Computer vision HW06 João Pinheiro

1\. a) $$\frac{\partial E}{\partial M} = \frac{\partial}{\partial M} \sum_{i=1}^{m} \| x_i' - M x_i - t \|^2$$

$$= \sum_{i=1}^{m} \frac{\partial}{\partial M} \| x_i' - M x_i - t \|^2$$

$$\| x_i' - M x_i - t \|^2 = (X_i' - MX - t)^T (X' - MX - t)$$

$$= X'^T X' - 2X'^T Mx + X^T M^T MX - 2X'^T t + 2t^T MX + t^T t$$

$$\frac{\partial E}{\partial M} = \sum_{i=1}^{m} \left(-2x_i' X_i^T + 2M x_i x_i^T + 2t x_i^T \right)$$

$$\frac{\partial E}{\partial t} = \sum_{i=1}^{m} \left(-2x_i' + 2M x_i + 2t \right)$$

b) $M_{2\times 2}$ $\quad t_{2\times 1}$

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \qquad \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$X' = MX + t \iff \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$x_1' = m_1 x_1 + m_2 x_2 + t_1$$

$$x_2' = m_3 x_1 + m_4 x_2 + t_2$$

For each point we have these 2 transformations equations

For n points, we'll have $2n$ equations

We can rewrite the equation in matrix form

$$A_i = \underset{2\times 6}{\begin{pmatrix} x_{i_1} & x_{i_2} & 0 & 0 & 1 & 0 \\ 0 & 0 & x_{i_1} & x_{i_2} & 0 & 1 \end{pmatrix}}$$

and $h = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{pmatrix}^{6\times 1}$

So for n points we will have matrix A $\begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_n \end{pmatrix}^{2n\times 6}$

$$\underset{(2n\times 6)}{A}\ \underset{(6\times 1)}{h} = \underset{(2n\times 1)}{b}$$

Normal equations: least squares solution to

$$A h = b$$

$$\downarrow$$

$$\underbrace{\underset{(6\times 2n)}{A^T}\ \underset{(2n\times 6)}{A}}_{\substack{(6\times 6)\\ S}}\ \underset{(6\times 1)}{h} = \underbrace{\underset{(6\times 2n)}{A^T}\ \underset{(2n\times 1)}{b}}_{\substack{(6\times 1)\\ \mu}}$$

$$S \quad \begin{pmatrix} \sum_{i=1}^{m} x_{i1}^2 & \sum x_{i1}x_{i2} & 0 & 0 & \sum x_{i1} & 0 \\ \sum x_{i1}x_{i2} & \sum x_{i2}^2 & 0 & 0 & \sum x_{i2} & 0 \\ 0 & 0 & \sum x_{i1}^2 & \sum x_{i1}x_{i2} & 0 & \sum x_{i1} \\ 0 & 0 & \sum x_{i1}x_{i2} & \sum x_{i2}^2 & 0 & \sum x_{i2} \\ \sum x_{i1} & \sum x_{i2} & 0 & 0 & m & 0 \\ 0 & 0 & \sum x_{i1} & \sum x_{i2} & 0 & m \end{pmatrix}$$

$$\mu \quad \begin{pmatrix} \sum x_{i1} x'_{i1} \\ \sum x_{i2} x'_{i1} \\ \sum x_{i1} x'_{i2} \\ \sum x_{i2} x'_{i2} \\ \sum x'_{i1} \\ \sum x'_{i2} \end{pmatrix}$$

c) $x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x'_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow x'_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$x_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow x'_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

By substitution

$$\begin{array}{ccc} S & h & u \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{pmatrix} & \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{pmatrix} & \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{pmatrix} \end{array}$$

$$S\,h = u$$

$$\begin{cases} m_1 + t_1 = 3 \\ m_2 + t_1 = 1 \\ m_3 + t_2 = 2 \\ m_4 + t_2 = 4 \\ m_1 + m_2 + 3t_1 = 5 \\ m_3 + m_4 + 3t_2 = 8 \end{cases} \quad (=) \quad \begin{cases} m_1 = 2 \\ t_1 = 1 \\ m_3 = 0 \\ t_2 = 2 \\ m_2 = 0 \\ m_4 = 2 \end{cases}$$

$$h = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

COMPUTER VISION HW 06

Yoão Pinheiro

Exercise 2.

$$x' = sRx + t \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = s \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

a) $$v' = x'_2 - x'_1 = \begin{pmatrix} x'_{2x} - x'_{1x} \\ x'_{2y} - x'_{1y} \end{pmatrix}$$

$$v = x_2 - x_1 = \begin{pmatrix} x_{2x} - x_{1x} \\ x_{2y} - x_{1y} \end{pmatrix}$$

The angle θ can be calculated using dot product and the cross product of the unit vectors:

$$\vec{v} \cdot \vec{v}' = \|v\| \, \|v'\| \cos\theta \Leftrightarrow$$

Unit vectors

$$\vec{v} = \frac{v}{\|v\|}$$

$$\Leftrightarrow \cos\theta = \frac{\vec{v} \cdot \vec{v}'}{\|v\| \, \|v'\|}$$

$$\Leftrightarrow \theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{v}'}{\|v\| \, \|v'\|} \right) \text{ where } \begin{cases} \vec{v} = \dfrac{v}{\|v\|} \\ \vec{v}' = \dfrac{v'}{\|v'\|} \end{cases}$$

b) $\|v\| = \sqrt{(x_{2x} - x_{1x})^2 + (x_{2y} - x_{1y})^2}$

$\|v'\| = \sqrt{(x'_{2x} - x'_{1x})^2 + (x'_{2y} - x'_{1y})^2}$

the scale factor s can be calculated through the ratio of the norms of $\vec{v}$ and $\vec{v}'$:

$$s = \frac{\|v'\|}{\|v\|}$$

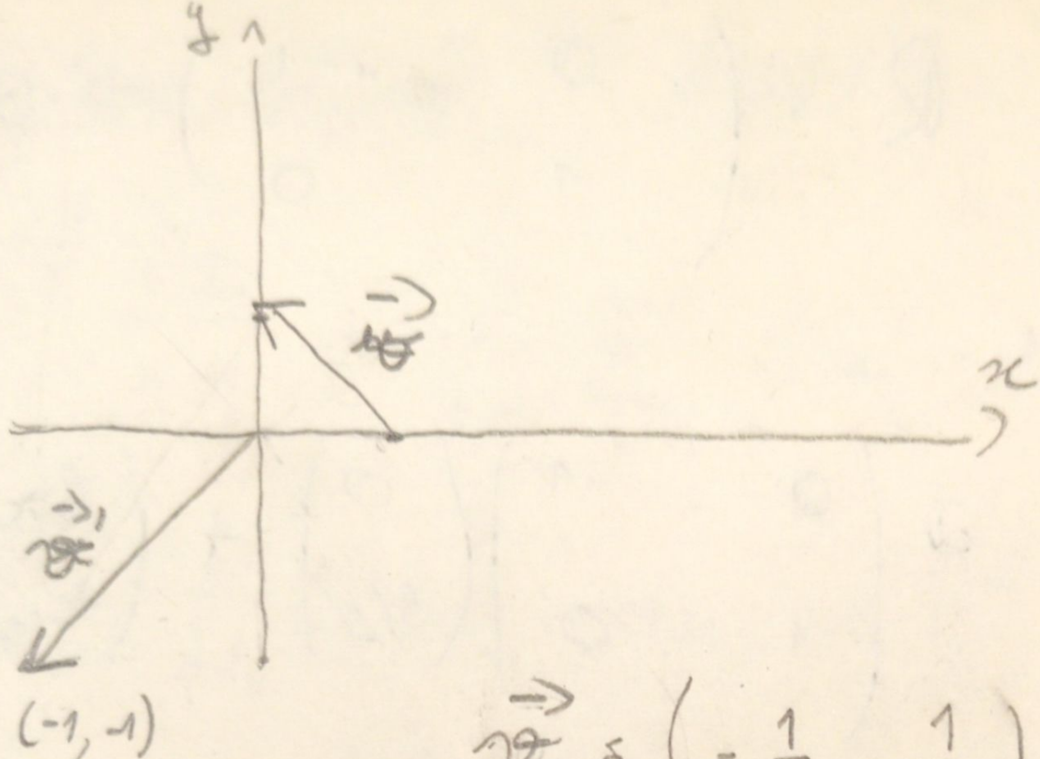
c) $t = x_1' - sRx_1$ (⇔)

(⇔) $t = x_1' - \dfrac{\|x_1\|}{\|x_1'\|} R x_1$

(⇔) $t = x_1' - \dfrac{\|x_1\|}{\|x'_1\|} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos\theta) \end{pmatrix} x_1$

Where $\theta = \cos^{-1}\left(\dfrac{\frac{x}{\|x\|} \cdot \frac{x'}{\|x'\|}}{\|x\| \, \|x'\|} \right)$

d)



$$\vec{v} = \left(-\frac{1}{2}, \frac{1}{2} \right)$$

$$\vec{v}' = \left(-1, -1 \right)$$

$$\lambda = \frac{\|\vec{v}'\|}{\|\vec{v}\|} = \frac{\sqrt{2}}{\sqrt{\frac{1}{2}}} = \frac{\sqrt{2}}{\frac{1}{\sqrt{2}}} = 2$$

$$\frac{\sqrt{1}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \cos^{-1}\left(\frac{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \cdot \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)}{\sqrt{2} \cdot \frac{1}{\sqrt{2}}} \right)$$

$$= \cos^{-1}\left(\frac{1}{2} + \left(-\frac{1}{2} \right) \right) = \cos^{-1}(0)$$

$$= 90^\circ = \theta$$

$$\cos(90) = 0$$

$$\text{sen}(90) = 1$$

$s = 2 \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \theta = 90^\circ$ $\frac{\pi}{2}$

$$(-1, -1) = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$(\cdot) \begin{pmatrix} t_x \\ t_y \end{pmatrix} = (-1, -1) - \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

$$(\cdot) \begin{pmatrix} t_x \\ t_y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$