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Christian-Albrechts-Universität zu Kiel **SS 2024**
Mathematisches Seminar
Prof. Dr. Mathias Vetter
Henrik Valett, Fan Yu, Ivo Richert, Anton Schellin Sheet 07

Computational Finance

Exercises for all participants

T-Exercise 25QF (for QF only) (4 points)

A *Poisson process* N with intensity $\lambda \in \mathbb{R}_+$ is a process with right-continuous, increasing paths such that for all $s,t \in \mathbb{R}_+$ the increments $N(t+s) - N(t)$ are independent of $N(t)$ and such that $N(t)$ follows a Poisson distribution with parameter λt. For $\rho, \mu \in \mathbb{R}$ and a Poisson process N with intensity $\lambda \in \mathbb{R}_+$, compute the characteristic function of $X(t) := \rho N(t) - \mu t$.

T-Exercise 25Math (Merton's jump-diffusion model) (4 points)

In Merton's jump-diffusion model the logarithm $X = \log(S)$ of the stock price is of the form

$$X(t) = X(0) + \mu t + \sigma W(t) + \sum_{j=1}^{N(t)} Y_j,$$

where W is a standard Brownian motion, $\mu \in \mathbb{R}$, $\sigma > 0$, $N(t)$ is a Poisson random variable with parameter λt for $\lambda > 0$ and $Y_1, Y_2, \ldots$ are normally distributed with mean $m \in \mathbb{R}$ and variance $s^2 > 0$. Moreover $W, N(t), Y_1, Y_2, \ldots$ are all independent. Calculate the characteristic function

$$\chi_t(u) := E[e^{iuX(t)}].$$

Hints:

(a) *The characteristic function of a normally distributed random variable Y with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}_+$ is given by* $E[\exp(iuY)] = \exp(iu\mu - u^2\sigma^2/2)$ *for $u \in \mathbb{R}$.*

(b) *It holds that* $\exp(\sum_{j=1}^{N_t} Y_j) = \sum_{m=0}^{\infty} 1_{\{N(t)=m\}} \exp(\sum_{j=1}^{m} Y_j)$ *for any $t \in \mathbb{R}_+$.*

Ex25

T-Exercise 25QF (for QF only) (4 points)

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Idea: We want to describe the law of $X(t)$ without knowing its exact probability density function.

1Step: Define characteristic function (ref. p.50)

$$\chi(u) = E\left[e^{iuX(t)}\right] \quad \text{(general definition)}$$

$$= E\left[e^{iu\,\rho N(t) - \mu t}\right] \quad \text{(Plug in } X(t))$$

2Step: Compute function

$$X(u) = E\left[e^{iu\overset{\text{stochastic}}{\boxed{\rho N(t)}}}\right] \cdot \overbrace{E\left[e^{-iu\mu t}\right]}^{\text{solely deterministic}}$$

$$= \underset{①}{E\left[e^{iu\rho N(t)}\right]} \cdot^{*} \underset{②}{e^{-iu\mu t}}$$

3Step: Apply properties of poisson on ① (from MGF - Appendix)

$$E\left[e^{iu\rho N(t)}\right] = e^{t\lambda\left(e^{iu\rho}-1\right)}$$

$E[e^{tx}] = Mgf \sim poish$

$= e^{t\lambda(x-1)}$

result combining ① and ②

$$\boxed{\varphi_t(u) = e^{t\lambda\left(e^{iu\mu}-1\right)} \cdot e^{-iu\mu t}}$$