Question 2

Joe Froelicher

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$$f_Y(y) = \frac{\lambda^y e^{-\lambda}}{y!},\ \lambda \geq 0,\ y \in \{0, 1, 2, ...\}$$
$$f_Y^*(y^*) = I(Y_i > 0)$$

0.1 Likelihood of Y_i^*

$$Y_i^* \sim bernoulli(p)$$
$$p = P(Y = 1)$$
$$f_Y^*(y^*) = P(Y_i = y|p) = p^y(1-p)^{1-y} I_{(0,1)}^{(p)}, y \in \{0, 1\}$$
$$L(p|Y_1^*, Y_2^*, ...Y_n^*) = \prod_{i=1}^{n} f_{Y_i^*}(y_i^*|p)$$
$$L(p|Y_1^*, Y_2^*, ...Y_n^*) = \prod_{i=1}^{n} \left[p_i^{y*}(1-p)^{1-y_i^*} \right]$$
$$L(p|Y_1^*, Y_2^*, ...Y_n^*) = p^{\sum_{i=1}^n y_i^*}(1-p)^{\sum_{i=1}^n 1-y_i^*}$$
$$L(p|Y_1^*, Y_2^*, ...Y_n^*) = p^{\sum_{i=1}^n y_i^*}(1-p)^{n-\sum_{i=1}^n y_i^*}$$
$$l(p|Y_i^*) = ln(L(p|Y_1^*, Y_2^*, ...Y_n^*))$$
$$l(p|Y_i^*) = \sum_{i=1}^{n} y_i^* ln(p) + (n - \sum_{i=1}^{n} y_i^*)(ln(1-p))$$

0.2 Maximum Likelihood Estimation

$$U(p) = \frac{\delta\, l(p|Y_i^*)}{\delta p}$$
$$U(p) = \frac{\delta}{\delta p} \left[\sum_{i=1}^{n} y_i^* ln(p) + (n - \sum_{i=1}^{n} y_i^*)(ln(1-p)) \right]$$
$$U(p) = \frac{\sum_{i=1}^n y_i^*}{p} - \frac{n - \sum_{i=1}^n y_i^*}{1-p}$$

Set $U(p) = 0$ and solve for p:

$$0 = \frac{\sum_{i=1}^{n} y_i^*}{p} - \frac{n - \sum_{i=1}^{n} y_i^*}{1-p}$$

$$\frac{\sum_{i=1}^{n} y_i^*}{p} = \frac{n - \sum_{i=1}^{n} y_i^*}{1-p}$$

$$\frac{1-p}{p} = \frac{n - \sum_{i=1}^{n} y_i^*}{\sum_{i=1}^{n} y_i^*}$$

$$\frac{1}{p} - \frac{p}{p} = \frac{n}{\sum_{i=1}^{n} y_i^*} - \frac{\sum_{i=1}^{n} y_i^*}{\sum_{i=1}^{n} y_i^*}$$

$$\frac{1}{p} = \frac{n}{\sum_{i=1}^{n} y_i^*}$$

$$p = \frac{\sum_{i=1}^{n} y_i^*}{n}$$

From a poisson distribution:

$$P(Y = 1) = 1 - P(Y = 0) = 1 - f_y(0)$$

$$f_Y(0) = \frac{\lambda^0 e^{-\lambda}}{0!}$$

$$1 - f_Y(0) = 1 - e^{-\lambda}$$

$$p = 1 - e^{-\lambda}$$

Hence:

$$1 - e^{-\lambda} = \frac{\sum_{i=1}^{n} y_i^*}{n}$$

$$1 - \frac{\sum_{i=1}^{n} y_i^*}{n} = e^{-\lambda}$$

$$ln\left(1 - \frac{\sum_{i=1}^{n} y_i^*}{n}\right) = -\lambda$$

$$\hat{\lambda}_{MLE} = -ln\left(1 - \frac{\sum_{i=1}^{n} y_i^*}{n}\right)$$

0.3 Information about λ

$$I(p) = Var[U(\lambda)] = E[U(\lambda)^2] = -E[U'(\lambda)]$$

$$I(p) = -E\left[\frac{\delta}{\delta p}\frac{\sum_{i=1}^{n} y_i^*}{p} - \frac{n - \sum_{i=1}^{n} y_i^*}{1-p}\right]$$

$$I(p) = E\left[\frac{\sum_{i=1}^{n} y_i^*}{p^2} + \frac{n - \sum_{i=1}^{n} y_i^*}{(1-p)^2}\right]$$

$$I(p) = \frac{np}{p^2} + \frac{n - np}{(1-p)^2}$$

$$I(p) = \frac{n}{p} + \frac{n(1-p)}{(1-p)^2}$$

$$I(p) = \frac{n}{p} + \frac{n}{(1-p)}$$

$$I(p) = \frac{n}{p(1-p)}$$

$$I(\lambda) = \frac{n}{(1-e^{-\lambda})(1-(1-e^{-\lambda}))}$$

$$I(\lambda) = \frac{n}{(1-e^{-\lambda})(-e^{-\lambda})}$$

0.4 Using the invariance property

Due to the invariance properties of MLE's, which implies that we know the mean of any function of λ. So $\tau(\hat{\lambda}_{MLE})$ is the MLE of $\tau(\lambda)$.

$$P(Y=1) = 1 - P(Y=0) = 1 - f_y(0)$$

$$f_Y(0) = \frac{\lambda^0 e^{-\lambda}}{0!}$$

$$1 - f_Y(0) = 1 - e^{-\lambda}$$

$$p = 1 - e^{-\lambda}$$

Hence:

$$1 - e^{-\lambda} = \frac{\sum_{i=1}^{n} y_i^*}{n}$$

$$1 - \frac{\sum_{i=1}^{n} y_i^*}{n} = e^{-\lambda}$$

$$ln\left(1 - \frac{\sum_{i=1}^{n} y_i^*}{n}\right) = -\lambda$$

$$\hat{\lambda}_{MLE} = -ln\left(1 - \frac{\sum_{i=1}^{n} y_i^*}{n}\right)$$

$$\hat{\lambda}_{MLE} = -ln(1-\bar{y})$$

Where $\bar{y}$ is the mean of $Y_i^* \sim bernoulli(p)$