Christian-Albrechts-Universität zu Kiel **SS 2024**
Mathematisches Seminar
Prof. Dr. Mathias Vetter
Henrik Valett, Fan Yu, Ivo Richert, Anton Schellin Sheet 02

Computational Finance

Exercises for all participants

T-Exercise 05 (4 points)
We want to price an American put option with strike price $K = 1.2$ and time to maturity being three years. For this purpose we want to utilize a CRR model with $M = 3$ equally spaced time periods, $S(0) = 1$, $\sigma^2 = 0.3$ and an annual interest rate of 5%.

a) Draw and calculate the corresponding CRR model by hand (of course you can still use a calculator) and write beneath each point the corresponding price of the option (please round on four position after the decimal point after each calculation).

b) Calculate the replicating portfolio $\varphi = (\varphi_0, \varphi_1)$ for all time periods.

C-Exercise 06 (Options in the CRR model) (4 points)

(a) Write a Python function

```
V_0 = CRR_AmEuPut (S_0, r, sigma, T, M, K, EU)
```

that computes and returns an approximation to the price of a European or an American put option with strike $K > 0$ and maturity $T > 0$ in the CRR model with initial stock price $S(0) > 0$, interest rate $r > 0$ and volatility $\sigma > 0$. The parameter "EU" is 1 if the price of an European put shall be computed or is 0 in the American case. Use the binomial method as presented in the course with $M \in \mathbb{N}$ time steps.

(b) As $M \to \infty$ we would expect convergence of the price in the binomial model towards the price in the Black-Scholes model. To show this implement the BS-Formula for European put options as a Python function:

```
V_0 = BlackScholes_EuPut (t, S_t, r, sigma, T, K)
```

(c) Test your algorithm with

$$S(0) = 100,\ r = 0.05,\ \sigma = 0.3,\ T = 1,\ M = 10,\ldots,500,\ K = 120.$$

- Plot the price of a European put option (EU = 1) in the binomial model in dependence on the number of steps M.
- Plot the fair price in the BS-model into the same plot using the same parameters.
- Print the price of an American put option (EU = 0) with the same $S(0), r, \sigma, T, K$ as above and $M = 500$ steps in the console.

Useful Python commands: `numpy.maximum`

T-Exercise 05:

Given: $K = 1{,}2$ $S(0) = 1$ $r = 0{,}05$

$M = 3$ $\sigma^2 = 0{,}3$

Group: QF2

Thi Ha Giang Le

Maris Leibold

Josef Fella

a)

1 Step: Calculate parameters

$$\Delta t = T/M = 3/3 = \boxed{1}$$

$$(1.6) \quad \beta = \frac{1}{2}\left(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right)$$

$$= \frac{1}{2}\left(e^{-0{,}05} + e^{(0{,}05+0{,}3)}\right) \approx \boxed{1{,}1851}$$

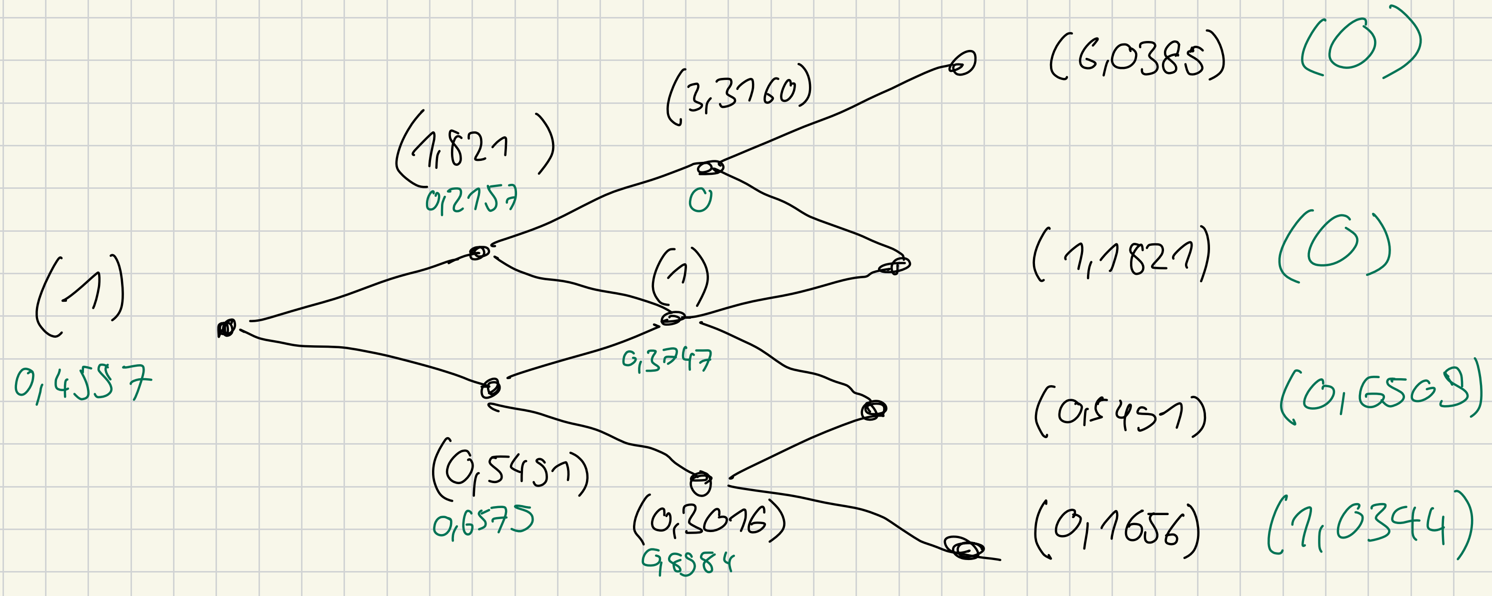
$$(1.4) \quad u = \beta + \sqrt{\beta^2 - 1} = 1{,}1851 + \sqrt{1{,}1851^2 - 1} \approx \boxed{1{,}821}$$

$$(1.5) \quad d = \frac{1}{u} = \frac{1}{1{,}821} \approx \boxed{0{,}5491}$$

$$(1.7) \quad q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0{,}05} - 0{,}5491}{1{,}821 - 0{,}5491} \approx \boxed{0{,}3948}$$

2 Step: Calculate S and claim X at T

$S_0\ V_0$ $S_1\ V_1$ $S_2\ V_2$ S_3 $V_3 = (K - S_3)^+$



3 Step Calculate the Value process of claim X

$\boxed{t_3}$

$$V_3 = X = \begin{cases} 0 & \omega \in \omega_1 \\ 0 & \vdots \\ 0{,}6509 & \vdots \\ 1{,}0344 & \omega \in \omega_4 \end{cases}$$

$\boxed{t_2}$ ~~By (1.15) we~~ get:

$$V_2 = \max\left\{ \max\left[K - S_{ji}\right]^+ ,\ \exp(-r\Delta t)\left(q V_{j+1,i+1} + (1-q) V_{j,i+1}\right)\right\}$$

① $$= \max\left\{ \max\{1{,}2 - 3{,}3..\}^+ ,\ \exp(-0{,}05)^{x} \cdot \left(0{,}3348 \cdot 0 + (1 - 0{,}3348)\cdot 0\right)\right\}$$

$$= \boxed{0}$$

② $$= \max\left\{ \underbrace{\max\{1{,}2 - 1\}^+}_{} ,\ \underbrace{\exp(-0{,}05)^{x} \cdot \left((1-0{,}3348) \cdot 0{,}6509\right)}_{}\right\}$$

$$= \max\{ 0{,}2 ,\ 0{,}3747 \} = \boxed{0{,}3747}$$

② $$= \max\left\{ \underbrace{\max\{1{,}2 - 0{,}3016\}^+}_{} ,\ \underbrace{\exp(-0{,}05)^{x} \cdot \left(0{,}3348 \cdot 0{,}6509 + (1-0{,}3348)\cdot 1{,}0344\right)}_{}\right\}$$

$$= \max\{ 0{,}8984 ,\ 0{,}8355 \} = \boxed{0{,}8984}$$

$\boxed{t_1}$

$$V_1 = \begin{cases} \max\left\{ \overset{0{,}0179}{\max\{1{,}2 - 1{,}1821\}^+} ;\ \overset{0{,}2157}{\exp(-0{,}05) \cdot \left((1-0{,}3348)\ 0{,}3747\right)}\right\} = \boxed{0{,}2157} \\ \max\left\{ \overset{0{,}6509}{\max\{1{,}2 - 0{,}5491\}^+} ;\ \overset{0{,}6579}{\exp(-0{,}05) \cdot \left(0{,}3348 \cdot 0{,}3747 + (1-0{,}3348)\cdot 0{,}8984\right)}\right\} = \boxed{0{,}6579} \end{cases}$$

$\boxed{t_0}$ 0,4557

$$V_1 = \max\{0,\ \exp(-0{,}05)\cdot(0{,}3548\cdot 0{,}2157 + (1-0{,}3548)\cdot 0{,}6575\}$$

$$= \boxed{0{,}4557}$$

5) Build replicating strategy:

and hence

$$\varphi_1(t_i) = \frac{V(S(t_{i-1})u, t_i) - V(S(t_{i-1})d, t_i)}{S(t_{i-1})(u-d)} \tag{1.12}$$

by taking differences. From (1.11) we get

$$\varphi_0(t_i) = \frac{V(S(t_{i-1}), t_{i-1}) - \varphi_1(t_i)S(t_{i-1})}{B(t_{i-1})}. \tag{1.13}$$

$\boxed{t_3}$:

$\boxed{\text{upper branch}}$ $\Rightarrow$ $\varphi_1, \varphi_0 = 0$ since no value here

$\boxed{\text{middle branch:}}$

$$\varphi_1(t_3) = \frac{0 - 0{,}6509}{(u-d)} \approx \boxed{-0{,}5118}$$

$$\varphi_0(t_3) = \frac{0{,}3747 - (-0{,}5118)\cdot 1}{\exp(r\cdot 2)} \approx \boxed{0{,}8027}$$

$\boxed{\text{lower branch}}$

$$\varphi_1(t_3) = \frac{0{,}6509 - 1{,}0344}{(u-d)} = \boxed{-0{,}3015}$$

$$\varphi_0(t_3) = \frac{0{,}8964 - (-0{,}3015\cdot 0{,}3016)}{\exp(0{,}05\cdot 2)} = \boxed{0{,}8972}$$

$\boxed{t_2}$:

upper branch:

$$\varphi_1(t_2) = \frac{0 - 0{,}3747}{1{,}821 \cdot (u-d)} = \boxed{-0{,}1618}$$

$$\varphi_0(t_2) = \frac{0{,}2157 - (-0{,}1618 \cdot 1{,}821)}{\exp(0{,}05)} = \boxed{0{,}4854}$$

lower branch:

$$\varphi_1(t_2) = \frac{0{,}3747 - 0{,}8984}{0{,}5481 \cdot (u-1)} = \boxed{-0{,}7493}$$

$$\varphi_0(t_2) = \frac{0{,}6579 - (-0{,}7493 \cdot 0{,}5491)}{\exp(0{,}05)} = \boxed{1{,}0175}$$

$\boxed{t_1}$:

$$\varphi_1(t_1) = \frac{0{,}2157 - 0{,}6579}{(u-d)} = \boxed{-0{,}3477}$$

$$\varphi_0(t_1) = 0{,}4597 - (-0{,}3477) = \boxed{0{,}8074}$$

$\boxed{t_0}$ By construction:

$$\varphi_1(t_1) = \varphi_1(t_0)$$

$$\varphi_0(t_1) = \varphi_0(t_0)$$