数学基础
向量、矩阵、变换

北京大学 前沿计算研究中心

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本节主要内容

- CG/CGI的数学基础
- 线性代数回顾
  - 三维向量与向量运算
  - 矩阵与矩阵运算
  - 坐标系与坐标变换
  - 三维旋转与表示

CG/CGI的数学基础

- 线性代数
- 微积分
- 优化
- 常微分方程
- 偏微分方程
- 数值计算
- 概率与统计
- 随机过程

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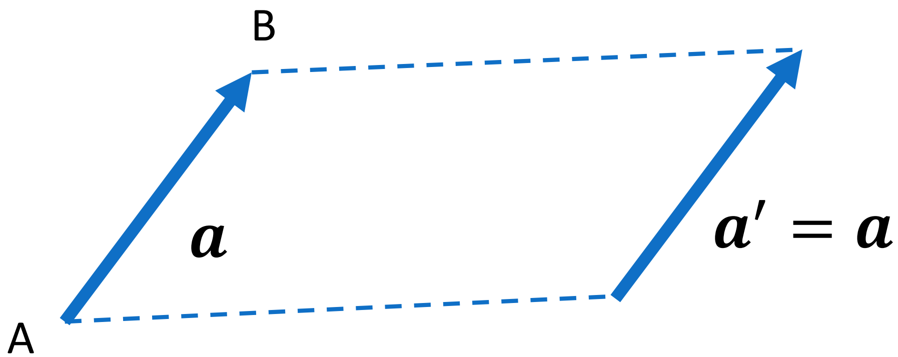
向量

Vector

向量

- 具有大小（长度）和方向的量



表示位置、方向、速度等

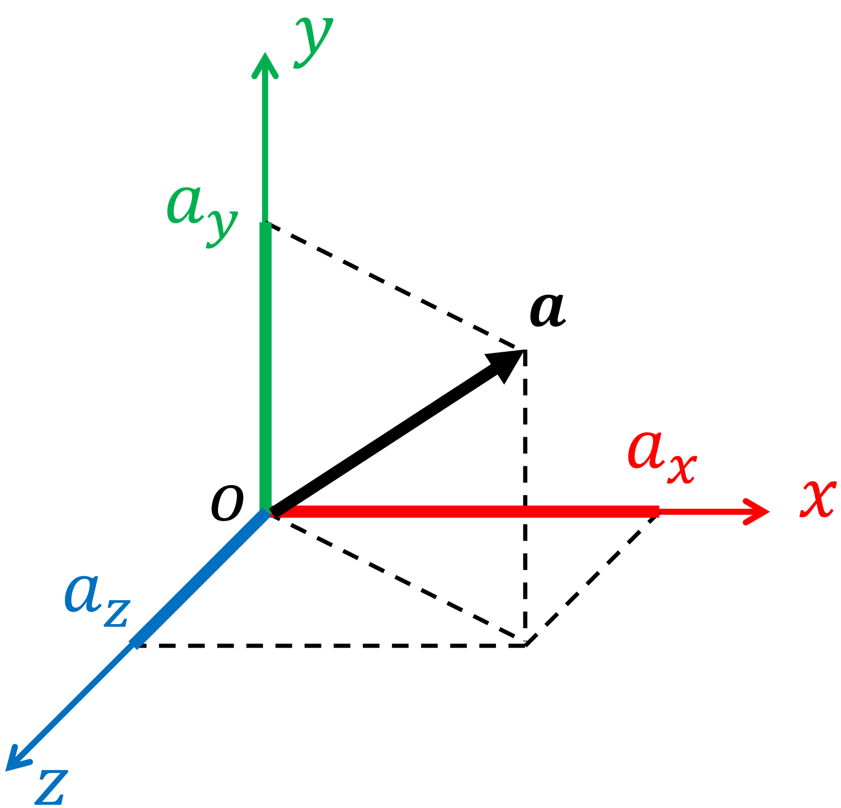
向量长度： $\|\boldsymbol{a}\|$

单位向量：长度为1的向量

$$\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

向量的表示

- 笛卡尔坐标系



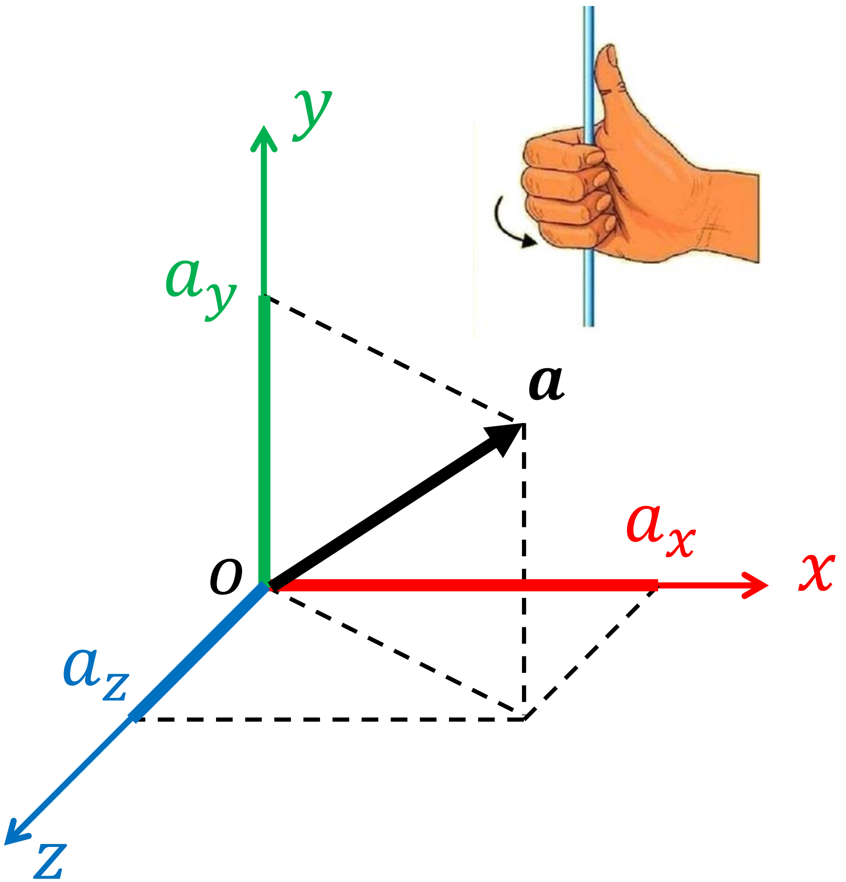
$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

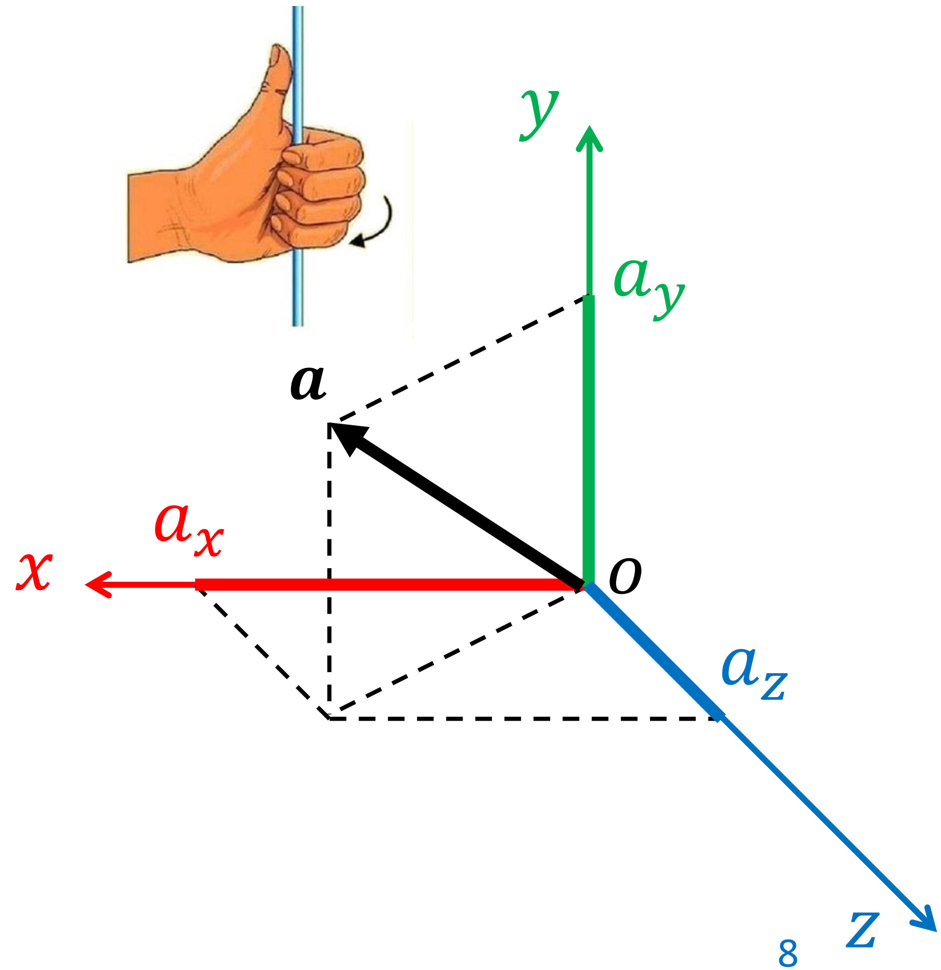
$$\|\boldsymbol{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$\boldsymbol{o} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

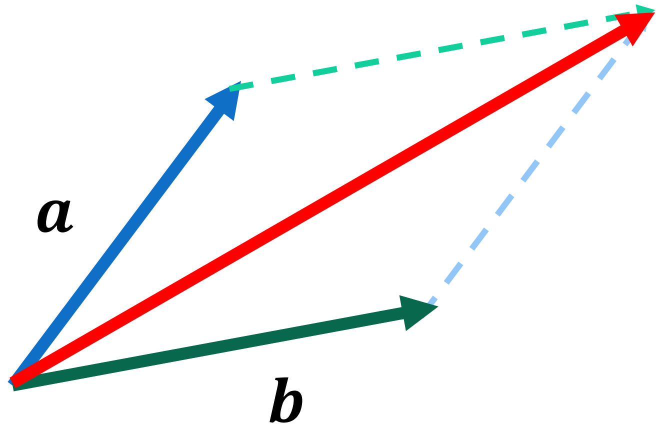
向量的表示

- 右手坐标系与左手坐标系





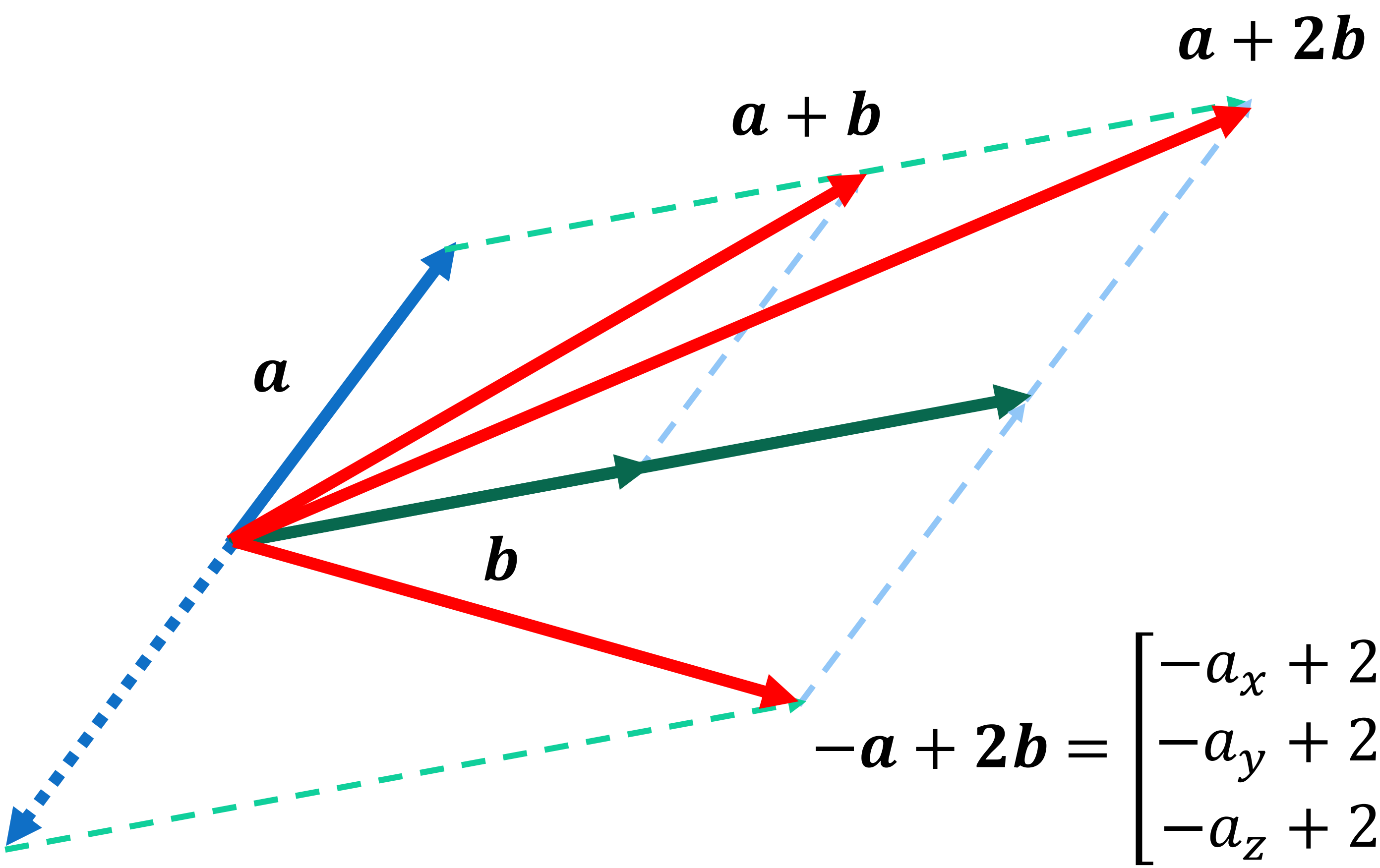
向量的数学运算



$$\boldsymbol{a}+\boldsymbol{b}=\begin{bmatrix} a_x+b_x \\ a_y+b_y \\ a_z+b_z \end{bmatrix}$$

向量的数学运算

$$\boldsymbol{a}+\boldsymbol{2b}=\begin{bmatrix} a_x+2b_x \\ a_y+2b_y \\ a_z+2b_z \end{bmatrix}$$

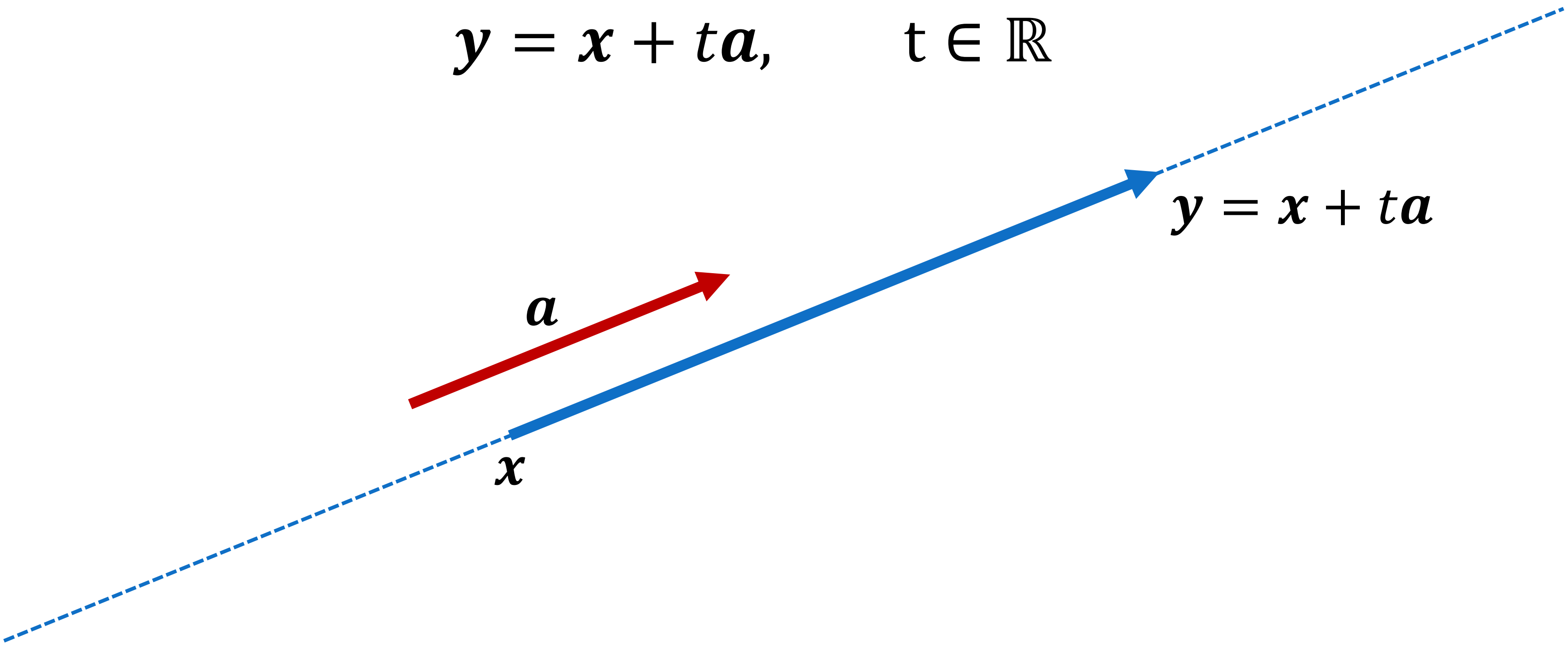


$$-\boldsymbol{a}+\boldsymbol{2b}=\begin{bmatrix} -a_x+2b_x \\ -a_y+2b_y \\ -a_z+2b_z \end{bmatrix}$$

向量的数学运算

- 直线的表示
  - 已知线上一点 $\boldsymbol{x}$，方向 $\boldsymbol{a}$，则线上任意一点 $\boldsymbol{y}$ 可表示为

$$\boldsymbol{y} = \boldsymbol{x} + t\boldsymbol{a}, \qquad \mathrm{t} \in \mathbb{R}$$



向量的点乘

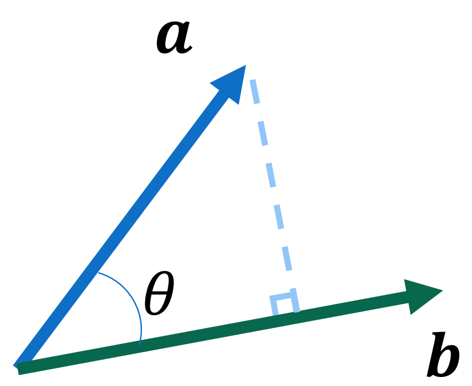
- 内积、标量积

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = a_x a_x + a_y a_y + a_z a_z = \|a\|^2$

向量的点乘的几何含义

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z = \boldsymbol{a}^T \boldsymbol{b}$$



$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\theta$$

向量的点乘的几何含义

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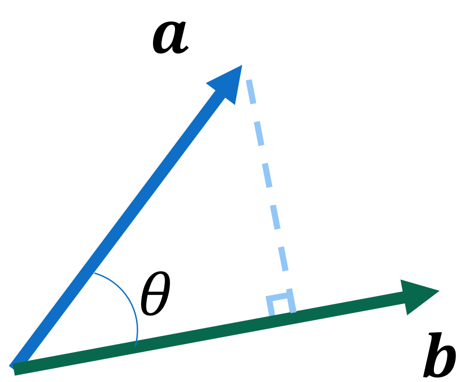


$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\theta$$

$$\theta = \arccos \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \|\boldsymbol{b}\|}$$

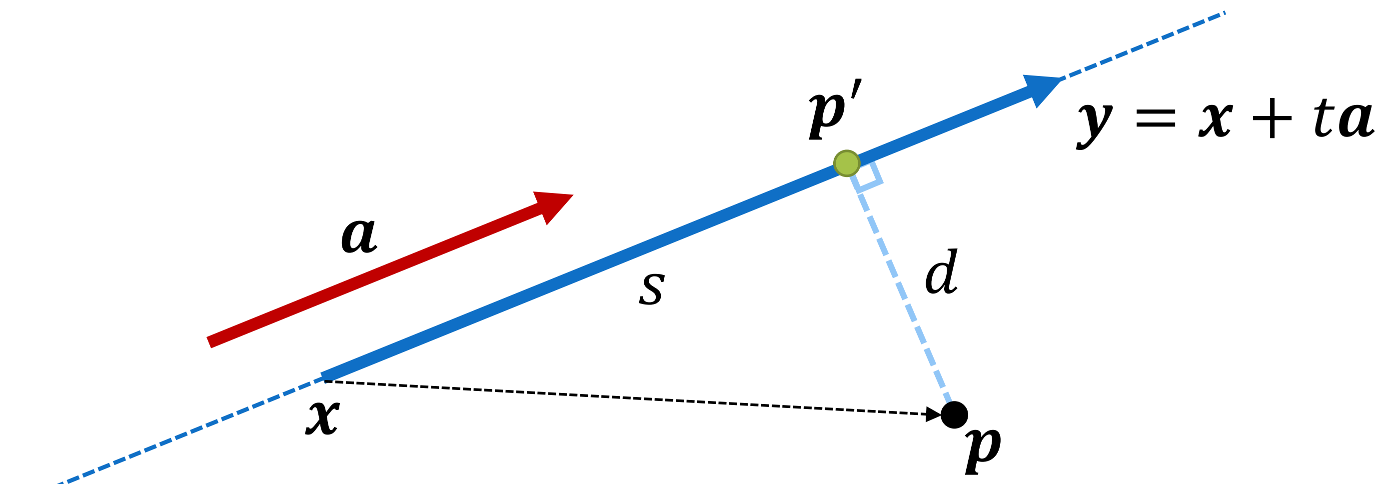
$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$

$\Rightarrow \cos\theta = 0 \Rightarrow \boldsymbol{\theta} = \boldsymbol{90}°$

$\Rightarrow \boldsymbol{a}, \boldsymbol{b}$ 垂直/正交

向量的点乘

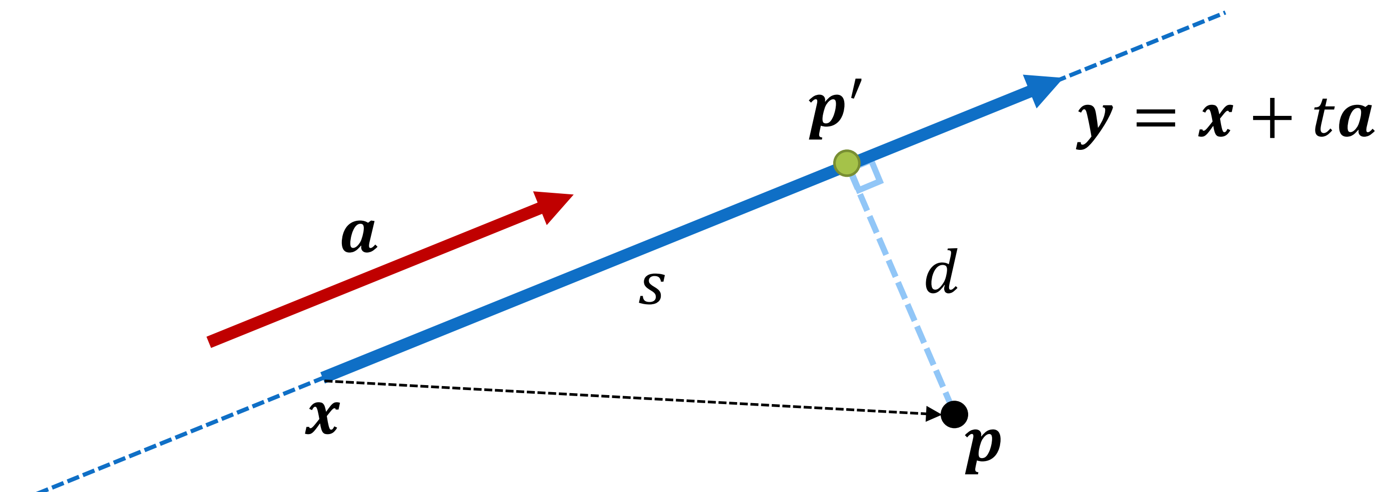
- 点乘的应用：点在线上的投影 - 最近点



$$s = (\boldsymbol{p} - \boldsymbol{x}) \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

向量的点乘

- 点乘的应用：点在线上的投影 - 最近点

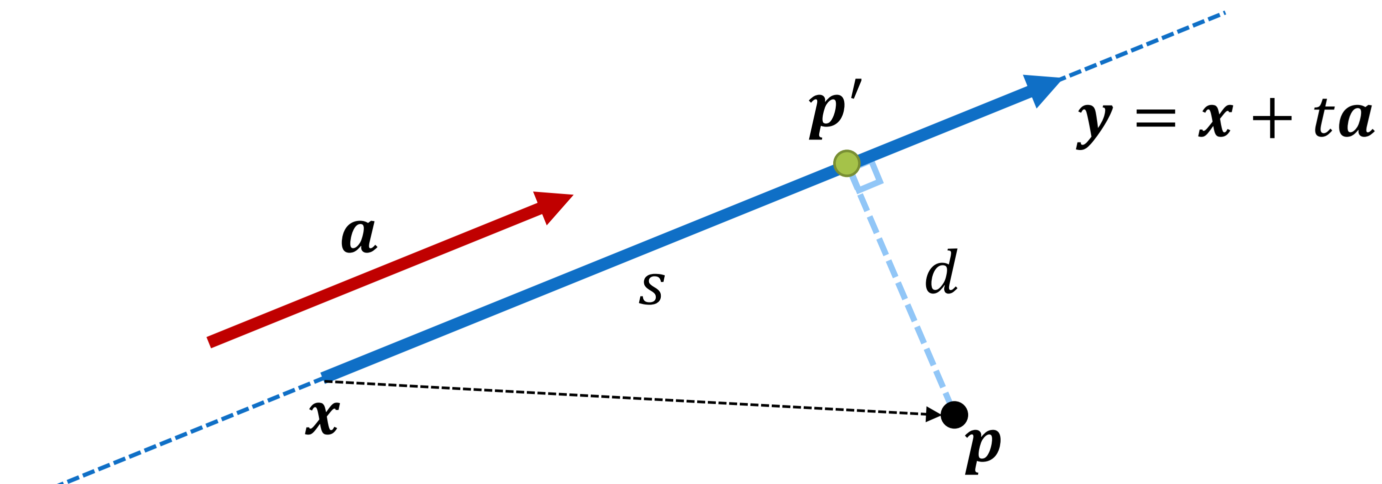


$$s = (\boldsymbol{p} - \boldsymbol{x}) \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

$$\boldsymbol{p}' = \boldsymbol{x} + s\boldsymbol{a}$$

向量的点乘

- 点乘的应用：点在线上的投影 - 最近点



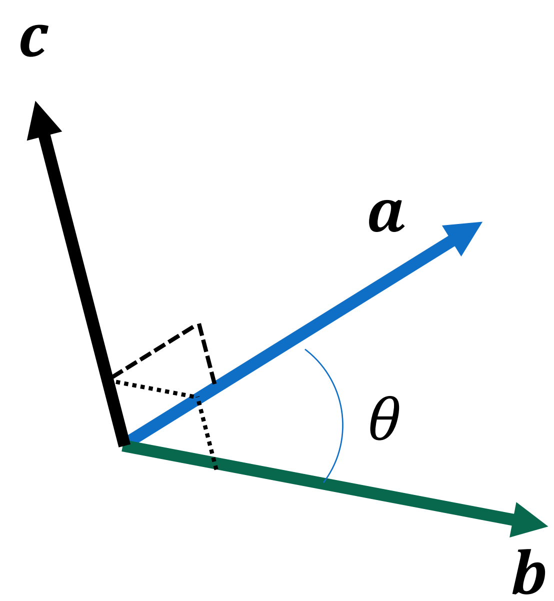
$$s = (\boldsymbol{p} - \boldsymbol{x}) \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

$$\boldsymbol{p}' = \boldsymbol{x} + s\boldsymbol{a}$$

$$d = ?$$

向量的叉乘

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$



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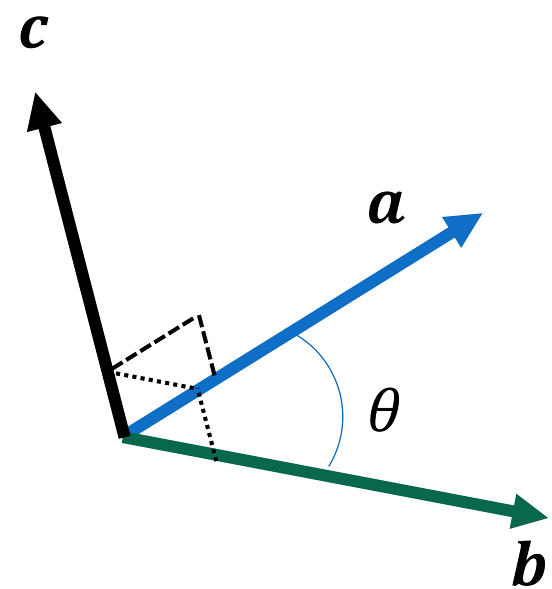
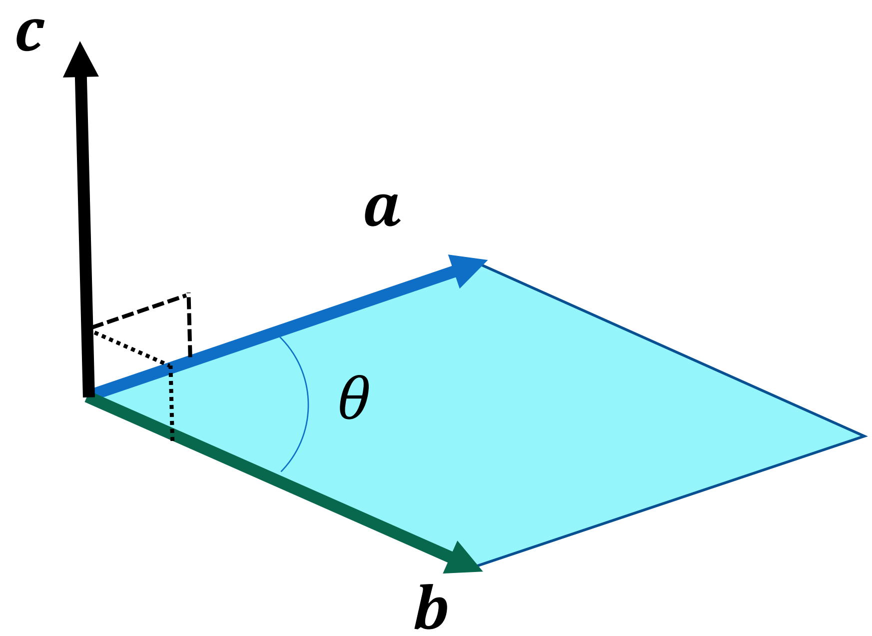


- $\boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{c} \cdot \boldsymbol{b} = \boldsymbol{0}$
  - 即 $\boldsymbol{c}$ 同时与 $\boldsymbol{a}$，$\boldsymbol{b}$ 垂直
- $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
- $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{d}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{d}$

向量叉乘的几何意义

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

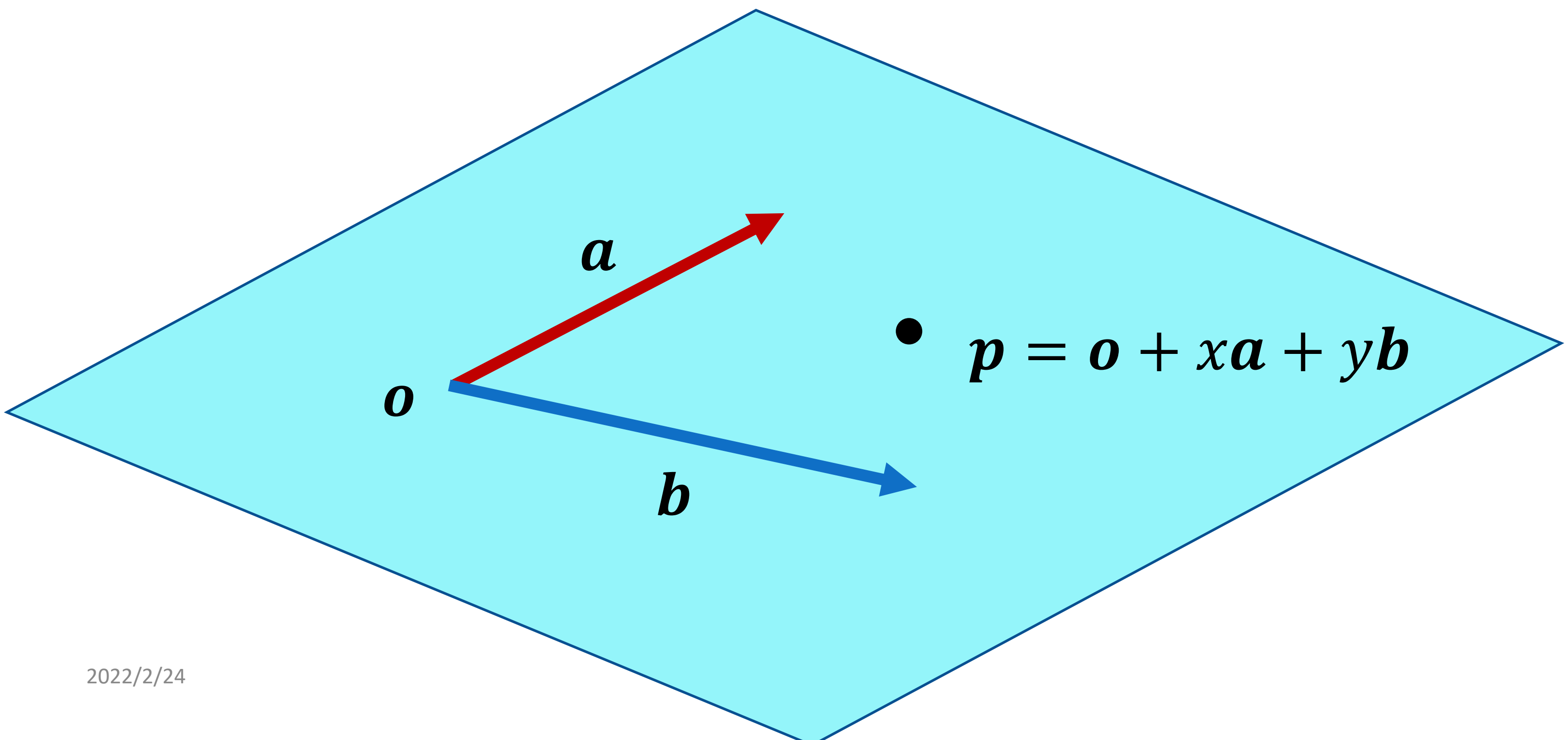


- $\|\boldsymbol{c}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\sin\theta$
- 平行四边形的面积
- $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0} \Rightarrow \boldsymbol{a}, \boldsymbol{b}$ 共线

平面表示

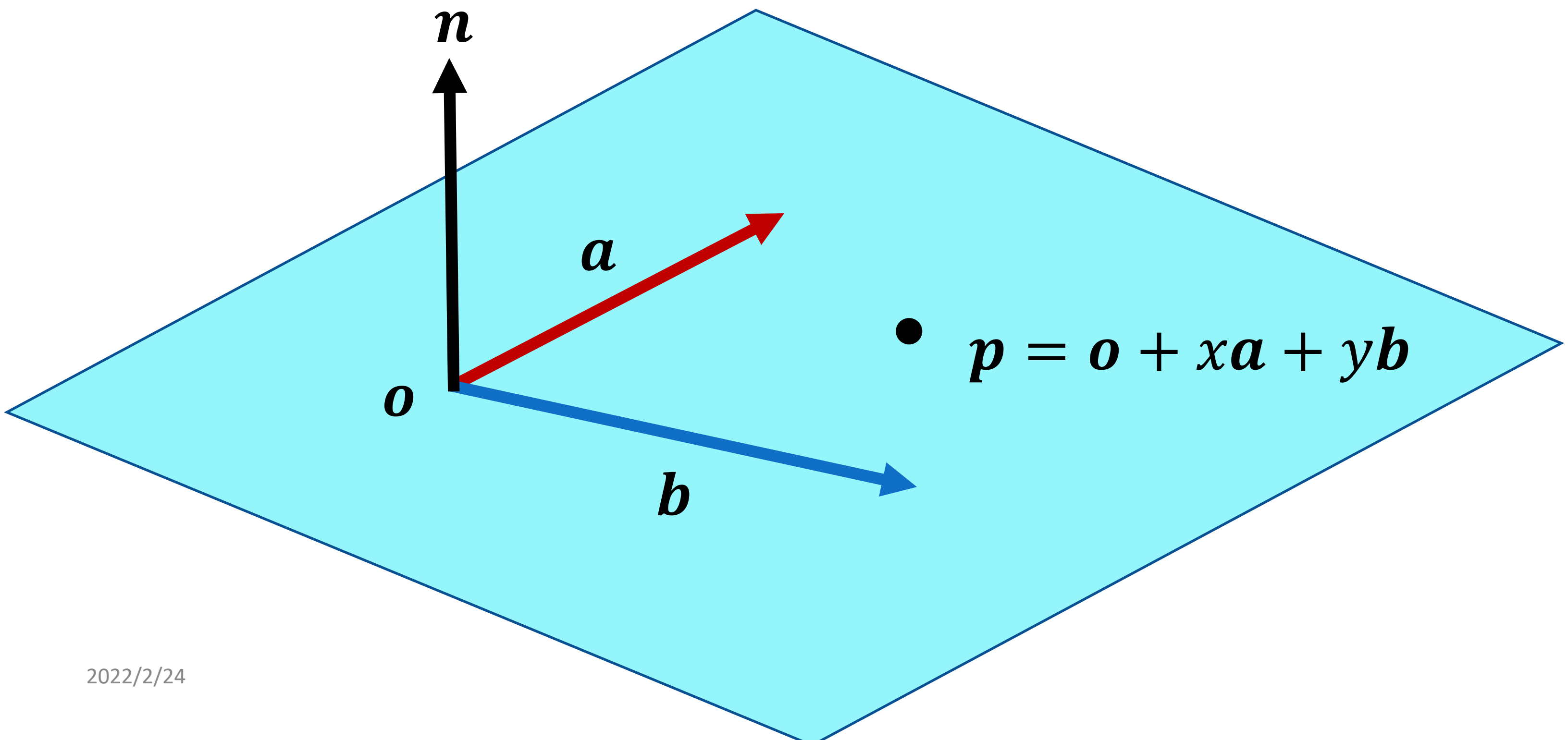
- 给出平面上一点 $\boldsymbol{o}$，以及两个不共线的向量 $\boldsymbol{a}, \boldsymbol{b}$ 则平面上任意一点 $\boldsymbol{p}$ 可表示为

$$\boldsymbol{p} = \boldsymbol{o} + x\boldsymbol{a} + y\boldsymbol{b}, \qquad x, y \in \mathbb{R}$$



平面表示

- 平面法向 $\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b} / \|\boldsymbol{a} \times \boldsymbol{b}\|$
- $\boldsymbol{n} \cdot (\boldsymbol{p} - \boldsymbol{o}) = 0,\ \forall \boldsymbol{p}$



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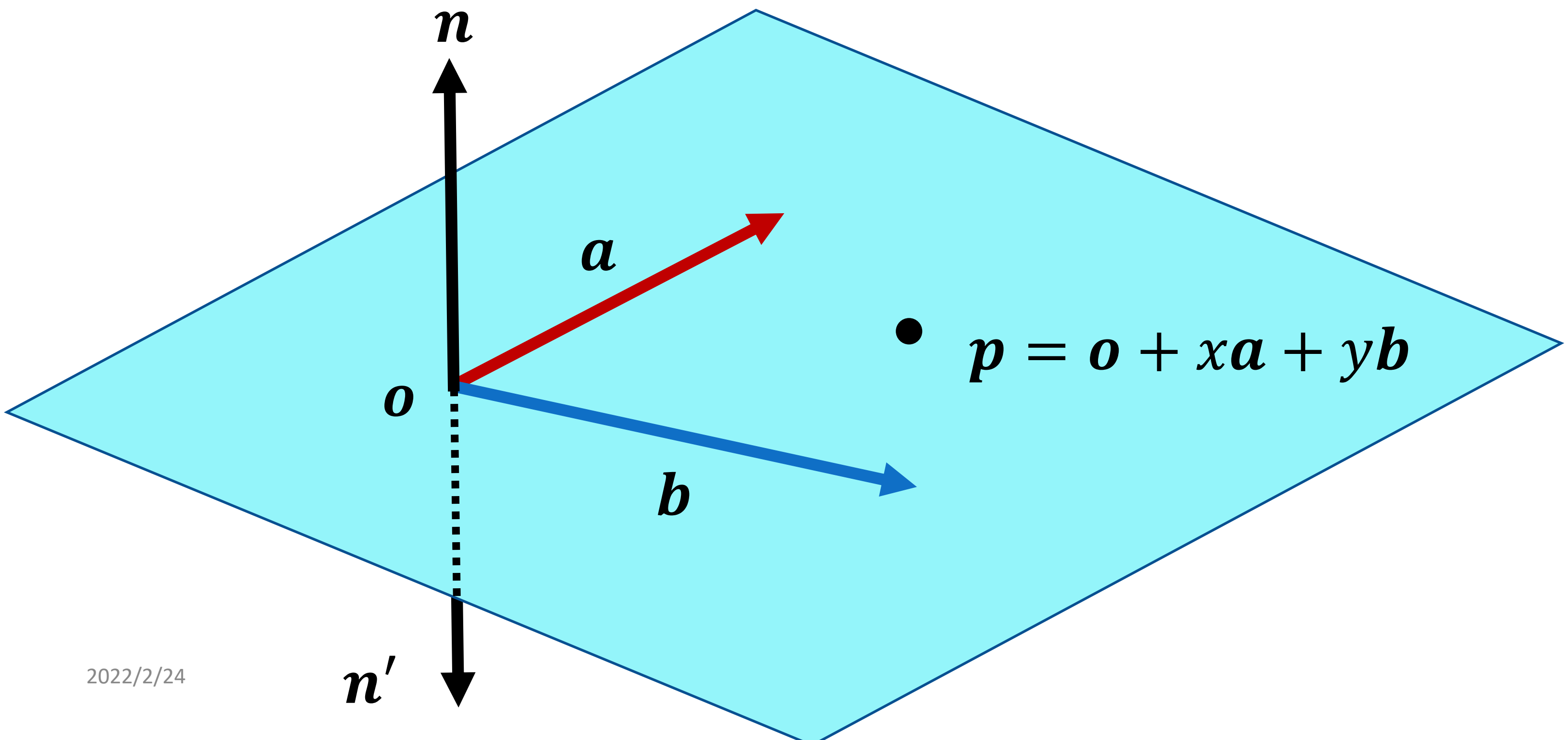
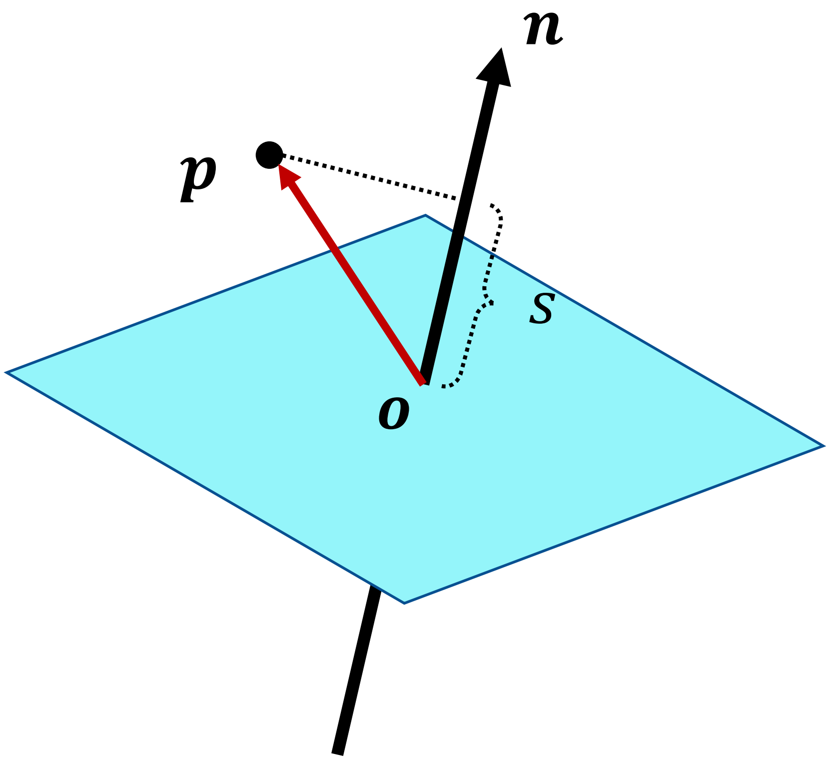


平面表示

- 判断空间一点 $\boldsymbol{p}$ 与平面的关系



$$s = \boldsymbol{n} \cdot (\boldsymbol{p} - \boldsymbol{o})$$

$$= \begin{cases} s > 0,\ \boldsymbol{p}\text{在平面上方} \\ s = 0,\ \boldsymbol{p}\text{在平面中} \\ s < 0,\ \boldsymbol{p}\text{在平面下方} \end{cases}$$

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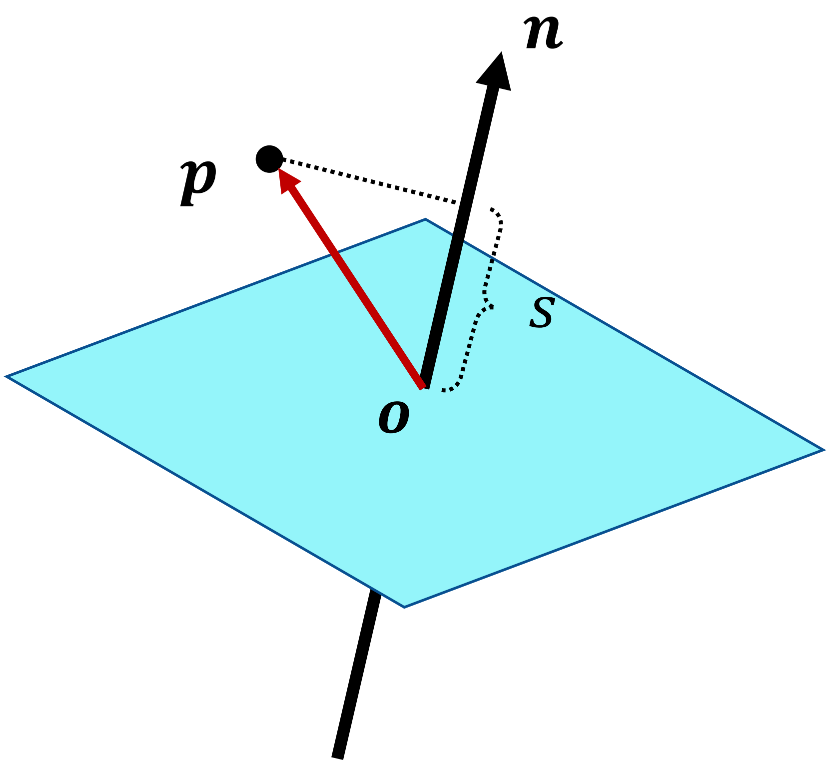

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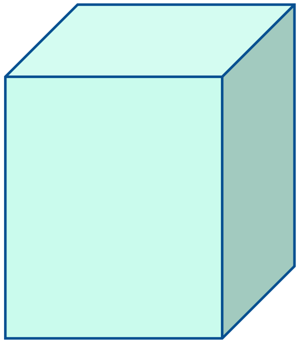
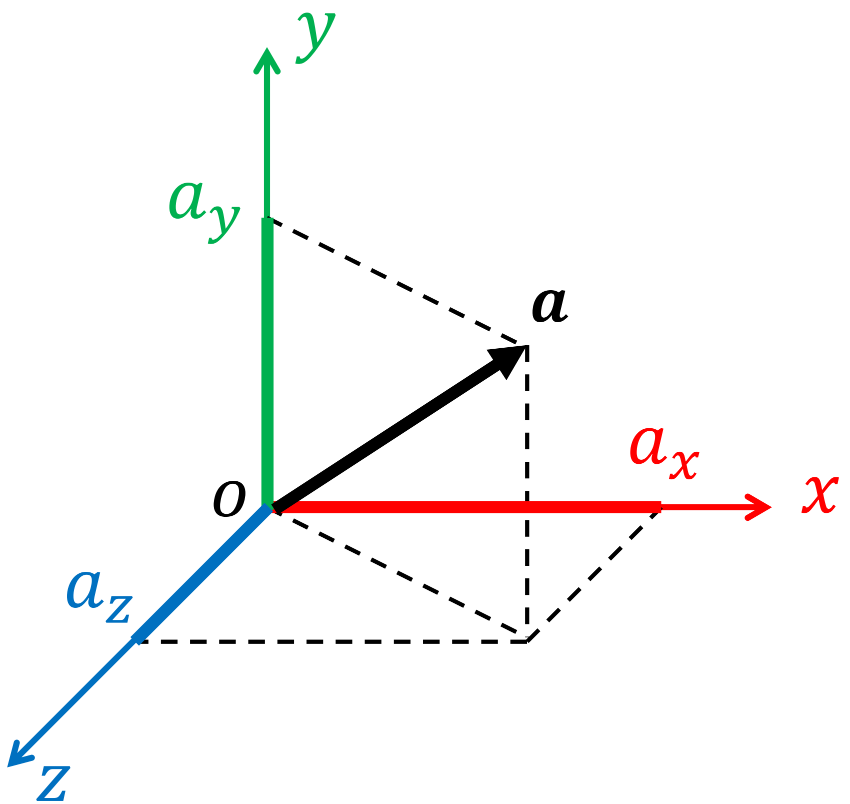


如何判断一点是否在立方体内部?

直角坐标系与正交基

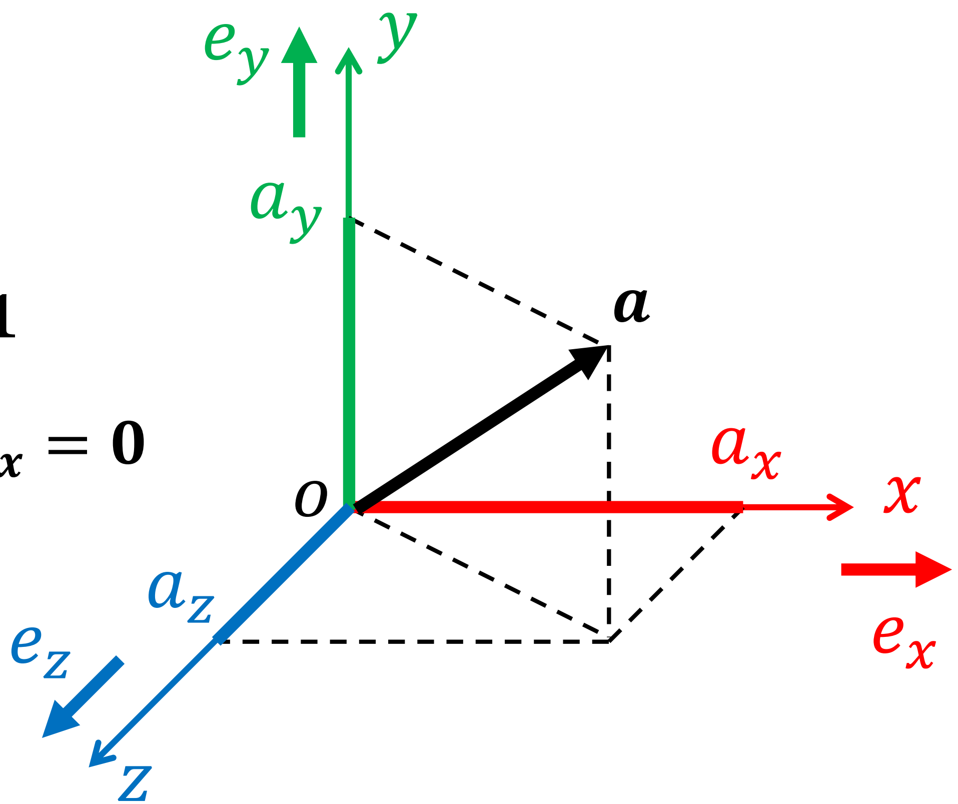
$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^{\mathbf{3}}$$



直角坐标系与正交基

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

- $\|\boldsymbol{e_x}\| = \|\boldsymbol{e_y}\| = \|\boldsymbol{e_z}\| = \mathbf{1}$
- $\boldsymbol{e_x} \cdot \boldsymbol{e_y} = \boldsymbol{e_y} \cdot \boldsymbol{e_z} = \boldsymbol{e_z} \cdot \boldsymbol{e_x} = \mathbf{0}$
- $\boldsymbol{e_x} \times \boldsymbol{e_y} = \boldsymbol{e_z}$



直角坐标系与正交基

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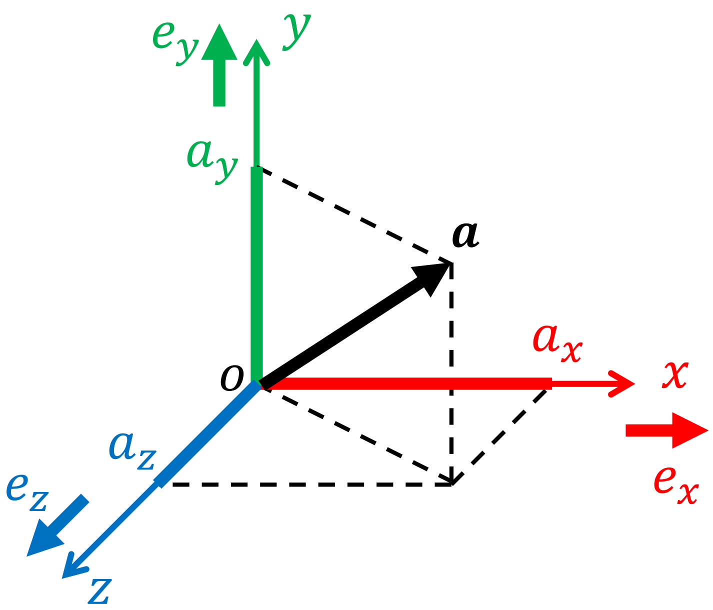
$$\boldsymbol{a} = a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}$$



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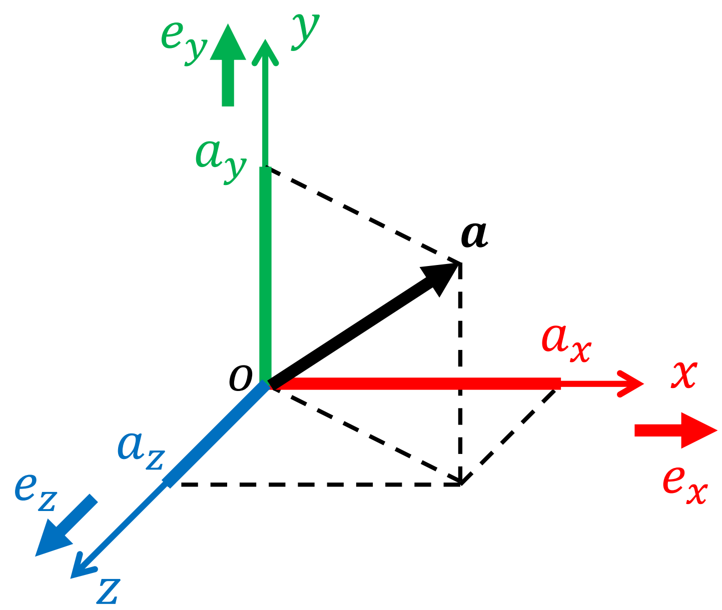


$$\boldsymbol{a} = a_x\boldsymbol{e_x} + a_y\boldsymbol{e_y} + a_z\boldsymbol{e_z}$$

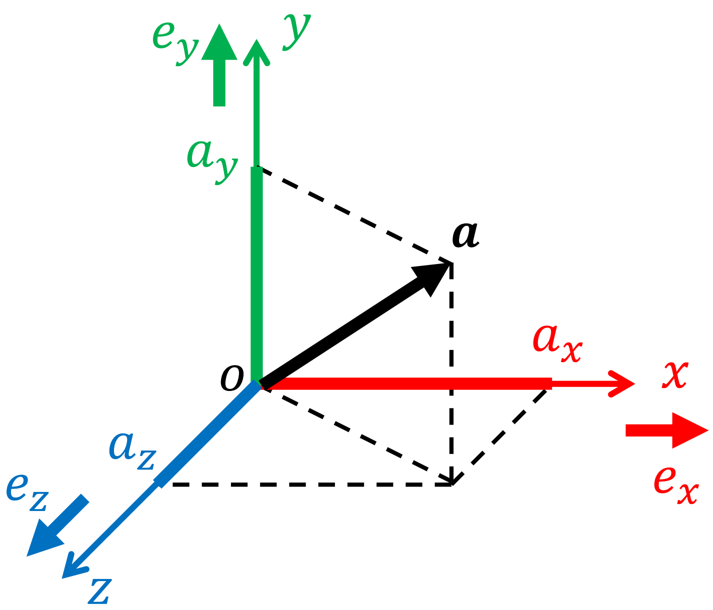
$$\boldsymbol{a} \cdot \boldsymbol{b} = \left(a_x\boldsymbol{e_x} + a_y\boldsymbol{e_y} + a_z\boldsymbol{e_z}\right) \cdot \left(b_x\boldsymbol{e_x} + b_y\boldsymbol{e_y} + b_z\boldsymbol{e_z}\right)$$

$$= a_x b_x \boldsymbol{e_x} \cdot \boldsymbol{e_x} + a_y b_y \boldsymbol{e_y} \cdot \boldsymbol{e_y} + a_z b_z \boldsymbol{e_z} \cdot \boldsymbol{e_z}$$

$$+ \sum_{i \neq j} a_i b_j \boldsymbol{e_i} \cdot \boldsymbol{e_j}$$

直角坐标系与正交基

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$



$$\boldsymbol{a} = a_x\boldsymbol{e_x} + a_y\boldsymbol{e_y} + a_z\boldsymbol{e_z}$$

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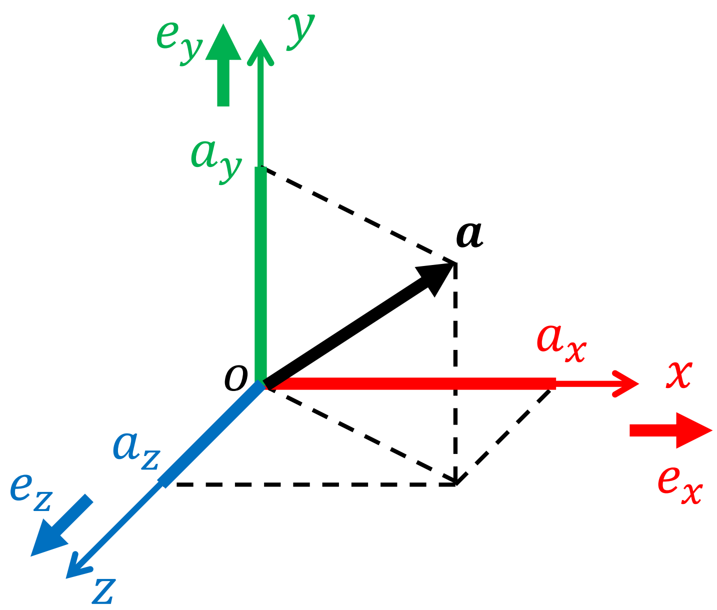
$$= a_x b_x \boldsymbol{e_x} \cdot \boldsymbol{e_x} + a_y b_y \boldsymbol{e_y} \cdot \boldsymbol{e_y} + a_z b_z \boldsymbol{e_z} \cdot \boldsymbol{e_z}$$

$$\boxed{+ \sum_{i \neq j} a_i b_j \boldsymbol{e_i} \cdot \boldsymbol{e_j}}$$

直角坐标系与正交基

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$



$$\boldsymbol{a} = a_x\boldsymbol{e_x} + a_y\boldsymbol{e_y} + a_z\boldsymbol{e_z}$$

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} = &\ a_x b_x \boldsymbol{e_x} \times \boldsymbol{e_x} + a_x b_y \boldsymbol{e_x} \times \boldsymbol{e_y} + a_x b_z \boldsymbol{e_x} \times \boldsymbol{e_z} \\ &+ a_y b_x \boldsymbol{e_y} \times \boldsymbol{e_x} + a_y b_y \boldsymbol{e_y} \times \boldsymbol{e_y} + a_y b_z \boldsymbol{e_y} \times \boldsymbol{e_z} \\ &+ a_z b_x \boldsymbol{e_z} \times \boldsymbol{e_x} + a_z b_y \boldsymbol{e_z} \times \boldsymbol{e_y} + a_z b_z \boldsymbol{e_z} \times \boldsymbol{e_z}\end{aligned}$$

直角坐标系与正交基

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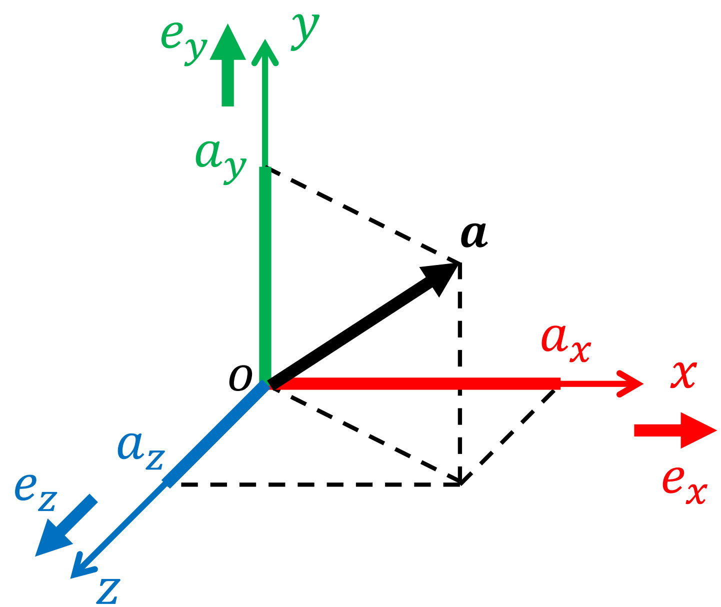

$$\boldsymbol{a} = a_x\boldsymbol{e_x} + a_y\boldsymbol{e_y} + a_z\boldsymbol{e_z}$$

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} = & \cancel{a_x b_x \boldsymbol{e_x} \times \boldsymbol{e_x}} + a_x b_y \boldsymbol{e_x} \times \boldsymbol{e_y} + a_x b_z \boldsymbol{e_x} \times \boldsymbol{e_z} \\ & + a_y b_x \boldsymbol{e_y} \times \boldsymbol{e_x} + \cancel{a_y b_y \boldsymbol{e_y} \times \boldsymbol{e_y}} + a_y b_z \boldsymbol{e_y} \times \boldsymbol{e_z} \\ & + a_z b_x \boldsymbol{e_z} \times \boldsymbol{e_x} + a_z b_y \boldsymbol{e_z} \times \boldsymbol{e_y} + \cancel{a_z b_z \boldsymbol{e_z} \times \boldsymbol{e_z}}\end{aligned}$$

直角坐标系与正交基

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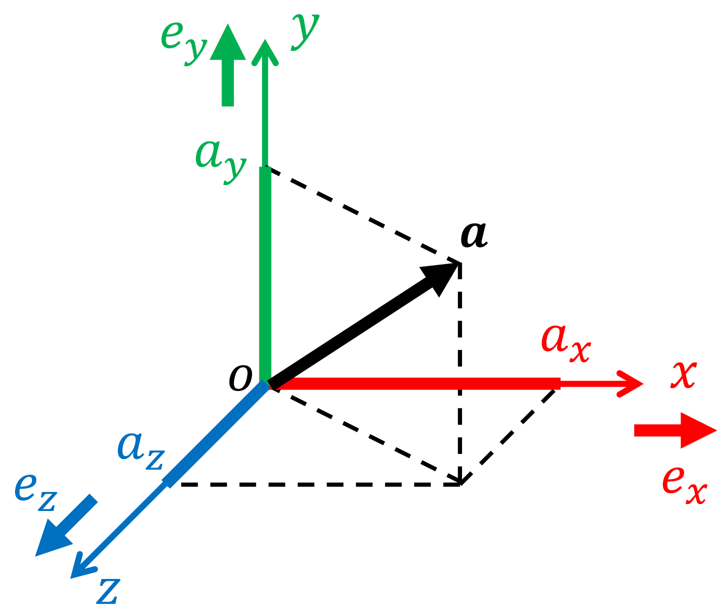


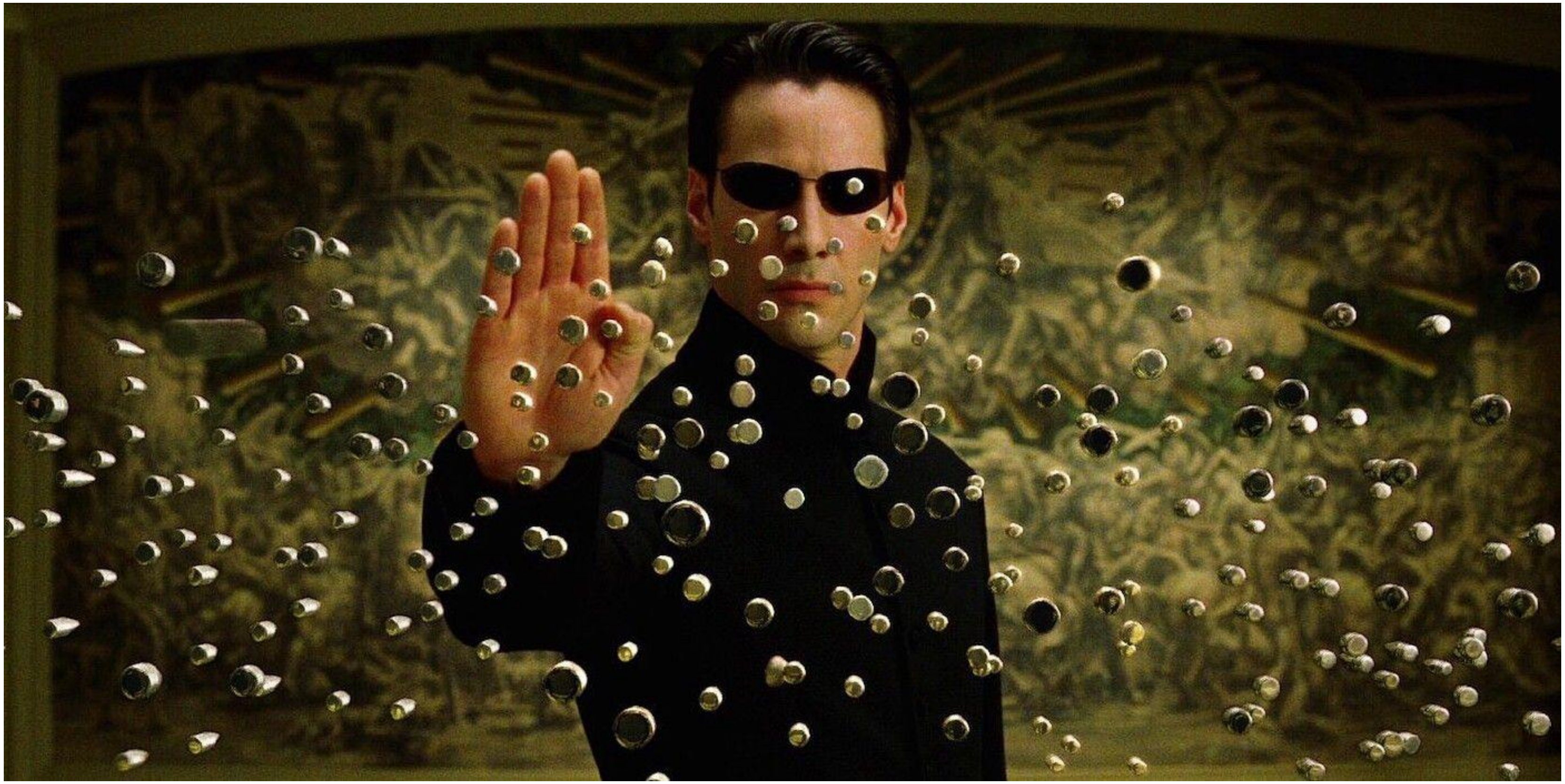
$$\boldsymbol{a} = a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}$$

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} = & \left(a_y b_z - a_z b_y\right) \boldsymbol{e_x} \\ & + \left(a_z b_x - a_x b_z\right) \boldsymbol{e_y} \\ & + \left(a_x b_y - a_y b_x\right) \boldsymbol{e_z} \end{aligned}$$

矩阵

Matrix

矩阵



The Matrix

矩阵

- 二维**矩**形的数字**阵**列

$$\mathrm{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$= \begin{bmatrix} \boldsymbol{a}_0 & \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^{3\times 1}$$

矩阵

- 二维**矩**形的数字**阵**列

$$
\mathrm{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}
$$

$$
= [\boldsymbol{a}_0 \quad \boldsymbol{a}_1 \quad \boldsymbol{a}_2]
$$

转置

$$
\mathrm{A}^{\mathrm{T}} = \begin{bmatrix} a_{00} & a_{10} & a_{20} \\ a_{01} & a_{11} & a_{21} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_0^T \\ \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \end{bmatrix}
$$

矩阵

- 二维**矩**形的数字**阵**列

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\begin{bmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix}$$

对角阵

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

单位阵

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

对称阵

$$A^T = A$$

矩阵运算

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\mathrm{sA} = \begin{bmatrix} sa_{00} & sa_{01} & sa_{02} \\ sa_{10} & sa_{11} & sa_{12} \\ sa_{20} & sa_{21} & sa_{22} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{00}+b_{00} & a_{01}+b_{01} & a_{02}+b_{02} \\ a_{10}+b_{10} & a_{11}+b_{11} & a_{12}+b_{12} \\ a_{20}+b_{20} & a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

矩阵乘法

$$C = AB = \begin{bmatrix} \boxed{a_{00} \quad a_{01} \quad a_{02}} \\ a_{10} \quad a_{11} \quad a_{12} \\ a_{20} \quad a_{21} \quad a_{22} \end{bmatrix} \begin{bmatrix} b_{00} & \boxed{b_{01}} & b_{02} \\ b_{10} & \boxed{b_{11}} & b_{12} \\ b_{20} & \boxed{b_{21}} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} * & \boxed{?} & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

c_{ij} = 矩阵 A 的第 i 行与矩阵 B 的第 j 列的点积

矩阵乘法

- 矩阵乘法规则

$$\boxed{AB \neq BA}$$

$$ABC = (AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(AB)^T = B^T A^T \qquad IA = A$$

- 矩阵的逆

$$M = A^{-1} \Leftrightarrow AM = MA = I$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

向量点乘

$$
\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} &= a_x b_x + a_y b_y + a_z b_z \\
&= \boldsymbol{a}^T \boldsymbol{b} \\
&= \boldsymbol{b}^T \boldsymbol{a}
\end{aligned}
$$

叉乘的矩阵表示

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= [\boldsymbol{a}]_\times \boldsymbol{b}$$

$$\boxed{[\boldsymbol{a}]_\times + [\boldsymbol{a}]_\times^{\boldsymbol{T}} = \boldsymbol{0} \quad \text{反对称阵}}$$

正交阵

- 由相互正交的单位向量构成的方阵

$$\mathrm{A} = [\boldsymbol{a}_0 \quad \boldsymbol{a}_1 \quad \boldsymbol{a}_2] \qquad \boldsymbol{a}_i^T \boldsymbol{a}_j = \begin{cases} 1, & \text{if } i = j \\ 0. & \text{if } i \neq j \end{cases}$$

$$\boxed{\mathrm{A}^{\mathrm{T}}\mathrm{A} = \begin{bmatrix} \boldsymbol{a}_0^T \\ \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \end{bmatrix} [\boldsymbol{a}_0 \quad \boldsymbol{a}_1 \quad \boldsymbol{a}_2] = \begin{bmatrix} \boldsymbol{a}_0^T\boldsymbol{a}_0 & \boldsymbol{a}_0^T\boldsymbol{a}_1 & \boldsymbol{a}_0^T\boldsymbol{a}_2 \\ \boldsymbol{a}_1^T\boldsymbol{a}_0 & \boldsymbol{a}_1^T\boldsymbol{a}_1 & \boldsymbol{a}_1^T\boldsymbol{a}_2 \\ \boldsymbol{a}_2^T\boldsymbol{a}_0 & \boldsymbol{a}_2^T\boldsymbol{a}_1 & \boldsymbol{a}_2^T\boldsymbol{a}_2 \end{bmatrix} = \mathbf{I}}$$

$$\boxed{\mathrm{A}^{\mathrm{T}} = \mathrm{A}^{-1}}$$

三维正交矩阵的自由度

- 三维矩阵, 9个变量

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

三维正交矩阵的自由度

- 三维矩阵, 9个变量

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

约束: 每个列向量长度为1

- $\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases}$

约束: 列向量两两正交

- $\begin{cases} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \end{cases}$

三维正交矩阵的自由度

- 三维矩阵, 9个变量

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

约束: 每个列向量长度为1

- $\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases}$

约束: 列向量两两正交

- $\begin{cases} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \end{cases}$

$$\boxed{\text{自由度} = 3}$$

矩阵的行列式

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times3}$$

$$\det A = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}$$

determinant

矩阵的行列式

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\det A = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}$$

$$\begin{matrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{matrix}$$

矩阵的行列式

$$
\mathrm{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}
$$

$$
\det \mathrm{A} = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}
$$

$$
\begin{matrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{matrix}
$$

矩阵的行列式

- $\det I = 1$
- $\det A * B = \det A * \det B$
- $\det A^T = \det A$
- 当 A 可逆时， $\det A^{-1} = (\det A)^{-1}$
- 当 U 是正交阵时， $\det U = \pm 1$

向量叉乘与行列式

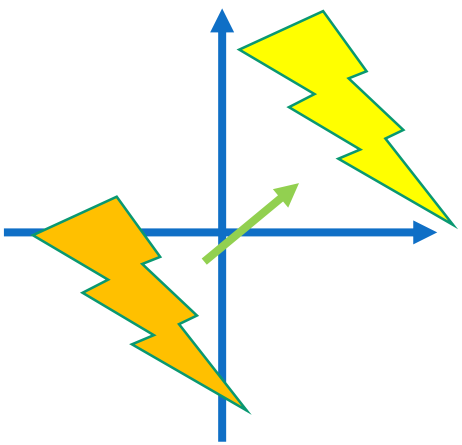
$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$= \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

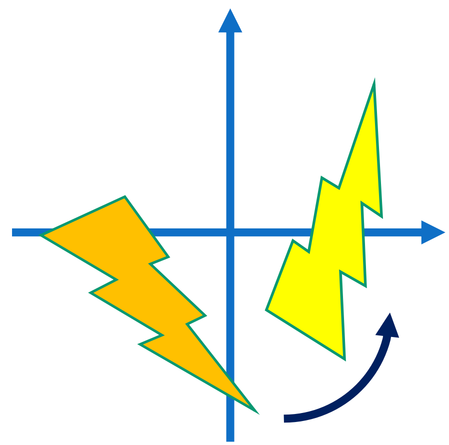
变换

Transform

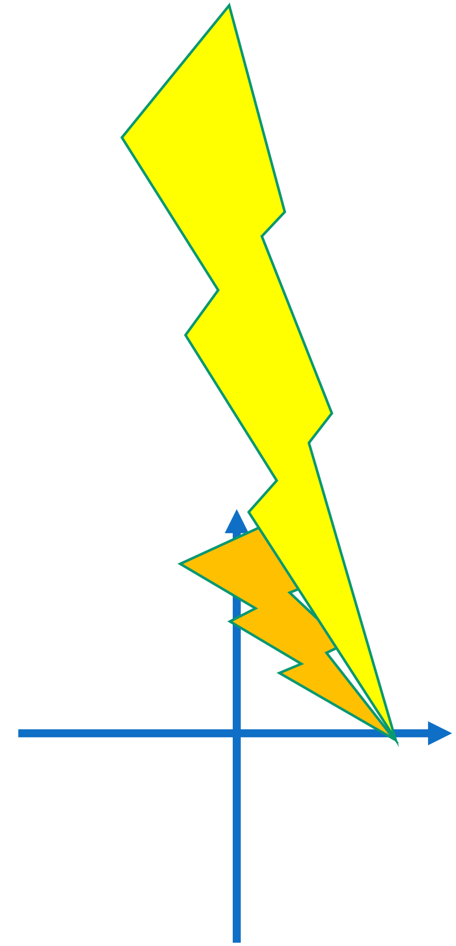
变换 = 平移 + 旋转 + 伸缩



平移

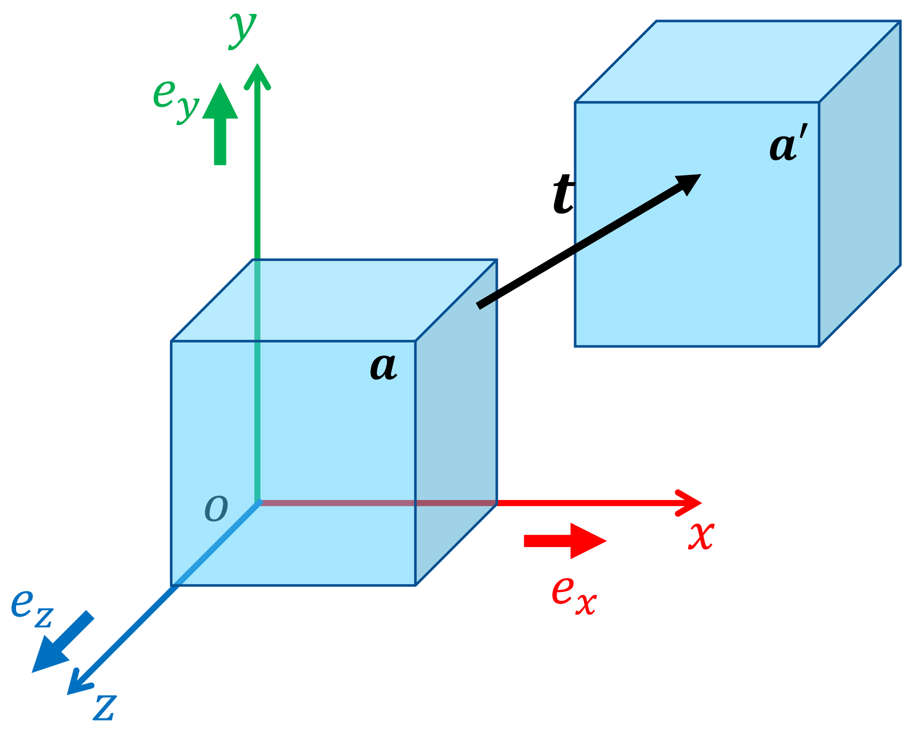


旋转



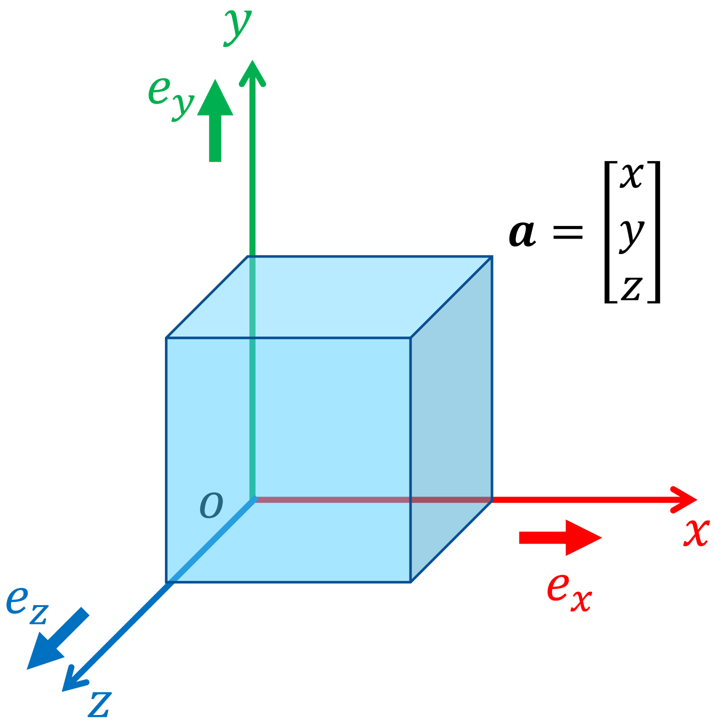
伸缩

平移



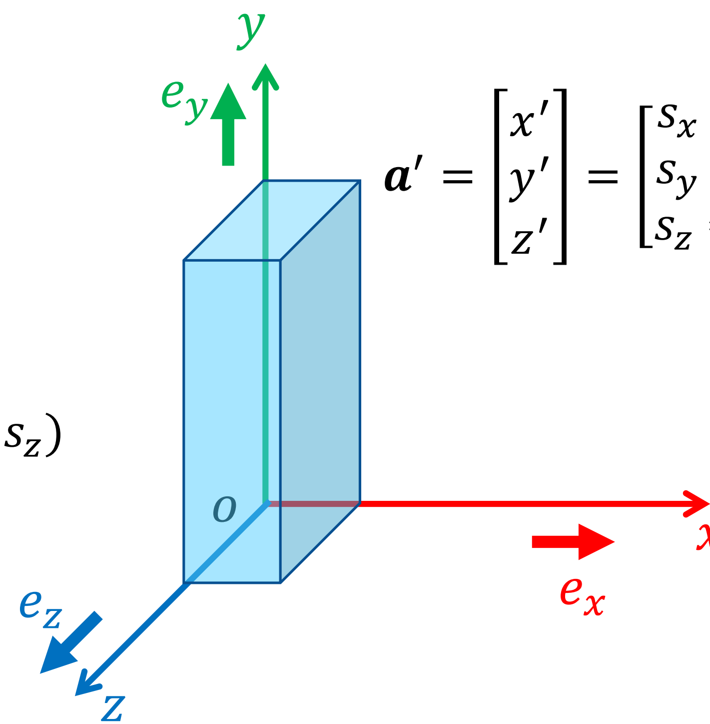
$$\boldsymbol{a}' = \boldsymbol{a} + \boldsymbol{t}$$

伸缩



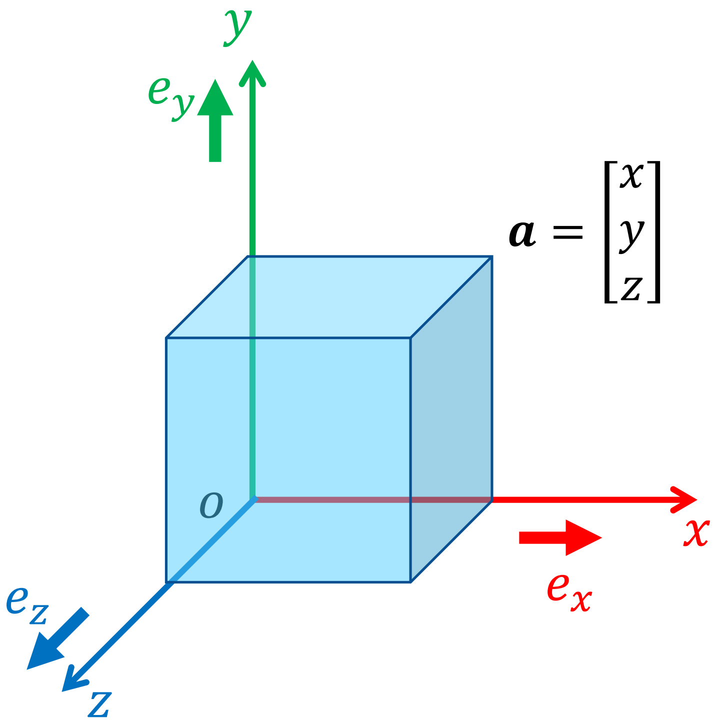
$$\boldsymbol{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$s = (s_x, s_y, s_z)$$

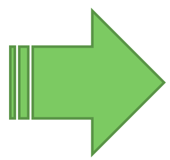


$$\boldsymbol{a}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x * x \\ s_y * y \\ s_z * z \end{bmatrix}$$

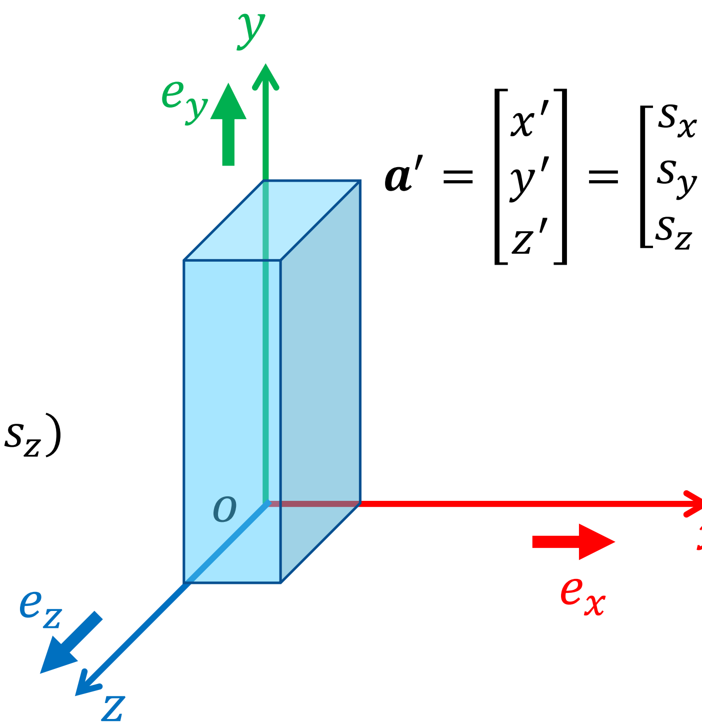
伸缩



$$\boldsymbol{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



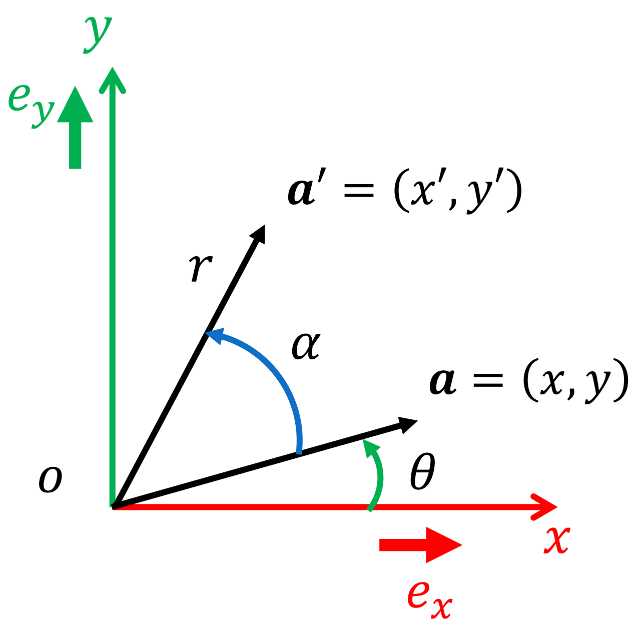
$$s = (s_x, s_y, s_z)$$



$$\boldsymbol{a}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x * x \\ s_y * y \\ s_z * \ z \end{bmatrix}$$

$$\boldsymbol{a}' = \begin{bmatrix} s_x & & \\ & s_y & \\ & & s_z \end{bmatrix} \boldsymbol{a}$$

二维旋转



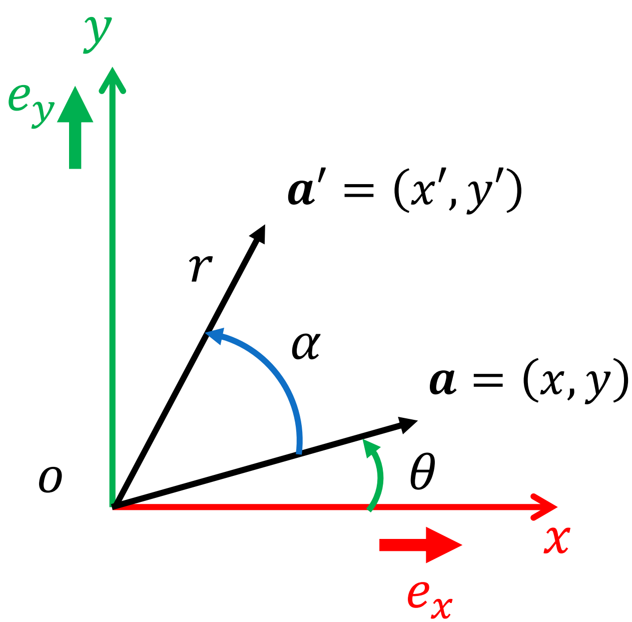
- 平面上向量$\boldsymbol{a} = (x, y)$, 绕原点逆时针旋转 α, 得到$\boldsymbol{a}' = (x', y')$

- 旋转前后坐标

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} x' = r\cos(\theta + \alpha) \\ y' = r\sin(\theta + \alpha) \end{cases}$$

二维旋转

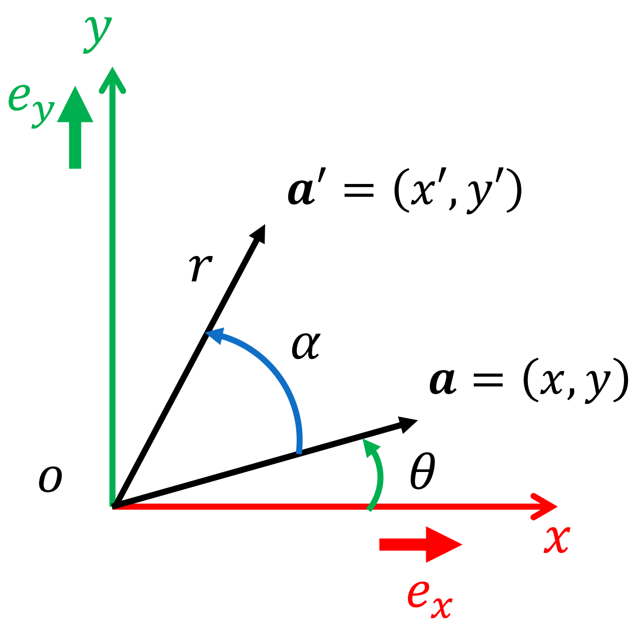


- 平面上向量$\boldsymbol{a} = (x, y)$, 绕原点逆时针旋转 α, 得到$\boldsymbol{a}' = (x', y')$

- 考虑到三角公式

$$
\begin{aligned}
x' &= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\
&= x\cos\alpha - y\sin\alpha \\
y' &= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \\
&= x\sin\alpha + y\cos\alpha
\end{aligned}
$$

二维旋转



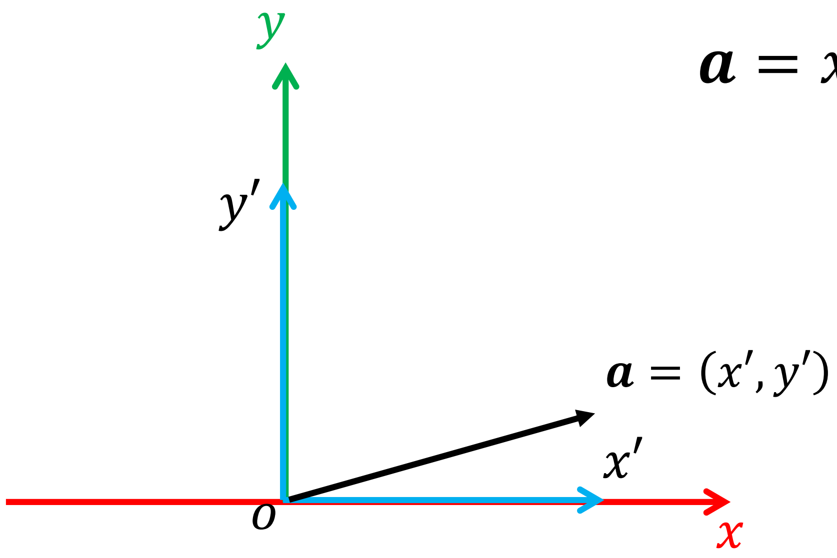
- 平面上向量$\boldsymbol{a} = (x, y)$, 绕原点逆时针旋转 α, 得到$\boldsymbol{a}' = (x', y')$
- 写成矩阵

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \boxed{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

或者

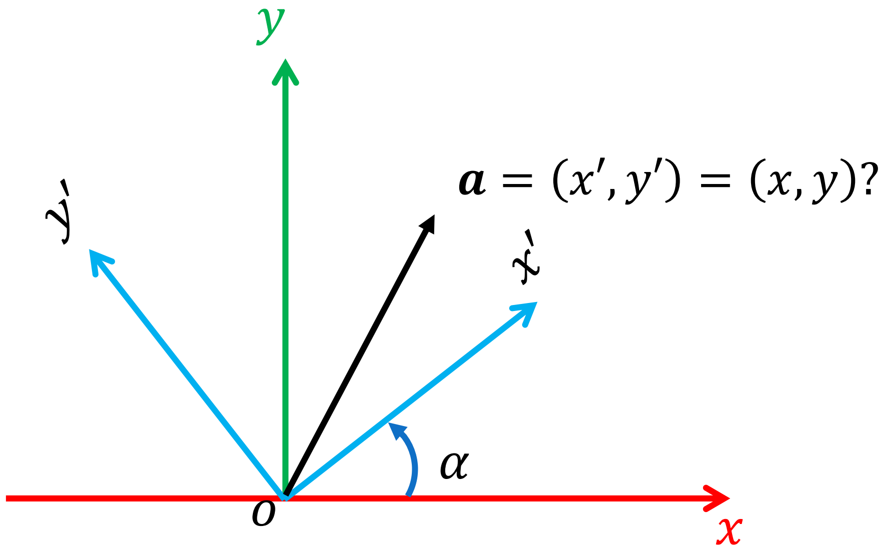
$$\boldsymbol{a}' = \mathrm{R}\boldsymbol{a}$$

二维旋转：坐标变换

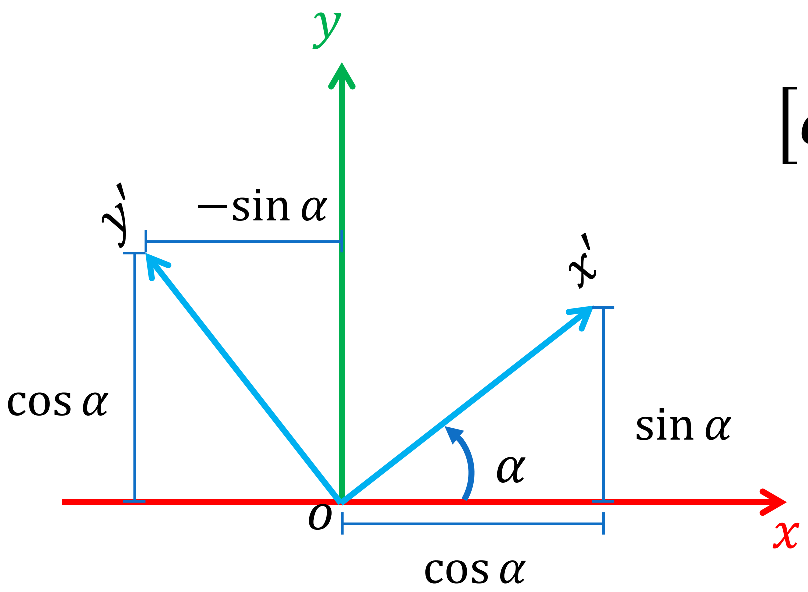


$$\boldsymbol{a} = x'\boldsymbol{e}_{x'} + y'\boldsymbol{e}_{y'} = \left[\boldsymbol{e}_{x'}, \boldsymbol{e}_{y'}\right]\begin{bmatrix} x' \\ y' \end{bmatrix}$$

二维旋转：坐标变换

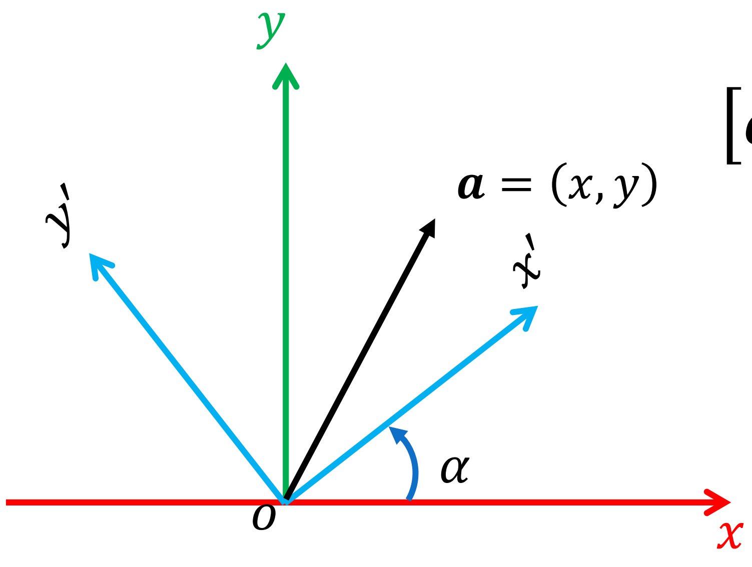


二维旋转：坐标变换



$$[\boldsymbol{e}'_x, \boldsymbol{e}'_y] = [\boldsymbol{e}_x, \boldsymbol{e}_y]\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}$$

二维旋转：坐标变换

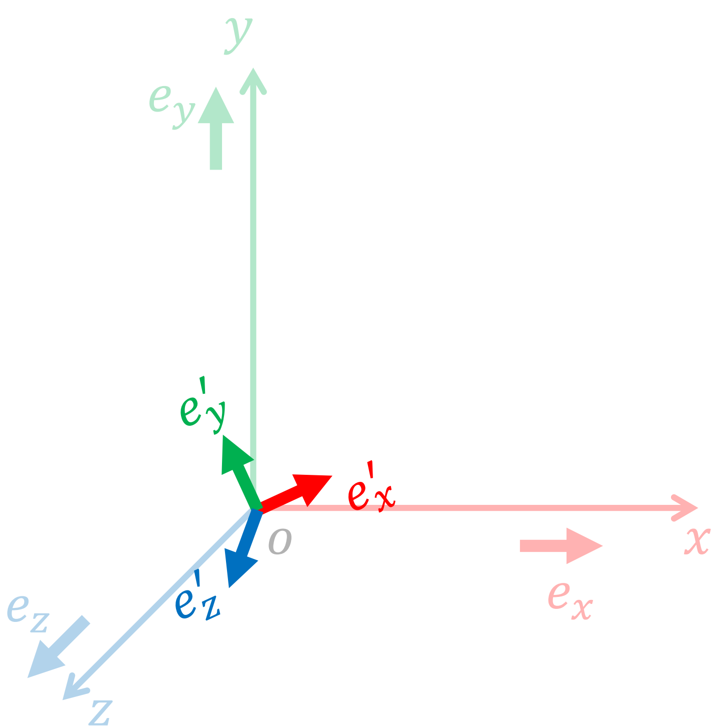


$$[\boldsymbol{e}'_x, \boldsymbol{e}'_y] = [\boldsymbol{e}_x, \boldsymbol{e}_y]\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$\boldsymbol{a} = [\boldsymbol{e}'_x, \boldsymbol{e}'_y]\begin{bmatrix}x' \\ y'\end{bmatrix}$$

$$= [\boldsymbol{e}_x, \boldsymbol{e}_y]\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}x' \\ y'\end{bmatrix}$$

三维旋转: 坐标变换

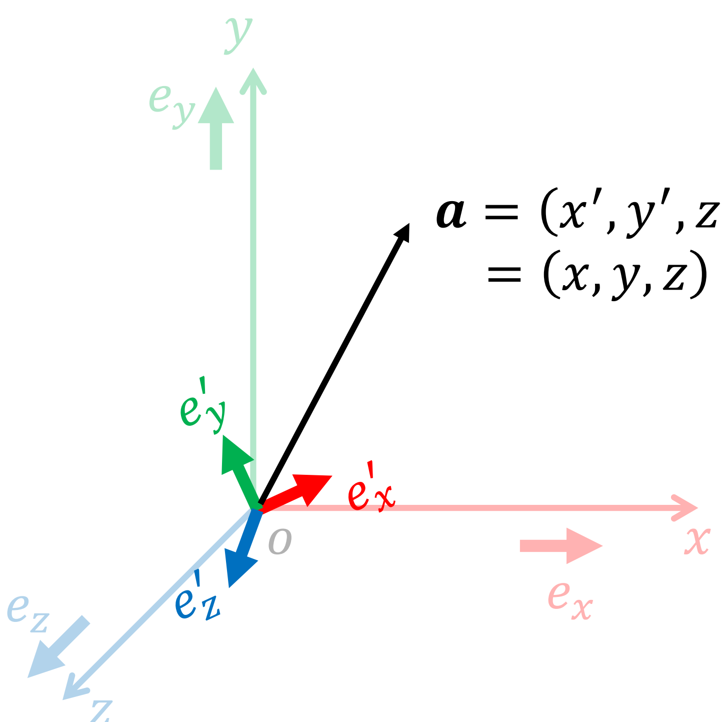


$$[\boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z] = [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z]\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

$$= [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z]\mathrm{R}$$

- 每一列代表变换后的坐标轴在原坐标系下的坐标值
- 每一个列向量都是单位向量
- 向量之间两两正交
- 旋转矩阵是正交矩阵
- $\det R = 1$

三维旋转: 坐标变换



$$\boldsymbol{a} = (x', y', z') = (x, y, z)$$

$$[\boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z] = [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z] \begin{bmatrix} \color{red}{\blacksquare} & \color{green}{\blacksquare} & \color{cyan}{\blacksquare} \\ \color{red}{\blacksquare} & \color{green}{\blacksquare} & \color{cyan}{\blacksquare} \\ \color{red}{\blacksquare} & \color{green}{\blacksquare} & \color{cyan}{\blacksquare} \end{bmatrix}$$

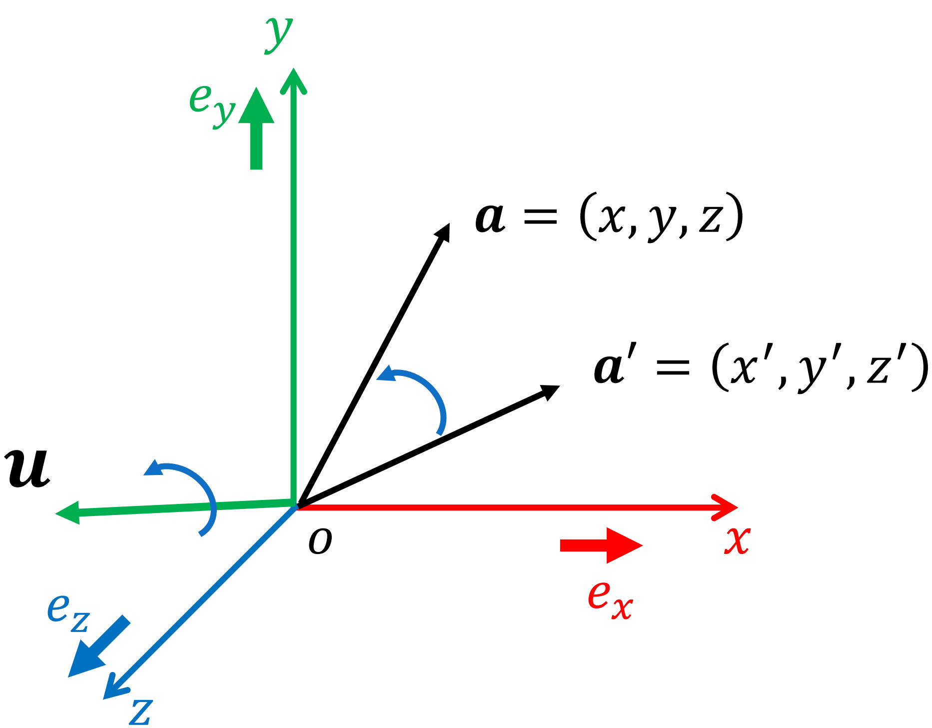
$$= [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z] \mathrm{R}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathrm{R} \begin{bmatrix} x' \\ y' \\ z \end{bmatrix}$$

三维旋转: 坐标变换

$$[\boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z] = [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z]\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

$$= [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z]\mathrm{R}$$



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathrm{R}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

三维旋转: 绕x, y, z轴旋转矩阵

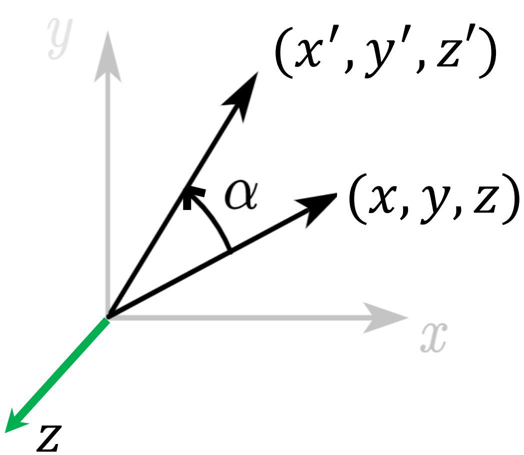


图: 绕z轴旋转

二维旋转 $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

绕z轴旋转 $R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

三维旋转: 绕x, y, z轴旋转矩阵

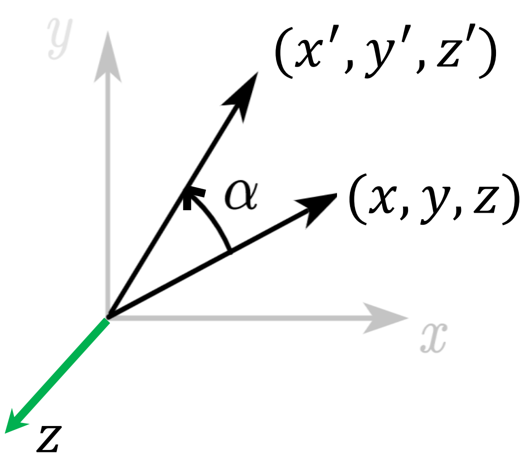


图: 绕z轴旋转

绕x轴旋转 $$R_x(\alpha)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}$$

绕y轴旋转 $$R_y(\alpha)=\begin{pmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{pmatrix}$$

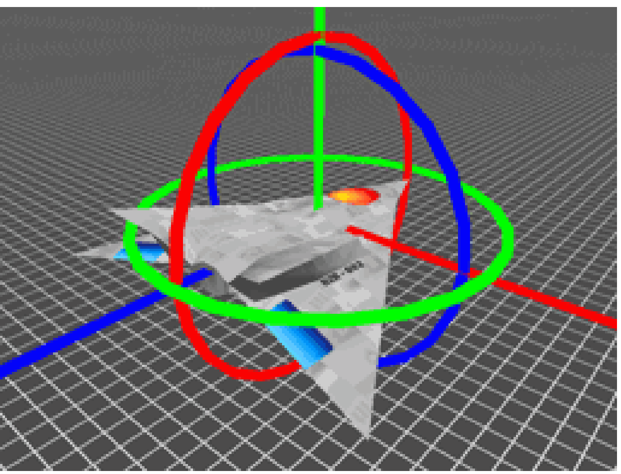
绕z轴旋转 $$R_z(\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}$$

三维旋转的表示

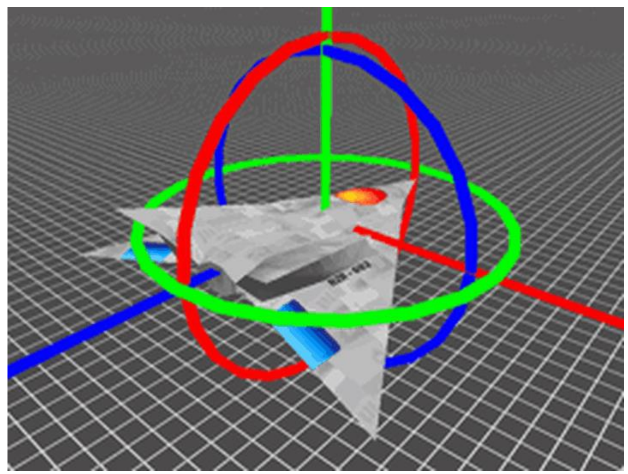
- 旋转矩阵 (Rotation Matrix)
- 欧拉角 (Euler Angle)
- 轴角表示法 (Axis-Angle)
- 四元数 (Quaternion)

欧拉角 (Euler Angle)

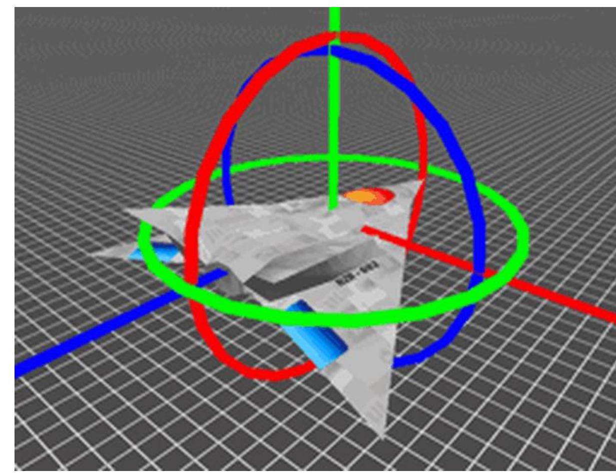
- 任何三维旋转可以分解成三个基本旋转的叠加
- 例如：



Pitch(俯仰)
绕x轴旋转 $R_x(\alpha)$



Yaw(偏航)
绕y轴旋转 $R_y(\beta)$



Roll(翻滚)
绕z轴旋转，$R_z(\gamma)$

可以用三个角度表示旋转，叫做欧拉角

欧拉角有不同旋转顺序

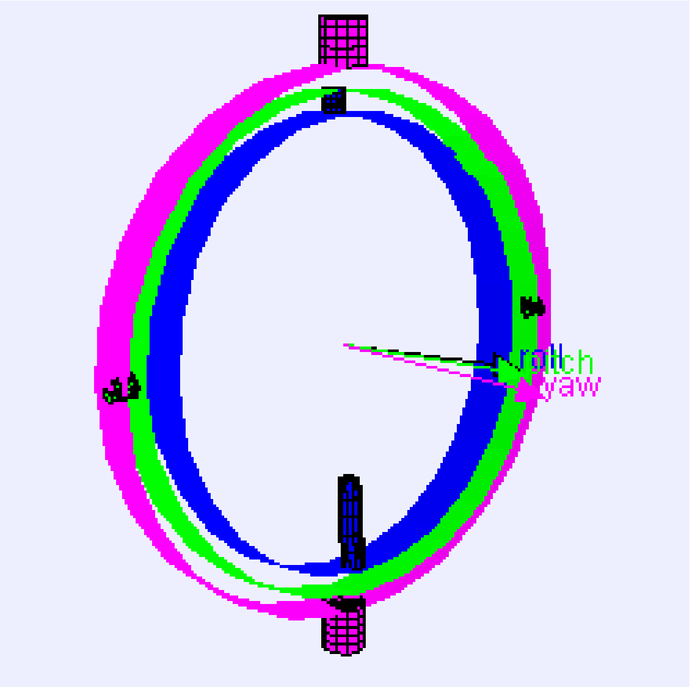
- 先绕 x 轴旋转，
  再绕 y 轴旋转，
  最后绕 z 轴旋转

$$\boldsymbol{p}' = R_z(\gamma)\, R_y(\beta)\, R_x(\alpha)\, \boldsymbol{p}$$

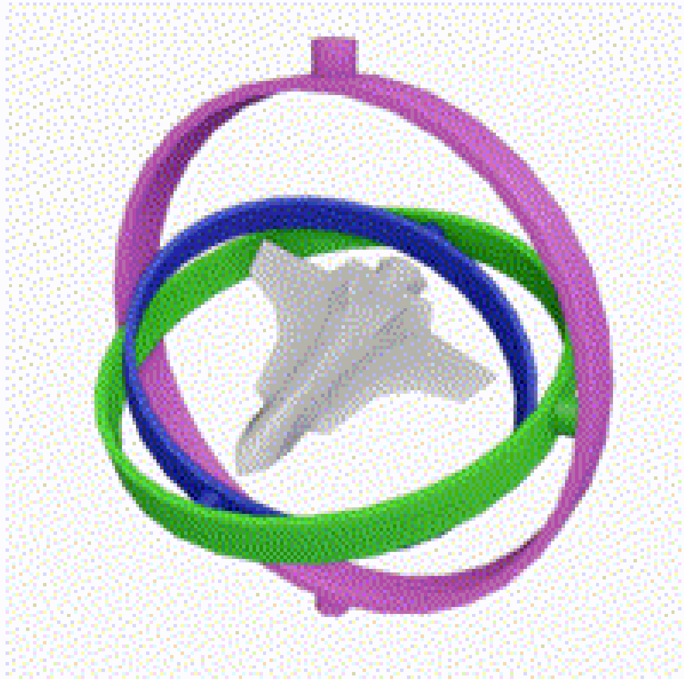
- 先绕 z 轴旋转，
  再绕 x 轴旋转，
  最后绕 y 轴旋转

$$\boldsymbol{p}'' = R_y(\beta)\, R_x(\alpha) R_z(\gamma) \boldsymbol{p}$$

欧拉角的万向锁(Gimbal lock)现象



正常旋转



两个轴平行，
丢失一个自由度

https://en.wikipedia.org/wiki/Gimbal_lock

三维旋转的欧拉角表示

- 三个角度, 三个自由度
- 万向锁问题
- 同一个旋转可以由多组不同的欧拉角表示
  - 要求连续两次旋转不共轴
  - 共12种组合 XYZ, ZYX, XYX, …

三维旋转的表示

- 旋转矩阵 (Rotation Matrix)
- 欧拉角 (Euler Angle)
- 轴角表示法 (Axis-Angle)
- 四元数 (Quaternion)

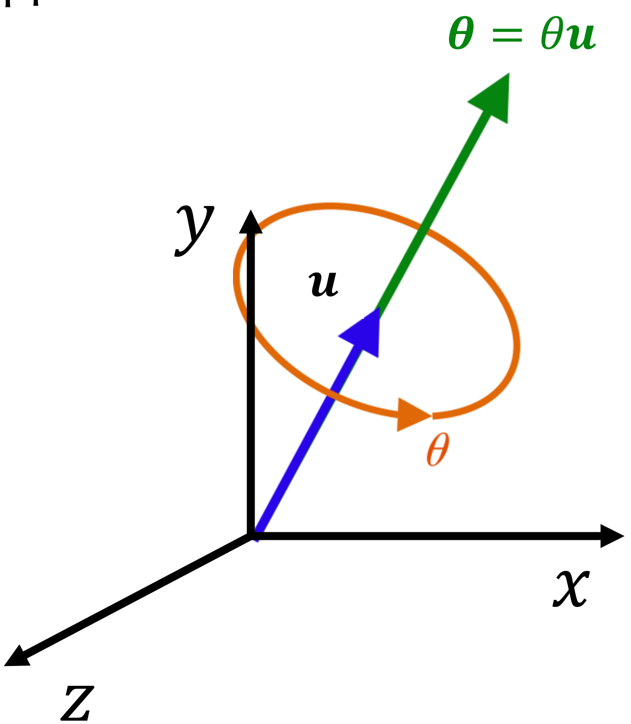
轴角表示

- 任意三维旋转可以表示成
  - 一个单位向量 $\boldsymbol{u}$，代表旋转轴
  - 一个数 θ，代表旋转角度

  的组合$(\boldsymbol{u},\theta)$

- 旋转向量（rotation vector）

  $$\boldsymbol{\theta}=\theta\boldsymbol{u}$$



轴角表示下的旋转

- Rodrigues' Rotation Formula
- 对于轴角表示 $(\boldsymbol{u}, \theta)$

$$\boldsymbol{R} = \boldsymbol{I} + \sin\theta\,[\boldsymbol{u}]_\times + (1 - \cos\theta)[\boldsymbol{u}]_\times^2$$

$$[\boldsymbol{u}]_\times = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

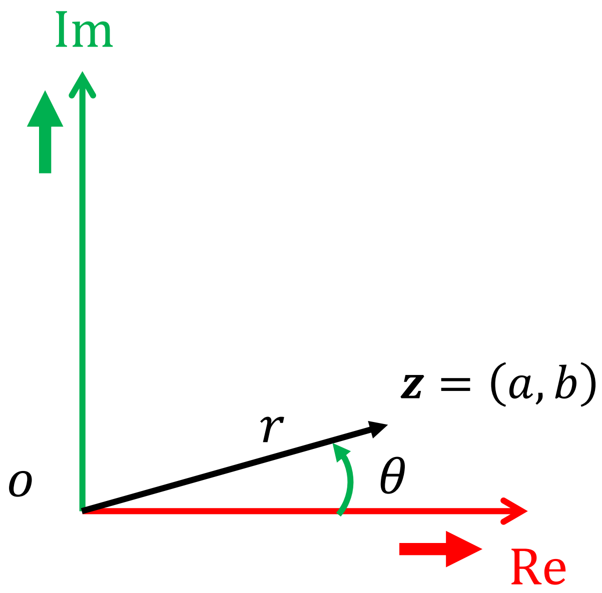
三维旋转的轴角表示

- 没有万向锁问题
- 同一个旋转可能有多种表示方式
  - $(\boldsymbol{u},\theta)$，$(-\boldsymbol{u},-\theta)$，$(\boldsymbol{u},\theta+2n\pi)$
- 多次旋转叠加，旋转轴较难直接计算

三维旋转的表示

- 旋转矩阵 (Rotation Matrix)
- 欧拉角 (Euler Angle)
- 轴角表示法 (Axis-Angle)
- 四元数 (Quaternion)

复数与二维旋转

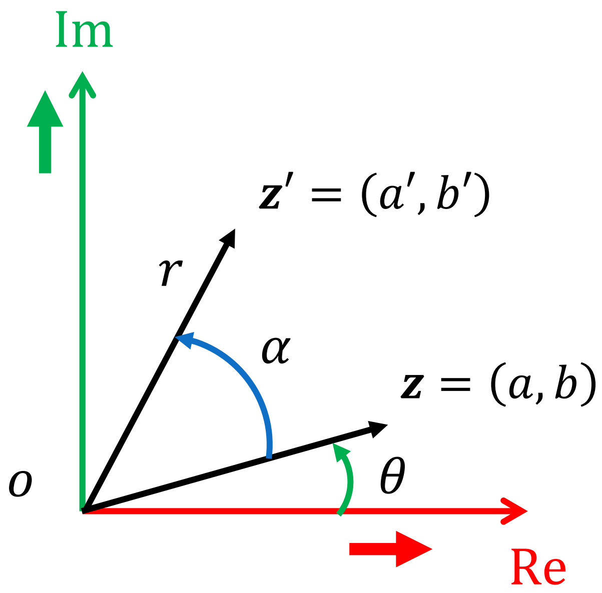


- 复数 $z = a + bi \in \mathbb{C}$
- 其中$a, b \in \mathbb{R}$， 且 $i^2 = -1$
- 欧拉公式

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{a^2 + b^2}$$
$$\theta = \arctan\frac{b}{a}$$

复数与二维旋转

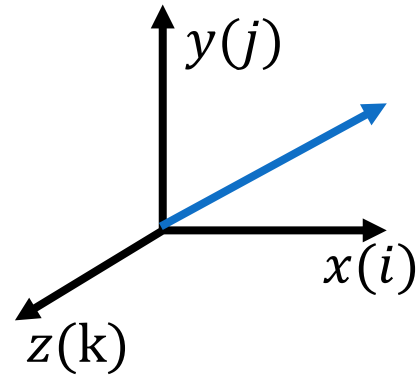


- 复数 $z = a + bi \in \mathbb{C}$
- 旋转 $\alpha, z' = a' + b'i$
- 欧拉公式

$$
\begin{aligned}
z' &= re^{i(\theta+\alpha)} \\
&= e^{i\alpha} \times re^{i\theta} \\
&= e^{i\alpha} z
\end{aligned}
$$

单位复数 ➔ 二维旋转

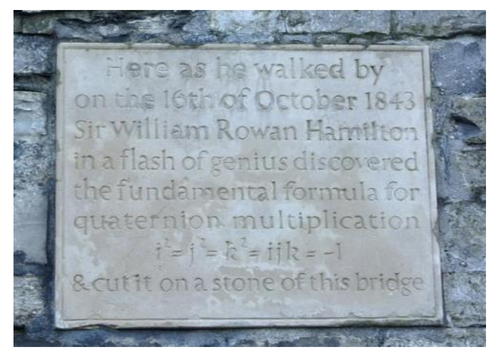
四元数 (Quaternion)



- “扩展”的复数

$i \neq j \neq k$

- 定义 $\boldsymbol{q} = a + bi + cj + dk \in \mathbb{H}, a, b, c, d \in \mathbb{R}$
  - 其中$i^2 = j^2 = k^2 = ijk = -1$
  - $ij = k, ji = -k$ (单位向量叉乘)
  - $jk = i, kj = -i$
  - $ki = j, ik = -j$
- 可以用一个三维的向量来表示虚部
  - $\boldsymbol{q} = [w, \boldsymbol{v}], \boldsymbol{v} = (x, y, z)^{\mathrm{T}}$
  - 纯四元数 $\boldsymbol{q} = [0, \boldsymbol{v}]$ 实部为0,
    只有虚部三维向量



William Rowan Hamilton
发明了四元数

四元数的性质

- 模长 $\|\boldsymbol{q}\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{w^2 + \boldsymbol{v} \cdot \boldsymbol{v}}$
- 加法

$$\boldsymbol{q}_1 = a_1 + b_1 i + c_1 j + d_1 k = w_1 + \boldsymbol{v}_1$$
$$\boldsymbol{q}_2 = a_2 + b_2 i + c_2 j + d_2 k = w_2 + \boldsymbol{v}_2$$

$$\boldsymbol{q_1} + \boldsymbol{q_2} = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$$

$$= (w_1 + w_2) + (\boldsymbol{v}_1 + \boldsymbol{v}_2)$$

- 标量乘法 $t\boldsymbol{q_1} = ta_1 + tb_1 i + tc_1 j + td_1 k$
- 点乘 $\boldsymbol{q_1} \cdot \boldsymbol{q_2} = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2$ (类比向量点乘)

四元数乘法

$$\boldsymbol{q_1}\ \boldsymbol{q_2} = (a_1 + b_1 i + c_1 j + d_1 k) * (a_2 + b_2 i + c_2 j + d_2 k)$$

展开一共$4 \times 4 = 16$项

$$\begin{aligned}\boldsymbol{q_1 q_2} = & a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ & +(b_1 a_2 + a_1 b_2 - d_1 c_2 + c_1 d_2) i \\ & +(c_1 a_2 + d_1 b_2 + a_1 c_2 - b_1 d_2) j \\ & +(d_1 a_2 - c_1 b_2 + b_1 c_2 + a_1 d_2) k\end{aligned}$$

注意:

$i^2 = j^2 = k^2 = ijk = -1$

$ij = k, ji = -k$ (单位向量叉乘)

$jk = i, kj = -i$

$ki = j, ik = -j$

四元数乘法

$$\boldsymbol{q_1}\ \boldsymbol{q_2} = (a_1 + b_1 i + c_1 j + d_1 k) * (a_2 + b_2 i + c_2 j + d_2 k)$$

- 写成矩阵形式

$$\boldsymbol{q_1 q_2} = \begin{pmatrix} a_1 & -b_1 & -c_1 & -d_1 \\ b_1 & a_1 & -d_1 & c_1 \\ c_1 & d_1 & a_1 & -b_1 \\ d_1 & -c_1 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}$$

四元数乘法

$$\boldsymbol{q}_1\,\boldsymbol{q}_2 = (a_1 + b_1 i + c_1 j + d_1 k) * (a_2 + b_2 i + c_2 j + d_2 k)$$

- 或者（Grassmann Inner Product)

$$\boldsymbol{q} = [w, \boldsymbol{v}], \boldsymbol{v} = (x, y, z)^{\mathrm{T}}$$

$$\boldsymbol{q_1 q_2} = [w_1 w_2 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2, w_1 \boldsymbol{v}_2 + w_2 \boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2]$$

四元数性质

- $\boldsymbol{q_1 q_2} \neq \boldsymbol{q_2 q_1}$，与矩阵乘法类似，不满足交换律
- 四元数的**逆** $\boldsymbol{q}\boldsymbol{q}^{-\mathbf{1}} = \boldsymbol{q}^{-\mathbf{1}}\boldsymbol{q} = 1$, 也即 $[1,(0,0,0)]$
- 四元数 $\boldsymbol{q} = a + bi + ck + dj$ 的**共轭四元数**

$$\boldsymbol{q}^* = a - bi - cj - dk$$

  - $\boldsymbol{q} = [w, \boldsymbol{v}]$的共轭可以写成 $\boldsymbol{q}^* = [w, -\boldsymbol{v}]$
- $\boldsymbol{q}\boldsymbol{q}^* = \boldsymbol{q}^*\boldsymbol{q} = \|\boldsymbol{q}\|^2$，也即 $\left[\|\boldsymbol{q}\|^2,(0,0,0)\right]$
- 四元数的逆 $\boldsymbol{q}^{-1} = \dfrac{\boldsymbol{q}^*}{\|\boldsymbol{q}\|^2}$

单位四元数

- **模长**为 **1** 的四元数

$$\boldsymbol{q}=\frac{\widetilde{\boldsymbol{q}}}{\|\widetilde{\boldsymbol{q}}\|}$$

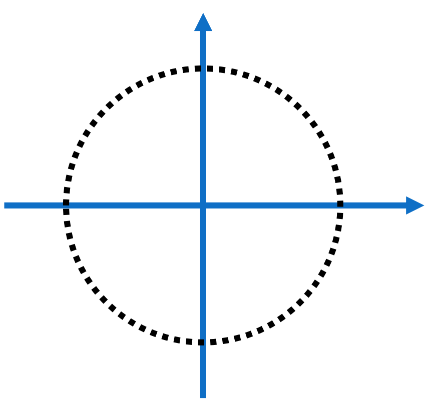
- 可记为 $q=[w,\boldsymbol{v}]=[\cos\frac{\theta}{2},\boldsymbol{u}\sin\frac{\theta}{2}]$， $\|\boldsymbol{u}\|=\mathbf{1},\theta\in\mathbb{R}$
- 单位四元数的逆 $\boldsymbol{q}^{-1}=\boldsymbol{q}^{*}$

单位四元数

- **模长为 1** 的四元数

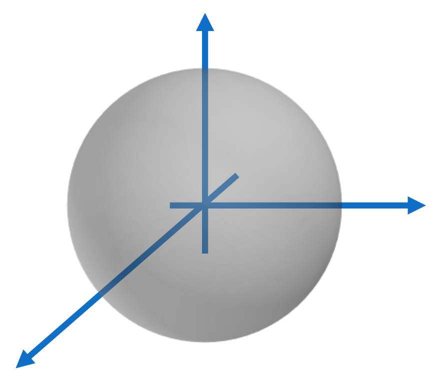
$$\boldsymbol{q} = \frac{\widetilde{\boldsymbol{q}}}{\|\widetilde{\boldsymbol{q}}\|}$$

- 可记为 $\boldsymbol{q} = [w, \boldsymbol{v}] = [\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}]$， $\|\boldsymbol{u}\| = \mathbf{1}, \theta \in \mathbb{R}$



单位复数

$z = \cos\theta + i\sin\theta$



单位四元数

$q = [\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}]$

用单位四元数表示旋转

- 任意三维旋转可以表示为**单位四元数**

$$\boldsymbol{q} = [w, \boldsymbol{v}] = [\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}]$$

  - $\boldsymbol{u}$为旋转轴，θ为旋转角
  - 对应轴角表示 $(\boldsymbol{u}, \theta)$ 或 $\boldsymbol{\theta} = \theta\boldsymbol{u}$

用单位四元数表示旋转

- 任意三维旋转可以表示为**单位四元数**

$$\boldsymbol{q} = [w, \boldsymbol{v}] = [\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}]$$

  - $\boldsymbol{u}$为旋转轴，θ为旋转角
  - 对应轴角表示 $(\boldsymbol{u}, \theta)$ 或 $\boldsymbol{\theta} = \theta\boldsymbol{u}$

- 单位四元数 $\boldsymbol{q}$ 与 $-\boldsymbol{q}$ 代表相同的旋转

四元数表示下的旋转

- 给出单位四元数 $\boldsymbol{q}$ 和任意三维向量 $\boldsymbol{v}$，则 $\boldsymbol{v}$ 在 $\boldsymbol{q}$ 作用下的旋转可以写为

$$\hat{\boldsymbol{v}}' = q\hat{v}q^*$$

其中 $\hat{v}$ 为纯四元数 $\hat{v} = [0, \boldsymbol{v}]$

运算结果仍为纯四元数 $\hat{v}' = [\mathbf{0}, \boldsymbol{v}']$

$\boldsymbol{v}'$ 即为旋转后的向量

两个旋转的复合

- 对于旋转 $\boldsymbol{q}_1, \boldsymbol{q}_2$, 向量 $\boldsymbol{v}$

$$\boldsymbol{v}' = \boldsymbol{q}_2(\boldsymbol{q}_1\boldsymbol{v}\boldsymbol{q}_1^*)\boldsymbol{q}_2^* = (\boldsymbol{q}_2\boldsymbol{q}_1)\boldsymbol{v}(\boldsymbol{q}_2\boldsymbol{q}_1)^*$$
$$= \boldsymbol{q}\boldsymbol{v}\boldsymbol{q}^*$$

其中 $\boldsymbol{q} = \boldsymbol{q}_2\boldsymbol{q}_1$ 表示 $\boldsymbol{q}_2, \boldsymbol{q}_1$ 的复合旋转

四元数总结

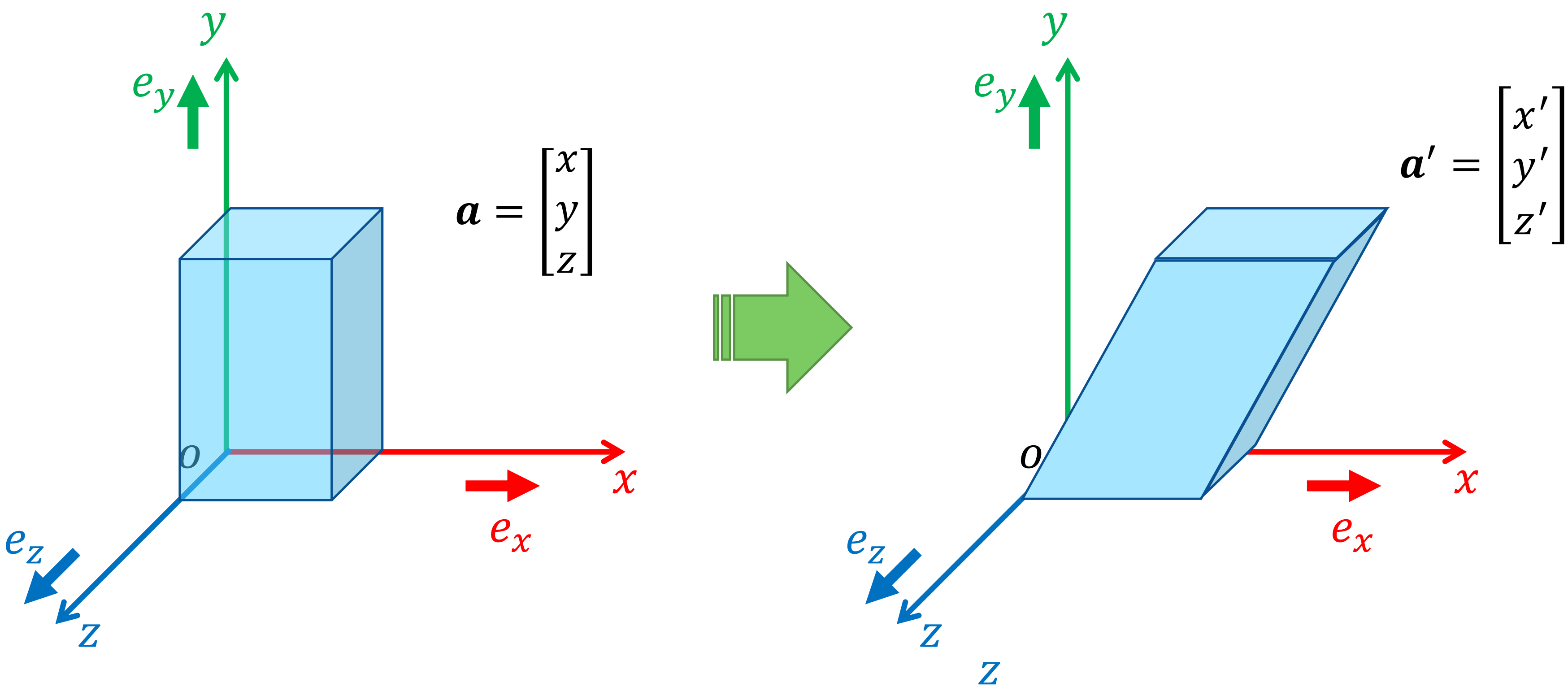
- 只有单位四元数才表示旋转
- $v' = qvq$
- 方便在运算时做归一化
- $\boldsymbol{q}$ 与 $-\boldsymbol{q}$ 代表相同的旋转
- 插值, 求逆等操作方便
- 较为常用(物理仿真等)

其他旋转表示

- 旋转表示的间断点问题
  - [Zhou et al. 2018 - *On the Continuity of Rotation Representations in Neural Networks*]

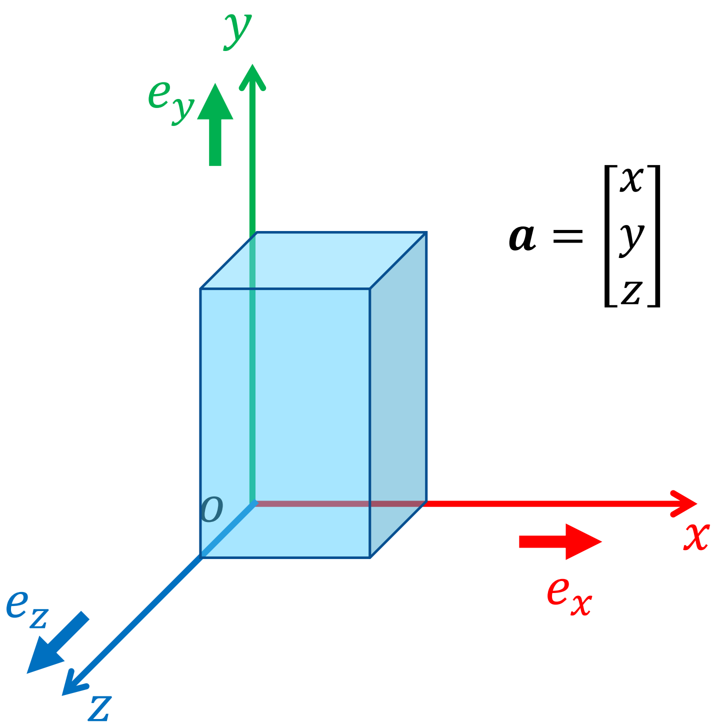
- 6D-vector
  - 使用旋转矩阵的前两列表示旋转
    - 第三列 ← 叉乘
  - 没有不连续点
  - 与旋转矩阵相似，较难直接修改

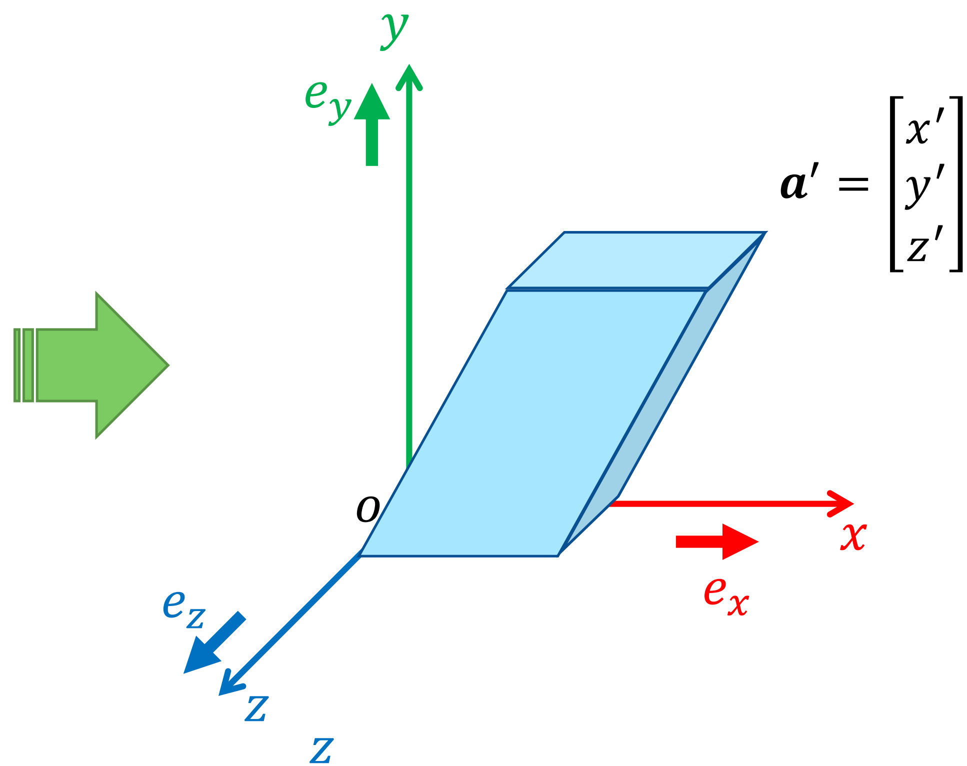
切变



y
e_y
$a = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
o
x
e_x
e_z
z
y
e_y
$a' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$
o
x
e_x
e_z
z
z

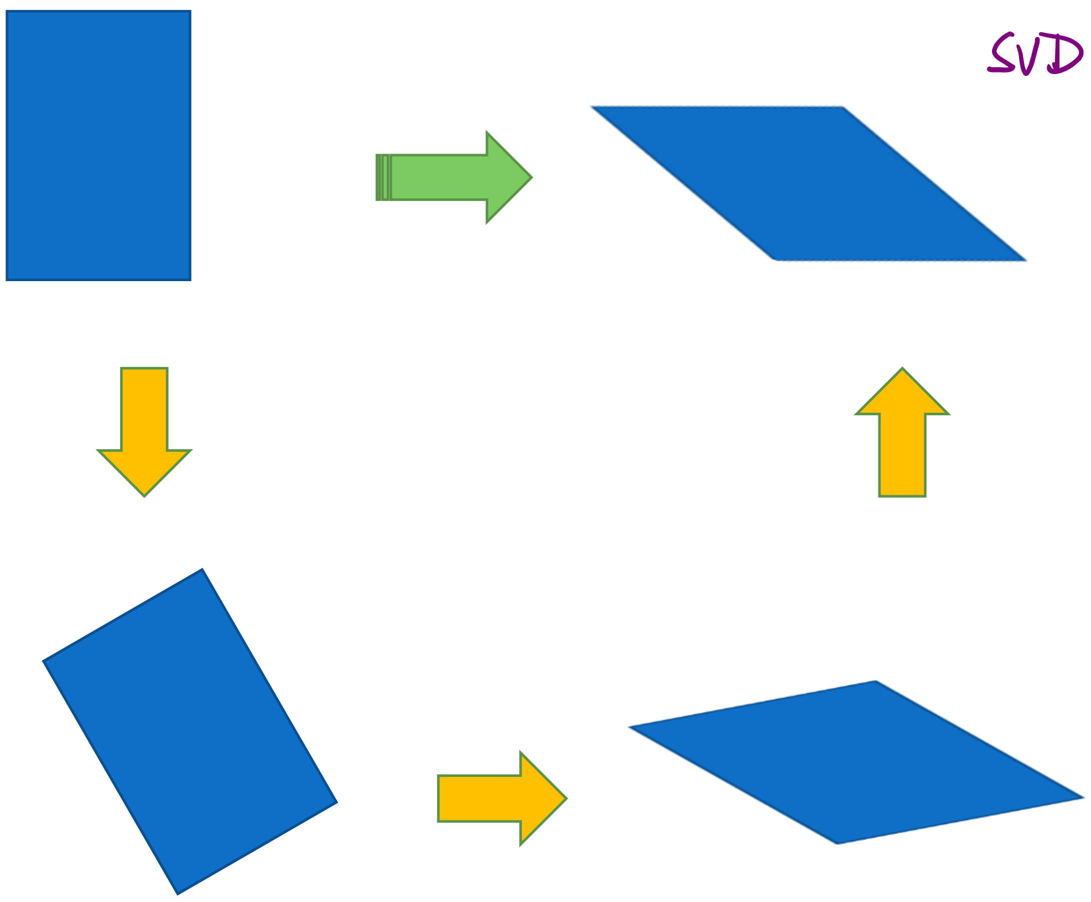
切变





$$\boldsymbol{a}' = \begin{bmatrix} 1 & h_{yx} & h_{zx} \\ h_{xy} & 1 & h_{zy} \\ h_{xz} & h_{yz} & 1 \end{bmatrix} \boldsymbol{a}$$

切变 ← 旋转 + 伸缩



本节主要内容

- CG/CGI的数学基础
- 线性代数回顾
  - 三维向量与向量运算
  - 矩阵与矩阵运算
  - 坐标系与坐标变换
  - 三维旋转与表示

Questions?