FEM Simulation of 3D Deformable Solids

Libin Liu

CFCS, Peking University

Outline

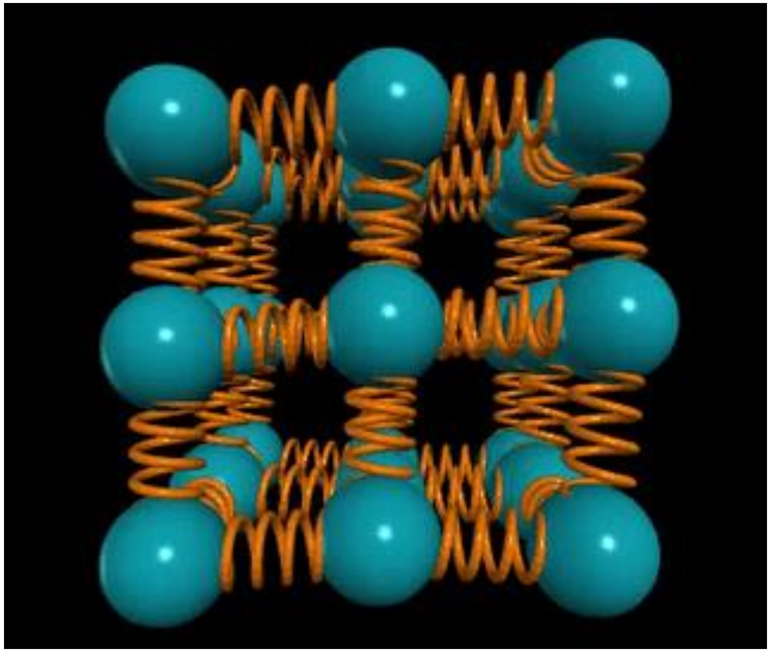
- Elasticity in 3D
- Discretization
- Constitutive models of materials
  - Linear elasticity
  - Corotated linear elasticity
  - Other materials (StVK, Neohookean)
- Modal analysis and model reduction

https://viterbi-web.usc.edu/~jbarbic/femdefo/
Eftychios Sifakis and Jernej Barbic. 2012. ***FEM simulation of 3D deformable solids: a practitioner's guide to theory, discretization and model reduction***. In *ACM SIGGRAPH 2012 Courses* (SIGGRAPH '12),

https://viterbi-web.usc.edu/~jbarbic/vega/ Vega FEM

Mass Spring Systems for Solids

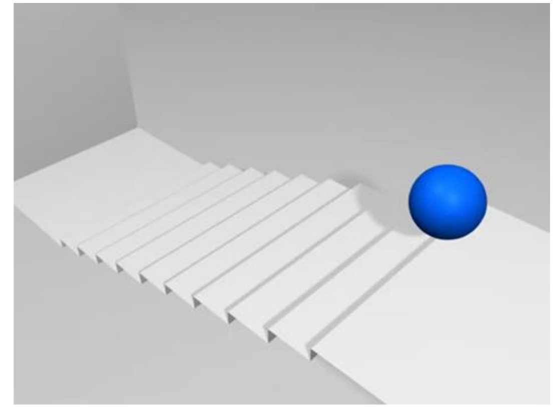
- Simple and faster
- Hard to simulate real materials



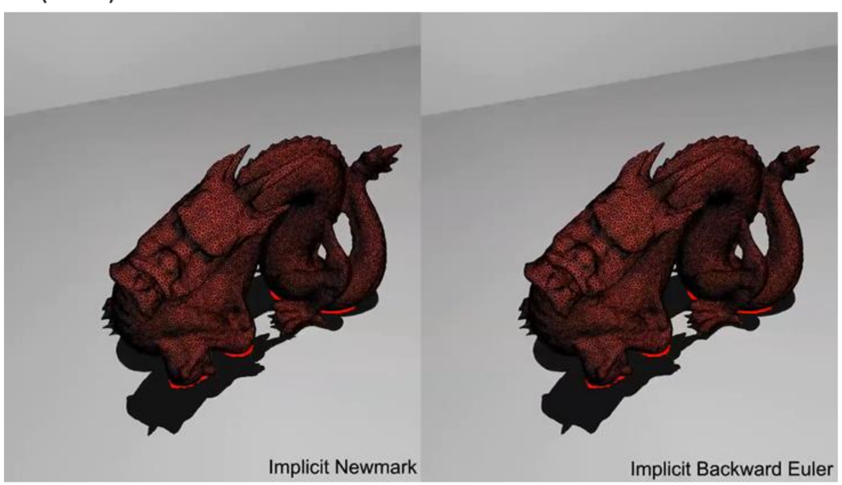
Deformable Solids

Tan, Jie, Greg Turk, and C. Karen Liu. "Soft body locomotion." *ACM Transactions on Graphics (TOG)* 31.4 (2012): 1-11.

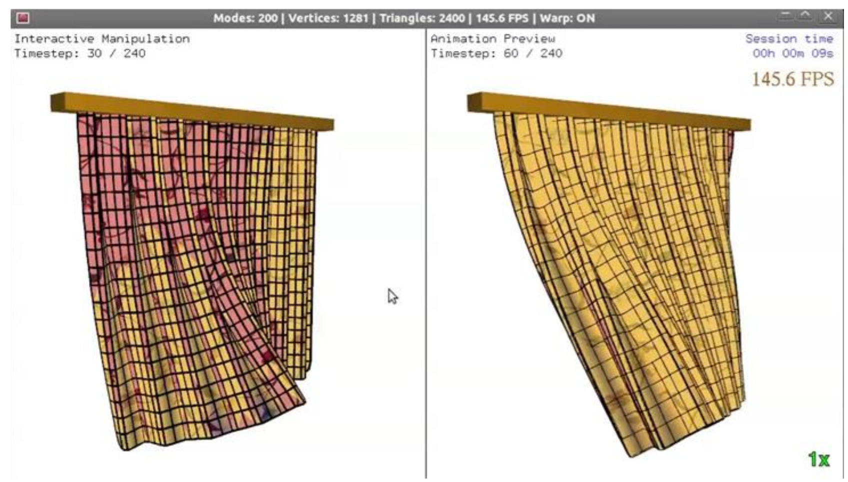
Barbič, Jernej, and Yili Zhao. "Real-time large-deformation substructuring." *ACM transactions on graphics (TOG)* 30.4 (2011): 1-8.



Coros, Stelian, et al. "Deformable objects alive!." *ACM Transactions on Graphics (TOG)* 31.4 (2012): 1-9.

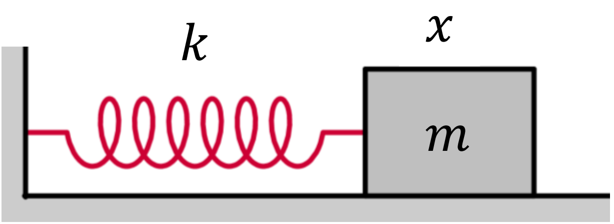


Sin, Fun Shing, Daniel Schroeder, and Jernej Barbič. "Vega: non-linear FEM deformable object simulator." *Computer Graphics Forum*. Vol. 32. No. 1.



Barbič, Jernej, Funshing Sin, and Eitan Grinspun. "Interactive editing of deformable simulations." *ACM Transactions on Graphics (TOG)* 31.4 (2012): 1-8.

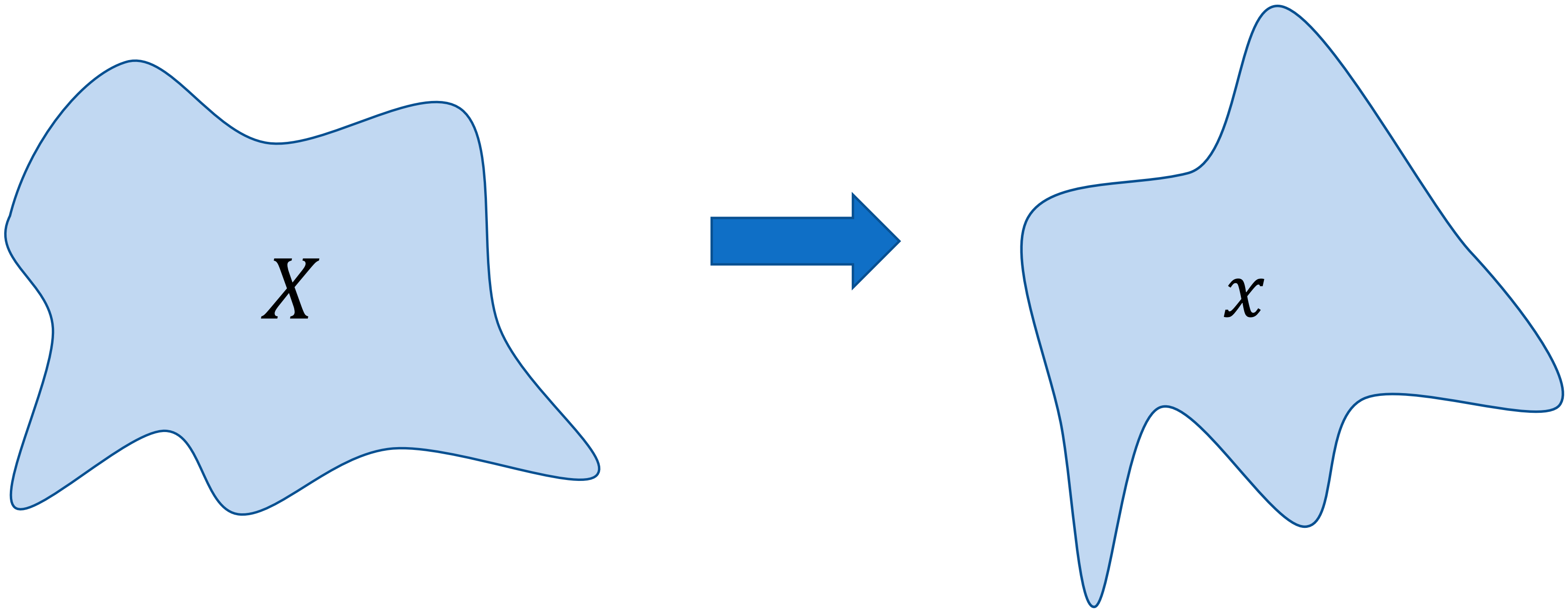
Hooke's Law



$$f = -k(x - x_0)$$

Elastic Force　　Displacement

Displacement Map

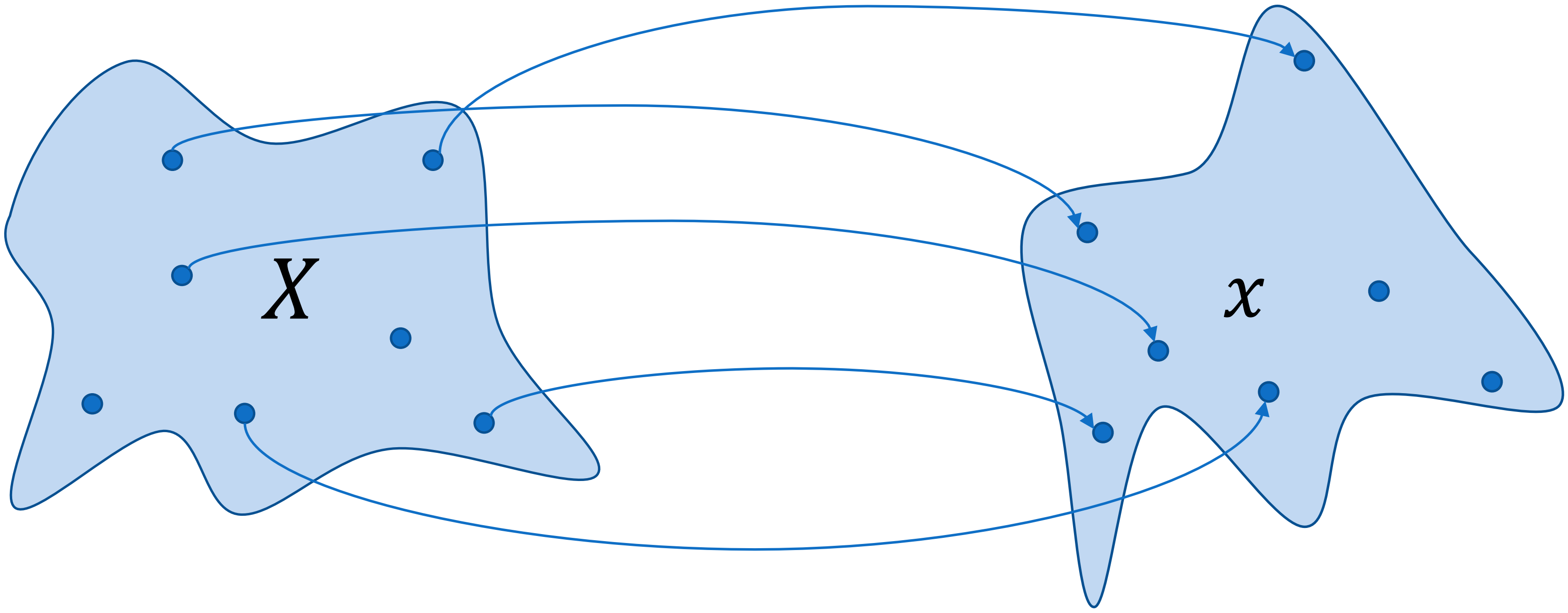


Reference Configuration/
Material Space

Deformed Configuration

Displacement Map

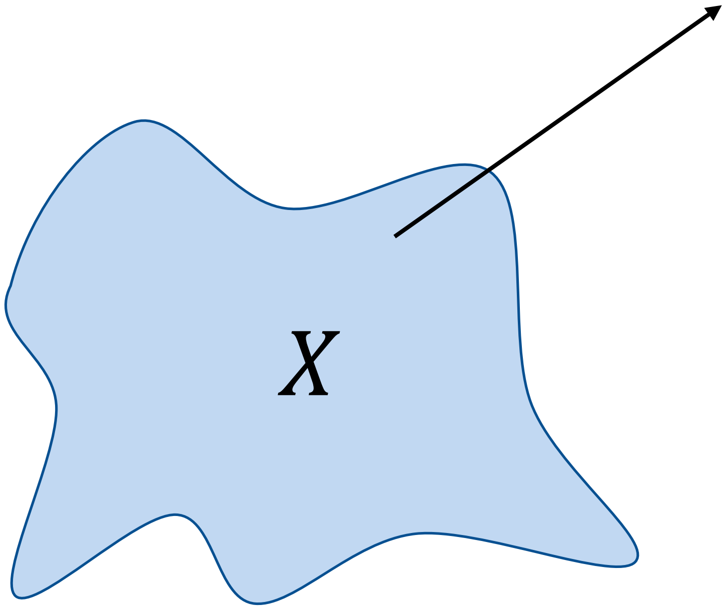
$$x = \varphi(X)$$



Reference Configuration/
Material Space

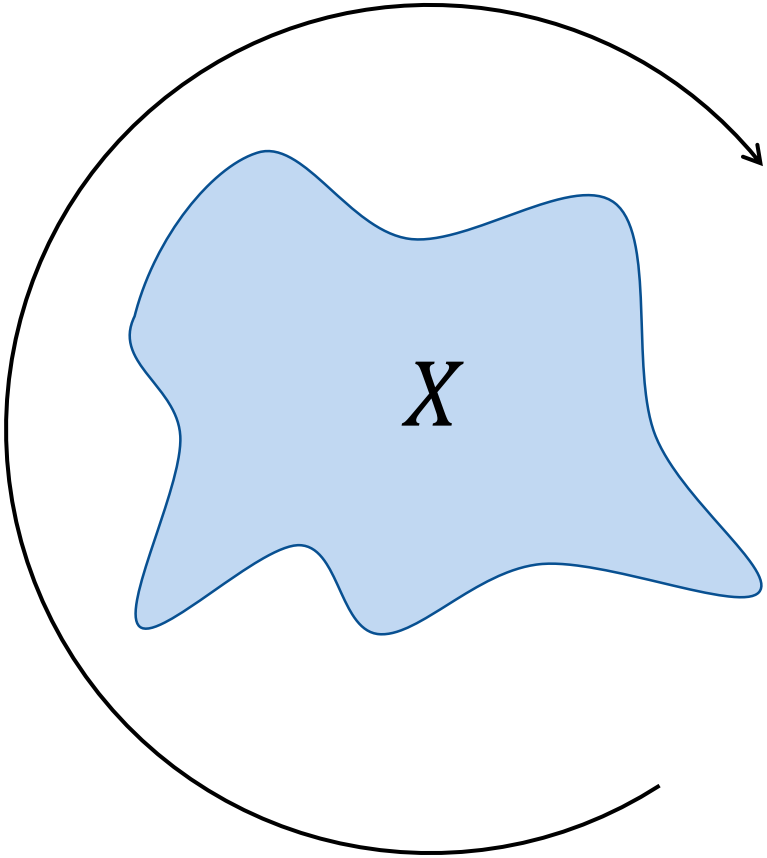
Deformed Configuration

Displacement Map



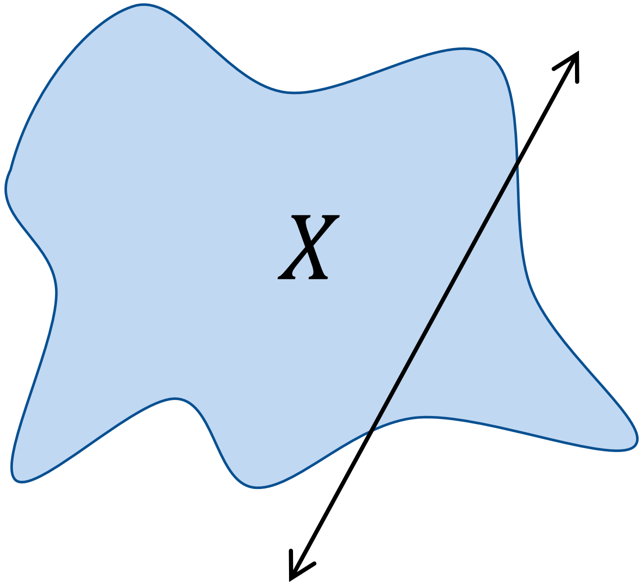
$$x = \varphi(X) = X + t$$

Displacement Map



$$x = \varphi(X) = RX$$

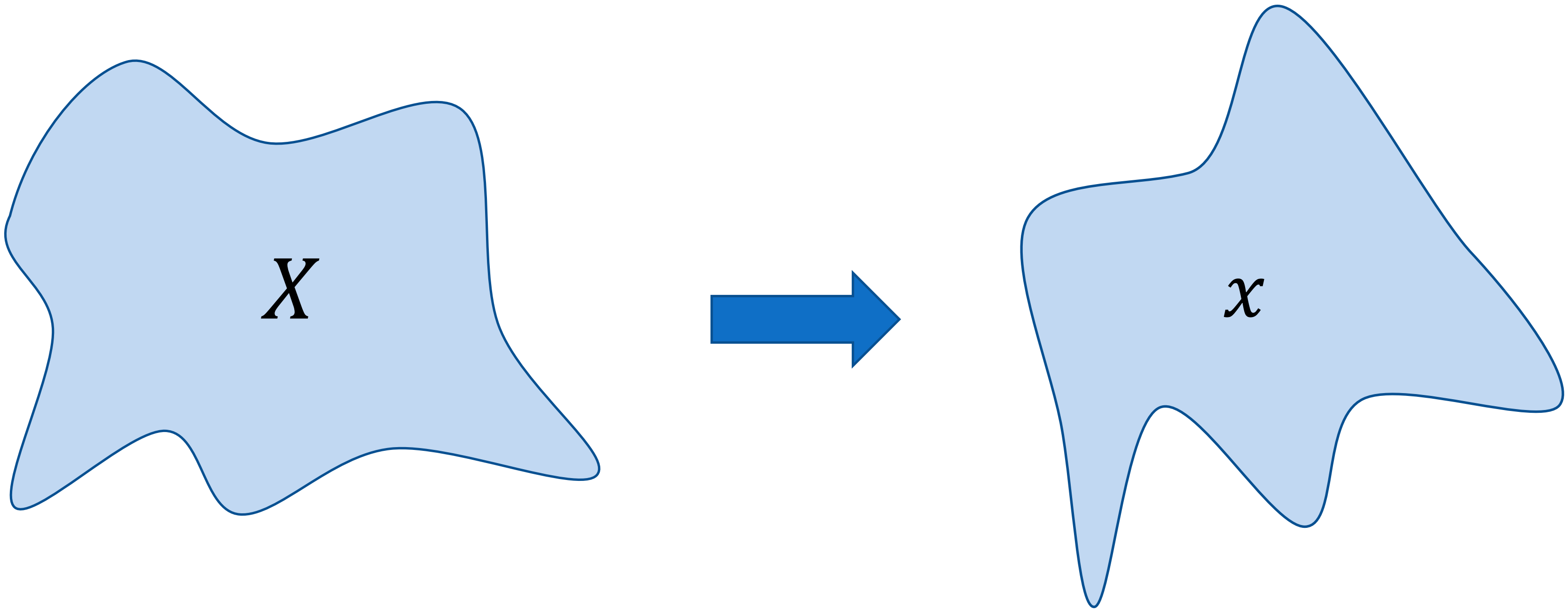
Displacement Map



$$x = \varphi(X) = SX$$

Deformation Gradient

$$x = \varphi(X) \qquad F = \frac{\partial \varphi(X)}{\partial X}$$



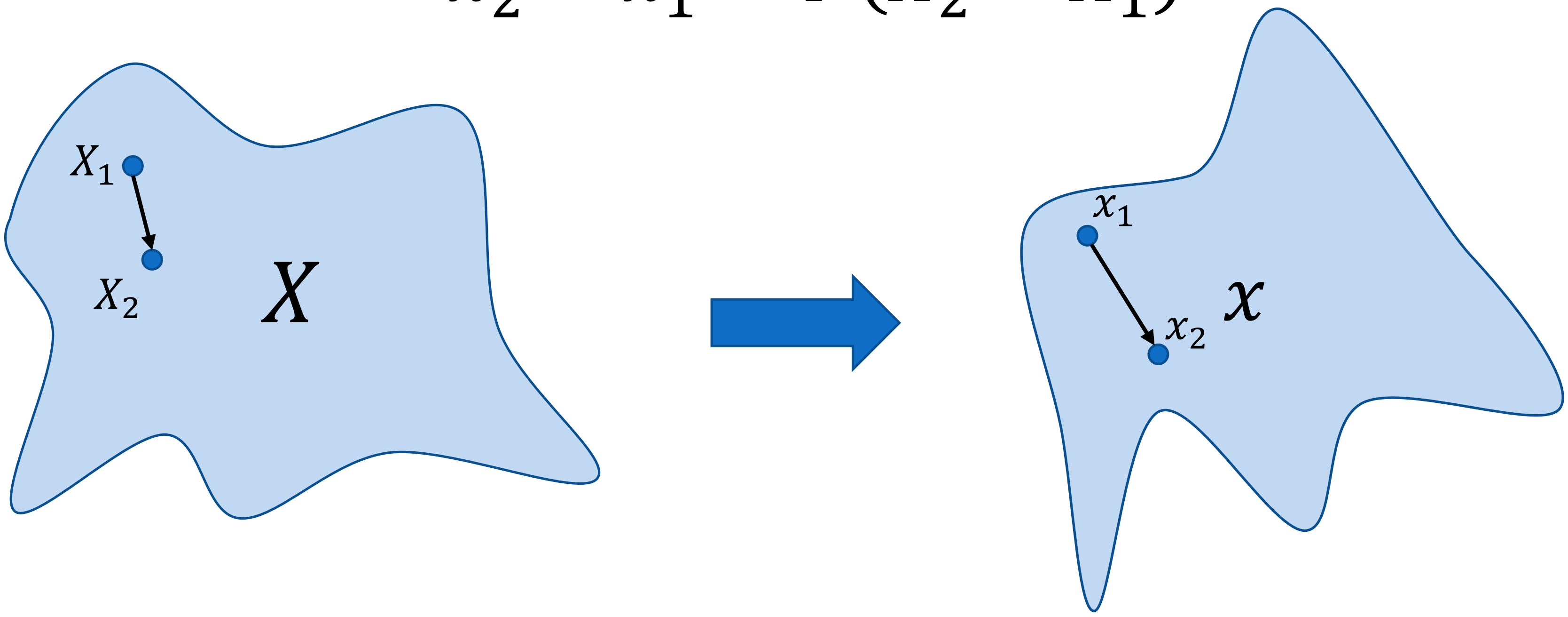
Reference Configuration/
Material Space

Deformed Configuration

Deformation Gradient

$$x = \varphi(X) \qquad F = \frac{\partial \varphi(X)}{\partial X}$$

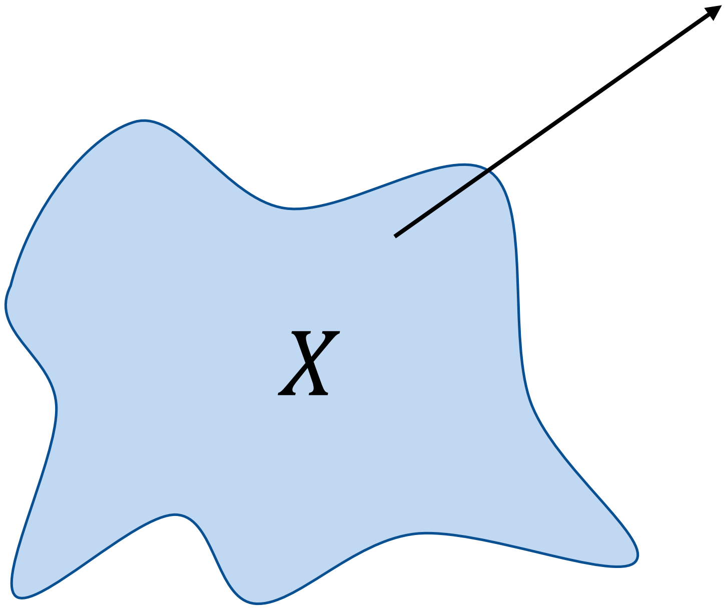
$$x_2 - x_1 \approx F(X_2 - X_1)$$



Reference Configuration/
Material Space

Deformed Configuration

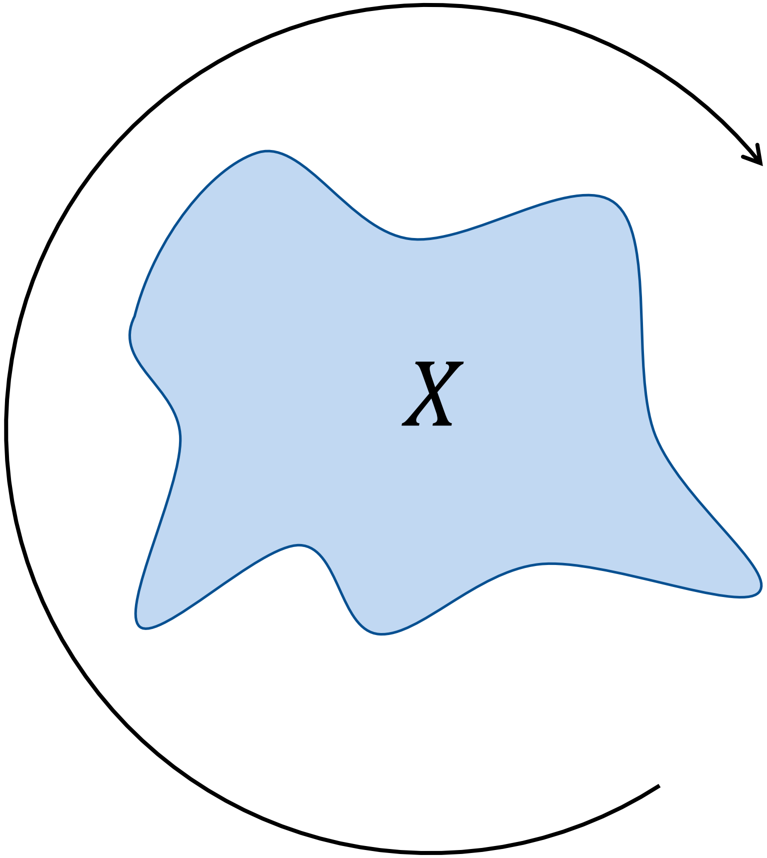
Displacement Map



$$x = \varphi(X) = X + t$$

$$F = \mathrm{I}$$

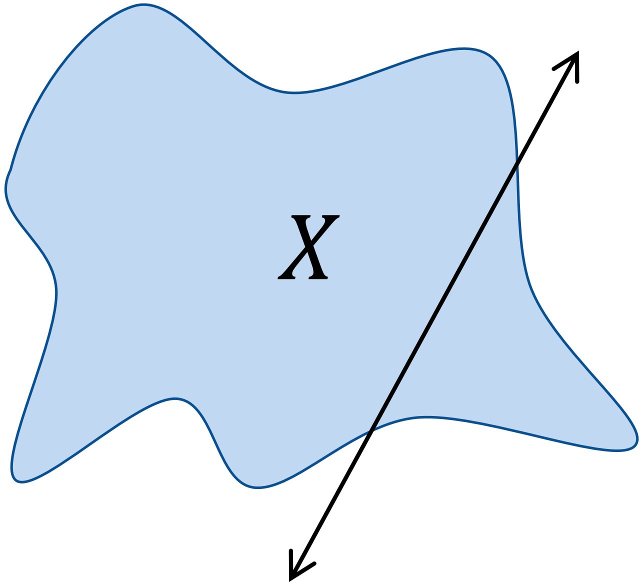
Displacement Map



$$x = \varphi(X) = RX$$

$$F = R$$

Displacement Map

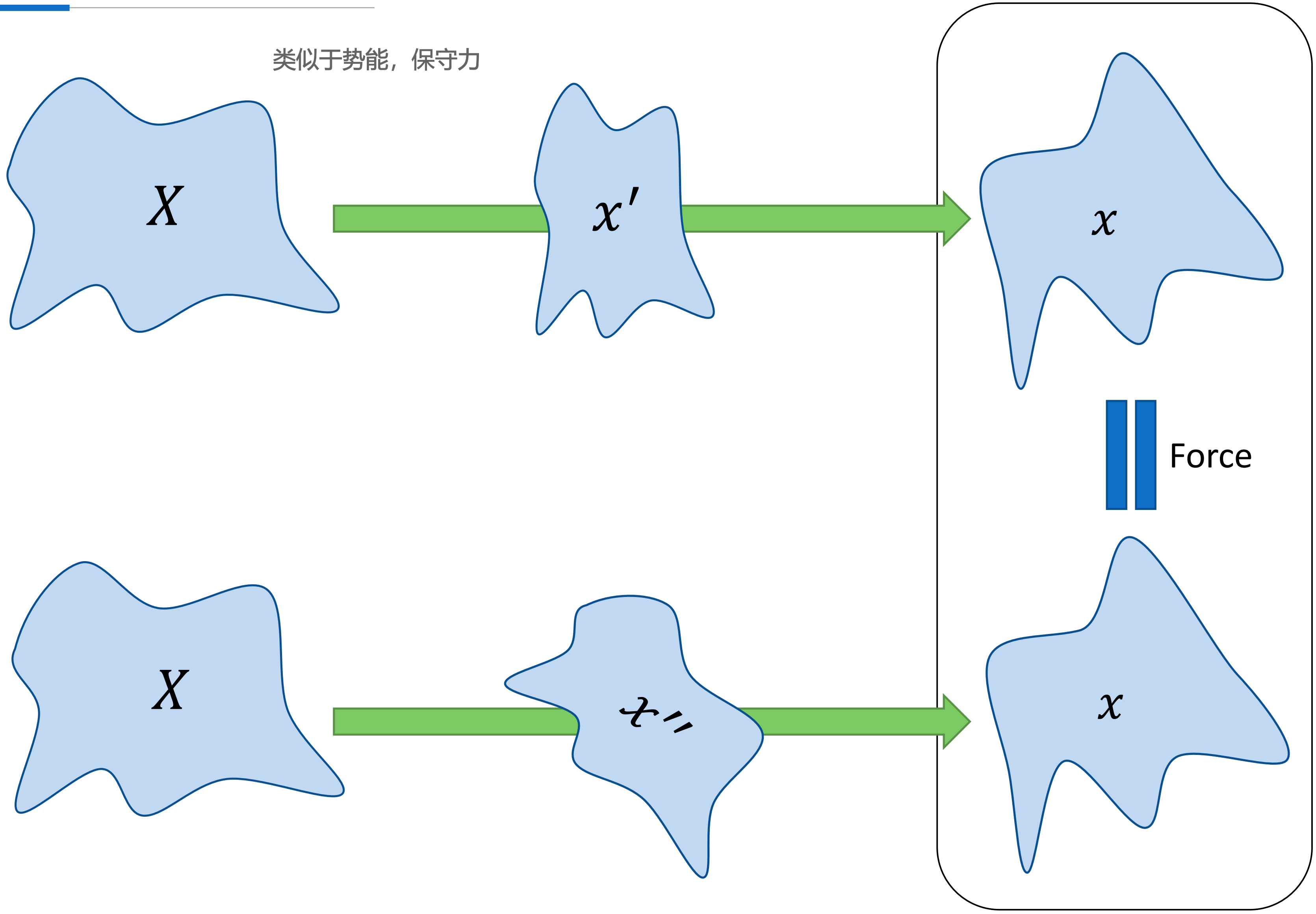


$$x = \varphi(X) = SX$$

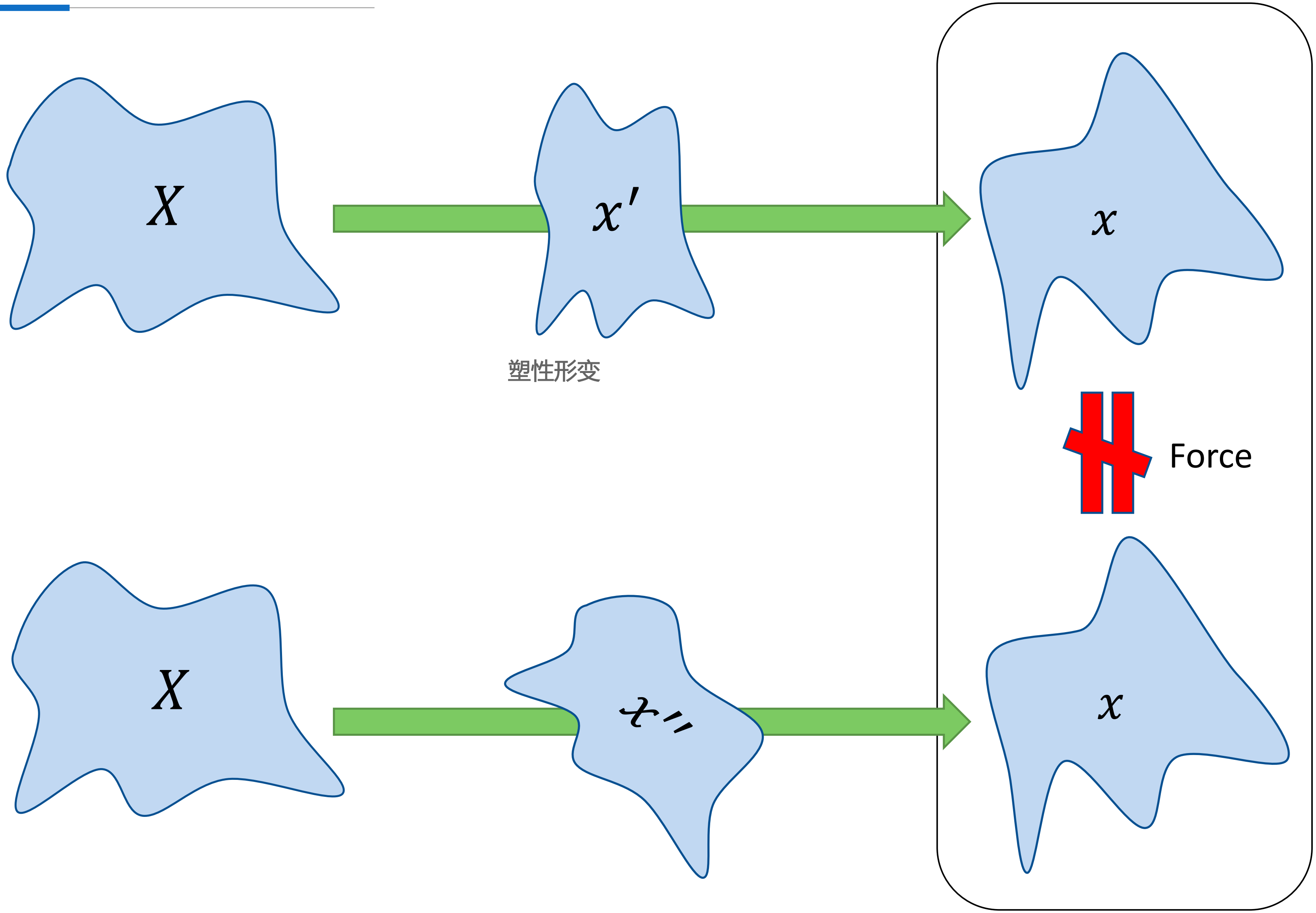
$$F = S$$

Hyperelasticity

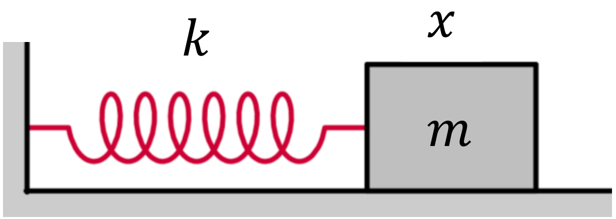
类似于势能，保守力



Not Hyperelastic?



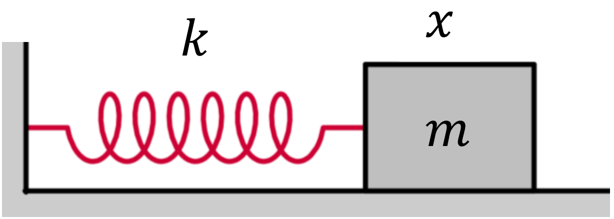
Energy and Force



$$f = -k(x - x_0)$$

$$E = \frac{1}{2}k(x - x_0)^2$$

Energy and Force



$$f = -k(x - x_0)$$

$$E = \frac{1}{2}k(x - x_0)^2$$

$$f = -\frac{dE}{dx}$$

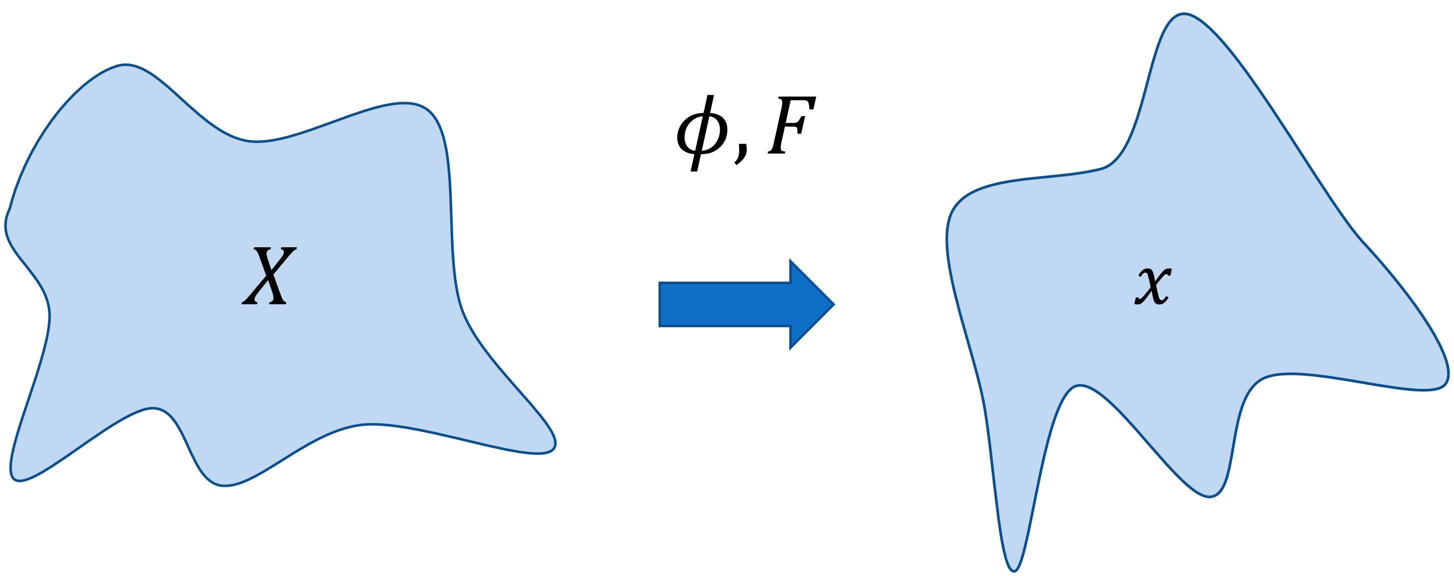
Energy and Force

$$f(x) = -\nabla_{\mathrm{x}} E(x)$$

$$E(x) = \int_{\Omega} \Psi(\phi; X) dX$$

Energy Density

拉格朗日视角，在material视角做

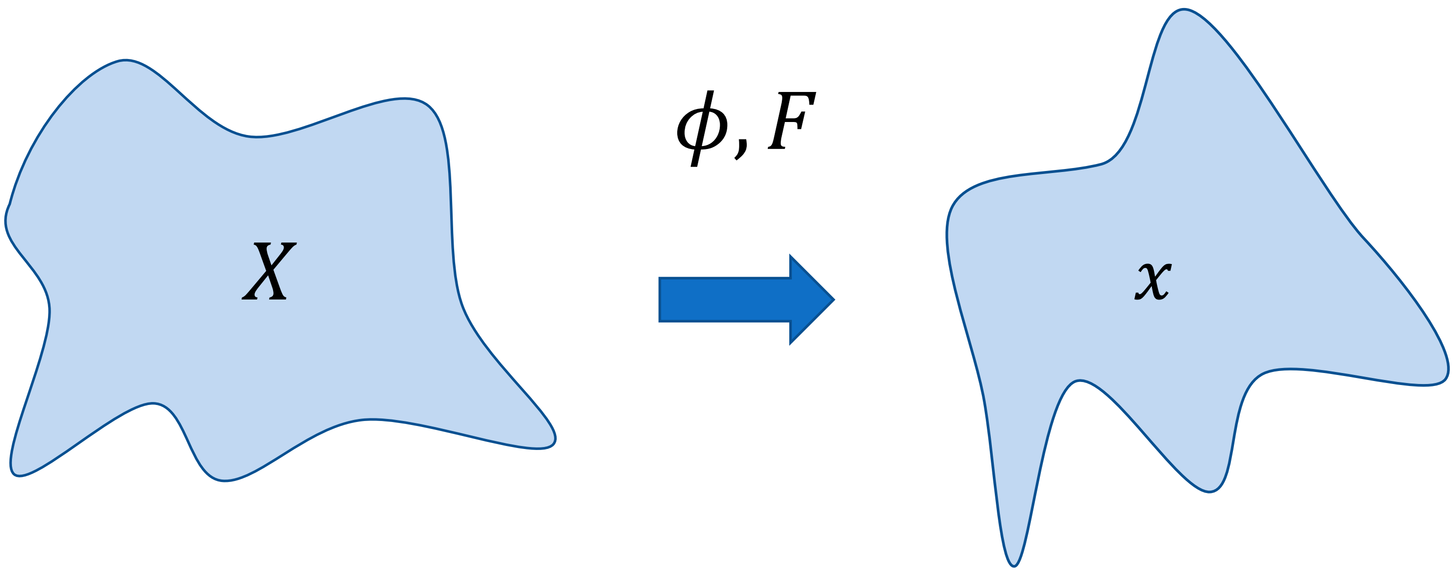


Energy and Force

$$f(x) = -\nabla_{\mathrm{x}} E(x)$$

$$E(x) = \int_{\Omega} \Psi(F)\, dX$$

Energy Density

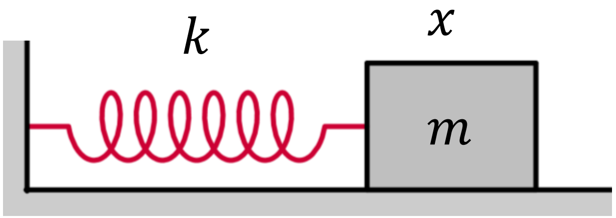


Energy Density

- What would a formula for $\Psi(\mathrm{F})$ look like?

- $\Psi(F) = \frac{k}{2} \|F\|_F^2$?

- $\Psi(F) = \frac{k}{2} \|F - I\|_F^2$



$$f = -k(x - x_0)$$

$$E = \frac{1}{2} k(x - x_0)^2$$

Stress

- A *fundamental descriptor* of force

- 1st *Piola-Kirchhoff* stress tensor
  - One of the variety of “stress” descriptors
  - Commonly used in graphics
  - For hyperelastic materials

$$P(F) = \frac{\partial \Psi(F)}{\partial F}$$

Strain

- $\epsilon(F)$: A measurement of severity of deformation
  - $\epsilon(I) = 0$ 没有形变 只有平移
  - $\epsilon(RF) = \epsilon(F)$ for $\forall R \in SO(n)$

- Example strain tensors:
  - Green strain tensors:

$$\epsilon(F) = \frac{1}{2}(F^T F - I)$$

$$\epsilon(F) = \frac{1}{2}U(\Sigma^2 V - I), \qquad F = U\Sigma V^T$$

  - Small (infinitesimal) strain tensors:

$$\epsilon(F) = \frac{1}{2}(F + F^T) - I$$

Constitutive Model of Material

- Relationship between

Force-quantities:
Ψ, P, E

Kinematic-quantities
F, ϵ, ϕ

Linear Elasticity

- Constitutive model:
  - Small (infinitesimal) strain tensors

$$\underset{3\times 3}{\epsilon(F)} = \frac{1}{2}(F + F^T) - I$$

  - Strain energy density

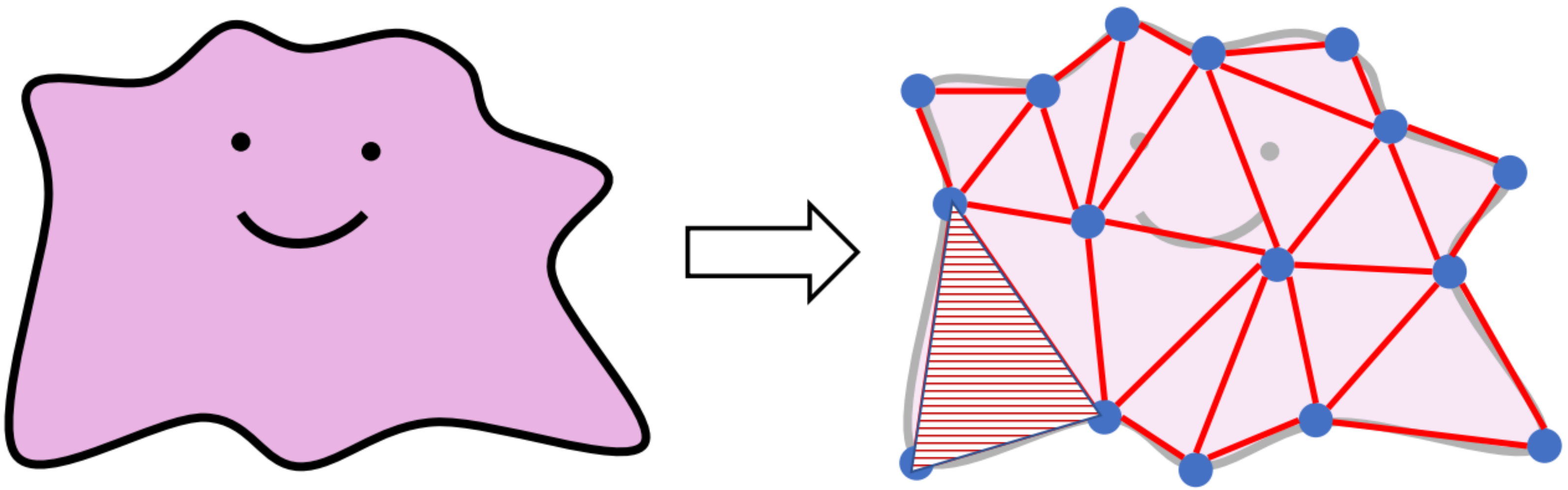
$$\boxed{A\!:\!B = \mathrm{tr}(AB)}$$

$$\Psi(F) = \mu\, \epsilon\!:\!\epsilon + \frac{\lambda}{2}\,\mathrm{tr}^2(\epsilon)$$

or

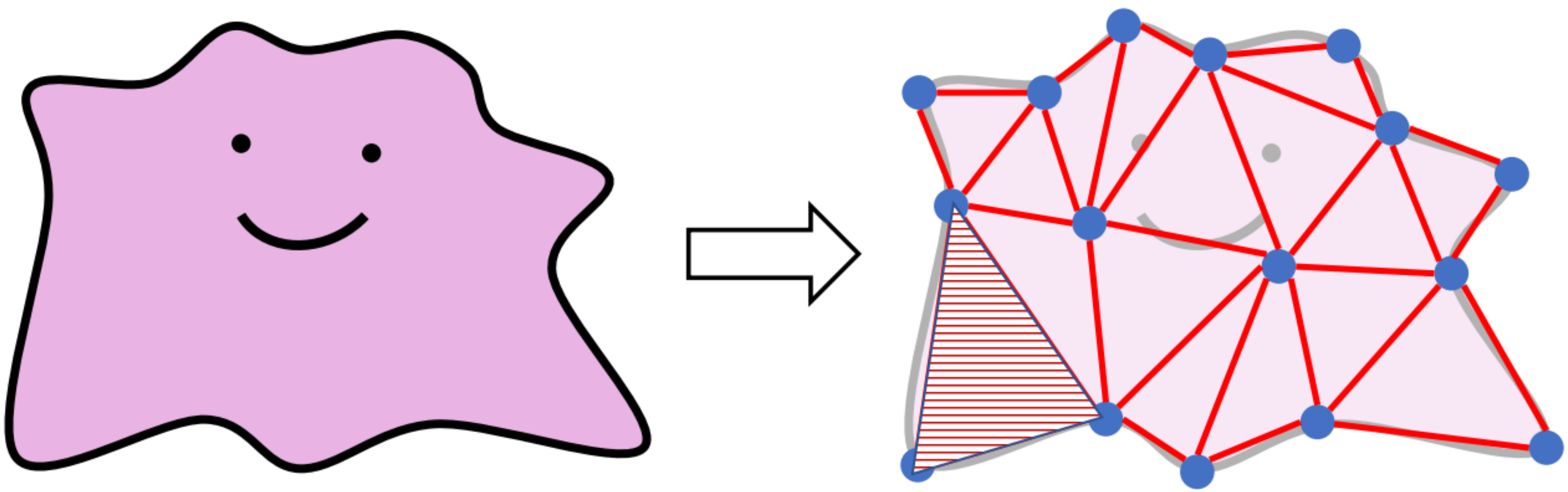
$$P = 2\mu\, \epsilon + \lambda\, \mathrm{tr}(\epsilon)$$

Discretization



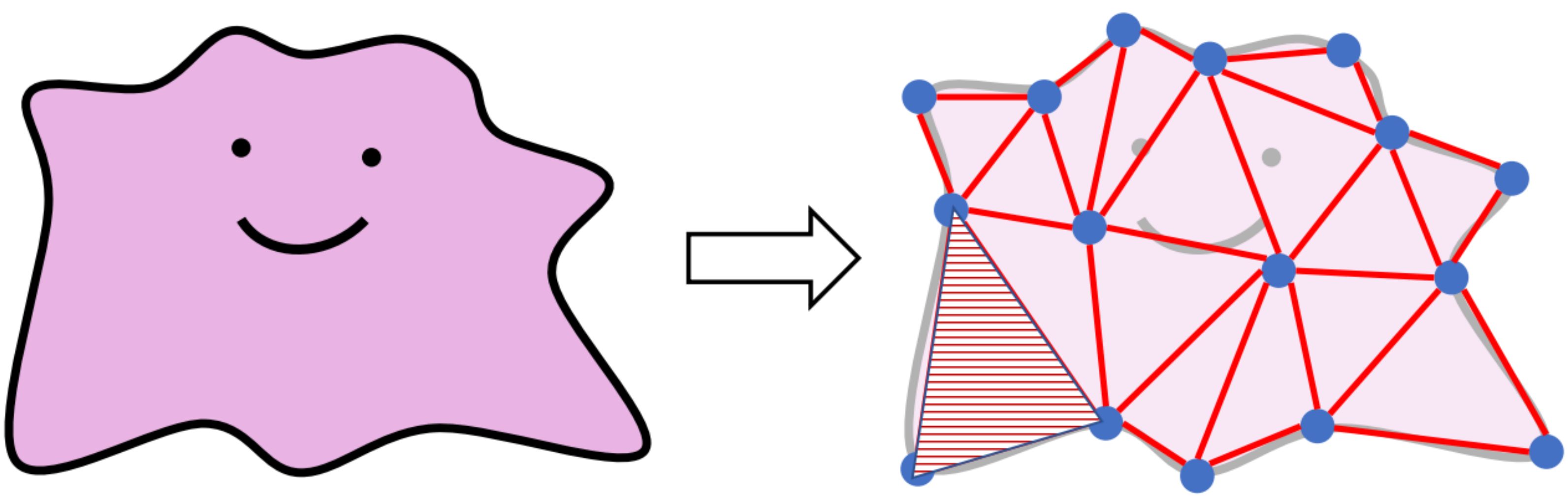
Energy

$$E(x) = \int_{\Omega} \Psi(F)\,dX \quad \Rightarrow \quad E(x) = \sum_{\Omega_i} \int_{\Omega_\mathrm{i}} \Psi(F)\,dX$$

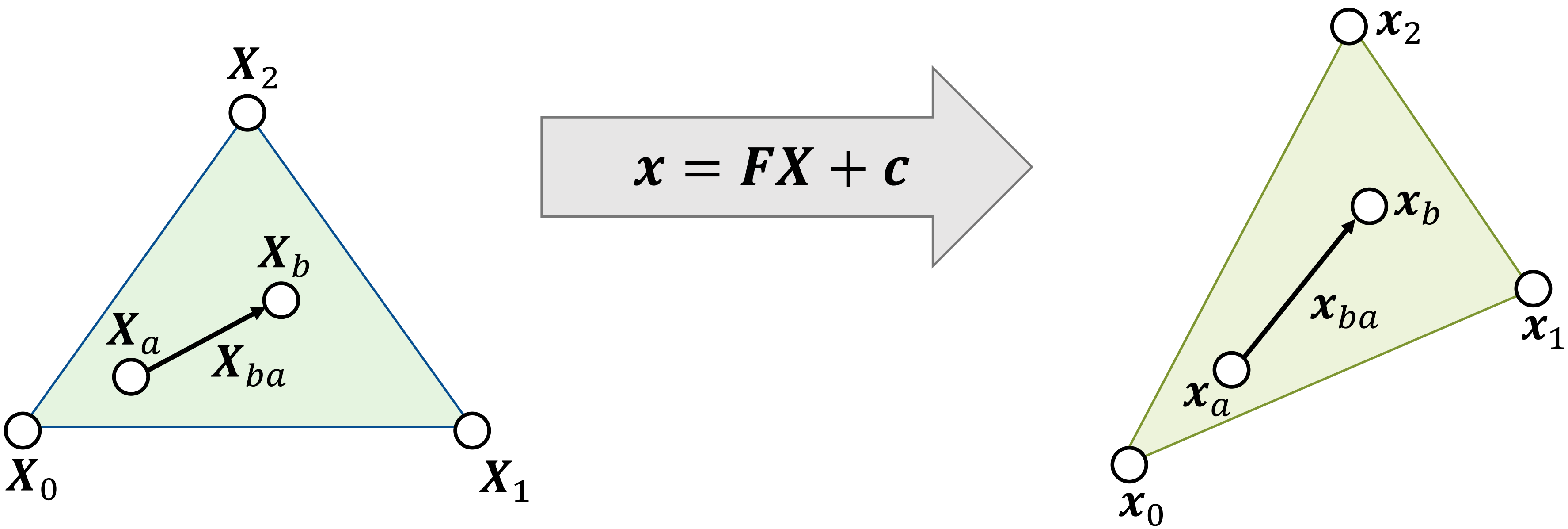


Energy

$$E(x) = \int_{\Omega} \Psi(F)\, dX \quad \Rightarrow \quad E(x) = \sum_{\Omega_i} W_i \Psi(F_i)$$



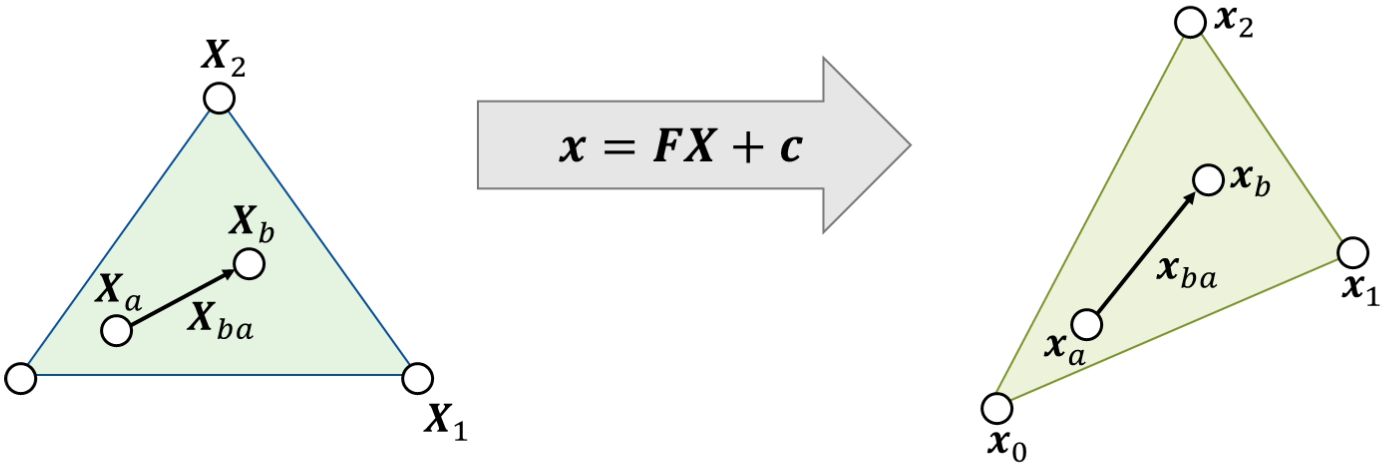
Linear Discretization



Reference Configuration
Material Space

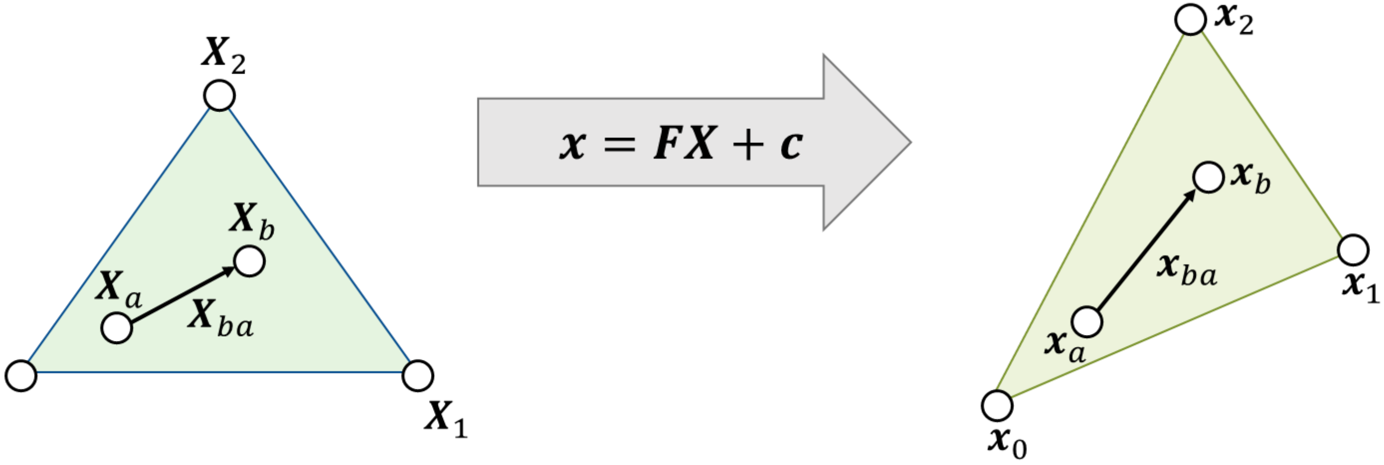
Deformed Configuration

Linear Triangular Elements



$$[x_1 - x_0 \quad x_2 - x_0] = F[X_1 - X_0 \quad X_2 - X_0]$$

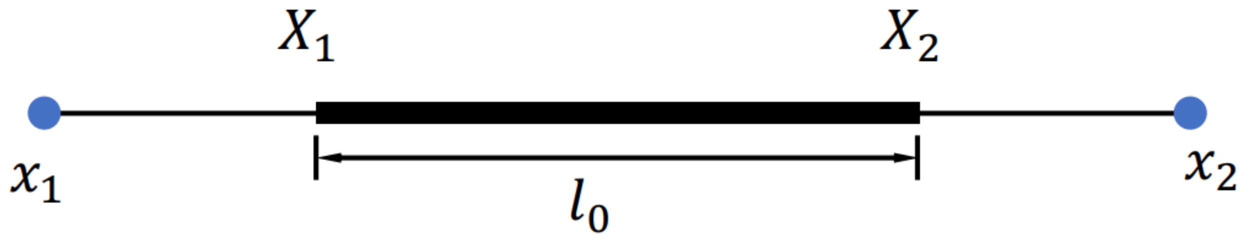
Linear Triangular Elements



$$[x_1 - x_0 \quad x_2 - x_0] = F[X_1 - X_0 \quad X_2 - X_0]$$

$$F = D_s D_m^{-1}$$

1D Example: spring



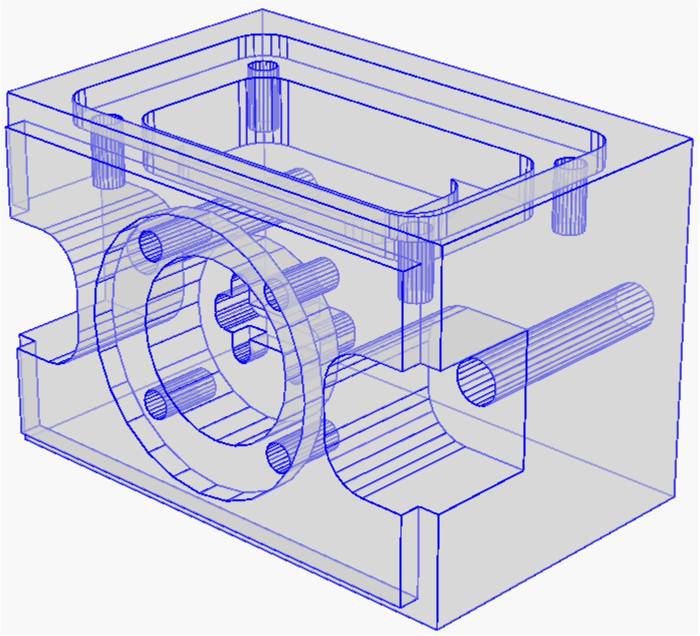
$$F = D_s D_m^{-1} = \frac{x_1 - x_2}{X_1 - X_2} = \frac{x_1 - x_2}{l_0}$$

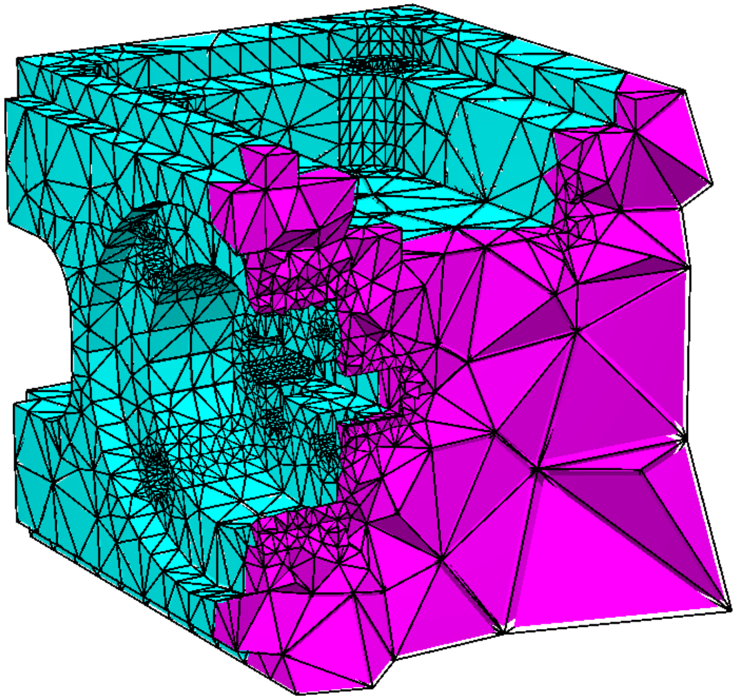
$$\epsilon = \|F\| - 1$$

$$\Psi = \mu \epsilon^2$$

$$\mathrm{E} = l_0 \Psi = \frac{1}{2} k (\|x_1 - x_2\| - l_0)^2$$

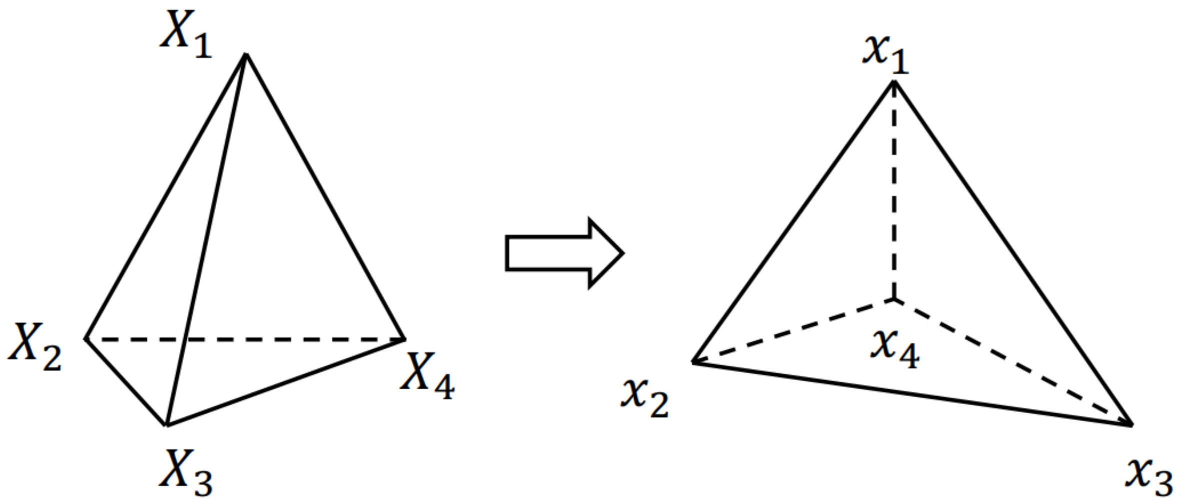
Tetrahedralization





TETGEN: https://wias-berlin.de/software/tetgen/features.html

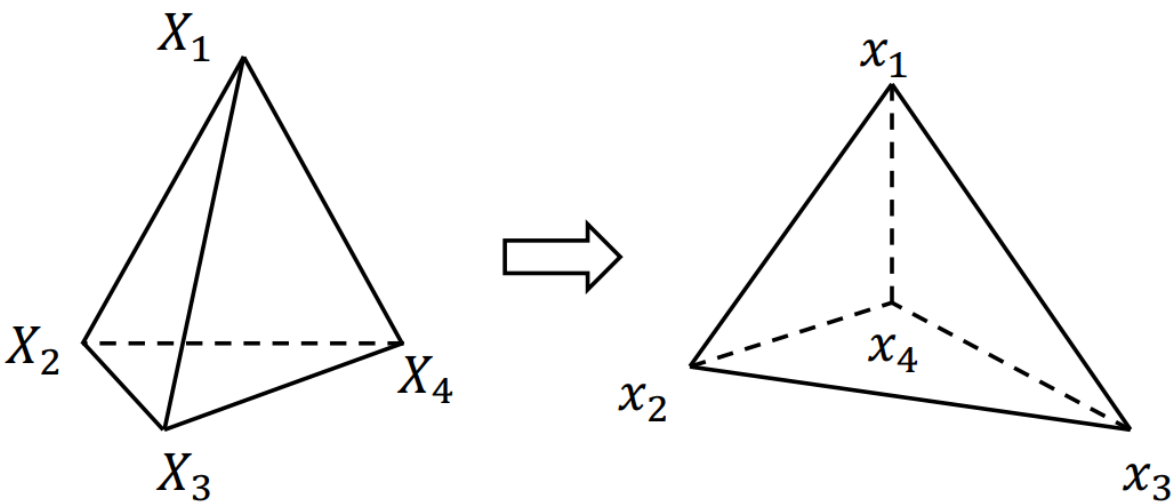
Linear Tetrahedral Elements



$$\underbrace{[x_1 - x_4 \quad x_2 - x_4 \quad x_3 - x_4]}_{D_s} = F\underbrace{[X_1 - X_4 \quad X_2 - X_4 \quad X_3 - X_4]}_{D_m}$$

$$F = D_s D_m^{-1}$$

Force Discretization



$$\vec{f}_i(\mathbf{x}) = -\frac{\partial E(\mathbf{x})}{\partial \vec{x}_i}$$

$$\mathbf{H} = \left[\vec{f}_1 \quad \vec{f}_2 \quad \vec{f}_3\right] = -W\mathbf{P}(\mathbf{F})\mathbf{D}_m^{-T} \quad \text{and} \quad \vec{f}_4 = -\vec{f}_1 - \vec{f}_2 - \vec{f}_3$$

Simulation of Deformable Solids

Algorithm 1 Batch computation of elastic forces on a tetrahedral mesh

procedure PRECOMPUTATION($\mathbf{x}, \mathbf{B}_m[1 \dots M], W[1 \dots M]$)

  for each $\mathcal{T}_e = (i, j, k, l) \in \mathcal{M}$ **do** ▷ M is the number of tetrahedra

    $\mathbf{D}_m \leftarrow \begin{bmatrix} X_i - X_l & X_j - X_l & X_k - X_l \\ Y_i - Y_l & Y_j - Y_l & Y_k - Y_l \\ Z_i - Z_l & Z_j - Z_l & Z_k - Z_l \end{bmatrix}$

    $\mathbf{B}_m[e] \leftarrow \mathbf{D}_m^{-1}$

    $W[e] \leftarrow \frac{1}{6}\det(\mathbf{D}_m)$ ▷ W is the undeformed volume of $\mathcal{T}_e$

  end for

end procedure

procedure COMPUTEELASTICFORCES($\mathbf{x}, \mathbf{f}, \mathcal{M}, \mathbf{B}_m[], W[]$)

  $\mathbf{f} \leftarrow \mathbf{0}$ ▷ $\mathcal{M}$ is a tetrahedral mesh

  for each $\mathcal{T}_e = (i, j, k, l) \in \mathcal{M}$ **do**

    $\mathbf{D}_s \leftarrow \begin{bmatrix} x_i - x_l & x_j - x_l & x_k - x_l \\ y_i - y_l & y_j - y_l & y_k - y_l \\ z_i - z_l & z_j - z_l & z_k - z_l \end{bmatrix}$

    $\mathbf{F} \leftarrow \mathbf{D}_s \mathbf{B}_m[e]$

    $\mathbf{P} \leftarrow \mathbf{P}(\mathbf{F})$ ▷ From the constitutive law

    $\mathbf{H} \leftarrow -W[e]\mathbf{P}\left(\mathbf{B}_m[e]\right)^T$

    $\vec{f}_i \mathrel{+}= \vec{h}_1,\ \vec{f}_j \mathrel{+}= \vec{h}_2,\ \vec{f}_k \mathrel{+}= \vec{h}_3$ ▷ $\mathbf{H} = \left[\vec{h}_1\ \vec{h}_2\ \vec{h}_3\right]$

    $\vec{f}_l \mathrel{+}= (-\vec{h}_1 - \vec{h}_2 - \vec{h}_3)$

  end for

end procedure