobilis' page: BVARs and factor models in a data-rich environment, TVP-VARs, Dynamic Model Averaging.
- Haroon Mumtaz's page: Markov-Switching VAR, TVP-VAR.
- Joshua Chan's page: Large BVARs, stochastic volatility and GARCH models.

Exercise 1

Use Monte Carlo integration to find π, the ratio of circumference to diameter of a circle with radius 1. Compare it with $\pi = 3.1416$. Hint: show that the ratio of the area of the circle to the area of a surrounding square equals $\pi/4$. Based on that do the Monte Carlo integration (a graphical sketch will probably help). Also calculate the numerical standard error and determine the number of replications such that the absolute difference between π and $\hat{\pi}$ is less than 0.01 with probability of 95%. By repeated Monte Carlo integration check your result.



- Area of the square: A = 2 . 2 = 4
- Area of the circle: B = $\pi . r^2 = \pi . 1^2 = \pi$

- Probability of choosing a point in the circle when drawing randomly from the square:
  $$P = \frac{B}{A} = \frac{\pi}{4} \Rightarrow \pi = 4P$$

Exercise 1: simulation setup

1. Generate $s = 1, ..., S$ random draws of a point $z^{(s)} = (x^{(s)}, y^{(s)})$ inside the square A.
2. Interpret $\pi = g(z) = 4P(z)$ with $z = (x, y)$ and $P(z^{(s)}) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$
3. Monte Carlo integration setup:
$E(\pi) = E(g(z)|x, y) = \int g(z) \underbrace{p(z|x, y)}_{\sim pdf?} dz \quad \Rightarrow \quad \hat{\pi} = \hat{g}_S = \frac{1}{S} \sum_{s=1}^{N} g(z^{(s)})$

Exercise 1: NSE and number of replications

- Note that $P(z^{(s)}) \sim \mathit{Bernoulli}(\pi/4)$.
- Hence, $P(z) = \sum_{s=1}^{S} z^{(s)} \sim \mathit{Bin}(S, \pi/4)$ with:

  $E(P(z)) = \frac{S\pi}{4}$ and $\text{Var}(P(z)) = S\frac{\pi}{4}(1 - \frac{\pi}{4})$.
- Use this to compute the mean and variance of $\hat{\pi} = \frac{4P(z)}{S}$:

  $E(\hat{\pi}) = \frac{4}{S}E(P(z)) = \frac{4}{S}\frac{S\pi}{4} = \pi$

  $\text{Var}(\hat{\pi}) = \frac{16}{S^2}\text{Var}(P(z)) = \frac{16}{S^2}S\frac{\pi}{4}(1 - \frac{\pi}{4}) = \frac{\pi(4-\pi)}{S}$
- Numerical standard error = $\frac{\sigma_g}{\sqrt{S}} = \frac{\sqrt{S\,\text{Var}(\hat{\pi})}}{\sqrt{S}} = \sqrt{\frac{\pi(4-\pi)}{S}}$.
- Number of replications? We need $P(|\hat{\pi} - \pi| < 1.96\frac{\sigma_g}{\sqrt{S}} = 0.01) = 0.95$.

  $\Rightarrow 1.96\sqrt{\frac{\pi(4-\pi)}{S}} = 0.01 \quad \Rightarrow \quad S = \frac{1.96^2\pi(4-\pi)}{0.01^2} \approx 103,599.$

Exercise 2

Suppose the posterior for a parameter θ is $N(0, 1)$.

- **(a) Create a MATLAB program which carries out Monte Carlo integration to estimate posterior mean and variance of θ.**
- **(b) How many replications are necessary to ensure that with a probability of 95% percent the Monte Carlo estimates of the posterior mean and variance are equal to their true values of 0 and 1 to three decimal places?**
- **(c) To your computer program, add code which calculates numerical standard errors. Experiment with calculating posterior means, standard deviations, and numerical standard errors for various values of S. Do the numerical standard errors give a reliable indication of the accuracy of approximation in the Monte Carlo integration estimates?**

Exercise 2 (b): posterior Mean $\bar{\theta}$

- Numerical standard error:

$$\frac{\sigma_g}{\sqrt{S}} = \frac{\sqrt{S\,\text{Var}(\hat{g}_S)}}{\sqrt{S}} = \sqrt{\text{Var}(\hat{g}_S)} = \sqrt{\text{Var}\left(\frac{1}{S}\sum_{s=1}^{S}\theta^{(s)}\right)}$$

$$= \sqrt{\frac{1}{S^2}\sum_{s=1}^{S}\underbrace{\text{Var}(\theta^{(s)})}_{=1 \text{ since } \theta\sim N(0,1)}} = \sqrt{\frac{1}{S^2}\sum_{s=1}^{S}1} = \frac{1}{\sqrt{S}}$$

- Number of replications? We need $P(|\bar{\theta} - E(\theta)| \leq 1.96\,\frac{\sigma_g}{\sqrt{S}} = 0.001) = 0.95$

$\Rightarrow \frac{1.96}{\sqrt{S}} = 0.001 \quad \Rightarrow \quad S = \frac{1.96^2}{0.001^2} \approx 3,841,600$

Exercise 2 (b): posterior variance $\bar{\sigma}^2$

- Numerical standard error:

$$\frac{\sigma_g}{\sqrt{S}} = \frac{\sqrt{S\,\mathrm{Var}(\hat{g}_S)}}{\sqrt{S}} = \sqrt{\mathrm{Var}(\hat{g}_S)} = \sqrt{\mathrm{Var}\left(\frac{1}{S-1}\sum_{s=1}^{S}(\theta^{(s)}-\bar{\theta})^2\right)}$$

$$= \sqrt{\frac{1}{(S-1)^2}\mathrm{Var}\underbrace{\sum_{s=1}^{S}\left[(\theta^{(s)}-\bar{\theta})^2\right]}_{\sim\chi^2_{(S-1)}}} = \sqrt{\frac{1}{(S-1)^2}2(S-1)} = \sqrt{\frac{2}{(S-1)}}$$

- Number of replications? We need $P(|\bar{\sigma}^2 - \mathrm{Var}(\theta)| \leq 1.96\,\frac{\sigma_g}{\sqrt{S}} = 0.001) = 0.95$

$\Rightarrow 1.96\sqrt{\frac{2}{(S-1)}} = 0.001 \quad \Rightarrow \quad S = 2\,\frac{1.96^2}{0.001^2} - 1 \approx 7,683,199$

Exercise 2 (c)

- Problem: typically, when we resort to Monte Carlo integration, the posterior distribution $p(\theta|y)$ is unknown. This implies that $g(\theta)$ and $\sigma_G^2 = \text{Var}(\hat{g}_S|y)$ are also unknown.
- Way out: repeat a Monte Carlo integration R times and estimate both $\text{E}(\hat{g}_S|y)$ and $\text{Var}(\hat{g}_S|y)$. Then $\hat{g}_S, r$ is the simulation estimate of $\text{E}(\hat{g}_S|y)$ in repetition $r = 1, \ldots, R$, and the best estimate is:

$$\widehat{\hat{g}_{S,R}} = \tfrac{1}{R}\textstyle\sum_{r=1}^{R} \hat{g}_{S,r} \qquad \text{with} \qquad \tfrac{\hat{\sigma}_G}{\sqrt{S}} = \sqrt{\widehat{\text{Var}}(\hat{g}_{S,r})} = \tfrac{1}{R-1}\textstyle\sum_{r=1}^{R}(\hat{g}_{S,r} - \widehat{\hat{g}_{S,R}})^2$$

- The numerical standard error estimate for $S \times R$ draws is then given by:

$$\frac{\hat{\sigma}_G}{\sqrt{S \times R}}$$