MS-E2132 - Laboratory Assignments in Operations Research II

Assignment 1: Optimal flight with a glider

1 Theory of flight

We examine the optimal control of the flight of a glider or some other aircraft. A glider is affected by the following forces:

- Gravity: $G = mg$
- Lift: $L(\alpha, h, v)$
- Drag: $D(\alpha, h, v)$

Gravity acts towards the center of the earth and is constant, which corresponds to the assumptions that the mass of the glider m and the gravitational acceleration g are constant. Drag acts opposite to the velocity vector of the glider and the lift is perpendicular to it. Lift is caused by the length of the glider not being parallel to the velocity vector, but differing by the angle of attack α. This angle causes a difference in pressure between the top and bottom sides of a wing. The lift can therefore be controlled by adjusting the angle of attack. Aircraft generally have a separate measuring device for the angle, but in a glider it is controlled simply by feel. The angle is usually adjusted by flaps or other actuators. Both the lift and the drag are affected by the altitude of the glider or the aircraft h and its velocity relative to the air v.

How we proceed from here is in principle a matter of choice. The following is a general convention.

For every angle of attack, there is a corresponding lift coefficient $C_L(\alpha)$. This is generally a nonlinear function of α. At small velocities (typically below 200 m/s) and angles close to 0, however, the linear approximation $C_L(\alpha) = C_{L_\alpha}\alpha$ is sufficient. The lift coefficient depends on the geometry of the aircraft. It is generally estimated using miniature models in wind tunnel tests. The lift is calculated by multiplying the lift coefficient by the reference area of the aircraft S and the kinetic energy density of the airflow $q(h, v) = 0.5\varrho(h)v^2$, which is also called kinetic pressure. Thus, the lift is

$$L(\alpha, h, v) = C_L(\alpha)Sq(h, v),$$

where $\varrho(h)$ is the air density at altitude h. The wing area of the aircraft is often used as the reference area.

The angle of attack cannot be freely chosen. At high velocities, a large angle leads to forces that the aircraft and/or a pilot cannot handle. At low velocities, a large angle causes the airflow on the top of the wing to become disrupted, which causes a dramatic drop in lift and the aircraft stalls.

The drag consists of two components: zero drag and induced drag. The zero drag is caused by the geometry of the aircraft and the flow of air around it. It is calculated by multiplying the kinetic pressure and the reference area with the aircraft dependent zero drag coefficient C_{D_0}, i.e.,

$$D_0 = C_{D_0}Sq(h, v).$$

The induced drag is mostly caused by the non-ideal structure of the wings. The pressure difference between the top and the bottom of the wing is evened out at the tip of the wing, which causes whirls to form at the tips. A portion of the aircraft's kinetic energy is used to maintain these whirls - hence the alternative naming parasitic drag. At large angles of attack, the flow of air changes. A good approximation for the induced drag is the so called quadratic polar, which assumes that the induced drag depends on the square of the lift coefficient and the induced drag coefficient K. Then, the induced drag is

$$D_I(\alpha, h, v) = KC_L(\alpha)^2 Sq(h, v),$$

and the total drag is

$$D(\alpha, h, v) = (C_{D_0} + KC_L(\alpha)^2)Sq(h, v).$$

The expression $C_D(\alpha) = C_{D_0} + KC_L(\alpha)^2$ is called the total drag coefficient.

Just like the lift coefficient, the zero drag coefficient and the induced drag coefficient are estimated based on wind tunnel or flight tests. Both coefficients are approximately constant at subsonic speeds (below the speed of sound), but at trans- and supersonic speeds, the coefficients depend on the Mach number, which is defined as the ratio of the aircraft's velocity to the speed of sound.

2 Modeling for optimization

A model with six degrees of freedom is required to fully describe the movement and rotation of an aircraft. Modeling the rotation of the aircraft would require knowledge of its moment of inertia along various axes as well as the forces affecting these. Fortunately, it is known that the dynamics of rotation are considerably more rapid than those of movement. As a consequence, the aircraft can be represented as a single point mass when ignoring rotations.

Now, flight is examined in a vertical plane. We select as state variables the x-coordinate, the altitude h, the velocity v, and the flight path angle (i.e., the angle formed between the aircraft's velocity vector and the x-axis) γ. The lift coefficient $C_L(\alpha)$, denoted by C_L from now on, is used as the control variable. The corresponding angle of attack α could be solved from this if required. Since the changes in altitude are small, we assume that the air density is constant.

Exercise 1. Draw a free body diagram of the glider, that includes the magnitude and direction of forces acting upon it and the state variables of the point mass model.

Exercise 2. Use the diagram to derive the state equations of the model. Use the state variables from above. Show your results to the course assistant before proceeding further.

Exercise 3. Investigate the model by simulating it using numerical integration and observing the trajectories of the glider, i.e., the histories of the state variables. Use MATLAB routines such as ode45 for simulation. It is recommended that you use controls that depend only on time.

i) Try out different starting positions, i.e., the initial values of the state variables, as well as alternative parameter values to identify the effects of the parameters on the behavior of the model.

ii) Use the parameter values given at the end of the instructions from here on. Calculate the stalling speed of the glider in horizontal flight ($\gamma = 0$). The stalling speed is the speed at which the greatest possible lift becomes less than the gravitational force.

iii) Simulate horizontal flight at speeds lower than the stalling speed. Is the model valid in this situation?

3 Optimization

The objective of the flight is to glide as far along the x-axis as possible for each unit of lost altitude. The theory of aerodynamics reveals that this ratio is maximized when the ratio between the total drag coefficient and the lift coefficient is as small as possible.

Exercise 4. Examine the basis of this result.

i) Assume that the flight path angle is close to 0 ($\sin \gamma \approx \gamma, \cos \gamma \approx 1$) and that the velocity and the flight path angle are constant. Modify the state equations according to these assumptions. You obtain a static, time-invariant optimization problem. Solve it.

ii) Show that the solution of the optimization problem, i.e., the time independent optimal control, can be obtained graphically by analyzing the (C_L, C_D) curve and its tangent passing through the origin.

iii) Simulate the point mass model with the resulting constant optimal control and compare the distance traveled to that provided by the static optimization problem.

Exercise 5. Formulate a dynamic optimization problem, with the final time free, such that the objective functional is to maximize the x-coordinate at the end of the flight, the state equations are fulfilled, the control is within its boundaries, the altitude at the end is h_f, and the velocity at the end is v_f.

Exercise 6. Derive the optimality conditions for the problem:

i) Form the Hamiltonian function $H(X, p, C_L)$, where the states $X = [x, h, v, \gamma]$ and the costates $p = [p_x, p_h, p_v, p_\gamma]$.

ii) Find the extreme value of C_L as a function of the states and the costates. Take the boundaries of C_L into consideration. Verify the nature of the extreme value by examining the second order derivative of H.

iii) Form the costate equations either by hand or using, e.g., Mathematica.

iv) Determine the initial and terminal conditions for the states and the costates.

The necessary conditions form a two point boundary problem. It can be solved using, e.g., a multiple shooting method. Methods, where the necessary conditions are solved, are generally referred to as indirect methods. In this assignment, we do not solve the conditions. Instead, we discretize the dynamic optimization problem in time and solve the resulting nonlinear optimization problem with nonlinear programming. Methods like this are referred to as direct methods. In such methods, it is only required that the state equations are fulfilled at certain points. In this assignment, the discretization method of choice is direct collocation. In the form used here, state variables are approximated by third degree piecewise polynomials and the control variable is assumed to be piecewise linear. The approximations of the state and control trajectories must be continuous. The states must also be smooth and fulfill the state equations at the centers of the discretized segments. In Exercise 7, we compare results obtained by this method to those obtained by indirect methods.

Exercise 7. The resulting nonlinear optimization problem is here solved using the sequential quadratic programming method in MATLAB's optimization toolbox. Assume that the glider is initially flying horizontally at an altitude of 50m and a velocity of 13m/s. Analyze and solve the problem of greatest increase along the x-axis for a drop of 10m in altitude, with the additional condition of a final velocity of at least 10m/s. This condition hopefully prevents the stalling of the glider.

i) Briefly explain the principles of sequential quadratic programming (SQP), see, e.g., Bazaraa, Sherali, Shetty: Nonlinear Programming, Theory and Applications.

ii) Briefly report the functionality of the MATLAB files `flight.m`, `collcon.m`, `objfun.m`, and `dy.m`. In addition, insert the state equations into `dy.m`. The MATLAB implementation first solves the problem with a small number of discretization points. Then it increases the number of points, interpolates the previous solution to provide a good initial solution, and solves the new problem. This is repeated until a specified number of discretization points is reached. This method is called continuation. It often allows for a quicker solution of the problem than starting directly with a large number of points would. In this case, the continuation parameter is the number of discretization points.

iii) Why are the state variables and the objective function scaled before the optimization? Find the answer by experimenting and from the supplemental material.

iv) Solve the problem and comment on the resulting optimal solution.

v) Compare your solution to that of the static problem in Exercise 4 and comment on the reasons for possible differences.

vi) The file `sol.txt` contains the values of the states and costates for an optimal solution found by indirect multiple shooting. The first row contains the time t, then in order the x-coordinate, the altitude h, the velocity v, the flight path angle γ, and their corresponding costates p_x, p_h, p_v, and p_γ. Compare the values of the state variables to those obtained with `flight.m`. Based on the values in the file, calculate the optimal control using the necessary conditions and compare it to the optimal control given by `flight.m`.

Exercise 8. Finally, examine how the glider should be controlled in thermal, referring to upward air flow. We describe the velocity of the rising air as a function of the x-coordinate as follows:

$$u(x) = 2.5e^{-(x-x_{A0})^2/R^2}(1-(x-x_{A0})^2/R^2).$$

Set $x_{A0} = 150$[m], $R = 100$[m]. We select the same state variables x and h as previously, but replace the other two with horizontal and vertical velocity denoted by v_x and v_h, respectively. These obey the state equations:

$$\dot{v_x} = (-L\sin\eta - D\cos\eta)/m,$$
$$\dot{v_h} = (L\cos\eta - D\sin\eta - mg)/m.$$

The total velocity relative to the air is $v_r = \sqrt{v_x^2 + (v_h - u(x))^2}$ and $\eta = \arctan((v_h - u(x))/v_x)$.

i) Show that when $u(x) = 0$, the state equations are equivalent to the earlier ones.

ii) Solve the problem using the MATLAB files `tflight.m`, `tcollcon.m`, `tobjfun.m`, and `tdy.m`. Notice that the addition of the thermal increases the time needed for solving the problem.

iii) Interpret the resulting optimal solution and comment on it.

4 Reporting

Your report must include

- An introduction containing the background and goal of the problem solving setting at hand.

- Answers to all the exercises supported by an appropriate amount of figures dealing with in particular simulations and optimal solutions. The report should be written in a professional and academic manner, including, e.g., references to the figures from the text as well as explanations of the figures. The format of the report is not as important as its content.
- A summary including comments on the methods and the models used in the assignment.
- Comments on the assignment itself and suggestions for future improvements.

Parameter values:

Combined mass of the glider and the pilot:	$m = 100\text{kg}$
Reference area of the glider:	$S = 14\text{m}^2$
Zero drag coefficient of the glider:	$C_{D_0} = 0.034$
Induced drag coefficient of the glider:	$K = 0.07$
Gravitational acceleration:	$g = 9.81\text{m/s}^2$
Air density:	$\varrho = 1.13\text{kg/m}^3$
Allowed lift coefficients:	$C_L \in [-1.4, 1.4]$