T-Exercise 03 (4 Points)
In the course we fixed the relation $u = \frac{1}{d}$ in the specification of the binomial model that is used as an approximation to the Black-Scholes model. For even $M \in \mathbb{N}$, this implies $S(0)u^{\frac{M}{2}}d^{\frac{M}{2}} = S(0)$. Replace the condition (1.3), i.e. $u = \frac{1}{d}$, by

(a) $q = 0.5$,

(b) $S(0)u^{\frac{M}{2}}d^{\frac{M}{2}} = S(0)e^{rT}$,

(c) $S(0)u^{\frac{M}{2}}d^{\frac{M}{2}} = K$.

and compute the value of the parameters u, d and q for each case (*maths students*) resp. only for (a) *(QF students)*. Discuss the potential benefit of the alternative conditions and in what scenario they may be useful.

T-Exercise 04 (Convergence of the binomial model to the Black-Scholes model) (for math only)
For $M \in \mathbb{N}$, denote by $\left(S_t^M\right)_{t\in\{t_0,\ldots,t_M\}}$ the stock price process in a binomial model with M timesteps that approximates the Black-Scholes model with parameters $r > 0$, $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and time horizon $T > 0$.

(a) Show that $\log\left(S_{t_M}^M\right) \xrightarrow[M\to\infty]{} Z$ in law, where the random variable Z is normally distributed with mean $\log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T$ and variance $\sigma^2 T$.

(b) Conclude that the initial prices of European put and call options with maturity T and strike $K > 0$ in the binomial model with M timesteps approximating the risk-neutral dynamics of the Black-Scholes model converge to the corresponding Black-Scholes prices as $M \to \infty$.

Without proof, you can use

Slutsky's Theorem: Let $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ be sequences of random variables such that $A_n \xrightarrow[n\to\infty]{} A$, $B_n \xrightarrow[n\to\infty]{} B$ in probability and $X_n \xrightarrow[n\to\infty]{} X$ in law. Then $A_nX_n + B_n \xrightarrow[n\to\infty]{} AX + B$ in law.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Thu, 02.05.2024, 12:00

T-Exercise 03:

(3a) Given we have to replace $u = \frac{1}{d}$ by $\boxed{q = 0{,}5}$

From theory: No arbitrage must hold,

$$d < \exp(r\Delta t) < u$$

$\Rightarrow$ (1.1) $$q = \frac{\exp(r\Delta t) - d}{u - d} = \boxed{0{,}5}$$

Hence:

$$= \exp(r\Delta t) - d = 0{,}5(u-d)$$

$$= 2\exp(r\Delta t) - 2d = u - d$$

$$= \boxed{2\exp(r\Delta t) = u + d}$$

Or if we solve for u and d:

1Step! Solve for d

$$0{,}5(u-d) = \exp(r\Delta t) - d$$

$$0{,}5u - 0{,}5d = \exp(r\Delta t) - d$$

$$0{,}5u - \exp(r\Delta t) = 0{,}5d \quad / : \tfrac{1}{2}$$

$$\boxed{d = u - 2\exp(r\Delta t)}$$

Step 2: Solve for u:

$$0{,}5 = \frac{\exp(r\Delta t) - (u - 2\exp(r\Delta t))}{u - (u - 2\exp(r\Delta t))}$$

$$0{,}5 = \frac{3\exp(r\Delta t) - u}{2\exp(r\Delta t)} \Rightarrow \boxed{u = \exp(r\Delta t)}$$

3Step: Solving d by plugging in u

$$d = u - 2\exp(r\Delta t)$$
$$= \exp(r\Delta t) - 2\exp(r\Delta t)$$
$$\boxed{d = -\exp(r\Delta t)}$$

Conclusion:

What we got? Basically a symmetric market where it is equally likely for the risky asset to go up or down.

$\Rightarrow$ $u = \exp(r\Delta t)$ so $d = -\exp(r\Delta t)$

Benefits? It is quick intuitive and easy to understand because of the symmetric construction.

Downsides? By assuming a market in some sort of equilibrium we might lose ability to accurately

model the true dynamics of our risky asset.
So it's a quiet hard assumption and most likely not coherent with true (financial) markets.

Application? In situations where we are likely to be in some sort of equilibrium it seems reasonable to use this assumption

or on very short term periods where factors like e.g. "trends" don't play a major role.