Christian-Albrechts-Universität zu Kiel **SS 2024**
Mathematisches Seminar
Prof. Dr. Mathias Vetter
Henrik Valett, Fan Yu, Ivo Richert, Anton Schellin Sheet 04

Computational Finance

Exercises for all participants

Application of Itô calculus → (Ref Math finance) (P.35)

T-Exercise 13 (Exchange rates) (4 points)
Assume that the exchange rate $D(t)$ of the US-Dollar in Euro at time $t > 0$ follows the equation

$$dD(t) = D(t)\mu \mathrm{d}t + D(t)\sigma \mathrm{d}W(t)$$

with $D(0) > 0$ and μ, $\sigma \in \mathbb{R}$. Hence, the exchange rate of the Euro in US-Dollar at time $t > 0$ is given by $E(t) := \frac{1}{D(t)}$. Represent the process E as Itō process, i.e. in the form

$$dE(t) = \ldots \mathrm{d}t + \ldots \mathrm{d}W(t).$$

Interpret your result economically in the case $\mu = \frac{1}{2}\sigma^2$.

T-Exercise 14 (Vasiček model for interest rates) (4 points)
Let W be a standard Brownian motion and let x, κ, λ and σ real numbers. Show as in the lecture that the process X with

$$dX(t) := (\kappa - \lambda X(t))dt + \sigma dW(t)$$

and $X(0) = x$ solves the equation

$$X(t) = xe^{-\lambda t} + \frac{\kappa}{\lambda}(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)}\sigma dW(s).$$

T-Exercise 15 (4 points) → standard Brown
Let W be a standard Brownian motion. Show that the process

$$X(t) := \mathscr{E}(W)(t)\left(1 + \int_0^t \frac{1}{\mathscr{E}(W)(s)} \mathrm{d}s\right), \quad t \in \mathbb{R}_+,$$

Ref. p.27

solves the stochastic differential equation

$$\mathrm{d}X(t) = 1\mathrm{d}t + X(t)\mathrm{d}W(t), \quad X(0) = 1.$$

↓

solve this $X(t) = \mathcal{E}(W)(t)\left(1 + \int_0^t \frac{1}{\mathcal{E}(W)(s)} ds\right)$

IT - Exercise 13

Marius Leibold, Josef Feller

1 Step: Collect given information

Ref. p.31

$$dD(t) = D(t)\mu dt + D(t)\sigma dW(t) \;\hat{=}\; D(t) = D(0) + \int_0^t D(s)\mu\, ds + \int_0^t D(s)\sigma\, dW(s)$$

marks change of $D(t)$

Differential form (3.6) — Integral form (3.7)

$\Rightarrow$ Process of these form = Itô process

GOAL: Get Itô for $\boxed{E(t) = \frac{1}{D(t)} = f(D(t))}$

basically the function we apply Itô on

$$df(X(t)) = \left(f'(X(t))\mu(t) + \frac{1}{2}f''(X(t))\sigma^2(t)\right)dt + f'(X(t))\sigma(t)dW(t)$$

① ② ③ — from initial equation

2 Step Get determinants for $f(D(t))$

$$\frac{\partial f}{\partial t} = -\frac{1}{D(t)^2} \quad ① \qquad \frac{\partial^2 f}{\partial t} = \frac{2}{D(t)^3} \quad ②$$

3 Step Get quadrate variation ③

p.30

$$d[X, X]_t = \sigma^2(t)\, dt \Rightarrow (\sigma d D(t) \cdot \sigma d D(t))\, dt$$

$$= \boxed{\sigma^2 d\, D(t)^2\, dt}$$

4. Step: Plug everything into Itô formula

$$dE(t) = \underbrace{-\frac{1}{D(t)^2}\left(\mu D(t)\,dt + \sigma D(t)\,dW(t)\right)}_{1} + \underbrace{\frac{1}{2}\,\frac{2}{D(t)^3}\cdot\boxed{\left(\sigma^2 D(t)^2\,dt\right)}}_{2}$$

$$\underbrace{-\frac{1}{D(t)^2}\cdot\sigma(t)\,dW(t)\cdot D(t)}_{3}\ ?$$

we get

1 $\Rightarrow$ $$-\frac{\mu\,\cancel{D(t)}}{D(t)^{\cancel{2}}}\,dt - \sigma\frac{\cancel{D(t)}}{D(t)^{\cancel{2}}}\,dW(t) = \boxed{-\frac{\mu}{D(t)}\,dt - \frac{\sigma}{D(t)}\,dW(t)}$$

2 $\Rightarrow$ $$\frac{\cancel{1}}{\cancel{2}}\,\frac{\cancel{2}}{D(t)^3}\,\sigma^2\cancel{D(t)^2}\,dt = \boxed{\frac{\sigma^2}{D(t)}\,dt}$$

supposed to become random part / Martingale

3 $$-\frac{1}{D(t)^{\cancel{2}}}\,\sigma(t)\,dW(t)\,\cancel{\cdot D(t)}\ ?$$

5. Step: Put everything together

$$dE(t) = -\frac{\mu}{D(t)}\boxed{dt} - \frac{\sigma}{D(t)}\,dW(t) + \frac{\sigma^2}{D(t)}\boxed{dt} - \frac{1}{D(t)^2}\,\sigma(t)\,dW(t)$$

$$= \frac{\sigma^2-\mu}{D(t)}\boxed{dt} - \frac{\sigma}{D(t)}\,dW(t) - \frac{1}{D(t)^2}\,\sigma(t)\,dW(t)$$

in terms of $E(t) = \frac{1}{D(t)}$

$$dE(t) = (\sigma^2-\mu)\,E(t)\,dt - \sigma E(t)\,dW(t) - E(t)^2\sigma(t)\,dW(t)$$

6. Step: Interpretation: with $\mu = \frac{1}{2}\sigma^2$ the deterministic part of our Itô formula would cancel out. This means on average on the long run we don't evolve. Only short term (randomness)