T19

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given:
$$dB_t = rB_t\,dt \qquad B_0 = 1$$
$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t \qquad S_0 > 0$$

(a) Calculate the Itô process representation of the cubic stock process $X_t := S_t^3$ and the associated quadratic variation process $[X,X]_t$.

function $\quad f(X_t) = S_t^3$

1Step Get derivatives for $f(X_t)$

$$f'(X_t) = 3S_t^2 \qquad f''(X_t) = 6S_t$$

no ∂t since function not depend on it

2Step Apply Itô's Lemma

$$d(f(X_t)) = f'(X_t)\cdot dX_t + \frac{1}{2}f''(X_t)\,d[X,X](t) \qquad \text{(Tutorial)}$$

$$df(X(t)) = \left(f'(X(t))\mu(t) + \frac{1}{2}f''(X(t))\sigma^2(t)\right)dt + f'(X(t))\sigma(t)dW(t) \qquad \boxed{p.31}$$

we get: substitute terms

$$dS_t^3 = 3S_t^2\mu(t) + 3S_t\sigma^2(t)dt + 3S_t^2\sigma(t)dW(t)$$
$$= (\mu(t) + \sigma(t)dW(t))3S_t^2 + 3S_t\sigma^2(t)dt$$

3Step Quadratic Variation

$$d[X,X]_t = X_t^2\sigma^2 dt$$