Question 1

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0.1 Log-Likelihood and Score Funtion

Log-likelihood:

$$f_Y(y) = \frac{\lambda^y e^{-\lambda}}{y!},\ \lambda \geq 0,\ y \in \{0, 1, 2, ...\}$$

$$L(\lambda|\mathbf{Y}) = \prod_{i=1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!}$$

$$l(\lambda|\mathbf{Y}) = ln\left(\prod_{i=1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!}\right)$$

$$l(\lambda|\mathbf{Y}) = \sum_{i=1}^{n} ln\left((e^{-\lambda})(\frac{1}{Y_i!})(\lambda \sum_{i=1}^{n} Y_i)\right)$$

$$l(\lambda|\mathbf{Y}) = \sum_{i=1}^{n} ln(e^{-\lambda}) - ln(Y_i!) + ln(\lambda) \sum_{i=1}^{n} Y_i$$

$$l(\lambda|\mathbf{Y}) = \sum_{i=1}^{n} -\lambda - ln(Y_i!) + ln(\lambda) \sum_{i=1}^{n} Y_i$$

$$l(\lambda|\mathbf{Y}) = -n\lambda - \sum_{i=1}^{n} ln(Y_i!) + ln(\lambda) \sum_{i=1}^{n} Y_i$$

Score Function:

$$U(\lambda) = \frac{\delta\, ln(L(\lambda|\mathbf{Y}))}{\delta\lambda}$$

$$U(\lambda) = \frac{\delta}{\delta\lambda}\left[-n\lambda - \sum_{i=1}^{n} ln(Y_i!) + ln(\lambda) \sum_{i=1}^{n} Y_i\right]$$

$$U(\lambda) = -n + \frac{\sum_{i=1}^{n} Y_i}{\lambda}$$

0.2 Maximum Likelihood Estimator

$$U(\lambda) = -n + \frac{\sum_{i=1}^{n} Y_i}{\lambda}$$

$$0 = -n + \frac{\sum_{i=1}^{n} Y_i}{\lambda}$$

$$n = \frac{\sum_{i=1}^{n} Y_i}{\lambda}$$

$$\frac{1}{n} = \frac{\lambda}{\sum_{i=1}^{n} Y_i}$$

$$\lambda = \frac{\sum_{i=1}^{n} Y_i}{n}$$

$$\hat{\lambda}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Check that it is a Maximum:

$$U'(\lambda) = \frac{\delta^2}{\delta^2\lambda}\left[\frac{\sum_{i=1}^{n} Y_i}{\lambda} - n\right]$$

$$U'(\lambda) = -\frac{\sum_{i=1}^{n} Y_i}{\lambda^2}$$

$$-\frac{\sum_{i=1}^{n} Y_i}{\lambda^2} < 0$$

The second derivative is negative, $\therefore$ this is a maximum.

0.3 Information about λ

$$I(\lambda) = Var[U(\lambda)] = E[U(\lambda)^2] = -E[U'(\lambda)]$$

$$I(\lambda) = -E\left[\frac{\delta^2 l(\lambda)}{\delta^2\lambda}\right]$$

$$I(\lambda) = -E\left[-\frac{\sum_{i=1}^{n} Y_i}{\lambda^2}\right]$$

$$I(\lambda) = n\lambda\left(\frac{1}{\lambda^2}\right)$$

$$I(\lambda) = \frac{n}{\lambda}$$

0.4 95% Wald Confidence Interval

$$Wald_{CI} = \left[\hat{\lambda} - Z_{\frac{\alpha}{2}} \hat{SE}(\hat{\lambda}), \hat{\lambda} + Z_{\frac{\alpha}{2}} \hat{SE}(\hat{\lambda})\right]$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} Y_i = 1.3;\ Z_{\frac{\alpha}{2}} = 1.96;\ \hat{SE}(\hat{\lambda}) = \sqrt{\frac{\hat{\lambda}}{n}} = \sqrt{\frac{1.3}{50}} = 0.1612452$$

$$Wald_{CI} = [1.3 - (1.96)(0.1612452), 1.3 + (1.96)(0.1612452)]$$

$$Wald_{CI} = [0.9839594, 1.616041]$$

0.5 Hypothesis Testing: Score Test

$$H_0 : \lambda = 1, H_A : \lambda \neq 1$$

Test Statistic:

$$T_s = \frac{U(\lambda)^2}{I(\lambda)}$$

$$T_s = \frac{(-n + \frac{\sum_{i=1}^{n} Y_i}{\lambda})^2}{\frac{n}{\lambda}}$$

$$T_s = \frac{(-50 + \frac{65}{1})^2}{\frac{50}{1}}$$

P-value:

```
> pchisq(4.5, 1, lower.tail = F)
[1] 0.03389485
```

$$4.5 \sim X^2_{1,0.95}$$

0.6 Hypothesis Testing: Likelihood Ratio Test

$$H_0 : \lambda = 1, H_A : \lambda \neq 1$$

Test Statistic:

$$T_s = -2\left[l(\lambda_0|\mathbf{Y}) - l(\hat{\lambda}|\mathbf{Y})\right]$$

$$T_s = -2\left[-n\lambda_0 - \sum_{i=1}^{n} ln(Y_i!) + ln(\lambda_0)\sum_{i=1}^{n} Y_i - (-n\hat{\lambda} - \sum_{i=1}^{n} ln(Y_i!) + ln(\hat{\lambda}))\right]$$

$$T_s = -2\left[-n\lambda_0 + ln(\lambda_0)\sum_{i=1}^{n} Y_i + n\hat{\lambda} - ln(\hat{\lambda})\sum_{i=1}^{n} Y_i\right]$$

$$T_s = -2\left[-(50)(1) - ln(1)(65) + (50)(1.3) - ln(1.3)(65)\right]$$

$$T_s = 4.107$$

P-value:

```
> pchisq(4.107, 1, lower.tail = F)
[1] 0.04270605
```

$$4.107 \sim X^2_{1,0.95}$$

0.7 Hypothesis Testing: Wald Test

$$H_0 : \lambda = 1, H_A : \lambda \neq 1$$

Test Statistic:

$$T_s = \frac{(\hat{\lambda} - \lambda_0)^2}{Var[\hat{\lambda}]} = I(\hat{\lambda})(\hat{\lambda} - \lambda_0)^2$$

$$T_s = \left(\frac{n}{\hat{\lambda}}\right)(\hat{\lambda} - \lambda_0)^2$$

$$T_s = \frac{50}{1.3}(1.3 - 1)^2$$

$$T_s = 3.461$$

P-value:

```
> pchisq(3.46, 1, lower.tail = F)
[1] 0.06287031
```

Fail to reject our the null hypothesis that lambda is distributed as Chi-Square.