Christian-Albrechts-Universität zu Kiel **SS 2024**
Mathematisches Seminar
Prof. Dr. Mathias Vetter
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Computational Finance

Exercises for all participants

T-Exercise 17 (4 points)
Let W_1, W_2 be independent standard Brownian motions. Consider a market with three assets S_0, S_1, S_2, which follow the equations

$$\begin{aligned} S_0(t) &= 1, \\ \mathrm{d}S_1(t) &= S_1(t)\left(4\mathrm{d}t + \mathrm{d}W_1(t) - \mathrm{d}W_2(t)\right), \\ \mathrm{d}S_2(t) &= S_2(t)\left(1\mathrm{d}t - \mathrm{d}W_1(t) + \mathrm{d}W_2(t)\right). \end{aligned}$$

Construct an arbitrage in this market.

T-Exercise 18 (Digital option in the Black-Scholes model) (4 points)
A *digital call option* with maturity $T > 0$ and strike $K > 0$ is a European option with payoff

$$V(T) = 1_{\{S(T) \geq K\}}.$$

Find a formula for the initial price of a digital call option in the Black-Scholes model, and compute the perfect hedging strategy.

Hint: To find the formula for the price, it is useful to work with the integral representation and not with the Black-Scholes PDE.

T-Exercise 19 (4 points)
For $\mu \in R$ and $\sigma, r > 0$ we consider the Black-Scholes market with bond B and stock price process S which evolve according to

$$\begin{aligned} dB_t &= rB_t dt, & B_0 &= 1, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t, & S_0 &> 0. \end{aligned}$$

(a) Calculate the Itô process representation of the cubic stock process $X_t := S_t^3$ and the associated quadratic variation process $[X,X]_t$.

(b) Consider a self-financing portfolio $\varphi = (\varphi_t^0, \varphi_t^1)_{t\geq 0}$ with initial value $V_0(\varphi) = 1$ that always invests half of the wealth into the stock, i.e. $\varphi_t^1 = \frac{V_t(\varphi)}{2S_t}$. Show that the value process $V_t(\varphi)$ is a geometric Brownian motion.

T-Exercise 20 (Hedging error in the BS-model) (for math only) (4 points)
Consider a stock with risk-neutral dynamics

$$\begin{aligned} B(t) &= e^{rt}, \\ S(t) &= S(0)\exp\left((r - \tfrac{\sigma^2}{2})t + \sigma W(t)\right). \end{aligned}$$

Denote by $V(t) = v(t, S(t))$ the Black-Scholes price of a European call if the volatility equals $\tilde{\sigma}$ instead of σ, i.e. with

$$\begin{aligned} v(t,x) = x\Phi &\left(\frac{\log\frac{x}{K} + r(T-t) + \frac{\tilde{\sigma}^2}{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}}\right) \\ &- Ke^{-r(T-t)}\Phi\left(\frac{\log\frac{x}{K} + r(T-t) - \frac{\tilde{\sigma}^2}{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}}\right). \end{aligned}$$

Suppose that the bank uses the incorrect volatility estimate $\tilde{\sigma}$. It sells a call option for the wrong price $V(0)$, and tries to hedge it with a self-financing portfolio $\varphi = (\varphi_0, \varphi_1)$ containing

$$\varphi_1(t) = \partial_2 v(t, S(t))$$

shares of stock. Determine the Itô process representation of the *observed hedging error* $\varepsilon(t) := V(t) - (V(0) + \int_0^t \varphi_0(s)dB(s) + \int_0^t \varphi_1(s)dS(s))$. What do you observe?

Hint: The computation of φ_0 can be avoided by working with discounted prices.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Thu, 06.06.2024, 12:00