FEM Simulation of 3D Deformable Solids

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Outline

- Elasticity in 3D
- Discretization
- Constitutive models of materials
  - Linear elasticity
  - Non-linear elasticity
    - Corotated linear elasticity
    - StVK, Neohookean, etc.
- Modal analysis and model reduction

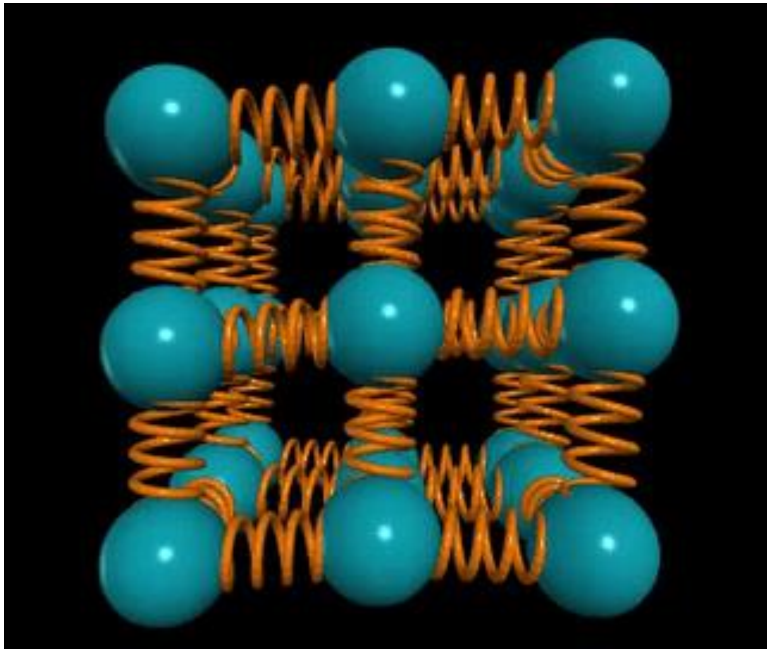
https://viterbi-web.usc.edu/~jbarbic/femdefo/

Eftychios Sifakis and Jernej Barbic. 2012. ***FEM simulation of 3D deformable solids: a practitioner's guide to theory, discretization and model reduction***. In *ACM SIGGRAPH 2012 Courses* (SIGGRAPH '12),

https://viterbi-web.usc.edu/~jbarbic/vega/ Vega FEM

Mass Spring Systems for Solids

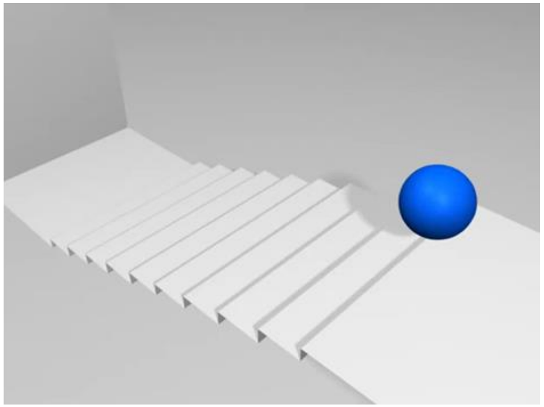
- Simple and faster
- Hard to simulate real materials



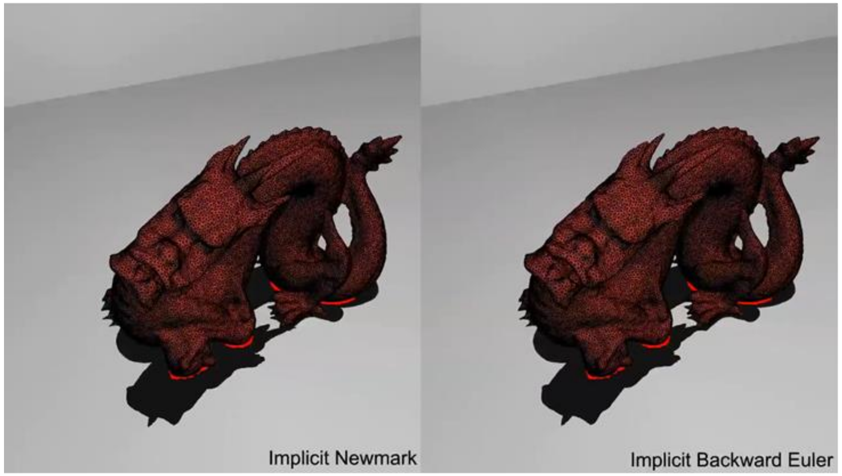
Deformable Solids

Tan, Jie, Greg Turk, and C. Karen Liu. "Soft body locomotion." *ACM Transactions on Graphics (TOG)* 31.4 (2012): 1-11.

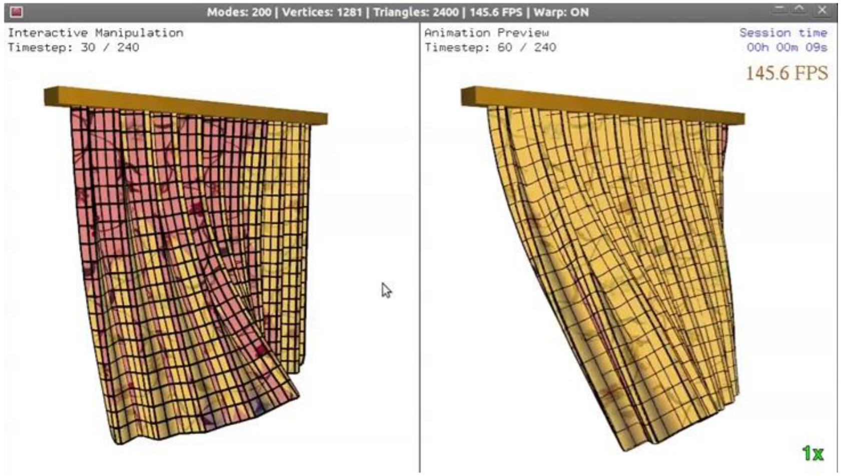
Barbič, Jernej, and Yili Zhao. "Real-time large-deformation substructuring." *ACM transactions on graphics (TOG)* 30.4 (2011): 1-8.



Coros, Stelian, et al. "Deformable objects alive!." *ACM Transactions on Graphics (TOG)* 31.4 (2012): 1-9.

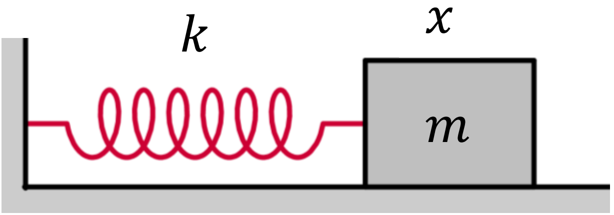


Sin, Fun Shing, Daniel Schroeder, and Jernej Barbič. "Vega: non-linear FEM deformable object simulator." *Computer Graphics Forum*. Vol. 32. No. 1.



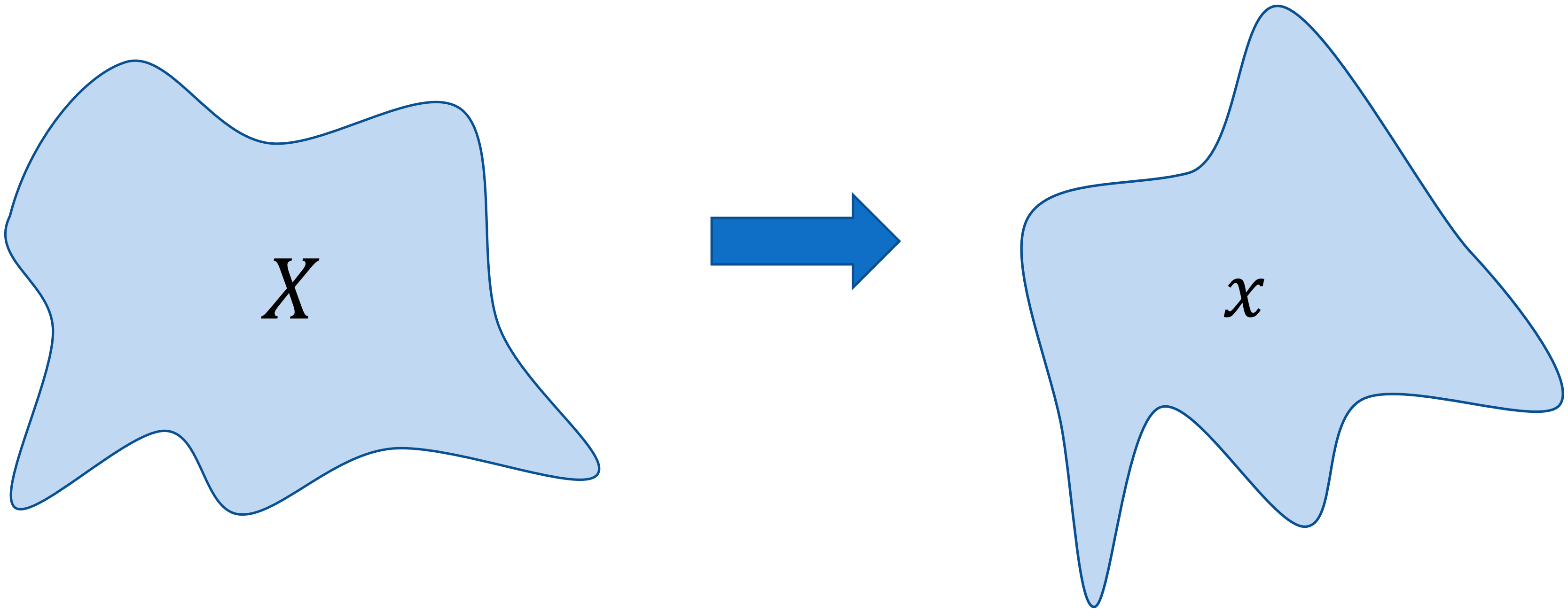
Barbič, Jernej, Funshing Sin, and Eitan Grinspun. "Interactive editing of deformable simulations." *ACM Transactions on Graphics (TOG)* 31.4 (2012): 1-8.

Hooke's Law



$$f = -k(x - x_0)$$

Displacement Map

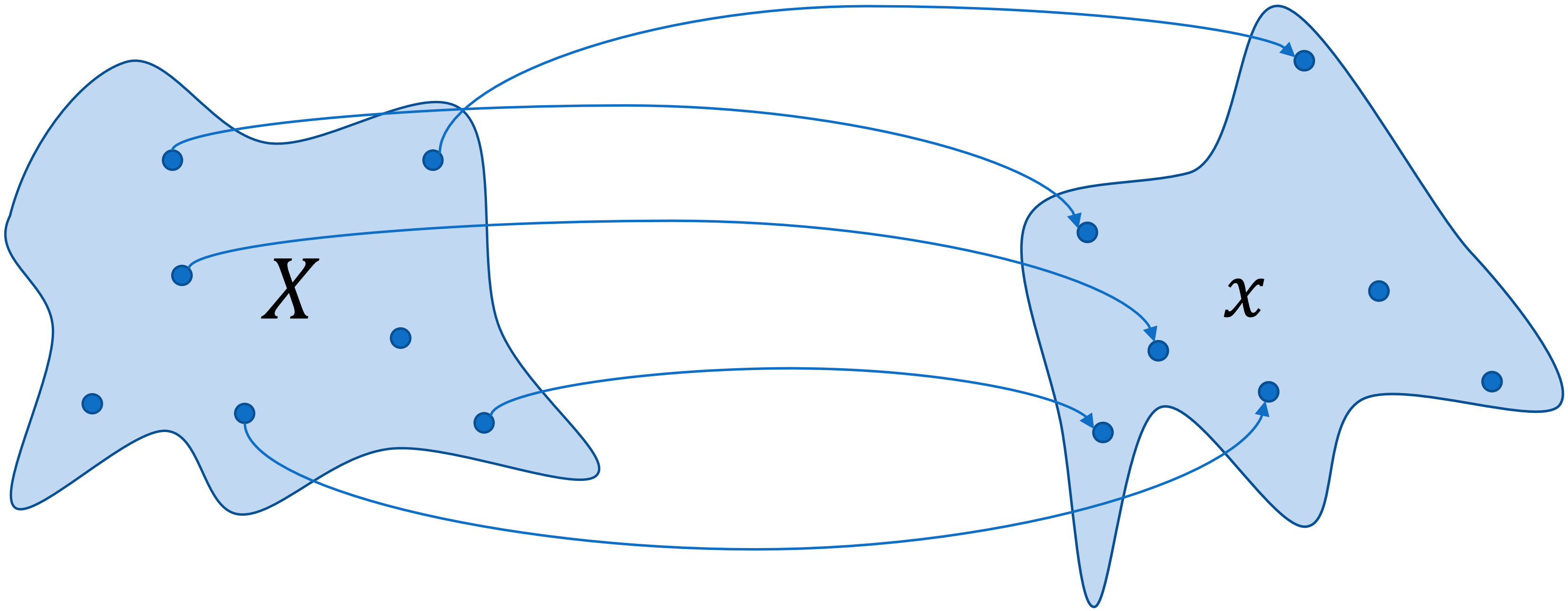


Reference Configuration/
Material Space

Deformed Configuration

Displacement Map

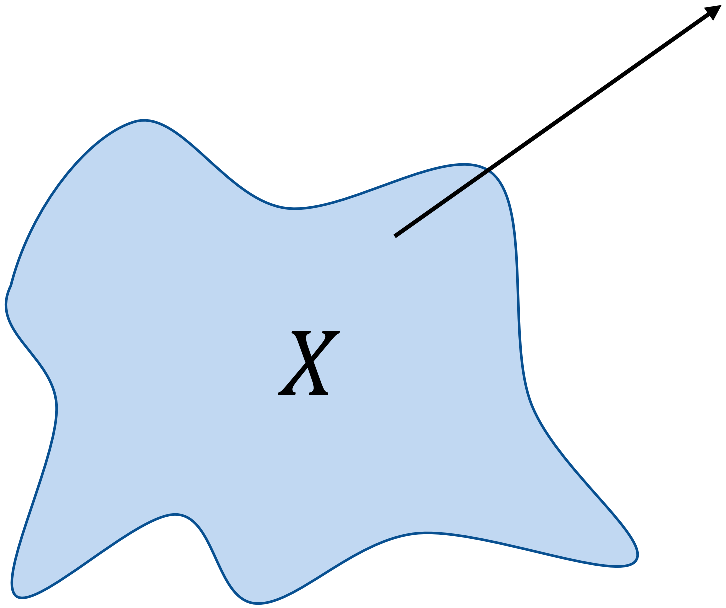
$$x = \varphi(X)$$



Reference Configuration/
Material Space

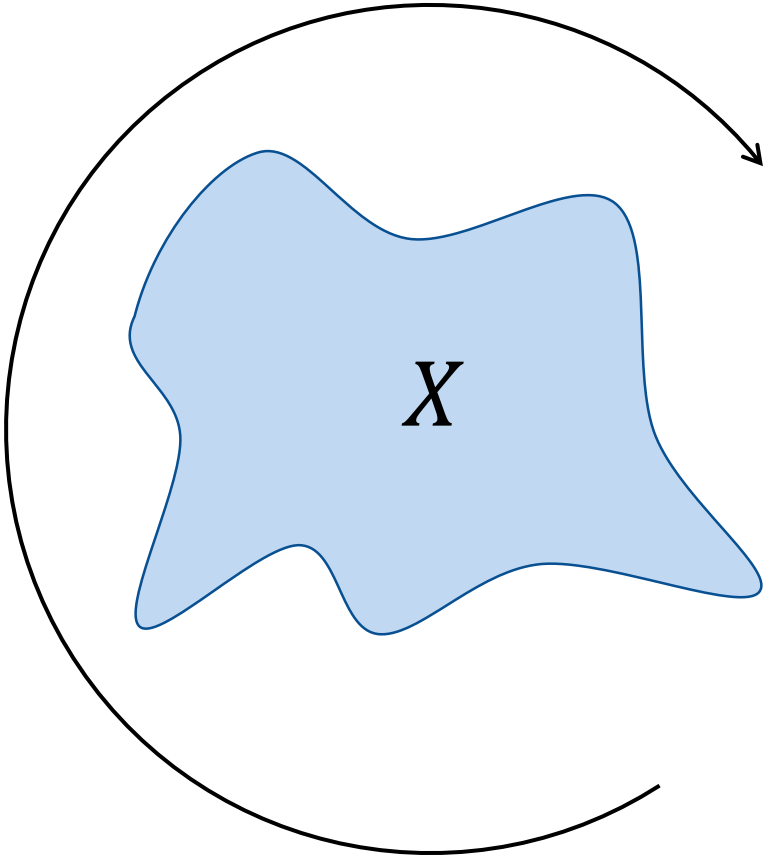
Deformed Configuration

Displacement Map



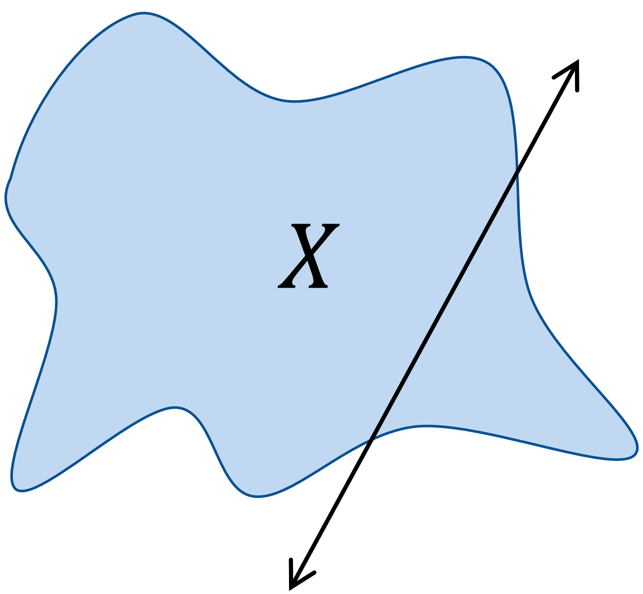
$$x = \varphi(X) = X + t$$

Displacement Map



$$x = \varphi(X) = RX$$

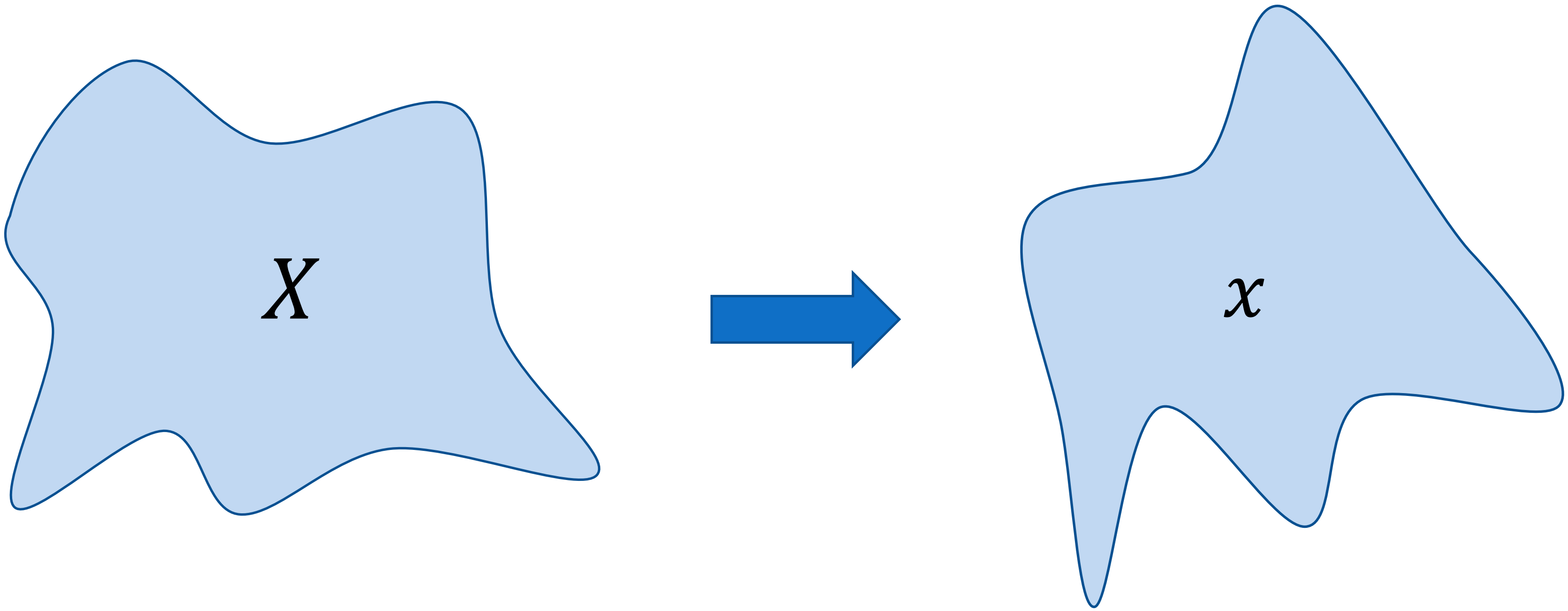
Displacement Map



$$x = \varphi(X) = SX$$

Deformation Gradient

$$x = \varphi(X) \qquad F = \frac{\partial \varphi(X)}{\partial X}$$



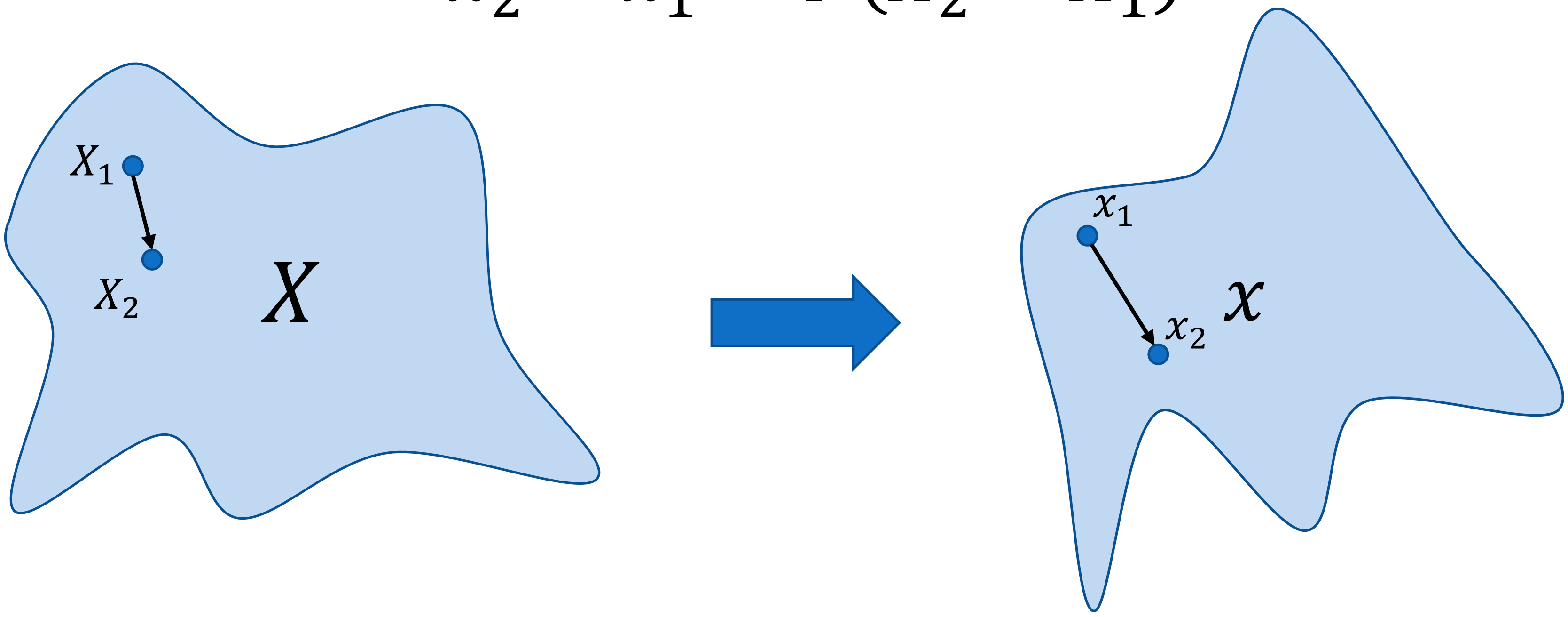
Reference Configuration/
Material Space

Deformed Configuration

Deformation Gradient

$$x = \varphi(X) \qquad F = \frac{\partial \varphi(X)}{\partial X}$$

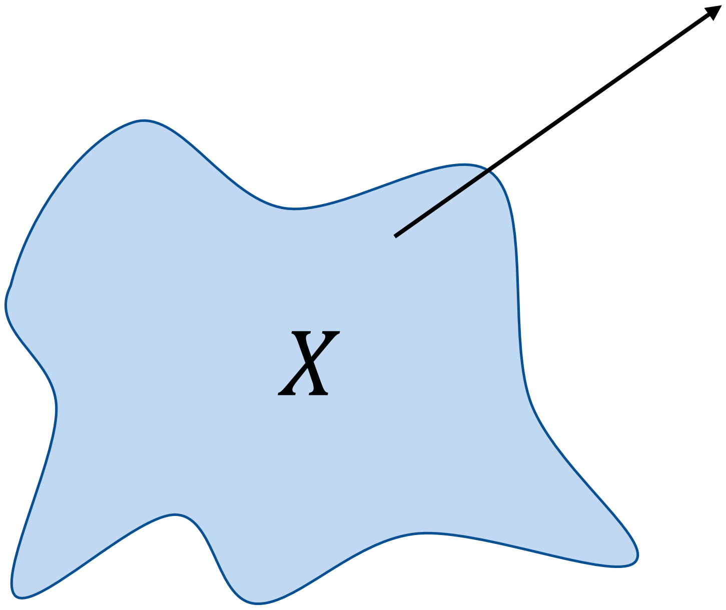
$$x_2 - x_1 \approx F(X_2 - X_1)$$



Reference Configuration/
Material Space

Deformed Configuration

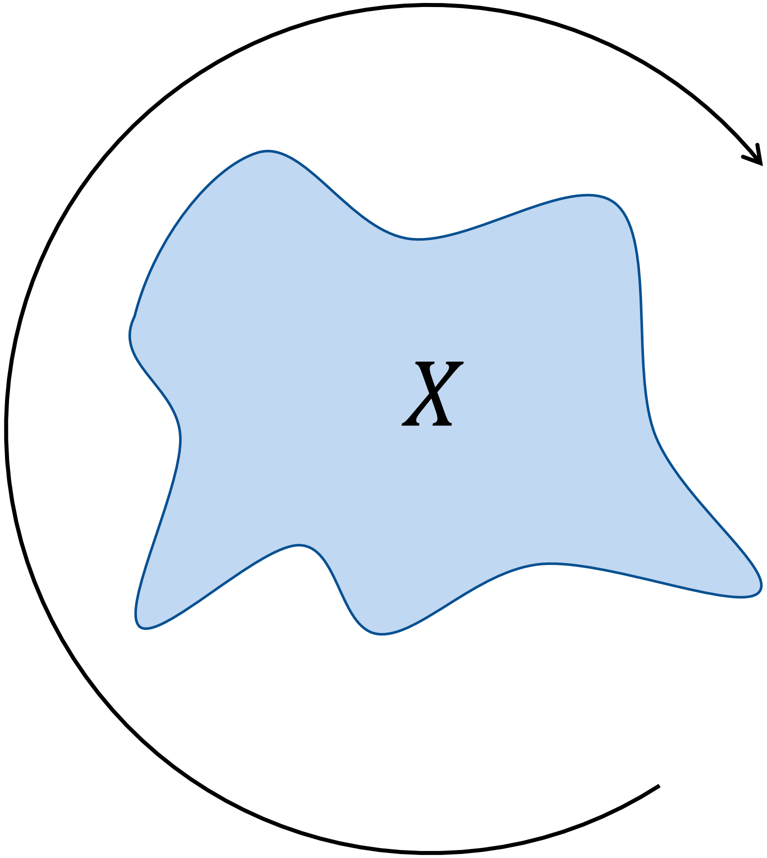
Displacement Map



$$x = \varphi(X) = X + t$$

$$F = \mathrm{I}$$

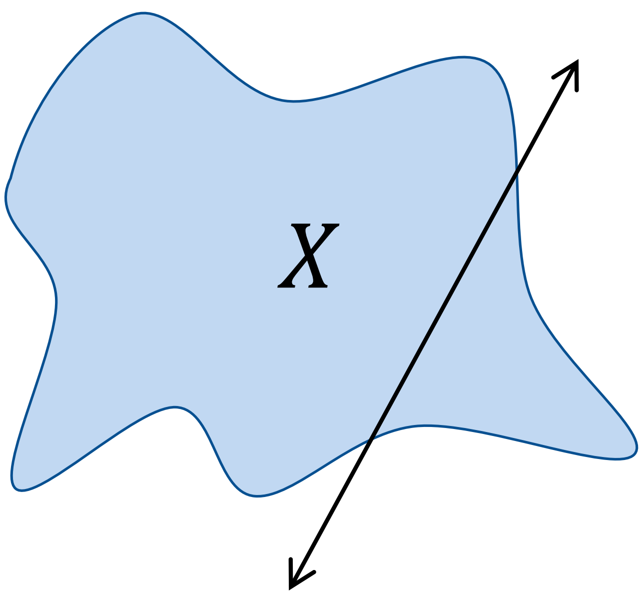
Displacement Map



$$x = \varphi(X) = RX$$

$$F = R$$

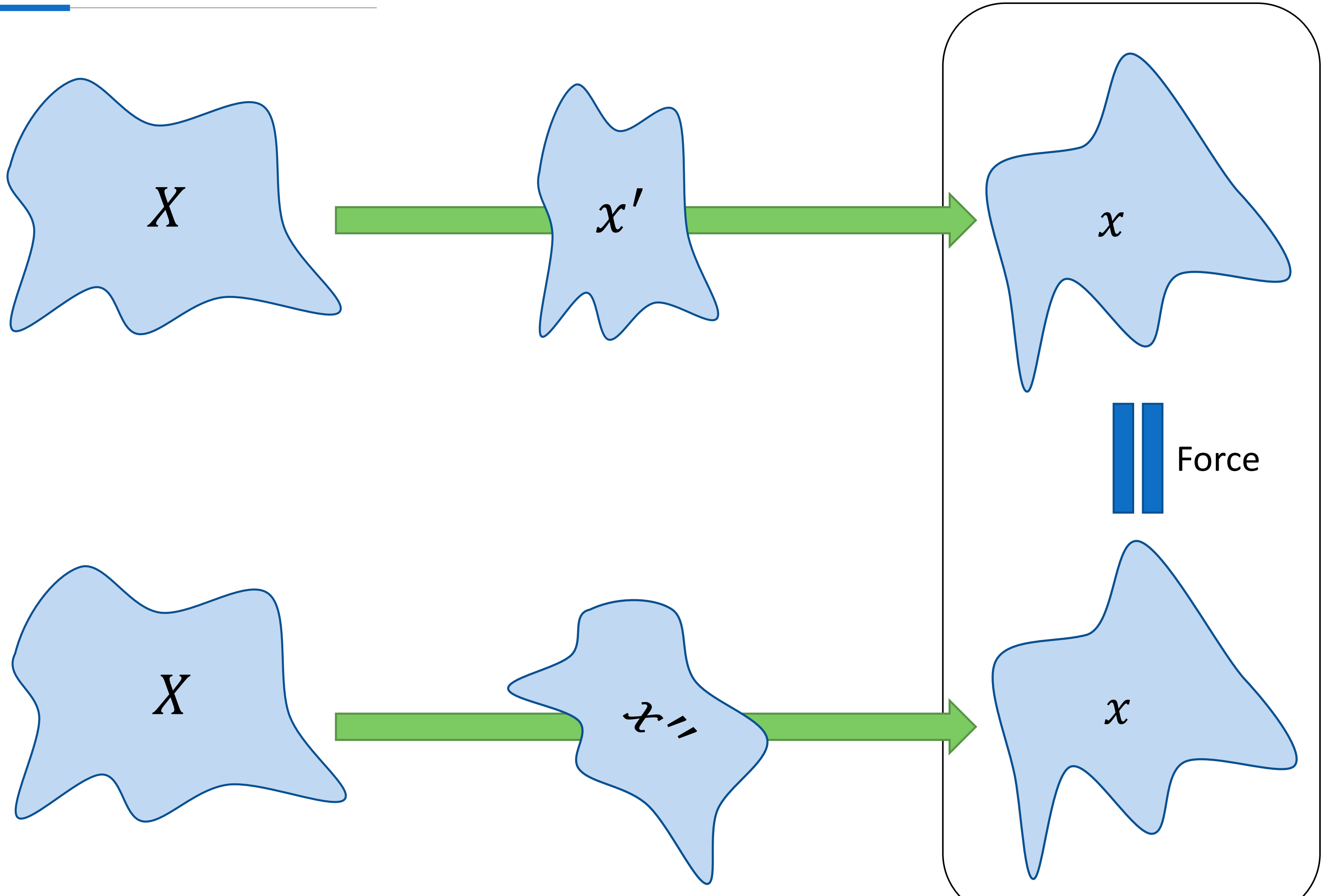
Displacement Map



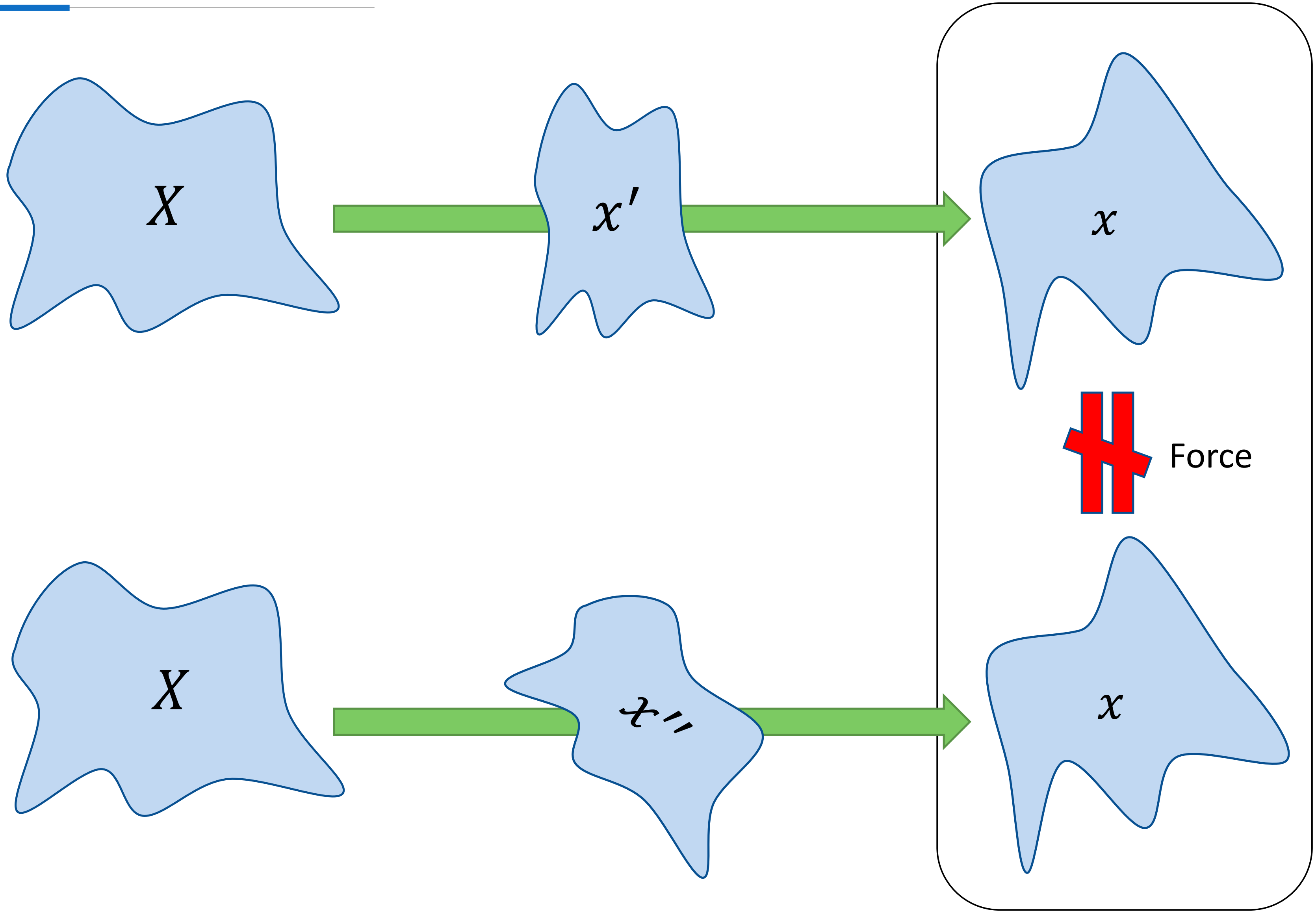
$$x = \varphi(X) = SX$$

$$F = S$$

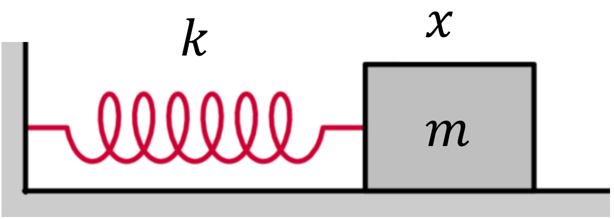
Hyperelasticity



Not Hyperelastic?



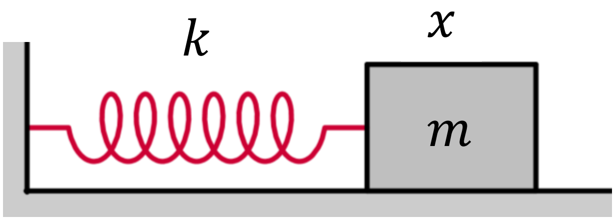
Energy and Force



$$f = -k(x - x_0)$$

$$E = \frac{1}{2}k(x - x_0)^2$$

Energy and Force



$$f = -k(x - x_0)$$

$$E = \frac{1}{2}k(x - x_0)^2$$

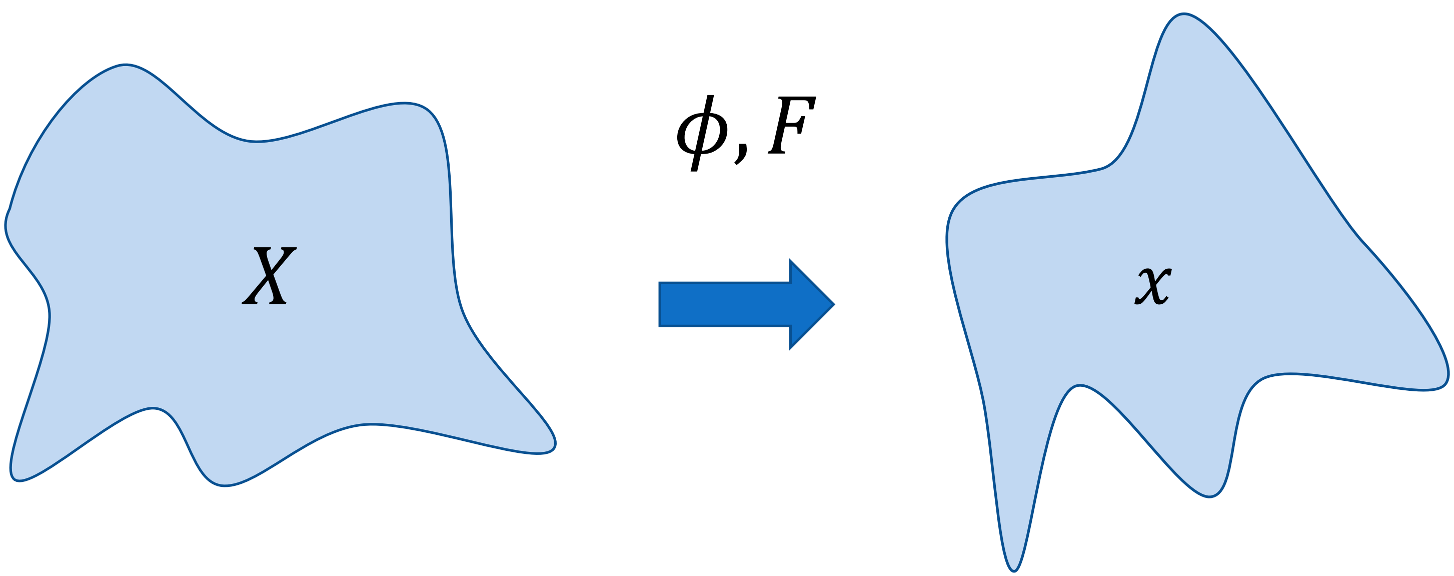
$$f = -\frac{dE}{dx}$$

Energy and Force

$$f(x) = -\nabla_{\mathrm{x}} E(x)$$

$$E(x) = \int_{\Omega} \Psi(\phi; X) dX$$

Energy Density



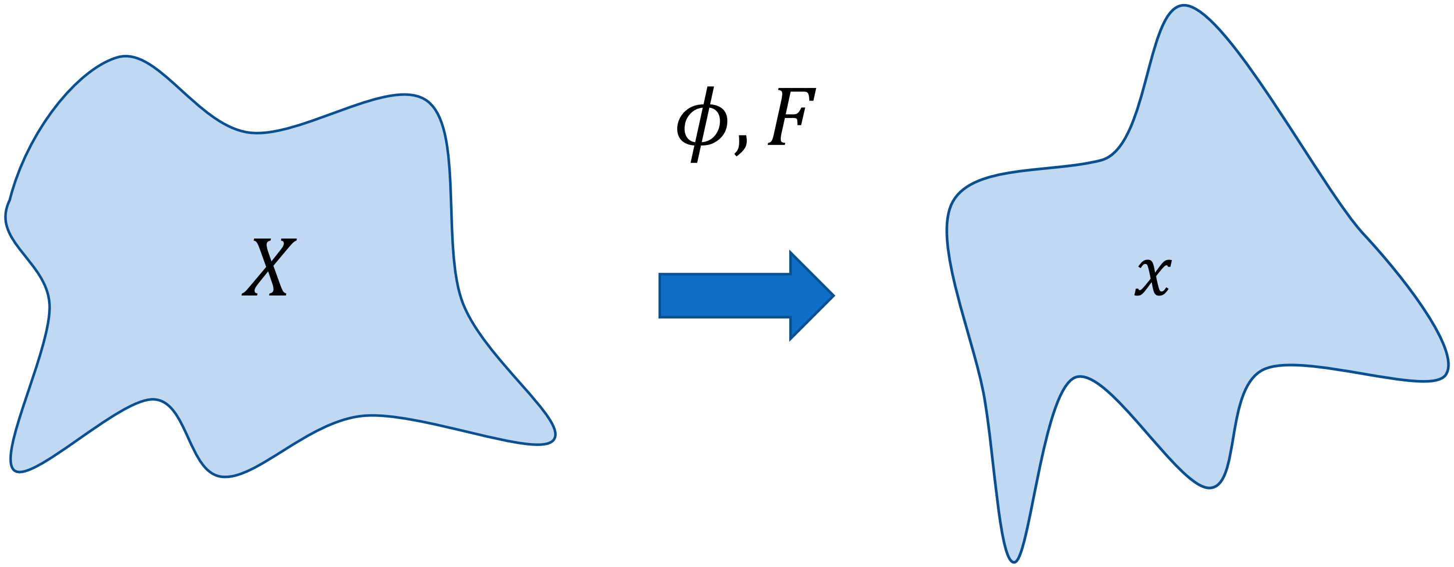
Energy and Force

$$f(x) = -\nabla_{\mathrm{x}} E(x)$$

$$E(x) = \int_{\Omega} \Psi(F)\, dX$$

小体积形变带来
的能量变化

Energy Density

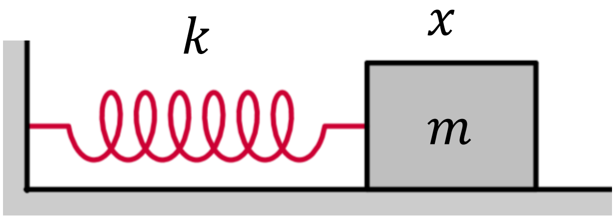


Energy Density

- What would a formula for $\Psi(\mathrm{F})$ look like?

- $\Psi(F) = \frac{k}{2}\|F\|_F^2$?

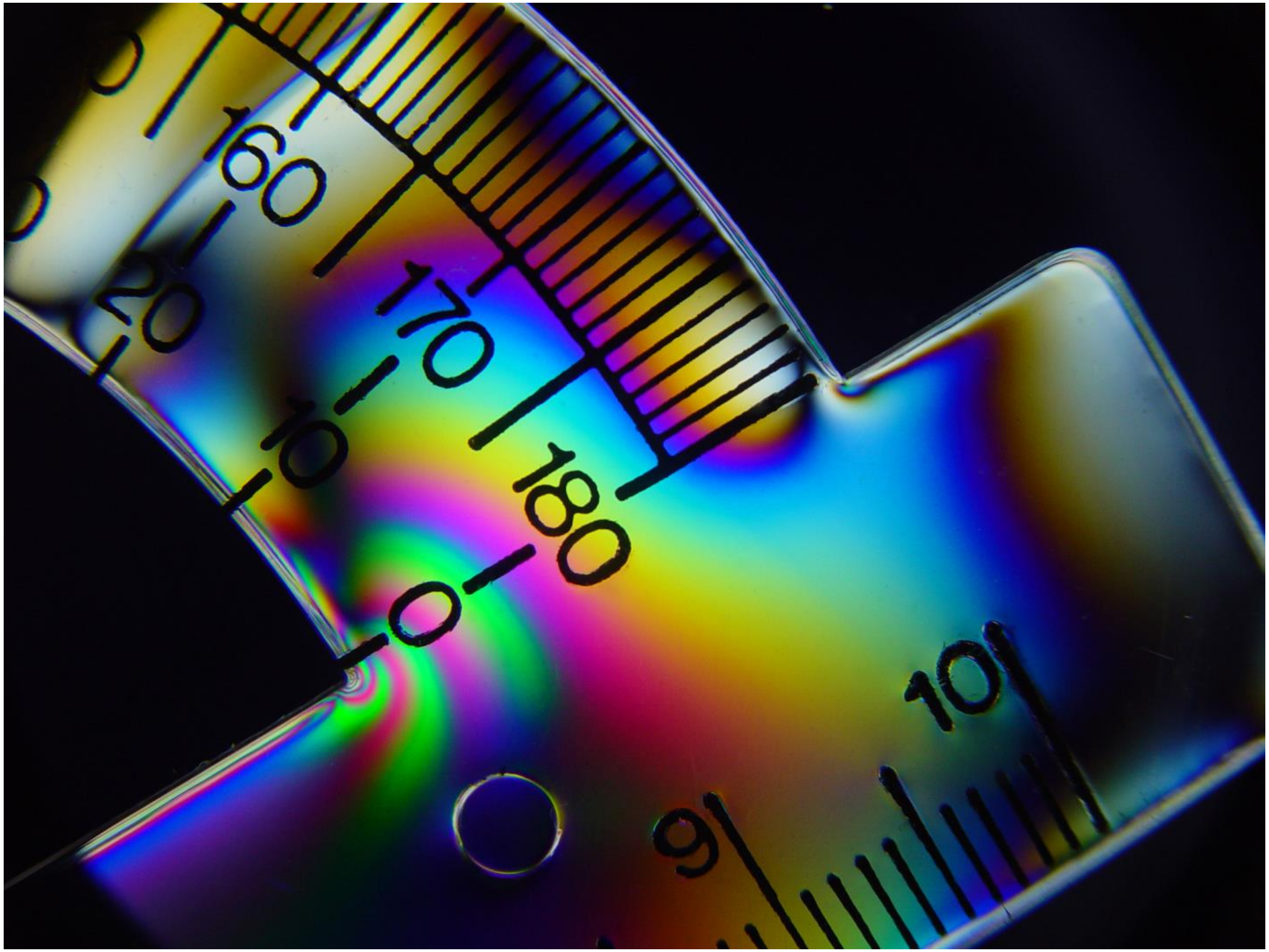
- $\Psi(F) = \frac{k}{2}\|F - I\|_F^2$



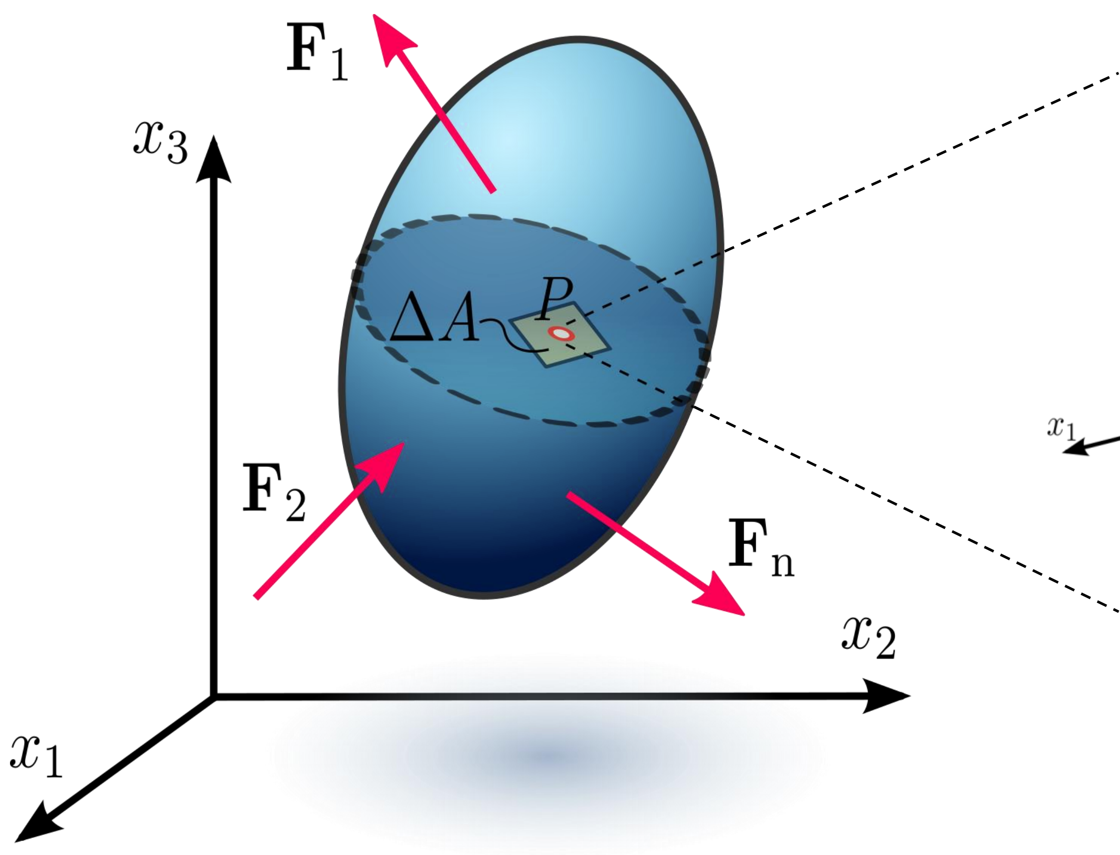
$$f = -k(x - x_0)$$

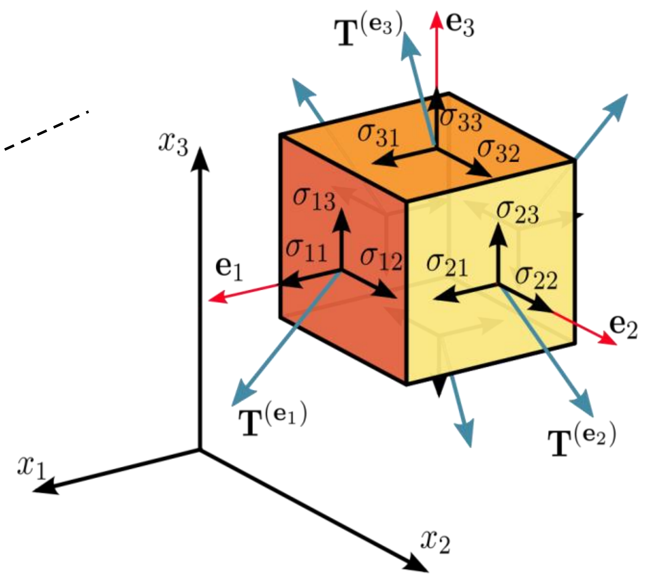
$$E = \frac{1}{2}k(x - x_0)^2$$

Stress 应力



Stress





$$\boldsymbol{T}^{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$$

$$\begin{bmatrix} T_1^{(\mathbf{n})} & T_2^{(\mathbf{n})} & T_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

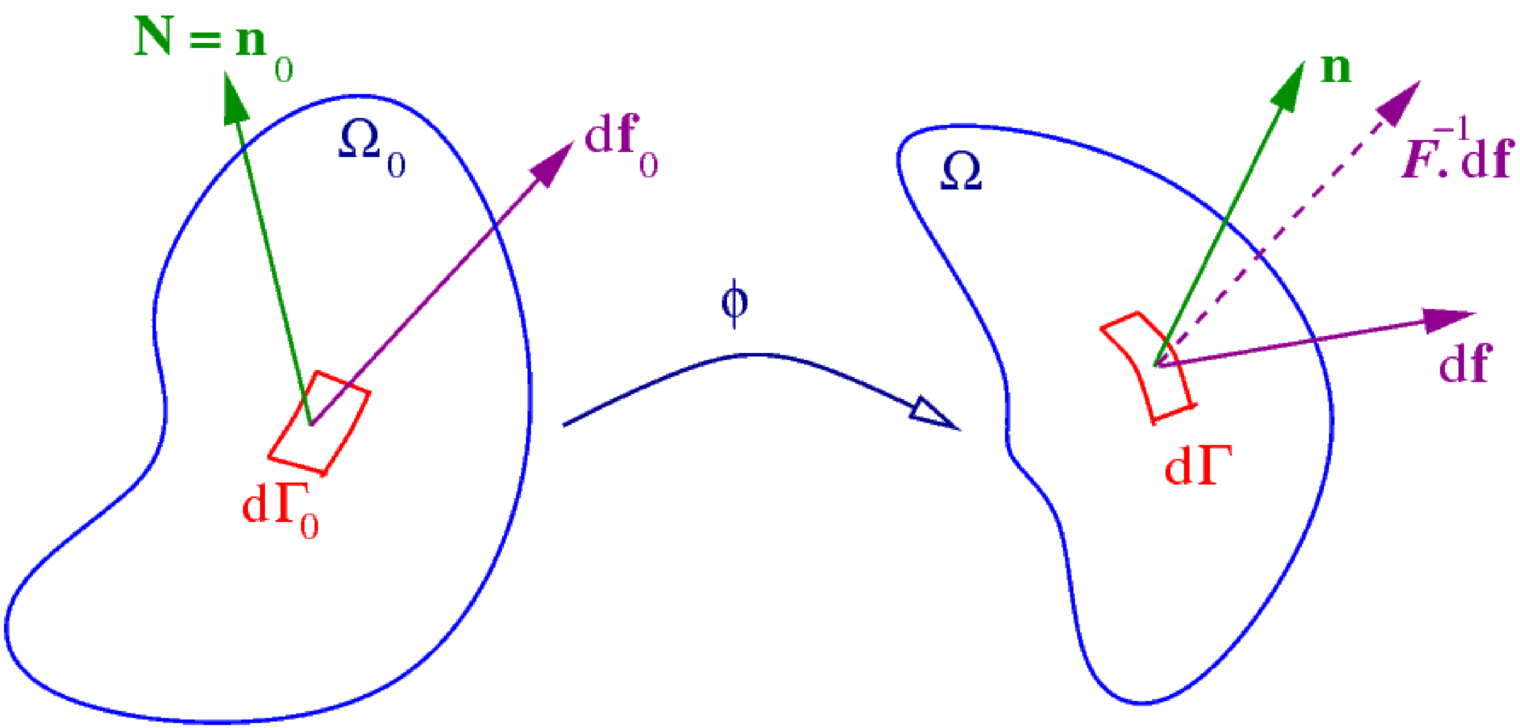
Stress

- A *fundamental descriptor* of force

- Cauchy stress tensor $\boldsymbol{\sigma}$
  - Stress tensor in *deformed space*
  - *Infinitesimal deformation*

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

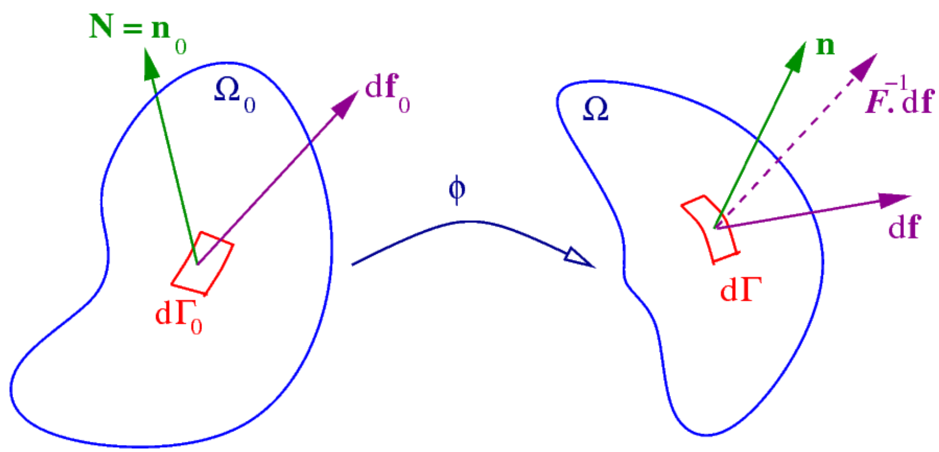
Different Forms of Stresses



Undeformed/Material/Reference
P, S

Deformed
σ

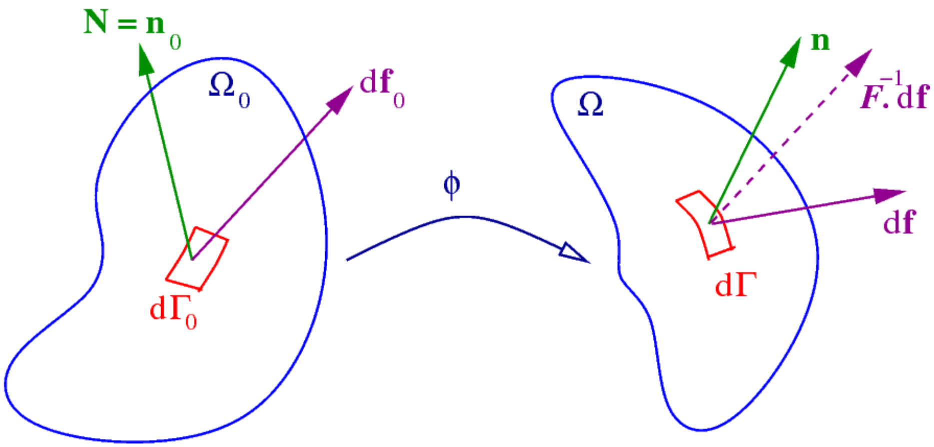
Different Forms of Stresses



Undeformed/Material/Reference   Deformed

Input / Output	Interface normal $\mathbf{N}$ in the *reference* state (unformed)	Interface normal $\mathbf{n}$ in the *current* state (deformed)
Traction in the *reference* state (unformed)	2nd Piola–Kirchhoff stress ($\mathbf{S}$)	
Traction in the *current* state (formed)	1st Piola–Kirchhoff stress ($\mathbf{P}$)	Cauchy Stress ($\boldsymbol{\sigma}$)

Different Forms of Stresses



Undeformed/Material/Reference

Deformed

Input / Output	Interface normal $\mathbf{N}$ in the *reference* state (unformed)	Interface normal $\mathbf{n}$ in the *current* state (deformed)
Traction in the *reference* state (unformed)	2nd Piola–Kirchhoff stress ($\mathbf{S}$)	
Traction in the *current* state (formed)	1st Piola–Kirchhoff stress ($\mathbf{P}$)	Cauchy Stress ($\boldsymbol{\sigma}$)

$$\mathbf{P} = \mathbf{FS}$$

$$\boldsymbol{\sigma} = \det{}^{-1}(\mathbf{F})\mathbf{FSF}^{\mathrm{T}}$$

$$\boldsymbol{\sigma} = \det{}^{-1}(\mathbf{F})\mathbf{PF}^{\mathrm{T}}$$

Stress

- A *fundamental descriptor* of force

- Cauchy stress tensor $\boldsymbol{\sigma}$
  - Stress tensor in *deformed space*
  - *Infinitesimal deformation*

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

- 1st *Piola-Kirchhoff* stress tensor $\boldsymbol{P}$
  - Stress tensor in *material space*
  - For hyperelastic materials

$$P(F) = \frac{\partial \Psi(F)}{\partial F}$$

Strain 应变

- $\epsilon(F)$: A measurement of severity of deformation
  - 1D case: $\epsilon = \frac{\delta l}{l_0}$
  - Property:

$$\epsilon(I) = 0$$

$$\epsilon(RF) = \epsilon(F) \text{ for } \forall R \in SO(n)$$

Strain 应变

- Example strain tensors:
  - Green strain tensors: (finite strain)

$$\epsilon(F) = \frac{1}{2}(F^T F - I)$$

$$\epsilon(F) = \frac{1}{2}(\Sigma^2 - I), \qquad F = U\Sigma V^T$$

  - Small (infinitesimal) strain tensors:

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I$$

Constitutive Model of Material

- Relationship between

Force-quantities:

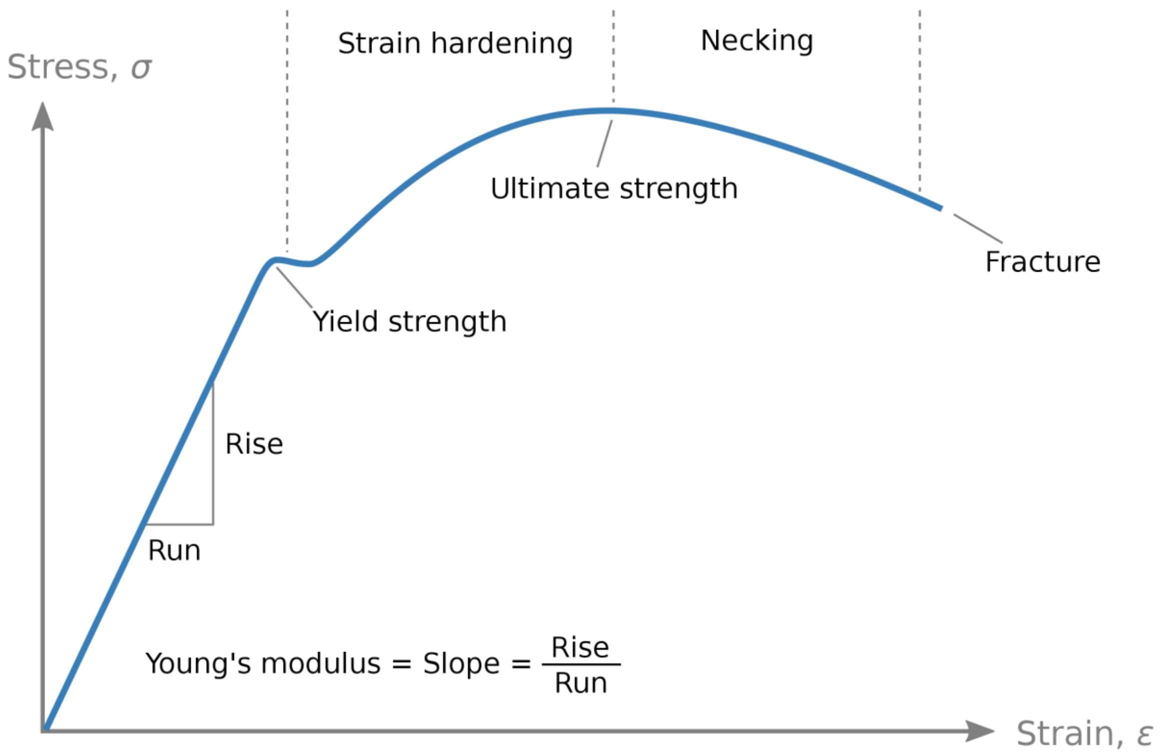
Ψ, P, S, σ, E

Kinematic-quantities

F, ϵ, ϕ

Constitutive Model of Material

- Particularly, the stress-strain relationship



Stress-strain curve typical of a low carbon steel.

https://en.wikipedia.org/wiki/Stress%E2%80%93strain_curve

Linear Elasticity (Hookean Model)

- Constitutive model:
  - Small (infinitesimal) strain tensors

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I$$

  - Strain energy density

$$\Psi(F) = \mu\, \epsilon : \epsilon + \frac{\lambda}{2}\, \mathrm{tr}^2(\epsilon)$$

$$\boxed{A : B = \mathrm{tr}(AB)}$$

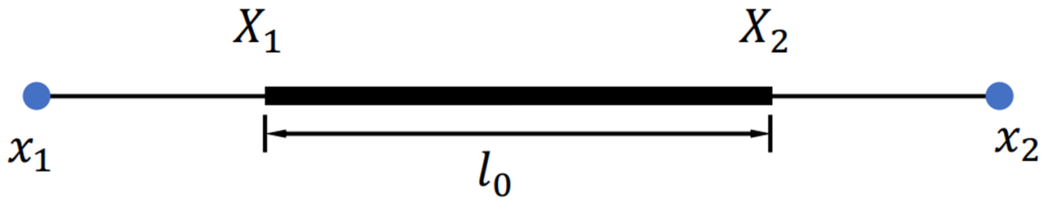
or

$$P = 2\mu\, \epsilon + \lambda\, \mathrm{tr}(\epsilon)$$

or

$$\sigma = C : \epsilon \qquad \boxed{\textbf{Hooke's law}}$$

1D Example: linear spring



$$F = \frac{x_2 - x_1}{X_2 - X_1} = \frac{x_2 - x_1}{l_0}$$

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I = F - 1 = \frac{x_2 - x_1 - l_0}{l_0}$$

$$\Psi(F) = \mu\epsilon\colon\epsilon + \frac{\lambda}{2}\mathrm{tr}^2(\epsilon) = \left(\mu + \frac{\lambda}{2}\right)\epsilon^2 = \frac{\kappa}{2}\epsilon^2$$

$$E = \int_{l_0} \Psi(F) = l_0\Psi = \frac{\kappa}{2}\frac{(x_2 - x_1 - l_0)^2}{l_0} = \frac{k}{2}(x_2 - x_1 - l_0)^2$$

Properties of Elastic Materials

$$\Psi(F) = \mu\, \epsilon : \epsilon + \frac{\lambda}{2}\, \mathrm{tr}^2(\epsilon)$$

- Lamé Coefficients (拉梅参数): μ, λ
- Young's Modulus (杨氏模量): κ
  Poisson's Ratio (泊松比): ν

$$\mu = \frac{\kappa}{2(1+\nu)}, \qquad \lambda = \frac{\kappa\nu}{(1+\nu)(1-2\nu)}$$

Young's Modulus of Materials

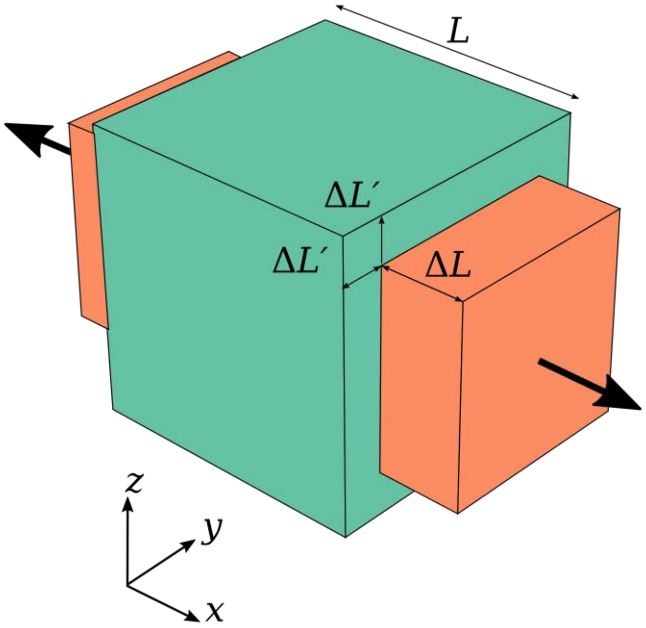
- Measure of material stiffness (unit: pascal, GPa)
  - Property of the material, not the shape
- Ratio between stress σ and strain ϵ

材料	杨氏模量 (GPa)
橡胶（微小应变）	0.01-0.1
木头	9 - 12
玻璃（所有种类）	71.7
铝	69
碳纤维强化塑料（单向，颗粒表面）	150
合金与钢	190-210
钨 (W)	400-410
钻石	1,050-1,200

https://en.wikipedia.org/wiki/Young%27s_modulus

Poisson's Ratio

- Measure of Poisson effect
- How the material resist to volume change
- Range： $[0.0, 0.5]$
  - 0.5: perfectly incompressible
    - E.g. rubber
  - 0.0: compressible
    - E.g. Cork



Linear Elasticity

- Constitutive model:
  - Small (infinitesimal) strain tensors

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I$$

  - Strain energy density

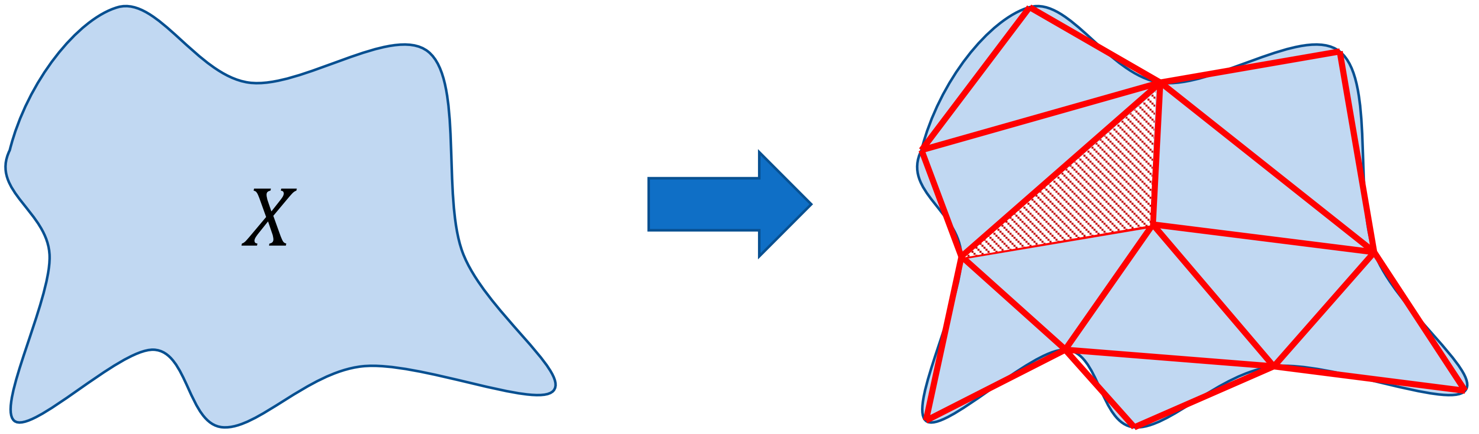
$$\boxed{A{:}B = \mathrm{tr}(AB)}$$

$$\Psi(F) = \mu\,\epsilon{:}\epsilon + \frac{\lambda}{2}\,\mathrm{tr}^2(\epsilon)$$

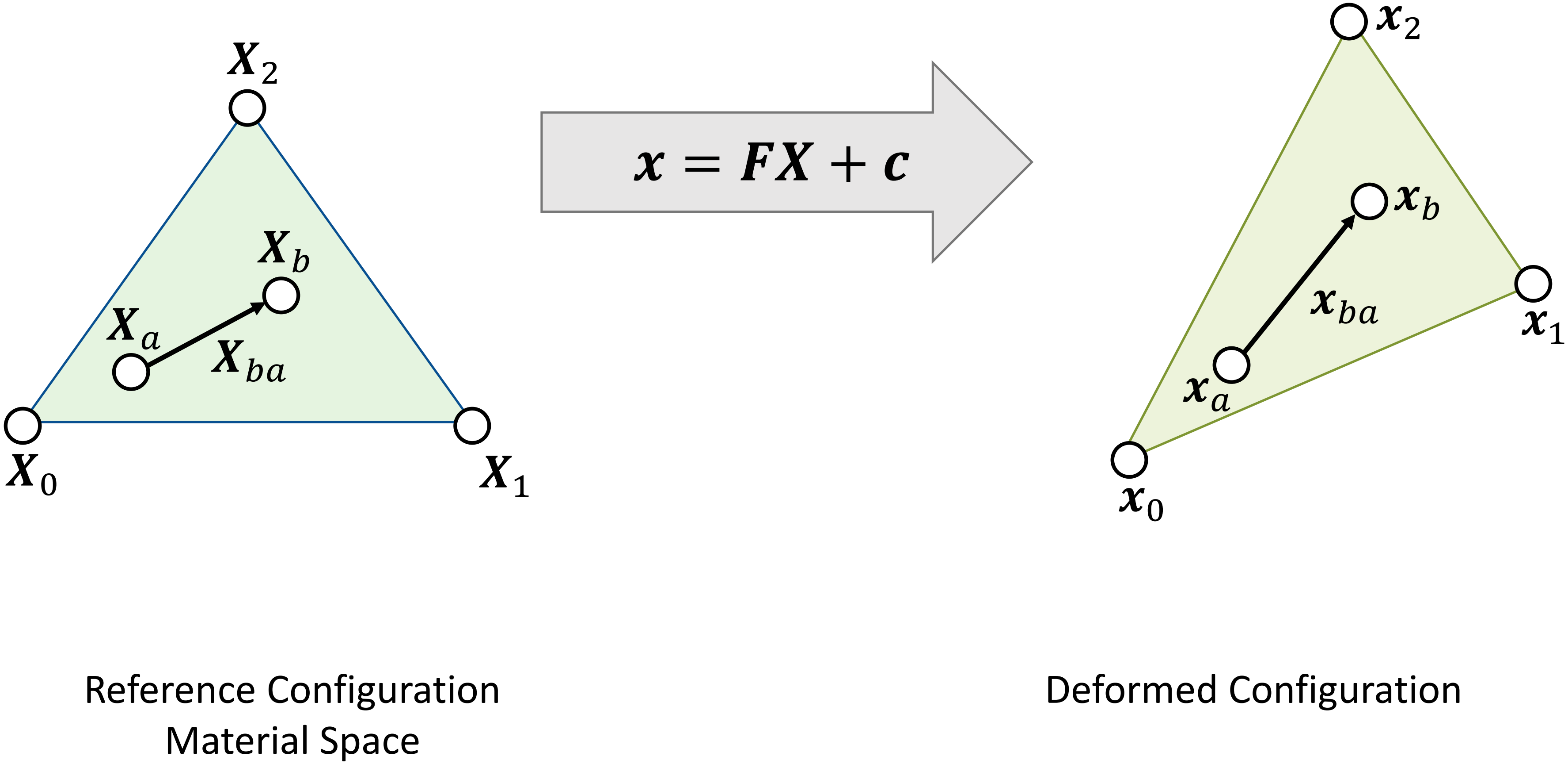
or

$$P = 2\mu\,\epsilon + \lambda\,\mathrm{tr}(\epsilon)$$

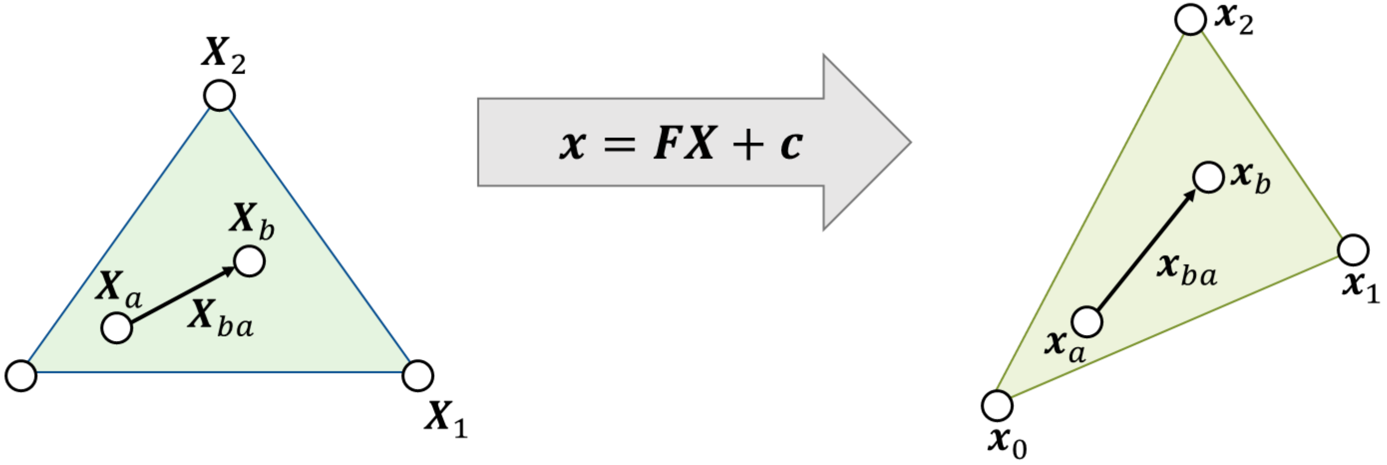
Discretization



Linear Discretization

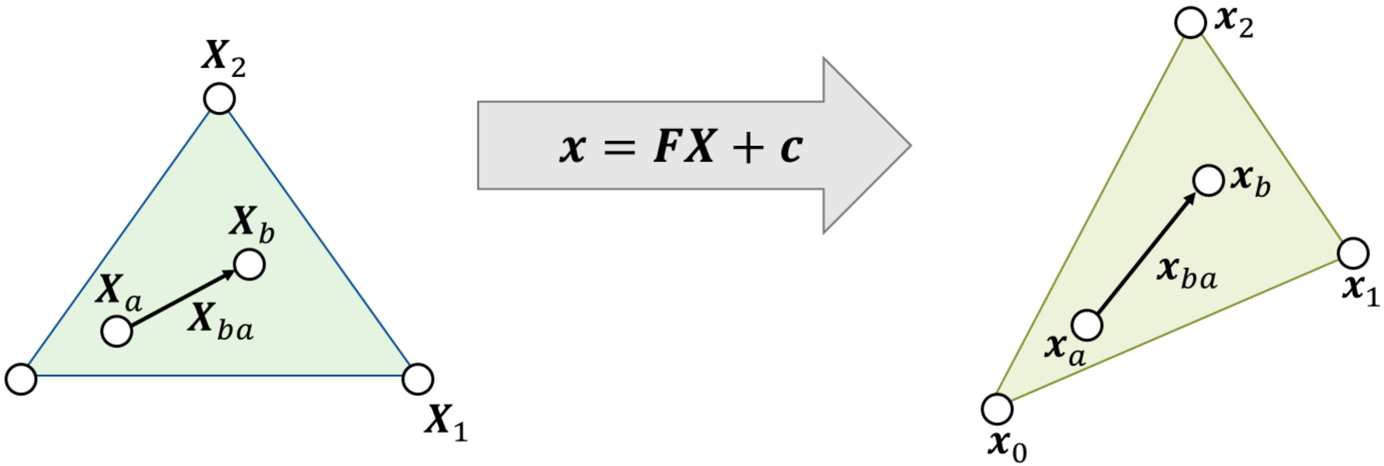


Linear Triangular Elements



$$[x_1 - x_0 \quad x_2 - x_0] = F[X_1 - X_0 \quad X_2 - X_0]$$

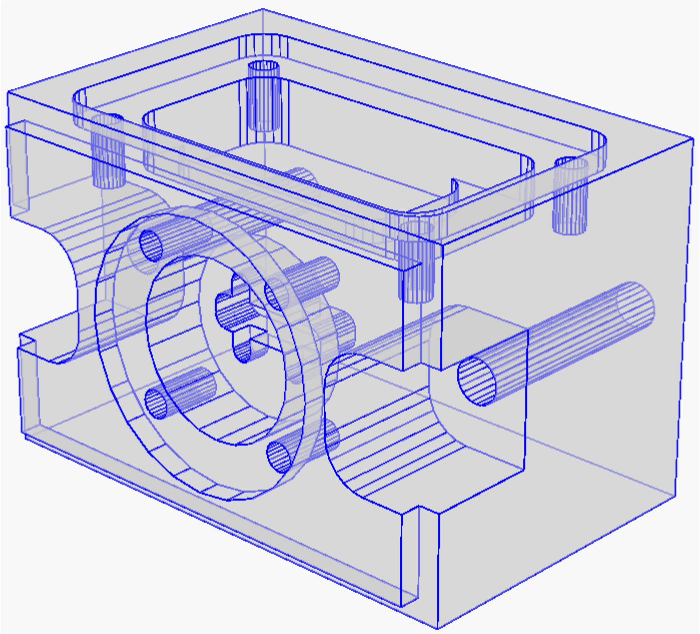
Linear Triangular Elements

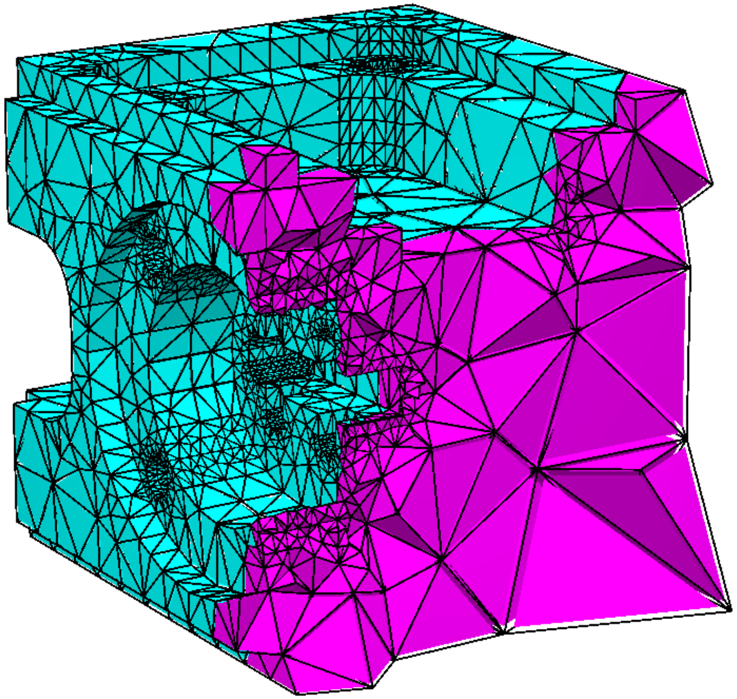


$$[x_1 - x_0 \quad x_2 - x_0] = F[X_1 - X_0 \quad X_2 - X_0]$$

$$F = D_s D_m^{-1}$$

Tetrahedralization



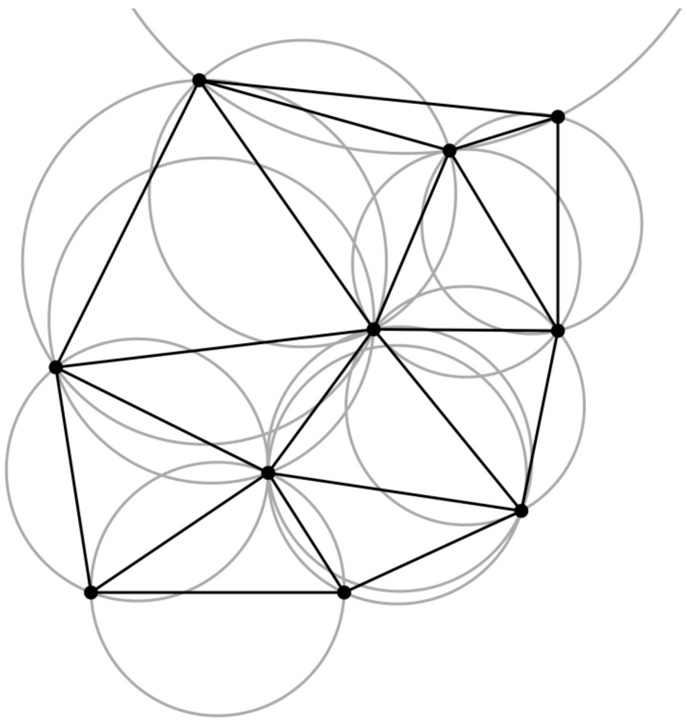


TETGEN: https://wias-berlin.de/software/tetgen/features.html

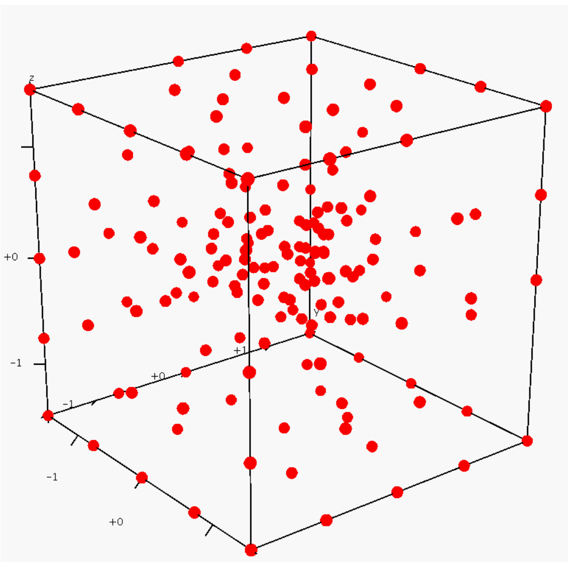
Delaunay Triangulation

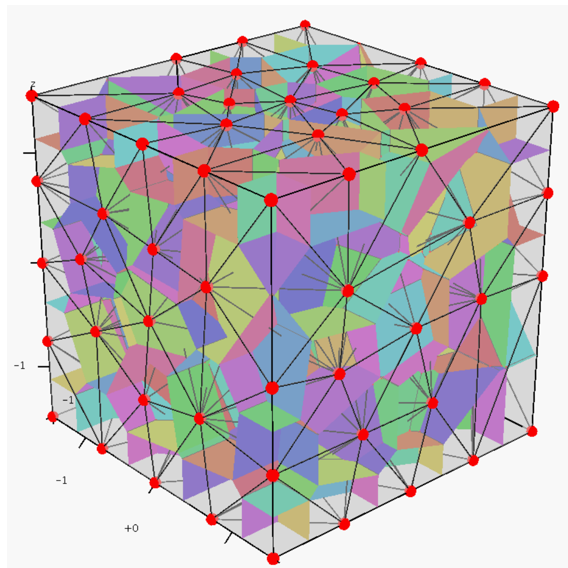
- A *triangulation* of point set P such that

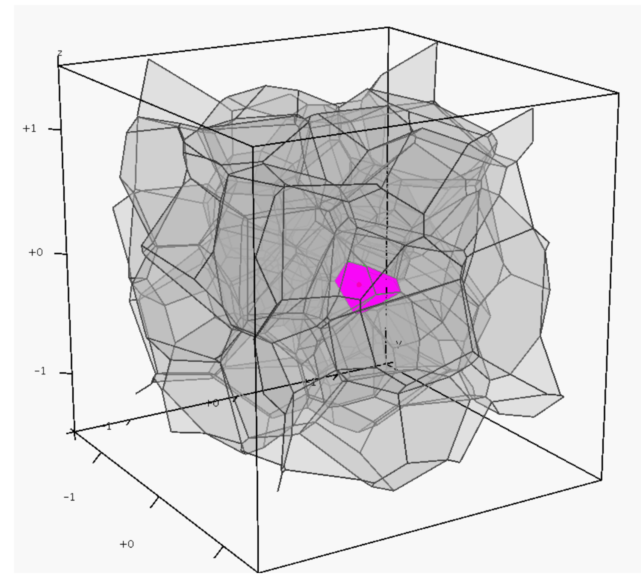
  no point in P is *inside*
  the circumcircle of any triangle
  in DT(P)



Delaunay Tetrahedralization







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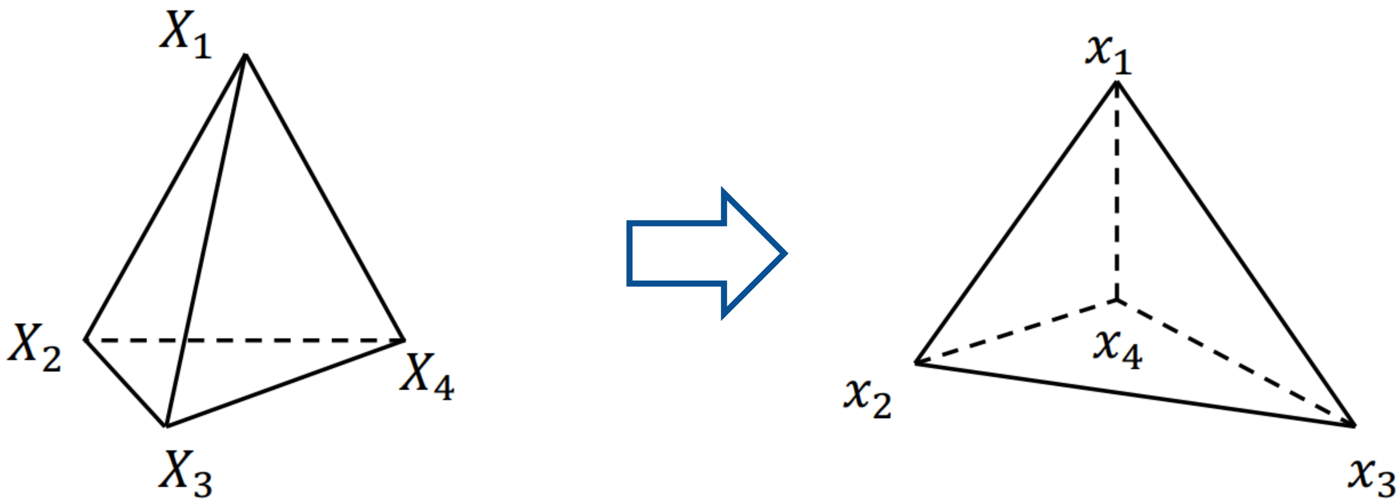
Linear Tetrahedral Elements

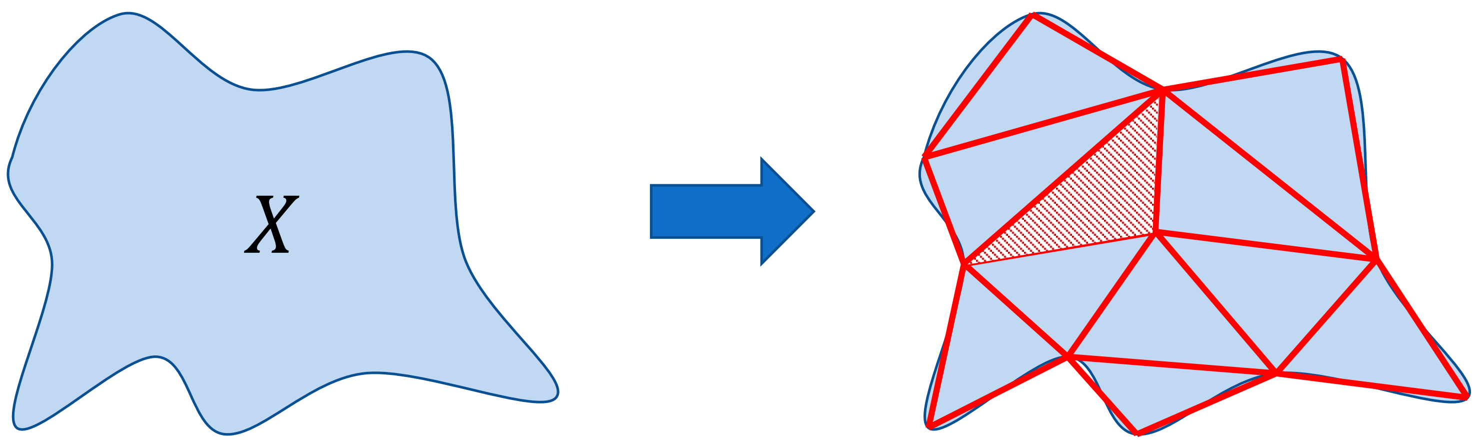


$$\underbrace{[x_1 - x_4 \quad x_2 - x_4 \quad x_3 - x_4]}_{D_s} = F \underbrace{[X_1 - X_4 \quad X_2 - X_4 \quad X_3 - X_4]}_{D_m}$$

$$F = D_s D_m^{-1}$$

Energy Discretization

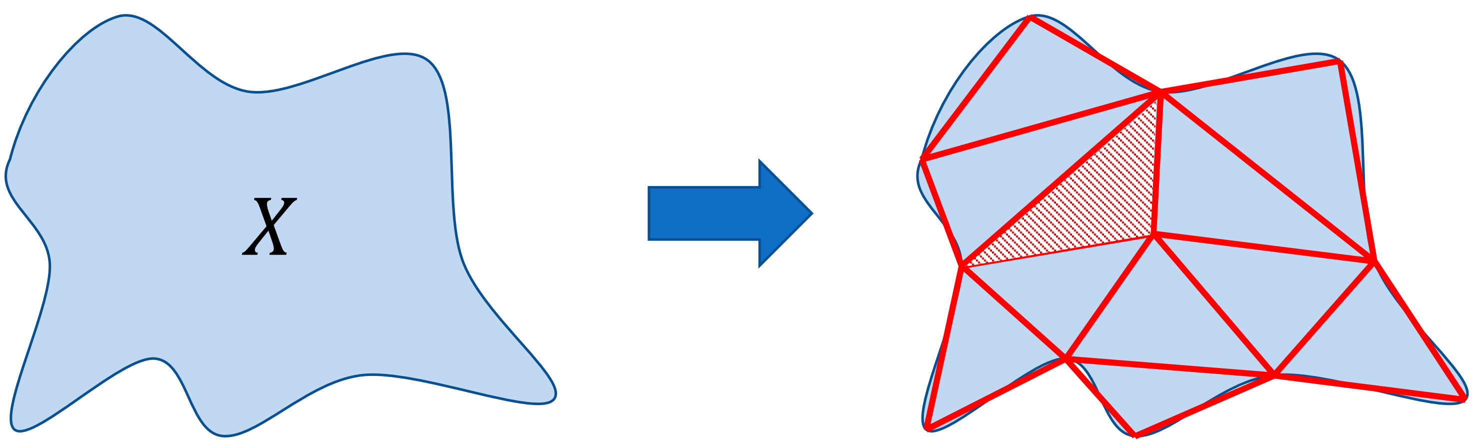
$$E(x) = \int_{\Omega} \Psi(F)\, dX \quad \Rightarrow \quad E(x) = \sum_{\Omega_i} \int_{\Omega_\mathrm{i}} \Psi(F)\, dX$$



Energy Discretization

$$E(x) = \int_{\Omega} \Psi(F)\, dX \quad \Rightarrow \quad E(x) = \sum_{\Omega_i} W_i \Psi(F_i)$$

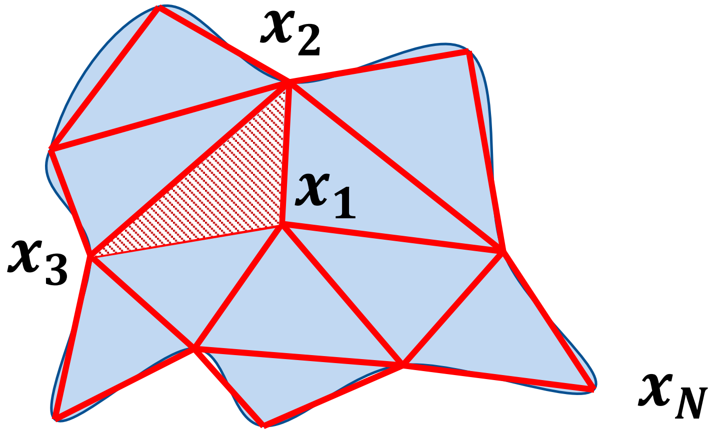
W_i: Volume of the element



Force Discretization

$$E(x) = \sum_{\Omega_i} W_i \Psi(F_i)$$

$$f_i(x) = -\frac{\partial E(x)}{\partial x_i}$$



$$f = [f_1, f_2, \ldots, f_N] = -\frac{\partial E(x)}{\partial x}$$

Force Discretization

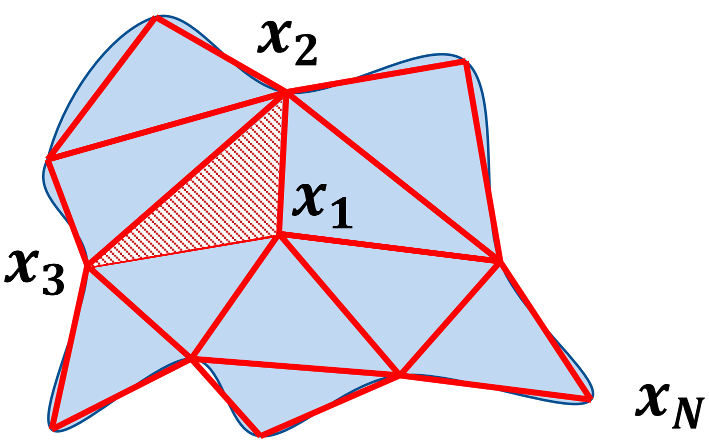
$$E(x) = \sum_{\Omega_i} W_i \Psi(F_i)$$

$$f_i(x) = -\sum_{\Omega_k} \frac{\partial E_k}{\partial x_i}$$

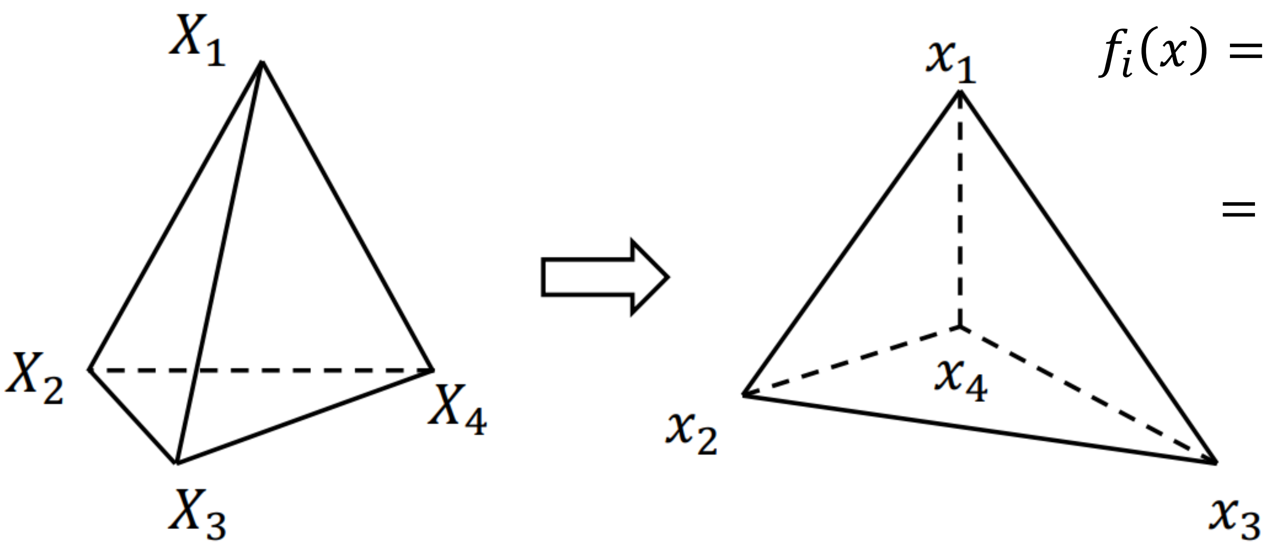
$$= -\sum_{\Omega_k} W_i \frac{\partial \Psi_k}{\partial x_i}$$

$$= -\sum_{\Omega_k} W_i \frac{\partial \Psi_k}{\partial F_k} \frac{\partial F_k}{\partial x_i}$$

$$= -\sum_{\Omega_k} W_i P(F_k) \frac{\partial F_k}{\partial x_i}$$



Force Discretization



$$f_i(x) = \frac{\partial E(x)}{\partial x_i}$$

$$= WP(F)\frac{\partial F}{\partial x_i}$$

$$\underbrace{[x_1 - x_4 \quad x_2 - x_4 \quad x_3 - x_4]}_{D_s} = F\underbrace{[X_1 - X_4 \quad X_2 - X_4 \quad X_3 - X_4]}_{D_m}$$

$$F = D_s D_m^{-1}$$

$$H = [f_1 \quad f_2 \quad f_3] = -WP(F)D_m^{-T}$$

$$f_4 = -f_1 - f_2 - f_3$$

"Internal" force should sum to zero

Linear Elasticity (Hookean Model)

- Constitutive model:
  - Small (infinitesimal) strain tensors

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I$$

  - Strain energy density

$$\Psi(F) = \mu\, \epsilon : \epsilon + \frac{\lambda}{2}\, \mathrm{tr}^2(\epsilon)$$

$$\boxed{A : B = \mathrm{tr}(AB)}$$

or

$$P = 2\mu\, \epsilon + \lambda\, \mathrm{tr}(\epsilon)$$

or

$$\sigma = C : \epsilon \quad \boxed{\textbf{Hooke's law}}$$

Computation of Nodal Forces

Algorithm 1 Batch computation of elastic forces on a tetrahedral mesh

procedure PRECOMPUTATION($\mathbf{x}, \mathbf{B}_m[1 \ldots M], W[1 \ldots M]$)

    for each $\mathcal{T}_e = (i, j, k, l) \in \mathcal{M}$ **do** ▷ M is the number of tetrahedra

$$\mathbf{D}_m \leftarrow \begin{bmatrix} X_i - X_l & X_j - X_l & X_k - X_l \\ Y_i - Y_l & Y_j - Y_l & Y_k - Y_l \\ Z_i - Z_l & Z_j - Z_l & Z_k - Z_l \end{bmatrix}$$

        $\mathbf{B}_m[e] \leftarrow \mathbf{D}_m^{-1}$

        $W[e] \leftarrow \frac{1}{6}\det(\mathbf{D}_m)$ ▷ W is the undeformed volume of $\mathcal{T}_e$

    end for

end procedure

procedure COMPUTEELASTICFORCES($\mathbf{x}, \mathbf{f}, \mathcal{M}, \mathbf{B}_m[], W[]$)

    $\mathbf{f} \leftarrow \mathbf{0}$ ▷ $\mathcal{M}$ is a tetrahedral mesh

    for each $\mathcal{T}_e = (i, j, k, l) \in \mathcal{M}$ **do**

$$\mathbf{D}_s \leftarrow \begin{bmatrix} x_i - x_l & x_j - x_l & x_k - x_l \\ y_i - y_l & y_j - y_l & y_k - y_l \\ z_i - z_l & z_j - z_l & z_k - z_l \end{bmatrix}$$

        $\mathbf{F} \leftarrow \mathbf{D}_s \mathbf{B}_m[e]$

        $\mathbf{P} \leftarrow \mathbf{P}(\mathbf{F})$ ▷ From the constitutive law

        $\mathbf{H} \leftarrow -W[e]\mathbf{P}\left(\mathbf{B}_m[e]\right)^T$

        $\vec{f}_i \mathrel{+}= \vec{h}_1,\ \vec{f}_j \mathrel{+}= \vec{h}_2,\ \vec{f}_k \mathrel{+}= \vec{h}_3$ ▷ $\mathbf{H} = \left[\vec{h}_1\ \vec{h}_2\ \vec{h}_3\right]$

        $\vec{f}_l \mathrel{+}= (-\vec{h}_1 - \vec{h}_2 - \vec{h}_3)$

    end for

end procedure

Eftychios Sifakis and Jernej Barbic. 2012. ***FEM simulation of 3D deformable solids: a practitioner's guide to theory, discretization and model reduction***. In *ACM SIGGRAPH 2012 Courses* (SIGGRAPH '12),

Simulation of Deformable Solid

- Simulation Loop
  - Clear forces
    - Prevent force accumulation
  - Calculate forces
    - Compute nodal forces based on the material model and the discretization
    - aka, *Algorithm 1*
  - Update
    - Loop over particles, update x_i and v_i using the corresponding integrator

x

v

f

$\boldsymbol{M}$

Equation of Motion

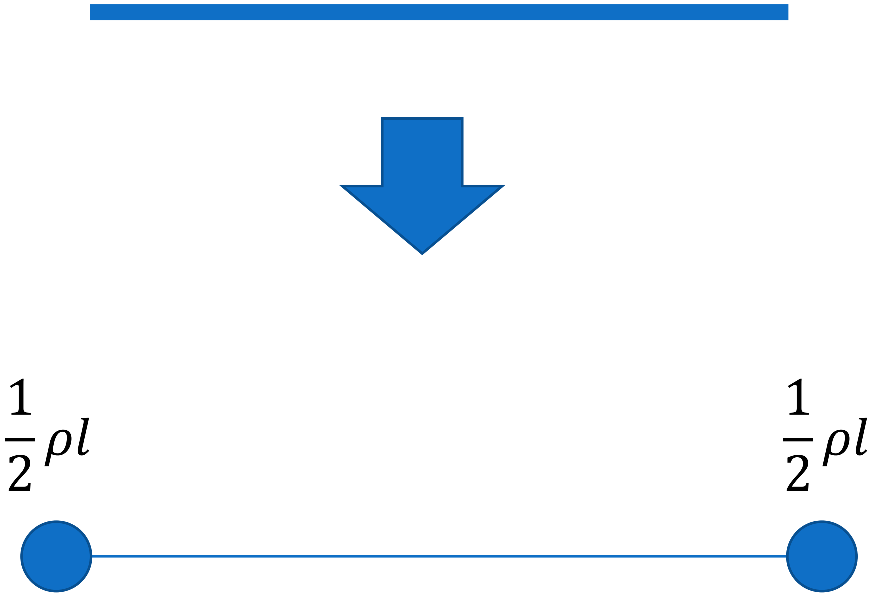
$$
\begin{aligned}
M\dot{v} &= f_{int} + f_{ext} \\
&= f_e(x) + f_d(x, v) + f_{ext}
\end{aligned}
$$

For linear material

$$f_e(x) = -K(x - X)$$

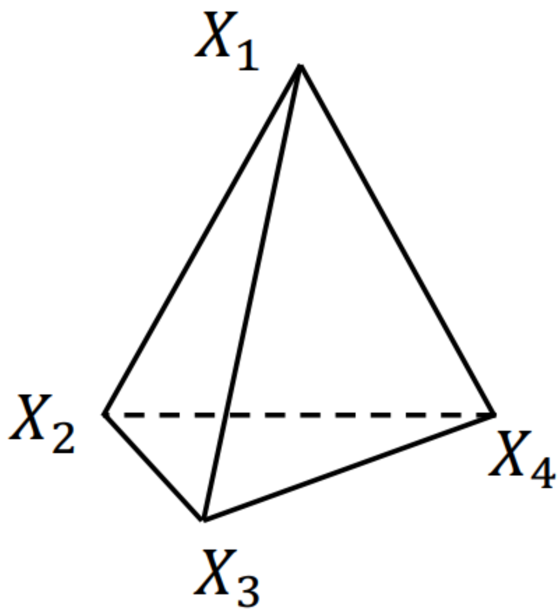
Lumped Mass Matrix

$$M = \mathrm{diag}(m_1, m_2, \dots, m_N)$$



Lumped Mass Matrix

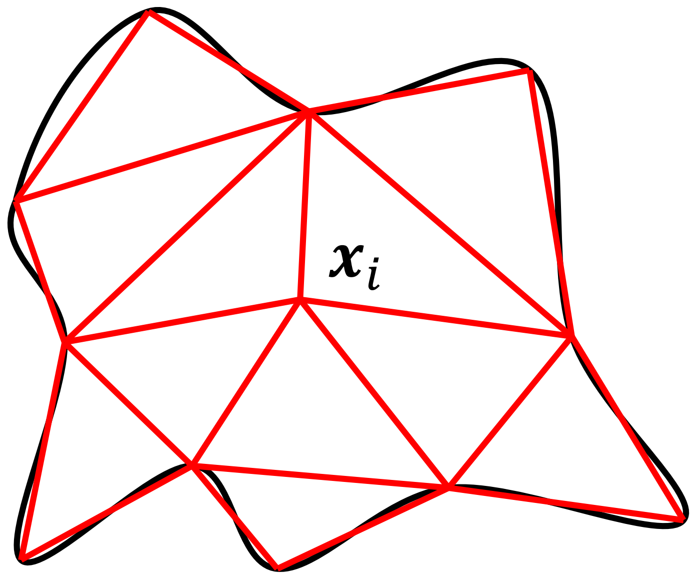
$$M = \mathrm{diag}(m_1, m_2, \ldots, m_N)$$



$$m_i = \frac{\rho W}{4}$$

Lumped Mass Matrix

$$M = \mathrm{diag}(m_1, m_2, \dots, m_N)$$



$$m_i = \sum_{j \in \mathcal{N}(i)} \frac{\rho W_i}{4}$$

Damping

$$M\dot{v} = f_{int} + f_{ext}$$

$$= f_e(x) + f_d(x, v) + f_{ext}$$

Damping stress $P_d = P_d(F, \dot{F})$

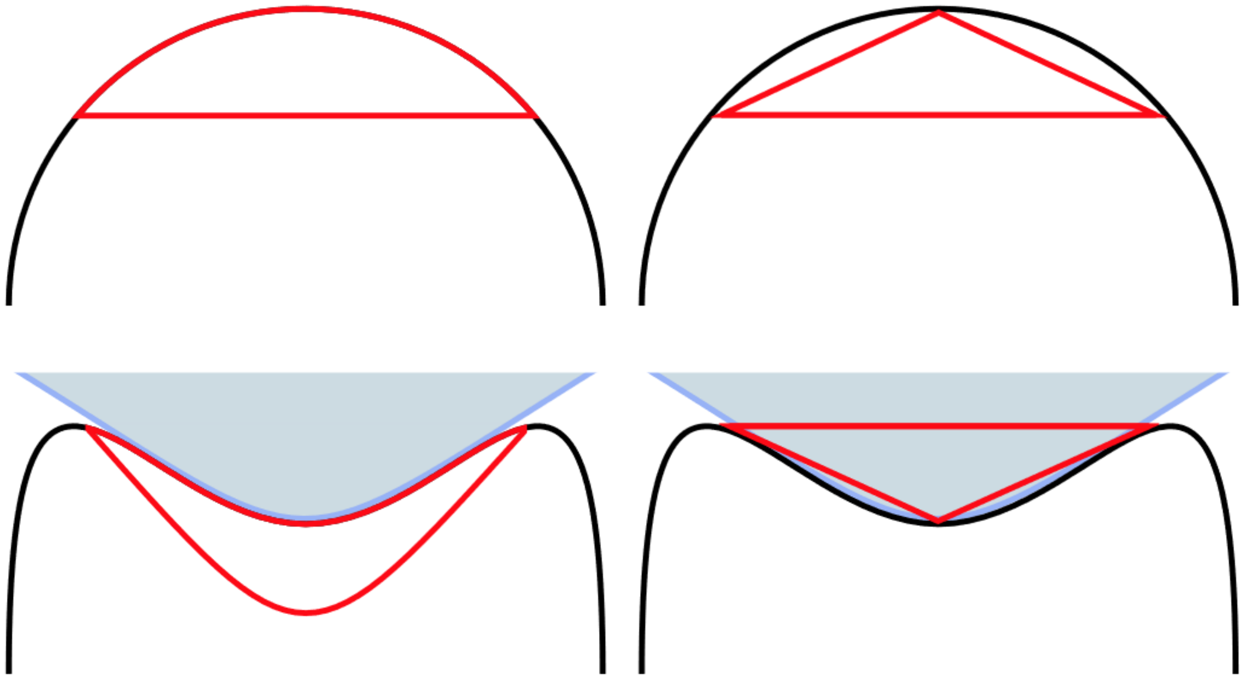
Rayleigh damping:

$$f_d(x, v) = -(\alpha M + \beta K)v$$

$$K = -\frac{\partial f_e(x)}{\partial x}$$

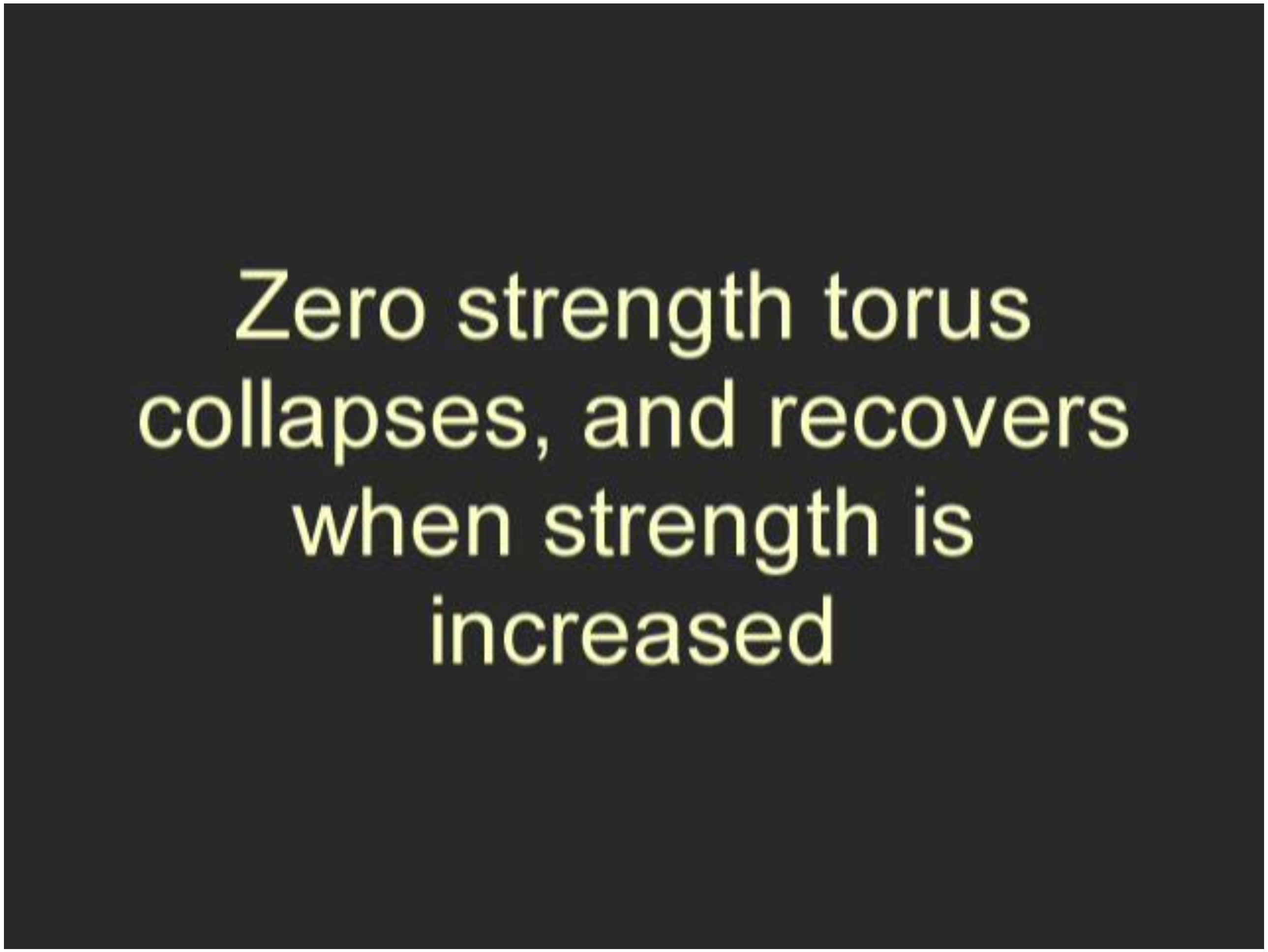
Other damping? Think about the mass spring system…

Inverted Tetrahedral



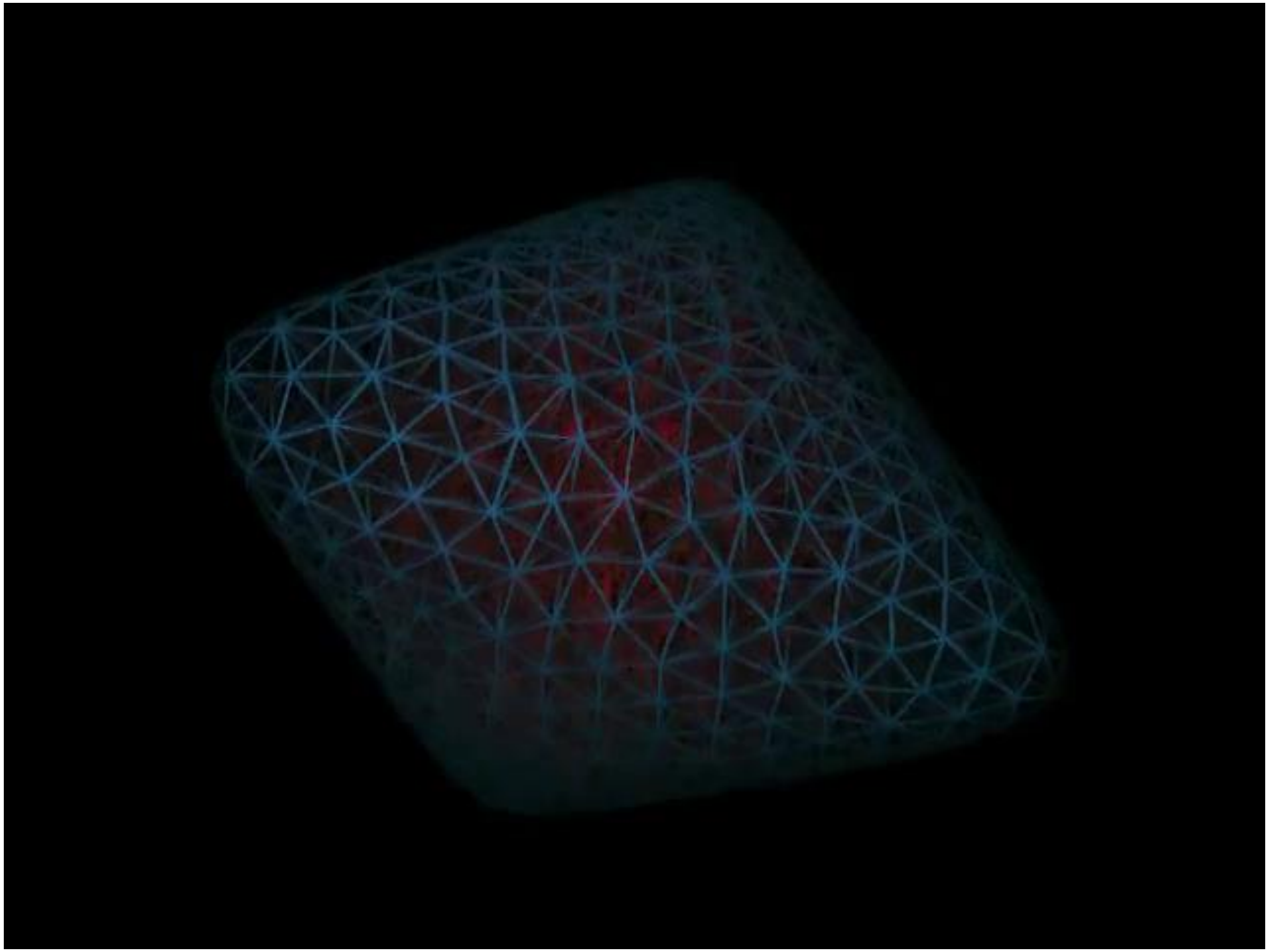
G. Irving, J. Teran, and R. Fedkiw. 2004. ***Invertible finite elements for robust simulation of large deformation***.
In *Proceedings of the 2004 ACM SIGGRAPH/Eurographics symposium on Computer animation* (SCA '04)

Example



G. Irving, J. Teran, and R. Fedkiw. 2004. ***Invertible finite elements for robust simulation of large deformation***. In *Proceedings of the 2004 ACM SIGGRAPH/Eurographics symposium on Computer animation* (SCA '04)

Example



Invertible Finite Elements Simulator – Stefan Zickler
https://www.youtube.com/watch?v=G2bEv_bPsDA

Plasticity & Ductile Fracture

James F. O'Brien, Adam W. Bargteil, and Jessica K. Hodgins. "**Graphical Modeling and Animation of Ductile Fracture**". In *Proceedings of ACM SIGGRAPH 2002*, pages 291–294. ACM Press, August 2002.

Plasticity

Linear Material	$\epsilon = \frac{1}{2}(F + F^T) - I$ $P = 2\mu\,\epsilon + \lambda\,\mathrm{tr}(\epsilon)$

- Additive plasticity model

$$\epsilon = \epsilon_e + \epsilon_p$$

$$\epsilon_p \leftarrow \epsilon_e$$

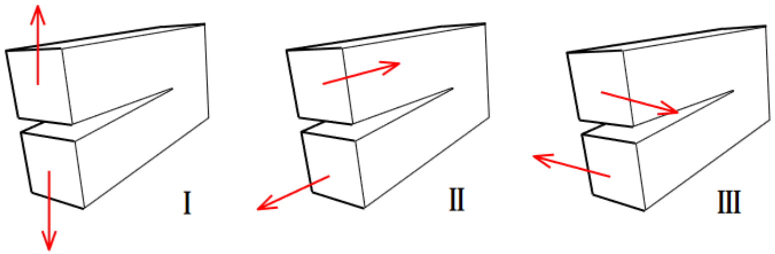
- Multiplicative plasticity model

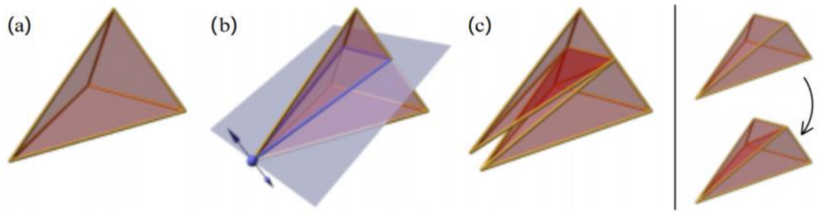
$$F = F_e F_p$$

$$F_p \leftarrow F_e$$

Fracture

- Key ideas:
  - Decompose force into tensile and compressive components, check if they are act to split the material
  - If the action is large enough, compute a fracture plane, split the elements, and remesh the object





James F. O'Brien, Adam W. Bargteil, and Jessica K. Hodgins. "**Graphical Modeling and Animation of Brittle Fracture**". In *Proceedings of ACM SIGGRAPH 2002*

More Recent Method

Joshuah Wolper, Yu Fang, Minchen Li, Jiecong Lu, Ming Gao, and Chenfanfu Jiang. 2019. ***CD-MPM: continuum damage material point methods for dynamic fracture animation.*** ACM Trans. Graph. 38, 4, Article 119 (August 2019)

Linear Elasticity (Hookean Model)

- Constitutive model:
  - Small (infinitesimal) strain tensors

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I$$

  - Strain energy density

$$\Psi(F) = \mu\, \epsilon : \epsilon + \frac{\lambda}{2}\, \mathrm{tr}^2(\epsilon)$$

$$\boxed{A : B = \mathrm{tr}(AB)}$$

or

$$P = 2\mu\, \epsilon + \lambda\, \mathrm{tr}(\epsilon)$$

or

$$\sigma = C : \epsilon$$

Hooke's law

Linear Elasticity (Hookean Model)

- Constitutive model:
  - Small (infinitesimal) strain tensors

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I$$

not rotationally invariant!

  - Strain energy density

$$\Psi(F) = \mu\, \epsilon : \epsilon + \frac{\lambda}{2}\, \mathrm{tr}^2(\epsilon)$$

$$A : B = \mathrm{tr}(AB)$$

or

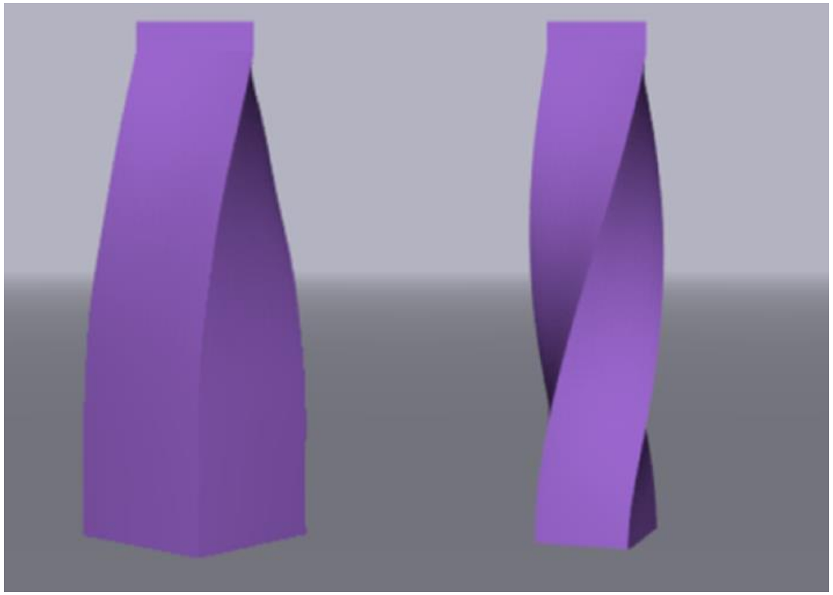
$$P = 2\mu\, \epsilon + \lambda\, \mathrm{tr}(\epsilon)$$

or

$$\sigma = C : \epsilon$$

Hooke's law

Problem of Linear Elasticity



Left: linear. Right: non-linear (co-rotated)

Wei Chen, Fei Zhu, Jing Zhao, Sheng Li, and Guoping Wang. 2018. ***Peridynamics-Based Fracture Animation for Elastoplastic Solids***. *Computer Graphics Forum* 37, 1 (2018)

St. Venant-Kirchhoff Model (StVK)

- Constitutive model:
  - Green strain tensors

$$E(F) = \frac{1}{2}(F^T F - I)$$

  - Strain energy density

$$\Psi(F) = \mu\, E\colon E + \frac{\lambda}{2}\, \mathrm{tr}^2(E)$$

or

$$P = F[2\mu E + \lambda\, \mathrm{tr}(E) I]$$

Corotated Linear Elasticity

- Constitutive model:
  - Small (infinitesimal) strain tensors

$$F = RS$$

$$\epsilon_c(F) = S - I$$

Polar decomposition

  - Strain energy density

$$\Psi(F) = \mu\, \epsilon_c : \epsilon_c + \frac{\lambda}{2}\,\mathrm{tr}^2(\epsilon_c)$$

or

$$P = 2\mu(F - R) + \lambda\,\mathrm{tr}(R^T F - I)R$$

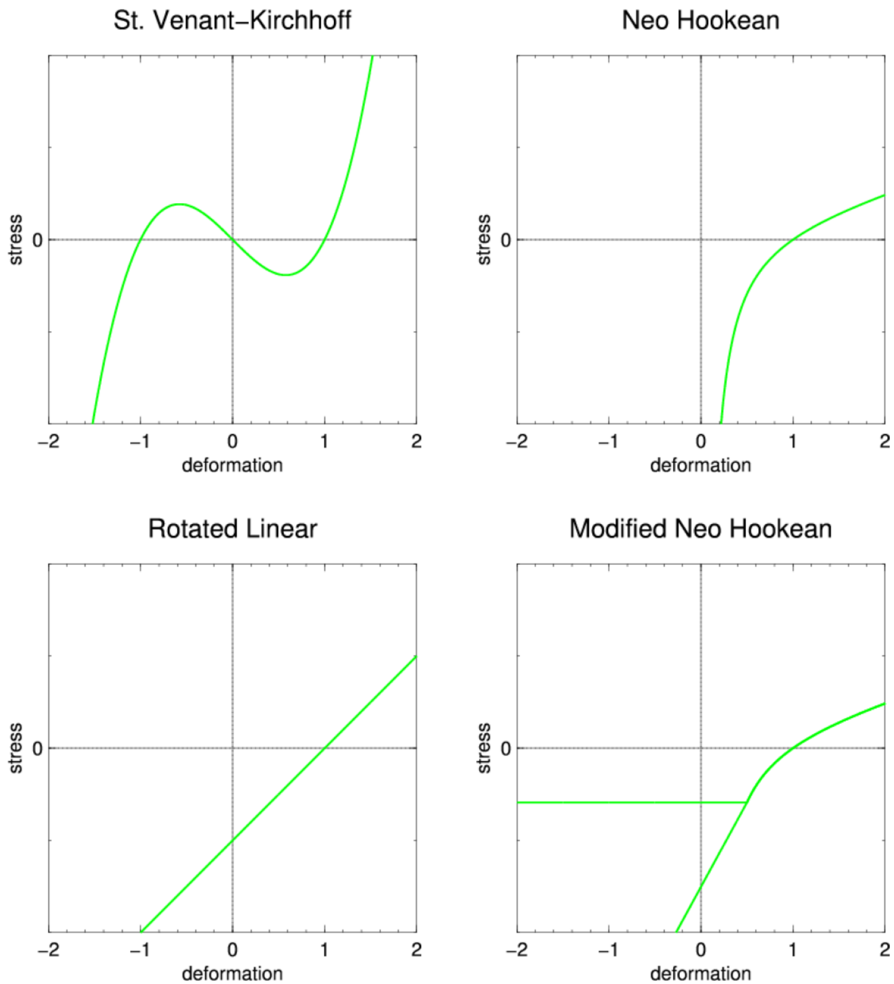
Neohookean Model

- A constitutive model defined based on the *isotropic invariants* of F, roughly, the singular values of F
  - $I = tr(F^T F), II = tr\left((F^T F)^2\right), III = \det(F^T F)$
- Strongly incompressible, volume-preserving

$$P = \mu(F - F^{-T}) + \lambda \log\left(\det(\mathrm{F})\right)F^{-T}$$

- Similarly defined materials:
  - Mooney-Rivlin, Fung, ...

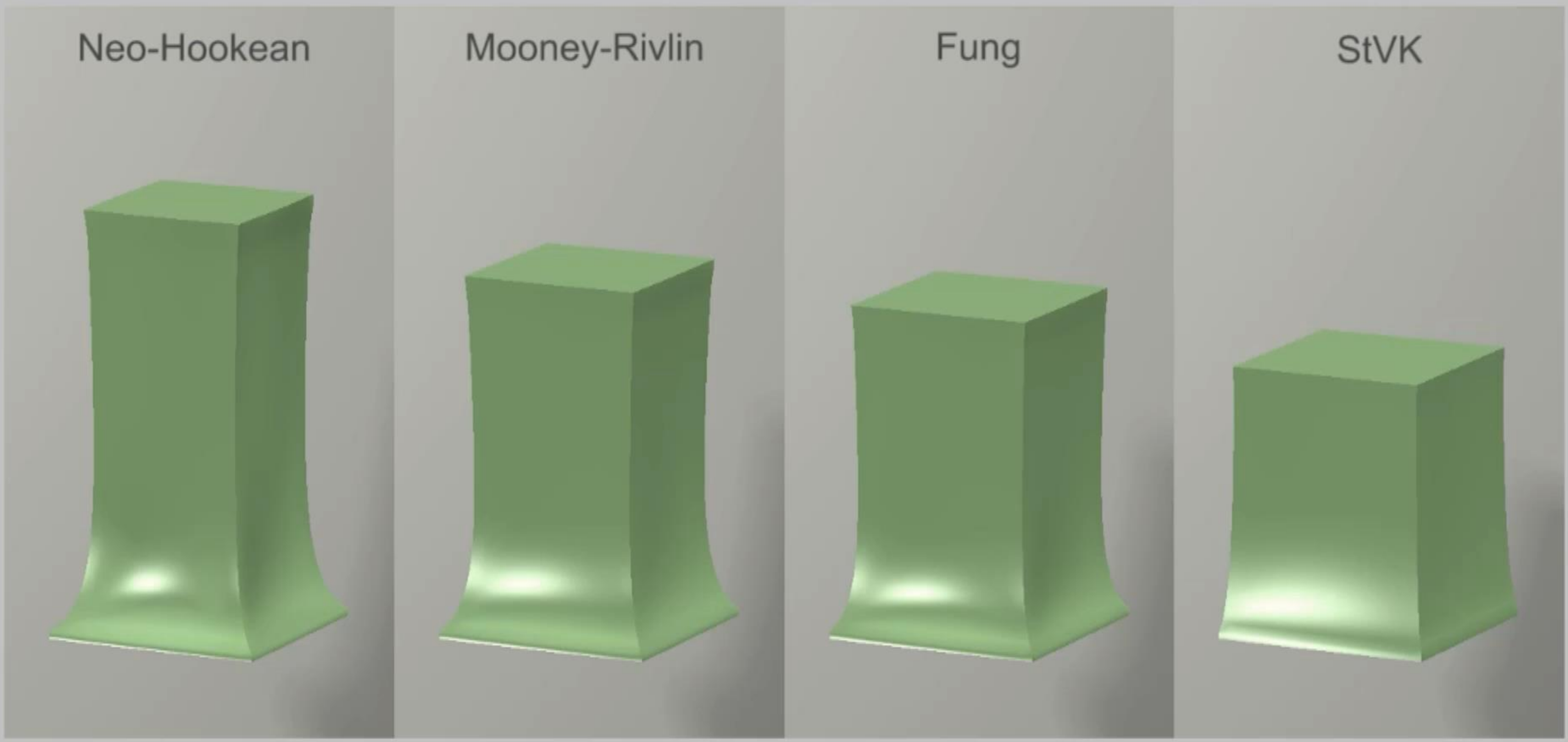
Non-linear Elasticity



G. Irving, J. Teran, and R. Fedkiw. 2004. ***Invertible finite elements for robust simulation of large deformation***. In *Proceedings of the 2004 ACM SIGGRAPH/Eurographics symposium on Computer animation* (SCA ’04)

Non-linear Elasticity

Descent Methods for Elastic Body Simulation on the GPU (SIGGRAPH Asia 2016)



Some Notes

- Constitutive models are not real materials
  - But good approximation in some sense
- We rarely use linear elasticity in graphs
  - Infinitesimal deformation is usually boring
  - Corotated linear, StVK, Neohooken are often seen in literature
- We did not talk about implicit integrator
  - Recall: it needs to compute $\frac{\partial f}{\partial x}$
  - Not very hard in fact when using linear elements

Towards Fast Simulation

Model Simplification

- Degrees of freedom of N vertices: $3N$
  - Computationally costly, aka. Slow…

- Modal analysis and model reduction

- Mesh embedding

- Parameter space approaches
  - Rig-space physics

Equation of Motion

$$M\dot{v} = f_{int} + f_{ext}$$

$$= f_e(x) + f_d(x, v) + f_{ext}$$

For linear material

$$f_e(x) = -K(x - X)$$

Rayleigh damping:

$$f_d(x, v) = -(\alpha M + \beta K)v$$

Equation of Motion

$$M\ddot{x} + D\dot{x} + Ku = f_{ext}$$

$$u = (x - X)$$

Generalized Eigenvalue Problem

Eigenvalue Problem

$$Av = \lambda v$$

$$\Downarrow$$

$$\boxed{Av = \lambda Bv}$$

Generalized Eigenvalue Problem

Generalized Eigenvalue Problem

$$\boxed{Av = \lambda Bv}$$

Similar to eigenvalue problem, we can find λ by considering

$$\det(A - \lambda B) = 0$$

Or equivalently, when B is invertible, consider the eigenvalue problem

$$B^{-1}Av = \lambda v$$

Generalized Eigenvalue Problem

$$\boxed{Av = \lambda Bv}$$

When A, B are SPD

$$A = BP \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1} = BP\Sigma P^{-1}$$

$$P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

$$v_i^T B v_j = \delta_{ij} \text{ or } P^T BP = I$$

Modal Analysis

$$M\ddot{x} + D\dot{x} + Ku = f_{ext}$$

Generalized Eigenvalue Problem

$$\boxed{Kv = \lambda M v}$$

$$K = MP \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$$

Modal Analysis

$$M\ddot{x} + D\dot{x} + Ku = f_{ext}$$

$$u = Pz$$

$$MP\ddot{z} + DP\dot{z} + KPz = f_{ext}$$

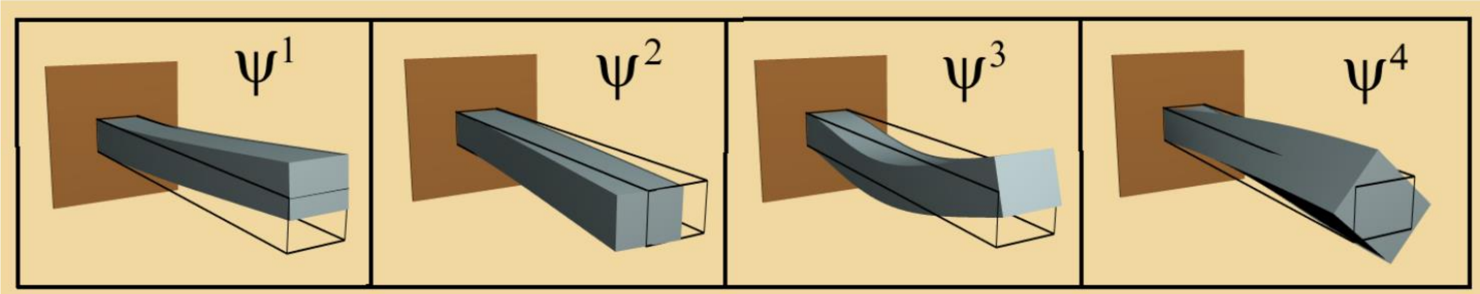
$$P^T \cdot$$

$$\ddot{z} + (\alpha I + \beta \Sigma)\dot{z} + \Sigma z = P^T f_{ext}$$

Modal Analysis

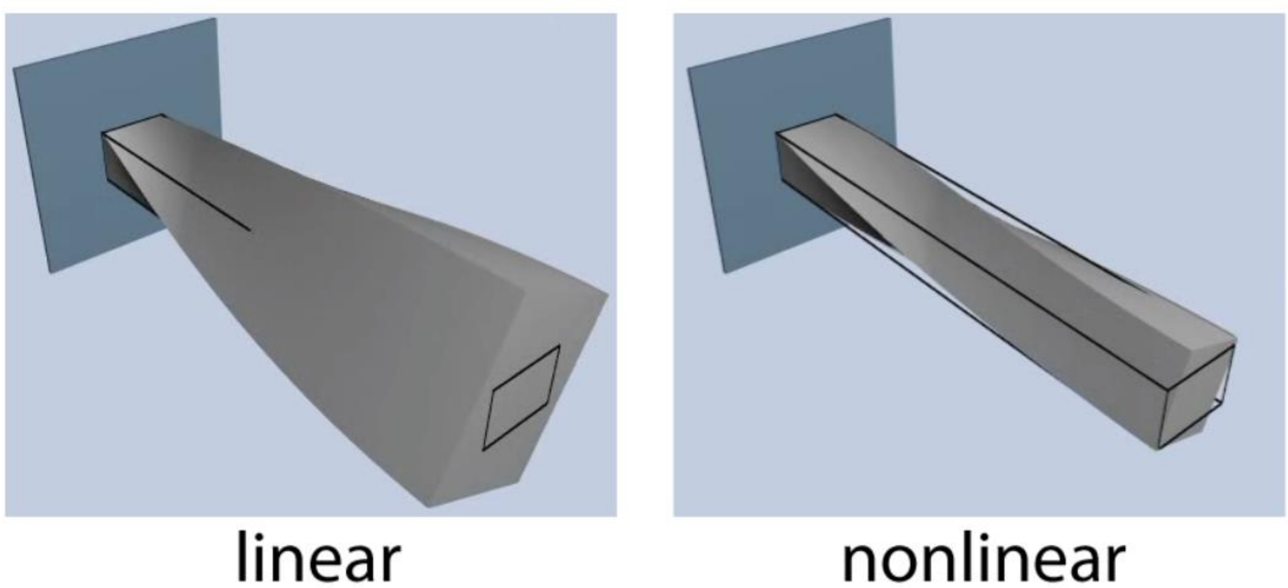
$$\ddot{z}_i + (\alpha + \beta\lambda_i)\dot{z}_i + \lambda_i z_i = v_i^T f_{ext}$$

- Decoupled/independent modes
  - Determined by the shape/structure of the object
  - Only need to consider $r \ll 3n$ modes
    - Corresponds to the largest r eigen values



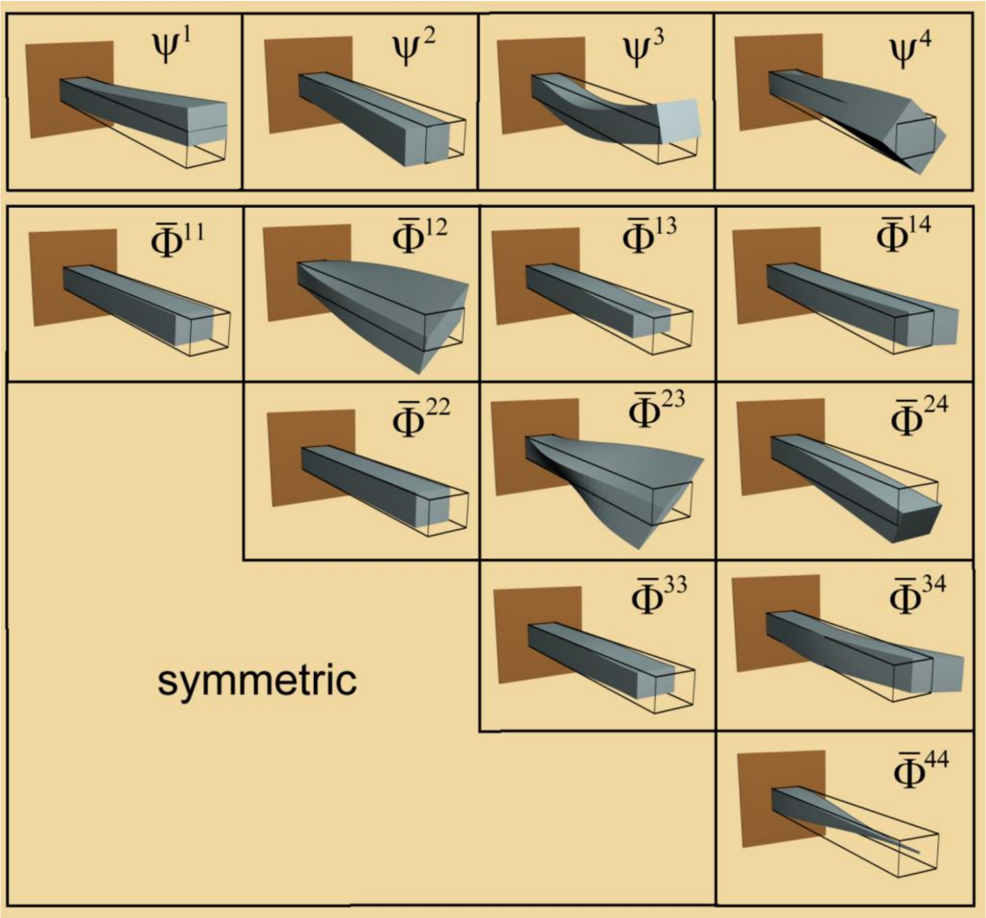
Jernej Barbič and Doug L. James. 2005. ***Real-Time subspace integration for St. Venant-Kirchhoff deformable models***. *ACM Trans. Graph.* 24, 3 (July 2005)

Extension to Non-linear Materials



Extension to Non-linear Materials

Key idea: consider the second-order Taylor expansion



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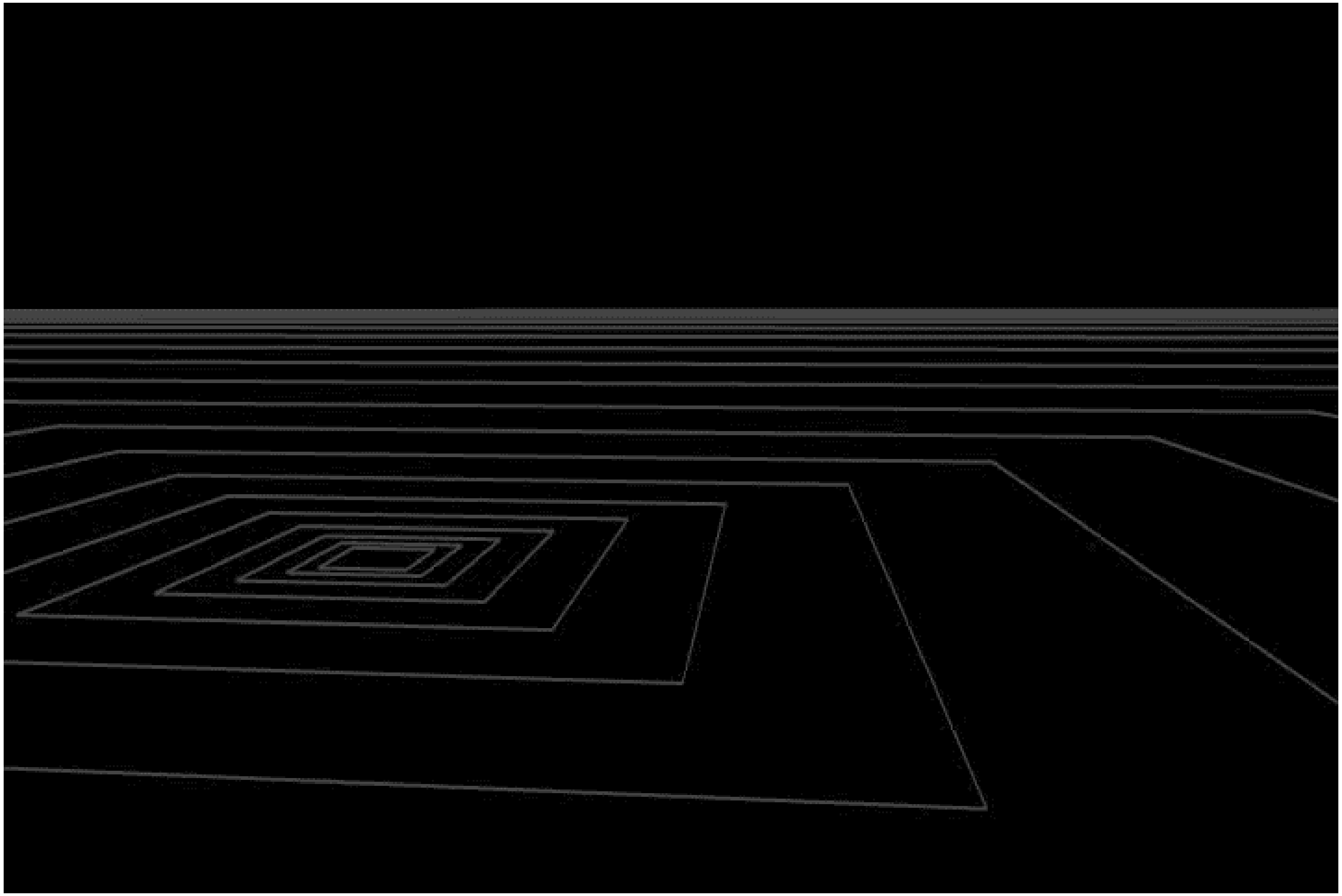
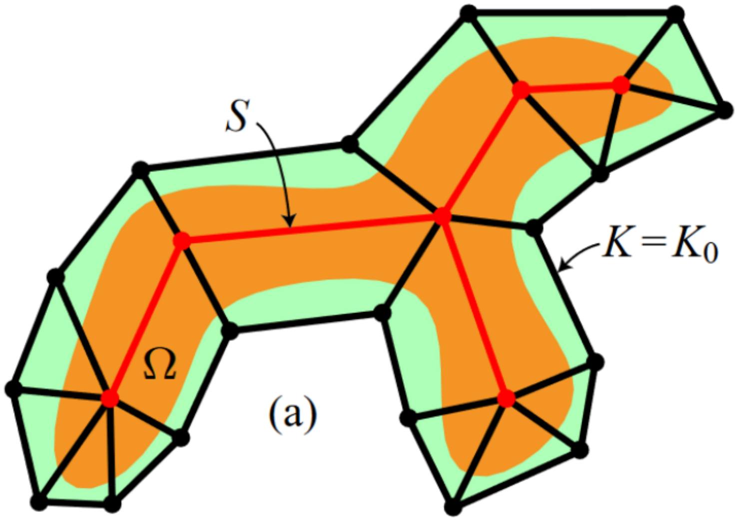
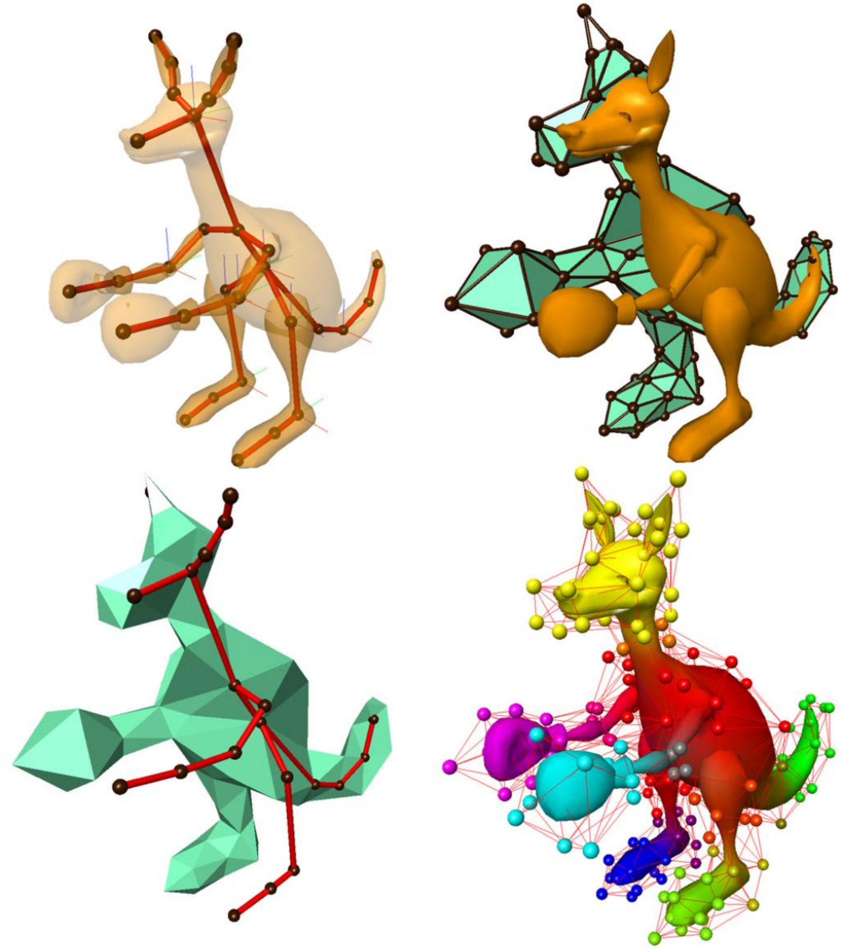
Extension to Non-linear Materials



Jernej Barbič and Doug L. James. 2005. ***Real-Time subspace integration for St. Venant-Kirchhoff deformable models***. *ACM Trans. Graph.* 24, 3 (July 2005)

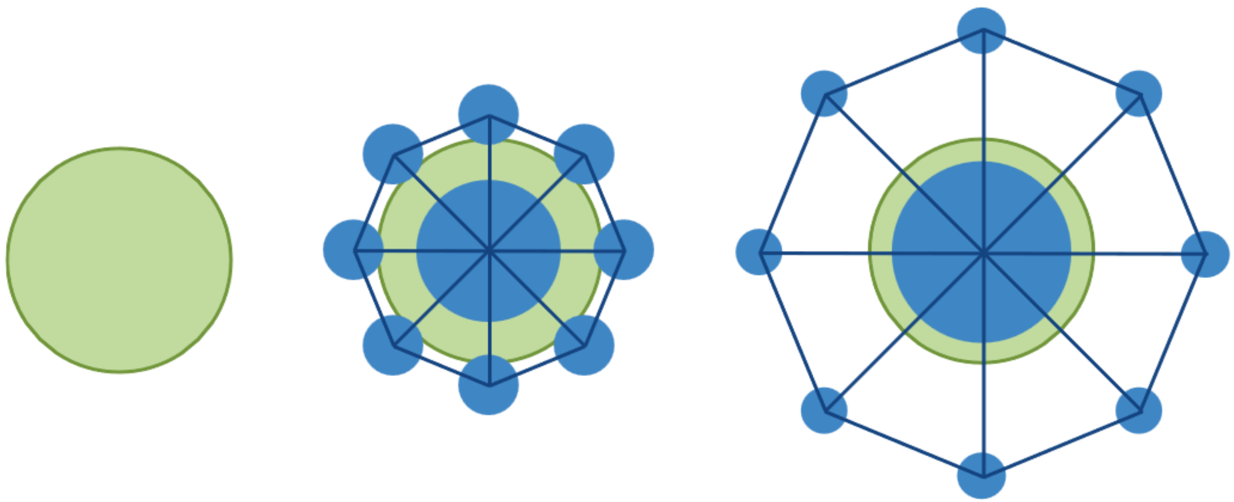
Mesh Embedding





Steve Capell, Seth Green, Brian Curless, Tom Duchamp, and Zoran Popović. 2002. ***Interactive skeleton-driven dynamic deformations***. *ACM Trans. Graph.* 21, 3 (July 2002),

Dealing with Partially Filled Elements

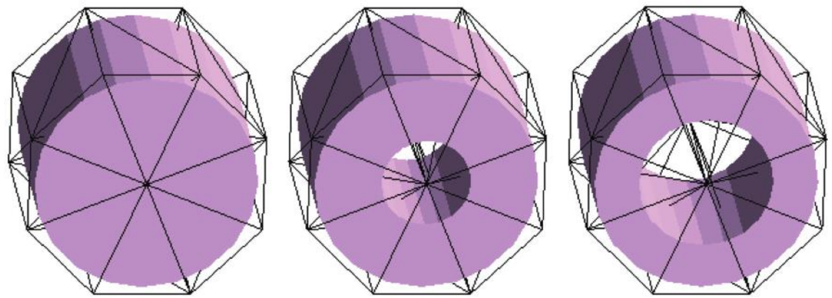


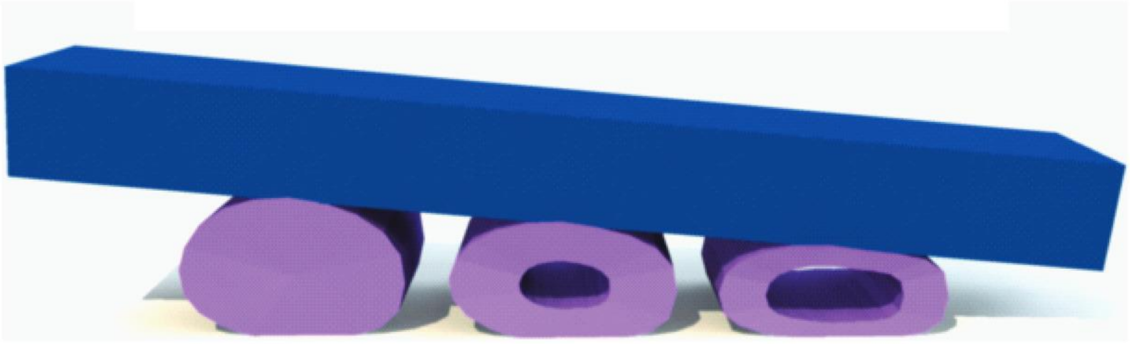
$$m_i = \int_V \rho \phi_i(x) dV$$

$\phi_i(x)$: nodal coordinates (barycentric) of x w.r.t i

Junggon Kim and Nancy S. Pollard. 2011. Fast simulation of skeleton-driven deformable body characters. *ACM Trans. Graph.* 30, 5 (October 2011),

Dealing with Partially Filled Elements



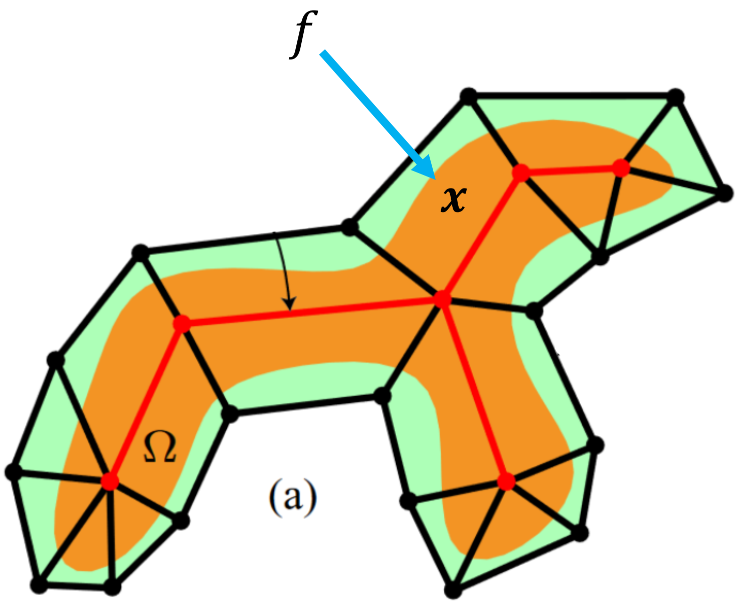


$$\tilde{f}_i = w_i f_i \qquad w_i = \frac{V_{\text{filled}}}{V}$$

Junggon Kim and Nancy S. Pollard. 2011. Fast simulation of skeleton-driven deformable body characters. *ACM Trans. Graph.* 30, 5 (October 2011),

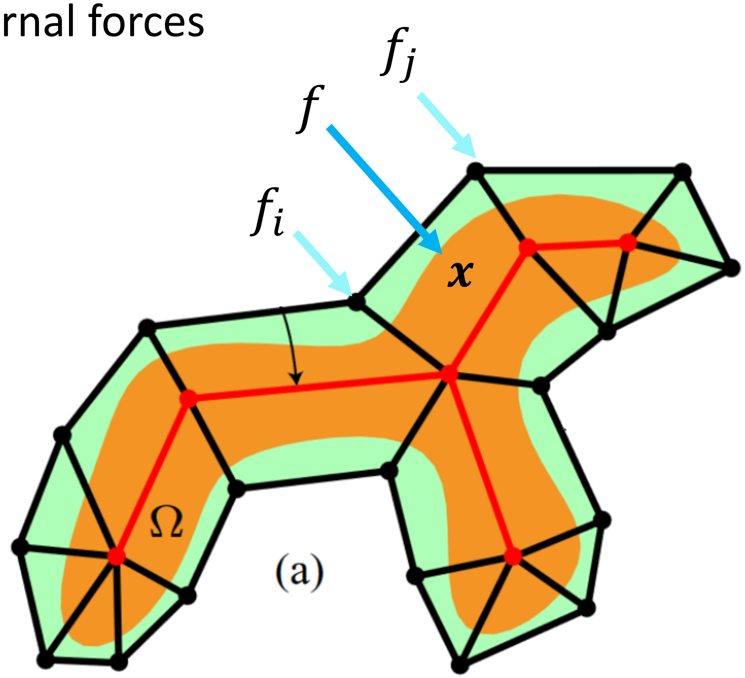
Dealing with Partially Filled Elements

Contact and external forces



Dealing with Partially Filled Elements

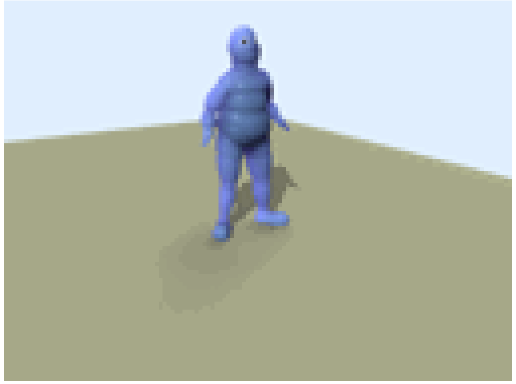
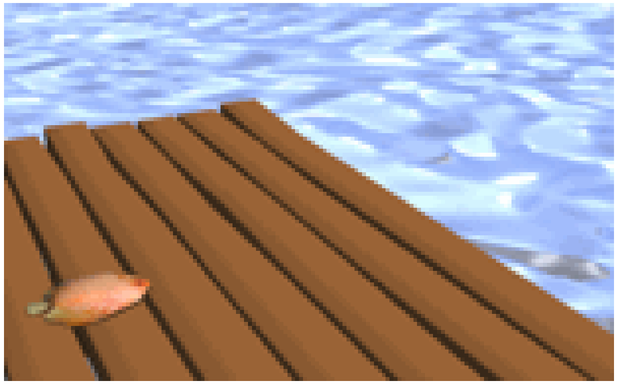
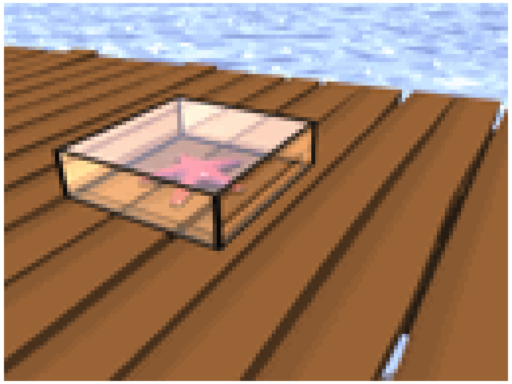
Contact and external forces



$$f_i = \phi_i f$$

$\phi_i(x)$: nodal coordinates (barycentric) of x w.r.t i

Mesh Embedding Example

Junggon Kim and Nancy S. Pollard. 2011. ***Fast simulation of skeleton-driven deformable body characters***. *ACM Trans. Graph.* 30, 5 (October 2011),

Parameter Space Approaches

$$M\dot{v} = f_{int} + f_{ext}$$

$$= f_e(x) + f_d(x, v) + f_{ext}$$

$$x = h(z)$$

$$\dim z \ll \dim x$$

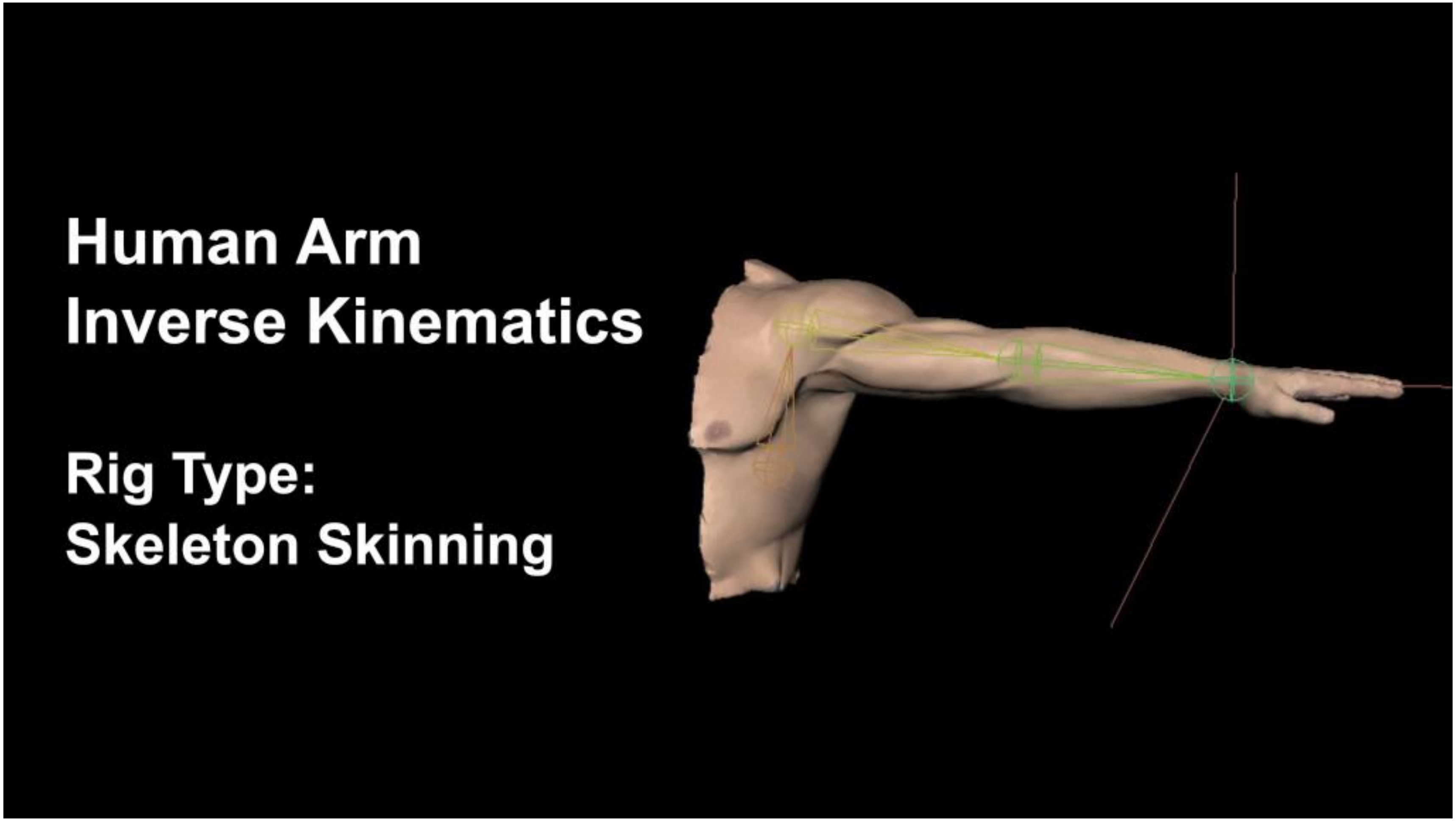
$$\widetilde{M}\ddot{z} = \widetilde{f}_e(z) + \widetilde{f}_d(z, \dot{z}) + \widetilde{f_{ext}}$$

*Hint: Lagrangian Mechanics

Rig-Space Physics

Fabian Hahn, Sebastian Martin, Bernhard Thomaszewski, Robert Sumner, Stelian Coros, and Markus Gross. 2012. ***Rig-space physics***. *ACM Trans. Graph.* 31, 4 (July 2012)

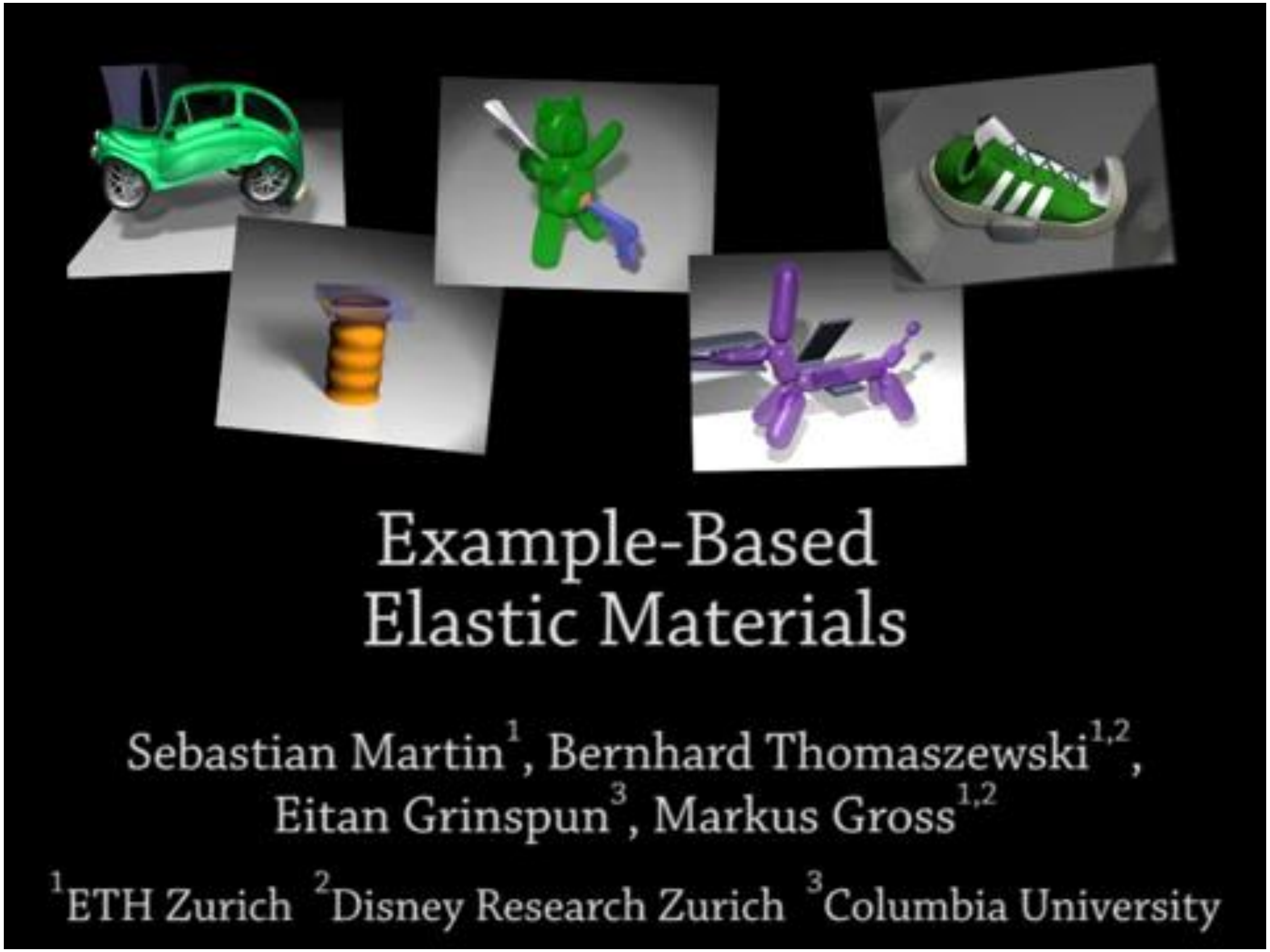
Pose-Space Subspace Dynamics



Hongyi Xu and Jernej Barbič. 2016***. Pose-space subspace dynamics***. *ACM Trans. Graph.* 35, 4 (July 2016)

Control of Deformable Solids

Control the Deformation Behavior

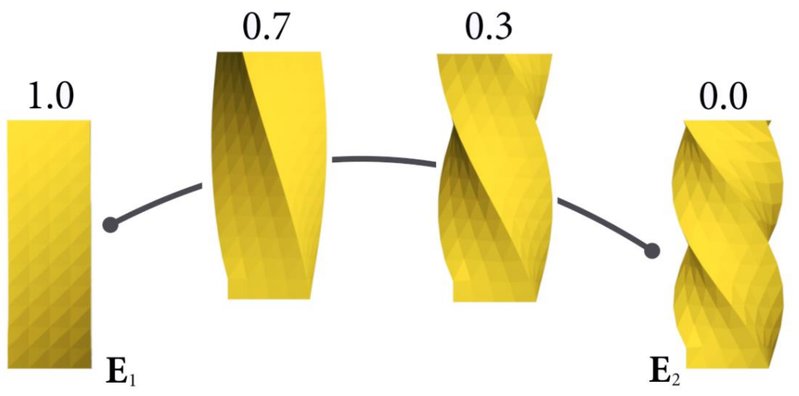


Sebastian Martin, Bernhard Thomaszewski, Eitan Grinspun, and Markus Gross. 2011. ***Example-based elastic materials***. *ACM Trans. Graph.* 30, 4 (July 2011)

Example-based Elasticity

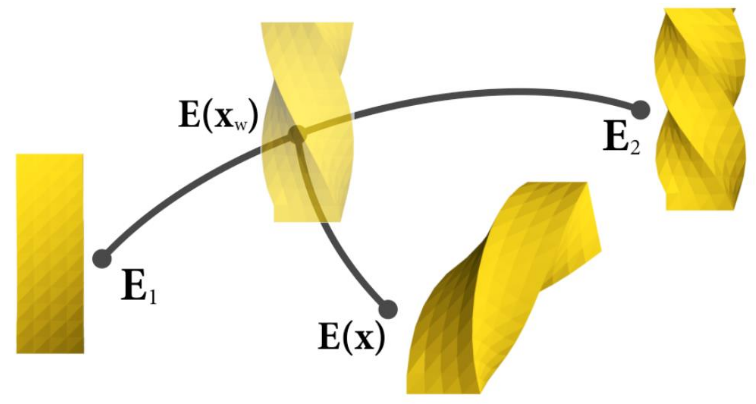
Blend space defined by examples

$$\mathbf{E}(w) = (1-w)\mathbf{E}_1 + w\mathbf{E}_2$$



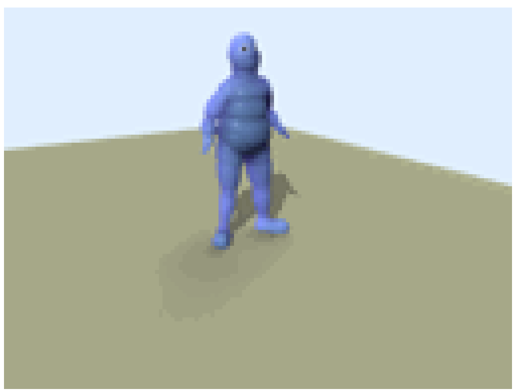
Project deformation onto the space

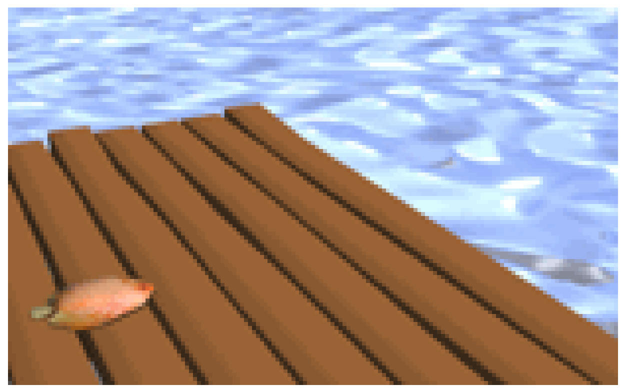
$$\min_{\mathbf{x}_w} \frac{1}{2} |\mathbf{E}(\mathbf{x}_w) - \mathbf{E}(w)|_F^2$$

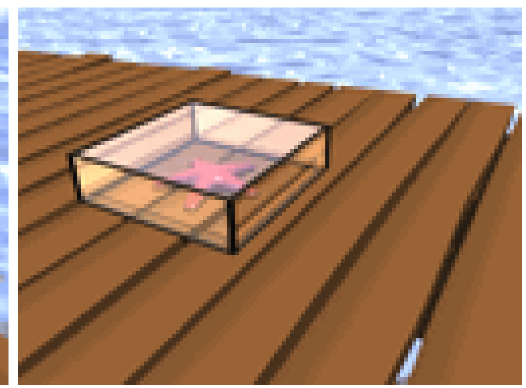


“As-Rigid-As-Possible”

Control of Deformable Characters

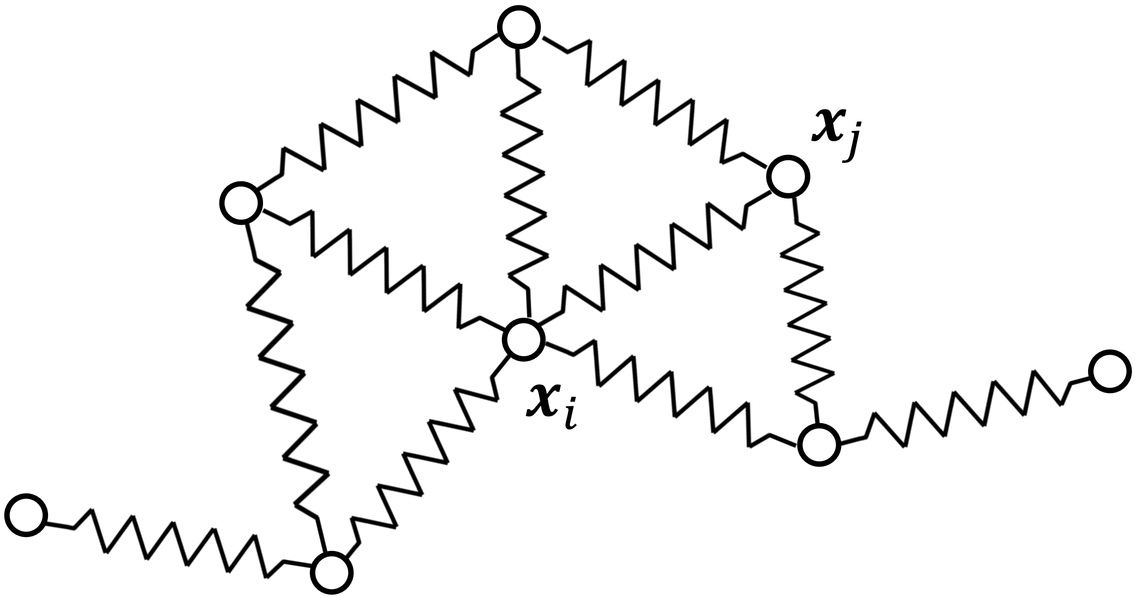






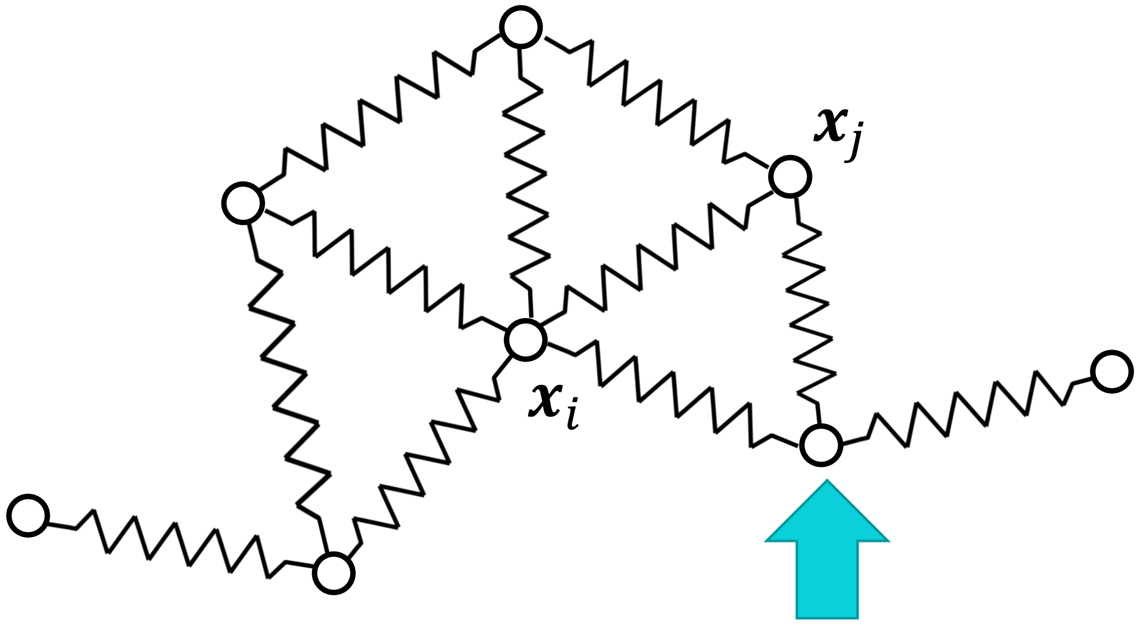
Junggon Kim and Nancy S. Pollard. 2011. ***Fast simulation of skeleton-driven deformable body characters***. *ACM Trans. Graph.* 30, 5 (October 2011),

Mass Spring System



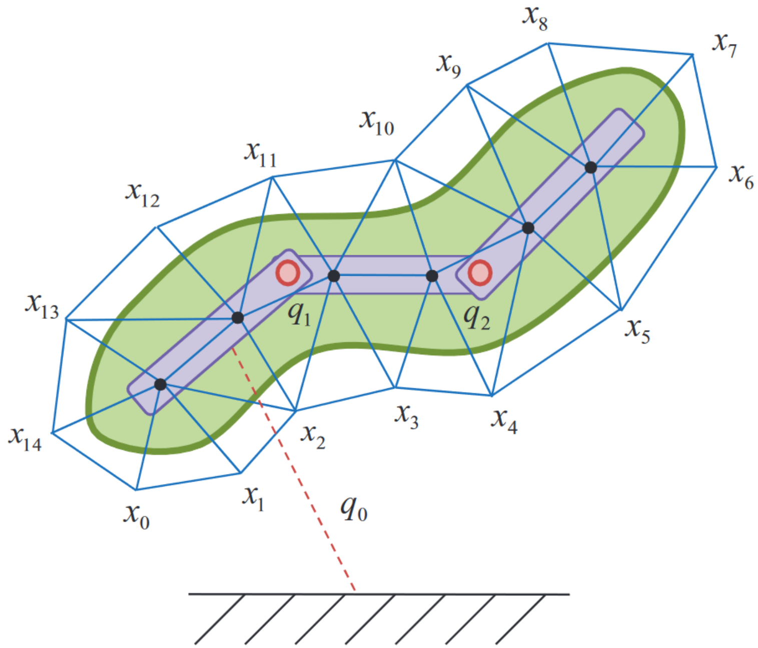
$$f_i = \sum_{j \in N(i)} -k\left(\|x_i - x_j\| - l_0\right) \frac{x_i - x_j}{\|x_i - x_j\|}$$

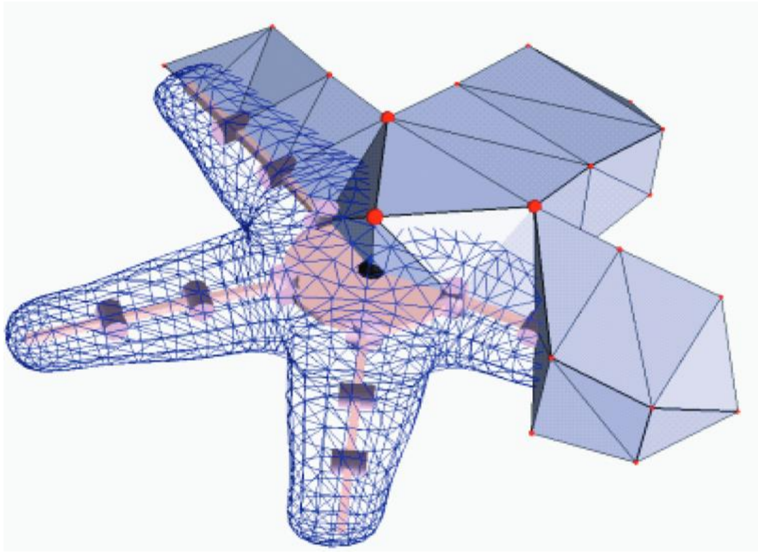
Control via Constraints/Nodal Forces



$$f_i = \sum_{j \in N(i)} -k\left(\|x_i - x_j\| - l_0\right) \frac{x_i - x_j}{\|x_i - x_j\|}$$

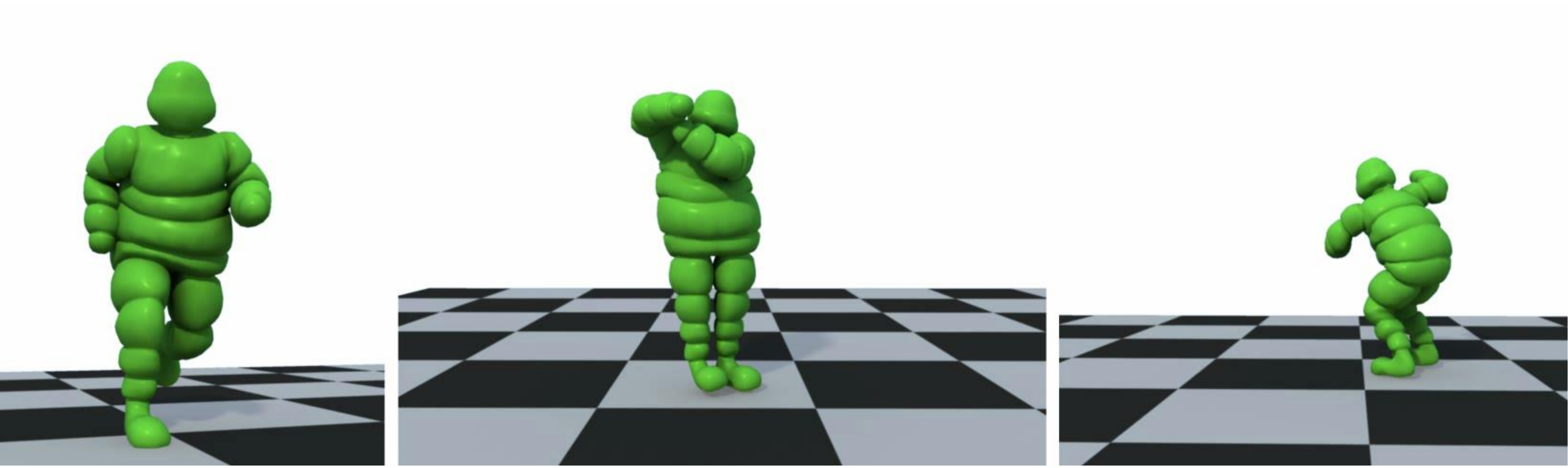
Control via Constraints/Nodal Forces



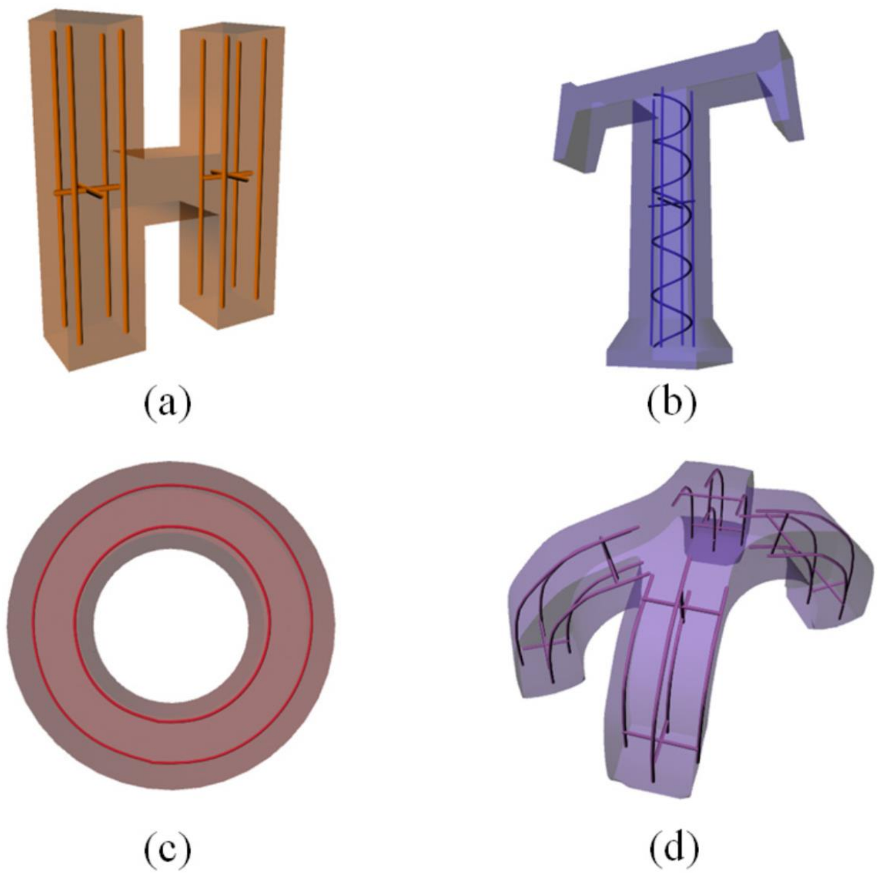


Bone-driven Nodes

Control via Constraints/Nodal Forces



Control via Constraints/Nodal Forces



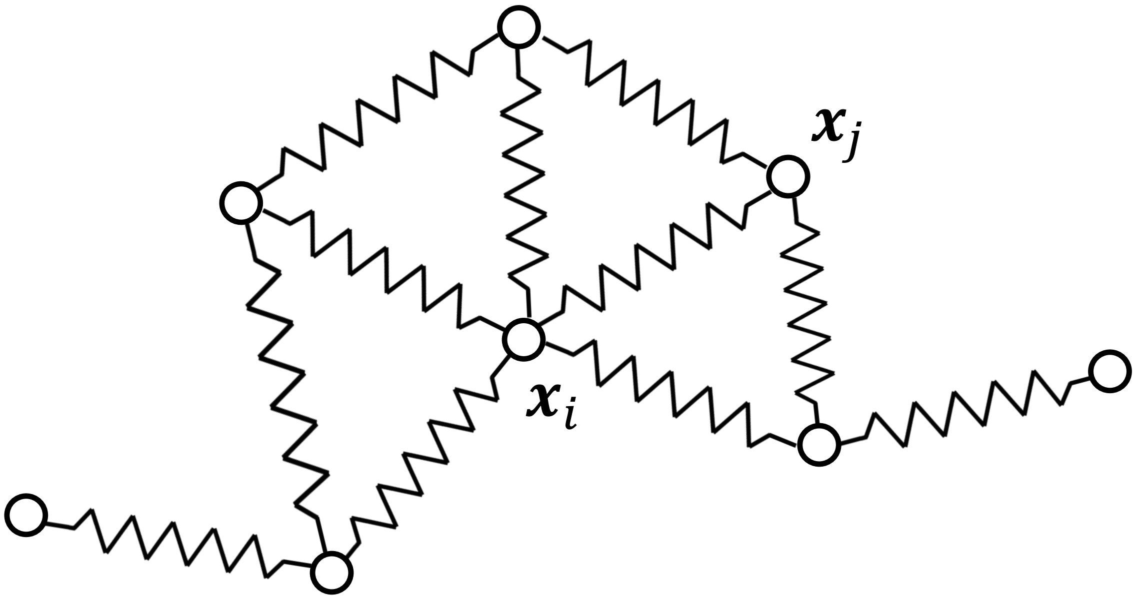
Fiber-driven Nodes

Jie Tan, Greg Turk, and C. Karen Liu. 2012. ***Soft body locomotion***. *ACM Trans. Graph.*

Control via Constraints/Nodal Forces

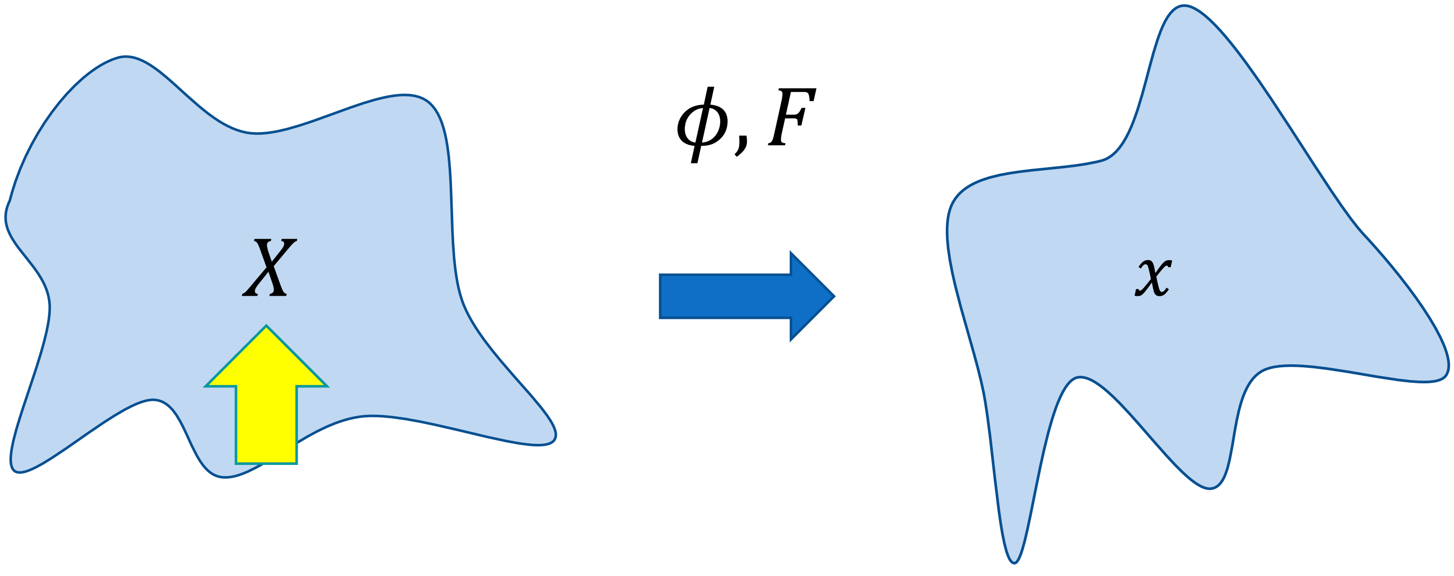
Jie Tan, Greg Turk, and C. Karen Liu. 2012. ***Soft body locomotion***. *ACM Trans. Graph.*

Control via Rest Length

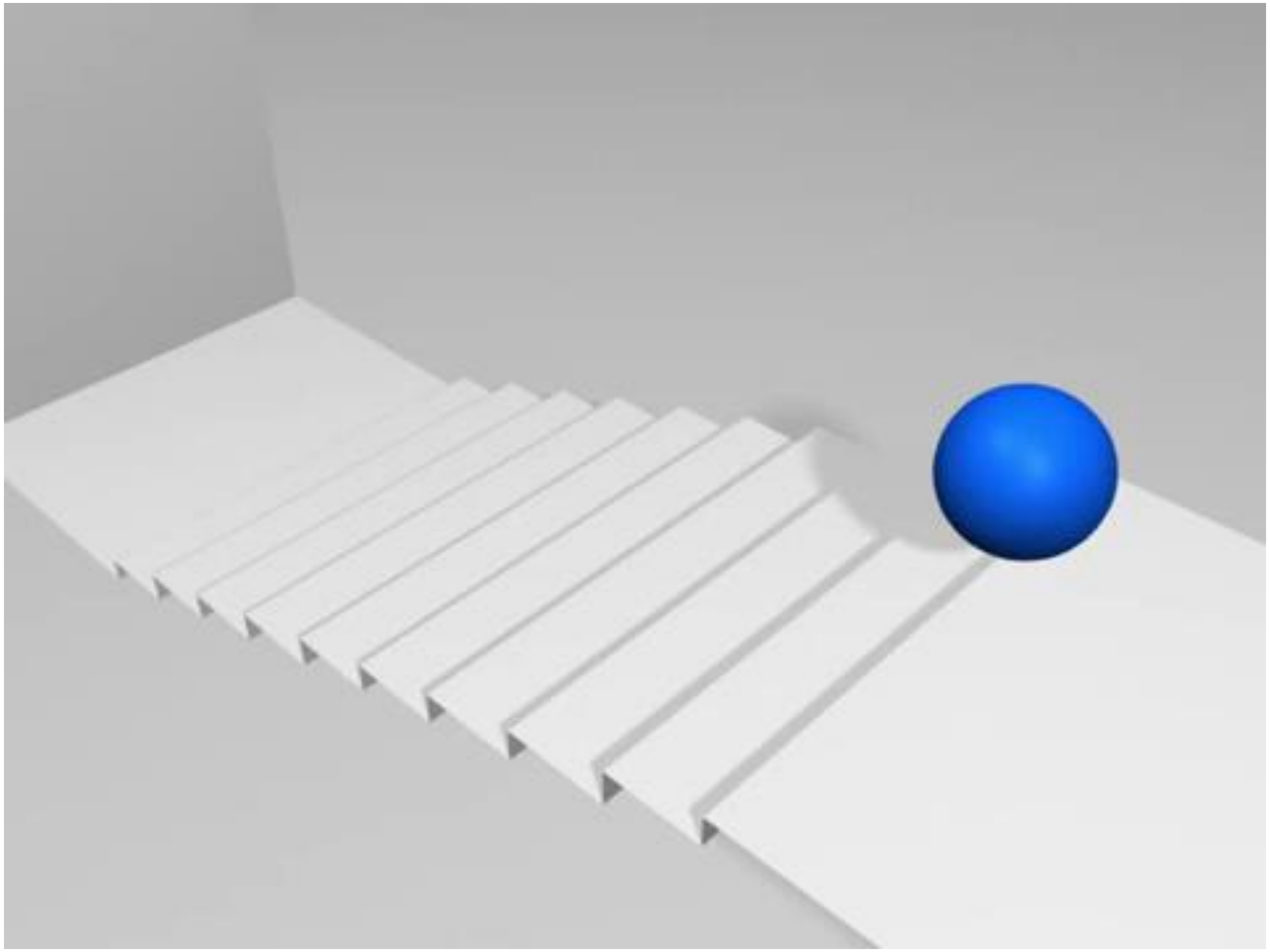


$$f_i = \sum_{j \in N(i)} -k\left(\|x_i - x_j\| - l_0\right)\frac{x_i - x_j}{\|x_i - x_j\|}$$

Control via Rest Configuration



Control via Rest Configuration



Stelian Coros, Sebastian Martin, Bernhard Thomaszewski, Christian Schumacher, Robert Sumner, and Markus Gross. 2012. ***Deformable objects alive!*** *ACM Trans. Graph.*

Questions?